Serrin Type Criterion for the Three-Dimensional Viscous Compressible Flows

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Abstract

We extend the well-known Serrin's blowup criterion for the three-dimensional (3D) incompressible Navier-Stokes equations to the 3D viscous compressible cases. It is shown that for the Cauchy problem of the 3D compressible Navier-Stokes system in the whole space, the strong or smooth solution exists globally if the velocity satisfies the Serrin's condition and either the supernorm of the density or the $L^1(0,T;L^\infty)$-norm of the divergence of the velocity is bounded. Furthermore, in the case that either the shear viscosity coefficient is suitably large or there is no vacuum, the Serrin's condition on the velocity can be removed in this criteria.

1 Introduction

The time evolution of the density and the velocity of a general viscous compressible barotropic fluid occupying a domain $\Omega \subset R^3$ is governed by the compressible Navier-Stokes equations

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\triangle u - (\mu+\lambda)\nabla(\operatorname{div} u) + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

where ρ, u, and P are the density, velocity and pressure respectively. The equation of state is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \gamma > 1).$$

The constants μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0. \tag{1.2}$$

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Let $\Omega = R^3$ and we consider the Cauchy problem to the equations (1.1) with initial data:

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x). \tag{1.3}$$

There are huge literatures on the large time existence and behavior of solutions to (1.1). The one-dimensional problem has been studied extensively by many people, see [10, 18, 26, 27] and the references therein. For the multidimensional problem (1.1), the local existence and uniqueness of classical solutions are known in [23, 28] in the absence of vacuum and recently, for strong solutions also, in [3, 4, 6, 25] for the case where the initial density need not be positive and may vanish in an open set. The global existence of classical solutions was first investigated by Matsumura-Nishida [22], who proved global existence of smooth solutions for data close to a non-vacuum equilibrium, and later by Hoff [11, 12] for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [21] (see also Feireisl [9]), where he obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish.

However, the regularity and uniqueness of such weak solutions remains open. In particular, Xin first showed in [32] that in the case that the initial density has compact support, any nontrivial smooth solution to the Cauchy problem of the non-barotropic compressible Navier-Stokes system without heat conduction blows up in finite time for any space dimension, and the same holds for the isentropic case (1.1), at least in one-dimension. See also the recent generalizations to the cases for the non-barotropic compressible Navier-Stokes system with heat conduction ([5]) and for non-compact but rapidly decreasing at far field initial densities ([24]).

In this paper, we are concerned with the main mechanism for possible breakdown of strong (or smooth) solutions to the 3-D compressible Navier-Stokes equations.

We will use the following conventions throughout this paper. Set

$$\int f dx = \int_{R^3} f dx.$$

For $1 < r < \infty$, the standard homogeneous and inhomogeneous Sobolev spaces are denoted as follows:

$$\begin{cases} L^r = L^r(R^3), \quad D^{k,r} = \left\{ u \in L^1_{loc}(R^3) \left| \|\nabla^k u\|_{L^r} < \infty \right. \right\}, \quad \|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \left\{ u \in L^6 \left| \|\nabla u\|_{L^2} < \infty \right. \right\}. \end{cases}$$

Next, the strong solutions to the Cauchy problem, (1.1)-(1.3), are defined as:

Definition 1.1 (Strong solutions) (ρ, u) *is called a strong solution to (1.1) in* $R^3 \times (0, T)$, *if for some* $q_0 \in (3, 6]$,

$$\begin{aligned} & 0 \le \rho \in C([0, T], W^{1,q_0}), \quad \rho_t \in C([0, T], L^{q_0}), \\ & u \in C([0, T], D^1 \cap D^2) \cap L^2(0, T; D^{2,q_0}) \\ & \rho^{1/2} u_t \in L^\infty(0, T; L^2), \quad u_t \in L^2(0, T; D^1), \end{aligned} \tag{1.4}$$

and (ρ, u) *satisfies (1.1) a.e. in* $R^3 \times (0, T)$.

There are several recent works ([3,7,8,14–17]) concerning blowup criteria for strong (or smooth) solutions to the compressible Navier-Stokes equations. In particular, it is proved in [8] for two dimensions, if $7\mu > 9\lambda$, then

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\|\rho\|_{L^\infty}+\int_0^T(\|\rho\|_{W^{1,q_0}}+\|\nabla\rho\|_{L^2}^4)dt\right)=\infty,$$

where $T^* < \infty$ is the maximal time of existence of a strong solution and $q_0 > 3$ is a constant. Later, we [14, 17] first establish a blowup criterion, analogous to the Beal-Kato-Majda criterion [1] for the ideal incompressible flows, for strong (or classical) solutions to (1.1) in three spatial dimensions, by assuming that if T^* is the maximal time for the existence of a strong (or classical) solution (ρ, u) and $T^* < \infty$, then

$$\lim_{T\to T^*}\int_0^T\|\nabla u\|_{L^\infty}dt=\infty, \tag{1.5}$$

under the condition on viscosity coefficients:

