Blowup Criterion for the Compressible Flows with Vacuum States

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Abstract

We prove that the maximum norm of the deformation tensor of velocity gradients controls the possible breakdown of smooth(strong) solutions for the 3-dimensional compressible Navier-Stokes equations, which will happen, for example, if the initial density is compactly supported [33]. More precisely, if a solution of the compressible Navier-Stokes equations is initially regular and loses its regularity at some later time, then the loss of regularity implies the growth without bound of the deformation tensor as the critical time approaches. Our result is the same as Ponce's criterion for 3-dimensional incompressible Euler equations ([26]). Moreover, our method can be generalized to the full Compressible Navier-Stokes system which improve the previous results. In addition, initial vacuum states are allowed in our cases.

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1 Introduction

The time evolution of the density and the velocity of a general viscous compressible barotropic fluid occupying a domain $\Omega \subset R^3$ is governed by the compressible Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\operatorname{div} u) + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

where ρ, u, P denotes the density, velocity and pressure respectively. The equation of state is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \gamma > 1),$$

μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$\mu > 0, \lambda + \frac{2}{3}\mu \geq 0. \tag{1.2}$$

The equations (1.1) will be studied with initial conditions:

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \tag{1.3}$$

and three types of boundary conditions:

1) Cauchy problem:

$$\Omega = R^3 \text{ and (in some weak sense) } \rho, u \text{ vanish at infinity;} \tag{1.4}$$

2) Dirichlet problem: in this case, Ω is a bounded smooth domain in R^3, and

$$u = 0 \text{ on } \partial\Omega; \tag{1.5}$$

3) Navier-slip boundary condition: in this case, Ω is a bounded smooth domain in R^3, and

$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0 \text{ on } \partial\Omega \tag{1.6}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\Omega$. The first condition in (1.6) is the non-penetration boundary condition, while the second one is also known in the form

$$(\mathcal{D}(u) \cdot n)_\tau = -\kappa_\tau u_\tau, \tag{1.7}$$

where $\mathcal{D}(u)$ is the deformation tensor:

$$\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t), \tag{1.8}$$

and κ_τ is the corresponding principal curvature of $\partial\Omega$. Condition (1.7) implies the tangential component of $\mathcal{D}(u)\cdot n$ vanishes on flat portions of the boundary $\partial\Omega$. Note that ∇u can be decomposed as

$$\nabla u = \mathcal{D}(u) + \mathcal{S}(u), \tag{1.9}$$

where $\mathcal{D}(u)$ is the deformation tensor defined by (1.8) and

$$\mathcal{S}(u) = \frac{1}{2}\left(\nabla u - \nabla u^t\right), \tag{1.10}$$

known as the rigid body rotation tensor. The tensors $\mathcal{D}(u)$ and $\mathcal{S}(u)$ are respectively the symmetric and skew-symmetric parts of ∇u.

There are huge literatures on the large time existence and behavior of solutions to (1.1). The one-dimensional problem has been studied extensively by many people, see [14, 21, 29, 30] and the references therein. The multidimensional problem (1.1) was investigated by Matsumura-Nishida [24], who proved global existence of smooth solutions for data close to a non-vacuum equilibrium, and later by Hoff [14–16] for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [22, 23] (see also Feireisl [13]), where he obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish.

However, the regularity and uniqueness of such weak solutions remains completely open. It should be noted that one cannot expect too much regularity of Lions's weak solutions in general because of the results of Xin([33]), who showed that there is no global smooth solution (ρ, u) to Cauchy problem for (1.1) with a nontrivial compactly supported initial density. Xin's blowup result ([33]) raises the questions of the mechanism of blowup and structure of possible singularities: What kinds of singularities will form in finite time? What is the main mechanism for possible breakdown of smooth solutions for the 3-D compressible equations?

We begin with the local existence of strong (or classical) solutions. In the absence of vacuum, the local existence and uniqueness of classical solutions are known [25, 31]. In the case where the initial density need not be positive and may vanish in an open set, the existence and uniqueness of local strong (or classical) solutions are proved recently in [4–6, 8, 28]. Before stating their local existence results, we first give the definition of strong solutions.

Definition 1.1 (Strong solutions) *(ρ, u) is called a strong solution to (1.1) in $\Omega\times(0,T)$, if for some $q_0\in(3,6]$,*

$$\begin{aligned}
&0\le\rho\in C([0,T],W^{1,q_0}(\Omega)), \quad \rho_t\in C([0,T],L^{q_0}(\Omega)),\\
&u\in C([0,T],D_0^1\cap D^2(\Omega))\cap L^2(0,T;D^{2,q_0}(\Omega))\\
&u_t\in L^\infty(0,T_*;L^2(\Omega))\cap L^2(0,T;D_0^1(\Omega)),
\end{aligned} \tag{1.11}$$

and (ρ, u) satisfies (1.1) a.e. in $\Omega\times(0,T)$.

