Analyticity of the Semigroup Associated with the Fluid-Rigid Body Problem and Local Existence of Strong Solutions*

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Abstract: In this paper, we consider the linear operator associated with fluid-rigid body problem. The operator was first introduced by T.Takahashi and M. Tucsnak [23]. For a general 3-dimensional solid body, we prove that the corresponding semigroup is analytic in $L^2(\mathbb{R}^3)$ and $L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) (p \geq 2)$. In particular, when the solid is a ball of $\mathbb{R}^3$, the corresponding semigroup is analytic in $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) (p \geq 6)$. This yields a local in time existence and uniqueness of strong solutions in a non-Hilbert space.

Keywords: fluid-rigid body system, semigroup, Navier-Stokes equations, exterior domain

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1 Introduction

Many physical phenomena involve interactions between moving structures and fluids. An interesting problem arising in fluid mechanics is the motion of a rigid body immersed in a viscous incompressible fluid. The motion of the fluid is governed by the classical Navier-Stokes equations with the non-slip boundary condition. The motion of the rigid body consisting of a translation part and a rotation part, is ruled by the conservation of linear and angular momentum.

Let the region occupied by the homogeneous rigid body at time t be denoted by $O(t)$, and the domain occupied by the homogeneous fluid be $\Omega(t) = \mathbb{R}^3 \setminus \overline{O(t)}$. Let $O(0) = O$, and $\Omega(0) = \Omega$. For the sake of simplicity, we assume that both the fluid and the solid are homogeneous with density 1. Then the system modeling the motion of the fluid and the rigid body is the following,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad (x,t) \in \Omega(t) \times (0,T), \\ \text{div } u = 0, \quad (x,t) \in \Omega(t) \times [0,T), \\ u(x,t) = h'(t) + \omega(t) \times (x - h(t)), \quad (x,t) \in \partial\Omega(t) \times [0,T), \\ mh''(t) = -\int_{\partial\Omega(t)} (\sigma(u,p)\vec{n}) d\Gamma, \quad t \in [0,T), \\ J\omega'(t) = -\int_{\partial\Omega(t)} (x - h(t)) \times \sigma(u,p)\vec{n} d\Gamma, \quad t \in [0,T), \\ u(x,0) = a(x), \quad x \in \Omega, \\ h(0) = 0 \in \mathbb{R}^3, \quad h'(0) = b \in \mathbb{R}^3, \quad \omega(0) = c \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Where $u = (u_1, u_2, u_3)$ and p denote the velocity field and the pressure of the fluid respectively; $\vec{n}(t)$ is the unit outward normal vector to $\partial\Omega(t)$; $h(t)$ and $\omega(t)$ denote the position of the center and the angular velocity of the solid at the time t respectively; m is the mass of the rigid body, i.e.,

$$m = \int_{O(t)} dx = \int_O dx,$$

$J = (J_{kl})$ is the moment of inertia related to the mass center of the rigid body,

$$J_{kl} = \int_{O(t)} \left[|x - h(t)|^2 \delta_{kl} - (x - h(t))_k (x - h(t))_l \right] dx = \int_O |x|^2 \delta_{kl} - x_k x_l dx,$$

where δ_{kl} is the Kronecker symbol, and $\sigma(u,p)$ is the Cauchy stress tensor field,

$$\sigma(u,p) = -pId + 2\nu D(u),$$

where Id is the identity matrix and $D(u)$ is the deformation tensor

$$D(u) = \frac{1}{2} \left[\nabla u + (\nabla u)^T \right].$$

There have been extensive researches on the problem (1.1) in recent years. The global existence of weak solutions to (1.1) has already been proved by [12] and [22]. When the fluid-rigid body system occupies a bounded domain, the existence of weak solutions has been studied by many mathematicians, see [2, 3, 6, 7, 8, 17]. Furthermore, the collision between the solid and the domain's boundary has been investigated, see [9, 10] and references therein.

However, only a few results are available on the existence and uniqueness of strong solutions. For the case that the rigid body is a disk in $\mathbb{R}^2$, T.Takahashi and M.Tucsnak [23] showed the existence and uniqueness of global strong solutions. Later, P.C.Santiago and T.Takahashi [21] extended the global existence result to general rigid body in $\mathbb{R}^2$. For 3-dimensional case, they proved the local existence and uniqueness of strong solutions in $C[0,T;W^{1,2}(\mathbb{R}^3))$ for general smooth rigid bodies, see also [5] for the local existence of strong solutions. The research methods in [21, 23] are totally different from that of the weak solution.