$$7\mu > \lambda. \tag{1.6}$$

Recently, for the initial density away from vacuum, that is,

$$\inf_{x\in R^3}\rho_0(x) > 0, \tag{1.7}$$

We [15] succeeded in removing the crucial condition (1.6) of [14, 17] and established the blowup criterion (1.5) under the physical restrictions (1.2). More recently, we [16] improve the results in [14,15,17] by allowing vacuum states initially and replacing (1.5) by

$$\lim_{T\to T^*}\int_0^T\|\mathcal{D}u\|_{L^\infty}dt=\infty, \tag{1.8}$$

where $\mathcal{D}(u)$ is the deformation tensor:

$$\mathcal{D}(u)=\frac{1}{2}(\nabla u+\nabla u^t).$$

Motivated by the well-known Serrin's criterion on the Leray-Hopf weak solutions to the 3D incompressible Navier-Stokes equations, which can be stated that if the velocity $u \in L^s(0,T;L^r)$ is a weak solution of 3D incompressible Navier-Stokes system, with r, s satisfying

$$\frac{2}{s}+\frac{3}{r}\le 1, \quad 3 < r \le \infty, \tag{1.9}$$

then u is regular (see ([2, 19, 29, 30]) and references therein), we try to extend Serrin's blow-up criterion to the compressible Navier-Stokes equations. More precisely, we have the following main result in this paper:

Theorem 1.1 *Let (ρ, u) be a strong solution to the Cauchy problem (1.1) (1.3) satisfying (1.4) while the initial data (ρ_0, u_0) satisfy*

$$0 \le \rho_0 \in L^1\cap H^1\cap W^{1,\tilde q}, \quad u_0 \in D^1\cap D^2, \tag{1.10}$$

for some $\tilde{q} \in (3, \infty)$ and the compatibility condition:

$$-\mu \triangle u_0 - (\lambda + \mu)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g \quad \textit{for some } g \in L^2. \tag{1.11}$$

If $T^* < \infty$ is the maximal time of existence, then both

$$\lim_{T \to T^*} (\|divu\|_{L^1(0,T;L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0,T;L^r)}) = \infty, \tag{1.12}$$

and

$$\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0,T;L^r)}) = \infty, \tag{1.13}$$

where r and s satisfy (1.9).

A few remarks are in order:

Remark 1.1 *If* $\mathrm{div} u \equiv 0$, *(1.12) and (1.13) reduce to the well-known Serrin's blowup criterion for 3D incompressible Navier-Stokes equations. Therefore, Theorem 1.1 can be regarded as the Serrin type blowup criterion on 3D compressible Navier-Stokes equations.*

Remark 1.2 *Theorem 1.1 also holds for classical solutions to the 3D compressible viscous flows.*

Remark 1.3 *These results can be generalized to viscous heat-conductive flows, which will be reported in a forthcoming paper.*

In the following two theorems, we will show that (1.12) and (1.13) can be in fact replaced by

$$\lim_{T \to T^*} \|\mathrm{div} u\|_{L^1(0,T;L^\infty)} = \infty, \tag{1.14}$$

and by

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty, \tag{1.15}$$

respectively, provided either the viscous coefficients satisfy the additional condition (1.6) besides (1.2) or the initial density is away from vacuum. The first is:

Theorem 1.2 *Under the conditions of Theorem 1.1, assume that (1.6) holds in addition. Then both (1.14) and (1.15) hold true.*

Remark 1.4 *The result in [2] shows that a Leray-Hopf's weak solution to the 3D incompressible Navier-Stokes equations becomes smooth for bounded pressure. Therefore, Corollary 1.2 seems reasonable since the pressure P here is bounded from above provided either (1.14) or (1.15) fails. Thus Theorem 1.2 can be considered as a generalization of the corresponding results in [2].*

Remark 1.5 *The conclusion in Theorem 1.2 have been obtained independently in [31].*

For the initial density away form vacuum, we have

Theorem 1.3 *In addition to the conditions of Theorem 1.1, assume that the initial density ρ_0 satisfies (1.7). Then (1.14) holds.*

Remark 1.6 *Theorem 1.3 improves the previous results in [15] and the ones in [16] in the absence of vacuum where the criteria (1.5) and (1.8) have been replaced by (1.14).*

