Optimal Hölder Continuity for A Class of Degenerate Elliptic Problems with An Application to Subsonic-Sonic Flows

Chunpeng Wang*

Department of Mathematics, Jilin University, Changchun 130012, P. R. China

The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong

(email: wangcp@jlu.edu.cn)

Zhouping Xin*

The Institute of Mathematical Sciences and Department of Mathematics,

The Chinese University of Hong Kong, Shatin, NT, Hong Kong

(email: zpxin@ims.cuhk.edu.hk)

Abstract

In this paper, we first study a class of elliptic equations with anisotropic boundary degeneracy. Besides establishing the existence, uniqueness and comparison principle, we obtain the optimal Hölder estimates for weak solutions by the estimates in the Campanato space. Based on such Hölder estimates, we then investigate subsonic-sonic flows with singularities at the sonic curves in a symmetric convergent nozzle with straight wall for an approximate model of the potential flow equation. It is proved that the perturbation problem of the symmetric subsonic-sonic flow is solvable and the symmetric subsonic-sonic flow is stable.

Keywords: Subsonic-sonic flow, elliptic equation with boundary degeneracy, Hölder continuity.
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1 Introduction

The motivation of this paper is to study continuous subsonic-sonic compressible Euler steady flows in a three-dimensional symmetric convergent nozzle with straight solid wall. If the flow is irrotational and is symmetric with respect to the z-axis, it satisfies the following potential flow equation in polar coordinates

$$\Big(r\rho\Big(\phi_r^2+\frac{\phi_\theta^2}{r^2}\Big)\phi_r\Big)_r+\frac{1}{r}\Big(\rho\Big(\phi_r^2+\frac{\phi_\theta^2}{r^2}\Big)\phi_\theta\Big)_\theta=0,\quad (r,\theta)\in G=(r_1,r_2)\times(0,\theta_0),\tag{1.1}$$

where $r_1<r_2<0$, $0<\theta_0<\pi/2$,

$$\rho(q^2)=\Big(1-\frac{\gamma-1}{2}q^2\Big)^{1/(\gamma-1)},\quad 0<q^2<\frac{2}{\gamma-1}\qquad(\gamma>1).$$

If the angular velocity ϕ_θ is relatively small compared with the linear velocity ϕ_r, (1.1) may be approximated by

$$\big(r\rho\big(\phi_r^2\big)\phi_r\big)_r+\frac{\rho\big(\phi_r^2\big)}{r}\phi_{\theta\theta}=0,\quad (r,\theta)\in G.\tag{1.2}$$

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We are interested in the boundary value problem of (1.2) prescribed the following boundary conditions

$$\phi_\theta(r,0)=\phi_\theta(r,\theta_0)=0, \qquad r_1<r<r_2, \tag{1.3}$$
$$\phi(r_1,\theta)=b(\theta), \qquad 0<\theta<\theta_0, \tag{1.4}$$
$$\phi_r(r_2,\theta)=c_*, \qquad 0<\theta<\theta_0, \tag{1.5}$$

where $c_*=\sqrt{2/(\gamma+1)}$ is the sonic speed. As the potential flow equation (1.1), (1.2) is also elliptic at subsonic state (where $0<\phi_r<c_*$), while degenerate at sonic state (where $\phi_r=c_*$).

If the incoming flow enters the nozzle with right angle so that b is a constant b_0, there is a unique subsonic-sonic flow ϕ_0, whose angular velocity is zero, to the problem (1.2)–(1.5). This symmetric flow is subsonic apart from the outlet and sonic at the outlet; further, it is singular at the sonic curve in the sense that the speed of the flow is only Hölder continuous and the acceleration of the flow is infinite at the sonic curve. We are interested in whether this flow is stable. That is to say, if b is a small perturbation of b_0, whether there is a subsonic-sonic flow ϕ of the problem (1.2)–(1.5). Moreover, if such a subsonic-sonic flow ϕ exists, is it singular at the sonic curve and what is the singularity, and is it sufficiently close to the symmetric flow ϕ_0? We will prove that the symmetric flow is stable and the perturbed flow possesses the same singularity at the sonic curve as the symmetric flow. The key to prove these conclusions is to establish some suitable estimates for the perturbations. Particularly, we need to establish the precise regularity of ϕ near the sonic curve $r=r_2$. To do end, we should study Hölder estimates of solutions to the following degenerate elliptic equation

$$\frac{\partial}{\partial\tau}\Big(\tau\frac{\partial V}{\partial\tau}\Big)+2\frac{\partial V}{\partial\tau}+\tau^2\frac{\partial^2 V}{\partial\theta^2}=\frac{\partial}{\partial\tau}F_1(\tau,\theta)+\tau^2\frac{\partial}{\partial\theta}F_2(\tau,\theta), \quad (\tau,\theta)\in(0,1)\times(0,\theta_0). \tag{1.6}$$

Although there are many studies on regularities of solutions to degenerate elliptic equations, Hölder continuity of solutions to (1.6) is still open. Thus, we will establish the Hölder estimates for (1.6) in the first part of this paper. More generally, we consider the following degenerate elliptic equation

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big)+\Lambda\frac{\partial u}{\partial x}(x,y)+x^\lambda\Delta_y u(x,y)=\frac{\partial}{\partial x}f_1(x,y)+x^\lambda\nabla_y\vec{f_2}(x,y),$$
$$(x,y)\in\Omega=(0,1)\times B_{R_0}(0), \tag{1.7}$$

where $\Lambda>0$, $\lambda\ge 0$, $B_{R_0}(0)$ is the ball in $\mathbb{R}^n$ centered at the origin with radius R_0, Δ_y and ∇_y are the Laplacian and gradient operators in $\mathbb{R}^n$. The equation (1.7) is degenerate on $\{0\}\times B_{R_0}(0)$, a portion of the boundary, with an anisotropic degeneracy. Corresponding to the boundary conditions (1.3)–(1.5) for subsonic-sonic flows, we prescribe the following mixed boundary conditions for (1.7)

$$\frac{\partial u}{\partial\nu}(x,y)=0, \qquad (x,y)\in(0,1)\times\partial B_{R_0}(0), \tag{1.8}$$
$$u(1,y)=g(y), \qquad y\in B_{R_0}(0), \tag{1.9}$$

where ν is the unit outer normal to $\partial\Omega$. It is noted that there is no boundary condition on $\{0\}\times B_{R_0}(0)$, where (1.7) is degenerate, according to the general theory on the linear equations of second order with nonnegative characteristic form ([25]).

The investigation of linear equations degenerating on the boundary began in the beginning of the last century. The 1951 paper of M. V. Keldyš [15], initiating a long series of papers, played a significant role in the development of the theory. It was this paper that first brought to light the fact that in the case of elliptic equations degenerating on the boundary, under definite assumptions a portion of the boundary may be free from the prescription of boundary conditions. Later, G. Fichera and O. A. Oleĭnik established general theory on the following second order linear elliptic equations with nonnegative characteristic form ([25])

$$\sum_{i,j=1}^n a_{ij}(y)\frac{\partial^2 u}{\partial y_i\partial y_j}(y)+\sum_{i=1}^n b_i(y)\frac{\partial u}{\partial y_i}(y)+c(y)u(y)=f(y), \quad y\in D,$$

where $a_{ij} \in W^{2,\infty}(D)$, $b_i \in W^{1,\infty}(D)$, $c \in L^\infty(D)$ and

$$\sum_{i,j=1}^n a_{ij}(y)\xi_i\xi_j \geq 0, \quad \xi \in \mathbb{R}^n,\ y \in D.$$

The general theory can be applied to

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big) + \Lambda\frac{\partial u}{\partial x}(x,y) + x^\lambda \Delta_y u(x,y) + c(x,y)u = f(x,y), \quad (x,y) \in \Omega \tag{1.10}$$

in the case $\lambda = 0, 1$ or $\lambda \geq 2$. It is shown that there exists a negative constant c_0 such that for any $c \leq c_0$, the weak solution of the problem (1.10), (1.8), (1.9) in some Hilbert space exists uniquely if $f \in L^2(\Omega)$. The Hilbert space is defined as follows. For $w, v \in C^1(\overline{\Omega})$, define the inner product

$$\langle w, v\rangle_{\mathcal{H}(\Omega)} = \iint_\Omega \Big(x\frac{\partial w}{\partial x}(x,y)\frac{\partial v}{\partial x}(x,y) + x^\lambda \nabla_y w(x,y)\cdot\nabla_y v(x,y) + w(x,y)v(x,y)\Big)dxdy$$
$$+\Lambda\int_{B_{R_0}(0)} w(0,y)v(0,y)dy.$$

The closure of $C^1(\overline{\Omega})$ with respect to this inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}(\Omega)}$ is denoted by $\mathcal{H}(\Omega)$, which is an Hilbert space. Therefore, the general theory can not be applied to (1.7), which is just (1.10) with $c = 0$. As to the regularity, it is shown in the general theory that the weak solution is Lipschitz continuous if c and f are Lipschitz continuous ([25]). Such a regularity is much different from the Schauder theory on uniformly elliptic equations and insufficient to study (1.2).

After the general theory by G. Fichera and O. A. Oleǐnik, many authors investigated the following second order linear elliptic equation with isotropic boundary degeneracy and with divergence form

$$\sum_{i,j=1}^n \frac{\partial}{\partial y_j}\Big(\omega(y)a_{ij}(y)\frac{\partial u}{\partial y_i}(y)\Big) + \sum_{i=1}^n b_i(y)\frac{\partial u}{\partial y_i}(y) + c(y)u(y) = f(y), \quad y \in D \subset \mathbb{R}^n, \tag{1.11}$$

where ω is a nonnegative function in D and

$$C_1|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(y)\xi_i\xi_j \leq C_2|\xi|^2, \quad \xi \in \mathbb{R}^n,\ y \in D \quad (0 < C_1 \leq C_2).$$

If $\omega \in C(\overline{D})$ and is positive in D, A. Nakaoka [23, 24] established the well-posedness for (1.11) under some structure conditions on c, similar to the L^2 theory on uniformly elliptic equations. If ω belongs to the Muckenhoupt class A_2, i.e.

$$\int_{B\bigcap D} w(y)dy \int_{B\bigcap D} w^{-1}(y)dy \leq C\big(\text{meas}(B)\big)^2 \quad \text{for any ball } B \subset \mathbb{R}^n \qquad (C > 0),$$

many authors established the Harnack's inequality for (1.11) and showed its solutions are Hölder continuous ([4, 5, 6, 7, 17, 18, 19, 26, 29]). Among these, [4, 17, 26] studied the following special case of (1.11)

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big) + x\Delta_y u(x,y) + b(x,y)\frac{\partial u}{\partial x}(x,y) = f(x,y), \quad (x,y) \in \Omega.$$

Particularly, [26] investigated the optimal Hölder continuity of solutions in the case that b is a constant greater than -1. There are few studies on degenerate elliptic equations with anisotropic degeneracy ([8, 10, 14]). In [8] and [10], the authors established the Harnack's inequality for a class of anisotropic degenerate elliptic equations related with the Sobolev-Poincaré inequality, whose typical example is the continuous Grushin-type equation

$$\frac{\partial}{\partial x}\Big((|x|^{\sigma+1} + |y|)^k\frac{\partial u}{\partial x}(x,y)\Big) + \frac{\partial}{\partial y}\Big((|x|^{\sigma+1} + |y|)^k|x|^\sigma\frac{\partial u}{\partial y}(x,y)\Big) = f(x,y), \quad (x,y) \in \mathbb{R}^2, \quad (k, \sigma > 0).$$

In [14], J. X. Hong and C. Zuily studied

$$\frac{\partial^2 u}{\partial x^2}(x,y)+x\Delta_y u(x,y)=f(x,y),\quad (x,y)\in(0,+\infty)\times\mathbb{R}^n \tag{1.12}$$

and proved that $u\in C^\infty([0,+\infty)\times\mathbb{R}^n)$ if $f\in C^\infty([0,+\infty)\times\mathbb{R}^n)$. It is noted that the boundary $\{0\}\times\mathbb{R}^n$, where (1.12) is degenerate, is noncharacteristic.

In the first part of this paper, we consider the problem of the anisotropic degenerate elliptic equation (1.7) with the boundary conditions (1.8) and (1.9). After establishing the existence, uniqueness and comparison principle, we will focus on the Hölder continuity of weak solutions. Here, since the degeneracy of (1.7) is anisotropic, the optimal Hölder continuity of weak solutions is also anisotropic. Let us clarify this anisotropy via a selfsimilar transformation. Assume that u is a weak solution of the homogeneous equation of (1.7). Then, for any $R>0$, the function

$$w(x,y)=u(R^{-1}x,R^{-(\lambda+1)/2}y)$$

is also a weak solution of this homogeneous equation. Therefore, we should choose the following seminorm to describe the Hölder continuity of weak solutions to (1.7)

$$[u]^*_{\alpha;\Omega}=\sup_{\substack{(\hat x,\hat y),(\check x,\check y)\in\Omega\\(\hat x,\hat y)\neq(\check x,\check y)}}\frac{\big(\max\{\hat x,\check x\}\big)^{(\lambda-1)\alpha/2}|u(\hat x,\hat y)-u(\check x,\check y)|}{\big(\max\{\hat x,\check x\}\big)^{(\lambda-1)\alpha/2}|\hat x-\check x|^\alpha+|\hat y-\check y|^\alpha}\qquad(0<\alpha<1).$$

By using the estimates in the Campanato space, we establish the optimal Hölder estimates of weak solutions by the above anisotropic seminorm. Based on such Hölder estimates, we then investigate the subsonic-sonic flow problem (1.2)–(1.5) in the second part of this paper. It is proved that the perturbation problem of the symmetric subsonic-sonic flow is solvable and the symmetric subsonic-sonic flow is stable in the sense that if b is a small perturbation of b_0, there is a subsonic-sonic flow ϕ of the problem (1.2)–(1.5), which is sufficiently close to ϕ_0 and possesses the same singularity at the sonic curve as ϕ_0.

The paper is arranged as follows. In §2 and §3, we prove the well-posedness and establish the optimal Hölder estimates for the problem (1.7)–(1.9). Subsequently, we solve the problem (1.2)–(1.5) in §4.

2 Well-posedness of boundary value problem

In this section, we investigate the well-posedness of the boundary value problem (1.7)–(1.9).

2.1 Definition of weak solutions

For (1.7), we may also define weak solutions in $\mathcal{H}(\Omega)$ as for (1.10). Indeed, set

$$\mathcal{H}_\omega(\Omega)=\Big\{w\in H^1_{\rm loc}(\Omega):\iint_\Omega\Big(x^\omega\Big(\frac{\partial w}{\partial x}(x,y)\Big)^2+x^{\omega+\lambda-1}|\nabla_y w(x,y)|^2+x^{\omega-2}w^2(x,y)\Big)dxdy<+\infty\Big\}$$

with $\omega>1$. This is an Hilbert space with the inner product

$$\langle w,v\rangle_{\mathcal{H}_\omega(\Omega)}=\iint_\Omega\Big(x^\omega\frac{\partial w}{\partial x}(x,y)\frac{\partial v}{\partial x}(x,y)+x^{\omega+\lambda-1}\nabla_y w(x,y)\cdot\nabla_y v(x,y)+x^{\omega-2}w(x,y)v(x,y)\Big)dxdy.$$

We can define weak solutions in $\mathcal{H}_\omega(\Omega)$ as follows.

Definition 2.1 *Assume that $f_1,\vec f_2\in L^1_{\rm loc}(0,1;L^1(B_{R_0}(0)))$ and $\vec f_2$ is piecewise continuous near $(0,1)\times\partial B_{R_0}(0)$. A function $u\in\mathcal{H}_\omega(\Omega)$ with $\omega>1$ is said to be a weak solution to the problem (1.7)–(1.9) if*

$$\begin{aligned}&\iint_\Omega\Big(x\frac{\partial u}{\partial x}(x,y)\frac{\partial\psi}{\partial x}(x,y)-\Lambda\frac{\partial u}{\partial x}(x,y)\psi(x,y)+x^\lambda\nabla_y u(x,y)\cdot\nabla_y\psi(x,y)\Big)dxdy\\=&\iint_\Omega\Big(f_1(x,y)\frac{\partial\psi}{\partial x}(x,y)+x^\lambda\vec f_2(x,y)\cdot\nabla_y\psi(x,y)\Big)dxdy-\int_0^1\int_{\partial B_{R_0}(0)}x^\lambda\vec f_2(x,y)\cdot\nu\psi(x,y)dSdx\end{aligned}$$

for any $\psi\in C^1(\overline{\Omega})$ vanishing near $x=0$ and $x=1$, and (1.9) holds in the sense of trace.

2.2 Uniqueness and existence of weak solutions

We first establish the uniqueness of weak solutions for the problem (1.7)–(1.9).

Theorem 2.1 *The problem* (1.7)–(1.9) *admits at most one weak solution in* $\mathcal{H}_{2\Lambda+1}(\Omega)$.

Proof. Let $u_1, u_2 \in \mathcal{H}_{2\Lambda+1}(\Omega)$ be two weak solutions of the problem (1.7)–(1.9). Set

$$u(x,y) = u_1(x,y) - u_2(x,y), \quad (x,y) \in \overline{\Omega}.$$

Then, $u \in \mathcal{H}_{2\Lambda+1}(\Omega)$ with $u(1,\cdot)\Big|_{B_{R_0}(0)} = 0$ in the sense of trace and

$$\iint_\Omega \Big(x\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x} - \Lambda \frac{\partial u}{\partial x}\psi + x^\lambda \nabla_y u \cdot \nabla_y \psi \Big) dxdy = 0 \tag{2.1}$$

for any $\psi \in C^1(\overline{\Omega})$ vanishing near $x=0$ and $x=1$.

For any $0<\delta<1$, let $\eta_\delta \in C^\infty([0,1])$ satisfying $\eta_\delta \equiv 1$ on $[\delta,1]$, $\eta_\delta \equiv 0$ on $[0,\delta/2]$ and

$$0 \le \eta_\delta(x) \le 1, \quad 0 \le \eta'_\delta(x) \le \frac{4}{\delta}, \quad 0 \le x \le 1.$$

Taking $\psi = x^{2\Lambda}\eta_\delta u$ in (2.1), one can get by an approximating process that

$$\begin{aligned} &\iint_\Omega \Big(x^{2\Lambda+1}\eta_\delta \Big(\frac{\partial u}{\partial x}\Big)^2 - \frac{\Lambda}{2} x^{2\Lambda}\eta_\delta \frac{\partial (u^2)}{\partial x} + x^{2\Lambda+\lambda}\eta_\delta |\nabla_y u|^2 \Big) dxdy \\ &+ 2\Lambda \iint_\Omega x^{2\Lambda}\eta_\delta u \frac{\partial u}{\partial x} dxdy + \iint_\Omega x^{2\Lambda+1}\eta'_\delta u \frac{\partial u}{\partial x} dxdy = 0. \end{aligned}$$

Integrating by parts leads to

$$\iint_\Omega x^{2\Lambda}\eta_\delta \frac{\partial (u^2)}{\partial x} dxdy = -\iint_\Omega \big(x^{2\Lambda}\eta'_\delta + 2\Lambda x^{2\Lambda-1}\eta_\delta\big) u^2 dxdy \le -2\Lambda \iint_\Omega x^{2\Lambda-1}\eta_\delta u^2 dxdy.$$

Therefore,

$$\begin{aligned} &\iint_\Omega \Big(x^{2\Lambda+1}\eta_\delta \Big(\frac{\partial u}{\partial x}\Big)^2 + \Lambda^2 x^{2\Lambda-1}\eta_\delta u^2 + x^{2\Lambda+\lambda}\eta_\delta |\nabla_y u|^2 \Big) dxdy \\ \le & - 2\Lambda \iint_\Omega x^{2\Lambda}\eta_\delta u \frac{\partial u}{\partial x} dxdy - \iint_\Omega x^{2\Lambda+1}\eta'_\delta u \frac{\partial u}{\partial x} dxdy. \end{aligned} \tag{2.2}$$

It follows from the Cauchy inequality, the Hölder inequality and $u \in \mathcal{H}_{2\Lambda+1}(\Omega)$ that

$$\Big| 2\Lambda \iint_\Omega x^{2\Lambda}\eta_\delta u \frac{\partial u}{\partial x} dxdy \Big| \le \iint_\Omega x^{2\Lambda+1}\eta_\delta \Big(\frac{\partial u}{\partial x}\Big)^2 dxdy + \Lambda^2 \iint_\Omega x^{2\Lambda-1}\eta_\delta u^2 dxdy \tag{2.3}$$

and

$$\begin{aligned} &\Big| \iint_\Omega x^{2\Lambda+1}\eta'_\delta u \frac{\partial u}{\partial x} dxdy \Big| \\ \le & \|x\eta'_\delta(x)\|_{L^\infty(\delta/2,\delta)} \Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)} x^{2\Lambda+1}\Big(\frac{\partial u}{\partial x}\Big)^2 dxdy \Big)^{1/2} \Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)} x^{2\Lambda-1} u^2 dxdy \Big)^{1/2} \\ \le & 4 \Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)} x^{2\Lambda+1}\Big(\frac{\partial u}{\partial x}\Big)^2 dxdy \Big)^{1/2} \Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)} x^{2\Lambda-1} u^2 dxdy \Big)^{1/2} \\ \longrightarrow & 0, \qquad \text{as} \quad \delta \to 0^+. \end{aligned} \tag{2.4}$$

Letting $\delta \to 0^+$ in (2.2) and using (2.3) and (2.4), we get that

$$\iint_\Omega x^{2\Lambda+\lambda}|\nabla_y u|^2 dxdy \le 0,$$

which implies

$$u(x,y) = w(x), \quad \text{a.e. } (x,y) \in \Omega$$

with w being the solution of

$$(xw'(x))' + \Lambda w'(x) = 0, \quad 0 < x < 1, \qquad w(1) = 0.$$

Therefore,

$$u(x,y) = C(1 - x^{-\Lambda}), \quad \text{a.e. } (x,y) \in \Omega \qquad (C \in \mathbb{R}).$$

Due to $u \in \mathcal{H}_{2\Lambda+1}(\Omega)$, we get that $C = 0$ and

$$u(x,y) = 0, \quad \text{a.e. } (x,y) \in \Omega.$$

The proof is complete. □

Remark 2.1 *The space $\mathcal{H}_{2\Lambda+1}(\Omega)$ is optimal for the uniqueness of the weak solution to the problem* (1.7)–(1.9)*. For example,*

$$u(x,y) = C(1 - x^{-\Lambda}), \quad (x,y) \in \overline{\Omega} \qquad (C \in \mathbb{R}),$$

which belongs to $\mathcal{H}_\omega(\Omega)$ for $\omega > 2\Lambda + 1$, is a weak solution to the problem (1.7)–(1.9) *with*

$$g \equiv 0, \quad f_1 \equiv 0, \quad \vec{f_2} \equiv 0.$$

Now, let us turn to the existence theorem for the problem (1.7)–(1.9). According to Remark 2.1, there may exists infinite weak solutions in $\mathcal{H}_\omega(\Omega)$ with $\omega > 2\Lambda+1$. Therefore, we just study the existence of weak solutions in $\mathcal{H}_\omega(\Omega)$ with $1 < \omega \le 2\Lambda + 1$.

Theorem 2.2 *Let $1 < \omega < 2\Lambda + 1$. Assume that $g \in H^1(B_{R_0}(0))$, $x^{(\omega-2)/2} f_1, x^{(\omega+\lambda-1)/2}\vec{f_2} \in L^2(\Omega)$, $\vec{f_2}$ is piecewise continuous near $(0,1) \times \partial B_{R_0}(0)$ and $x^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f_2} \in L^2((0,1) \times \partial B_{R_0}(0))$, where $(\lambda-1)_- = 0$ if $\lambda \ge 1$ while $(\lambda-1)_- = 1-\lambda$ if $0 \le \lambda < 1$. Then there exists at least one weak solution $u \in \mathcal{H}_\omega(\Omega)$ to the problem* (1.7)–(1.9)*.*

Proof. Without loss of generality, we assume that $g \equiv 0$.

For any $0 < \varepsilon < 1$, consider the following approximating problem

$$\frac{\partial}{\partial x}\Big((x+\varepsilon)\frac{\partial u_\varepsilon}{\partial x}\Big) + \Lambda\frac{\partial u_\varepsilon}{\partial x} + (x+\varepsilon)^\lambda \Delta_y u_\varepsilon = \frac{\partial}{\partial x} f_{1,\varepsilon}(x,y) + (x+\varepsilon)^\lambda \nabla_y \vec{f}_{2,\varepsilon}(x,y), \quad (x,y) \in \Omega, \tag{2.5}$$

$$\frac{\partial u_\varepsilon}{\partial \nu}(x,y) = 0, \qquad (x,y) \in (0,1) \times \partial B_{R_0}(0), \tag{2.6}$$

$$u_\varepsilon(1,y) = 0, \qquad y \in B_{R_0}(0), \tag{2.7}$$

$$\frac{\partial u_\varepsilon}{\partial x}(0,y) = 0, \qquad y \in B_{R_0}(0), \tag{2.8}$$

where $f_{1,\varepsilon} \in C_0^\infty(\Omega)$ and $\vec{f}_{2,\varepsilon} \in C^\infty(\overline{\Omega})$ satisfy

$$\|(x+\varepsilon)^{(\omega-2)/2} f_{1,\varepsilon}\|_{L^2(\Omega)} \le \|x^{(\omega-2)/2} f_1\|_{L^2(\Omega)}, \quad \|(x+\varepsilon)^{(\omega+\lambda-1)/2}\vec{f}_{2,\varepsilon}\|_{L^2(\Omega)} \le \|x^{(\omega+\lambda-1)/2}\vec{f_2}\|_{L^2(\Omega)}, \tag{2.9}$$

$$\|(x+\varepsilon)^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f}_{2,\varepsilon}\|_{L^2((0,1)\times\partial B_{R_0}(0))} \le \|x^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f_2}\|_{L^2((0,1)\times\partial B_{R_0}(0))}, \tag{2.10}$$

$$\lim_{\varepsilon\to 0^+}\iint_\Omega x^{\omega-2}(f_{1,\varepsilon}-f_1)^2dxdy=0,\quad \lim_{\varepsilon\to 0^+}\iint_\Omega x^{\omega+\lambda-1}|\vec{f}_{2,\varepsilon}-\vec{f}_2|^2dxdy=0 \tag{2.11}$$

and

$$\lim_{\varepsilon\to 0^+}\int_0^1\int_{\partial B_{R_0}(0)} x^{\omega+\lambda-1-(\lambda-1)_-}|\vec{f}_{2,\varepsilon}-\vec{f}_2|^2dSdx=0. \tag{2.12}$$

It follows from the classical theory on the uniformly elliptic equations that the problem (2.5)–(2.8) admits a unique classical solution $u_\varepsilon\in C^\infty(\overline{\Omega})$. Multiplying (2.5) on both sides by $-(x+\varepsilon)^{\omega-1}u_\varepsilon$, then integrating by parts over Ω with (2.6)–(2.8), we obtain that

$$\begin{aligned}
&\iint_\Omega \Big((x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2-\frac{\Lambda}{2}(x+\varepsilon)^{\omega-1}\frac{\partial(u_\varepsilon^2)}{\partial x}+(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2\Big)dxdy\\
=&\iint_\Omega\Big(-(\omega-1)(x+\varepsilon)^{\omega-1}u_\varepsilon\frac{\partial u_\varepsilon}{\partial x}+(x+\varepsilon)^{\omega-1}f_{1,\varepsilon}\frac{\partial u_\varepsilon}{\partial x}\\
&\qquad+(\omega-1)(x+\varepsilon)^{\omega-2}f_{1,\varepsilon}u_\varepsilon+(x+\varepsilon)^{\omega+\lambda-1}\vec{f}_{2,\varepsilon}\cdot\nabla_y u_\varepsilon\Big)dxdy\\
&\qquad-\int_0^1\int_{\partial B_{R_0}(0)}(x+\varepsilon)^{\omega+\lambda-1}\vec{f}_{2,\varepsilon}\cdot\nu u_\varepsilon dSdx.
\end{aligned} \tag{2.13}$$

Integrating by parts gives

$$\begin{aligned}
\iint_\Omega(x+\varepsilon)^{\omega-1}\frac{\partial(u_\varepsilon^2)}{\partial x}dxdy=&\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega-1}u_\varepsilon^2(x,y)dy\Big|_{x=0}^{x=1}-(\omega-1)\iint_\Omega(x+\varepsilon)^{\omega-2}u_\varepsilon^2dxdy\\
\le&-(\omega-1)\iint_\Omega(x+\varepsilon)^{\omega-2}u_\varepsilon^2dxdy.
\end{aligned} \tag{2.14}$$

It follows from the Hölder inequality and the trace theorem that

$$\begin{aligned}
&\Big|\int_0^1\int_{\partial B_{R_0}(0)}(x+\varepsilon)^{\omega+\lambda-1}\vec{f}_{2,\varepsilon}\cdot\nu u_\varepsilon dSdx\Big|\\
\le&\big\|(x+\varepsilon)^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f}_{2,\varepsilon}\big\|_{L^2((0,1)\times\partial B_{R_0}(0))}\big\|(x+\varepsilon)^{(\omega+\lambda-1)/2+(\lambda-1)_-/2}u_\varepsilon\big\|_{L^2((0,1)\times\partial B_{R_0}(0))}\\
\le&M_1\big\|(x+\varepsilon)^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f}_{2,\varepsilon}\big\|_{L^2((0,1)\times\partial B_{R_0}(0))}\big\|(x+\varepsilon)^{(\omega+\lambda-1)/2+(\lambda-1)_-/2}u_\varepsilon\big\|_{H^1((0,1)\times\partial B_{R_0}(0))}\\
\le&M_2\big\|(x+\varepsilon)^{(\omega+\lambda-1)/2-(\lambda-1)_-/2}\vec{f}_{2,\varepsilon}\big\|_{L^2((0,1)\times\partial B_{R_0}(0))}\\
&\cdot\Big(\iint_\Omega\Big((x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2+(x+\varepsilon)^{\omega-2}u_\varepsilon^2\Big)dxdy\Big)^{1/2},
\end{aligned} \tag{2.15}$$

where $M_1,M_2>0$ depend only on ω, λ and R_0. Substitute (2.14) and (2.15) into (2.13) and use the Cauchy inequality to get

$$\begin{aligned}
&\iint_\Omega\Big((x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+\frac{\Lambda}{2}(\omega-1)(x+\varepsilon)^{\omega-2}u_\varepsilon^2+(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2\Big)dxdy\\
\le&(1-3\tau)\iint_\Omega(x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2dxdy+\frac{(\omega-1)^2}{4(1-3\tau)}\iint_\Omega(x+\varepsilon)^{\omega-2}u_\varepsilon^2dxdy\\
&+\tau\iint_\Omega(x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2dxdy+\frac{1}{4\tau}\iint_\Omega(x+\varepsilon)^{\omega-2}f_{1,\varepsilon}^2dxdy\\
&+\tau(\omega-1)\iint_\Omega(x+\varepsilon)^{\omega-2}u_\varepsilon^2dxdy+\frac{\omega-1}{4\tau}\iint_\Omega(x+\varepsilon)^{\omega-2}f_{1,\varepsilon}^2dxdy\\
&+\frac12\iint_\Omega(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2dxdy+\frac12\iint_\Omega(x+\varepsilon)^{\omega+\lambda-1}|\vec{f}_{2,\varepsilon}|^2dxdy
\end{aligned}$$

$$+\tau\iint_\Omega\Big((x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2+(x+\varepsilon)^{\omega-2}u_\varepsilon^2\Big)dxdy$$
$$+\frac{M_2^2}{4\tau}\int_0^1\int_{\partial B_{R_0}(0)}(x+\varepsilon)^{\omega+\lambda-1-(\lambda-1)_-}|\vec{f}_{2,\varepsilon}|^2dSdx \tag{2.16}$$

for any $0<\tau<1/3$. Owing to $1<\omega<2\Lambda+1$, there exists $\tau\in(0,1/3)$ such that

$$\frac{(\omega-1)^2}{4(1-3\tau)}+\tau(\omega-1)+\tau<\frac{\Lambda}{2}(\omega-1).$$

Then, combining (2.16) with (2.9) and (2.10) yields that

$$\iint_\Omega\Big((x+\varepsilon)^\omega\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\omega+\lambda-1}|\nabla_y u_\varepsilon|^2+(x+\varepsilon)^{\omega-2}u_\varepsilon^2\Big)dxdy$$
$$\le M_0\iint_\Omega\Big((x+\varepsilon)^{\omega-2}f_{1,\varepsilon}^2+(x+\varepsilon)^{\omega+\lambda-1}|\vec{f}_{2,\varepsilon}|^2\Big)dxdy+M_0\int_0^1\int_{\partial B_{R_0}(0)}(x+\varepsilon)^{\omega+\lambda-1-(\lambda-1)_-}|\vec{f}_{2,\varepsilon}|^2dSdx$$
$$\le M_0\iint_\Omega\Big(x^{\omega-2}f_1^2+x^{\omega+\lambda-1}|\vec{f}_2|^2\Big)dxdy+M_0\int_0^1\int_{\partial B_{R_0}(0)}x^{\omega+\lambda-1-(\lambda-1)_-}|\vec{f}_2|^2dSdx \tag{2.17}$$

with $M_0>0$ depending only on Λ, ω, λ and R_0.