Since the domain occupied by the fluid is varying with time and not a priori known, it's a free boundary problem. This can be transformed into an equivalent fixed boundary problem by moving along the center of the solid body. For example, consider the case that O is a ball in $\mathbb{R}^3$. In this case, let

$$y = x - h(t), \qquad v(y,t) = u(y + h(t), t),$$

$$q(y,t) = p(y + h(t), t), \qquad l(t) = h'(t),$$

$$\sigma(v,q) = -q(y,t)Id + 2\nu D(v)(y,t).$$

Then the problem (1.1) becomes

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v - (l\cdot\nabla)v + \nabla q = 0, \quad (y,t)\in\Omega\times(0,T), \\ \text{div } v = 0, \qquad (y,t)\in\Omega\times[0,T), \\ v(y,t) = l(t) + \omega(t)\times y, \qquad (y,t)\in\partial\Omega\times[0,T), \\ ml'(t) = -\int_{\partial\Omega}\sigma(v,q)\vec{n}d\Gamma, \quad t\in[0,T), \\ J\omega'(t) = -\int_{\partial\Omega} y\times[\sigma(v,q)\vec{n}]d\Gamma, \quad t\in[0,T), \\ v(y,0) = a(y), \qquad y\in\Omega, \\ l(0) = b\in\mathbb{R}^3, \quad \omega(0) = c\in\mathbb{R}^3. \end{cases} \tag{1.2}$$

Remark 1.1 If O is not a ball in $\mathbb{R}^3$, but a general solid, similar linear transformations can be found to fix the domain. For references, see [19] or [21].

To study the new system (1.2), the authors of [23] and [21] applied the method of semigroups. They extended v to a function defined on the whole space by letting $v(y,t) = l(t) + \omega(t) \times y$ in O and defined a new linear operator A_2 as follows,

$$D(A_2) = \{v \in W^{1,2}(\mathbb{R}^3) : \text{div } v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O,\ v|_\Omega \in W^{2,2}(\Omega)\},$$

$$\mathcal{A}_2 v = \begin{cases} -\nu \Delta v, & \text{in } \Omega \\ \frac{2\nu}{m} \int_{\partial O} D(v)\vec{n} d\Gamma + 2\nu J^{-1} \left[\int_{\partial O} y \times D(v)\vec{n} d\Gamma \right] \times y, & \text{in } O \end{cases}$$

and

$$A_2 v = \mathbb{P} \mathcal{A}_2 v,$$

where $\mathbb{P}$ is the orthogonal projector from $L^2(\mathbb{R}^3)$ onto its subspace H_1^2, where

$$H_1^2 = \{v \in L^2(\mathbb{R}^3) : \text{div } v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O\}.$$

Omitting the nonlinear terms $(v \cdot \nabla)v$ and $(l \cdot \nabla)v$ in the first equation of (1.2), one can get the corresponding linearized system. Then A_2 is the linear operator associated with this linearized system, since [23] has proved that the linearized system equals to the following abstract equation in some sense,

$$\begin{cases} \partial_t v + A_2 v = 0, \\ v(y,0) = \begin{cases} a(y), & y \in \Omega \\ b + c \times y, & y \in O \end{cases} \end{cases}$$

In [21], it was proved that $-A_2$ is the generator of a contraction strongly continuous semigroup on H_1^2. In our paper, we will prove that $-A_2$ is the generator of an analytic semigroup $\{e^{-tA_2}\}$ in H_1^2. Moreover, the corresponding operator in $H_1^{\frac{6}{5}} \cap H_1^p (p \geq 2)$ is also the generator of an analytic semigroup. When the solid is a ball in $\mathbb{R}^3$, we can even prove its analyticity in the space $H_1^2 \cap H_1^p (p \geq 6)$.

As an application, we apply the Fujita-Kato approach to get the local existence and uniqueness of strong solutions in $H_1^{\frac{6}{5}} \cap H_1^p (p > 3)$ space when the solid is a ball in $\mathbb{R}^3$. Similar results hold in $H_1^2 \cap H_1^p (p \geq 6)$. Note that the local strong solution derived in [5] and [21] required the initial data at least belongs to $W^{1,2}(\mathbb{R}^3)$, hence we extend the class of initial data. Their proof relies strongly on the properties of Hilbert spaces, while our proof applies to more general setting. When the rigid body is a general solid in $\mathbb{R}^3$, the estimates about the semigroup in section 6, combined with the linear transformation in [21], help to establishing the local existence of local strong solutions. Furthermore, we believe that the properties of the linear operator derived here is useful for exploring more information about the original problem.

2 Main results and Preliminaries

Before stating the main results in this paper, we introduce some function spaces and notations. Let O be a bounded, simply connected domain of C^2 in $\mathbb{R}^3$, and Ω be its exterior domain, $\Omega = \mathbb{R}^3 \setminus \overline{O}$. In this paper, without loss of generality, the center of O is supposed to be the origin, i.e.,

$$\int_O y dy = 0 \in \mathbb{R}^3,$$

otherwise, one can translate coordinates system to achieve this. $\vec{n}$ denotes the outer unit normal of the boundary $\partial\Omega$. Let $m = \int_O dy$, and $J = (J_{kl})$,

$$J_{kl} = \int_O \left[|y|^2 \delta_{kl} - y_k y_l\right] dy.$$

$B_R(0)$ is the ball in $\mathbb{R}^3$ centered at 0 and with the radius R. For any linear operator A, denote the domain of A by $D(A)$ and the range of A by $R(A)$. Denote the complex conjugate of a function f by $\overline{f}$. In the case of non-confusion, we do not distinguish the notation of vector-valued function space and scalar function space. $L^p(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev spaces defined in the domain Ω. While $L^p(\mathbb{R}^3)$ and $W^{k,p}(\mathbb{R}^3)$ are the usual Sobolev spaces defined on $\mathbb{R}^3$. $C^\infty_{0,\sigma}(\Omega)$ consists of smooth functions defined on Ω with compact support and divergence free.