We now comment on the analysis of this paper. The key step in proving Theorem 1.1 is to derive the $L^\infty(0,T;L^p)$-estimate on the gradient of the density. Note that in all previous works [14,16,17], their methods depend crucially on the $L^1(0,T;L^\infty)$-norm of either the gradient of the velocity or its symmetry part instead of the divergence. Thus, under the assumption of the left hand side of either (1.12) or (1.13) is finite, we need to derive the upper bound for the $L^1(0,T;L^\infty)$-norm of the velocity gradient. Some new ideas are needed for this. Take the case (1.13) for example, and assume the left hand side of is finite. We first obtain the estimate on the $L^\infty(0,T;L^2)$-norm of ∇u by using the a priori assumptions. Next we deduce the estimates on the $L^2(0,T;L^\infty)$-norm of both the divergence and the vorticity of the velocity by combining the basic estimates on the material derivatives of the velocity developed by Hoff [11] and a priori estimate on $L^\infty(0,T;L^p)$-norm of the density gradient $\nabla\rho$. These estimates can be obtained simultaneously by solving a logarithm Gronwall inequality based on a Beal-Kato-Majda type inequality (see Lemma 2.3) and the a priori estimates we have just derived.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed later. The main results, Theorem 1.1, Corollaries1.2 and 1.3 are proved in Section 3 and Section 4 respectively.

2 Preliminaries

In this section, we recall some known facts and elementary inequalities which will be used later.

We begin with the local existence and uniqueness of strong solutions when the initial density may not be positive and may vanish in an open set obtained in [3].

Lemma 2.1 *If the initial data (ρ_0, u_0) satisfy (1.10) and (1.11), then there exists a positive time $T_1 \in (0,\infty)$ and a unique strong solution (ρ, u) to the Cauchy problem (1.1)(1.3) in $R^3 \times (0, T_1]$.*

Next, the following well-known Gagliardo-Nirenberg inequality which will be used later frequently (see [20]).

Lemma 2.2 (Gagliardo-Nirenberg) *For $p \in [2,6], q \in (1,\infty)$, and $r \in (3,\infty)$, there exists some generic constant $C > 0$ which may depend on q, r such that for $f \in H^1$ and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

Finally, we state the following Beal-Kato-Majda type inequality which was proved in [1] when $\mathrm{div} u \equiv 0$ and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^6}$.

Lemma 2.3 *For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2 \cap D^{1,q}$,*

$$\begin{aligned} \|\nabla u\|_{L^\infty} \le\ & C\left(\|\mathrm{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}\right)\log(e + \|\nabla^2 u\|_{L^q}) \\ & + C\|\nabla u\|_{L^2} + C. \end{aligned} \tag{2.3}$$

Proof. The proof is similar to that of (15) in [1] and is sketched here for completeness. It follows from Poisson's formula that

$$
\begin{aligned}
u(x) &= -\frac{1}{4\pi}\int \frac{\Delta u(y)}{|x-y|}dy \\
&\equiv \int \mathrm{div}u(y)K(x-y)dy - \int K(x-y)\times(\nabla\times u)(y)dy \\
&\triangleq v + w,
\end{aligned}
\tag{2.4}
$$

where

$$
K(x-y) \triangleq \frac{x-y}{4\pi|x-y|^3},
$$

satisfies

$$
|K(x-y) \le C|x-y|^{-2}, \quad |\nabla K(x-y)| \le C|x-y|^{-3}. \tag{2.5}
$$

It suffices to estimate the term ∇v since ∇w can be handled similarly (see [1]). Let $\delta \in (0,1]$ be a constant to be chosen and introduce a cut-off function $\eta_\delta(x)$ satisfying $\eta_\delta(x) = 1$ for $|x| < \delta, \eta_\delta(x) = 0$ for $|x| > 2\delta$, and $|\nabla\eta_\delta(x)| \le C\delta^{-1}$. Then ∇v can be rewritten as

$$
\begin{aligned}
\nabla v = &\int \eta_\delta(y)K(y)\nabla\mathrm{div}u(x-y)dy - \int \nabla\eta_\delta(x-y)K(x-y)\mathrm{div}u(y)dy \\
&+ \int(1-\eta_\delta(x-y))\nabla K(x-y)\mathrm{div}u(y)dy.
\end{aligned}
\tag{2.6}
$$

Each term on the righthand side of (2.6) can be estimated by (2.5) as follows:

$$
\begin{aligned}
&\left|\int \eta_\delta(y)K(y)\nabla\mathrm{div}u(x-y)dy\right| \\
&\le C\|\eta_\delta(y)K(y)\|_{L^{q/(q-1)}}\|\nabla^2 u\|_{L^q} \\
&\le C\left(\int_0^{2\delta} r^{-2q/(q-1)}r^2dr\right)^{(q-1)/q}\|\nabla^2 u\|_{L^q} \\
&\le C\delta^{(q-3)/q}\|\nabla^2 u\|_{L^q},
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\left|\int \nabla\eta_\delta(x-y)K(x-y)\mathrm{div}u(y)dy\right| \\
&\le \int |\nabla\eta_\delta(z)||K(z)|dz\|\mathrm{div}u\|_{L^\infty} \\
&\le C\int_\delta^{2\delta} \delta^{-1}r^{-2}r^2dr\|\mathrm{div}u\|_{L^\infty} \\
&\le C\|\mathrm{div}u\|_{L^\infty},
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\left|\int(1-\eta_\delta(x-y))\nabla K(x-y)\mathrm{div}u(y)dy\right| \\
&\le C\left(\int_{\delta\le|x-y|\le 1} + \int_{|x-y|>1}\right)|\nabla K(x-y)||\mathrm{div}u(y)|dy \\
&\le C\int_\delta^1 r^{-3}r^2dr\|\mathrm{div}u\|_{L^\infty} + C\left(\int_1^\infty r^{-6}r^2dr\right)^{1/2}\|\mathrm{div}u\|_{L^2} \\
&\le -C\ln\delta\|\mathrm{div}u\|_{L^\infty} + C\|\nabla u\|_{L^2}.
\end{aligned}
\tag{2.9}
$$