Due to (2.17) and (2.7), there exist a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$, denoted by itself for convenience, and a function $u\in\mathcal{H}_\omega(\Omega)$ with $u(1,\cdot)\Big|_{B_{R_0}(0)}=0$ in the sense of trace such that

$$(x+\varepsilon)^{\omega/2}\frac{\partial u_\varepsilon}{\partial x}\rightharpoonup x^{\omega/2}\frac{\partial u}{\partial x}\text{ and }(x+\varepsilon)^{(\omega+\lambda-1)/2}\nabla_y u_\varepsilon\rightharpoonup x^{(\omega+\lambda-1)/2}\nabla_y u\text{ in }L^2(\Omega). \tag{2.18}$$

For any $\psi\in C^1(\overline{\Omega})$ vanishing near $x=0$ and $x=1$, multiplying (2.5) by $-\psi$ and then integrating over Ω by parts with (2.6), one gets that

$$\iint_\Omega\Big((x+\varepsilon)\frac{\partial u_\varepsilon}{\partial x}\frac{\partial\psi}{\partial x}-\Lambda\frac{\partial u_\varepsilon}{\partial x}\psi+(x+\varepsilon)^\lambda\nabla_y u_\varepsilon\cdot\nabla_y\psi\Big)dxdy$$
$$=\iint_\Omega\Big(f_{1,\varepsilon}\frac{\partial\psi}{\partial x}+(x+\varepsilon)^\lambda\vec{f}_{2,\varepsilon}\cdot\nabla_y\psi\Big)dxdy-\int_0^1\int_{\partial B_{R_0}(0)}(x+\varepsilon)^\lambda\vec{f}_{2,\varepsilon}\cdot\nu\psi dSdx.$$

Letting $\varepsilon\to0^+$, together with (2.18), (2.11) and (2.12), yields

$$\iint_\Omega\Big(x\frac{\partial u}{\partial x}\frac{\partial\psi}{\partial x}-\Lambda\frac{\partial u}{\partial x}\psi+x^\lambda\nabla_y u\cdot\nabla_y\psi\Big)dxdy$$
$$=\iint_\Omega\Big(f_1\frac{\partial\psi}{\partial x}+x^\lambda\vec{f}_2\cdot\nabla_y\psi\Big)dxdy-\int_0^1\int_{\partial B_{R_0}(0)}x^\lambda\vec{f}_2\cdot\nu\psi dSdx.$$

Therefore, u is a solution of the problem (1.7)–(1.9) with $g\equiv0$. The proof is complete. □

Remark 2.2 *Theorem 2.2 is invalid for $\omega=2\Lambda+1$. For example, if*

$$f_1(x,y)=x^{-\Lambda}\Big(-\ln\frac{x}{2}\Big)^{-3/2},\quad \vec{f}_2(x,y)=0,\quad (x,y)\in\Omega,$$

then all solutions of (1.7) are of the form

$$u(x,y)=2x^{-\Lambda}\Big(-\ln\frac{x}{2}\Big)^{-1/2}+C_1x^{-\Lambda}+C_2,\quad (x,y)\in\overline{\Omega}\qquad (C_1,C_2\in\mathbb{R})$$

Here, $x^{(2\Lambda-1)/2}f_1\in L^2(\Omega)$, while $u\notin\mathcal{H}_{2\Lambda+1}(\Omega)$ for any $C_1,C_2\in\mathbb{R}$.

2.3 Comparison principle

Consider the problem

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big)+\Lambda\frac{\partial u}{\partial x}(x,y)+x^{\lambda}\Delta_y u(x,y)\ge 0,\qquad (x,y)\in\Omega, \tag{2.19}$$

$$\frac{\partial u}{\partial \nu}(x,y)\le 0,\qquad (x,y)\in(0,1)\times\partial B_{R_0}(0), \tag{2.20}$$

$$u(1,y)\le 0,\qquad y\in B_{R_0}(0), \tag{2.21}$$

Definition 2.2 *A function $u\in\mathcal{H}_\omega(\Omega)$ with $\omega>1$ is said to be a weak solution of the problem* (2.19)–(2.21) *if*

$$\iint_\Omega\Big(x\frac{\partial u}{\partial x}(x,y)\frac{\partial \psi}{\partial x}(x,y)-\Lambda\frac{\partial u}{\partial x}(x,y)\psi(x,y)+x^{\lambda}\nabla_y u(x,y)\cdot\nabla_y\psi(x,y)\Big)dxdy\le 0$$

for any nonnegative function $\psi\in C^1(\overline{\Omega})$ vanishing near $x=0$ and $x=1$, and (2.21) *holds in the sense of trace.*

Theorem 2.3 *Let $u\in\mathcal{H}_\omega(\Omega)$ with $1<\omega<2\Lambda+1$ be a weak solution of the problem* (2.19)–(2.21)*. Then*

$$u(x,y)\le 0,\quad a.e.\ (x,y)\in\Omega. \tag{2.22}$$

Proof. The proof is based on a duality argument (see for example [28] Theorem 3.2.1). For any nonpositive function $f\in C_0^\infty(\Omega)$, consider the problem

$$\frac{\partial}{\partial x}\Big((x+\varepsilon)\frac{\partial \psi_\varepsilon}{\partial x}\Big)-\Lambda\frac{\partial \psi_\varepsilon}{\partial x}+(x+\varepsilon)^{\lambda}\Delta_y\psi_\varepsilon=f(x,y),\qquad (x,y)\in\Omega, \tag{2.23}$$

$$\frac{\partial \psi_\varepsilon}{\partial \nu}(x,y)=0,\qquad (x,y)\in(0,1)\times\partial B_{R_0}(0), \tag{2.24}$$

$$\psi_\varepsilon(1,y)=0,\qquad y\in B_{R_0}(0), \tag{2.25}$$

$$\psi_\varepsilon(0,y)=0,\qquad y\in B_{R_0}(0), \tag{2.26}$$

where $0<\varepsilon<1$. From the classical theory on the uniformly elliptic equations, the problem (2.23)–(2.26) admits a unique classical solution $\psi_\varepsilon\in C^\infty(\overline{\Omega})$.

Below we derive uniform estimates on ψ_ε and denote by M a positive constant depending only on Λ, λ, n and f, but independent of ε. Since $f\in C_0^\infty(\Omega)$ is nonpositive, it follows from the classical maximum principle that

$$0\le\psi_\varepsilon(x,y)\le M,\quad (x,y)\in\Omega. \tag{2.27}$$

This, together with the classical boundary estimate, leads to

$$-M\le\frac{\partial \psi_\varepsilon}{\partial x}(1,y)\le 0,\quad y\in B_{R_0}(0). \tag{2.28}$$

Multiplying (2.23) on both sides by $-(x+\varepsilon)^{1-\omega}\psi_\varepsilon$ and then integrating by parts over Ω with (2.24)–(2.26), we get that

$$\begin{aligned}&\iint_\Omega\Big((x+\varepsilon)^{2-\omega}\Big(\frac{\partial \psi_\varepsilon}{\partial x}\Big)^2+\frac{\Lambda}{2}(x+\varepsilon)^{1-\omega}\frac{\partial(\psi_\varepsilon^2)}{\partial x}+(x+\varepsilon)^{\lambda+1-\omega}|\nabla_y\psi_\varepsilon|^2\Big)dxdy\\ =&(\omega-1)\iint_\Omega(x+\varepsilon)^{1-\omega}\psi_\varepsilon\frac{\partial \psi_\varepsilon}{\partial x}dxdy-\iint_\Omega(x+\varepsilon)^{1-\omega}f\psi_\varepsilon dxdy.\end{aligned} \tag{2.29}$$

Integrating by parts gives

$$\iint_\Omega (x+\varepsilon)^{1-\omega}\frac{\partial(\psi_\varepsilon^2)}{\partial x}dxdy=\int_{B_{R_0}(0)}(x+\varepsilon)^{1-\omega}\psi_\varepsilon^2(x,y)dy\Big|_{x=0}^{x=1}+(\omega-1)\iint_\Omega(x+\varepsilon)^{-\omega}\psi_\varepsilon^2dxdy$$
$$=(\omega-1)\iint_\Omega(x+\varepsilon)^{-\omega}\psi_\varepsilon^2dxdy. \tag{2.30}$$

Substituting (2.30) into the left side of (2.29) and using the Cauchy inequality, one gets that

$$\begin{aligned}&\iint_\Omega\Big((x+\varepsilon)^{2-\omega}\Big(\frac{\partial\psi_\varepsilon}{\partial x}\Big)^2+\frac{\Lambda}{2}(\omega-1)(x+\varepsilon)^{-\omega}\psi_\varepsilon^2+(x+\varepsilon)^{\lambda+1-\omega}|\nabla_y\psi_\varepsilon|^2\Big)dxdy\\ \le&\frac{2(\omega-1)}{2\Lambda+\omega-1}\iint_\Omega(x+\varepsilon)^{2-\omega}\Big(\frac{\partial\psi_\varepsilon}{\partial x}\Big)^2dxdy+\frac{2\Lambda+\omega-1}{8}(\omega-1)\iint_\Omega(x+\varepsilon)^{-\omega}\psi_\varepsilon^2dxdy\\ &+\sup_\Omega|\psi_\varepsilon|\iint_\Omega(x+\varepsilon)^{1-\omega}|f|dxdy,\end{aligned}$$

which, together with (2.27), $1<\omega<2\Lambda+1$ and $f\in C_0^\infty(\Omega)$, yields

$$\iint_\Omega\Big((x+\varepsilon)^{2-\omega}\Big(\frac{\partial\psi_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{-\omega}\psi_\varepsilon^2+(x+\varepsilon)^{\lambda+1-\omega}|\nabla_y\psi_\varepsilon|^2\Big)dxdy\le M. \tag{2.31}$$

For any $0<\delta<1$, let η_δ be the function defined in the proof of Theorem 2.1. Taking $\psi=\eta_\delta\psi_\varepsilon$ in the definition of the weak solution to the problem (2.19)–(2.21) by a standard approximating process, one can get that

$$\begin{aligned}&\iint_\Omega\Big((x+\varepsilon)\eta_\delta\frac{\partial u}{\partial x}\frac{\partial\psi_\varepsilon}{\partial x}-\Lambda\eta_\delta\frac{\partial u}{\partial x}\psi_\varepsilon+(x+\varepsilon)^\lambda\eta_\delta\nabla_yu\cdot\nabla_y\psi_\varepsilon\Big)dxdy\\ \le&-\iint_\Omega x\eta_\delta'\frac{\partial u}{\partial x}\psi_\varepsilon dxdy+\varepsilon\iint_\Omega\eta_\delta\frac{\partial u}{\partial x}\frac{\partial\psi_\varepsilon}{\partial x}dxdy+\iint_\Omega((x+\varepsilon)^\lambda-x^\lambda)\eta_\delta\nabla_yu\cdot\nabla_y\psi_\varepsilon dxdy,\end{aligned}$$

which, together with (2.23)–(2.25), implies after integration by parts that

$$\begin{aligned}\iint_\Omega\eta_\delta ufdxdy=&\iint_\Omega\eta_\delta u\Big(\frac{\partial}{\partial x}\Big((x+\varepsilon)\frac{\partial\psi_\varepsilon}{\partial x}\Big)-\Lambda\frac{\partial\psi_\varepsilon}{\partial x}+(x+\varepsilon)^\lambda\Delta_y\psi_\varepsilon\Big)dxdy\\ \ge&\int_{B_{R_0}(0)}(1+\varepsilon)u(1,y)\frac{\partial\psi_\varepsilon}{\partial x}(1,y)dy-\iint_\Omega(x+\varepsilon)\eta_\delta'u\frac{\partial\psi_\varepsilon}{\partial x}dxdy\\ &+\Lambda\iint_\Omega\eta_\delta'u\psi_\varepsilon dxdy+\iint_\Omega x\eta_\delta'\frac{\partial u}{\partial x}\psi_\varepsilon dxdy-\varepsilon\iint_\Omega\eta_\delta\frac{\partial u}{\partial x}\frac{\partial\psi_\varepsilon}{\partial x}dxdy\\ &-\iint_\Omega((x+\varepsilon)^\lambda-x^\lambda)\eta_\delta\nabla_yu\cdot\nabla_y\psi_\varepsilon dxdy.\end{aligned} \tag{2.32}$$

It follows from (2.21) and (2.28) that

$$\int_{B_{R_0}(0)}(1+\varepsilon)u(1,y)\frac{\partial\psi_\varepsilon}{\partial x}(1,y)dy\ge0. \tag{2.33}$$

Using the Hölder inequality and (2.31) yields that

$$\begin{aligned}&\Big|\iint_\Omega(x+\varepsilon)\eta_\delta'u\frac{\partial\psi_\varepsilon}{\partial x}dxdy\Big|+\Big|\iint_\Omega\eta_\delta'u\psi_\varepsilon dxdy\Big|+\Big|\iint_\Omega x\eta_\delta'\frac{\partial u}{\partial x}\psi_\varepsilon dxdy\Big|\\ \le&\|(x+\varepsilon)\eta_\delta'(x)\|_{L^\infty(\delta/2,\delta)}\Big(\int_{\delta/2}^\delta\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega-2}u^2dxdy\Big)^{1/2}\Big(\int_{\delta/2}^\delta\int_{B_{R_0}(0)}(x+\varepsilon)^{2-\omega}\Big(\frac{\partial\psi_\varepsilon}{\partial x}\Big)^2dxdy\Big)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&+\|(x+\varepsilon)\eta_\delta'(x)\|_{L^\infty(\delta/2,\delta)}\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega-2}u^2dxdy\Big)^{1/2}\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{-\omega}\psi_\varepsilon^2dxdy\Big)^{1/2}\\
&+\|x\eta_\delta'(x)\|_{L^\infty(\delta/2,\delta)}\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega}\Big(\frac{\partial u}{\partial x}\Big)^2dxdy\Big)^{1/2}\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{-\omega}\psi_\varepsilon^2dxdy\Big)^{1/2}\\
\le& M\Big(1+\frac{\varepsilon}{\delta}\Big)\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega-2}u^2dxdy\Big)^{1/2}+M\Big(\int_{\delta/2}^{\delta}\int_{B_{R_0}(0)}(x+\varepsilon)^{\omega}\Big(\frac{\partial u}{\partial x}\Big)^2dxdy\Big)^{1/2} \qquad (2.34)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\varepsilon\iint_\Omega \eta_\delta\frac{\partial u}{\partial x}\frac{\partial \psi_\varepsilon}{\partial x}dxdy\Big|+\Big|\iint_\Omega((x+\varepsilon)^\lambda-x^\lambda)\eta_\delta\nabla_y u\cdot\nabla_y\psi_\varepsilon dxdy\Big|\\
\le&\sup_{(0,1)}\frac{\varepsilon(x+\varepsilon)^{\omega/2}\eta_\delta(x)}{x^{\omega/2}(x+\varepsilon)}\Big(\iint_\Omega x^\omega\Big(\frac{\partial u}{\partial x}\Big)^2dxdy\Big)^{1/2}\Big(\iint_\Omega(x+\varepsilon)^{2-\omega}\Big(\frac{\partial\psi_\varepsilon}{\partial x}\Big)^2dxdy\Big)^{1/2}\\
&+\sup_{(0,1)}\frac{((x+\varepsilon)^\lambda-x^\lambda)(x+\varepsilon)^{(\lambda+1-\omega)/2}\eta_\delta(x)}{x^{(\omega+\lambda-1)/2}}\Big(\iint_\Omega x^{\omega+\lambda-1}|\nabla_y u|^2dxdy\Big)^{1/2}\\
&\cdot\Big(\iint_\Omega(x+\varepsilon)^{\lambda+1-\omega}|\nabla_y\psi_\varepsilon|^2dxdy\Big)^{1/2}\\
\le& M\sup_{(0,1)}\frac{\varepsilon(x+\varepsilon)^{\omega/2}\eta_\delta(x)}{x^{\omega/2}(x+\varepsilon)}\Big(\iint_\Omega x^\omega\Big(\frac{\partial u}{\partial x}\Big)^2dxdy\Big)^{1/2}\\
&+M\sup_{(0,1)}\frac{((x+\varepsilon)^\lambda-x^\lambda)(x+\varepsilon)^{(\lambda+1-\omega)/2}\eta_\delta(x)}{x^{(\omega+\lambda-1)/2}}\Big(\iint_\Omega x^{\omega+\lambda-1}|\nabla_y u|^2dxdy\Big)^{1/2}. \qquad (2.35)
\end{aligned}
$$

Letting $\varepsilon\to0^+$ and then $\delta\to0^+$ in (2.32) and using (2.33)–(2.35), we get that

$$\iint_\Omega uf\,dxdy\ge0,$$

which implies (2.22) since the nonpositive function $f\in C_0^\infty(\Omega)$ is arbitrary. The proof is complete. □

Based on Theorem 2.3, one can get the global boundedness of weak solutions to the problem (1.7)–(1.9).

Theorem 2.4 *Under the conditions of Theorem 2.2, if*

$$\Big|\frac{\partial f_1}{\partial x}(x,y)+x^\lambda\nabla_y\vec{f_2}(x,y)\Big|\le Mx^{\beta-1},\quad (x,y)\in\Omega$$

with $M>0$ and $\beta>0$, then the weak solution $u\in\mathcal{H}_\omega(\Omega)$ with $1<\omega<2\Lambda+1$ of the problem (1.7)–(1.9) *is bounded and satisfies*

$$\|u\|_{L^\infty(\Omega)}\le\|g\|_{L^\infty(B_{R_0}(0))}+\frac{M}{\beta(\Lambda+\beta)}. \qquad (2.36)$$

Proof. It follows from Theorem 2.3 that

$$u(x,y)\le\|g\|_{L^\infty(B_{R_0}(0))}+\frac{M}{\beta(\Lambda+\beta)}-\frac{M}{\beta(\Lambda+\beta)}x^\beta,\quad (x,y)\in\Omega$$

and

$$u(x,y)\ge-\|g\|_{L^\infty(B_{R_0}(0))}-\frac{M}{\beta(\Lambda+\beta)}+\frac{M}{\beta(\Lambda+\beta)}x^\beta,\quad (x,y)\in\Omega.$$

These two estimates imply (2.36) and complete the proof of the theorem. □

3 Hölder continuity of weak solutions

We now concentrate on the regularity of weak solutions to the problem (1.7)–(1.9) in this section. As an example, we choose $\mathcal{H}_2(\Omega)$ as the class of weak solutions. According to Remark 2.1, Λ should satisfy $\Lambda > 1/2$.

3.1 Estimates in Campanato space for weak solutions to the homogeneous equation

In this subsection, we do estimates in Campanato space for weak solutions to the homogeneous equation. First consider the regularity near the boundary $\{0\} \times B_{R_0}(0)$, where (1.7) is degenerate. Moreover, the degeneracy along the x direction and the degeneracy along the y direction is different. That is to say, the degeneracy is anisotropic. Let us clarify this difference via a selfsimilar transformation. Assume that u is a weak solution of the homogeneous equation

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big) + \Lambda\frac{\partial u}{\partial x}(x,y) + x^\lambda \Delta_y u(x,y) = 0. \tag{3.1}$$

Then, for any $R > 0$, the function

$$w(x,y) = u(R^{-1}x, R^{-(\lambda+1)/2}y)$$

is also a weak solution of (3.1). Therefore, the Hölder continuity of weak solutions to (3.1) and (1.7) is also anisotropic, and the standard rectangles should be substituted by the following anisotropic ones in studying this continuity

$$Q_R(0,\bar y) = (-R,R) \times B_{R^{(\lambda+1)/2}}(\bar y), \quad \Omega_R(0,\bar y) = \Omega \cap Q_R(0,\bar y), \tag{3.2}$$

where $\bar y \in \mathbb{R}^n$ and $R > 0$. However, it is not convenient to investigate (3.1) directly since the equation is degenerate. Thus, we consider the approximating equations.

Lemma 3.1 *Assume that $\Lambda > 1/2$, $\lambda \geq 0$, $\varepsilon > 0$, $\bar y \in B_{R_0/2}(0)$ and $0 < \bar R \leq \min\big\{1, (R_0/2)^{2/(\lambda+1)}\big\}$. Let $u_\varepsilon \in C^\infty(\overline{\Omega}_{\bar R}(0,\bar y))$ be a solution of*

$$\frac{\partial}{\partial x}\Big((x+\varepsilon)\frac{\partial u_\varepsilon}{\partial x}(x,y)\Big) + \Lambda\frac{\partial u_\varepsilon}{\partial x}(x,y) + (x+\varepsilon)^\lambda \Delta_y u_\varepsilon(x,y) = 0, \quad (x,y) \in \Omega_{\bar R}(0,\bar y) \tag{3.3}$$

satisfying

$$\frac{\partial u_\varepsilon}{\partial x}(0,y) = 0, \quad y \in B_{\bar R^{(\lambda+1)/2}}(0,\bar y). \tag{3.4}$$

Then for any $0 < \varrho < R \leq \bar R$,

$$\begin{aligned}
&\iint_{\Omega_\varrho(0,\bar y)} \Big(\big(u_\varepsilon - (u_\varepsilon)_{\varrho,(0,\bar y)}\big)^2 + (x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2 + (x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon - (\nabla_y u_\varepsilon)_{\varrho,(0,\bar y)}\big|^2\Big)dxdy \\
\leq & M\Big(1+\frac{\varepsilon}{\varrho}\Big)^\kappa\Big(1+\frac{\varepsilon}{R}\Big)^\kappa\Big(\frac{\varrho}{R}\Big)^{2+n(\lambda+1)/2+\min\{1,\lambda\}} \iint_{\Omega_R(0,\bar y)} \Big(\big(u_\varepsilon - (u_\varepsilon)_{R,(0,\bar y)}\big)^2 + (x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2 \\
&\qquad + (x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon - (\nabla_y u_\varepsilon)_{R,(0,\bar y)}\big|^2\Big)dxdy,
\end{aligned} \tag{3.5}$$

where $\kappa > 0$ and $M > 0$ depend only on Λ, λ, n, but not on ε, and

$$(w)_{R,(0,\bar y)} = \frac{1}{\operatorname{meas}\big(\Omega_R(0,\bar y)\big)}\iint_{\Omega_R(0,\bar y)} w(x,y)dxdy.$$

Proof. For convenience, set

$$Q_R = Q_R(0,\bar{y}), \quad \Omega_R = \Omega_R(0,\bar{y}), \quad (w)_R = (w)_{R,(0,\bar{y})},$$

M to be a generic positive constant depending only on Λ, λ and n, while $M(\cdot)$ depending also on the variables in the parentheses. Additionally, for nonnegative integer k, ∂_y^k will denote the k-th order partial derivatives with respect to the variation y. The proof of the lemma falls into four steps.

Step I. Weighted Caccioppoli inequalities.

For any $0<\varrho<R\le\bar{R}$, let $\xi\in C_0^\infty(Q_R)$ satisfying $\xi\equiv 1$ in Q_ϱ,

$$0\le \xi(x,y)\le 1, \quad \Big|\frac{\partial\xi}{\partial x}(x,y)\Big|\le \frac{2}{R-\varrho}, \quad (x,y)\in Q_R,$$

$$|\nabla_y\xi(x,y)|\le \frac{2}{R^{(\lambda+1)/2}-\varrho^{(\lambda+1)/2}}\le \frac{4}{R^{(\lambda-1)/2}(R-\varrho)}, \quad (x,y)\in Q_R$$

and

$$\frac{\partial\xi}{\partial x}(x,y)=0, \quad (x,y)\in(-\varrho,\varrho)\times B_{R^{(\lambda+1)/2}}(\bar{y}).$$

First, we estimate u_ε. For any $L\in\mathbb{R}$, multiplying (3.3) on both sides by $-(x+\varepsilon)\xi^2(u_\varepsilon-L)$, then integrating by parts over Ω_R and using (3.4) and the Cauchy inequality yield

$$\begin{aligned}
&\iint_{\Omega_R}\Big((x+\varepsilon)^2\xi^2\Big(\frac{\partial(u_\varepsilon-L)}{\partial x}\Big)^2+\frac{\Lambda}{2}\xi^2(u_\varepsilon-L)^2+(x+\varepsilon)^{\lambda+1}\xi^2|\nabla_y(u_\varepsilon-L)|^2\Big)dxdy\\
&\quad+\frac{\Lambda}{2}\varepsilon\int_{B_{R^{(\lambda+1)/2}}(\bar{y})}\xi^2(0,y)(u_\varepsilon(0,y)-L)^2dy\\
=&-2\iint_{\Omega_R}(x+\varepsilon)^2\xi(u_\varepsilon-L)\frac{\partial(u_\varepsilon-L)}{\partial x}\frac{\partial\xi}{\partial x}dxdy-\iint_{\Omega_R}(x+\varepsilon)\xi^2(u_\varepsilon-L)\frac{\partial(u_\varepsilon-L)}{\partial x}dxdy\\
&\quad-\Lambda\iint_{\Omega_R}(x+\varepsilon)\xi(u_\varepsilon-L)^2\frac{\partial\xi}{\partial x}dxdy-2\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}\xi(u_\varepsilon-L)\nabla_y(u_\varepsilon-L)\cdot\nabla_y\xi dxdy\\
\le&\tau\iint_{\Omega_R}(x+\varepsilon)^2\xi^2\Big(\frac{\partial(u_\varepsilon-L)}{\partial x}\Big)^2dxdy+\frac{1}{\tau}\iint_{\Omega_R}(x+\varepsilon)^2\Big(\frac{\partial\xi}{\partial x}\Big)^2(u_\varepsilon-L)^2dxdy\\
&\quad+(1-\tau)\iint_{\Omega_R}(x+\varepsilon)^2\xi^2\Big(\frac{\partial(u_\varepsilon-L)}{\partial x}\Big)^2dxdy+\frac{1}{4(1-\tau)}\iint_{\Omega_R}\xi^2(u_\varepsilon-L)^2dxdy\\
&\quad+\tau\iint_{\Omega_R}\xi^2(u_\varepsilon-L)^2dxdy+\frac{\Lambda^2}{4\tau}\iint_{\Omega_R}(x+\varepsilon)^2\Big(\frac{\partial\xi}{\partial x}\Big)^2(u_\varepsilon-L)^2dxdy\\
&\quad+\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}\xi^2|\nabla_y(u_\varepsilon-L)|^2dxdy+\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}|\nabla_y\xi|^2(u_\varepsilon-L)^2dxdy
\end{aligned}$$

for any $0<\tau<1$. Due to $\Lambda>1/2$, there exists $\tau\in(0,1)$ such that

$$\frac{1}{4(1-\tau)}+\tau<\frac{\Lambda}{2}.$$

Therefore, for any $0<\varrho<R\le\bar{R}$, it holds that

$$\begin{aligned}
&\iint_{\Omega_\varrho}(u_\varepsilon-L)^2dxdy\\
\le&M\int_\varrho^R\int_{B_{R^{(\lambda+1)/2}}(\bar{y})}(x+\varepsilon)^2\Big(\frac{\partial\xi}{\partial x}\Big)^2(u_\varepsilon-L)^2dxdy+M\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}|\nabla_y\xi|^2(u_\varepsilon-L)^2dxdy\\
\le&\frac{M(R+\varepsilon)}{(\varrho+\varepsilon)^\lambda(R-\varrho)^2}\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}(u_\varepsilon-L)^2dxdy+\frac{M}{R^{\lambda-1}(R-\varrho)^2}\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}(u_\varepsilon-L)^2dxdy
\end{aligned}$$

$$\leq \frac{M}{(R-\varrho)^2}\Big(\frac{(R+\varepsilon)}{(\varrho+\varepsilon)^\lambda}+\frac{1}{R^{\lambda-1}}\Big)\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}(u_\varepsilon-L)^2dxdy. \tag{3.6}$$

Next, we estimate $\dfrac{\partial u_\varepsilon}{\partial x}$. Set

$$v_\varepsilon(x,y)=\frac{\partial u_\varepsilon}{\partial x}(x,y),\quad (x,y)\in\overline{\Omega}_{\bar R}.$$