Let

$$H_1^p = \{u \in L^p(\mathbb{R}^3) : \ \text{div}\ u = 0 \text{ in } \mathbb{R}^3, \quad D(u) = 0 \text{ in } O\},$$

$$G_1^p = \{u \in L^p(\mathbb{R}^3) : \ u = \nabla q_1, \ q_1 \in L^1_{loc}(\mathbb{R}^3)\},$$

$$G_2^p = \left\{ u \in L^p(\mathbb{R}^3) \left| \begin{array}{l} \text{div}\ u = 0 \text{ in } \mathbb{R}^3, \quad u = \nabla q_2 \text{ in } \Omega, \ q_2 \in L^1_{loc}(\Omega) \\ u = \phi \ \text{ in } O, \text{ and } \displaystyle\int_O \phi \times y dy = -\int_{\partial\Omega} q_2 \vec{n} \times y d\Gamma \end{array} \right. \right\}.$$

We have the following characterization of functions in H_1^p.

Lemma 2.1 Let $1 \le p \le \infty$, and $u \in H_1^p$. Then

$$u(y) = l_u + \omega_u \times y, \quad \text{in } O,$$

where

$$l_u = \frac{1}{m}\int_O u dy \quad \text{and} \quad \omega_u = -J^{-1}\int_O u \times y dy.$$

Proof Since div $u = 0$ and $D(u) = 0$ in O, u must be a rigid body motion. It means that there exist some vectors l_u and ω_u, such that

$$u(y) = l_u + \omega_u \times y \quad \text{in } O. \tag{2.3}$$

Integrating both sides of (2.3) over the domain O to get

$$\int_O u(y)dy = \int_O l_u dy + \int_O \omega_u \times y dy.$$

As $\int_O y dy = 0$,

$$l_u = \frac{1}{m}\int_O u dy.$$

On the other hand, multiplying both sides of (2.3) by y and integrating over O yield

$$\begin{aligned} \textstyle\int_O u(y) \times y dy &= \textstyle\int_O (\omega_u \times y) \times y dy \\ &= \textstyle\int_O y(\omega_u \cdot y) - \omega_u |y|^2 dy \\ &= -J\omega_u, \end{aligned}$$

hence,

$$\omega_u = -J^{-1}\int_O u \times y dy.$$

Recall a theorem about the decomposition of $L^p(\mathbb{R}^3)$, which was proved in [24],

Lemma 2.2 For $1 < p < \infty$,

$$L^p(\mathbb{R}^3) = H_1^p \oplus G_1^p \oplus G_2^p.$$

Thus, for any $u \in L^p(\mathbb{R}^3)$, one has

$$u = v + \nabla q_1 + w \in H_1^p \oplus G_1^p \oplus G_2^p.$$

Set $v = \mathbb{P}_p u$, where $\mathbb{P}_p$ is the projection operator from $L^p(\mathbb{R}^3)$ onto H_1^p.

As indicated in the proof of Lemma 2.2, for every $u \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p \neq q$, $\mathbb{P}_p u = \mathbb{P}_q u$. Hence, we will omit the subindex of $\mathbb{P}_p$, and just write $\mathbb{P}$ instead in this paper.

Set

$$D(A_{\frac{6}{5}\cap p}) = \left\{ v \in W^{1,\frac{6}{5}}(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3) \;\middle|\; \begin{array}{l} \operatorname{div} v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O, \\ v|_\Omega \in W^{2,\frac{6}{5}}(\Omega) \cap W^{2,p}(\Omega) \end{array} \right\}. \tag{2.4}$$

For any $v \in D(A_{\frac{6}{5}\cap p})$, define

$$\mathcal{A}_{\frac{6}{5}\cap p} v = \begin{cases} -\nu\Delta v, & \text{in } \Omega \\ \dfrac{2\nu}{m}\displaystyle\int_{\partial O} D(v)\vec{n} d\Gamma + 2\nu J^{-1}\left[\int_{\partial O} y \times D(v)\vec{n} d\Gamma\right] \times y, & \text{in } O \end{cases} \tag{2.5}$$

and

$$A_{\frac{6}{5}\cap p} v = \mathbb{P}\mathcal{A}_{\frac{6}{5}\cap p} v. \tag{2.6}$$

Similarly, one can define the space $D(A_{2\cap p})$, the linear operator $\mathcal{A}_{2\cap p}$ and operator $A_{2\cap p}$, by replacing $\frac{6}{5}$ with 2 in (2.4), (2.5) and (2.6) respectively.

Now our main results read as

Theorem 2.1 The linear operator $-A_2$(defined in section 1) is the infinitesimal generator of an analytic semigroup $\{e^{-tA_2}\}$ of operators on the space H_1^2.

For non-Hilbert spaces, we have the following generalizations.