Here and throughout this paper, we use the following notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$\begin{cases} D^{k,r}(\Omega) = \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^r(\Omega)} < \infty\}, \\ W^{k,r}(\Omega) = L^r(\Omega) \cap D^{k,r}(\Omega), \quad H^k(\Omega) = W^{k,2}(\Omega), \quad D^k(\Omega) = D^{k,2}(\Omega), \\ D^1_0(\Omega) = \left\{u \in L^6(\Omega) : \|\nabla u\|_{L^2(\Omega)} < \infty, \text{ and (1.4) or (1.5) or (1.6) holds}\right\}, \\ H^1_0(\Omega) = L^2(\Omega) \cap D^1_0(\Omega), \quad \|u\|_{D^{k,r}(\Omega)} = \|\nabla^k u\|_{L^r(\Omega)}. \end{cases}$$

In particular, Cho etc [4] proved the following result.

Theorem 1.1 *If the initial data ρ_0 and u_0 satisfy*

$$0 \le \rho_0 \in L^1(\Omega) \cap W^{1,\tilde{q}}(\Omega), \quad u_0 \in D^1_0 \cap D^2(\Omega), \tag{1.12}$$

for some $\tilde{q} \in (3, \infty)$ and the compatibility condition:

$$-\mu \triangle u_0 - (\lambda + \mu)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g \quad \textit{for some } g \in L^2(\Omega), \tag{1.13}$$

then there exists a positive time $T_1 \in (0, \infty)$ and a unique strong solution (ρ, u) to the initial boundary value problem (1.1)(1.3) together with (1.4) or (1.5) or (1.6) in $\Omega \times (0, T_1]$. Furthermore, the following blow-up criterion holds: if T^* is the maximal time of existence of the strong solution (ρ, u) and $T^* < \infty$, then

$$\sup_{t \to T^*} (\|\rho\|_{H^1 \cap W^{1,q_0}} + \|u\|_{D^1_0}) = \infty, \tag{1.14}$$

with $q = \min(6, \tilde{q})$.

There are several works ([9,11,12,17,20]) trying to establish blow up criterions for the strong (smooth) solutions to the compressible Navier-Stokes equations. In particular, it is proved in [11] for two dimensions, if $7\mu > 9\lambda$, then

$$\lim_{T \to T^*} \left(\sup_{0 \le t \le T} \|\rho\|_{L^\infty} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla \rho\|^4_{L^2}) dt \right) = \infty,$$

where $T^* < \infty$ is the maximal time of existence of a strong solution and $q_0 > 3$ is a constant.

Later, we [17, 19, 20] first establish a blowup criterion, analogous to the Beal-Kato-Majda criterion [1] for the ideal incompressible flows, for the strong and classical solutions to the isentropic compressible flows in three-dimension:

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty} dt = \infty, \tag{1.15}$$

under the stringent condition on viscous coefficients:

$$7\mu > \lambda. \tag{1.16}$$

Recently, the ideas of [17, 19, 20] has been generalized in [12] to establish a blowup criterion similar to (1.15), under the same assumption (1.16), for the non-isentropic fluids, that is,

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\|\theta\|_{L^\infty}+\int_0^T\|\nabla u\|_{L^\infty}dt\right)=\infty. \tag{1.17}$$

Very recently, in the absence of vacuum, Huang-Li [18] succeeded in removing the crucial condition (1.16) of [12, 17, 19, 20] and established blowup criterions (1.15) and

$$\lim_{T\to T^*}\int_0^T\left(\|\theta\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}\right)dt=\infty, \tag{1.18}$$

for isentropic and non-isentropic compressible Navier-Stokes equations, under the physical restrictions (1.2), respectively.

It should be noted that for ideal incompressible flows, Beal-Kato-Majda [1] established a well-known blowup criterion for the 3-Dimensional incompressible Euler equations that a solution remains smooth if

$$\int_0^T\|\mathcal{S}(u)\|_{L^\infty}dt \tag{1.19}$$

is bounded, where $\mathcal{S}(u)$ is the rigid body rotation tensor defined by (1.10). Later, Ponce [26] rephrased the Beal-Kato-Majda's theorem in terms of deformation tensor $\mathcal{D}(u)$, that is, the same results in [1] hold if

$$\int_0^T\|\mathcal{D}(u)\|_{L^\infty}dt \tag{1.20}$$

remains bounded. Moreover, as pointed out by Constantin [10], the solution is smooth if and only if

$$\int_0^T\|((\nabla u)\xi)\cdot\xi\|_{L^\infty}$$

is bounded, where ξ is the unit vector in the direction of vorticity curl$u$. All these facts in [1, 10, 26] show that the solution becomes smooth either the skew-symmetric or symmetric part of ∇u is controlled.

Note that the results in [18] are not so satisfactory in two-fold: one is that the results exclude initial vacuum states; moreover, nothing is known from (1.15) about the natural question: which part of ∇u, the symmetric part $\mathcal{D}(u)$ or the skew-symmetric part $\mathcal{S}(u)$, will become arbitrarily large as the critical time approaches?