Then, $v_\varepsilon\in C^\infty(\overline{\Omega}_{\bar R})$ is a solution of

$$(x+\varepsilon)\frac{\partial^2 v_\varepsilon}{\partial x^2}+(x+\varepsilon)^\lambda\Delta_y v_\varepsilon+(\Lambda+2-\lambda)\frac{\partial v_\varepsilon}{\partial x}-\lambda(\Lambda+1)(x+\varepsilon)^{-1}v_\varepsilon=0,\quad (x,y)\in\Omega_{\bar R} \tag{3.7}$$

satisfying

$$v_\varepsilon(0,y)=0,\quad y\in B_{\bar R^{(\lambda+1)/2}}(\bar y). \tag{3.8}$$

Assume that $-3<l\leq 2\lambda-1$. Then,

$$l^2<2\big(\lambda(l+2)+\Lambda(2\lambda-l)-l\big) \tag{3.9}$$

owing to $\Lambda>1/2$ and $\lambda\geq 0$. Multiplying (3.7) on both sides by $-(x+\varepsilon)^{-l}\xi^2v_\varepsilon$ and then integrating by parts over Ω_R with (3.8) lead to

$$\begin{aligned}&\iint_{\Omega_R}\Big((x+\varepsilon)^{-l+1}\xi^2\Big(\frac{\partial v_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{-l+\lambda}\xi^2|\nabla_y v_\varepsilon|^2\\&\qquad+\frac12\big(\lambda(l+2)+\Lambda(2\lambda-l)-l\big)(x+\varepsilon)^{-l-1}\xi^2v_\varepsilon^2\Big)dxdy\\=&-2\iint_{\Omega_R}(x+\varepsilon)^{-l+1}\xi v_\varepsilon\frac{\partial v_\varepsilon}{\partial x}\frac{\partial\xi}{\partial x}dxdy-2\iint_{\Omega_R}(x+\varepsilon)^{-l+\lambda}\xi v_\varepsilon\nabla_y v_\varepsilon\cdot\nabla_y\xi dxdy\\&+(\lambda-\Lambda-1)\iint_{\Omega_R}(x+\varepsilon)^{-l}\xi\frac{\partial\xi}{\partial x}v_\varepsilon^2dxdy+l\iint_{\Omega_R}(x+\varepsilon)^{-l}\xi^2v_\varepsilon\frac{\partial v_\varepsilon}{\partial x}dxdy.\end{aligned}$$

Similar to the proof of (3.6), using the Cauchy inequality and (3.9), one can get that for any $0<\varrho<R\leq\bar R$,

$$\begin{aligned}&\iint_{\Omega_\varrho}\Big((x+\varepsilon)^{-l+1}\Big(\frac{\partial v_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{-l+\lambda}|\nabla_y v_\varepsilon|^2+(x+\varepsilon)^{-l-1}v_\varepsilon^2\Big)dxdy\\\leq&M\iint_{\Omega_R}(x+\varepsilon)^{-l+1}\Big(\frac{\partial\xi}{\partial x}\Big)^2v_\varepsilon^2dxdy+M\iint_{\Omega_R}(x+\varepsilon)^{-l+\lambda}|\nabla_y\xi|^2v_\varepsilon^2dxdy\\\leq&\frac{M(R+\varepsilon)}{(R-\varrho)^2}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)\iint_{\Omega_R}(x+\varepsilon)^{-l}v_\varepsilon^2dxdy.\end{aligned} \tag{3.10}$$

Denote by m the integer satisfying

$$2\lambda+2\leq m<2\lambda+3.$$

For any $0<\varrho<R\leq\bar R$, taking $l=2\lambda-1,\ 2\lambda-2,\ \cdots,\ 2\lambda-m$ in (3.10) with suitable ϱ and R and iterating, one can get

$$\begin{aligned}&\iint_{\Omega_\varrho}\Big((x+\varepsilon)^{-2\lambda+2}\Big(\frac{\partial v_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{-\lambda+1}|\nabla_y v_\varepsilon|^2+(x+\varepsilon)^{-2\lambda}v_\varepsilon^2\Big)dxdy\\\leq&\frac{M(R+\varepsilon)^m}{(R-\varrho)^{2m}}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}(x+\varepsilon)^{m-2\lambda}v_\varepsilon^2dxdy\end{aligned}$$

$$\leq \frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy. \tag{3.11}$$

Finally, we estimate $\Delta_y u_\varepsilon$ and $\nabla_y u_\varepsilon$. Rewrite (3.3) as

$$\Delta_y u_\varepsilon = -(x+\varepsilon)^{-\lambda+1}\frac{\partial v_\varepsilon}{\partial x} - (\Lambda+1)(x+\varepsilon)^{-\lambda} v_\varepsilon, \quad (x,y)\in\Omega_R. \tag{3.12}$$

For any $0<\varrho<R\leq \bar{R}$, it follows from (3.12) and (3.11) that

$$\iint_{\Omega_\varrho} (\Delta_y u_\varepsilon)^2 dxdy \leq \frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy. \tag{3.13}$$

For any $L\in\mathbb{R}$, multiply (3.12) on both sides by $-\xi^2(u_\varepsilon - L)$ and then integrate over Ω_R to get

$$-\iint_{\Omega_R} \xi^2(u_\varepsilon - L)\Delta_y u_\varepsilon dxdy = \iint_{\Omega_R} \xi^2(u_\varepsilon-L)\Big((x+\varepsilon)^{-\lambda+1}\frac{\partial v_\varepsilon}{\partial x} - (\Lambda+1)(x+\varepsilon)^{-\lambda} v_\varepsilon\Big)dxdy,$$

which implies after integration by parts and direct estimates that

$$\begin{aligned}\iint_{\Omega_\varrho} |\nabla_y u_\varepsilon|^2 dxdy \leq & M\Big(\frac{1}{R^{\lambda-1}(R-\varrho)^2}+1\Big)\iint_{\Omega_R}(u_\varepsilon-L)^2 dxdy \\ & + M\iint_{\Omega_R}\Big((x+\varepsilon)^{-2\lambda+2}\Big(\frac{\partial v_\varepsilon}{\partial x}\Big)^2 + (x+\varepsilon)^{-2\lambda}v_\varepsilon^2\Big)dxdy.\end{aligned}$$

Then, for any $0<\varrho<R\leq\bar{R}$, it follows from this and (3.11) that

$$\begin{aligned}\iint_{\Omega_\varrho} |\nabla_y u_\varepsilon|^2 dxdy \leq & M\Big(\frac{1}{R^{\lambda-1}(R-\varrho)^2}+1\Big)\iint_{\Omega_R}(u_\varepsilon-L)^2 dxdy \\ & + \frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy.\end{aligned} \tag{3.14}$$

Step II. L^2 estimates.

Since $\partial_y^k v_\varepsilon$ satisfies the same equation (3.7) and the boundary condition (3.8) as v_ε for each positive integer k, it follows from (3.11) that

$$\begin{aligned}&\iint_{\Omega_\varrho}\Big((x+\varepsilon)^{-2\lambda+2}\Big|\frac{\partial}{\partial x}\partial_y^k v_\varepsilon\Big|^2 + (x+\varepsilon)^{-\lambda+1}|\nabla_y\partial_y^k v_\varepsilon|^2 + (x+\varepsilon)^{-2\lambda}|\partial_y^k v_\varepsilon|^2\Big)dxdy \\ \leq & \frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^2 |\partial_y^k v_\varepsilon|^2 dxdy \\ \leq & \frac{M(R+\varepsilon)^{2m-(\lambda+1)}}{(R-\varrho)^{2m}} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^{-\lambda+1} |\partial_y^k v_\varepsilon|^2 dxdy \\ \leq & M(\rho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m-(\lambda+1)} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m \iint_{\Omega_R} (x+\varepsilon)^{-\lambda+1} |\partial_y^k v_\varepsilon|^2 dxdy.\end{aligned}$$

This, together with (3.11) and an iteration process, yields that

$$\begin{aligned}&\iint_{\Omega_\varrho}\Big((x+\varepsilon)^{-2\lambda+2}\Big|\frac{\partial}{\partial x}\partial_y^k v_\varepsilon\Big|^2 + (x+\varepsilon)^{-\lambda+1}|\nabla_y\partial_y^k v_\varepsilon|^2 + (x+\varepsilon)^{-2\lambda}|\partial_y^k v_\varepsilon|^2\Big)dxdy \\ \leq & M(k,\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m(k+1)-(\lambda+1)(k+2)} \Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(k+1)} \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy, \quad k=0,1,2,\cdots,\end{aligned}$$

which yields

$$\iint_{\Omega_\varrho}|\partial_y^k v_\varepsilon|^2dxdy \le (R+\varepsilon)^{2\lambda}\iint_{\Omega_\varrho}(x+\varepsilon)^{-2\lambda}|\partial_y^k v_\varepsilon|^2dxdy$$
$$\le M(k,\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m(k+1)-\lambda k-(k+2)}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(k+1)}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy,$$
$$k=0,1,2,\cdots. \qquad (3.15)$$

Since $\partial_y^k u_\varepsilon$ satisfies the same equation (3.3) and the boundary condition (3.4) as u_ε for each nonnegative integer k, it follows from (3.13) and (3.14) with $L=0$ that for any $0<\varrho<R\le\bar{R}$,

$$\iint_{\Omega_\varrho}|\Delta_y\partial_y^k u_\varepsilon|^2dxdy \le \frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}(x+\varepsilon)^2|\partial_y^k v_\varepsilon|^2dxdy$$
$$\le M(\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m-2\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}|\partial_y^k v_\varepsilon|^2dxdy \qquad (3.16)$$

and

$$\iint_{\Omega_\varrho}|\nabla_y\partial_y^k u_\varepsilon|^2dxdy \le M\Big(\frac{1}{R^{\lambda-1}(R-\varrho)^2}+1\Big)\iint_{\Omega_R}|\partial_y^k u_\varepsilon|^2dxdy$$
$$+\frac{M(R+\varepsilon)^{2m-2(\lambda+1)}}{(R-\varrho)^{2m}}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}(x+\varepsilon)^2|\partial_y^k v_\varepsilon|^2dxdy$$
$$\le M(\varrho,R)\iint_{\Omega_R}|\partial_y^k u_\varepsilon|^2dxdy$$
$$+M(\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m-2\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}|\partial_y^k v_\varepsilon|^2dxdy. \qquad (3.17)$$

Due to (3.14) and (3.6) with $L=(\partial_y^1 u_\varepsilon)_R$, one can get that for any $0<\varrho<R\le\bar{R}$,

$$\iint_{\Omega_\varrho}|\nabla_y\partial_y^1 u_\varepsilon|^2dxdy \le M(\varrho,R)\Big(1+\Big(1+\frac{\varepsilon}{\varrho}\Big)^{-\lambda}\Big(1+\frac{\varepsilon}{R}\Big)\Big)\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}\big|\partial_y^1 u_\varepsilon-(\partial_y^1 u_\varepsilon)_R\big|^2dxdy$$
$$+M(\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{2m-2\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^m\iint_{\Omega_R}|\partial_y^1 v_\varepsilon|^2dxdy, \qquad (3.18)$$

Combining (3.16) and (3.15) leads to

$$\iint_{\Omega_\varrho}|\Delta_y\partial_y^k u_\varepsilon|^2dxdy \le M(k,\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(k+2)}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(k+2)}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy,$$
$$k=0,1,2,\cdots. \qquad (3.19)$$

Similarly, it follows from (3.17), (3.14) and (3.15) that

$$\iint_{\Omega_\varrho}|\partial_y^{k+1}u_\varepsilon|^2dxdy$$
$$\le M(k,\varrho,R)\iint_{\Omega_R}\big(u_\varepsilon-(u_\varepsilon)_R\big)^2dxdy$$
$$+M(k,\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(k+2)}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(k+2)}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy,$$
$$k=0,1,2,\cdots; \qquad (3.20)$$

while (3.17), together with (3.18) and (3.15), shows that

$$\begin{aligned}
&\iint_{\Omega_\varrho} |\partial_y^{k+2} u_\varepsilon|^2 dxdy \\
\le &M(k,\varrho,R)\Big(1+\Big(1+\frac{\varepsilon}{\varrho}\Big)^{-\lambda}\Big(1+\frac{\varepsilon}{R}\Big)\Big)\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2 dxdy \\
&+M(k,\varrho,R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(k+3)}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(k+3)}\iint_{\Omega_R}(x+\varepsilon)^2 v_\varepsilon^2 dxdy, \\
&\qquad k=0,1,2,\cdots.
\end{aligned} \tag{3.21}$$

Step III. L^∞ estimates.

These will follow from the L^2 estimates and the Sobolev embedding theorem. Indeed, it follows from (3.15) and (3.20) that

$$\begin{aligned}
\sup_{\Omega_{R/2}}|\nabla_y u_\varepsilon|^2 \le &M(R)\sum_{0\le k\le n}\iint_{\Omega_{R/2}}(|\partial_y^{k+1}v_\varepsilon|^2+|\partial_y^{k+1}u_\varepsilon|^2)dxdy \\
\le &M(R)\iint_{\Omega_R}\big(u_\varepsilon-(u_\varepsilon)_R\big)^2 dxdy \\
&+M(R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(n+2)+\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(n+2)}\iint_{\Omega_R}(x+\varepsilon)^2 v_\varepsilon^2 dxdy.
\end{aligned} \tag{3.22}$$

Similarly, it follows from (3.15), (3.19) and (3.21) that

$$\begin{aligned}
\sup_{\Omega_{R/2}}|\Delta_y u_\varepsilon|^2 \le &M(R)\sum_{0\le k\le n}\iint_{\Omega_{R/2}}(|\partial_y^{k+2}v_\varepsilon|^2+|\Delta_y\partial_y^{k}u_\varepsilon|^2)dxdy \\
\le &M(R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(n+3)+\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(n+3)}\iint_{\Omega_R}(x+\varepsilon)^2 v_\varepsilon^2 dxdy,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\sup_{\Omega_{R/2}}|\Delta_y\nabla_y u_\varepsilon|^2 \le &M(R)\sum_{0\le k\le n}\iint_{\Omega_{R/2}}(|\partial_y^{k+3}v_\varepsilon|^2+|\Delta_y\partial_y^{k+1}u_\varepsilon|^2)dxdy \\
\le &M(R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(n+4)+\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(n+4)}\iint_{\Omega_R}(x+\varepsilon)^2 v_\varepsilon^2 dxdy
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
\sup_{\Omega_{R/2}}|\partial_y^2 u_\varepsilon|^2 \le &M(R)\sum_{0\le k\le n}\iint_{\Omega_{R/2}}(|\partial_y^{k+2}v_\varepsilon|^2+|\partial_y^{k+2}u_\varepsilon|^2)dxdy \\
\le &M(R)\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{1-\lambda}\Big)\iint_{\Omega_R}(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2 dxdy \\
&+M(R)\Big(1+\frac{\varepsilon}{R}\Big)^{(2m-\lambda-1)(n+3)+\lambda}\Big(1+\Big(1+\frac{\varepsilon}{R}\Big)^{\lambda-1}\Big)^{m(n+3)}\iint_{\Omega_R}(x+\varepsilon)^2 v_\varepsilon^2 dxdy.
\end{aligned} \tag{3.25}$$

The constants $M(R)$ in (3.22)–(3.25) will be determined by a rescaling technique. Set

$$\tilde{Q}_{1/2}=(0,1/2)\times B_{(1/2)^{(\lambda+1)/2}}(\bar{y}),\quad \tilde{Q}_1=(0,1)\times B_1(\bar{y})$$

and assume that $u_0\in C^\infty(\overline{\tilde{Q}_1})$ is a solution of

$$\frac{\partial}{\partial x}\Big((x+\tilde{\varepsilon})\frac{\partial u_0}{\partial x}(x,y)\Big)+\Lambda\frac{\partial u_0}{\partial x}(x,y)+(x+\tilde{\varepsilon})^\lambda\Delta_y u_0(x,y)=0,\quad (x,y)\in\tilde{Q}_1 \tag{3.26}$$

with $\tilde\varepsilon > 0$ and satisfying the boundary condition

$$\frac{\partial u_0}{\partial x}(0,y) = 0, \qquad\qquad y \in B_1(\bar y). \tag{3.27}$$

Then, it follows from (3.22)–(3.25) that

$$\sup_{\tilde Q_{1/2}} |\nabla_y u_0|^2 \le M_0(1+\tilde\varepsilon)^{\kappa_0} \iint_{\tilde Q_1} \Big(\big(u_0-(u_0)_1\big)^2 + (x+\tilde\varepsilon)^2 v_0^2\Big)dxdy, \tag{3.28}$$

$$\sup_{\tilde Q_{1/2}} |\Delta_y u_0|^2 \le M_0(1+\tilde\varepsilon)^{\kappa_0} \iint_{\tilde Q_1} (x+\tilde\varepsilon)^2 v_0^2 dxdy, \tag{3.29}$$

$$\sup_{\tilde Q_{1/2}} |\Delta_y \nabla_y u_0|^2 \le M_0(1+\tilde\varepsilon)^{\kappa_0} \iint_{\tilde Q_1} (x+\tilde\varepsilon)^2 v_0^2 dxdy, \tag{3.30}$$

$$\sup_{\tilde Q_{1/2}} |\partial_y^2 u_0|^2 \le M_0(1+\tilde\varepsilon)^{\kappa_0} \iint_{\tilde Q_1} \Big((x+\tilde\varepsilon)^{\lambda+1}\big|\nabla_y u_0-(\nabla_y u_0)_1\big|^2 + (x+\tilde\varepsilon)^2 v_0^2\Big)dxdy \tag{3.31}$$

with $M_0 > 0$ depending only on Λ, λ and n, $\kappa_0 = (2m-\lambda-1)(n+4)+\lambda+\lambda m(n+4)$ and

$$v_0(x,y) = \frac{\partial u_0}{\partial x}(x,y), \quad (x,y) \in \tilde Q_1.$$

Now, for any $0 < R \le \bar R$, set

$$\tilde u(x,y) = u_\varepsilon(Rx, \bar y + R^{(\lambda+1)/2}(y-\bar y)), \quad (x,y) \in \tilde Q_1.$$

It follows from (3.3) and (3.4) that $\tilde u \in C^\infty(\overline{\tilde Q}_1)$ is a solution of (3.26) with $\tilde\varepsilon = \varepsilon/R$ and satisfying the boundary condition (3.27). Therefore, $\tilde u$ satisfies (3.28)–(3.31) with $\tilde\varepsilon = \varepsilon/R$. Coming back to u_ε we finally arrive at that for any $0 < R \le \bar R$,

$$\sup_{\Omega_{R/2}} |\nabla_y u_\varepsilon|^2 \le M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+2)(\lambda+1)/2} \iint_{\Omega_R} \Big(\big(u_\varepsilon-(u_\varepsilon)_R\big)^2 + (x+\varepsilon)^2 v_\varepsilon^2\Big)dxdy, \tag{3.32}$$

$$\sup_{\Omega_{R/2}} |\Delta_y u_\varepsilon|^2 \le M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+4)(\lambda+1)/2} \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy, \tag{3.33}$$

$$\sup_{\Omega_{R/2}} |\Delta_y \nabla_y u_\varepsilon|^2 \le M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+6)(\lambda+1)/2} \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy. \tag{3.34}$$

$$\begin{aligned}\sup_{\Omega_{R/2}} |\partial_y^2 u_\varepsilon|^2 \le{}& M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+4)(\lambda+1)/2} \\ &\cdot \iint_{\Omega_R} \Big((x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2 + (x+\varepsilon)^2 v_\varepsilon^2\Big)dxdy.\end{aligned} \tag{3.35}$$

The equation (3.3) gives

$$\frac{\partial}{\partial x}\big((x+\varepsilon)^{\Lambda+1}v_\varepsilon\big) = -(x+\varepsilon)^{\Lambda+\lambda}\Delta_y u_\varepsilon, \quad \frac{\partial}{\partial x}\big((x+\varepsilon)^{\Lambda+1}\nabla_y v_\varepsilon\big) = -(x+\varepsilon)^{\Lambda+\lambda}\Delta_y\nabla_y u_\varepsilon, \quad (x,y)\in\Omega_R,$$

which, together with (3.4), (3.33) and (3.34), imply that

$$\sup_{\Omega_{R/2}} \frac{v_\varepsilon^2}{(x+\varepsilon)^{2\lambda}} \le M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+4)(\lambda+1)/2} \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy, \tag{3.36}$$

$$\sup_{\Omega_{R/2}} \frac{|\nabla_y v_\varepsilon|^2}{(x+\varepsilon)^{2\lambda}} \le M_0\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0} R^{-1-(n+6)(\lambda+1)/2} \iint_{\Omega_R} (x+\varepsilon)^2 v_\varepsilon^2 dxdy. \tag{3.37}$$

Step IV. Estimates in the Campanato space.
Fix $0<\varrho<R/2<R\le\bar{R}$. From (3.36), (3.32) and the Poincaré inequality, we get that

$$\begin{aligned}
&\iint_{\Omega_\varrho}\big(u_\varepsilon-(u_\varepsilon)_\varrho\big)^2dxdy\\
\le&M\varrho^2\iint_{\Omega_\varrho}{v_\varepsilon}^2dxdy+M\varrho^{\lambda+1}\iint_{\Omega_\varrho}|\nabla_y u_\varepsilon|^2dxdy\\
\le&M\varrho^{3+n(\lambda+1)/2}\sup_{\Omega_{R/2}}v_\varepsilon^2+M\varrho^{1+(n+2)(\lambda+1)/2}\sup_{\Omega_{R/2}}|\nabla_y u_\varepsilon|^2\\
\le&M\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0+2\lambda}\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy\\
&\quad+M\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0}\Big(\frac{\varrho}{R}\Big)^{1+(n+2)(\lambda+1)/2}\iint_{\Omega_R}\Big(\big(u_\varepsilon-(u_\varepsilon)_R\big)^2+(x+\varepsilon)^2v_\varepsilon^2\Big)dxdy\\
\le&M\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0+2\lambda}\Big(\frac{\varrho}{R}\Big)^{2+n(\lambda+1)/2+\min\{1,\lambda\}}\iint_{\Omega_R}\Big(\big(u_\varepsilon-(u_\varepsilon)_R\big)^2+(x+\varepsilon)^2v_\varepsilon^2\Big)dxdy.
\end{aligned}$$

Similarly, (3.37) and (3.35) yield that

$$\begin{aligned}
&\iint_{\Omega_\varrho}(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_\varrho\big|^2dxdy\\
\le&\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\varrho^{\lambda+1}\iint_{\Omega_\varrho}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_\varrho\big|^2dxdy\\
\le&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\varrho^{\lambda+3}\iint_{\Omega_\varrho}|\nabla_y v_\varepsilon|^2dxdy+M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\varrho^{2(\lambda+1)}\iint_{\Omega_\varrho}|\partial_y^2u_\varepsilon|^2dxdy\\
\le&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0+2\lambda}\Big(\frac{\varrho}{R}\Big)^{3+(n+2)(\lambda+1)/2}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy\\
&\quad+M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0}\Big(\frac{\varrho}{R}\Big)^{1+(n+4)(\lambda+1)/2}\\
&\qquad\cdot\iint_{\Omega_R}\Big((x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2+(x+\varepsilon)^2v_\varepsilon^2\Big)dxdy\\
\le&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\lambda+1}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0+2\lambda}\Big(\frac{\varrho}{R}\Big)^{2+(n+2)(\lambda+1)/2+\min\{1,\lambda\}}\\
&\quad\cdot\iint_{\Omega_R}\Big((x+\varepsilon)^2v_\varepsilon^2+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2\Big)dxdy.
\end{aligned}$$

Additionally, it follows from (3.36) that

$$\iint_{\Omega_\varrho}(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2dxdy\le M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{2(\lambda+1)}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0}\Big(\frac{\varrho}{R}\Big)^{1+(n+4)(\lambda+1)/2}\iint_{\Omega_R}(x+\varepsilon)^2v_\varepsilon^2dxdy.$$

Collecting these three estimates, one gets that for any $0<\varrho<R/2<R\le\bar{R}$,

$$\begin{aligned}
&\iint_{\Omega_\varrho}\Big(\big(u_\varepsilon-(u_\varepsilon)_\varrho\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_\varrho\big|^2\Big)dxdy\\
\le&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{2(\lambda+1)}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa_0+2\lambda}\Big(\frac{\varrho}{R}\Big)^{2+n(\lambda+1)/2+\min\{1,\lambda\}}\\
&\quad\cdot\iint_{\Omega_R}\Big(\big(u_\varepsilon-(u_\varepsilon)_R\big)^2+(x+\varepsilon)^2v_\varepsilon^2+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_R\big|^2\Big)dxdy,
\end{aligned}$$

which is just (3.5) in the case $0 < \varrho < R/2 < R \le \bar{R}$ if we take $\kappa \ge \kappa_0 + 2(\lambda+1)$. The estimate (3.5) is trivial in the case $0 < R/2 \le \varrho < R \le \bar{R}$ provided that $M \ge 2^{2+n(\lambda+1)/2+\min\{1,\lambda\}}$. The proof is complete. □

Now we turn to the inner regularity of solutions of the homogeneous equation (3.1). Apart from the boundary $\{0\} \times B_{R_0}(0)$, (3.1) is uniformly elliptic. Thus we can investigate this equation directly. However, the upper and lower bounds of the coefficients of the second order terms depend on the distance to the boundary $\{0\} \times B_{R_0}(0)$ and are anisotropic. This leads to the following definition of anisotropic rectangles to replace the standard ones. For $(\bar{x}, \bar{y}) \in \mathbb{R}^{1+n}$ and $R > 0$, denote

$$Q_R(\bar{x},\bar{y}) = (\bar{x} - \bar{x}R, \bar{x} + \bar{x}R) \times B_{\bar{x}^{(\lambda+1)/2}R}(\bar{y}). \tag{3.38}$$

The reason of this definition may be found in the transformation (3.43) in the following lemma.

Lemma 3.2 *Assume that* $\Lambda > 1/2$, $\lambda \ge 0$, $(\bar{x},\bar{y}) \in (0, \min\{1/2, R_0^{2/(\lambda+1)}\}) \times B_{R_0/2}(0)$. *Let* $u \in C^\infty(\overline{Q}_{1/2}(\bar{x},\bar{y}))$ *be a solution of the equation*

$$\frac{\partial}{\partial x}\Big(x\frac{\partial u}{\partial x}(x,y)\Big) + \Lambda\frac{\partial u}{\partial x}(x,y) + x^\lambda \Delta_y u(x,y) = 0, \quad (x,y) \in Q_{1/2}(\bar{x},\bar{y}). \tag{3.39}$$

Then for any $0 < \varrho < R \le 1/2$,

$$\begin{aligned}&\iint_{Q_\varrho(\bar{x},\bar{y})} \Big(\big(u - (u)_{\varrho,(\bar{x},\bar{y})}\big)^2 + \Big(x\frac{\partial u}{\partial x} - \Big(x\frac{\partial u}{\partial x}\Big)_{\varrho,(\bar{x},\bar{y})}\Big)^2 + \bar{x}^{\lambda+1}\big|\nabla_y u - (\nabla_y u)_{\varrho,(\bar{x},\bar{y})}\big|^2\Big)dxdy\\ \le& M\Big(\frac{\varrho}{R}\Big)^{n+3} \iint_{Q_R(\bar{x},\bar{y})} \Big(\big(u - (u)_{R,(\bar{x},\bar{y})}\big)^2 + \Big(x\frac{\partial u}{\partial x} - \Big(x\frac{\partial u}{\partial x}\Big)_{R,(\bar{x},\bar{y})}\Big)^2 + \bar{x}^{\lambda+1}\big|\nabla_y u - (\nabla_y u)_{R,(\bar{x},\bar{y})}\big|^2\Big)dxdy,\end{aligned} \tag{3.40}$$

where $M > 0$ *depends only on* Λ, λ, n *and* R_0, *and*

$$(u)_{R,(\bar{x},\bar{y})} = \frac{1}{meas\big(Q_R(\bar{x},\bar{y})\big)} \iint_{Q_R(\bar{x},\bar{y})} u(x,y)dxdy.$$

Proof. First, it suffices to prove that for any solution $w \in C^\infty(\overline{Q}_{1/2}(1,\bar{y}))$ of

$$\frac{\partial}{\partial x}\Big(x\frac{\partial w}{\partial x}(x,y)\Big) + \Lambda\frac{\partial w}{\partial x}(x,y) + x^\lambda \Delta_y w(x,y) = 0, \quad (x,y) \in Q_{1/2}(1,\bar{y}) \tag{3.41}$$

and any $0 < \varrho < R \le 1/2$, it holds that

$$\begin{aligned}&\iint_{Q_\varrho(1,\bar{y})} \Big(\big(w - (w)_{\varrho,(1,\bar{y})}\big)^2 + \Big(x\frac{\partial w}{\partial x} - \Big(x\frac{\partial w}{\partial x}\Big)_{\varrho,(1,\bar{y})}\Big)^2 + \big|\nabla_y w - (\nabla_y w)_{\varrho,(1,\bar{y})}\big|^2\Big)dxdy\\ \le& M\Big(\frac{\varrho}{R}\Big)^{n+3} \iint_{Q_R(1,\bar{y})} \Big(\big(w - (w)_{R,(1,\bar{y})}\big)^2 + \Big(x\frac{\partial w}{\partial x} - \Big(x\frac{\partial w}{\partial x}\Big)_{R,(1,\bar{y})}\Big)^2 + \big|\nabla_y w - (\nabla_y w)_{R,(1,\bar{y})}\big|^2\Big)dxdy,\end{aligned} \tag{3.42}$$

where $M > 0$ depends only on Λ, λ, n and R_0. This is due to that if

$$w(x,y) = u(\bar{x}x, \bar{y} + \bar{x}^{(\lambda+1)/2}(y - \bar{y})), \quad (x,y) \in Q_{1/2}(1,\bar{y}), \tag{3.43}$$

then $w \in C^\infty(\overline{Q}_{1/2}(1,\bar{y}))$ is a solution of (3.41) and (3.40) follows from (3.42).

In what follows, we will prove (3.42). For convenience, we use M to denote the positive constant depending only on Λ and λ, while $M(\cdot)$ depends also on the variables in the parentheses.