Theorem 2.2 For any $2 \le p < \infty$, the linear operator $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$. And for every $t > 0$, it holds that

$$\|e^{-tA_{\frac{6}{5}\cap p}}\| \le M_1, \qquad \left\|A^k_{\frac{6}{5}\cap p} e^{-tA_{\frac{6}{5}\cap p}}\right\| \le \frac{M_1}{|t|^k} \tag{2.7}$$

with $M_1 = M_1(p,\Omega) > 0$. Then it follows that for all $u \in H_1^{\frac{6}{5}} \cap H_1^p$,

$$\lim_{t\to+\infty} \left\|e^{-tA_{\frac{6}{5}\cap p}} u\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} = 0.$$

The corresponding result for the classical Stokes operator $\tilde{A}_p$ in the domain Ω was proved in [1], which reads as

Proposition 2.1 Let $1 < p < \infty$, $0 < \theta < \frac{\pi}{2}$. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| > 0$, $|\arg\lambda| \le \frac{\pi}{2} + \theta$, the resolvent $(\lambda I + \tilde{A}_p)^{-1}$ of the operator $\tilde{A}_p$ exists and it holds

$$\|(\lambda I + \tilde{A}_p)^{-1}\| \le \frac{C}{|\lambda|} \quad \text{for all} \quad |\lambda| > 0, \quad |\arg\lambda| \le \frac{\pi}{2} + \theta,$$

where $C = C(p,\theta,\Omega) > 0$. And it follows that the semigroup $\{e^{-t\tilde{A}_p}\}$ is analytic for $t \in \mathbb{C}$, $t \neq 0$, and $|\arg t| < \theta$.

More concretely, taking into accout the result in [15] on the stokes operator on exterior domains, we can restate Proposition 2.1 as follows:

Proposition 2.1′ Let $1 < p < \infty$, $0 < \theta < \frac{\pi}{2}$. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| > 0$, $|\arg\lambda| \le \frac{\pi}{2} + \theta$, and every $f \in L^p(\Omega)$, the system

$$\begin{cases} \lambda u - \nu\Delta u + \nabla p = f, \quad \text{in } \Omega \\ \text{div } u = 0, \quad \text{in } \Omega \\ u(y) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ with the following estimates,

$$|\lambda| \cdot \|u\|_{L^p(\Omega)} \le C(p,\Omega)\|f\|_{L^p(\Omega)}, \tag{2.8}$$

$$\|D^2u\|_{L^p(\Omega)} + \|\nabla p\|_{L^p(\Omega)} \le C(p,\Omega)\left[\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right], \tag{2.9}$$

where $C(p,\Omega)$ is some constant depending only on p and Ω.

Remark 2.1 Comparing Theorem 2.2 with Proposition 2.1, we would like to prove that $-A_p$ is analytic on H_1^p. Yet, we can not achieve this at this moment. However, when O is a ball in $\mathbb{R}^3$, we have some refined results.

Theorem 2.3 Suppose O is a ball of radius 1 in $\mathbb{R}^3$. For any $6 \le p < \infty$, the linear operator $-A_{2\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{2\cap p}}\}$ of operators on $H_1^2 \cap H_1^p$. And for every $t > 0$, it holds that

$$\|e^{-tA_{2\cap p}}\| \le M, \qquad \|A_{2\cap p}^k e^{-tA_{2\cap p}}\| \le \frac{M}{|t|^k},$$

with $M = M(p,\Omega) > 0$. Then it follows that for all $u \in H_1^2 \cap H_1^p$,

$$\lim_{t\to+\infty} \|e^{-tA_{2\cap p}}u\|_{L^2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} = 0.$$

Remark 2.2 Theorem 2.2 and Theorem 2.3 are the key estimates for establishing the local strong solution in $H_1^{\frac{6}{5}}) \cap H_1^p$ and $H_1^2 \cap H_1^p$ respectively. The assumption that the initial data of system (1.2) belongs to H_1^2 is necessary in some sense, otherwise we may not get the uniform bound of the velocity of the solid. Hence Theorem 2.3 seems better and more reasonable. However, whether the conclusion of Theorem 2.3 holds for $2 < p < 6$ is open.

Remark 2.3 Although there are some differences between 3-dimensional case and 2-dimensional case, our proof of Theorem 2.1 and Theorem 2.3 also applies to the corresponding cases in 2-dimensional space.

As an application of the above properties, we will give the local well-posed results in the particular case that the solid is the unit ball in $\mathbb{R}^3$ for example. Indeed, we have

Theorem 2.4 Assume that O is a unit ball in $\mathbb{R}^3$ and $p > 3$. Let the initial data

$$v_0(y) = \begin{cases} a(y), & y \in \Omega, \\ b + c \times y, & y \in O. \end{cases}$$

Suppose $v_0 \in H_1^{\frac{6}{5}} \cap H_1^p$, then there exists a unique local strong solution $v \in C([0,T_0]; H_1^{\frac{6}{5}} \cap H_1^p)$ to the system (1.2), and v satisfies

$$t^{\frac{1}{2}} v(y,t) \in C([0,T_0]; W^{1,\frac{6}{5}}(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3)).$$

3 Proof of Theorem 2.1

To prove Theorem 2.1, we need some basic lemmas in functional analysis.

Lemma 3.1 Suppose X is a Banach space and X^* is its dual space. Assume that A is a closed linear operator with dense domain $D(A)$ in X. Let $S(A)$ be the numerical range of A, that is

$$S(A) = \{\langle x^*, Ax\rangle : \ x \in D(A), \ \|x\| = 1, \ x^* \in X^*, \ \|x^*\| = 1, \ \langle x^*, x\rangle = 1\},$$

and let Σ be the complement of $\overline{S(A)}$ in $\mathbb{C}$. If Σ_0 is a component of Σ satisfying $\rho(A)\cap\Sigma_0 \neq \varnothing$, then the spectrum of A is contained in the complement S_0 of Σ_0, and $\forall \ \lambda \in \Sigma_0$,

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{d(\lambda, \overline{S(A)})}.$$

Here $\rho(A)$ is the resolvent set of A, and $d(\lambda, \overline{S(A)})$ is the distance of λ from $\overline{S(A)}$.