The aim of this paper is to improve all the previous blowup criterion results for compressible Navier-Stokes equations by removing the stringent condition (1.16), and allowing initial vacuum states, and furthermore, instead of (1.15), describing the blowup mechanism in terms of the deformation tensor $\mathcal{D}(u)$. Our main result can be stated as follows:

Theorem 1.2 *Let (ρ, u) be a strong solution of the initial boundary value problem (1.1)(1.3) together with (1.4) or (1.5) or (1.6) satisfying* (1.11). *Assume that the initial data (ρ_0, u_0) satisfies (1.12) and (1.13). If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} \int_0^T \|\mathcal{D}(u)\|_{L^\infty(\Omega)} dt = \infty, \tag{1.21}$$

where $\mathcal{D}(u)$ is the deformation tensor defined by (1.8).

A few remarks are in order:

Remark 1.1 *Theorem 1.2 also holds for classical solutions to the compressible flows with initial vacuum, which improves the results of [18] to the case where the initial density need not be positive and may vanish in an open set. In addition, Theorem 1.2 holds for all μ and λ satisfying the physical restrictions (1.2), which removed the condition (1.16) which is essential in the analysis in [12, 17, 19, 20].*

Remark 1.2 *In 1998, Xin [33] gave an life span estimate of classical solutions to the compactly supported initial density of the Cauchy problem (1.1)(1.3)(1.4) at least in one dimension. However, it's unclear which quantity becomes infinite as the critical time approaches. Theorem 1.2 shows that instabilities can develop only if the size of the deformation tensor becomes arbitrarily large.*

Remark 1.3 *Theorem 1.2 gives a counter part of Ponce's result in [26] for the incompressible flows.*

Next, we indicate that the results in Theorem 1.2 can be generalized to the non-isentropic fluids described by

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla P = 0, \\ c_v[\partial_t(\rho\theta) + \mathrm{div}(\rho\theta u)] - \kappa\triangle\theta + P\mathrm{div} u = \dfrac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\mathrm{div} u)^2, \end{cases} \tag{1.22}$$

where θ is the absolute temperature, $P = R\rho\theta (R > 0)$, and $\kappa \geq 0, R > 0, c_v > 0$ are physical constants.

The local existence of strong solutions with initial vacuum is established in [5], where it is essentially shown that for initial data (ρ_0, u_0, θ_0) satisfying

$$\begin{aligned} &0 \leq \rho_0 \in W^{1,\tilde{q}}(\Omega) \quad \text{for} \quad \text{some} \quad 3 < \tilde{q} \leq 6 \\ &u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \theta_0 \in H^2(\Omega), \theta_0 \geq 0, \end{aligned} \tag{1.23}$$

and the compatibility conditions

$$\begin{aligned} &\mu\triangle u_0 + (\mu + \lambda)\nabla\mathrm{div} u_0 - R\nabla(\rho_0\theta_0) = \rho_0^{\frac{1}{2}} g_1, \\ &\kappa\triangle\theta_0 + \frac{\mu}{2}|\nabla u_0 + \nabla u_0^t|^2 + \lambda(\mathrm{div} u_0)^2 - R\rho_0\theta_0\mathrm{div} u_0 = \rho_0^{\frac{1}{2}} g_2, \end{aligned} \tag{1.24}$$

for some $g_1, g_2 \in L^2(\Omega)$, furthermore, $\{x \in \Omega | \rho_0(x) = 0\}$ being an open subset of Ω, there exist a $T_* > 0$ and a unique strong solution (ρ, u, θ) on $[0, T_*]$ to the Cauchy problem of (1.22), such that for any $q_0 \in (3, \tilde{q})$,

$$\begin{aligned}
&0 \le \rho \in C([0,T_*], W^{1,q_0}), \quad \rho_t \in C([0,T_*], L^{q_0}),\\
&u \in C([0,T_*], D_0^1 \cap D^2) \cap L^2(0,T_*; D^{2,q_0}),\\
&u_t \in L^\infty(0,T_*; L^2) \cap L^2(0,T_*; D_0^1),\\
&\theta \in C([0,T_*]; H^2) \cap L^2(0,T_*; D^{2,q_0}), \theta > 0,\\
&\theta_t \in L^\infty(0,T_*; L^2) \cap L^2(0,T_*; H^1).
\end{aligned} \tag{1.25}$$

By modifying the analysis for Theorem 1.2 and in [18], one can obtain the following blowup criterion for the full compressible Navier-Stokes system (1.22).

Theorem 1.3 *Assume that the initial data satisfy* (1.23) *and* (1.24). *Let* (ρ, u, θ) *be a strong solution to the Cauchy problem of* (1.22) *satisfying* (1.25). *If* $T^* < \infty$ *is the maximal time of existence, then*

$$\lim_{T \to T^*} \int_0^T \left(\|\theta\|_{L^\infty}^2 + \|\mathcal{D}(u)\|_{L^\infty} \right) dt = \infty. \tag{1.26}$$

As aforementioned [7, 33], there are no global smooth solutions for the compressible Navier-Stokes when the initial density is compactly supported. We have the following corollary immediately.