It follows from (2.6)-(2.9) that

$$\|\nabla v\|_{L^\infty} \le C\left(\delta^{(q-3)/q}\|\nabla^2 u\|_{L^q} + (1-\ln\delta)\|\mathrm{div}u\|_{L^\infty} + \|\nabla u\|_{L^2}\right). \tag{2.10}$$

Set $\delta = \min\left\{1, \|\nabla^2 u\|_{L^q}^{-q/(q-3)}\right\}$. Then (2.10) becomes

$$\|\nabla v\|_{L^\infty} \le C(q)\left(1 + \ln(e + \|\nabla^2 u\|_{L^q})\|\mathrm{div}u\|_{L^\infty} + \|\nabla u\|_{L^2}\right).$$

Therefore (2.3) holds.

3 Proof of Theorem 1.1

Let (ρ, u) be a strong solution to the problem (1.1)-(1.2) as described in Theorem 1.1. Then the standard energy estimate yields

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u(t)\|_{L^2}^2 + \|\rho\|_{L^1} + \|\rho\|_{L^\gamma}^\gamma\right) + \int_0^T \|\nabla u\|_{L^2}^2 dt \le C, \quad 0 \le T < T^*. \tag{3.1}$$

We first prove (1.13). Otherwise, there exists some constant $M_0 > 0$ such that

$$\lim_{T\to T^*}(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r)}) \le M_0. \tag{3.2}$$

The first key estimate on ∇u will be given in the following lemma.

Lemma 3.1 *Under the condition (3.2), it holds that for* $0 \le T < T^*$,

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 + \int_0^T\int \rho u_t^2 dxdt \le C, \tag{3.3}$$

where and in what follows, C *denotes a generic constant depending only on* $\mu, \lambda, a, \gamma, M_0, T$, *and the initial data.*

Proof. It follows from the momentum equations in (1.1) that

$$\triangle G = \mathrm{div}(\rho\dot{u}), \quad \mu\triangle\omega = \nabla\times(\rho\dot{u}), \tag{3.4}$$

where

$$\dot{f} \triangleq f_t + u\cdot\nabla f, \quad G \triangleq (2\mu+\lambda)\mathrm{div}u - P(\rho), \quad \omega \triangleq \nabla\times u, \tag{3.5}$$

are the material derivative of f, the effective viscous flux and the vorticity respectively.

The standard L^p-estimate for the elliptic system (3.4), and (2.1) give directly that

$$\|\nabla G\|_{L^2} + \|\nabla\omega\|_{L^2} \le C(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2}), \tag{3.6}$$

and

$$\|\nabla G\|_{L^6} + \|\nabla\omega\|_{L^6} \le C\|\rho\dot{u}\|_{L^6} \le C\|\nabla\dot{u}\|_{L^2}. \tag{3.7}$$

Multiplying the momentum equation $(1.1)_2$ by u_t and integrating the resulting equation over R^3 gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2\right)dx + \int\rho u_t^2 dx\\ &= \int P\mathrm{div}u_t dx - \int\rho u\cdot\nabla u\cdot u_t dx.\end{aligned} \tag{3.8}$$

For the first term on the righthand side of (3.8), one has

$$
\begin{aligned}
&\int P\mathrm{div}u_t dx\\
&=\frac{d}{dt}\int P\mathrm{div}u dx-\int P_t\mathrm{div}u dx\\
&=\frac{d}{dt}\int P\mathrm{div}u dx+\int \mathrm{div}(Pu)\mathrm{div}u dx+(\gamma-1)\int P(\mathrm{div}u)^2dx\\
&=\frac{d}{dt}\int P\mathrm{div}u dx-\int (Pu)\cdot\nabla\mathrm{div}u dx+(\gamma-1)\int P(\mathrm{div}u)^2dx \qquad (3.9)\\
&=\frac{d}{dt}\int P\mathrm{div}u dx-\frac{1}{2\mu+\lambda}\int Pu\cdot\nabla G dx-\frac{1}{2(2\mu+\lambda)}\int P^2\mathrm{div}u dx\\
&\quad+(\gamma-1)\int P(\mathrm{div}u)^2dx\\
&\le\frac{d}{dt}\int P\mathrm{div}u dx+\varepsilon\|\nabla G\|_{L^2}^2+C(\varepsilon)\|\nabla u\|_{L^2}^2+C(\varepsilon),
\end{aligned}
$$

due to

$$
P_t+\mathrm{div}(Pu)+(\gamma-1)P\mathrm{div}u=0,
$$

which comes from $(1.1)_1$.