Lemma 3.2 Suppose X is a Banach space. Let A be a linear dissipative operator with $R(I - A) = X$. If X is a reflexive, then $\overline{D(A)} = X$.

The above two lemmas can be found in [20].

Next, some basic properties of the operator A_2 are studied.

Proposition 3.1 The linear operator A_2 is closed and $\overline{D(A_2)} = H_1^2$, $R(I+A_2) = H_1^2$.

Proof This is proved by Lemma 3.2. We show first that $-A_2$ is dissipative. For any $u \in D(A_2)$, one can suppose $u = V_u + \omega_u \times y$ in O. Then

$$\begin{aligned}
\langle A_2 u, u\rangle &= \int_\Omega (-\nu\Delta u)\cdot \overline{u}dy + \int_O \frac{2\nu}{m}\int_{\partial O} D(u)\vec{n}d\Gamma \cdot \overline{u}dy \\
&\quad + \int_O \left[\left(2\nu J^{-1}\int_{\partial O} y\times D(u)\vec{n}d\Gamma\right)\times y\right]\cdot \overline{u}dy \\
&= \int_\Omega -2\nu \mathrm{div}\ (D(u))\cdot \overline{u}dy + \frac{2\nu}{m}\int_{\partial O} D(u)\vec{n}d\Gamma\cdot (m\overline{V_u}) \\
&\quad + \left(2\nu J^{-1}\int_{\partial O} y\times D(u)\vec{n}d\Gamma\right)\cdot \int_O y\times(\overline{\omega_u}\times y)dy \\
&= 2\nu\int_\Omega |D(u)|^2 dy - 2\nu\int_{\partial O} D(u)\overline{u}\cdot\vec{n}d\Gamma + 2\nu\int_{\partial O} D(u)\vec{n}d\Gamma\cdot\overline{V_u} \\
&\quad +2\nu\int_{\partial O} y\times D(u)\vec{n}d\Gamma\cdot J^{-1}\int_O y\times(\overline{\omega_u}\times y)dy
\end{aligned}$$

where the second equality follows from $\int_O ydy = 0$, and the third equality is due to the fact that J^{-1} is a symmetric matrix.

Since
$$\int_O y\times(\overline{\omega_u}\times y)dy=\int_O \overline{\omega_u}|y|^2-y(y\cdot\overline{\omega_u})dy=J\overline{\omega_u},$$
and $D(u)$ is symmetric, then

$$\begin{aligned}\langle A_2u,u\rangle &= 2\nu\int_\Omega |D(u)|^2dy-2\nu\int_{\partial O} D(u)\overline{u}\cdot\vec{n}d\Gamma\\ &\quad +2\nu\int_{\partial O} D(u)\vec{n}d\Gamma\cdot\overline{V_u}+2\nu\int_{\partial O} y\times D(u)\vec{n}d\Gamma\cdot\overline{\omega_u}\\ &= 2\nu\int_\Omega |D(u)|^2dy-2\nu\int_{\partial O} D(u)\overline{u}\cdot\vec{n}d\Gamma\\ &\quad +2\nu\int_{\partial O} D(u)\vec{n}d\Gamma\cdot\overline{V_u}+2\nu\int_{\partial O} D(u)\vec{n}\cdot(\overline{\omega_u}\times y)d\Gamma\\ &= 2\nu\int_{\mathbb{R}^3}|D(u)|^2dy\geq 0.\end{aligned}$$

Hence $-A_2$ is dissipative.

Next, we show that $R(I+A_2)=H^2_1$. For any $f\in H^2_1$, it suffices to show that there exists some function $u\in D(A_2)$ such that

$$(I+A_2)u=f, \tag{3.10}$$

which means
$$(I+A_2)Reu=Ref,\quad (I+A_2)Imu=Imf. \tag{3.11}$$

Here Reu and Imu are the real and imaginary part of u respectively.

Let
$$V_2=\{v\in W^{1,2}(\mathbb{R}^3):\ \text{div } v=0 \text{ in } \mathbb{R}^3,\quad D(v)=0 \text{ in } O\},$$

and
$$ReV_2=\{v\in V_2:\forall\ y\in\mathbb{R}^3,\ v(y)\in\mathbb{R}^3\}.$$

Define the bilinear functional $a:\ ReV_2\times ReV_2\to\mathbb{R}$,

$$a(u,\varphi)=\int_{\mathbb{R}^3} u\cdot\varphi dy+2\nu\int_\Omega D(u):D(\varphi)dy.$$

Obviously, a is a bounded bilinear functional on $ReV_2\times ReV_2$. And for any $u\in ReV_2$, div $u=0$ in $\mathbb{R}^3$, $D(u)=0$ in O, one has

$$\begin{aligned}a(u,u) &= \|u\|^2_{L^2(\mathbb{R}^3)}+2\nu\|D(u)\|^2_{L^2(\mathbb{R}^3)}\\ &= \|u\|^2_{L^2(\mathbb{R}^3)}+\nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)}\\ &\geq \min\{\nu,1\}\|u\|^2_{W^{1,2}(\mathbb{R}^3)},\end{aligned}$$

which implies a is coercive.