Corollary 1.4 *Assume that* $(\rho_0, u_0, \theta_0) \in H^4(R^3)$ *satisfy the initial compatibility condition* (1.24) *such that there exists a finite number* $0 < r < \infty$ *with* $supp \rho_0 \subset B_r$. *Let* (ρ, u, θ) *be the corresponding classical solutions. Then there exists a time* $T_* < \infty$*, such that (1.26) holds.*

We now comment on the analysis of this paper. Note that in all previous works [17–20](see also [12]), either the assumption (1.16) or the absence of vacuum played an important role in their analysis in order to obtain an improved energy estimate which is essential not only for bounding the L^2 norm of the convection term $F = \rho u_t + \rho u \cdot \nabla u$ but also for improving the regularity of the solutions. Their method depends on the L^∞-norm of ∇u also. It is thus difficult to adapt their analysis here. To proceed, some new ideas are needed. The key step in proving Theorem 1.2 is to derive the L^2-estimate on gradients of both the density ρ and the velocity u. Observe that there are two main difficulties here: one is due to the possible vacuum states, the other is the strong nonlinearities of convection terms. In order to overcome these difficulties, we will use the simple observation that the momentum equations $(1.1)_2$ become “more” diffusive near vacuum if divided on both sides by ρ as long as ρ remains bounded above which is guaranteed by the boundedness of the temporal integral of the super-norm in space of the deformation tensor. Thus a new energy estimate by using the effective stress tensor will lead to a prior estimates on the L^2-norms of gradients of both the density and the velocity. A simple combination of the above facts with the ideas used in [18] then yields the blowup criterion for the full compressible Navier-Stokes system (1.22). The details of the proof of Theorems 1.2 and 1.3 are given in Sections 2 and 3 respectively.

2 Proof of Theorem 1.2

Let (ρ, u) be a strong solution to the problem (1.1)-(1.2) as described in Theorem 1.2. First, the standard energy estimate yields

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u(t)\|_{L^2}^2+\|\rho\|_{L^\gamma}^\gamma\right)+\int_0^T\|\nabla u\|_{L^2}^2dt\le C,\quad 0\le T<T^*. \tag{2.1}$$

To prove the theorem, we assume otherwise that

$$\lim_{T\to T^*}\int_0^T\|\mathcal{D}(u)\|_{L^\infty(\Omega)}dt\le C<\infty. \tag{2.2}$$

Then (2.2), together with $(1.1)_1$, immediately yields the following L^∞ bound of the density ρ. Indeed, on has

Lemma 2.1 *Assume that*

$$\int_0^T\|divu\|_{L^\infty}dt\le C,\quad 0\le T<T^*.$$

Then

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le C,\quad 0\le T<T^*. \tag{2.3}$$

Proof. It follows from $(1.1)_1$ that for $\forall p\ge\gamma$,

$$\partial_t(\rho^p)+\operatorname{div}(\rho^p u)+(p-1)\rho^p\operatorname{div}u=0. \tag{2.4}$$

Integrating (2.4) over Ω leads to

$$\partial_t\int_\Omega\rho^p dx\le(p-1)\|\operatorname{div}u\|_{L^\infty(\Omega)}\int_\Omega\rho^p dx,$$

that is,

$$\partial_t\|\rho\|_{L^p}\le\frac{p-1}{p}\|\operatorname{div}u\|_{L^\infty(\Omega)}\|\rho\|_{L^p},$$

which implies immediately

$$\|\rho\|_{L^p}(t)\le C,$$

with C independent of p, so our lemma follows.

The key estimates on $\nabla\rho$ and ∇u will be given in the following lemma.

Lemma 2.2 *Under* (2.2), *it holds that for any* $T<T^*$,

$$\sup_{0\le t\le T}\left(\nabla\rho\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\int_0^T\|\nabla u\|_{H^1}^2dt\le C. \tag{2.5}$$

To prove Lemma 2.2, we need the following lemma (see [3]), which gives the estimate of ∇u by div$u$ and curl$u$.

Lemma 2.3 *Let $u \in H^s(\Omega)$ be a vector-valued function satisfying $u \cdot n|_{\partial\Omega} = 0$, where n is the unit outer normal of $\partial\Omega$. Then*

$$\|u\|_{H^s} \leq C(\|\mathrm{div} u\|_{H^{s-1}} + \|\mathrm{curl} u\|_{H^{s-1}} + \|u\|_{H^{s-1}}), \tag{2.6}$$

for $s \geq 1$ and the constant C depends only on s and Ω.