Assume that $u_\varepsilon \in C^\infty(\overline{Q}_{1/2}(1,\bar{y}))$ solves

$$\frac{\partial}{\partial x}\Big((1+\varepsilon(x-1))\frac{\partial u_\varepsilon}{\partial x}(x,y)\Big)+\Lambda\varepsilon\frac{\partial u_\varepsilon}{\partial x}(x,y)+(1+\varepsilon(x-1))^\lambda\Delta_y u_\varepsilon(x,y)=0,\quad (x,y)\in Q_{1/2}(1,\bar{y}), \tag{3.44}$$

where $0\le\varepsilon\le 1$. Note that

$$\frac{1}{2}<1+\varepsilon(x-1)<\frac{3}{2},\quad \text{for each } 0\le\varepsilon\le 1 \text{ and each } (x,y)\in Q_{1/2}(1,\bar{y}). \tag{3.45}$$

Thus (3.44) is uniformly elliptic. For any $0<\varrho<R\le 1/2$, let $\xi\in C_0^\infty(Q_R(1,\bar{y}))$ satisfying $\xi\equiv 1$ in $Q_\varrho(1,\bar{y})$ and

$$0\le\xi(x,y)\le 1,\quad \Big|\frac{\partial\xi}{\partial x}(x,y)\Big|\le\frac{2}{R-\varrho},\quad |\nabla_y\xi(x,y)|\le\frac{2}{R-\varrho},\quad (x,y)\in Q_R(1,\bar{y}).$$

For any $L\in\mathbb{R}$, multiplying (3.44) on both sides by $-\xi^2(u_\varepsilon-L)$ and then integrating by parts over $Q_R(1,\bar{y})$ show that

$$\begin{aligned}&\iint_{Q_R(1,\bar{y})}\Big((1+\varepsilon(x-1))\xi^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(1+\varepsilon(x-1))^\lambda\xi^2|\nabla_y u_\varepsilon|^2\Big)dxdy\\
=&-2\iint_{Q_R(1,\bar{y})}(1+\varepsilon(x-1))\xi(u_\varepsilon-L)\frac{\partial u_\varepsilon}{\partial x}\frac{\partial\xi}{\partial x}dxdy+\Lambda\varepsilon\iint_{Q_R(1,\bar{y})}\xi^2(u_\varepsilon-L)\frac{\partial u_\varepsilon}{\partial x}dxdy\\
&-2\iint_{Q_R(1,\bar{y})}(1+\varepsilon(x-1))^\lambda\xi(u_\varepsilon-L)\nabla_y u_\varepsilon\cdot\nabla_y\xi dxdy.\end{aligned}$$

Using the Cauchy inequality and (3.45) leads to

$$\begin{aligned}\iint_{Q_\varrho(1,\bar{y})}\Big(\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+|\nabla_y u_\varepsilon|^2\Big)dxdy\le& M\iint_{Q_R(1,\bar{y})}\Big(\Big(\frac{\partial\xi}{\partial x}\Big)^2+1+|\nabla_y\xi|^2\Big)(u_\varepsilon-L)^2dxdy\\
\le&\frac{M}{(R-\varrho)^2}\iint_{Q_R(1,\bar{y})}(u_\varepsilon-L)^2dxdy.\end{aligned}$$

Since $\partial_y^k u_\varepsilon$ satisfies the same equation (3.44) as u_ε for each nonnegative integer k, one gets via an iteration process that

$$\iint_{Q_\varrho(1,\bar{y})}\Big(\Big(\frac{\partial}{\partial x}\partial_y^k u_\varepsilon\Big)^2+|\partial_y^{k+1}u_\varepsilon|^2\Big)dxdy\le M(k,\varrho,R)\iint_{Q_R(1,\bar{y})}\big(u_\varepsilon-(u_\varepsilon)_{R,(1,\bar{y})}\big)^2dxdy,\quad k=0,1,2\cdots. \tag{3.46}$$

Rewrite (3.44) as

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}=-(\Lambda+1)\varepsilon(1+\varepsilon(x-1))^{-1}\frac{\partial u_\varepsilon}{\partial x}-(1+\varepsilon(x-1))^{\lambda-1}\Delta_y u_\varepsilon,\quad (x,y)\in Q_{1/2}(1,\bar{y}),$$

which, together with (3.45) and (3.46), implies that

$$\iint_{Q_\varrho(1,\bar{y})}\Big|\frac{\partial^2}{\partial x^2}\partial_y^k u_\varepsilon\Big|^2dxdy\le M(k,\varrho,R)\iint_{Q_R(1,\bar{y})}\big(u_\varepsilon-(u_\varepsilon)_{R,(1,\bar{y})}\big)^2dxdy,\quad k=0,1,2\cdots. \tag{3.47}$$

It follows from (3.46), (3.47) and the Sobolev embedding theorem that

$$\sup_{Q_{1/4}(1,\bar{y})}\Big(\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+|\nabla_y u_\varepsilon|^2\Big)\le M\iint_{Q_{1/2}(1,\bar{y})}\big(u_\varepsilon-(u_\varepsilon)_{1/2,(1,\bar{y})}\big)^2dxdy. \tag{3.48}$$

Now, let $w \in C^\infty(\overline{Q}_{1/2}(1,\bar{y}))$ solve (3.41). For any $0 < R \le 1/2$, define

$$u_\varepsilon(x,y) = w(1+2R(x-1), \bar{y}+2R(y-\bar{y})), \quad (x,y) \in Q_{1/2}(1,\bar{y}).$$

Then, $u_\varepsilon \in C^\infty(\overline{Q}_{1/2}(1,\bar{y}))$ solves (3.44) with $\varepsilon = 2R$. Thus, it follows from (3.48) that

$$\sup_{Q_{R/2}(1,\bar{y})} \Big(\Big(\frac{\partial w}{\partial x}\Big)^2 + |\nabla_y w|^2\Big) \le MR^{-(n+3)} \iint_{Q_R(1,\bar{y})} \big(w - (w)_{R,(1,\bar{y})}\big)^2 dxdy. \tag{3.49}$$

Since $\nabla_y w$ satisfies the same equation (3.41) as w, (3.49) yields that

$$\sup_{Q_{R/2}(1,\bar{y})} \Big(\Big|\frac{\partial}{\partial x}\nabla_y w\Big|^2 + |\partial_y^2 w|^2\Big) \le MR^{-(n+3)} \iint_{Q_R(1,\bar{y})} \big|\nabla_y w - (\nabla_y w)_{R,(1,\bar{y})}\big|^2 dxdy. \tag{3.50}$$

Rewrite (3.41) as

$$\frac{\partial^2 w}{\partial x^2} = -(\Lambda+1)x^{-1}\frac{\partial w}{\partial x} - x^{\lambda-1}\Delta_y w, \quad (x,y) \in Q_{1/2}(1,\bar{y}),$$

which, together with (3.49) and (3.50), implies that

$$\sup_{Q_{R/2}(1,\bar{y})} \Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 \le MR^{-(n+3)} \iint_{Q_R(1,\bar{y})} \Big(\big(w - (w)_{R,(1,\bar{y})}\big)^2 dxdy + \big|\nabla_y w - (\nabla_y w)_{R,(1,\bar{y})}\big|^2\Big) dxdy. \tag{3.51}$$

For any $0 < \varrho < R/2 \le 1/4$, we can prove by using the Poincaré inequality and (3.49)–(3.51) that

$$\begin{aligned}
\iint_{Q_\varrho(1,\bar{y})} \big(w - (w)_{\varrho,(1,\bar{y})}\big)^2 dxdy \le& M\Big(\frac{\varrho}{R}\Big)^{n+3} \iint_{Q_R(1,\bar{y})} \big(w - (w)_{R,(1,\bar{y})}\big)^2 dxdy,\\
\iint_{Q_\varrho(1,\bar{y})} \big|\nabla_y w - (\nabla_y w)_{\varrho,(1,\bar{y})}\big|^2 dxdy \le& M\Big(\frac{\varrho}{R}\Big)^{n+3} \iint_{Q_R(1,\bar{y})} \big|\nabla_y w - (\nabla_y w)_{R,(1,\bar{y})}\big|^2 dxdy
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q_\varrho(1,\bar{y})} \Big(x\frac{\partial w}{\partial x} - \Big(x\frac{\partial w}{\partial x}\Big)_{\varrho,(1,\bar{y})}\Big)^2 dxdy\\
\le& M\Big(\frac{\varrho}{R}\Big)^{n+3} \iint_{Q_R(1,\bar{y})} \Big(\big(w - (w)_{R,(1,\bar{y})}\big)^2 dxdy + \big|\nabla_y w - (\nabla_y w)_{R,(1,\bar{y})}\big|^2\Big) dxdy.
\end{aligned}$$

From these three estimates, we obtain (3.42) in the case $0 < \varrho < R/2 \le 1/4$. Additionally, the estimate (3.42) is trivial in the case $0 < R/2 \le \varrho < R \le 1/2$ provided that $M \ge 2^{n+3}$. The proof is complete. □

3.2 Regularity of weak solutions to the nonhomogeneous equation

In this subsection, we establish the regularity of weak solutions to the problem (1.7)–(1.9). As shown in Lemma 3.1, the optimal Hölder continuity of weak solutions depends on the sign of $\lambda-1$. For convenience, we just consider the case $\lambda \ge 1$, when the degeneracy along y direction is not weaker than the degeneracy along x direction. For another case $0 \le \lambda < 1$, we can estimate the optimal Hölder continuity of weak solutions in the same way.

Assume that $\lambda \ge 1$. For the exponent $0 < \alpha < 1$ and the function w defined in Ω, set

$$\begin{aligned}
|w|_{0;\Omega} =& \sup_{(x,y)\in\Omega} |w(x,y)|,\\
[w]^*_{\alpha;\Omega} =& \sup_{\substack{(\hat{x},\hat{y}),(\check{x},\check{y})\in\Omega\\ (\hat{x},\hat{y})\ne(\check{x},\check{y})}} \frac{\big(\max\{\hat{x},\check{x}\}\big)^{(\lambda-1)\alpha/2}|w(\hat{x},\hat{y}) - w(\check{x},\check{y})|}{\big(\max\{\hat{x},\check{x}\}\big)^{(\lambda-1)\alpha/2}|\hat{x}-\check{x}|^\alpha + |\hat{y}-\check{y}|^\alpha},
\end{aligned}$$

$$
\begin{aligned}
\|w\|^*_{\alpha;\Omega} &= |w|_{0;\Omega} + [w]^*_{\alpha;\Omega},\\
|w|^{**}_{0;\Omega} &= \sup_{(x,y)\in\Omega} \big(x^{(\lambda+1)/2}|w(x,y)|\big),\\
[w]^{**}_{\alpha;\Omega} &= \sup_{\substack{(\hat{x},\hat{y}),(\check{x},\check{y})\in\Omega\\ (\hat{x},\hat{y})\neq(\check{x},\check{y})}} \frac{\big(\min\{\hat{x},\check{x}\}\big)^{(\lambda+1)/2}\big(\max\{\hat{x},\check{x}\}\big)^{(\lambda-1)\alpha/2}|w(\hat{x},\hat{y})-w(\check{x},\check{y})|}{\big(\max\{\hat{x},\check{x}\}\big)^{(\lambda-1)\alpha/2}|\hat{x}-\check{x}|^\alpha + |\hat{y}-\check{y}|^\alpha},\\
\|w\|^{**}_{\alpha;\Omega} &= |w|^{**}_{0;\Omega} + [w]^{**}_{\alpha;\Omega}.
\end{aligned}
$$

Define the Hölder spaces

$$
C^\alpha_*(\overline{\Omega}) = \{w\in C(\Omega) : \|w\|^*_{\alpha;\Omega} < +\infty\},\quad C^\alpha_{**}(\overline{\Omega}) = \{w\in C^1(\Omega) : \|w\|^{**}_{\alpha;\Omega} < +\infty\}.
$$

It can be verified easily that these anisotropic Hölder spaces possess the following property.

Lemma 3.3 *Assume that $w_1, w_2 \in C^\alpha_*(\overline{\Omega})$ and $w_3 \in C^\alpha_{**}(\overline{\Omega})$. Then $w_1w_2 \in C^\alpha_*(\overline{\Omega})$, $w_1w_3 \in C^\alpha_{**}(\overline{\Omega})$, and*

$$
\begin{gathered}
[w_1w_2]^*_{\alpha;\Omega} \le |w_1|_{0;\Omega}[w_2]^*_{\alpha;\Omega} + [w_1]^*_{\alpha;\Omega}|w_2|_{0;\Omega},\\
|w_1w_3|^{**}_{0;\Omega} \le |w_1|_{0;\Omega}|w_3|^{**}_{0;\Omega},\quad [w_1w_3]^{**}_{\alpha;\Omega} \le |w_1|_{0;\Omega}[w_3]^{**}_{\alpha;\Omega} + [w_1]^*_{\alpha;\Omega}|w_3|^{**}_{0;\Omega}.
\end{gathered}
$$

According to the discussion at the beginning of §3.1, it is reasonable to choose $C^\alpha_*(\overline{\Omega})$ as the Hölder space for u, $x\dfrac{\partial u}{\partial x}$ and $x^{(\lambda+1)/2}\nabla_y u$. It is noted that $x^{(\lambda+1)/2}\nabla_y u \in C^\alpha_*(\overline{\Omega})$ is equivalent to $\nabla_y u \in C^\alpha_{**}(\overline{\Omega})$. Therefore, to get the optimal Hölder estimates, we choose $C^\alpha_*(\overline{\Omega})$ and $C^\alpha_{**}(\overline{\Omega})$ as the Hölder spaces for f_1 and $\vec{f}_2$, respectively.

Theorem 3.1 *Assume that $g\in H^1(B_{R_0}(0))$, $f_1\in C^\alpha_*(\overline{\Omega})$ and $\vec{f}_2 \in C^\alpha_{**}(\overline{\Omega})$. Let $u\in\mathcal{H}_2(\Omega)\cap L^\infty(\Omega)$ be the weak solution of the problem* (1.7)–(1.9)*. Then, $u, x\dfrac{\partial u}{\partial x}\in C^\alpha_*(\overline{\tilde{\Omega}})$ and $\nabla_y u\in C^\alpha_{**}(\overline{\tilde{\Omega}})$ with*

$$
\tilde{\Omega} = (0,1/3)\times B_{R_0}(0).
$$

Furthermore, there exists a constant $M>0$ depending only on Λ, λ, n and R_0 such that

$$
\|u\|^*_{\alpha;\tilde{\Omega}} + \Big\|x\frac{\partial u}{\partial x}\Big\|^*_{\alpha;\tilde{\Omega}} + \|\nabla_y u\|^{**}_{\alpha;\tilde{\Omega}} \le M\Big([f_1]^*_{\alpha;\Omega} + [\vec{f}_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big). \tag{3.52}
$$

Proof. As shown in Theorems 2.1 and 2.2, the problem (1.7)–(1.9) admits a unique solution $u\in\mathcal{H}_2(\Omega)$, which is the limit of a convergent subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$, where $u_\varepsilon\in C^\infty([0,1)\times\overline{B}_{R_0}(0))\cap H^1(\Omega)$ is the solution of the problem

$$
\begin{cases}
\dfrac{\partial}{\partial x}\Big((x+\varepsilon)\dfrac{\partial u_\varepsilon}{\partial x}\Big) + \Lambda\dfrac{\partial u_\varepsilon}{\partial x} + (x+\varepsilon)^\lambda\Delta_y u_\varepsilon = \dfrac{\partial}{\partial x} f_{1,\varepsilon}(x,y) + x^m(x+\varepsilon)^{\lambda-m}\nabla_y\vec{f}_{2,\varepsilon}(x,y), & (x,y)\in\Omega,\\
\dfrac{\partial u_\varepsilon}{\partial\nu}(x,y) = 0, & (x,y)\in(0,1)\times\partial B_{R_0}(0),\\
u_\varepsilon(1,y) = g(y), & y\in B_{R_0}(0),\\
\dfrac{\partial u_\varepsilon}{\partial x}(0,y) = 0, & y\in B_{R_0}(0)
\end{cases}
$$

with $f_{1,\varepsilon},\vec{f}_{2,\varepsilon}\in C^\infty(\overline{\Omega})$ satisfying $\|f_{1,\varepsilon}\|^*_{\alpha;\Omega}\le\|f_1\|^*_{\alpha;\Omega}$, $\|\vec{f}_{2,\varepsilon}\|^{**}_{\alpha;\Omega}\le\|\vec{f}_2\|^{**}_{\alpha;\Omega}$ and

$$
\lim_{\varepsilon\to0^+}\|f_{1,\varepsilon}-f_1\|_{L^\infty(\Omega)} = 0,\quad \lim_{\varepsilon\to0^+}\|(x+\varepsilon)^{(\lambda+1)/2}|\vec{f}_{2,\varepsilon}-\vec{f}_2|\|_{L^\infty(\Omega)} = 0,
$$

and m being the positive integer satisfying $m-1\le\lambda<m$. As for the proof of (2.17) in Theorem 2.2, one can show that

$$
\iint_\Omega\Big(u_\varepsilon^2 + (x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2 + (x+\varepsilon)^{\lambda+1}|\nabla_y u_\varepsilon|^2\Big)dxdy \le C_1 \tag{3.53}
$$

with $C_1 > 0$ independent of ε. Additionally, for any $0 < \delta < 1/2$, it follows from the classical Hölder estimates that

$$\|u_\varepsilon\|_{1,\alpha;(\delta,1-\delta)\times B_{R_0}(0)} \leq C_2(\delta) \tag{3.54}$$

with $C_2(\delta) > 0$ depending on δ but independent of ε. By even extensions, u and u_ε can be regarded as the solutions to the corresponding problems in $(0,1)\times\mathbb{R}^n$. The extended functions will still be denoted by u and u_ε.

For convenience, in the proof we use M to denote a generic positive constant depending only on Λ, λ, n and R_0. The proof of the theorem is given in the following three steps.

Step I. Boundary Campanato estimate.

Fix $\bar{y} \in B_{R_0}(0)$. For any $0 < R < 1$, let $u_{\varepsilon,1} \in C^\infty(\overline{\Omega}_R(0,\bar{y}))$ be the solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial x}\Big((x+\varepsilon)\dfrac{\partial u_{\varepsilon,1}}{\partial x}\Big) + \Lambda\dfrac{\partial u_{\varepsilon,1}}{\partial x} + (x+\varepsilon)^\lambda \Delta_y u_{\varepsilon,1} = 0, & (x,y) \in \Omega_R(0,\bar{y}),\\ u_{\varepsilon,1}(x,y) = u_\varepsilon(x,y), & (x,y) \in \partial\Omega_R(0,\bar{y}) \cap \Omega,\\ \dfrac{\partial u_{\varepsilon,1}}{\partial x}(x,y) = 0, & (x,y) \in \partial\Omega_R(0,\bar{y}) \cap Q_R(0,\bar{y}), \end{cases}$$

where $Q_R(0,\bar{y})$ and $\Omega_R(0,\bar{y})$ are the domains given by (3.2). Then, $u_{\varepsilon,2} = u_\varepsilon - u_{\varepsilon,1} \in C^\infty(\overline{\Omega}_R(0,\bar{y}))$ is the solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial x}\Big((x+\varepsilon)\dfrac{\partial u_{\varepsilon,2}}{\partial x}\Big) + \Lambda\dfrac{\partial u_{\varepsilon,2}}{\partial x} + (x+\varepsilon)^\lambda \Delta_y u_{\varepsilon,2} = \dfrac{\partial}{\partial x} f_{1,\varepsilon}(x,y) + x^m (x+\varepsilon)^{\lambda-m}\nabla_y \vec{f}_{2,\varepsilon}(x,y) \\ \qquad = \dfrac{\partial}{\partial x}\big(f_{1,\varepsilon}(x,y) - f_{1,\varepsilon}(R,\bar{y})\big) + x^m(x+\varepsilon)^{\lambda-m}\nabla_y\big(\vec{f}_{2,\varepsilon}(x,y) - \vec{f}_{2,\varepsilon}(R,\bar{y})\big), & (x,y) \in \Omega_R(0,\bar{y}),\\ u_{\varepsilon,2}(x,y) = 0, & (x,y) \in \partial\Omega_R(0,\bar{y}) \cap \Omega,\\ \dfrac{\partial u_{\varepsilon,2}}{\partial x}(x,y) = 0, & (x,y) \in \partial\Omega_R(0,\bar{y}) \cap Q_R(0,\bar{y}). \end{cases}$$

According to Lemma 3.1, for any $0 < \varrho < R$,

$$\begin{aligned} &\iint_{\Omega_\varrho(0,\bar{y})} \Big(\big(u_{\varepsilon,1} - (u_{\varepsilon,1})_{\varrho,(0,\bar{y})}\big)^2 + (x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,1}}{\partial x}\Big)^2 + (x+\varepsilon)^{\lambda+1}\big|\nabla_y u_{\varepsilon,1} - (\nabla_y u_{\varepsilon,1})_{\varrho,(0,\bar{y})}\big|^2\Big)dxdy\\ \leq& M\Big(1+\frac{\varepsilon}{\varrho}\Big)^\kappa\Big(1+\frac{\varepsilon}{R}\Big)^\kappa\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar{y})}\Big(\big(u_{\varepsilon,1} - (u_{\varepsilon,1})_{R,(0,\bar{y})}\big)^2 + (x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,1}}{\partial x}\Big)^2\\ &\qquad + (x+\varepsilon)^{\lambda+1}\big|\nabla_y u_{\varepsilon,1} - (\nabla_y u_{\varepsilon,1})_{R,(0,\bar{y})}\big|^2\Big)dxdy. \end{aligned} \tag{3.55}$$

As to $u_{\varepsilon,2}$, multiplying the equation for $u_{\varepsilon,2}$ on both sides by $-(x+\varepsilon)u_{\varepsilon,2}$ and then integrating by parts over $\Omega_R(0,\bar{y})$ yield that

$$\begin{aligned} &\iint_{\Omega_R(0,\bar{y})}\Big((x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2 - \frac{\Lambda}{2}(x+\varepsilon)\frac{\partial(u_{\varepsilon,2}^2)}{\partial x} + (x+\varepsilon)^{\lambda+1}|\nabla_y u_{\varepsilon,2}|^2\Big)dxdy\\ =&\iint_{\Omega_R(0,\bar{y})}\Big(-(x+\varepsilon)u_{\varepsilon,2}\frac{\partial u_{\varepsilon,2}}{\partial x} + (x+\varepsilon)(f_{1,\varepsilon} - f_{1,\varepsilon}(R,\bar{y}))\frac{\partial u_{\varepsilon,2}}{\partial x}\\ &\qquad + (f_{1,\varepsilon} - f_{1,\varepsilon}(R,\bar{y}))u_{\varepsilon,2} + x^m(x+\varepsilon)^{\lambda-m+1}(\vec{f}_{2,\varepsilon} - \vec{f}_{2,\varepsilon}(R,\bar{y}))\cdot\nabla_y u_{\varepsilon,2}\Big)dxdy\\ &+\varepsilon\int_{B_{R^{(\lambda+1)/2}}(\bar{y})}(f_{1,\varepsilon}(0,y) - f_{1,\varepsilon}(R,\bar{y}))u_{\varepsilon,2}(0,y)dy. \end{aligned} \tag{3.56}$$

Integrating by parts gives

$$\iint_{\Omega_R(0,\bar{y})}(x+\varepsilon)\frac{\partial(u_{\varepsilon,2}^2)}{\partial x}dxdy = \int_{B_{R^{(\lambda+1)/2}}(\bar{y})}(x+\varepsilon)u_{\varepsilon,2}^2(x,y)dy\Big|_{x=0}^{x=R} - \iint_{\Omega_R(0,\bar{y})}u_{\varepsilon,2}^2dxdy$$

$$\leq -\iint_{\Omega_R(0,\bar y)} u_{\varepsilon,2}^2 dxdy. \tag{3.57}$$

It follows from the Hölder inequality and $u_{\varepsilon,2}(R,\cdot)\Big|_{B_{R^{(\lambda+1)/2}}(\bar y)}=0$ that

$$\begin{aligned}
&\varepsilon\Big|\int_{B_{R^{(\lambda+1)/2}}(\bar y)}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(R,\bar y))u_{\varepsilon,2}(0,y)dy\Big|\\
\leq&\Big(\int_{B_{R^{(\lambda+1)/2}}(\bar y)}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(R,\bar y))^2dy\Big)^{1/2}\Big(\int_{B_{R^{(\lambda+1)/2}}(\bar y)}\varepsilon^2u_{\varepsilon,2}^2(0,y)dy\Big)^{1/2}\\
\leq&\Big(\int_{B_{R^{(\lambda+1)/2}}(\bar y)}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(R,\bar y))^2dy\Big)^{1/2}\Big(R\iint_{\Omega_R(0,\bar y)}\Big(u_{\varepsilon,2}+(x+\varepsilon)\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2dxdy\Big)^{1/2}.
\end{aligned}\tag{3.58}$$

Substituting (3.57) and (3.58) into (3.56) and using the Cauchy inequality and $\Lambda>1/2$, one can get

$$\begin{aligned}
&\iint_{\Omega_R(0,\bar y)}\Big(u_{\varepsilon,2}^2+(x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}|\nabla_y u_{\varepsilon,2}|^2\Big)dxdy\\
\leq&M\iint_{\Omega_R(0,\bar y)}\Big((f_{1,\varepsilon}-f_{1,\varepsilon}(R,\bar y))^2+x^{2m}(x+\varepsilon)^{\lambda-2m+1}|\vec f_{2,\varepsilon}-\vec f_{2,\varepsilon}(R,\bar y)|^2\Big)dxdy\\
&\quad+MR\int_{B_{R^{(\lambda+1)/2}}(\bar y)}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(R,\bar y))^2dy\\
\leq&M\iint_{\Omega_R(0,\bar y)}\Big((f_{1,\varepsilon}-f_{1,\varepsilon}(R,\bar y))^2+x^{\lambda+1}|\vec f_{2,\varepsilon}-\vec f_{2,\varepsilon}(R,\bar y)|^2\Big)dxdy\\
&\quad+MR\int_{B_{R^{(\lambda+1)/2}}(\bar y)}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(R,\bar y))^2dy\\
\leq&M\Big(\big([f_{1,\varepsilon}]^*_{\alpha;\Omega}\big)^2+\big([\vec f_{2,\varepsilon}]^{**}_{\alpha;\Omega}\big)^2\Big)R^{1+n(\lambda+1)/2+2\alpha}\\
\leq&M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec f_2]^{**}_{\alpha;\Omega}\big)^2\Big)R^{1+n(\lambda+1)/2+2\alpha}.
\end{aligned}\tag{3.59}$$

It follows from (3.55) and (3.59) that for any $0<\varrho<R<1$,

$$\begin{aligned}
&\iint_{\Omega_\varrho(0,\bar y)}\Big(\big(u_\varepsilon-(u_\varepsilon)_{\varrho,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_{\varrho,(0,\bar y)}\big|^2\Big)dxdy\\
\leq&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^\kappa\Big(1+\frac{\varepsilon}{R}\Big)^\kappa\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar y)}\Big(\big(u_{\varepsilon,1}-(u_{\varepsilon,1})_{R,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,1}}{\partial x}\Big)^2\\
&\qquad+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_{\varepsilon,1}-(\nabla_y u_{\varepsilon,1})_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\iint_{\Omega_\varrho(0,\bar y)}\Big(u_{\varepsilon,2}^2+(x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}|\nabla_y u_{\varepsilon,2}|^2\Big)dxdy\\
\leq&M\Big(1+\frac{\varepsilon}{\varrho}\Big)^\kappa\Big(1+\frac{\varepsilon}{R}\Big)^\kappa\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar y)}\Big(\big(u_\varepsilon-(u_\varepsilon)_{R,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2\\
&\qquad+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\Big(1+\frac{\varepsilon}{\varrho}\Big)^\kappa\Big(1+\frac{\varepsilon}{R}\Big)^\kappa\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar y)}\Big(\big(u_{\varepsilon,2}-(u_{\varepsilon,2})_{R,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2\\
&\qquad+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_{\varepsilon,2}-(\nabla_y u_{\varepsilon,2})_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\iint_{\Omega_R(0,\bar y)}\Big(u_{\varepsilon,2}^2+(x+\varepsilon)^2\Big(\frac{\partial u_{\varepsilon,2}}{\partial x}\Big)^2+(x+\varepsilon)^3|\nabla_y u_{\varepsilon,2}|^2\Big)dxdy
\end{aligned}$$

$$
\begin{aligned}
\leq & M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\kappa}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa}\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar y)}\Big(\big(u_\varepsilon-(u_\varepsilon)_{R,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2\\
&\qquad+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\Big(1+\frac{\varepsilon}{\varrho}\Big)^{\kappa}\Big(1+\frac{\varepsilon}{R}\Big)^{\kappa}\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec f_2]^{**}_{\alpha;\Omega}\big)^2\Big)R^{1+n(\lambda+1)/2+2\alpha}. \qquad (3.60)
\end{aligned}
$$

For any $0<\varrho<1$ and any $0<\varepsilon\leq\varrho$, taking $R=\varrho^{1/2}$ in (3.60), we get from (3.53) that

$$
\iint_{\Omega_\varrho(0,\bar y)}\Big(\big(u_\varepsilon-(u_\varepsilon)_{\varrho,(0,\bar y)}\big)^2+(x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}\big|\nabla_y u_\varepsilon-(\nabla_y u_\varepsilon)_{\varrho,(0,\bar y)}\big|^2\Big)dxdy\leq C_3\rho
$$

with C_3 independent of ϱ and ε. From this and (3.54), it is not hard to prove that there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$, denoted by itself for convenience, such that

$$
u_\varepsilon\to u,\ (x+\varepsilon)\frac{\partial u_\varepsilon}{\partial x}\to x\frac{\partial u}{\partial x} \text{ and } (x+\varepsilon)^{(\lambda+1)/2}\nabla_y u_\varepsilon\to x^{(\lambda+1)/2}\nabla_y u \text{ in } L^2((0,R)\times B_{R_0}(0))
$$

for any $0<R<1$. Then, letting $\varepsilon\to0^+$ in (3.60) yields that for any $0<\varrho<R<1$,

$$
\begin{aligned}
&\iint_{\Omega_\varrho(0,\bar y)}\Big(\big(u-(u)_{\varrho,(0,\bar y)}\big)^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{\varrho,(0,\bar y)}\big|^2\Big)dxdy\\
\leq & M\Big(\frac{\varrho}{R}\Big)^{3+n(\lambda+1)/2}\iint_{\Omega_R(0,\bar y)}\Big(\big(u-(u)_{R,(0,\bar y)}\big)^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec f_2]^{**}_{\alpha;\Omega}\big)^2\Big)R^{1+n(\lambda+1)/2+2\alpha}.
\end{aligned}
$$

Due to the iteration lemma (see [12, 13]), for any $0<\varrho<R<1$,

$$
\begin{aligned}
&\varrho^{-(1+n(\lambda+1)/2+2\alpha)}\iint_{\Omega_\varrho(0,\bar y)}\Big(\big(u-(u)_{\varrho,(0,\bar y)}\big)^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{\varrho,(0,\bar y)}\big|^2\Big)dxdy\\
\leq & MR^{-(1+n(\lambda+1)/2+2\alpha)}\iint_{\Omega_R(0,\bar y)}\Big(\big(u-(u)_{R,(0,\bar y)}\big)^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{R,(0,\bar y)}\big|^2\Big)dxdy\\
&+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec f_2]^{**}_{\alpha;\Omega}\big)^2\Big). \qquad (3.61)
\end{aligned}
$$

We now estimate the first term on the right side of (3.61) with $R=1/2$. Fix the cutoff function $\zeta\in C^\infty(\overline\Omega_1(0,\bar y))$ satisfying $\zeta\equiv1$ in $\Omega_{1/2}(0,\bar y)$, $\zeta\equiv0$ in $\Omega_1(0,\bar y)\setminus\Omega_{3/4}(0,\bar y)$ and

$$
0\leq\zeta(x,y)\leq1,\quad \Big|\frac{\partial\zeta}{\partial x}(x,y)\Big|\leq M,\quad \big|\nabla_y\zeta(x,y)\big|\leq M,\qquad (x,y)\in\Omega_1(0,\bar y).
$$

Rewrite the equation of u_ε as

$$
\begin{aligned}
&\frac{\partial}{\partial x}\Big((x+\varepsilon)\frac{\partial u_\varepsilon}{\partial x}\Big)+\Lambda\frac{\partial u_\varepsilon}{\partial x}+(x+\varepsilon)^\lambda\Delta_y u_\varepsilon\\
=&\frac{\partial}{\partial x}\big(f_{1,\varepsilon}(x,y)-f_{1,\varepsilon}(1,\bar y)\big)+x^m(x+\varepsilon)^{\lambda-m}\nabla_y\big(\vec f_{2,\varepsilon}(x,y)-\vec f_{2,\varepsilon}(1,\bar y)\big),\quad (x,y)\in\Omega_1(0,\bar y), \qquad (3.62)
\end{aligned}
$$

Multiplying above equation on both sides by $-(x+\varepsilon)\zeta^2u_\varepsilon$ and then integrating by parts over Ω and using the Cauchy inequality, one can prove that

$$
\begin{aligned}
&\iint_{\Omega_{1/2}(0,\bar y)}\Big((x+\varepsilon)^2\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2+(x+\varepsilon)^{\lambda+1}|\nabla_y u_\varepsilon|^2\Big)dxdy\\
\leq & M\iint_{\Omega_1(0,\bar y)}\Big(\big(f_{1,\varepsilon}(x,y)-f_{1,\varepsilon}(1,\bar y)\big)^2+x^{\lambda+1}|\vec f_{2,\varepsilon}(x,y)-\vec f_{2,\varepsilon}(1,\bar y)|^2+u_\varepsilon^2\Big)dxdy
\end{aligned}
$$

$$+ M\int_{B_1(\bar{y})}(f_{1,\varepsilon}(0,y)-f_{1,\varepsilon}(1,\bar{y}))^2dy$$
$$\le M\Big(\big([f_{1,\varepsilon}]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_{2,\varepsilon}]^{**}_{\alpha;\Omega}\big)^2+\iint_{\Omega_1(0,\bar{y})}u_\varepsilon^2dxdy\Big).$$

Letting $\varepsilon\to 0^+$ yields

$$\iint_{\Omega_{1/2}(0,\bar{y})}\Big(u^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}|\nabla_y u|^2\Big)dxdy\le M\Big(\big([f_1]_{0;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{0;\Omega}\big)^2+|u|^2_{0;\Omega}\Big). \tag{3.63}$$

It follows from (3.61) and (3.63) that for any $0<\varrho\le 1/2$,

$$\iint_{\Omega_\varrho(0,\bar{y})}\Big(\big(u-(u)_{\varrho,(0,\bar{y})}\big)^2+x^2\Big(\frac{\partial u}{\partial x}\Big)^2+x^{\lambda+1}\Big(\nabla_y u-\Big(\nabla_y u\Big)_{\varrho,(0,\bar{y})}\Big)^2\Big)dxdy$$
$$\le M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2+|u|^2_{0;\Omega}\Big)\varrho^{1+n(\lambda+1)/2+2\alpha}. \tag{3.64}$$

Step II. Global Campanato estimate.