On the other hand, given any $f \in L^2(\mathbb{R}^3)$, the mapping

$$\varphi \longmapsto \int_\Omega Ref \cdot \varphi dy$$

is linear and bounded on ReV_2.

Therefore, by the Lax-Milgram Theorem, there exists a unique $v_1 \in ReV_2$ which satisfies

$$a(v_1, \varphi) = \int_\Omega Ref \cdot \varphi dy, \tag{3.12}$$

for every $\varphi \in ReV_2$.

Since (3.12) holds for any $\varphi \in C^\infty_{0,\sigma}(\Omega)$, there exists $p_1 \in \mathcal{D}'(\Omega)$, such that

$$v_1 - \nu\Delta v_1 + \nabla p_1 = Ref, \quad \text{in } \mathcal{D}'(\Omega)$$

As in the proof of Proposition 4.2 in [23], we deduce that $v_1 \in W^{2,2}(\Omega) \cap V_2$ and

$$\|v_1\|_{W^{2,2}(\Omega)} \le C\|Ref\|_{L^2(\mathbb{R}^3)}. \tag{3.13}$$

Take v_1 itself as a test function in (3.12),

$$\|v_1\|_{W^{1,2}(\mathbb{R}^3)} \le C\|Ref\|_{L^2(\mathbb{R}^3)}. \tag{3.14}$$

In the same way, we can prove that there exists some function $v_2 \in D(A_2)$, such that

$$v_2 + A_2 v_2 = Imf. \tag{3.15}$$

Let $u = v_1 + iv_2$. Then u satisfies that

$$u + A_2 u = f,$$

with the following estimates,

$$\|v_2\|_{W^{2,2}(\Omega)} \le C\|Imf\|_{L^2(\mathbb{R}^3)}, \quad \|v_2\|_{W^{1,2}(\mathbb{R}^3)} \le C\|Imf\|_{L^2(\mathbb{R}^3)}. \tag{3.16}$$

It follows from to Lemma 3.2 that $D(A_2)$ is dense in H^2_1, i.e., $\overline{D(A_2)} = H^2_1$.

For an arbitrary sequence $\{u_n\}$ in $D(A_2)$ satisfying that

$$u_n \longrightarrow u \ \text{ in } H^2_1, \quad A_2 u_n \longrightarrow f \ \text{ in } H^2_1,$$

we'll prove $u \in D(A_2)$ and $A_2 u = f$. Hence A_2 is a closed operator.

Let $g_n = u_n + A_2 u_n$, then $g_n \longrightarrow g = u + f$ in H^2_1. As indicated by the proof above, for $g = u + f$, there exists a unique $\tilde{u}$ in $D(A_2)$ such that

$$\tilde{u} + A_2 \tilde{u} = g = u + f.$$

Since $\{g_n\}$ converges to g in H_1^2, then by the estimates (3.13)(3.14)(3.16), $\{u_n\}$ converges to $\tilde{u}$ in $W^{1,2}(\mathbb{R}^3)$ and $W^{2,2}(\Omega)$. Therefore $\tilde{u} = u$, and $A_2 u = f$.

Proof of Theorem 2.1 Theorem 2.1 will be proved by using Lemma 3.1. It suffices to verify all the assumptions posed in Lemma 3.1 for A_2. For each $u \in D(A_2)$,

$$\langle u, A_2 u\rangle = \nu \int_{\mathbb{R}^3} |\nabla u|^2 dy \geq 0,$$

hence $S(A_2) \subseteq \mathbb{R}^+$.

Choose some θ_0 such that $0 < \theta_0 < \pi/2$. Let $\Sigma_0 = \{\lambda \in \mathbb{C} :\ |arg\lambda| \geq \theta_0,\ |\lambda| \neq 0\}$. Following almost the same proof of $R(I + A_2) = H_1^2$, one can easily get that

$$\{\lambda \in \mathbb{R} :\ \lambda < 0\} \subseteq \rho(A_2).$$

Hence $\rho(A_2) \cap \Sigma_0 \neq \varnothing$.

According to Lemma 3.1, $\Sigma_0 \subseteq \rho(A_2)$, and for any $\lambda \in \Sigma_0$,

$$\|(\lambda I - A_2)^{-1}\| \leq \frac{1}{d(\lambda, \overline{S(A_2)})}.$$

When $Re\lambda \leq 0$,

$$d(\lambda, \overline{S(A_2)}) \geq |\lambda|.$$

When $Re\lambda > 0$,

$$d(\lambda, \overline{S(A_2)}) \geq |Im\lambda|.$$

And since $\lambda \in \Sigma_0$, $|arg\lambda| \geq \theta_0$, if $Re\lambda > 0$, then

$$\frac{|Re\lambda|}{|Im\lambda|} \leq \cot\theta_0,$$

$$d(\lambda, \overline{S(A_2)}) \geq \frac{1}{\sqrt{1 + (\cot\theta_0)^2}}|\lambda|.$$

Therefore, for any $\lambda \in \Sigma_0$,

$$\|(\lambda I - A_2)^{-1} u\|_{L^2(\mathbb{R}^3)} \leq C(\Sigma_0)|\lambda|^{-1}\|u\|_{L^2(\mathbb{R}^3)},$$

which implies the conclusion of Theorem 2.1.