Proof of Lemma 2.2. Multiplying $\rho^{-1}(\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla P)$ on both sides of the momentum equations $(1.1)_2$, integrating the resulting equation over Ω, one has after integration by parts

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\mu}{2}|\nabla u|^2 + \frac{\mu+\lambda}{2}(\mathrm{div} u)^2 dx + \int_\Omega \rho^{-1}(\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla P)^2 dx \\
&\quad = -\mu\int_\Omega u\cdot\nabla u \cdot \nabla\times \mathrm{curl} u dx + (2\mu+\lambda)\int_\Omega u\cdot\nabla u\cdot\nabla \mathrm{div} u dx \\
&\qquad - \int_\Omega u\cdot\nabla u\cdot\nabla P dx - \int_\Omega u_t\cdot\nabla P dx,
\end{aligned} \tag{2.7}$$

due to $\Delta u = \nabla \mathrm{div} u - \nabla\times \mathrm{curl} u$. When u satisfies boundary condition (1.4) or (1.5), we deduce from standard L^2-theory of elliptic system that

$$\begin{aligned}
&\|\nabla^2 u\|_{L^2}^2 - C\|\nabla P\|_{L^2}^2 \\
&\quad \leq C\|\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u\|_{L^2}^2 - C\|\nabla P\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad \leq C\|\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla P\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad \leq C\int_\Omega \rho^{-1}(\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla P)^2 dx + C\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{2.8}$$

due to $\rho^{-1} \geq C^{-1} > 0$. Lemma 2.3 yields that (2.8) also holds for u satisfying boundary condition (1.6) due to the following simple fact by (1.6):

$$\|(2\mu+\lambda)\nabla \mathrm{div} u\|_{L^2}^2 + \|\mu\nabla\times \mathrm{curl} u\|_{L^2}^2 = \|\mu\triangle u + (\mu+\lambda)\nabla \mathrm{div} u\|_{L^2}^2.$$

Next, we shall treat each term on the righthand side of (2.7) under the boundary condition (1.6), since the estimate here is more subtle than the other cases, (1.4) or (1.5), due to the effect of boundary.

Using (1.6) and the facts that $u\times \mathrm{curl} u = \frac{1}{2}\nabla(|\nabla u|^2) - u\cdot\nabla u$ and $\nabla\times(a\times b) = (b\cdot\nabla)a - (a\cdot\nabla)b + (\mathrm{div} b)a - (\mathrm{div} a)b$, one gets after integration by parts and direct computations that

$$\begin{aligned}
&\left|\int_\Omega (u\cdot\nabla)u\cdot\nabla\times \mathrm{curl} u dx\right| \\
&\quad = \left|\int_\Omega \mathrm{curl} u\cdot\nabla\times((u\cdot\nabla)u)\, dx\right| \\
&\quad = \left|\int_\Omega \mathrm{curl} u\cdot\nabla\times(u\times \mathrm{curl} u) dx\right| \\
&\quad = \left|\frac{1}{2}\int_\Omega |\mathrm{curl} u|^2 \mathrm{div} u dx - \int_\Omega \mathrm{curl} u\cdot\mathcal{D}(u)\cdot \mathrm{curl} u dx\right| \\
&\quad \leq C\|\nabla u\|_{L^2(\Omega)}^2\|\mathcal{D}(u)\|_{L^\infty},
\end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
&\left|\int_\Omega u\cdot\nabla u\cdot\nabla \mathrm{div}u dx\right| \\
&\quad= \left|\int_{\partial\Omega} u^i\partial_i u^j n_j \mathrm{div}u dS - \int_\Omega \nabla u : \nabla u^t \mathrm{div}u dx + \frac{1}{2}\int_\Omega (\mathrm{div}u)^3 dx\right| \\
&\quad\le \varepsilon\|\nabla^2 u\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2 + \|\mathcal{D}(u)\|_{L^\infty}\right),
\end{aligned}
\tag{2.10}
$$

due to the following simple fact:

$$
\begin{aligned}
\left|\int_{\partial\Omega} u^i\partial_i u^j n_j \mathrm{div}u dS\right| &= \left|\int_{\partial\Omega} u^i\partial_i (u\cdot n)\mathrm{div}u dS - \int_{\partial\Omega} u^i u^j\partial_i n_j \mathrm{div}u dS\right| \\
&= \left|\int_{\partial\Omega} u^i u^j\partial_i n_j \mathrm{div}u dS\right| \\
&\le C\|u\|_{L^4(\partial\Omega)}^2\|\mathrm{div}u\|_{L^2(\partial\Omega)} \\
&\le C\|\nabla u\|_{L^2(\Omega)}^2\|\nabla u\|_{H^1(\Omega)} \\
&\le C(\varepsilon)\|\nabla u\|_{L^2(\Omega)}^4 + \varepsilon\|\nabla^2 u\|_{L^2(\Omega)}^2 + C(\varepsilon),
\end{aligned}
$$

where (1.6) and the Poincaré type inequality and the Ehrling inequality have been used. Similarly,