For the second term on the righthand side of (3.8), Cauchy's inequality yields

$$
\left|\int\rho u\cdot\nabla u\cdot u_t dx\right|\le\frac{1}{4}\int\rho u_t^2dx+C\int\rho|u\cdot\nabla u|^2dx. \tag{3.10}
$$

Substituting (3.9) and (3.10) into (3.8), one has by choosing ε suitably small,

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx+\frac{1}{2}\int\rho u_t^2dx\\
&\quad\le C\|\nabla u\|_{L^2}^2+C\int\rho|u\cdot\nabla u|^2dx+C.
\end{aligned} \tag{3.11}
$$

Gagliardo-Nirenberg's inequality (2.1) yields that for r,s satisfying (1.9),

$$
\begin{aligned}
&\|\rho^{\frac12}u\cdot\nabla u\|_{L^2}\\
&\le C\|\rho^{\frac12}u\|_{L^r}\|\nabla u\|_{L^{\frac{2r}{r-2}}}\\
&\le C\|\rho^{\frac12}u\|_{L^r}(\|G\|_{L^{\frac{2r}{r-2}}}+\|\omega\|_{L^{\frac{2r}{r-2}}}+1)\\
&\le C\|\rho^{\frac12}u\|_{L^r}(\|G\|_{L^2}^{1-\frac3r}\|\nabla G\|_{L^2}^{\frac3r}+\|\omega\|_{L^2}^{1-\frac3r}\|\nabla\omega\|_{L^2}^{\frac3r}+1)\\
&\le\varepsilon(\|\nabla G\|_{L^2}+\|\nabla\omega\|_{L^2})+C(\varepsilon)\|\rho^{\frac12}u\|_{L^r}^{\frac s2}(\|G\|_{L^2}+\|\omega\|_{L^2}+1)+C(\varepsilon)\\
&\le C\varepsilon(\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2})+C(\varepsilon)\|\rho^{\frac12}u\|_{L^r}^{\frac s2}(\|\nabla u\|_{L^2}+1)+C(\varepsilon),
\end{aligned}
$$

where in the last inequality we have used (3.6). Thus, for ε small enough,

$$
\|\rho^{\frac12}u\cdot\nabla u\|_{L^2}\le C\varepsilon\|\rho u_t\|_{L^2}+C(\varepsilon)\|\rho^{\frac12}u\|_{L^r}^{\frac s2}(\|\nabla u\|_{L^2}+1)+C(\varepsilon). \tag{3.12}
$$

Substituting (3.12) into (3.11), one has

$$\begin{aligned}
&\frac{d}{dt}\int_{R^3}\left(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx+\frac{1}{4}\int\rho u_t^2dx\\
&\quad\le C(\|\rho^{1/2}u\|_{L^r}^s+1)(\|\nabla u\|_{L^2}^2+1),
\end{aligned}$$

which, together with (3.2) and Gronwall's inequality, gives (3.3). The proof of Lemma 3.1 is completed.

Next, we improve the regularity estimates on ρ and u. Motivated by Hoff [11], we start with the basic bounds on the material derivatives of u.

Lemma 3.2 *Under the condition (3.2), it holds that for* $0\le T<T^*$,

$$\sup_{0\le t\le T}\int\rho|\dot u|^2dx+\int_0^T\int|\nabla\dot u|^2dxdt\le C. \tag{3.13}$$

Proof. We will follow the idea due to Hoff [11]. Applying $\dot u^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ to $(1.2)^j$ and integrating by parts give

$$\begin{aligned}
&\int\rho|\dot u|^2dx\\
&=\int_0^t\int[-\dot u^j[\partial_jP_t+\mathrm{div}(\partial_jPu)]+\mu\dot u^j[\triangle u_t^j+\mathrm{div}(u\triangle u^j)]\\
&\quad+(\lambda+\mu)\dot u^j[\partial_t\partial_j\mathrm{div}u+\mathrm{div}(u\partial_j\mathrm{div}u)]]dxds\\
&=\sum_{i=1}^3N_i.
\end{aligned}\tag{3.14}$$

One gets after integration by parts and using (1.1)

$$\begin{aligned}
N_1&=-\int_0^t\int\dot u^j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dxds\\
&=\int_0^t\int[\partial_j\dot u^jP'\rho_t+\partial_k\dot u^j\partial_jPu^k]dxds\\
&=\int_0^t\int[-P'\rho\mathrm{div}u\partial_j\dot u^j-\partial_j\dot u^ju^k\partial_kP+\partial_k\dot u^ju^k\partial_jP]dxds\\
&=\int_0^t\int[-P'\rho\mathrm{div}u\partial_j\dot u^j+\partial_k(\partial_j\dot u^ju^k)P-P\partial_j(\partial_k\dot u^ju^k)]dxds\\
&\le C(\int_0^t\int|\nabla u|^2dxds)^{\frac12}(\int_0^t\int|\nabla\dot u|^2dxds)^{\frac12}\\
&\le C(\int_0^t\int|\nabla\dot u|^2dxds)^{\frac12}.
\end{aligned}\tag{3.15}$$