For any $(\bar{x},\bar{y})\in\tilde{\Omega}$ and any $0<R\le 1/2$, decompose $u=u_1+u_2$ with $u_1\in C^\infty(\overline{Q}_R(\bar{x},\bar{y}))$ being the solution of the problem

$$\begin{cases}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial u_1}{\partial x}\Big)+\Lambda\dfrac{\partial u_1}{\partial x}+x^\lambda\dfrac{\partial^2 u_1}{\partial y^2}=0, & (x,y)\in Q_R(\bar{x},\bar{y}),\\ u_1(x,y)=u(x,y), & (x,y)\in\partial Q_R(\bar{x},\bar{y}),\end{cases}$$

while $u_2\in H^1_0(Q_R(\bar{x},\bar{y}))$ solving the problem

$$\begin{cases}\dfrac{\partial}{\partial x}\Big(x\dfrac{\partial u_2}{\partial x}\Big)+\Lambda\dfrac{\partial u_2}{\partial x}+x^{\lambda+1}\dfrac{\partial^2 u_2}{\partial y^2}=\dfrac{\partial}{\partial x}f_1(x,y)+x^\lambda\nabla_y\vec{f}_2(x,y)\\ \qquad=\dfrac{\partial}{\partial x}\big(f_1(x,y)-f_1(\bar{x},\bar{y})\big)+x^{\lambda+1}\dfrac{\partial}{\partial y}\big(\vec{f}_2(x,y)-\vec{f}_2(\bar{x},\bar{y})\big), & (x,y)\in Q_R(\bar{x},\bar{y}),\\ u_2(x,y)=0, & (x,y)\in\partial Q_R(\bar{x},\bar{y})\end{cases}$$

and $Q_R(\bar{x},\bar{y})$ is given by (3.38). Due to Lemma 3.2, it holds that for any $0<\varrho<R\le 1/2$,

$$\iint_{Q_\varrho(\bar{x},\bar{y})}\Big(\big(u_1-(u_1)_{\varrho,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u_1}{\partial x}-\Big(x\frac{\partial u_1}{\partial x}\Big)_{\varrho,(\bar{x},\bar{y})}\Big)^2+\bar{x}^{\lambda+1}\big|\nabla_y u_1-(\nabla_y u_1)_{\varrho,(\bar{x},\bar{y})}\big|^2\Big)dxdy$$
$$\le M\Big(\frac{\varrho}{R}\Big)^{n+3}\iint_{Q_R(\bar{x},\bar{y})}\Big(\big(u_1-(u_1)_{R,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u_1}{\partial x}-\Big(x\frac{\partial u_1}{\partial x}\Big)_{R,(\bar{x},\bar{y})}\Big)^2$$
$$+\bar{x}^{\lambda+1}\big|\nabla_y u_1-(\nabla_y u_1)_{R,(\bar{x},\bar{y})}\big|^2\Big)dxdy. \tag{3.65}$$

As to u_2, multiplying the equation for u_2 on both sides by $-xu_2$ and then integrating by parts over $\Omega_R(\bar{x},\bar{y})$ show that

$$\iint_{Q_R(\bar{x},\bar{y})}\Big(x^2\Big(\frac{\partial u_2}{\partial x}\Big)^2-\frac{\Lambda}{2}x\frac{\partial(u_2^2)}{\partial x}+x^{\lambda+1}|\nabla_y u_2|^2\Big)dxdy$$
$$=\iint_{Q_R(\bar{x},\bar{y})}\Big(-xu_2\frac{\partial u_2}{\partial x}+x(f_1-f_1(\bar{x},\bar{y}))\frac{\partial u_2}{\partial x}+(f_1-f_1(\bar{x},\bar{y}))u_2+x^{\lambda+1}(\vec{f}_2-\vec{f}_2(\bar{x},\bar{y}))\cdot\nabla_y u_2\Big)dxdy.$$

Similar to the proof of (3.59), one can get that for any $0<R\le 1/2$,

$$\iint_{Q_R(\bar{x},\bar{y})}\Big(u_2^2+x^2\Big(\frac{\partial u_2}{\partial x}\Big)^2+x^{\lambda+1}|\nabla_y u_2|^2\Big)dxdy\le M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}R^{n+1+2\alpha}. \tag{3.66}$$

Using (3.65) and (3.66), and repeating the argument for (3.60), we derive that for any $0<\varrho<R\le 1/2$,

$$\begin{aligned}
&\iint_{Q_\varrho(\bar{x},\bar{y})}\Big(\big(u-(u)_{\varrho,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{\varrho,(\bar{x},\bar{y})}\Big)^2+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{\varrho,(\bar{x},\bar{y})}\big|^2\Big)dxdy\\
\le& M\Big(\frac{\varrho}{R}\Big)^{n+3}\iint_{Q_R(\bar{x},\bar{y})}\Big(\big(u-(u)_{R,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{R,(\bar{x},\bar{y})}\Big)^2\\
&\qquad+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{R,(\bar{x},\bar{y})}\big|^2\Big)dxdy+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}R^{n+1+2\alpha}.
\end{aligned}$$

Then, it follows from the iteration lemma that for any $0<\varrho<R\le 1/2$,

$$\begin{aligned}
&\iint_{Q_\varrho(\bar{x},\bar{y})}\Big(\big(u-(u)_{\varrho,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{\varrho,(\bar{x},\bar{y})}\Big)^2+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{\varrho,(\bar{x},\bar{y})}\big|^2\Big)dxdy\\
\le& M\Big(\frac{\varrho}{R}\Big)^{n+1+2\alpha}\iint_{Q_R(\bar{x},\bar{y})}\Big(\big(u-(u)_{R,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{R,(\bar{x},\bar{y})}\Big)^2\\
&\qquad+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{R,(\bar{x},\bar{y})}\big|^2\Big)dxdy+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}\varrho^{n+1+2\alpha}.
\end{aligned}$$

This inequality with $R=1/2$ and (3.64) with $\varrho=3\bar{x}/2$ lead to that for any $0<\varrho\le 1/2$,

$$\begin{aligned}
&\iint_{Q_\varrho(\bar{x},\bar{y})}\Big(\big(u-(u)_{\varrho,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{\varrho,(\bar{x},\bar{y})}\Big)^2+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{\varrho,(\bar{x},\bar{y})}\big|^2\Big)dxdy\\
\le& M\Big(\frac{\varrho}{1/2}\Big)^{n+1+2\alpha}\iint_{Q_{1/2}(\bar{x},\bar{y})}\Big(\big(u-(u)_{1/2,(\bar{x},\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{1/2,(\bar{x},\bar{y})}\Big)^2\\
&\qquad+\bar{x}^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{1/2,(\bar{x},\bar{y})}\big|^2\Big)dxdy+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}\varrho^{n+1+2\alpha}\\
\le& M\varrho^{n+1+2\alpha}\iint_{\Omega_{3\bar{x}/2}(0,\bar{y})}\Big(\big(u-(u)_{3\bar{x}/2,(0,\bar{y})}\big)^2+\Big(x\frac{\partial u}{\partial x}-\Big(x\frac{\partial u}{\partial x}\Big)_{3\bar{x}/2,(0,\bar{y})}\Big)^2\\
&\qquad+x^{\lambda+1}\big|\nabla_y u-(\nabla_y u)_{3\bar{x}/2,(\bar{x},\bar{y})}\big|^2\Big)dxdy+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}\varrho^{n+1+2\alpha}\\
\le& M\varrho^{n+1+2\alpha}\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2+|u|^2_{0;\Omega}\Big)(3\bar{x}/2)^{1+n(\lambda+1)/2+2\alpha}\\
&\quad+M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}\varrho^{n+1+2\alpha}\\
\le& M\Big(\big([f_1]^*_{\alpha;\Omega}\big)^2+\big([\vec{f}_2]^{**}_{\alpha;\Omega}\big)^2+|u|^2_{0;\Omega}\Big)\bar{x}^{1+n(\lambda+1)/2+2\alpha}\varrho^{n+1+2\alpha}.
\end{aligned}\tag{3.67}$$

Here we also use the fact

$$Q_{1/2}(\bar{x},\bar{y})\subset\Omega_{3\bar{x}/2}(0,\bar{y})\quad\text{and}\quad 3\bar{x}/2\le 1/2.$$

Step III. Hölder estimates.

We will use the Campanato theorem (see [12, 13]) to get the desired results. It follows from (3.64) that

$$|u(0,\hat{y})-u(0,\check{y})|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f}_2]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^{2\alpha/(\lambda+1)},\quad \hat{y},\check{y}\in B_{R_0}(0)\tag{3.68}$$

and

$$\lim_{x\to 0^+}x\frac{\partial u}{\partial x}(x,\cdot)\Big|_{B_{R_0}(0)}=0.\tag{3.69}$$

In addition, for any $(\hat{x},\hat{y}),(\check{x},\check{y})\in\tilde{\Omega}\cap Q_{1/2}(\bar{x},\bar{y})$ with some $(\bar{x},\bar{y})\in\tilde{\Omega}$, (3.67) yields

$$|u(\hat{x},\hat{y})-u(\check{x},\check{y})|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f}_2]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\Big(|\hat{x}-\check{x}|^\alpha+\bar{x}^{-(\lambda-1)\alpha/2}|\hat{y}-\check{y}|^\alpha\Big),\tag{3.70}$$

$$\left|\hat{x}\frac{\partial u}{\partial x}(\hat{x},\hat{y})-\check{x}\frac{\partial u}{\partial x}(\check{x},\check{y})\right|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\Big(|\hat{x}-\check{x}|^\alpha+\bar{x}^{-(\lambda-1)\alpha/2}|\hat{y}-\check{y}|^\alpha\Big),\tag{3.71}$$

$$|\nabla_y u(\hat{x},\hat{y})-\nabla_y u(\check{x},\check{y})|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\bar{x}^{-(\lambda+1)/2}\Big(|\hat{x}-\check{x}|^\alpha+\bar{x}^{-(\lambda-1)\alpha/2}|\hat{y}-\check{y}|^\alpha\Big).\tag{3.72}$$

On the one hand, fix $(\hat{x},\bar{y}),(\check{x},\bar{y})\in\tilde{\Omega}$ with $\hat{x}\le\check{x}$. There exists a nonnegative integer k such that

$$2^k\hat{x}\le\check{x}<2^{(k+1)}\hat{x}.$$

One gets from (3.70) that

$$|u(\hat{x},\bar{y})-u(\check{x},\bar{y})|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{x}-\check{x}|^\alpha\tag{3.73}$$

in the case $k=0$, while that

$$\begin{aligned}|u(\hat{x},\bar{y})-u(\check{x},\bar{y})|\le&\sum_{j=1}^{k}|u(2^{j-1}\hat{x},\bar{y})-u(2^j\hat{x},\bar{y})|+|u(2^k\hat{x},\bar{y})-u(\check{x},\bar{y})|\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\sum_{j=1}^{k+1}(2^{j-1}\hat{x})^\alpha\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\sum_{j=1}^{k+1}(2^{j-k}|\hat{x}-\check{x}|)^\alpha\\ =&M\sum_{i=0}^{k}2^{-(i-1)\alpha}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{x}-\check{x}|^\alpha\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{x}-\check{x}|^\alpha\end{aligned}\tag{3.74}$$

in the case $k\ge1$. On the other hand, fix $(\bar{x},\hat{y}),(\bar{x},\check{y})\in\tilde{\Omega}$. If $|\hat{y}-\check{y}|<\bar{x}^{(\lambda+1)/2}$, then it follows from (3.70) that

$$|u(\bar{x},\hat{y})-u(\bar{x},\check{y})|\le M\bar{x}^{-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha;\tag{3.75}$$

while if $|\hat{y}-\check{y}|\ge\bar{x}^{(\lambda+1)/2}$, then it follows from (3.73), (3.74) and (3.68) that

$$\begin{aligned}|u(\bar{x},\hat{y})-u(\bar{x},\check{y})|\le&|u(\bar{x},\hat{y})-u(0,\hat{y})|+|u(\bar{x},\check{y})-u(0,\check{y})|+|u(0,\hat{y})-u(0,\check{y})|\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\bar{x}^\alpha+M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^{2\alpha/(\lambda+1)}\\ \le&M\bar{x}^{-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\bar{x}^{(\lambda+1)\alpha/2}\\ &+M|\hat{y}-\check{y}|^{-(\lambda-1)\alpha/(\lambda+1)}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha\\ \le&M\bar{x}^{-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha.\end{aligned}\tag{3.76}$$

Therefore, for any $(\hat{x},\hat{y}),(\check{x},\check{y})\in\tilde{\Omega}$ with $\hat{x}\le\check{x}$, it follows from (3.73)–(3.76) that

$$\begin{aligned}&|u(\hat{x},\hat{y})-u(\check{x},\check{y})|\\ \le&|u(\hat{x},\hat{y})-u(\check{x},\hat{y})|+|u(\check{x},\hat{y})-u(\check{x},\check{y})|\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{x}-\check{x}|^\alpha+M\check{x}^{-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha\\ \le&M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\Big(|\hat{x}-\check{x}|^\alpha+\check{x}^{-(\lambda-1)\alpha/2}|\hat{y}-\check{y}|^\alpha\Big),\end{aligned}$$

i.e.

$$[u]^*_{\alpha;\tilde\Omega} \le M\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big).$$

Similarly, it follows from (3.69) and (3.71) that

$$\Big[x\frac{\partial u}{\partial x}\Big]^*_{\alpha;\tilde\Omega} \le M\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big).$$

Now we consider $\nabla_y u$, which belongs to a different Hölder space. On the one hand, fix $(\hat x, \bar y), (\check x, \bar y) \in \tilde\Omega$ with $\hat x \le \check x$. There exists a nonnegative integer k such that

$$2^k \hat x \le \check x < 2^{(k+1)} \hat x.$$

Similar to the proof of (3.73) and (3.74), we get from (3.72) that

$$|\nabla_y u(\hat x, \bar y) - \nabla_y u(\check x, \bar y)| \le M \hat x^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat x - \check x|^\alpha \tag{3.77}$$

in the case $k = 0$, while that

$$\begin{aligned}|\nabla_y u(\hat x, \bar y) - \nabla_y u(\check x, \bar y)| \le& \sum_{j=1}^{k} |\nabla_y u(2^{j-1}\hat x, \bar y) - \nabla_y u(2^j \hat x, \bar y)| + |\nabla_y u(2^k \hat x, \bar y) - \nabla_y u(\check x, \bar y)| \\ \le& M\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big) \sum_{j=1}^{k+1} (2^j \hat x)^{-(\lambda+1)/2} (2^{j-1}\hat x)^\alpha \\ \le& M \hat x^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat x - \check x|^\alpha \end{aligned} \tag{3.78}$$

in the case $k \ge 1$. These two estimates, together with the classical Hölder estimate, yield also

$$|\nabla_y u(\hat x, \bar y)| \le M \hat x^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big). \tag{3.79}$$

On the other hand, fix $(\bar x, \hat y), (\bar x, \check y) \in \tilde\Omega$. If $|\hat y - \check y| < \bar x^{(\lambda+1)/2}$, then it follows from (3.72) that

$$|\nabla_y u(\bar x, \hat y) - \nabla_y u(\bar x, \check y)| \le M \bar x^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat y - \check y|^\alpha; \tag{3.80}$$

if $|\hat y - \check y| \ge 3^{-(\lambda+1)/2}$, then it follows from (3.79) that

$$\begin{aligned}\Big|\nabla_y u(\bar x, \hat y) - \nabla_y u(\bar x, \check y)\Big| \le& M \bar x^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big) \\ \le& M \bar x^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat y - \check y|^\alpha;\end{aligned} \tag{3.81}$$

while if $\bar x^{(\lambda+1)/2} \le |\hat y - \check y| < 3^{-(\lambda+1)/2}$, then it follows from (3.77), (3.78) and (3.72) that

$$\begin{aligned}&|\nabla_y u(\bar x, \hat y) - \nabla_y u(\bar x, \check y)| \\ \le& |\nabla_y u(\bar x, \hat y) - \nabla_y u(|\hat y - \check y|^{2/(\lambda+1)}, \hat y)| + |\nabla_y u(\bar x, \check y) - \nabla_y u(|\hat y - \check y|^{2/(\lambda+1)}, \check y)| \\ &\quad + |\nabla_y u(|\hat y - \check y|^{2/(\lambda+1)}, \hat y) - \nabla_y u(|\hat y - \check y|^{2/(\lambda+1)}, \check y)| \\ \le& M \bar x^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)\big(|\hat y - \check y|^{2/(\lambda+1)} - \bar x\big)^\alpha \\ &\quad + M\big(|\hat y - \check y|^{2/(\lambda+1)}\big)^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat y - \check y|^\alpha \\ \le& M \bar x^{-(\lambda+1)/2} |\hat y - \check y|^{-(\lambda-1)\alpha/(\lambda+1)}\Big([f_1]^*_{\alpha;\Omega} + [\vec f_2]^{**}_{\alpha;\Omega} + |u|_{0;\Omega}\Big)|\hat y - \check y|^\alpha\end{aligned}$$

$$
\begin{aligned}
&+ M\big(|\hat{y}-\check{y}|^{2/(\lambda+1)}\big)^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha\\
\le& M\bar{x}^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha.
\end{aligned}\tag{3.82}
$$

Therefore, for any $(\hat{x},\hat{y}),(\check{x},\check{y})\in\tilde{\Omega}$ with $\hat{x}\le\check{x}$, it follows from (3.77), (3.78) and (3.80)–(3.82) that

$$
\begin{aligned}
|\nabla_y u(\hat{x},\hat{y})-\nabla_y u(\check{x},\check{y})|\le&|\nabla_y u(\hat{x},\hat{y})-\nabla_y u(\check{x},\hat{y})|+|\nabla_y u(\check{x},\hat{y})-\nabla_y u(\check{x},\check{y})|\\
\le& M\hat{x}^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{x}-\check{x}|^\alpha\\
&+M\check{x}^{-(\lambda+1)/2-(\lambda-1)\alpha/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)|\hat{y}-\check{y}|^\alpha\\
\le& M\hat{x}^{-(\lambda+1)/2}\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)\Big(|\hat{x}-\check{x}|^\alpha+\check{x}^{-(\lambda-1)\alpha/2}|\hat{y}-\check{y}|^\alpha\Big),
\end{aligned}
$$

i.e.

$$
[\nabla_y u]^{**}_{\alpha;\tilde{\Omega}}\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big).
$$

Summing up, one gets that $u, x\dfrac{\partial u}{\partial x}\in C^\alpha_*(\overline{\tilde{\Omega}})$ and $\nabla_y u\in C^\alpha_{**}(\overline{\tilde{\Omega}})$ satisfying the estimate (3.52). The proof of the theorem is complete. □

Remark 3.1 *Under the assumption of Theorem 3.1, it follows from* (3.64) *that*

$$
x\frac{\partial u}{\partial x}(x,y)\Big|_{x=0}=\lim_{x\to 0^-}x\frac{\partial u}{\partial x}(x,y)=0,\quad y\in B_{R_0}(0).
$$

Therefore

$$
\Big|\frac{\partial u}{\partial x}(x,y)\Big|\le M\Big([f_1]^*_{\alpha;\Omega}+[\vec{f_2}]^{**}_{\alpha;\Omega}+|u|_{0;\Omega}\Big)x^{-1+\alpha},\quad (x,y)\in\tilde{\Omega}.
$$

4 Application to subsonic-sonic flows in convergent nozzles

Based on the Hölder estimates in Theorem 3.1, we investigate the problem (1.2)–(1.5) in this section. As mentioned in the introduction, the main motivation of the current paper lies in the study on continuous subsonic-sonic compressible Euler steady flows in a three-dimensional symmetric convergent nozzle with straight solid wall, which will be described in more details below.

Consider the compressible Euler system of steady flow in a three-dimensional nozzle

$$
\begin{cases}
\operatorname{div}(\rho\vec{u})=0, & \text{in } \mathcal{D},\\
\operatorname{div}(\rho\vec{u}\otimes\vec{u})+\nabla P=0, & \text{in } \mathcal{D},
\end{cases}
$$

where $\vec{u}$, P and ρ represent the velocity, pressure and density of the flow, respectively, and $\mathcal{D}$ is the nozzle. The flow is assumed to be isentropic so that $P=P(\rho)$ is a smooth function. In particular, for a polytropic gas with adiabatic exponent $\gamma>1$,

$$
P(\rho)=\frac{1}{\gamma}\rho^\gamma
$$

is the normalized pressure. Assume further that the flow is irrotational, i.e.

$$
curl\,\vec{u}=0,\quad \text{in } \mathcal{D}.
$$

Then the Euler system is transformed into the full potential equation

$$
\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi)=0,\quad \text{in } \mathcal{D},\tag{4.1}
$$

where

$$\rho(q^2) = \Big(1 - \frac{\gamma-1}{2}q^2\Big)^{1/(\gamma-1)}, \quad 0 < q^2 < \frac{2}{\gamma-1}$$

and φ is a velocity potential so that

$$\nabla\varphi = \vec{u}, \qquad \text{in } \mathcal{D}.$$

The sound speed c is defined as

$$c^2 = P'(\rho) = \rho^{\gamma-1} = 1 - \frac{\gamma-1}{2}|\nabla\varphi|^2.$$

At the sonic state, the sound speed is

$$c_* = \Big(\frac{2}{\gamma+1}\Big)^{1/2},$$

which is also the sonic speed. Then, the flow is subsonic when $|\nabla\varphi| < c_*$, sonic when $|\nabla\varphi| = c_*$ and supersonic when $|\nabla\varphi| > c_*$.

There have been extensive studies on two kinds of problems on subsonic-sonic flows. One involves subsonic-sonic flows past a profile ([1, 3, 9, 11]) and the other concerns subsonic-sonic flows in an infinite nozzle [2, 27]. It was shown that these two kinds of subsonic-sonic flows both can be realized as weak limits of sequences of strictly subsonic flows associated with some physical quantity, which is the freestream Mach number for flows past a profile and is the incoming mass flux for flows in an infinite nozzle, increasing to the critical value [3, 27]. It should be noted that the subsonic-sonic flows in [3, 27] both are weak solutions and the regularities of these solutions are unknown. In particular, it is unclear about the location of sonic states. There are also some studies on smooth transonic flows. It is well known that smooth transonic flows past a profile and smooth transonic flows of Taylor type in a nozzle do not exist in general and are unstable even they exist ([20, 21, 22] and the books [2, 16]). Some examples of smooth transonic flows of Meyer type were constructed by using the hodograph plane in which the governing equations become linear (the books [2, 16]). Moreover, Kuz'min[16] formulated the perturbation problems of accelerating smooth transonic flows in the nearsonic approximation and solved these perturbation problems by using the principle of contracting mappings.

It is known that there exist symmetric subsonic-sonic flows in a symmetric convergent nozzle with straight wall; furthermore, these flows are singular at sonic curves and cannot flow over sonic curves. We are interested in the perturbation problems of these symmetric flows. More precisely, assume that the nozzle is symmetric with respect to the z-axis and is converging with straight solid wall whose vertex is the origin, i.e.

$$\mathcal{D} = \big\{(r\sin\theta\cos\vartheta, r\sin\theta\sin\vartheta, r\cos\theta) : r_1 < r < r_2, 0 < \theta < \theta_0, 0 < \vartheta < 2\pi\big\}$$

with $r_1 < r_2 < 0$ and $0 < \theta_0 < \pi/2$. Assume further that the flow is symmetric with respect to the z-axis such that

$$\varphi(x,y,z) = \phi(r,\theta), \quad r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z}.$$

Then ϕ satisfies (1.1). If the angular velocity ϕ_θ is relatively small compared with the linear velocity ϕ_r, (1.1) may be approximated by (1.2). Since the flow is symmetric and the wall is solid, ϕ should satisfy the symmetric and slip conditions (1.3).

In this section we consider the subsonic-sonic solutions of the equation (1.2) with relatively small angular velocity satisfying the symmetric and slip conditions (1.3), i.e. for a given incoming flow with small angular velocity, we seek a subsonic-sonic flow whose angular velocity is small and which is sonic at the outlet and satisfies the symmetric and slip conditions (1.3). Furthermore, if such a subsonic-sonic flow exists, is it stable with respect to the perturbation of the incoming flow? Note that for a fixed incoming flow $\phi_0(r_1) = b_0$, it will be clear that there is a unique symmetric subsonic-sonic flow ϕ_0 with zero angular velocity and speeding up to sonic at the outlet. This symmetric flow is singular at the sonic curve in the sense that the speed of the flow is only Hölder continuous and the acceleration of the flow is infinite at the sonic curve. We are interested in whether this symmetric flow is stable, i.e. for a given incoming flow satisfying (1.4), which is a small perturbation of b_0, whether there is a subsonic-sonic flow ϕ of the equation (1.2) satisfying the symmetric and slip conditions (1.3) such that it is sonic at the outlet, i.e. (1.5) holds. Moreover, if such a subsonic-sonic flow ϕ exists, is it singular at the sonic curve and what is the singularity, and is it sufficiently close to the symmetric flow ϕ_0? We will prove that the symmetric flow is stable and the perturbed flow possesses the same singularity at the sonic curve as the symmetric flow. The key in our analysis is to establish some suitable estimates and regularities for the perturbations. It is noted that (1.2) is degenerate along the r direction at the sonic curve while not degenerate along the θ direction. Furthermore, if ϕ_0 is stable as we expected, then the rate of the degeneracy along the r direction is $O\big((r_2-r)^{1/2}\big)$, which is just Hölder continuous. Therefore, we choose a weighted Hölder space as the class of the subsonic-sonic flow ϕ. However, it seems to be difficult to get the desired upper and lower bound estimates on the speed ϕ_r just in this weighted Hölder space. To overcome this difficulty, we need establish the precise regularity of the flow near the sonic curve, which is done by using the Hölder continuity and the Hölder estimates in §3.