$$
\begin{aligned}
&\left|\int_\Omega u\cdot\nabla u\cdot\nabla P dx\right| \\
&\quad= \left|\int_{\partial\Omega} u^i\partial_i u^j n_j P dS - \int_\Omega \partial_j u^i\partial_i u^j P dx - \int_\Omega u^i\partial_i \mathrm{div}u P dx\right| \\
&\quad= \left|\int_{\partial\Omega} u^i u^j\partial_i n_j P dS + \int_\Omega \partial_j u^i\partial_i u^j P dx - \int_\Omega (\mathrm{div}u)^2 P dx - \int_\Omega u\cdot\nabla P \mathrm{div}u dx\right| \\
&\quad\le C\|\nabla u\|_{L^2}^2 + \left|\int_\Omega u\cdot\nabla P \mathrm{div}u dx\right| \\
&\quad\le C\|\nabla u\|_{L^2}^2 + \|u\|_{L^6}\|\mathrm{div}u\|_{L^3}\|\nabla P\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{5}{3}}\|\mathcal{D}(u)\|_{L^\infty}^{\frac{1}{3}}\|\nabla\rho\|_{L^2} \\
&\quad\le C\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\left(\|\mathcal{D}(u)\|_{L^\infty}+1\right) + C,
\end{aligned}
\tag{2.11}
$$

which yields also that

$$
\begin{aligned}
&-\int_\Omega u_t\cdot\nabla P dx \\
&\quad= \frac{d}{dt}\int_\Omega P\mathrm{div}u dx - \int_\Omega P_t \mathrm{div}u dx \\
&\quad= \frac{d}{dt}\int_\Omega P\mathrm{div}u dx + \int_\Omega u\cdot\nabla P\mathrm{div}u dx + (\gamma-1)\int_\Omega P(\mathrm{div}u)^2 dx \\
&\quad\le \frac{d}{dt}\int_\Omega P\mathrm{div}u dx + C\|\nabla u\|_{L^2}^2\left(\|\mathcal{D}(u)\|_{L^\infty}+1\right) \\
&\qquad + C\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C.
\end{aligned}
\tag{2.12}
$$

Substituting (2.8)-(2.12) into (2.7) gives that for ε suitably small,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx+C_0\|\nabla^2 u\|_{L^2}^2\\
&\quad\le C\left(\|\nabla\rho\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\left(\|\nabla u\|_{L^2}^2+\|\mathcal{D}(u)\|_{L^\infty}+1\right)+C. \qquad (2.13)
\end{aligned}$$

It remains to bound the L^2-norm of $\nabla\rho$. To this end, one can differentiate $(1.1)_1$ and then multiply the resulting equation by $2\nabla\rho$ to get

$$\begin{aligned}
&\partial_t|\nabla\rho|^2+\mathrm{div}(|\nabla\rho|^2u)+|\nabla\rho|^2\mathrm{div}u\\
&\quad=-2(\nabla\rho)^t\nabla u\nabla\rho-2\rho\nabla\rho\cdot\nabla\mathrm{div}u\\
&\quad=-2(\nabla\rho)^t\mathcal{D}(u)\nabla\rho-2\rho\nabla\rho\cdot\nabla\mathrm{div}u. \qquad (2.14)
\end{aligned}$$

Integrating (2.14) over Ω yields

$$\partial_t\left(\|\nabla\rho\|_{L^2}^2\right)\le C\|\nabla\rho\|_{L^2}^2\|\mathcal{D}(u)\|_{L^\infty}+\varepsilon\|\nabla^2u\|_{L^2}^2+C(\varepsilon)\|\nabla\rho\|_{L^2}^2. \qquad (2.15)$$

Adding (2.15) to (2.13), we deduce, after choosing ε suitably small and using Gronwall's inequality, that (2.5) holds. The proof of Lemma 2.2 is completed.

Next step is to improve the regularity of ρ and u. We start with some bounds on derivatives of u based on above estimates.

Lemma 2.4 *Under the condition* (2.2), *it holds that*

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u_t(t)\|_{L^2}+\|\nabla u\|_{H^1}\right)+\int_0^T\|\nabla u_t\|_{L^2}^2dt\le C,\quad 0\le T<T^*. \qquad (2.16)$$

Proof. Differentiating the momentum equations $(1.1)_2$ with respect to t yields

$$\rho u_{tt}+\rho u\cdot\nabla u_t-\mu\triangle u_t-(\mu+\lambda)\nabla\mathrm{div}u_t=-\nabla P_t-\rho_tu_t-\rho u_t\cdot\nabla u-\rho_tu\cdot\nabla u. \qquad (2.17)$$

Taking the inner product of the above equation with u_t in $L^2(\Omega)$ and integrating by parts, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho u_t^2dx+\int_\Omega\left(\mu|\nabla u_t|^2+(\lambda+\mu)(\mathrm{div}u_t)^2\right)dx\\
&\quad=\int_\Omega P_t\mathrm{div}u_tdx-\int_\Omega\rho(u_t\cdot\nabla u)\cdot u_tdx-\int_\Omega\rho u\cdot\nabla\left(|u_t|^2+u\cdot\nabla u\cdot u_t\right)dx\\
&\quad\le C\int_\Omega\left(|u||\nabla\rho|+|\nabla u|\right)|\nabla u_t|dx+C\int_\Omega\left(\rho|u_t|^2|\nabla u|+\rho|u||u_t||\nabla u_t|\right)dx\\
&\qquad+C\int_\Omega\left(|u||u_t||\nabla u|^2+|u|^2|u_t||\nabla^2u|+|u|^2||\nabla u||\nabla u_t|\right)dx\\
&\quad=\sum_{i=1}^3 I_i. \qquad (2.18)
\end{aligned}$$