Integration by parts leads to

$$
\begin{aligned}
N_2 &= \int_0^t \int \mu \dot{u}^j [\triangle u_t^j + \mathrm{div}(u \triangle u^j)] dxds \\
&= -\int_0^t \int \mu [\partial_i \dot{u}^j \partial_i u_t^j + \triangle u^j u \cdot \nabla \dot{u}^j] dxds \\
&= -\int_0^t \int \mu [|\nabla \dot{u}|^2 - \partial_i \dot{u}^j u^k \partial_k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j + \triangle u^j u \cdot \nabla \dot{u}^j] dxds \\
&= -\int_0^t \int \mu [|\nabla \dot{u}|^2 + \partial_i \dot{u}^j \partial_k u^k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_i u^j \partial_i u^k \partial_k \dot{u}^j] dxds \\
&\le -\frac{\mu}{2} \int_0^t \int |\nabla \dot{u}|^2 dxds + C \int_0^t \int |\nabla u|^4 dxds.
\end{aligned} \tag{3.16}
$$

Similarly,

$$
N_3 \le -\frac{\mu+\lambda}{2} \int_0^t \int (\mathrm{div} \dot{u})^2 dxds + C \int_0^t \int |\nabla u|^4 dxds. \tag{3.17}
$$

Substituting (3.15)-(3.17) into (3.14), we obtain immediately by (3.5), (3.7), (2.1) and (2.2) that

$$
\begin{aligned}
&\sup_{0 \le s \le t} \int \rho |\dot{u}|^2 dx + \int_0^t \int |\nabla \dot{u}|^2 dxds \\
&\quad \le C \int_0^t \|\nabla u\|_{L^4}^4 ds + C \\
&\quad \le C \int_0^t \left(\|G\|_{L^4}^4 + \|\omega\|_{L^4}^4 \right) ds + C \\
&\quad \le C \int_0^t \left(\|G\|_{L^2}^{5/2} \|\nabla G\|_{L^6}^{3/2} + \|\omega\|_{L^2}^{5/2} \|\nabla \omega\|_{L^6}^{3/2} \right) ds + C \\
&\quad \le C \int_0^t \|\nabla \dot{u}\|_{L^2}^{3/2} ds + C \\
&\quad \le \delta \int_0^t \|\nabla \dot{u}\|_{L^2}^2 ds + C_\delta,
\end{aligned}
$$

which gives directly (3.13). The proof of Lemma 3.2 is completed.

The next lemma is used to bound the density gradient and $L^1(0,T;L^\infty)$-norm of ∇u.

Lemma 3.3 *Under the condition* (3.2)*, it holds that for any* $q \in (3,6]$

$$
\sup_{0 \le t \le T} (\|\rho\|_{H^1 \cap W^{1,q}} + \|\nabla u\|_{H^1}) \le C, \quad 0 \le T < T^*.
$$

Proof. In fact, for $2 \le p \le 6$, $|\nabla \rho|^p$ satisfies

$$
\begin{aligned}
&(|\nabla \rho|^p)_t + \mathrm{div}(|\nabla \rho|^p u) + (p-1)|\nabla \rho|^p \mathrm{div} u \\
&+ p|\nabla \rho|^{p-2} (\nabla \rho)^t \nabla u (\nabla \rho) + p\rho |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \mathrm{div} u = 0,
\end{aligned}
$$

which together with (2.1), (3.5), (3.7) and (3.13) gives

$$
\begin{aligned}
\partial_t \|\nabla \rho\|_{L^p} &\le C(1 + \|\nabla u\|_{L^\infty} + \|\nabla G\|_{L^p}) \|\nabla \rho\|_{L^p} \\
&\le C(1 + \|\nabla u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2}) \|\nabla \rho\|_{L^p}.
\end{aligned} \tag{3.18}
$$

Rewrite the momentum equations $(1.1)_2$ as

$$\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u=\rho\dot{u}+\nabla P. \tag{3.19}$$

The standard L^p-estimate for the elliptic system (3.19), (3.7) and (3.13) yield that for $q\in(3,6]$

$$\begin{aligned}\|\nabla u\|_{W^{1,q}}&\le C\left(\|\rho\dot{u}\|_{L^2}+\|\nabla u\|_{L^2}+\|P\|_{L^2}+\|\rho\dot{u}\|_{L^q}+\|\nabla\rho\|_{L^q}\right)\\&\le C\left(1+\|\nabla\dot{u}\|_{L^2}+\|\nabla\rho\|_{L^q}\right),\end{aligned}$$

which, combining with Lemmas 2.2 and 2.3, leads to

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C+C\left(\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\right)\ln(e+\|\nabla u\|_{W^{1,q}})\\&\le C+C\left(\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\right)\ln(e+\|\nabla\dot{u}\|_{L^2})\\&\quad+C\left(\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\right)\ln(e+\|\nabla\rho\|_{L^q}).\end{aligned} \tag{3.20}$$