4 Proof of Theorem 2.2

Let θ_0 and Σ_0 be the same as those in the proof of Theorem 2.1. In order to show that $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$, it suffices to show that $\Sigma_0 \subseteq \rho(A_{\frac{6}{5}\cap p})$, and for any $\lambda \in \Sigma_0$ and $f \in H_1^{\frac{6}{5}} \cap H_1^p$, there exists some constant $C = C(\Sigma_0, p, \Omega)$ such that

$$\|(\lambda I - A_{\frac{6}{5}\cap p})^{-1} f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C|\lambda|^{-1}\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.$$

It follows from the proof of Theorem 2.1 that $\Sigma_0 \subseteq \rho(A_2)$. Thus, for every $\lambda \in \Sigma_0$ and every $f \in H_1^2$, there exists a function $u \in D(A_2)$, such that

$$(\lambda I - A_2)u = f. \tag{4.17}$$

Suppose $f \in H_1^{\frac{6}{5}} \cap H_1^p$, we will prove that the solution $u \in D(A_{\frac{6}{5}\cap p})$, and

$$\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|u\|_{L^p(\mathbb{R}^3)} \leq C(\Sigma_0, p, \Omega)|\lambda|^{-1}[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}],$$

with some constant $C(\Sigma_0, p, \Omega)$ independent of f. Note that in the proof of Theorem 2.1, it was shown that

$$\|u\|_{L^2(\mathbb{R}^3)} \leq C(\Sigma_0)|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}.$$

Suppose that $u = V_u + \omega_u \times y$ in O. Then by Lemma 2.1, one has

$$|V_u| = \frac{1}{m}\left|\int_O u dy\right| \leq C\|u\|_{L^2(O)} \leq C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{4.18}$$

and

$$|\omega_u| = \left|J^{-1}\int_O u \times y dy\right| \leq C\|u\|_{L^2(O)} \leq C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}. \tag{4.19}$$

On the other hand, it follows from (4.17) that

$$Re\lambda\|u\|_{L^2(\mathbb{R}^3)}^2 - \nu\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = Re\langle f, u\rangle,$$

$$Im\lambda\|u\|_{L^2(\mathbb{R}^3)}^2 = Im\langle f, u\rangle.$$

If $Re\lambda \leq 0$, then by the Hölder's inequality and the Sobolev imbedding inequality,

$$\begin{aligned}\nu\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 &\leq -Re\langle f, u\rangle \\ &\leq \|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|u\|_{L^6(\mathbb{R}^3)} \\ &\leq C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|\nabla u\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

While for $Re\lambda > 0$, then $Re\lambda \le \cot\theta_0 \cdot |Im\lambda|$, and

$$\begin{aligned}
\nu\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 &= Re\lambda\|u\|_{L^2(\mathbb{R}^3)}^2 - Re\langle f, u\rangle \\
&\le C|Im\lambda|\|u\|_{L^2(\mathbb{R}^3)}^2 - Re\langle f, u\rangle \\
&= C|Im\langle f, u\rangle| - Re\langle f, u\rangle \\
&\le C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|u\|_{L^6(\mathbb{R}^3)} \\
&\le C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|\nabla u\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Hence, in both cases we have

$$\|u\|_{L^6(\mathbb{R}^3)} \le C\|\nabla u\|_{L^2(\mathbb{R}^3)} \le C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)},$$

and

$$|V_u| = \frac{1}{m}\left|\int_O u dy\right| \le C\|u\|_{L^6(O)} \le C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \tag{4.20}$$

$$|\omega_u| = \left|J^{-1}\int_O u \times y dy\right| \le C\|u\|_{L^6(O)} \le C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{4.21}$$

Based on the relationship between u and f in (4.17), it was shown in [23] that there exists some $p \in \mathcal{D}'$ such that (u, p) satisfies the Stokes type system:

$$\begin{cases}
\lambda u + \nu\Delta u + \nabla p = f, \quad \text{in } \Omega, \\
\text{div } u = 0, \quad \text{in } \Omega, \\
u(y) = V_u + \omega_u \times y, \quad \text{on } \partial\Omega.
\end{cases}$$

Take $\psi \in C_0^\infty(\mathbb{R})$ with

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| \ge 2. \end{cases}$$

Set $\psi_R(y) = \psi(|y|/R)$, with R large enough such that $O \subset B_{R/2}(0)$. Let

$$v = \text{curl } \left[\frac{1}{2}V_u \times y\psi_R(y)\right] - \text{curl } \left[\frac{1}{2}\omega_u|y|^2\psi_R(y)\right],$$

and $w = u - v$. It's easy to verify that

$$\begin{cases}
\lambda w + \nu\Delta w + \nabla p = f - \lambda v - \nu\Delta v, \quad \text{in } \Omega, \\
\text{div } w = 0, \quad \text{in } \Omega, \\
w(y) = 0, \quad \text{on } \partial\Omega.
\end{cases} \tag{4.22}$$