4.1 Formulation of the perturbation problem and its solvability

We now study the boundary value problem (1.2)–(1.5) in this section. For the special case

$$b(\theta)=b_0,\quad 0<\theta<\theta_0$$

with $b_0\in\mathbb{R}$, the problem (1.2)–(1.5) may be reduced into a two-points boundary value problem of an ordinary differential equation

$$\big(r\rho\big((\phi_0')^2\big)\phi_0'\big)'=0,\qquad r_1<r<r_2, \tag{4.2}$$

$$\phi_0(r_1)=b_0,\quad \phi_0'(r_2)=c_*. \tag{4.3}$$

It is clear that the problem (4.2), (4.3) admits a unique solution $\phi_0\in C^\infty([r_1,r_2))\cap C^1([r_1,r_2])$, such that

$$\phi_0(r)=b_0+\int_{r_1}^r\phi_0'(t)dt,\quad r\rho((\phi_0'(r))^2)\phi_0'(r)=r_2\rho(c_*^2)c_*,\qquad r_1\le r\le r_2,$$

$$\phi_0''(r)=\frac{(-r_2)^{\gamma-1}c_*^{\gamma+3}}{(-r)^\gamma(\phi_0'(r))^{\gamma-2}(c_*+\phi_0'(r))(c_*-\phi_0'(r))},\quad r_1\le r<r_2.$$

It is not hard to verify that ϕ_0 satisfies

$$0<\phi_0'(r)<c_*,\quad \phi_0''(r)>0,\qquad r_1\le r<r_2$$

$$c_*-\phi_0'(r)=\frac{c_*^2}{\sqrt{|r_2|}}(r_2-r)^{1/2}+O\big((r_2-r)\big),\quad r_2-r=\frac{|r_2|}{c_*^4}(c_*-\phi_0'(r))^2+O\big((r_2-r)^{3/2}\big),$$

$$\phi_0''(r)=\frac{c_*^4}{2|r_2|}(c_*-\phi_0'(r))^{-1}+O\big(1\big),\quad \phi_0'''(r)=\frac{c_*^8}{4|r_2|^2}(c_*-\phi_0'(r))^{-3}+O\big((r_2-r)^{-1}\big).$$

For the general case that b is a small perturbation of b_0, let ϕ be a solution to the problem (1.2)–(1.5). Set

$$w(r,\theta)=\phi_0(r)-\phi(r,\theta),\qquad (r,\theta)\in\overline{G}.$$

It follows from (1.2), (4.2) and some tedious calculations that w satisfies

$$\big(h_0(r)w_r\big)_r+h_1(r,w_r)w_rw_{rr}+h_2(r,w_r)w_rw_r+h_3(r,w_r)w_{\theta\theta}=0,\quad (r,\theta)\in G, \tag{4.4}$$

where $h_0(r)$ and $h_i(r,w_r)\,(i=1,2,3)$ are given explicitly in the Appendix satisfying the following important properties

$$-C_2(r_2-r)^{1/2}\le h_0(r)\le -C_1(r_2-r)^{1/2},\quad -C_2\le h_3(r,0)\le -C_1,\quad r_1<r<r_2$$

with some $0 < C_1 < C_2$, and

$$h_0(r) = -\frac{2|r_2|}{c_*^3}(c_* - \phi_0'(r)) + O\big((r_2 - r)\big), \quad h_0'(r) = c_*(c_* - \phi_0'(r))^{-1} + O\big(1\big),$$
$$h_0''(r) = \frac{c_*^5}{2|r_2|}(c_* - \phi_0'(r))^{-3} + O\big((r_2 - r)^{-1}\big), \quad h_1(r,0) = O\big(1\big), \quad \frac{\partial h_1}{\partial r}(r,0) = O\big((r_2 - r)^{-1/2}\big),$$
$$h_2(r,0) = O\big((r_2 - r)^{-1/2}\big), \quad \frac{\partial h_2}{\partial r}(r,0) = O\big((r_2 - r)^{-3/2}\big), \quad h_3(r,0) = -\frac{1}{|r_2|} + O\big((r_2 - r)^{1/2}\big),$$
$$\frac{\partial h_3}{\partial r}(r,0) = \frac{c_*^3}{2|r_2|^2}(c_* - \phi_0'(r))^{-1} + O\big(1\big), \quad \frac{\partial^2 h_3}{\partial r^2}(r,0) = \frac{c_*^7}{4|r_2|^3}(c_* - \phi_0'(r))^{-3} + O\big((r_2 - r)^{-1}\big).$$

Remark 4.1 *In the case $\gamma = 2$, one has the simplified expressions as*

$$h_0(r) = \frac{r}{\rho(c_*^2)c_*^2}\Big[c_*^2 - (\phi_0'(r))^2\Big], \quad h_1(r, w_r) = \frac{3r}{\rho(c_*^2)}\phi_0'(r) - \frac{3r}{2\rho(c_*^2)}w_r,$$
$$h_2(r, w_r) = \frac{3}{2\rho(c_*^2)}\phi_0'(r) + \frac{3r}{2\rho(c_*^2)}\phi_0''(r) - \frac{1}{2\rho(c_*^2)}w_r, \quad h_3(r, w_r) = \frac{1}{r\rho(c_*^2)}\rho((\phi_0'(r) - w_r)^2)$$

with $c_*^2 = 2/3$.

For convenience, (4.4) will be rewritten as

$$\frac{\partial}{\partial r}\Big(a_0(r)\frac{\partial w}{\partial r}\Big) + q_1(r, w_r)\frac{\partial^2 w}{\partial r^2} + q_2(r, w_r)\frac{\partial w}{\partial r} + a_3(r)\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} = 0, \quad (r,\theta) \in G, \tag{4.5}$$

with

$$a_0(r) = \frac{h_0(r)}{h_3(r,0)}, \quad a_3(r) = \frac{h_0(r)}{h_3^2(r,0)}\frac{\partial h_3}{\partial r}(r,0), \qquad r_1 < r < r_2,$$
$$q_i(r, w_r) = \frac{h_i(r, w_r)}{h_3(r, w_r)}w_r + \Big(\frac{1}{h_3(r, w_r)} - \frac{1}{h_3(r,0)}\Big)\partial_r^{i-1}h_0(r), \quad r_1 < r < r_2,\ w_r \in \mathbb{R},\ i = 1, 2.$$

Clearly,

$$C_3(r_2 - r)^{1/2} \le a_0(r) \le C_4(r_2 - r)^{1/2}, \quad r_1 < r < r_2 \tag{4.6}$$
$$-C_4(c_* - \phi_0'(r))^{-1} \le \frac{1}{2}a_0'(r) + a_3(r) \le -C_3(c_* - \phi_0'(r))^{-1}, \quad r_1 < r < r_2 \tag{4.7}$$

with some $0 < C_3 < C_4$, and

$$a_0(r) = \frac{2|r_2|^2}{c_*^3}(c_* - \phi_0'(r)) + O\big((r_2 - r)\big), \quad a_0'(r) = -|r_2|c_*(c_* - \phi_0'(r))^{-1} + O\big(1\big),$$
$$a_0''(r) = -\frac{c_*^5}{2}(c_* - \phi_0'(r))^{-3} + O\big((r_2 - r)^{-1}\big),\ a_3(r) = -|r_2| + O\big((r_2 - r)^{1/2}\big),\ a_3'(r) = O\big((r_2 - r)^{-1/2}\big).$$

Due to (1.3)–(1.5) and (4.3), w satisfies the following boundary conditions

$$w_\theta(r,0) = w_\theta(r,\theta_0) = 0, \qquad r_1 < r < r_2, \tag{4.8}$$
$$w(r_1,\theta) = g(\theta), \qquad 0 < \theta < \theta_0, \tag{4.9}$$
$$w_r(r_2,\theta) = 0, \qquad 0 < \theta < \theta_0, \tag{4.10}$$

where

$$g(\theta) = b_0 - b(\theta), \quad \theta \in [0, \theta_0].$$

To study the possible singularities near the sonic curve of solutions to the problem (4.5), (4.8)–(4.10), we introduce some weighted Hölder spaces.

Let $\alpha \in (0,1)$ and $k, \mu \in \mathbb{R}$. Define

$$|u|_{0;G}^{(k)} = \sup_{(r,\theta)\in G} \big((r_2 - r)^k |u(r,\theta)|\big),$$

$$[u]_{\alpha;r,G}^{(k;\mu)} = \sup_{\theta\in(0,\theta_0)} \sup_{\substack{\hat r,\check r\in(r_1,r_2)\\ \hat r\neq\check r}} \Big(\big(\min\{r_2-\hat r, r_2-\check r\}\big)^{\max\{k,0\}}\big(\max\{r_2-\hat r, r_2-\check r\}\big)^{\min\{k,0\}}$$

$$\cdot\big(\max\{r_2-\hat r, r_2-\check r\}\big)^{\mu\alpha}\frac{|u(\hat r,\theta)-u(\check r,\theta)|}{|\hat r-\check r|^\alpha}\Big),$$

$$[u]_{\alpha;\theta,G}^{(k;\mu)} = \sup_{r\in(r_1,r_2)} \sup_{\substack{\hat\theta,\check\theta\in(0,\theta_0)\\ \hat\theta\neq\check\theta}} \Big((r_2-r)^{k+\mu\alpha}\frac{|u(r,\hat\theta)-u(r,\check\theta)|}{|\hat\theta-\check\theta|^\alpha}\Big),$$

$$[u]_{\alpha;G}^{(k;\star)} = [u]_{\alpha;r,G}^{(k;1)} + [u]_{\alpha;\theta,G}^{(k;3/4)}, \qquad \|u\|_{\alpha;G}^{(k;\star)} = |u|_{0;G}^{(k)} + [u]_{\alpha;G}^{(k;\star)}.$$

Define a weighted Hölder space $C^\alpha_{(k;\star)}(\overline{G})$ as

$$C^\alpha_{(k;\star)}(\overline{G}) = \{u \in C^1(G) : \|u\|_{\alpha;G}^{(k;\star)} < +\infty\}.$$

Here, since the degeneracy of (4.5) just occurs along the r direction, the weight for the r direction and the θ direction is different. For this weighted Hölder space, we can prove the following property.

Lemma 4.1 *Assume that $u_1 \in C^\alpha_{(k_1;\star)}(\overline{G})$ and $u_2 \in C^\alpha_{(k_2;\star)}(\overline{G})$. Then $u_1u_2 \in C^\alpha_{(k_1+k_2;\star)}(\overline{G})$ and*

$$|u_1u_2|_{0;G}^{(k_1+k_2)} \le |u_1|_{0;G}^{(k_1)}|u_2|_{0;G}^{(k_2)}, \quad [u_1u_2]_{\alpha;G}^{(k_1+k_2;\star)} \le |u_1|_{0;G}^{(k_1)}[u_2]_{\alpha;G}^{(k_2;\star)} + [u_1]_{\alpha;G}^{(k_1;\star)}|u_2|_{0;G}^{(k_2)}.$$

We also denote

$$[u]_{\alpha;r,G} = \sup_{\theta\in[0,\theta_0]} \sup_{\substack{\hat r,\check r\in[r_1,r_2]\\ \hat r\neq\check r}} \Big(\frac{|u(\hat r,\theta)-u(\check r,\theta)|}{|\hat r-\check r|^\alpha}\Big), \quad [u]_{\alpha;\theta,G} = \sup_{r\in[r_1,r_2]} \sup_{\substack{\hat\theta,\check\theta\in[0,\theta_0]\\ \hat\theta\neq\check\theta}} \Big(\frac{|u(r,\hat\theta)-u(r,\check\theta)|}{|\hat\theta-\check\theta|^\alpha}\Big).$$

Let us give some explanations on the choices of the weighted Hölder spaces. Near the sonic curve, (4.5) can be approximated as

$$A(r_2-r)^{1/2}\frac{\partial^2 w}{\partial r^2} - B(r_2-r)^{-1/2}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial\theta^2} = 0, \quad (A>0,\ B>0) \tag{4.11}$$

which is degenerate along the r direction while not along the θ direction. In order to study the Hölder estimates of solutions to (4.11), we transform it into an isotropic and uniformly elliptic equation by the coordinates transformation

$$\hat r = \delta^{-1}(r-r_2), \quad \hat\theta = \delta^{-3/4}\theta.$$

Via this transformation, (4.11) in $(r_2-2\delta, r_2-\delta)\times(-\delta^{3/4},\delta^{3/4})$ is transformed to an isotropic and uniformly elliptic equation in $(-2,-1)\times(-1,1)$. See Proposition 4.3 for more details. According to this analysis, it seems reasonable to choose the following space as the space of solutions of the problem (4.5), (4.8)–(4.10)

$$\mathscr{G}(G) = \Big\{u \in C^2(G) : u \in C^\alpha(\overline{G}),\ \frac{\partial u}{\partial r} \in C^\alpha_{(-1/2;\star)}(\overline{G}),\ \frac{\partial u}{\partial\theta} \in C^\alpha_{(0;\star)}(\overline{G}),\ \frac{\partial^2 u}{\partial r^2} \in C^\alpha_{(1/2;\star)}(\overline{G}),$$

$$\frac{\partial^2 u}{\partial r\partial\theta} \in C^\alpha_{(1/4;\star)}(\overline{G}),\ \frac{\partial^2 u}{\partial\theta^2} \in C^\alpha_{(0;\star)}(\overline{G}) \text{ and } \frac{\partial u}{\partial\theta}(\cdot,0)\Big|_{(r_1,r_2)} = \frac{\partial u}{\partial\theta}(\cdot,\theta_0)\Big|_{(r_1,r_2)} = 0\Big\},$$

which is a Banach space with the norm

$$\|u\|_{\mathscr{G}(G)} =\|u\|_{\alpha;G} + \Big\|\frac{\partial u}{\partial r}\Big\|^{(-1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial \theta}\Big\|^{(0;\star)}_{\alpha;G} + \Big\|\frac{\partial^2 u}{\partial r^2}\Big\|^{(1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial^2 u}{\partial r\partial\theta}\Big\|^{(1/4;\star)}_{\alpha;G} + \Big\|\frac{\partial^2 u}{\partial \theta^2}\Big\|^{(0;\star)}_{\alpha;G}.$$

However, as mentioned in the beginning of this section, in order to get the existence of solutions in this space, we need to give precise regularity of solutions near the sonic curve (see Remark 4.3). For $u \in \mathscr{G}(G)$, the asymptotic behavior of $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial^2}{\partial r^2}$ near r_2 is like $(r_2-r)^{1/2}$ and $(r_2-r)^{-1/2}$, respectively. Therefore, it is more convenient to investigate the precise regularity of solutions in a new space (τ,θ) with some suitable coordinate transformation $\tau(r) = O\big((r_2-r)^{1/2}\big)$. We can choose this transformation as

$$\tau(r) = \Big(\int_{r_2}^{r_1} \frac{1}{a_0(t)}dt\Big)^{-1}\int_{r_2}^{r}\frac{1}{a_0(t)}dt, \quad r_1 \le r \le r_2, \tag{4.12}$$

whose inverse transformation is denoted by $r(\tau)$. Simple computations yield

$$\begin{aligned}
\tau(r) =& \frac{1}{|r_2|c_*}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^{-1}(c_* - \phi_0'(r)) + O\big((r_2-r)\big),\\
\tau'(r) =& -\frac{c_*^3}{2|r_2|^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^{-1}(c_* - \phi_0'(r))^{-1} + O\big(1\big),\\
\tau''(r) =& -\frac{c_*^7}{4|r_2|^3}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^{-1}(c_* - \phi_0'(r))^{-3} + O\big((r_2-r)^{-1}\big).
\end{aligned}$$

Via this transformation, the degeneracy of (4.5) near the sonic curve is transformed to the degeneracy with a linear rate along the normal direction of the boundary where the equation is degenerate. However, the degeneracy rate along another direction is superlinear (Proposition 4.4). More precisely, (4.5) is transformed into (1.7) with $\lambda = 2$. In this section, we denote

$$\Omega = (0,1)\times(0,\theta_0)$$

and $C_*^\alpha(\overline{\Omega})$ and $C_{**}^\alpha(\overline{\Omega})$ are the spaces defined in § 3.2 with $\lambda = 2$. Set

$$\mathscr{B}(G) = \Big\{u \in C^1(G) : V, \tau\frac{\partial V}{\partial \tau} \in C_*^\alpha(\overline{\Omega}), \frac{\partial V}{\partial\theta} \in C_{**}^\alpha(\overline{\Omega}), \text{ and } \lim_{\tau\to 0^+}\tau\frac{\partial V}{\partial\tau}(\tau,\cdot)\Big|_{(0,\theta_0)} = 0 \quad \text{for } V(\tau,\theta) = \tau^{-1}u(r(\tau),\theta)\Big\},$$

which is a Banach space with the norm

$$\|u\|_{\mathscr{B}(G)} = \|V\|^*_{\alpha;\Omega} + \Big\|\tau\frac{\partial V}{\partial\tau}\Big\|^*_{\alpha;\Omega} + \Big\|\frac{\partial V}{\partial\theta}\Big\|^{**}_{\alpha;\Omega} \quad \text{with } V(\tau,\theta) = \tau^{-1}u(r(\tau),\theta), \quad (\tau,\theta)\in\Omega$$

and will be used to describe the precise regularity of solutions of the problem (4.5), (4.8)–(4.10) near the sonic curve.

The main results in this section are as follows.

Theorem 4.1 (Existence and Stability) *Let $0 < \alpha < 1$ and $b_0 \in \mathbb{R}$. Assume that $b \in C^{2,\alpha}([0,\theta_0])$ with $b'(0) = b'(\theta_0) = 0$. Then there exists a positive constant δ_0 depending only on γ, r_1, r_2, θ_0 and α such that if $\|b - b_0\|_{2,\alpha;(0,\theta_0)} \le \delta_0$, the problem (1.2)–(1.5) has at least one solution $\phi \in C^2([r_1,r_2)\times[0,\theta_0])\cap C^1([r_1,r_2]\times[0,\theta_0])$ satisfying*

(i) *$\phi \in C^{2,\alpha}([r_1,r_2)\times[0,\theta_0])$ with $\phi - \phi_0 \in \mathscr{G}(G)$ and $\dfrac{\partial\phi}{\partial r} - \phi_0' \in \mathscr{B}(G)$;*

(ii) *there exists a constant $M > 0$ depending only on γ, r_1, r_2, θ_0 and α such that*

$$\|\phi - \phi_0\|_{\mathscr{G}(G)} + \Big\|\frac{\partial\phi}{\partial r} - \phi_0'\Big\|_{\mathscr{B}(G)} \le M\|b - b_0\|_{2,\alpha;(0,\theta_0)}.$$

Theorem 4.2 (Uniqueness) *Let $0 < \alpha < 1$ and $b_0 \in \mathbb{R}$. There exists a positive constant δ_0 depending only on γ, r_1, r_2, θ_0 and α such that the problem (1.2)–(1.5) admits at most one solution $\phi \in C^{2,\alpha}([r_1, r_2) \times [0, \theta_0])$ with $\phi - \phi_0 \in \mathscr{G}(G)$ and satisfying $\|\phi - \phi_0\|_{\mathscr{G}(G)} \le \delta_0$.*

Remark 4.2 *Theorem 4.1 shows that the singularity of ϕ is the same as ϕ_0 near the sonic curve.*

4.2 Proof of the solvability

We start with the solvability, while the uniqueness will be proved by energy estimates given at the end of the subsection. According to the computations in §4.1, the problem (1.2)–(1.5) is equivalent to the problem (4.5), (4.8)–(4.10). In order to use the Schauder fixed point theorem, we should define a mapping by solving the corresponding linearized problem in a suitable space. This leads to the following consideration.

Given $\tilde{w} \in \mathscr{G}(G)$ with $\tilde{w}_r \in \mathscr{B}(G)$ and

$$\|\tilde{w}\|_{\mathscr{G}(G)} + \|\tilde{w}_r\|_{\mathscr{B}(G)} \le \sigma. \tag{4.13}$$

Consider the linear equation

$$\frac{\partial}{\partial r}\Big(a_0(r)\frac{\partial w}{\partial r}\Big) + a_1(r,\theta)\frac{\partial^2 w}{\partial r^2} + \big(a_2(r,\theta) + a_3(r)\big)\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} = 0, \qquad (r,\theta) \in G \tag{4.14}$$

with the following boundary conditions

$$\frac{\partial w}{\partial \theta}(r,0) = \frac{\partial w}{\partial \theta}(r,\theta_0) = 0, \qquad r_1 < r < r_2, \tag{4.15}$$

$$w(r_1,\theta) = g(\theta), \qquad 0 < \theta < \theta_0, \tag{4.16}$$

$$\frac{\partial w}{\partial r}(r_2,\theta) = 0, \qquad 0 < \theta < \theta_0, \tag{4.17}$$

where

$$a_1(r,\theta) = q_1(r, \tilde{w}_r(r,\theta)), \quad a_2(r,\theta) = q_2(r, \tilde{w}_r(r,\theta)), \qquad (r,\theta) \in G.$$

Since

$$a_0(r_2) = a_1(r_2,\theta) = 0, \quad 0 < \theta < \theta_0,$$

(4.14) is degenerate on $\{r_2\} \times (0, \theta_0)$, a portion of the boundary.

For notional convenience, throughout this subsection, M and M_i denote generic positive constants depending only on γ, r_1, r_2, θ_0 and α, while $M(\cdot)$ depends also on the variables in the parentheses.

In order to describe the properties of a_0, a_1, a_2 and a_3, we set

$$\mathscr{R}_1(G) = \Big\{u \in C^1(G) : u \in C^{\alpha}_{(-1/2;\star)}(\overline{G}),\ \frac{\partial u}{\partial r} \in C^{\alpha}_{(1/2;\star)}(\overline{G}),\ \frac{\partial u}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(\overline{G})\Big\},$$

$$\mathscr{R}_2(G) = \Big\{u \in C^1(G) : u \in C^{\alpha}_{(0;\star)}(\overline{G}),\ \frac{\partial u}{\partial r} \in C^{\alpha}_{(1;\star)}(\overline{G}),\ \frac{\partial u}{\partial \theta} \in C^{\alpha}_{(3/4;\star)}(\overline{G})\Big\}$$

with the norms

$$\|u\|_{\mathscr{R}_1(G)} = \|u\|^{(-1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial r}\Big\|^{(1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial \theta}\Big\|^{(1/4;\star)}_{\alpha;G}, \quad \|u\|_{\mathscr{R}_2(G)} = \|u\|^{(0;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial r}\Big\|^{(1;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial \theta}\Big\|^{(3/4;\star)}_{\alpha;G}.$$

Proposition 4.1 *The coefficients of (4.14) possess the following properties*

(i) *$a_0 \in \mathscr{R}_1(G)$ and $a_3 \in \mathscr{R}_2(G)$;*

(ii) *$a_1 \in \mathscr{R}_1(G) \cap \mathscr{B}(G)$, $a_2 \in \mathscr{R}_2(G)$ and*

$$\|a_1\|_{\mathscr{R}_1(G)} + \|a_1\|_{\mathscr{B}(G)} \le M\sigma, \quad \|a_2\|_{\mathscr{R}_2(G)} \le M\sigma. \tag{4.18}$$

Proof. We just prove (ii) since (i) is clear. First, it follows from $\tilde{w} \in \mathscr{G}(G)$ that

$$\Big\|\frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\Big\|^{(0;\star)}_{\alpha;G} \le M, \quad \Big\|\frac{\partial}{\partial r}\Big(\frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\Big)\Big\|^{(1/2;\star)}_{\alpha;G} \le M, \quad \Big\|\frac{\partial}{\partial \theta}\Big(\frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\Big)\Big\|^{(1/4;\star)}_{\alpha;G} \le M.$$

These, $\tilde{w} \in \mathscr{G}(G)$ and Lemma 4.1 yield that

$$\begin{aligned} a_1(r,\theta) &= \frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\tilde{w}_r(r,\theta) \in C^{\alpha}_{(-1/2;\star)}(G), \\ \frac{\partial a_1}{\partial r}(r,\theta) &= \frac{\partial}{\partial r}\Big(\frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\Big)\tilde{w}_r(r,\theta) + \frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\frac{\partial \tilde{w}_r(r,\theta)}{\partial r} \in C^{\alpha}_{(1/2;\star)}(G), \\ \frac{\partial a_1}{\partial \theta}(r,\theta) &= \frac{\partial}{\partial \theta}\Big(\frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\Big)\tilde{w}_r(r,\theta) + \frac{a_1(r,\theta)}{\tilde{w}_r(r,\theta)}\frac{\partial \tilde{w}_r(r,\theta)}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(G). \end{aligned}$$

Hence $a_1 \in \mathscr{R}_1(G)$ with

$$\|a_1\|_{\mathscr{R}_1(G)} \le M\|\tilde{w}\|_{\mathscr{G}(G)} \le M\sigma.$$

Similarly, $a_2 \in \mathscr{R}_2(G)$ with

$$\|a_2\|_{\mathscr{R}_2(G)} \le M\|\tilde{w}\|_{\mathscr{G}(G)} \le M\sigma.$$

Finally, it follows from $\tilde{w} \in \mathscr{G}(G)$ and $\tilde{w}_r \in \mathscr{B}(G)$ that

$$\Big\|\frac{a_1(r(\tau),\theta)}{\tilde{w}_r(r(\tau),\theta)}\Big\|^{*}_{\alpha;\Omega} \le M, \quad \Big\|\tau\frac{\partial}{\partial \tau}\Big(\frac{a_1(r(\tau),\theta)}{\tilde{w}_r(r(\tau),\theta)}\Big)\Big\|^{*}_{\alpha;\Omega} \le M, \quad \Big\|\frac{\partial}{\partial \theta}\Big(\frac{a_1(r(\tau),\theta)}{\tilde{w}_r(r(\tau),\theta)}\Big)\Big\|^{**}_{\alpha;\Omega} \le M.$$

Due to these, $\tilde{w}_r \in \mathscr{B}(G)$ with (4.13) and Lemma 3.3, one can conclude from

$$\frac{a_1(r(\tau),\theta)}{\tau} = \frac{a_1(r(\tau),\theta)}{\tilde{w}_r(r(\tau),\theta)}\frac{\tilde{w}_r(r(\tau),\theta)}{\tau}, \quad (\tau,\theta) \in \Omega$$

that $a_1 \in \mathscr{B}(G)$ with

$$\|a_1\|_{\mathscr{B}(G)} \le M\|\tilde{w}_r\|_{\mathscr{B}(G)} \le M\sigma.$$

The proof is complete. □

Proposition 4.2 *There exists $\sigma_1 > 0$ depending only on γ, r_1, r_2, θ_0 and α, such that for any $0 < \sigma \le \sigma_1$ and any $g \in C^{2,\alpha}([0,\theta_0])$ with $g'(0) = g'(\theta_0) = 0$, the problem (4.14)–(4.17) admits a solution $w \in C^{2,\alpha}([r_1,r_2) \times [0,\theta_0])$ with $w_r \in C(\overline{G})$, and the solution also satisfies the following three estimates*

$$\|w\|_{2,\alpha;(r_1,r)\times(0,\theta_0)} \le M(r)\|g\|_{2,\alpha;(0,\theta_0)}, \quad r \in (r_1,r_2), \tag{4.19}$$

$$\Big|\frac{\partial w}{\partial r}(r,\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r)^{1/2}, \quad (r,\theta) \in G, \tag{4.20}$$

$$\int_G \Big((r_2-r)^{-3/2}\Big(\frac{\partial w}{\partial r}\Big)^2 + (r_2-r)^{1/2}\Big(\frac{\partial^2 w}{\partial r^2}\Big)^2 + \Big(\frac{\partial^2 w}{\partial r\partial \theta}\Big)^2\Big)drd\theta \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.21}$$

Proof. Extend the functions a_0, a_1, a_2 and a_3 to the domain $G_0 = (r_1-1,r_2) \times (0,\theta_0)$ such that the extended functions, denoted by themselves for convenience, belong to $C^{1,\alpha}([r_1-1,r_2) \times [0,\theta_0])$ and possess the same properties and estimates as the ones of the original functions.

It can be shown that there exist $\bar{\sigma}_1 \in (0,1)$, $\kappa > 0$ and $r_3 \in (r_1,r_2)$ such that for any $0 < \sigma \le \bar{\sigma}_1$,

$$a_0(r) + a_1(r,\theta) \ge \kappa(r_2-r)^{1/2}, \quad (r,\theta) \in G, \tag{4.22}$$

$$\frac{\partial}{\partial r}\Big(a_0'(r)+a_2(r,\theta)+a_3(r)\Big)\le -\kappa(r_2-r)^{-3/2},\quad (r,\theta)\in(r_3,r_2)\times(0,\theta_0), \tag{4.23}$$

where $\bar\sigma_1$, κ and r_3 depend only on γ, r_1, r_2, θ_0 and α. For $0<\varepsilon<1$, consider the following approximating problem

$$\frac{\partial}{\partial r}\Big(a_{0,\varepsilon}(r)\frac{\partial w_\varepsilon}{\partial r}\Big)+a_{1,\varepsilon}(r,\theta)\frac{\partial^2 w_\varepsilon}{\partial r^2}+\big(a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)\frac{\partial w_\varepsilon}{\partial r}+\frac{\partial^2 w_\varepsilon}{\partial\theta^2}=0,\quad (r,\theta)\in G, \tag{4.24}$$

$$\frac{\partial w_\varepsilon}{\partial\theta}(r,0)=\frac{\partial w_\varepsilon}{\partial\theta}(r,\theta_0)=0,\qquad r_1<r<r_2, \tag{4.25}$$

$$w_\varepsilon(r_1,\theta)=g(\theta),\qquad 0<\theta<\theta_0, \tag{4.26}$$

$$\frac{\partial w_\varepsilon}{\partial r}(r_2,\theta)=0,\qquad 0<\theta<\theta_0, \tag{4.27}$$

where $a_{0,\varepsilon},a_{1,\varepsilon},a_{2,\varepsilon},a_{3,\varepsilon}\in C^{1,\alpha}(\overline{G})$ with

$$a_{0,\varepsilon}(r)=a_0(r-\varepsilon),\quad a_{3,\varepsilon}(r)=a_3(r-\varepsilon),\qquad r_1<r<r_2,$$
$$a_{1,\varepsilon}(r,\theta)=a_1(r-\varepsilon,\theta),\quad a_{2,\varepsilon}(r,\theta)=a_2(r-\varepsilon,\theta),\qquad (r,\theta)\in G.$$

Then, (4.24) is uniformly elliptic in G due to (4.22). According to the classical theory, the problem (4.24)–(4.27) admits a unique classical solution $w_\varepsilon\in C^{2,\alpha}(\overline{G})$.

We now estimate the solution w_ε of the problem (4.24)–(4.27).