Noticing that by (2.5) and Sobolev's inequality, one has

$$\begin{aligned}
I_1&\le C\left(\|u\|_{L^\infty}\|\nabla\rho\|_{L^2}+\|\nabla u\|_{L^2}\right)\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}\\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2+C(\varepsilon)\|\nabla u\|_{H^1}^2. \qquad (2.19)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_2 &\le C\|\rho^{1/2}u_t\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3} + C\|u\|_{L^\infty}\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le C\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1} \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\|\rho^{1/2}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2, \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\le C\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^3}^2 + C\|u^2\|_{L^3}\|u_t\|_{L^6}\|\nabla^2 u\|_{L^2} \\
&\quad + C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^6}\|u^2\|_{L^3} \\
&\le C\|\nabla u_t\|_{L^2}\left(\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} + \|\nabla^2 u\|_{L^2}\right) \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{H^1}^2. \qquad (2.21)
\end{aligned}
$$

We conclude from (2.18)-(2.21) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u_t^2 dx + \mu\int_\Omega |\nabla u_t|^2 dx \\
&\quad \le 6\varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{H^1}^2 + C(\varepsilon)\|\rho^{1/2}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

which, together with Gronwall's inequality, implies that for ε suitably small,

$$
\sup_{0\le t\le T}\|\rho^{1/2}u_t(t)\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \le C, \quad 0 \le T < T^*, \qquad (2.22)
$$

due to the fact that

$$
\rho_0(x)^{\frac{1}{2}}u_t(x, t=0) = \rho_0^{\frac{1}{2}}u_0\cdot\nabla u_0(x) - \rho_0^{\frac{1}{2}}g \in L^2(\Omega),
$$

which comes from the compatibility condition (1.13). Moreover, since u satisfies

$$
\begin{cases}
\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u = \rho u_t + \rho u\cdot\nabla u + \nabla P, \\
\text{(1.4) or (1.5) or (1.6) holds,}
\end{cases}
$$

similar to (2.8), one has

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2} &\le C(\|\rho^{\frac{1}{2}}u_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|\nabla P\|_{L^2}) \\
&\le C + C\|\nabla^2 u\|_{L^2}^{1/2}.
\end{aligned}
$$

Hence,

$$
\sup_{0\le T<T^*}\|\nabla u\|_{H^1} \le C. \qquad (2.23)
$$

Thus, Lemma 2.4 follows from (2.22) and (2.23) immediately.

Finally, the following lemma gives bounds of the first derivatives of the density ρ and the second derivatives of the velocity u.

Lemma 2.5 *Under the condition (2.2), it holds that for any $q \in (3,6]$*

$$\sup_{0\le t\le T} \|\rho\|_{W^{1,q}} \le C, \quad 0 \le T < T^*. \tag{2.24}$$

Proof. In fact, $(1.1)_1$ gives

$$\begin{aligned}&(|\nabla\rho|^q)_t + \mathrm{div}(|\nabla\rho|^q u) + (q-1)|\nabla\rho|^q \mathrm{div}u \\ &+ q|\nabla\rho|^{q-2}(\nabla\rho)^t\mathcal{D}(u)(\nabla\rho) + q\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla\mathrm{div}u = 0,\end{aligned}$$

which yields for the case that (1.4) or (1.5) holds,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C(\|\mathcal{D}(u)\|_{L^\infty}+1)\|\nabla\rho\|_{L^q} + C\|\nabla\mathrm{div}u\|_{L^q}, \tag{2.25}$$

and for the case that (1.6) holds,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C(\|\mathcal{D}(u)\|_{L^\infty} + \|\nabla G\|_{L^q}+1)\|\nabla\rho\|_{L^q}, \tag{2.26}$$

with $G \triangleq (2\mu+\lambda)\mathrm{div}u - P$. Using the L^p-estimate of elliptic system, we have for the case that (1.4) or (1.5) holds,

$$\begin{aligned}\|\nabla^2 u\|_{L^q} &\le C(\|\rho u_t\|_{L^q} + \|u\cdot\nabla u\|_{L^q} + \|\nabla P\|_{L^q}) \\ &\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^{(6-q)/(2q)}\|u_t\|_{L^6}^{(3q-6)/(2q)} + \|u\|_{L^\infty}\|\nabla u\|_{L^q} + \|\nabla\rho\|_{L^q}\right) \\ &\le C(\|\nabla u_t\|_{L^2} + \|\nabla\rho\|_{L^q}+1),\end{aligned} \tag{2.27}$$

due to (2.16). When the boundary condition (1.6) holds, noticing that (1.6) yields that $(\nabla\times\mathrm{curl}u)\cdot n = 0$ on the boundary $\partial\Omega$ (see [2]), and $(1.1)_1$ can be rewritten as

$$\nabla G = \rho u_t + \rho u\cdot\nabla u + \mu\nabla\times\mathrm{curl}u, \tag{2.28}$$

we have

$$\nabla G\cdot n|_{\partial\Omega} = \rho(u\cdot\nabla)u\cdot n|_{\partial\Omega} = -\rho(u\cdot\nabla)n\cdot u|_{\partial\Omega}. \tag{2.29}$$