Set $p=q$ in (3.18) and

$$f(t)\triangleq e+\|\nabla\rho\|_{L^q},\quad g(t)\triangleq\left(1+\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}+\|\nabla\dot{u}\|_{L^2}\right)\log(e+\|\nabla\dot{u}\|_{L^2}).$$

It follows from (3.18), (3.7), and (3.20) that

$$f'(t)\le Cg(t)f(t)+Cg(t)f(t)\ln f(t)+Cg(t),$$

which yields

$$(\ln f(t))'\le Cg(t)+Cg(t)\ln f(t), \tag{3.21}$$

due to $f(t)>1$.

We obtain from (3.5), (3.13), (2.2) and (3.7) that

$$\begin{aligned}&\int_0^T\left(\|\mathrm{div}u\|_{L^\infty}^2+\|\omega\|_{L^\infty}^2\right)dt\\&\le C\int_0^T\left(\|G\|_{L^\infty}^2+\|P\|_{L^\infty}^2+\|\omega\|_{L^\infty}^2\right)dt\\&\le C\int_0^T\left(\|G\|_{L^2}^2+\|\nabla G\|_{L^6}^2+\|\omega\|_{L^2}^2+\|\nabla\omega\|_{L^6}^2\right)dt+C\\&\le C\int_0^T\|\nabla\dot{u}\|_{L^2}^2dt+C\\&\le C,\end{aligned} \tag{3.22}$$

which together with (3.13), (3.21) and Gronwall's inequality yields that

$$\sup_{0\le t\le T}f(t)\le C.$$

Consequently,

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C, \tag{3.23}$$

which, combining with (3.20), (3.22) and (3.13), gives directly that

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C. \tag{3.24}$$

It thus follows from (3.18), (3.13) and (3.24) that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C. \tag{3.25}$$

The standard L^2-estimate for the elliptic system (3.19), (3.25) and (3.13) yield that

$$\sup_{0\le t\le T}\|\nabla^2 u\|_{L^2}\le C\sup_{0\le t\le T}\|\rho\dot{u}\|_{L^2}+C\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C, \tag{3.26}$$

which together with (3.3), (3.1), (3.23), and (3.25) finishes the proof of Lemma 3.3.

The combination of Lemma 3.2 with Lemma 3.3 is enough to extend the strong solutions of (ρ,u) beyond $t\ge T^*$. In fact, the functions $(\rho,u)(x,T^*)\triangleq\lim_{t\to T^*}(\rho,u)$ satisfy the conditions imposed on the initial data (1.10) at the time $t=T^*$. Furthermore,

$$-\mu\triangle u-(\mu+\lambda)\nabla(\mathrm{div}u)+\nabla P|_{t=T^*}=\lim_{t\to T^*}(\rho\dot{u})=\rho^{\frac12}(x,T^*)g(x),$$

with $g(x)\triangleq\lim_{t\to T^*}\left(\rho^{1/2}\dot{u}\right)(x,t)\in L^2$. Thus, $(\rho,u)(x,T^*)$ satisfies (1.11) also. Therefore, we can take $(\rho,u)(x,T^*)$ as the initial data and apply Lemma 2.1 to extend the local strong solution beyond T^*. This contradicts the assumption on T^*. We thus finish the proof of (1.13).

It remains to prove (1.12). Assume otherwise that

$$\lim_{T\to T^*}(\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}+\|\sqrt{\rho}u\|_{L^s(0,T;L^r)})\le C<\infty.$$

This together with $(1.1)_1$, yields immediately the following L^∞ bound of the density ρ, which contradicts (1.13). Indeed, one has

Lemma 3.4 *Assume that*

$$\int_0^T\|divu\|_{L^\infty}dt\le C,\quad 0<T<T^*.$$

Then

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le C,\quad 0<T<T^*. \tag{3.27}$$

Moreover, if in addition (1.7) holds, then

$$\sup_{0\le t\le T}\|\rho^{-1}\|_{L^\infty}\le C,\quad 0<T<T^*. \tag{3.28}$$

Proof. It follows from $(1.1)_1$ that for $\forall p\ge1$,

$$\partial_t(\rho^p)+\mathrm{div}(\rho^p u)+(p-1)\rho^p\mathrm{div}u=0. \tag{3.29}$$

Integrating (3.29) over R^3 leads to

$$\partial_t\int\rho^p dx\le(p-1)\|\mathrm{div}u\|_{L^\infty}\int\rho^p dx,$$

that is,

$$\partial_t\|\rho\|_{L^p}\le\frac{p-1}{p}\|\mathrm{div}u\|_{L^\infty}\|\rho\|_{L^p},$$

which implies immediately

$$\|\rho\|_{L^p}(t)\le C,$$

with C independent of p, so (3.27) follows. The same procedure works for ρ^{-1} provided (1.7) holds. The proof of Lemma 3.4 is finished.