First, the maximal principle yields

$$\|w_\varepsilon\|_{L^\infty(G)}\le\|g\|_{L^\infty(G)}. \tag{4.28}$$

Second, by even extension with respect to $\theta=0$ and $\theta=\theta_0$, w_ε may be regarded as a solution of the equation in the domain $(r_1,r_2)\times(-\theta_0,2\theta_0)$. Therefore, for any $r\in(r_1,r_2)$, it follows from the classical Schauder theory and (4.28) that

$$\|w_\varepsilon\|_{2,\alpha;(r_1,r)\times(0,\theta_0)}\le M(r)(\|w_\varepsilon\|_{L^\infty(G)}+\|g\|_{2,\alpha;(0,\theta_0)})\le M(r)\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.29}$$

Next, set

$$u_\varepsilon(r,\theta)=\frac{\partial w_\varepsilon}{\partial r}(r,\theta),\quad (r,\theta)\in\overline{G}\qquad\text{and}\qquad G_0=(r_0,r_2)\times(0,\theta_0),$$

where $r_0\in(r_3,r_2)$ will be determined below. Then, $u_\varepsilon\in C^{1,\alpha}(\overline{G}_0)$ is a solution of the problem

$$\frac{\partial}{\partial r}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\frac{\partial u_\varepsilon}{\partial r}\Big)+\frac{\partial}{\partial r}\Big(\big(a_{0,\varepsilon}'(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)u_\varepsilon\Big)+\frac{\partial^2 u_\varepsilon}{\partial\theta^2}=0,\quad (r,\theta)\in G_0, \tag{4.30}$$

$$\frac{\partial u_\varepsilon}{\partial\theta}(r,0)=\frac{\partial u_\varepsilon}{\partial\theta}(r,\theta_0)=0,\qquad r_0<r<r_2, \tag{4.31}$$

$$u_\varepsilon(r_0,\theta)=g_0(\theta),\qquad 0<\theta<\theta_0, \tag{4.32}$$

$$u_\varepsilon(r_2,\theta)=0,\qquad 0<\theta<\theta_0, \tag{4.33}$$

where

$$g_0(\theta)=u_\varepsilon(r_0,\theta)=\frac{\partial w_\varepsilon}{\partial r}(r_0,\theta),\quad 0\le\theta\le\theta_0.$$

For $0<\varepsilon<\varepsilon_0<\min\{1,r_0-r_3\}$, (4.22) and (4.23) guarantee that the comparison principle is valid for the problem (4.30)–(4.33). We estimate u_ε near $r=r_2$ by constructing barrier functions. Define

$$\overline{u}_\varepsilon(r,\theta)=M_0\|g\|_{2,\alpha;(0,\theta_0)}\Big(\frac{r_2-r+\varepsilon}{a_{0,\varepsilon}(r)}-\frac{(r_2-r+\varepsilon)^{1+\alpha/2}}{a_{0,\varepsilon}(r)}\Big),\quad (r,\theta)\in\overline{G}_0$$

with $M_0 > 0$ to be determined below. Then

$$\begin{aligned}
&\frac{\partial}{\partial r}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\frac{\partial \overline{u}_\varepsilon}{\partial r}\Big)+\frac{\partial}{\partial r}\Big(\big(a'_{0,\varepsilon}(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)\overline{u}_\varepsilon\Big)+\frac{\partial^2 \overline{u}_\varepsilon}{\partial \theta^2}\\
=&\frac{\partial^2}{\partial r^2}\big(a_{0,\varepsilon}(r)\overline{u}_\varepsilon\big)+\frac{\partial}{\partial r}\Big(a_{1,\varepsilon}(r,\theta)\frac{\partial \overline{u}_\varepsilon}{\partial r}+a_{2,\varepsilon}(r,\theta)\overline{u}_\varepsilon+a_{3,\varepsilon}(r)\overline{u}_\varepsilon\Big)+\frac{\partial^2 \overline{u}_\varepsilon}{\partial \theta^2}\\
=&-\frac{\alpha}{2}\Big(1+\frac{\alpha}{2}\Big)M_0\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r+\varepsilon)^{-1+\alpha/2}\\
&\quad+\frac{\partial}{\partial r}\Big(a_{1,\varepsilon}(r,\theta)\frac{\partial \overline{u}_\varepsilon}{\partial r}+a_{2,\varepsilon}(r,\theta)\overline{u}_\varepsilon+a_{3,\varepsilon}(r)\overline{u}_\varepsilon\Big),\qquad (r,\theta)\in G_0.
\end{aligned}\tag{4.34}$$

We now estimate the second term on the right side of (4.34). First, owing to $a_1 \in \mathscr{R}_1(G)\cap\mathscr{B}(G)$ with (4.18), we have that

$$\Big|\frac{a_1(r,\theta)}{\tau(r)}\Big|=\frac{(r_2-r)^{1/2}}{\tau(r)}\Big|\frac{a_1(r,\theta)}{(r_2-r)^{1/2}}\Big|\le M\|a_1\|_{\mathscr{R}_1(G)}\le M\sigma\le M\sigma_1,\quad (r,\theta)\in G \tag{4.35}$$

and

$$\Big|\tau\frac{\partial}{\partial \tau}\Big(\frac{a_1(r(\tau),\theta)}{\tau}\Big)\Big|\le\|a_1\|_{\mathscr{B}(G)}\tau^\alpha\le M\sigma\tau^\alpha\le M\sigma_1\tau^\alpha,\quad (\tau,\theta)\in\Omega. \tag{4.36}$$

It follows from (4.36) that

$$\begin{aligned}
\Big|\frac{\partial}{\partial r}\Big(\frac{a_1(r,\theta)}{\tau(r)}\Big)\Big|=&\big|\tau'(r)\big|\Big|\frac{\partial}{\partial \tau}\Big(\frac{a_1(r(\tau),\theta)}{\tau}\Big)\Big|_{\tau=\tau(r)}\Big|\\
\le& M\sigma_1(r_2-r)^{-1/2}\tau^{-1+\alpha}(r)\le M\sigma_1(r_2-r)^{-1+\alpha/2},\quad (r,\theta)\in G.
\end{aligned}\tag{4.37}$$

Combining (4.35) with (4.37) yields that

$$\Big|\frac{a_{1,\varepsilon}(r,\theta)}{\tau(r-\varepsilon)}\Big|\le M\sigma_1,\quad \Big|\frac{\partial}{\partial r}\Big(\frac{a_{1,\varepsilon}(r,\theta)}{\tau(r-\varepsilon)}\Big)\Big|\le M\sigma_1(r_2-r+\varepsilon)^{-1+\alpha/2},\quad (r,\theta)\in G_0. \tag{4.38}$$

Here $r(\cdot)$ and $\tau(\cdot)$ are the transformations defined in §4.1. Additionally, one has that

$$\begin{aligned}
\tau(r-\varepsilon)\frac{\partial \overline{u}_\varepsilon}{\partial r}=&-M_0\|g\|_{2,\alpha;(0,\theta_0)}\Big(\frac{\tau(r-\varepsilon)}{a_{0,\varepsilon}(r)}+\frac{\tau(r-\varepsilon)(r_2-r+\varepsilon)a'_{0,\varepsilon}(r)}{a^2_{0,\varepsilon}(r)}\Big)\\
&+M_0\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r+\varepsilon)^{\alpha/2}\\
&\cdot\Big(\Big(1+\frac{\alpha}{2}\Big)\frac{\tau(r-\varepsilon)}{a_{0,\varepsilon}(r)}+\frac{\tau(r-\varepsilon)(r_2-r+\varepsilon)a'_{0,\varepsilon}(r)}{a^2_{0,\varepsilon}(r)}\Big),\quad (r,\theta)\in G_0.
\end{aligned}$$

Note that

$$\Big|\frac{\tau(r)}{a_0(r)}\Big|\le M,\quad \Big|\frac{\tau(r)(r_2-r)a'_0(r)}{a^2_0(r)}\Big|\le M,\quad r_1<r<r_2$$

and

$$\Big|\frac{\partial}{\partial r}\Big(\frac{\tau(r)}{a_0(r)}\Big)\Big|\le M(r_2-r)^{-1/2},\quad \Big|\frac{\partial}{\partial r}\Big(\frac{\tau(r)(r_2-r)a'_0(r)}{a^2_0(r)}\Big)\Big|\le M(r_2-r)^{-1/2},\quad r_1<r<r_2.$$

Therefore,

$$\Big|\tau(r-\varepsilon)\frac{\partial \overline{u}_\varepsilon}{\partial r}\Big|\le MM_0\|g\|_{2,\alpha;(0,\theta_0)},\quad (r,\theta)\in G_0, \tag{4.39}$$

$$\Big|\frac{\partial}{\partial r}\Big(\tau(r-\varepsilon)\frac{\partial \overline{u}_\varepsilon}{\partial r}\Big)\Big|\le MM_0\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r+\varepsilon)^{-1+\alpha/2},\quad (r,\theta)\in G_0. \tag{4.40}$$

It follows from (4.38)–(4.40) that

$$\Big|\frac{\partial}{\partial r}\Big(a_{1,\varepsilon}(r,\theta)\frac{\partial\overline{u}_\varepsilon}{\partial r}\Big)\Big| \le \Big|\frac{a_{1,\varepsilon}(r,\theta)}{\tau(r-\varepsilon)}\frac{\partial}{\partial r}\Big(\tau(r-\varepsilon)\frac{\partial\overline{u}_\varepsilon}{\partial r}\Big)\Big| + \Big|\tau(r-\varepsilon)\frac{\partial\overline{u}_\varepsilon}{\partial r}\frac{\partial}{\partial r}\Big(\frac{a_{1,\varepsilon}(r,\theta)}{\tau(r-\varepsilon)}\Big)\Big|$$
$$\le M_1M_0\|g\|_{2,\alpha;(0,\theta_0)}\sigma_1(r_2-r+\varepsilon)^{-1+\alpha/2},\quad (r,\theta)\in G_0. \tag{4.41}$$

Second, due to $a_2\in\mathscr{R}_2(G)$ and (4.18), it holds that

$$\begin{aligned}\Big|\frac{\partial}{\partial r}\big(a_{2,\varepsilon}(r,\theta)\overline{u}_\varepsilon\big)\Big| \le& \Big|a_{2,\varepsilon}(r,\theta)\frac{\partial\overline{u}_\varepsilon}{\partial r}\Big| + \Big|\overline{u}_\varepsilon\frac{\partial a_{2,\varepsilon}}{\partial r}(r,\theta)\Big|\\ \le& MM_0\|g\|_{2,\alpha;(0,\theta_0)}\sigma(r_2-r+\varepsilon)^{-1/2}\\ \le& M_2M_0\|g\|_{2,\alpha;(0,\theta_0)}\sigma_1(r_2-r+\varepsilon)^{-1+\alpha/2},\quad (r,\theta)\in G_0.\end{aligned}$$

Third, direct calculations yield

$$\Big|\frac{\partial}{\partial r}\big(a_{3,\varepsilon}(r)\overline{u}_\varepsilon\big)\Big| \le \Big|a_{3,\varepsilon}(r)\frac{\partial\overline{u}_\varepsilon}{\partial r}\Big| + |a_{3,\varepsilon}'(r)\overline{u}_\varepsilon| \le M_3M_0\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r+\varepsilon)^{-1/2},\quad (r,\theta)\in G_0.$$

Now we determine σ_1, r_0, ε_0 and M_0 in turn. First fix $0<\sigma_1\le\bar\sigma_1$ and $\max\{r_3,r_2-1\}<r_0<r_2$ such that

$$\sigma_1\le\frac{\alpha}{4(M_1+M_2+1)}\Big(1+\frac{\alpha}{2}\Big),\qquad (r_2-r_0)^{-(1-\alpha)/2}\ge\frac{16M_3}{\alpha(2+\alpha)},$$

then fix $0<\varepsilon_0<\min\{r_0-r_3,r_0-r_2+1\}$ such that

$$(r_2-r_0+\varepsilon_0)^{-(1-\alpha)/2}\ge\frac{8M_3}{\alpha(2+\alpha)}.$$

With σ_1, r_0 and ε_0 so fixed, finally choose $M_0>0$ sufficiently large such that

$$\overline{u}_\varepsilon(r_0,\theta)\ge\|g_0\|_{L^\infty(0,\theta_0)},\quad 0<\theta<\theta_0$$

for all $0<\varepsilon<\varepsilon_0$. Then, for any $0<\varepsilon<\varepsilon_0$, $\overline{u}_\varepsilon$ is a supersolution of the problem (4.30)–(4.33). It is clear that $-\overline{u}_\varepsilon$ is a subsolution. The comparison principle leads to

$$-\overline{u}_\varepsilon(r,\theta)\le u_\varepsilon(r,\theta)\le\overline{u}_\varepsilon(r,\theta),\quad (r,\theta)\in G_0,$$

which yields

$$\Big|\frac{\partial w_\varepsilon}{\partial r}(r,\theta)\Big| = |u_\varepsilon(r,\theta)|\le M\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r+\varepsilon)^{1/2},\quad (r,\theta)\in G_0. \tag{4.42}$$

Finally, since $u_\varepsilon\in C^{1,\alpha}(G_0)$ is the solution of the problem (4.30)–(4.33),

$$\int_{G_0}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\frac{\partial u_\varepsilon}{\partial r}\frac{\partial\zeta}{\partial r}+\big(a_{0,\varepsilon}'(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)u_\varepsilon\frac{\partial\zeta}{\partial r}+\frac{\partial u_\varepsilon}{\partial\theta}\frac{\partial\zeta}{\partial\theta}\Big)drd\theta$$
$$+\int_0^{\theta_0}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\frac{\partial u_\varepsilon}{\partial r}+\big(a_{0,\varepsilon}'(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)u_\varepsilon\Big)\zeta\Big|_{r=r_0}d\theta=0 \tag{4.43}$$

for any $\zeta\in C^1(\overline{G}_0)$ with $\zeta(r_2,\cdot)\Big|_{(0,\theta_0)}=0$. Take $\zeta=u_\varepsilon$ in (4.43) to get

$$0=\int_{G_0}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2+\big(a_{0,\varepsilon}'(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)u_\varepsilon\frac{\partial u_\varepsilon}{\partial r}+\Big(\frac{\partial u_\varepsilon}{\partial\theta}\Big)^2\Big)drd\theta$$
$$+\int_0^{\theta_0}\Big(\big(a_{0,\varepsilon}(r)+a_{1,\varepsilon}(r,\theta)\big)\frac{\partial u_\varepsilon}{\partial r}u_\varepsilon+\big(a_{0,\varepsilon}'(r)+a_{2,\varepsilon}(r,\theta)+a_{3,\varepsilon}(r)\big)u_\varepsilon^2\Big)\Big|_{r=r_0}d\theta$$

$$= \int_{G_0} \Big(\big(a_{0,\varepsilon}(r) + a_{1,\varepsilon}(r,\theta)\big)\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2 - \frac{1}{2}\frac{\partial}{\partial r}\Big(a'_{0,\varepsilon}(r) + a_{2,\varepsilon}(r,\theta) + a_{3,\varepsilon}(r)\Big)u_\varepsilon^2 + \Big(\frac{\partial u_\varepsilon}{\partial \theta}\Big)^2\Big) dr d\theta$$
$$+ \int_0^{\theta_0} \Big(\big(a_{0,\varepsilon}(r) + a_{1,\varepsilon}(r,\theta)\big)\frac{\partial u_\varepsilon}{\partial r} u_\varepsilon + \frac{1}{2}\big(a'_{0,\varepsilon}(r) + a_{2,\varepsilon}(r,\theta) + a_{3,\varepsilon}(r)\big)u_\varepsilon^2\Big)\Big|_{r=r_0} d\theta.$$

This, together with (4.22), (4.23) and (4.29), leads to

$$\int_{G_0} \Big((r_2 - r + \varepsilon)^{1/2}\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2 + (r_2 - r + \varepsilon)^{-3/2} u_\varepsilon^2 + \Big(\frac{\partial u_\varepsilon}{\partial \theta}\Big)^2\Big) dr d\theta \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.44}$$

Using the a priori estimates (4.28), (4.29), (4.42) and (4.44), one may complete the proof of the proposition by a standard limiting process, so the details are omitted. □

Remark 4.3 *It is noted that in deriving* (4.42)*, one has used the fact that not only* $\tilde{w} \in \mathscr{R}(G)$ *but also* $\tilde{w}_r \in \mathscr{B}(G)$*. Indeed,* a_1 *just belongs to* $\mathscr{R}_1(G)$ *if* $\tilde{w} \in \mathscr{R}(G)$ *while* $\tilde{w}_r \notin \mathscr{B}(G)$*. Then instead of* (4.41)*, one just gets that*

$$\Big|\frac{\partial}{\partial r}\Big(a_{1,\varepsilon}(r,\theta)\frac{\partial \overline{u}_\varepsilon}{\partial r}\Big)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}\sigma_1 (r_2 - r + \varepsilon)^{-1}, \quad (r,\theta) \in G_0,$$

which is not sufficient to show that $\overline{u}_\varepsilon$ *is a supersolution of* (4.30).

Proposition 4.3 *Assume that* $0 < \sigma \le \sigma_1$*,* $g \in C^{2,\alpha}([0,\theta_0])$ *with* $g'(0) = g'(\theta_0) = 0$ *and* $w \in C^{2,\alpha}([r_1, r_2) \times [0,\theta_0])$ *with* $w_r \in C(\overline{G})$ *is the solution of the problem* (4.14)–(4.17) *obtained in Proposition 4.2. Then there exists* $\sigma_2 > 0$ *depending only on* γ*,* r_1*,* r_2*,* θ_0 *and* α*, such that for any* $0 < \sigma \le \min\{\sigma_1, \sigma_2\}$*,* $w \in \mathscr{G}(G)$ *with*

$$\|w\|_{\mathscr{G}(G)} \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.45}$$

Proof. Set

$$u(r,\theta) = \frac{\partial w}{\partial r}(r,\theta), \quad (r,\theta) \in \overline{G}.$$

Then, Proposition 4.2 implies that $u \in C^{1,\alpha}([r_1, r_2) \times [0,\theta_0]) \cap C(\overline{G})$ is a weak solution of

$$\frac{\partial}{\partial r}\Big(\big(a_0(r) + a_1(r,\theta)\big)\frac{\partial u}{\partial r}\Big) + \frac{\partial}{\partial r}\Big(\big(a'_0(r) + a_2(r,\theta) + a_3(r)\big)u\Big) + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad (r,\theta) \in G. \tag{4.46}$$

For $0 < \delta < \delta_1 = \min\big\{(r_2 - r_1)/2, (\theta_0/2)^{4/3}\big\}$ and $\delta^{3/4} < \bar\theta < \theta_0 - \delta^{3/4}$, denote

$$E = (r_2 - 2\delta, r_2 - \delta) \times (\bar\theta - \delta^{3/4}, \bar\theta + \delta^{3/4}).$$

Rewrite (4.46) as

$$\frac{\partial}{\partial r}\Big(p(r,\theta)\frac{\partial u}{\partial r}(r,\theta)\Big) + \frac{\partial^2 u}{\partial \theta^2}(r,\theta) = \frac{\partial f}{\partial r}(r,\theta), \quad (r,\theta) \in G, \tag{4.47}$$

where

$$p(r,\theta) = a_0(r) + a_1(r,\theta), \quad f(r,\theta) = -\big(a'_0(r) + a_2(r,\theta) + a_3(r)\big)u(r,\theta), \qquad (r,\theta) \in G.$$

Introduce the coordinates transformation

$$r = r_2 + \delta\tilde r, \quad \theta = \bar\theta + \delta^{3/4}\tilde\theta, \qquad (\tilde r, \tilde\theta) \in \tilde E = (-2,-1) \times (-1,1)$$

and the functions transformation

$$\tilde u(\tilde r,\tilde\theta) = u(r_2 + \delta\tilde r, \bar\theta + \delta^{3/4}\tilde\theta),\ \tilde p(\tilde r,\tilde\theta) = p(r_2 + \delta\tilde r, \bar\theta + \delta^{3/4}\tilde\theta),\ \tilde f(\tilde r,\tilde\theta) = f(r_2 + \delta\tilde r, \bar\theta + \delta^{3/4}\tilde\theta),\ (\tilde r,\tilde\theta) \in \tilde E.$$

Then, $\tilde{u} \in C^1(\overline{\tilde{E}})$ is a solution of the equation

$$\frac{\partial}{\partial \tilde{r}}\Big(\delta^{-1/2}\tilde{p}(\tilde{r},\tilde{\theta})\frac{\partial \tilde{u}}{\partial \tilde{r}}(\tilde{r},\tilde{\theta})\Big) + \frac{\partial^2 \tilde{u}}{\partial \tilde{\theta}^2}(\tilde{r},\tilde{\theta}) = \delta^{1/2}\frac{\partial \tilde{f}}{\partial \tilde{r}}(\tilde{r},\tilde{\theta}), \quad (\tilde{r},\tilde{\theta}) \in \tilde{E}.$$

From $a_1 \in \mathscr{R}_1(G)$ with (4.18), there exist $\sigma_2 > 0$ and $0 < \kappa_1 \leq \kappa_2$ depending only on γ, r_1, r_2, θ_0 and α, such that for any $0 < \sigma \leq \sigma_2$ and any $0 < \delta < \delta_1$,

$$\kappa_1 \leq \delta^{-1/2}\tilde{p}(\tilde{r},\tilde{\theta}) \leq \kappa_2, \quad \Big|\delta^{-1/2}\frac{\partial \tilde{p}}{\partial \tilde{r}}(\tilde{r},\tilde{\theta})\Big| + \Big|\delta^{-1/2}\frac{\partial \tilde{p}}{\partial \tilde{\theta}}(\tilde{r},\tilde{\theta})\Big| \leq \kappa_2, \qquad (\tilde{r},\tilde{\theta}) \in \tilde{E}.$$

Thus, the classical Hölder estimates yield

$$[\tilde{u}]_{\alpha;\tilde{E}} + \Big[\frac{\partial \tilde{u}}{\partial \tilde{r}}\Big]_{\alpha;\tilde{E}} + \Big|\frac{\partial \tilde{u}}{\partial \tilde{r}}\Big|_{0;\tilde{E}} + \Big[\frac{\partial \tilde{u}}{\partial \tilde{\theta}}\Big]_{\alpha;\tilde{E}} + \Big|\frac{\partial \tilde{u}}{\partial \tilde{\theta}}\Big|_{0;\tilde{E}} \leq M\Big(\delta^{1/2}[\tilde{f}]_{\alpha;\tilde{E}} + \delta^{1/2}|\tilde{f}|_{0;\tilde{E}} + |\tilde{u}|_{0;\tilde{E}}\Big).$$

Transform this estimate into the (r,θ) coordinates to get

$$\begin{aligned}
&\delta^{\alpha}[u]_{\alpha;r,E} + \delta^{3\alpha/4}[u]_{\alpha;\theta,E} + \delta^{1+\alpha}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;r,E} + \delta^{1+3\alpha/4}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;\theta,E} + \delta\Big|\frac{\partial u}{\partial r}\Big|_{0;E}\\
&\quad + \delta^{3/4+\alpha}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;r,E} + \delta^{3/4+3\alpha/4}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;\theta,E} + \delta^{3/4}\Big|\frac{\partial u}{\partial \theta}\Big|_{0;E}\\
\leq& M\Big(\delta^{1/2+\alpha}[f]_{\alpha;r,E} + \delta^{1/2+3\alpha/4}[f]_{\alpha;\theta,E} + \delta^{1/2}|f|_{0;E} + |u|_{0;E}\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\delta^{-1/2+\alpha}[u]_{\alpha;r,E} + \delta^{-1/2+3\alpha/4}[u]_{\alpha;\theta,E} + \delta^{1/2+\alpha}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;r,E} + \delta^{1/2+3\alpha/4}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;\theta,E} + \delta^{1/2}\Big|\frac{\partial u}{\partial r}\Big|_{0;E}\\
&\quad + \delta^{1/4+\alpha}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;r,E} + \delta^{1/4+3\alpha/4}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;\theta,E} + \delta^{1/4}\Big|\frac{\partial u}{\partial \theta}\Big|_{0;E}\\
\leq& M\Big(\delta^{\alpha}\big[(a_0' + a_2 + a_3)u\big]_{\alpha;r,E} + \delta^{3\alpha/4}\big[(a_0' + a_2 + a_3)u\big]_{\alpha;\theta,E} + \big|(a_0' + a_2 + a_3)u\big|_{0;E} + \delta^{-1/2}|u|_{0;E}\Big)\\
\leq& M\delta^{\alpha}\Big(|a_0' + a_2 + a_3|_{0;E}[u]_{\alpha;r,E} + [a_0' + a_2 + a_3]_{\alpha;r,E}|u|_{0;E}\Big)\\
&\quad + M\delta^{3\alpha/4}\Big(|a_0' + a_2 + a_3|_{0;E}[u]_{\alpha;\theta,E} + [a_0' + a_2 + a_3]_{\alpha;\theta,E}|u|_{0;E}\Big)\\
&\quad + M|a_0' + a_2 + a_3|_{0;E}|u|_{0;E} + M\delta^{-1/2}|u|_{0;E}\\
\leq& M\Big(\delta^{-1/2+\alpha}[u]_{\alpha;r,E} + \delta^{-1/2+3\alpha/4}[u]_{\alpha;\theta,E} + \delta^{-1/2}|u|_{0;E}\Big).
\end{aligned}$$

Additionally, it follows from the interpolation inequalities that

$$\delta^{\alpha}[u]_{\alpha;r,E} \leq \varepsilon\delta^{1+\alpha}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;r,E} + M(\varepsilon)|u|_{0;E}, \quad \delta^{3\alpha/4}[u]_{\alpha;\theta,E} \leq \varepsilon\delta^{3/4+3\alpha/4}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;\theta,E} + M(\varepsilon)|u|_{0;E}$$

with $\varepsilon > 0$. These, together with (4.20), yield

$$\begin{aligned}
&\delta^{-1/2+\alpha}[u]_{\alpha;r,E} + \delta^{-1/2+3\alpha/4}[u]_{\alpha;\theta,E} + \delta^{1/2+\alpha}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;r,E} + \delta^{1/2+3\alpha/4}\Big[\frac{\partial u}{\partial r}\Big]_{\alpha;\theta,E} + \delta^{1/2}\Big|\frac{\partial u}{\partial r}\Big|_{0;E}\\
&\quad + \delta^{1/4+\alpha}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;r,E} + \delta^{1/4+3\alpha/4}\Big[\frac{\partial u}{\partial \theta}\Big]_{\alpha;\theta,E} + \delta^{1/4}\Big|\frac{\partial u}{\partial \theta}\Big|_{0;E} \leq M\delta^{-1/2}|u|_{0;E} \leq M\|g\|_{2,\alpha;(0,\theta_0)}.
\end{aligned} \tag{4.48}$$

Based on this estimate, on can claim that $u \in C^{\alpha}_{(-1/2;\star)}(\overline{G})$, $\dfrac{\partial u}{\partial r} \in C^{\alpha}_{(1/2;\star)}(\overline{G})$, $\dfrac{\partial u}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(\overline{G})$ and

$$\|u\|^{(-1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial r}\Big\|^{(1/2;\star)}_{\alpha;G} + \Big\|\frac{\partial u}{\partial \theta}\Big\|^{(1/4;\star)}_{\alpha;G} \leq M\|g\|_{2,\alpha;(0,\theta_0)}, \tag{4.49}$$

which is assumed to hold for this moment. Thus,

$$\frac{\partial w}{\partial r} = u \in C^{\alpha}_{(-1/2;\star)}(\overline{G}), \quad \frac{\partial^2 w}{\partial r^2} = \frac{\partial u}{\partial r} \in C^{\alpha}_{(1/2;\star)}(\overline{G}), \quad \frac{\partial^2 w}{\partial r \partial \theta} = \frac{\partial u}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(\overline{G}),$$

and

$$w(r,\theta) = g(\theta) + \int_{r_1}^{r} \frac{\partial w}{\partial r}(t,\theta)dt \in C^{\alpha}(\overline{G}).$$

Additionally, (4.14) and Lemma 4.1 imply that

$$\frac{\partial^2 w}{\partial \theta^2} = -\big(a_0(r) + a_1(r,\theta)\big)\frac{\partial^2 w}{\partial r^2} - \big(a_0'(r) + a_2(r,\theta) + a_3(r)\big)\frac{\partial w}{\partial r} \in C^{\alpha}_{(0;\star)}(\overline{G}),$$

and thus

$$\frac{\partial w}{\partial \theta}(r,\theta) = \int_0^{\theta} \frac{\partial^2 w}{\partial \theta^2}(r,t)dt \in C^{\alpha}_{(0;\star)}(\overline{G}).$$

These, together with (4.15) and (4.49), yield $w \in \mathscr{G}(G)$ with (4.45).

Finally, it remains to prove the claim that we prove that $u \in C^{\alpha}_{(-1/2;\star)}(\overline{G})$, $\dfrac{\partial u}{\partial r} \in C^{\alpha}_{(1/2;\star)}(\overline{G})$, $\dfrac{\partial u}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(\overline{G})$ and (4.49) holds. By even extension with respect to $\theta = 0$ and $\theta = \theta_0$, u may be regarded as a solution of the equation in the domain $(r_1, r_2) \times \mathbb{R}$. Therefore, (4.48) holds for all $0 < \bar{\theta} < \theta_0$ and we may assume that $\theta_0 \geq 2((r_2 - r_1)/2)^{3/4}$. Under this assumption, $\delta_1 = (r_2 - r_1)/2$. Here, we note that the weight of u has negative exponent, while the ones for $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$ are positive. This difference leads to different proofs.