Therefore, (2.28) yields that G satisfies

$$\begin{cases}\Delta G = \mathrm{div}(\rho u_t + \rho u\cdot\nabla u) \\ \nabla G\cdot n|_{\partial\Omega} = -\rho(u\cdot\nabla)n\cdot u|_{\partial\Omega}.\end{cases}$$

Using the L^q-estimate for Neumann problem to the elliptic equation, we have

$$\|\nabla G\|_{L^q} \le C\left(\|\rho u_t\|_{L^q} + \|u\cdot\nabla u\|_{L^q} + \|\rho|u|^2\|_{C(\bar\Omega)}\right) \le C(\|\nabla u_t\|_{L^2}+1). \tag{2.30}$$

We deduce from (2.25)(resp. (2.26)), (2.27)(resp. (2.30)), (2.16), and Gronwall's inequality that

$$\sup_{0\le t\le T}\|\rho\|_{W^{1,q}} \le C. \tag{2.31}$$

We complete the proof of Lemma 2.5.

The combination of Lemmas 2.4 and 2.5 is enough to extend the classical solutions of (ρ, u) beyond $t \geq T^*$. In fact, in view of (2.16) and (2.24), the functions $(\rho, u)|_{t=T^*} = \lim_{t\to T^*}(\rho, u)$ satisfy the conditions imposed on the initial data (1.12) at the time $t = T^*$. Furthermore,

$$-\mu\triangle u - (\mu+\lambda)\nabla(\mathrm{div}u) + \nabla P|_{t=T^*} = \lim_{t\to T^*}(\rho u_t + \rho u\cdot\nabla u) \triangleq \rho^{\frac{1}{2}} g|_{t=T^*},$$

whith $g|_{t=T^*} \in L^2(\Omega)$. Thus, $(\rho, u)|_{t=T^*}$ satisfies (1.13) also. Therefore, we can take $(\rho, u)|_{t=T^*}$ as the initial data and apply the local existence theorem [4, 5] to extend our local strong solution beyond T^*. This contradicts the assumption on T^*.

3 Generalization to the heat-conductive flows

We can modify the previous proof to be fit for the heat-conductive flows. First, following the proof of Lemma 2.2 and noticing that

$$P_t + u\cdot\nabla P + \gamma P\mathrm{div}u = (\gamma-1)\kappa\triangle\theta + (\gamma-1)(\frac{\mu}{2}(\nabla u + \nabla u^t)^2 + \lambda(\mathrm{div}u)^2), \tag{3.1}$$

one gets

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\mu}{2}|\nabla u|^2 + \frac{\mu+\lambda}{2}(\mathrm{div}u)^2 - P\mathrm{div}u dx \\
&\quad + \int_\Omega \rho^{-1}(\mu\triangle u + (\mu+\lambda)\nabla\mathrm{div}u - \nabla P)^2 dx \\
&\le C(\|\mathcal{D}u\|_{L^\infty} + 1)\|\nabla u\|_{L^2}^2 + |\int_\Omega P_t \mathrm{div}u dx| \\
&\le C(\|\mathcal{D}u\|_{L^\infty} + 1)\|\nabla u\|_{L^2}^2 + \varepsilon\|\nabla u\|_{H^1}^2 + C(\varepsilon)\|\nabla\theta\|_{L^2}^2 \\
&+ C(\varepsilon)(1 + \|\mathcal{D}u\|_{L^\infty} + \|\theta\|_{L^\infty}^2)\|\nabla u\|_{L^2}^2, \quad \forall 0 < \varepsilon < 1.
\end{aligned} \tag{3.2}$$

Multiplying θ on both sides of the energy equation, one has after integration by parts that

$$\partial_t\int_\Omega \rho\theta^2 dx + 2\kappa\int_\Omega |\nabla\theta|^2 dx \le C\|\theta\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \tag{3.3}$$

Combining (2.15), (3.2), (3.3) and applying Gronwall's inequality, we easily deduce that

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\rho^{\frac{1}{2}}\theta\|_{L^2}^2) + \int_0^T(\|\nabla u\|_{H^1}^2 + \|\nabla\theta\|_{L^2}^2)dt \le C.$$

The higher regularity of (ρ, u, θ) can be obtained following the proof of [18].

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