4 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Theorem 1.2 is a consequence of Theorem 1.1, Lemma 3.4 and the following Lemma 4.1.

Lemma 4.1 *Assume that (1.6) holds and (1.15) fails. Then there exist some $q > 3$ and $C > 0$ such that*

$$\sup_{0\le t<T^*}\int \rho|u|^q(x,t)dx \le C. \tag{4.1}$$

Proof. This follows from an argument due to Hoff [13] (see [14, 17]). Setting $q > 3$ and multiplying $(1.1)_2$ by $q|u|^{q-2}u$, and integrating the resulting equation over R^3, we obtain by Lemma 3.4 that

$$\begin{aligned}\frac{d}{dt}\int \rho|u|^q dx + \int F dx &= q\int \mathrm{div}(|u|^{q-2}u)P dx\\ &\le C\int \rho^{\frac12}|u|^{q-2}|\nabla u|dx\\ &\le \varepsilon\int |u|^{q-2}|\nabla u|^2dx + C(\varepsilon)\int \rho|u|^{q-2}dx\\ &\le \varepsilon\int |u|^{q-2}|\nabla u|^2dx + C(\varepsilon)(\int \rho|u|^q dx)^{\frac{q-2}{q}},\end{aligned} \tag{4.2}$$

for F being defined by

$$\begin{aligned}F \triangleq\ & q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2 + \mu(q-2)|\nabla|u||^2]\\ &+ q(\lambda+\mu)\mathrm{div}u u\cdot\nabla|u|^{q-2}\\ \ge\ & q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2 + \mu(q-2)|\nabla|u||^2\\ &- (\lambda+\mu)(q-2)|\nabla|u||\cdot|\mathrm{div}u|]\\ =\ & q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u - \frac12|\nabla|u||)^2]\\ &+ q|u|^{q-2}[\mu(q-2) - \frac14(\lambda+\mu)(q-2)^2]|\nabla|u||^2\\ \ge\ & C|u|^{q-2}|\nabla u|^2,\end{aligned} \tag{4.3}$$

where we have used $|\nabla|u|| \le |\nabla u|$ and the following simple fact

$$\mu(q-1) - \frac14(\lambda+\mu)(q-2)^2 > 0,$$

due to (1.6).

Inserting (4.3) into (4.2) and taking ε small enough, we may apply Gronwall's inequality to conclude (4.1) and thus complete the proof of Lemma 4.1.

Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 1.1 and the next Lemma.

Lemma 4.2 *It holds that for $0 < T < T^*$,*

$$\sup_{0\le t\le T}\int \rho|u|^4dx \le C, \tag{4.4}$$

provided (1.7) holds and (1.14) fails.

Proof. The main idea is due to [15]. Indeed, multiplying $(1.1)_2$ by $4|u|^2u$, and integrating the resulting equation over R^3, we obtain by using (3.28) and (3.27) that

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^4dx + 4\int_{R^3}|u|^2\left(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2 + 2\mu|\nabla|u||^2\right)dx \\
&= -4(\lambda+\mu)\int u\cdot\nabla|u|^2\mathrm{div}u dx + 4\int \mathrm{div}(|u|^2u)Pdx \\
&\le C\int |u|^2|\nabla u||\mathrm{div}u|dx + \varepsilon\int |u|^2|\nabla u|^2dx + C(\varepsilon)\int \rho|u|^2dx \\
&\le C\|\mathrm{div}u\|_{L^\infty}\left(\int \rho|u|^4dx + \|\nabla u\|_{L^2}^2\right) + \varepsilon\int |u|^2|\nabla u|^2dx + C(\varepsilon). \qquad (4.5)
\end{aligned}$$

Combining (3.11) with (4.5), we conclude by choosing a positive ε_0 suitably small that

$$\begin{aligned}
&\frac{d}{dt}\int \left(\frac{\varepsilon_0\mu}{2}|\nabla u|^2 + \frac{\varepsilon_0(\lambda+\mu)}{2}(\mathrm{div}u)^2 - \varepsilon_0 P\mathrm{div}u + \rho|u|^4\right)dx \\
&\quad + \int\left(\frac{\varepsilon_0}{2}\rho u_t^2 + \mu|u|^2|\nabla u|^2\right)dx \\
&\le C\|\nabla u\|_{L^2}^2 + C\|\mathrm{div}u\|_{L^\infty}\left(\int \rho|u|^4dx + \|\nabla u\|_{L^2}^2\right) + C,
\end{aligned}$$

which, together with Gronwall's inequality gives (4.4). We finish the proof of Lemma 4.2.

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