As for u, first, (4.20) implies

$$|u(r,\theta)| \leq M\|g\|_{2,\alpha;(0,\theta_0)}(r_2 - r)^{1/2}, \quad (r,\theta) \in G. \tag{4.50}$$

Second, let $\hat{r}, \check{r}$ and $\bar{\theta}$ satisfy $r_1 < \hat{r} < \check{r} < r_2$ and $0 < \bar{\theta} < \theta_0$. If $r_2 - \check{r} > (r_2 - \hat{r})/2$, then it follows from (4.48) with $\delta = (r_2 - \hat{r})/2$ that

$$|u(\hat{r},\bar{\theta}) - u(\check{r},\bar{\theta})| \leq M(r_2 - \hat{r})^{1/2-\alpha}\|g\|_{2,\alpha;(0,\theta_0)}|\hat{r} - \check{r}|^{\alpha};$$

while if $r_2 - \check{r} \leq (r_2 - \hat{r})/2$, then $r_2 - \hat{r} \leq 2|\hat{r} - \check{r}|$ and thus (4.50) yields

$$\begin{aligned} |u(\hat{r},\bar{\theta}) - u(\check{r},\bar{\theta})| \leq & |u(\hat{r},\bar{\theta})| + |u(\check{r},\bar{\theta})| \leq M\|g\|_{2,\alpha;(0,\theta_0)}\Big((r_2 - \hat{r})^{1/2} + (r_2 - \check{r})^{1/2}\Big) \\ \leq & M(r_2 - \hat{r})^{1/2-\alpha}\|g\|_{2,\alpha;(0,\theta_0)}|\hat{r} - \check{r}|^{\alpha}. \end{aligned}$$

Hence

$$[u]^{(-1/2;1)}_{\alpha;r,G_0} \leq M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.51}$$

Third, let $\bar{r}$, $\hat{\theta}$ and $\check{\theta}$ be given such that $r_1 < \bar{r} < r_2$ and $\hat{\theta}, \check{\theta} \in (0,\theta_0)$. If $|\hat{\theta} - \check{\theta}| < 2(r_2 - \bar{r})^{3/4}$, then (4.48) yields that

$$|u(\bar{r},\hat{\theta}) - u(\bar{r},\check{\theta})| \leq M(r_2 - \bar{r})^{1/2-3\alpha/4}\|g\|_{2,\alpha;(0,\theta_0)}|\hat{\theta} - \check{\theta}|^{\alpha};$$

while if $|\hat{\theta} - \check{\theta}| \geq 2(r_2 - \bar{r})^{3/4}$, then it follows from (4.50) that

$$\begin{aligned} |u(\bar{r},\hat{\theta}) - u(\bar{r},\check{\theta})| \leq & |u(\bar{r},\hat{\theta})| + |u(\bar{r},\check{\theta})| \leq M\|g\|_{2,\alpha;(0,\theta_0)}(r_2 - \bar{r})^{1/2} \\ \leq & M(r_2 - \bar{r})^{1/2-3\alpha/4}\|g\|_{2,\alpha;(0,\theta_0)}|\hat{\theta} - \check{\theta}|^{\alpha}. \end{aligned}$$

Thus

$$[u]^{(-1/2;3/4)}_{\alpha;\theta,G_0} \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.52}$$

It follows from (4.50)–(4.52) that $u \in C^{\alpha}_{(-1/2;\star)}(\overline{G})$ and satisfies

$$\|u\|^{(-1/2;\star)}_{\alpha;G} \le M\|g\|_{2,\alpha;(0,\theta_0)}.$$

Next we consider $\dfrac{\partial u}{\partial r}$. For given $(r_1+r_2)/2 < r < r_2$, there exists a nonnegative integer k such that

$$r_1 < r_2 - 2^{k+1}(r_2 - r) \le \frac{1}{2}(r_1 + r_2).$$

For any integer j satisfying $0 \le j \le k$, it follows from (4.48) that

$$\Big|\frac{\partial u}{\partial r}(r_2 - 2^j(r_2-r),\theta) - \frac{\partial u}{\partial r}(r_2 - 2^{j+1}(r_2-r),\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}2^{-j/2}(r_2-r)^{-1/2}, \quad 0<\theta<\theta_0.$$

Summing up leads to

$$\Big|\frac{\partial u}{\partial r}(r,\theta) - \frac{\partial u}{\partial r}(r_2 - 2^{k+1}(r_2-r),\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r)^{-1/2}, \quad 0<\theta<\theta_0.$$

This, together with (4.19), shows

$$\Big|\frac{\partial u}{\partial r}(r,\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}(r_2-r)^{-1/2}, \quad (r,\theta)\in G. \tag{4.53}$$

On the one hand, consider $\hat r$, $\check r$ and $\bar\theta$ such that $r_1 < \hat r \le \check r < r_2$ and $0 < \bar\theta < \theta_0$. If $r_2 - \check r > (r_2 - \hat r)/2$, then it follows from (4.48) that

$$\Big|\frac{\partial u}{\partial r}(\hat r,\bar\theta) - \frac{\partial u}{\partial r}(\check r,\bar\theta)\Big| \le M(r_2-\check r)^{-1/2}(r_2-\hat r)^{-\alpha}\|g\|_{2,\alpha;(0,\theta_0)}|\hat r - \check r|^{\alpha};$$

while if $r_2 - \check r \le (r_2-\hat r)/2$, then $r_2 - \hat r \le 2|\hat r - \check r|$ and thus (4.53) yields

$$\begin{aligned}\Big|\frac{\partial u}{\partial r}u(\hat r,\bar\theta) - \frac{\partial u}{\partial r}(\check r,\bar\theta)\Big| \le& \Big|\frac{\partial u}{\partial r}(\hat r,\bar\theta)\Big| + \Big|\frac{\partial u}{\partial r}(\check r,\bar\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}\Big((r_2-\hat r)^{-1/2} + (r_2-\check r)^{-1/2}\Big)\\ \le& M(r_2-\check r)^{-1/2}(r_2-\hat r)^{-\alpha}\|g\|_{2,\alpha;(0,\theta_0)}|\hat r-\check r|^{\alpha}.\end{aligned}$$

Hence

$$\Big[\frac{\partial u}{\partial r}\Big]^{(1/2;1)}_{\alpha;r,G_0} \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.54}$$

On the other hand, let $\bar r$, $\hat\theta$ and $\check\theta$ be given satisfying $r_1 < \bar r < r_2$ and $\hat\theta,\check\theta \in (0,\theta_0)$. If $|\hat\theta - \check\theta| < 2(r_2-\bar r)^{3/4}$, then it follows from (4.48) that

$$\Big|\frac{\partial u}{\partial r}(\bar r,\hat\theta) - \frac{\partial u}{\partial r}(\bar r,\check\theta)\Big| \le M(r_2-\bar r)^{-1/2-3\alpha/4}\|g\|_{2,\alpha;(0,\theta_0)}|\hat\theta-\check\theta|^{\alpha};$$

while if $|\hat\theta-\check\theta| \ge 2(r_2-\bar r)^{3/4}$, then (4.53) yields that

$$\begin{aligned}\Big|\frac{\partial u}{\partial r}(\bar r,\hat\theta) - \frac{\partial u}{\partial r}(\bar r,\check\theta)\Big| \le& \Big|\frac{\partial u}{\partial r}(\bar r,\hat\theta)\Big| + \Big|\frac{\partial u}{\partial r}(\bar r,\check\theta)\Big| \le M\|g\|_{2,\alpha;(0,\theta_0)}(r_2-\bar r)^{-1/2}\\ \le& M(r_2-\bar r)^{-1/2-3\alpha/4}\|g\|_{2,\alpha;(0,\theta_0)}|\hat\theta-\check\theta|^{\alpha}.\end{aligned}$$

Thus

$$\left[\frac{\partial u}{\partial r}\right]^{(1/2;3/4)}_{\alpha;\theta,G_0} \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.55}$$

As a consequence of (4.53)–(4.55), $\dfrac{\partial u}{\partial r} \in C^{\alpha}_{(1/2;\star)}(\overline{G}_0)$ and satisfies

$$\left\|\frac{\partial u}{\partial r}\right\|^{(1/2;\star)}_{\alpha;G_0} \le M\|g\|_{2,\alpha;(0,\theta_0)}.$$

Similarly, one can prove that $\dfrac{\partial u}{\partial \theta} \in C^{\alpha}_{(1/4;\star)}(\overline{G}_0)$ and satisfies

$$\left\|\frac{\partial u}{\partial \theta}\right\|^{(1/4;\star)}_{\alpha;G_0} \le M\|g\|_{2,\alpha;(0,\theta_0)}.$$

The proof is complete. $\square$

Proposition 4.4 *Assume that* $0 < \sigma \le \min\{\sigma_1, \sigma_2\}$, $g \in C^{2,\alpha}([0,\theta_0])$ *with* $g'(0) = g'(\theta_0) = 0$ *and* $w \in C^{2,\alpha}([r_1,r_2]\times[0,\theta_0])\cap\mathscr{G}(G)$ *is the solution of the problem (4.14)–(4.17) obtained in Proposition 4.2. Then there exists* $\sigma_3 > 0$ *depending only on* γ, r_1, r_2, θ_0 *and* α, *such that for any* $0 < \sigma \le \min\{\sigma_1,\sigma_2,\sigma_3\}$, $\dfrac{\partial w}{\partial r} \in \mathscr{B}(G)$ *with*

$$\left\|\frac{\partial w}{\partial r}\right\|_{\mathscr{B}(G)} \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.56}$$

Proof. Set

$$u(r,\theta) = \frac{\partial w}{\partial r}(r,\theta), \quad (r,\theta)\in\overline{G}.$$

Then, Propositions 4.2 and 4.3 imply $u \in \mathscr{R}_1(G)$ is a weak solution of

$$\frac{\partial}{\partial r}\Big(a_0(r)\frac{\partial u}{\partial r}\Big) + \frac{\partial}{\partial r}\Big(a_1(r,\theta)\frac{\partial u}{\partial r}\Big) + \frac{\partial}{\partial r}\Big(\big(a_0'(r)+a_2(r,\theta)+a_3(r)\big)u\Big) + \frac{\partial^2 u}{\partial\theta^2} = 0, \quad (r,\theta)\in G.$$

Set

$$v(\tau,\theta) = u(r(\tau),\theta) = \frac{\partial w}{\partial r}(r(\tau),\theta), \quad (\tau,\theta)\in\overline{\Omega},$$

where $r(\cdot)$ is the transformation defined in §4.1. Then, v is a weak solution of

$$e_0(\tau)\frac{\partial^2 v}{\partial\tau^2} + e_0(\tau)\frac{\partial}{\partial\tau}\Big(e_1(\tau,\theta)\frac{\partial v}{\partial\tau}\Big) + e_0(\tau)\frac{\partial}{\partial\tau}\big(\tau^{-1}v\big) + e_0(\tau)\frac{\partial}{\partial\tau}\big(e_2(\tau,\theta)v\big) + \frac{\partial^2 v}{\partial\theta^2} = 0, \quad (\tau,\theta)\in\Omega, \tag{4.57}$$

where

$$e_0(\tau) = \Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^{-2}\frac{1}{a_0(r(\tau))}, \quad 0<\tau<1$$

and

$$e_1(\tau,\theta) = \frac{a_1(r(\tau),\theta)}{a_0(\tau)}, \quad e_2(\tau,\theta) = \int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big(a_0'(r(\tau)) + a_2(r(\tau),\theta) + a_3(r(\tau))\Big) - \frac{1}{\tau}, \quad (\tau,\theta)\in\Omega.$$

Set

$$V(\tau,\theta) = \frac{v(\tau,\theta)}{\tau}, \quad (\tau,\theta)\in\Omega.$$

Then, it follows from (4.19)–(4.21) that

$$\|V\|_{1,\alpha;(\tau,1)\times(0,\theta_0)} \le M(\tau)\|g\|_{2,\alpha;(0,\theta_0)}, \quad \tau\in(0,1), \tag{4.58}$$

$$\|V\|_{L^\infty(\Omega)} \le M\|g\|_{2,\alpha;(0,\theta_0)}, \qquad \int_\Omega \Big(V^2+\tau^2\Big(\frac{\partial V}{\partial\tau}\Big)^2+\tau^3\Big(\frac{\partial V}{\partial\theta}\Big)^2\Big)d\tau d\theta \le M\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.59}$$

Furthermore, (4.57) can be rewritten as

$$\frac{\partial}{\partial\tau}\Big(\tau\frac{\partial V}{\partial\tau}\Big)+2\frac{\partial V}{\partial\tau}+\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3\tau^2\frac{\partial^2 V}{\partial\theta^2}=\frac{\partial}{\partial\tau}F_1(\tau,\theta)+\tau^2\frac{\partial}{\partial\theta}F_2(\tau,\theta), \quad (\tau,\theta)\in\Omega, \tag{4.60}$$

where

$$\begin{aligned} F_1(\tau,\theta)=&-e_1(\tau,\theta)\tau\frac{\partial V}{\partial\tau}(\tau,\theta)-\big(e_1(\tau,\theta)+\tau e_2(\tau,\theta)\big)V(\tau,\theta), && (\tau,\theta)\in\Omega,\\ F_2(\tau,\theta)=&\Big(\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\Big)\frac{\partial V}{\partial\theta}(\tau,\theta), && (\tau,\theta)\in\Omega. \end{aligned}$$

Thus, $V\in\mathcal{H}_2(\Omega)\cap L^\infty(\Omega)$ is a weak solution of (4.60). Further, V satisfies the boundary condition

$$\frac{\partial V}{\partial\theta}(\tau,0)=\frac{\partial V}{\partial\theta}(\tau,\theta_0)=0, \quad \tau\in(0,1).$$

By Theorem 3.1 and Remark 3.1, one gets that

$$\|V\|^*_{\alpha;\tilde\Omega}+\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|^*_{\alpha;\tilde\Omega}+\Big\|\frac{\partial V}{\partial\theta}\Big\|^{**}_{\alpha;\tilde\Omega}\le M\Big([F_1]^*_{\alpha;\Omega}+[F_2]^{**}_{\alpha;\Omega}+|V|_{0;\Omega}\Big) \tag{4.61}$$

and

$$\tau\frac{\partial V}{\partial\tau}(\tau,\theta)\Big|_{\tau=0}=\lim_{\tau\to0^+}\tau\frac{\partial V}{\partial\tau}(\tau,\theta)=0, \quad \theta\in(0,\theta_0), \tag{4.62}$$

where $\tilde\Omega=(0,1/3)\times(0,\theta_0)$. For $0<\tilde\tau<1/3$, set

$$\Omega_1=(0,\tilde\tau)\times(0,\theta_0)\subset\tilde\Omega, \quad \Omega_2=(\tilde\tau,1)\times(0,\theta_0).$$

It follows from (4.61) and Lemma 3.3 that

$$\begin{aligned} &\|V\|^*_{\alpha;\Omega_1}+\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|^*_{\alpha;\Omega_1}+\Big\|\frac{\partial V}{\partial\theta}\Big\|^{**}_{\alpha;\Omega_1}\\ \le& M_0\Big([F_1]^*_{\alpha;\Omega_1}+[F_1]^*_{\alpha;\Omega_2}+[F_2]^{**}_{\alpha;\Omega_1}+[F_2]^{**}_{\alpha;\Omega_2}+|V|_{0;\Omega}\Big)\\ \le& M_0\|e_1\|^*_{\alpha;\Omega_1}\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|^*_{\alpha;\Omega_1}+M_0\|e_1\|^*_{\alpha;\Omega_2}\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|^*_{\alpha;\Omega_2}+M_0\big(\|e_1\|^*_{\alpha;\Omega_1}+\|\tau e_2\|^*_{\alpha;\Omega_1}\big)\|V\|^*_{\alpha;\Omega_1}\\ &+M_0\big(\|e_1\|^*_{\alpha;\Omega_2}+\|\tau e_2\|^*_{\alpha;\Omega_2}\big)\|V\|^*_{\alpha;\Omega_2}+M_0\Big\|\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\Big\|^*_{\alpha;\Omega_1}\Big\|\frac{\partial V}{\partial\theta}\Big\|^{**}_{\alpha;\Omega_1}\\ &+M_0\Big\|\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\Big\|^*_{\alpha;\Omega_2}\Big\|\frac{\partial V}{\partial\theta}\Big\|^{**}_{\alpha;\Omega_2}+M_0|V|_{0;\Omega}. \end{aligned} \tag{4.63}$$

One can verify that

$$\tau a_0'(r(\tau))\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt-1,\ \tau a_3(r(\tau)),\ \frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\in C^1([0,1])$$

and all vanish at $\tau=0$, while

$$\frac{\tau}{a_0(r(\tau))}\in C^1([0,1]), \quad \tau^2\frac{a_2(r(\tau),\theta)}{\tilde w_r(r(\tau),\theta)}\in C^1(\overline{\Omega}).$$

These, together with (4.13) and (4.18), lead to

$$\Big\|\tau a_0'(r(\tau))\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt-1\Big\|_{\alpha;\Omega_1}^{*}+\|\tau a_3(r(\tau))\|_{\alpha;\Omega_1}^{*}+\Big\|\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\Big\|_{\alpha;\Omega_1}^{*}\le M\tilde{\tau}^{1-\alpha},$$

$$\|e_1\|_{\alpha;\Omega_1}^{*}+\|\tau a_2(r(\tau),\theta)\|_{\alpha;\Omega_1}^{*}=\Big\|\frac{a_1(r(\tau),\theta)}{\tau}\frac{\tau}{a_0(r(\tau))}\Big\|_{\alpha;\Omega_1}^{*}+\Big\|\tau^2\frac{a_2(r(\tau),\theta)}{\tilde{w}_r(r(\tau),\theta)}\frac{\tilde{w}_r(r(\tau),\theta)}{\tau}\Big\|_{\alpha;\Omega_1}^{*}\le M\sigma.$$

Therefore, there exist $\sigma_3>0$ and $\tilde{\tau}>0$ sufficiently small such that

$$\|e_1\|_{\alpha;\Omega_1}^{*}+\|\tau e_2\|_{\alpha;\Omega_1}^{*}+\Big\|\frac{2|r_2|^3}{c_*^2}\Big(\int_{r_1}^{r_2}\frac{1}{a_0(t)}dt\Big)^3-\frac{1}{\tau e_0(\tau)}\Big\|_{\alpha;\Omega_1}^{*}\le\frac{1}{2M_0}.$$

For such $\tilde{\tau}\in(0,1/3)$, (4.63), together with (4.58), leads to

$$\|V\|_{\alpha;\Omega_1}^{*}+\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|_{\alpha;\Omega_1}^{*}+\Big\|\frac{\partial V}{\partial\theta}\Big\|_{\alpha;\Omega_1}^{**}\le M(\|g\|_{2,\alpha;(0,\theta_0)}+|V|_{0;\Omega}). \tag{4.64}$$

Then, we get from (4.64), (4.58) and (4.59), we get that $V,\tau\frac{\partial V}{\partial\tau}\in C_*^{\alpha}(\overline{\Omega})$, $\frac{\partial V}{\partial\theta}\in C_{**}^{\alpha}(\overline{\Omega})$ and

$$\|V\|_{\alpha;\Omega}^{*}+\Big\|\tau\frac{\partial V}{\partial\tau}\Big\|_{\alpha;\Omega}^{*}+\Big\|\frac{\partial V}{\partial\theta}\Big\|_{\alpha;\Omega}^{**}\le M\|g\|_{2,\alpha;(0,\theta_0)}.$$

This, together with (4.62), shows $\frac{\partial w}{\partial r}\in\mathscr{B}(G)$ with the desired estimate (4.56). The proof is complete. □

Proposition 4.5 *Assume $\tilde{w}\in\mathscr{G}(G)$ with $\|\tilde{w}\|_{\mathscr{G}(G)}\le\sigma$. Then there exists $\sigma_4>0$ depending only on γ, r_1, r_2, θ_0 and α, such that for any $0<\sigma\le\sigma_4$ the problem (4.14)–(4.17) admits at most one solution $w\in\mathscr{G}(G)$.*

Proof. Assume that $w_1,w_2\in\mathscr{G}(G)$ are two solutions of the problem (4.14)–(4.17). Define

$$w(r,\theta)=w_1(r,\theta)-w_2(r,\theta),\quad (r,\theta)\in\overline{G}.$$

Then, $w\in\mathscr{G}(G)$ is the solution of

$$\frac{\partial}{\partial r}\Big(a_0(r)\frac{\partial w}{\partial r}\Big)+a_1(r,\theta)\frac{\partial^2 w}{\partial r^2}+\big(a_2(r,\theta)+a_3(r)\big)\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial\theta^2}=0,\qquad (r,\theta)\in G \tag{4.65}$$

satisfying the following boundary conditions

$$\frac{\partial w}{\partial\theta}(r,0)=\frac{\partial w}{\partial\theta}(r,\theta_0)=0,\qquad r_1<r<r_2, \tag{4.66}$$

$$w(r_1,\theta)=0,\qquad 0<\theta<\theta_0. \tag{4.67}$$

Multiplying (4.65) by $-\frac{\partial w}{\partial r}$ and then integrating over $(r_1,s)\times(0,\theta_0)$, we get that

$$\int_{r_1}^{s}\int_0^{\theta_0}\Big(-\frac{1}{2}\big(a_0(r)+a_1(r,\theta)\big)\frac{\partial}{\partial r}\Big(\Big(\frac{\partial w}{\partial r}\Big)^2\Big)-\big(a_0'(r)+a_2(r,\theta)+a_3(r)\big)\Big(\frac{\partial w}{\partial r}\Big)^2-\frac{\partial^2 w}{\partial\theta^2}\frac{\partial w}{\partial r}\Big)drd\theta=0,$$

where $r_1<s<r_2$. Integrating by parts and using (4.66) and (4.67) lead to

$$0=-\int_{r_1}^{s}\int_0^{\theta_0}\Big(\frac{1}{2}a_0'(r)-\frac{1}{2}\frac{\partial a_1}{\partial r}(r,\theta)+a_2(r,\theta)+a_3(r)\Big)\Big(\frac{\partial w}{\partial r}\Big)^2drd\theta$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^{\theta_0}\big(a_0(r)+a_1(r,\theta)\big)\Big(\frac{\partial w}{\partial r}\Big)^2 d\theta\Big|_{r=r_1}^{r=s}+\frac{1}{2}\int_0^{\theta_0}\Big(\frac{\partial w}{\partial \theta}\Big)^2 d\theta\Big|_{r=r_1}^{r=s}\\
\geq &-\int_{r_1}^{s}\int_0^{\theta_0}\Big(\frac{1}{2}a_0'(r)-\frac{1}{2}\frac{\partial a_1}{\partial r}(r,\theta)+a_2(r,\theta)+a_3(r)\Big)\Big(\frac{\partial w}{\partial r}\Big)^2 drd\theta\\
&-\frac{1}{2}\int_0^{\theta_0}\big(a_0(r)+a_1(r,\theta)\big)\Big(\frac{\partial w}{\partial r}\Big)^2 d\theta\Big|_{r=r_1}^{r=s}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&-\int_{r_1}^{s}\int_0^{\theta_0}\Big(\frac{1}{2}a_0'(r)-\frac{1}{2}\frac{\partial a_1}{\partial r}(r,\theta)+a_2(r,\theta)+a_3(r)\Big)\Big(\frac{\partial w}{\partial r}\Big)^2 drd\theta\\
\leq &\frac{1}{2}\int_0^{\theta_0}\big(a_0(r)+a_1(r,\theta)\big)\Big(\frac{\partial w}{\partial r}\Big)^2 d\theta\Big|_{r=s}-\frac{1}{2}\int_0^{\theta_0}\big(a_0(r)+a_1(r,\theta)\big)\Big(\frac{\partial w}{\partial r}\Big)^2 d\theta\Big|_{r=r_1}.
\end{aligned}
\tag{4.68}
$$

From (4.6), (4.7) and $\tilde{w}\in\mathscr{G}(G)$ with $\|\tilde{w}\|_{\mathscr{G}(G)}\leq\sigma$, there exists $\sigma_4>0$ depending only on γ, r_1, r_2, θ_0 and α, such that for all $0<\sigma\leq\sigma_4$,

$$
\kappa_1(r_2-r)^{1/2}\leq a_0(r)+a_1(r,\theta)\leq\kappa_2(r_2-r)^{1/2},\quad (r,\theta)\in G \tag{4.69}
$$

and

$$
-\kappa_2(r_2-r)^{-1/2}\leq\frac{1}{2}a_0'(r)-\frac{1}{2}\frac{\partial a_1}{\partial r}(r,\theta)+a_2(r,\theta)+a_3(r)\leq-\kappa_1(r_2-r)^{-1/2},\quad (r,\theta)\in G, \tag{4.70}
$$

where $0<\kappa_1<\kappa_2$ are two constants depending only on γ, r_1, r_2, θ_0 and α. Additionally, $w\in\mathscr{G}(G)$ implies

$$
\Big|\frac{\partial w}{\partial r}(r,\theta)\Big|\leq\|w\|_{\mathscr{G}(G)}(r_2-r)^{1/2},\quad (r,\theta)\in G. \tag{4.71}
$$

By (4.69)–(4.71), letting $s\to r_2^-$ in (4.68) yields

$$
\int_G\Big(\frac{\partial w}{\partial r}\Big)^2(r,\theta)drd\theta\leq 0,
$$

which implies

$$
\frac{\partial w}{\partial r}(r,\theta)=0,\quad (r,\theta)\in G.
$$

Then, it follows from this and (4.67) that

$$
w(r,\theta)=0,\quad (r,\theta)\in G.
$$

The proof is complete. $\square$

Finally, we can prove the main results (Theorem 4.1 and Theorem 4.2).

Proof of Theorem 4.1. As shown at the beginning of this subsection, this theorem on the problem (1.2)–(1.5) is equivalent to the corresponding one on the problem (4.5), (4.8)–(4.10). Denote

$$
\mathcal{M}=\Big\{u\in C^2([r_1,r_2)\times[0,\theta_0]);\|u\|_{\mathscr{G}(G)}+\Big\|\frac{\partial u}{\partial r}\Big\|_{\mathscr{B}(G)}\leq\min\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}\Big\}.
$$

For any $\tilde{w}\in\mathcal{M}$, Propositions 4.2–4.5 imply that the problem (4.14)–(4.17) admits a unique solution $w\in\mathscr{G}(G)$ with $\dfrac{\partial w}{\partial r}\in\mathscr{B}(G)$ and

$$
\|w\|_{\mathscr{G}(G)}+\Big\|\frac{\partial w}{\partial r}\Big\|_{\mathscr{B}(G)}\leq M_0\|g\|_{2,\alpha;(0,\theta_0)}. \tag{4.72}
$$

Take

$$\delta_0 = \frac{1}{M_0}\min\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}.$$

Then, the theorem can be proved by using the Schauder fixed point theorem in a standard way, so the details are omitted. Thus, the proof of Theorem 4.1 is considered complete. □

Proof of Theorem 4.2. Assume that $\phi_1, \phi_2 \in C^{2,\alpha}([r_1, r_2) \times [0, \theta_0])$ are two solutions of the problem (1.2)–(1.5) satisfying

$$\|\phi_1 - \phi_0\|_{\mathscr{G}(G)} \le \delta, \quad \|\phi_2 - \phi_0\|_{\mathscr{G}(G)} \le \delta \tag{4.73}$$

with some $\delta > 0$ determined below. Let

$$w(r,\theta) = \phi_1(r) - \phi_2(r,\theta), \quad (r,\theta) \in \overline{G}.$$

Then, it follows from (1.2) and direct computations that

$$\frac{\partial}{\partial r}\Big(a_0(r)\frac{\partial w}{\partial r}\Big) + \frac{\partial}{\partial r}\Big(\big(\tilde{a}_0(r,\theta) + \tilde{a}_1(r,\theta)\big)\frac{\partial w}{\partial r}\Big) + \big(\tilde{a}_2(r,\theta) + \tilde{a}_3(r,\theta)\big)\frac{\partial w}{\partial r} + a_3(r)\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} = 0,\ (r,\theta) \in G,$$

where $a_0(r)$ and $a_3(r)$ are given as in (4.5), while $\tilde{a}_i(r,\theta)\,(i = 0, 1, 2, 3)$ can be given explicitly in terms of ϕ_1 and ϕ_2. In fact, the detailed expressions of $\tilde{a}_i\,(i = 0, 1, 2, 3)$ are not important. It suffices to note that they possess the following estimates

$$|\tilde{a}_0(r,\theta)| + |\tilde{a}_1(r,\theta)| \le M\delta(r_2 - r)^{1/2}, \quad \Big|\frac{\partial \tilde{a}_0}{\partial r}(r,\theta)\Big| + \Big|\frac{\partial \tilde{a}_1}{\partial r}(r,\theta)\Big| \le M\delta(r_2 - r)^{-1/2}, \qquad (r,\theta) \in G$$

and

$$|\tilde{a}_2(r,\theta)| + |\tilde{a}_3(r,\theta)| \le M\delta, \qquad (r,\theta) \in G$$

for some $M > 0$ depending only on γ, r_1, r_2, θ_0 and α, which can be verified directly. Then, via an energy estimate as in the proof of Proposition 4.5, we may prove

$$w(r,\theta) = 0, \quad (r,\theta) \in G,$$

provided that δ is sufficiently small. This completes the proof of the theorem and we omit the details. □

Appendix

The coefficients $h_0(r)$ and $h_i(r, w_r)\,(i = 1, 2, 3)$ in (4.4) can be given explicitly as follows

$$\begin{aligned}
h_0(r) =& \frac{r}{\rho(c_*^2)}\big[\rho((\phi_0')^2) + 2\rho'((\phi_0')^2)(\phi_0')^2\big], \quad r_1 < r < r_2, \\
h_1(r, w_r) =& -\frac{6r}{\rho(c_*^2)}\rho'((\phi_0')^2)\phi_0' + \frac{3r}{\rho(c_*^2)}\rho'((\phi_0')^2)w_r + \frac{r}{\rho(c_*^2)}j_1(r, w_r)\big(2(\phi_0')^2 - 6\phi_0' w_r + 3w_r^2\big) \\
& - \frac{2r}{\rho(c_*^2)}(\phi_0' - w_r)\int_0^1 (1-t)\rho''(t(\phi_0')^2 + (1-t)(\phi_0' - w_r)^2)dt \\
& \cdot \big(2(\phi_0')^2 - 3\phi_0' w_r + w_r^2\big), \quad r_1 < r < r_2,\ w_r \in \mathbb{R}, \\
h_2(r, w_r) =& -\frac{3}{\rho(c_*^2)}\big(r\rho'((\phi_0')^2)\phi_0'\big)' + \frac{1}{\rho(c_*^2)}\big(r\rho'((\phi_0')^2)\big)' w_r \\
& + \frac{1}{\rho(c_*^2)}j_1(r, w_r)\big(2(r(\phi_0')^2)' - 3(r\phi_0')' w_r + w_r^2\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2r}{\rho(c_*^2)}\phi_0'' \int_0^1 (1-t)\rho''(t(\phi_0')^2 + (1-t)(\phi_0' - w_r)^2)dt\big(2(\phi_0')^2 - 3\phi_0' w_r + w_r^2\big) \\
&+ \frac{2r}{\rho(c_*^2)} j_2(r, w_r)\phi_0'\phi_0''\big(2(\phi_0')^2 - 3\phi_0' w_r + w_r^2\big), \quad r_1 < r < r_2,\ w_r \in \mathbb{R},
\end{aligned}
$$

$$
h_3(r, w_r) = \frac{1}{r\rho(c_*^2)}\rho((\phi_0' - w_r)^2), \quad r_1 < r < r_2,\ w_r \in \mathbb{R},
$$

with

$$
j_1(r, w_r) = \begin{cases} \dfrac{1}{w_r}\Big[\displaystyle\int_0^1 \rho'(t(\phi_0')^2 + (1-t)(\phi_0' - w_r)^2)dt - \rho'((\phi_0')^2)\Big], & w_r \neq 0, \\ -\rho''((\phi_0')^2)\phi_0', & w_r = 0 \end{cases}
$$

and

$$
j_2(r, w_r) = \begin{cases} \dfrac{1}{w_r}\Big[\displaystyle\int_0^1 \rho''(t(\phi_0')^2 + (1-t)(\phi_0' - w_r)^2)dt - \rho''((\phi_0')^2)\Big], & w_r \neq 0, \\ -\rho'''((\phi_0')^2)\phi_0', & w_r = 0. \end{cases}
$$

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