According to Proposition 2.1′, one has

$$|\lambda|\|w\|_{L^{\frac{6}{5}}(\Omega)} \leq C(\Omega)\left[\|f\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda|\cdot(|V_u|+|\omega_u|)+|V_u|+|\omega_u|\right],$$

and

$$|\lambda|\|w\|_{L^p(\Omega)} \leq C(p,\Omega)\left[\|f\|_{L^p(\Omega)} + |\lambda|\cdot(|V_u|+|\omega_u|)+|V_u|+|\omega_u|\right].$$

Collecting all the estimates (4.18), (4.19), (4.20) and (4.21), leads to

$$\begin{aligned}|\lambda|\|w\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda|\|w\|_{L^p(\Omega)} &\leq C\left[\|f\|_{L^{\frac{6}{5}}(\Omega)} + \|f\|_{L^p(\Omega)} + \|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}\right]\\ &\leq C(\Sigma_0,p,\Omega)\left[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\right].\end{aligned}$$

Consequently,

$$\begin{aligned}|\lambda|\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + |\lambda|\|u\|_{L^p(\mathbb{R}^3)} &\leq |\lambda||V_u| + |\lambda||\omega_u| + |\lambda|\|w\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda|\|w\|_{L^p(\Omega)}\\ &\leq C(\Sigma_0,p,\Omega)\left[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\right].\end{aligned} \tag{4.23}$$

It follows that $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$, which completes the proof of Theorem 2.2.

5 Proof of Theorem 2.3

In this section, O is a unit ball in $\mathbb{R}^3$. The main difference between the proof of Theorem 2.3 and that of Theorem 2.2 is the choice of the cut-off function v.

Proof As before, let θ_0, Σ_0 be the same as in the proof of Theorem 2.1. For any $\lambda \in \Sigma_0$ and any $f \in H_1^2$, since $\Sigma_0 \subseteq \rho(A_2)$, there exists some function $u \in D(A_2)$, such that

$$(\lambda I - A_2)u = f, \tag{5.24}$$

and

$$\|u\|_{L^2(\mathbb{R}^3)} \leq C(\Sigma_0,\Omega)|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}.$$

Suppose that $f \in H_1^2 \cap H_1^p$, we will prove that $u \in D(A_{2\cap p})$, and

$$\|u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^p(\mathbb{R}^3)} \leq C(\Sigma_0,p,\Omega)|\lambda|^{-1}[\|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}]. \tag{5.25}$$

First, as in the proof of Theorem 2.1, one has

$$\|u\|_{L^2(\mathbb{R}^3)} \leq C(\Sigma_0)|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.26}$$

with some constant C depending only on Σ_0.

Suppose that $u = V_u + \omega_u \times y$ in O, then by Lemma 2.1,

$$|V_u| = \frac{1}{m}\left|\int_O u dy\right| \leq C\|u\|_{L^2(O)} \leq C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.27}$$

$$|\omega_u| = \left|J^{-1}\int_O u \times y dy\right| \leq C\|u\|_{L^2(O)} \leq C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}. \tag{5.28}$$

While (5.24) implies that

$$Re\lambda\|u\|^2_{L^2(\mathbb{R}^3)} - \nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = Re\langle f, u\rangle,$$

$$Im\lambda\|u\|^2_{L^2(\mathbb{R}^3)} = Im\langle f, u\rangle.$$

If $Re\lambda \leq 0$, then by the Hölder's and Sobolev's inequalities,

$$\begin{aligned}\nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} &\leq |Re\langle f, u\rangle| \\ &\leq C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|u\|_{L^2(\mathbb{R}^3)} \\ &\leq C|\lambda|^{-1}\|f\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Similarly, for $Re\lambda > 0$, one has

$$\begin{aligned}\nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} &= Re\lambda\|u\|^2_{L^2(\mathbb{R}^3)} - Re\langle f, u\rangle \\ &\leq C|Im\lambda|\|u\|^2_{L^2(\mathbb{R}^3)} - Re\langle f, u\rangle \\ &= C|Im\langle f, u\rangle| - Re\langle f, u\rangle \\ &\leq C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|u\|_{L^2(\mathbb{R}^3)} \\ &\leq C|\lambda|^{-1}\|f\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

where we used the fact that for every $\lambda \in \Sigma_0$, $Re\lambda \leq C(\Sigma_0)|Im\lambda|$.

Hence, in both cases we have $\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}$.

According to the Sobolev imbedding theorem,

$$\|u\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.29}$$

and it follows that

$$|V_u| = \frac{1}{m}\left|\int_O u dy\right| \leq C\|u\|_{L^6(O)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.30}$$

$$|\omega_u| = \left|J^{-1}\int_O u \times y dy\right| \leq C\|u\|_{L^6(O)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}. \tag{5.31}$$

Then we now consider two separate cases: $|\lambda| < \frac{1}{2}$ and $|\lambda| \geq \frac{1}{2}$. When $|\lambda| < \frac{1}{2}$, set

$$v(y) = \text{curl } \left[\frac{1}{2} V_u \times y\psi_{\mu_1}(y)\right] + \text{curl } \left[\omega_u |y|^{-1}\psi_{\mu_2}(y)\right],$$

with some constants $\mu_1, \mu_2 > 1$ to be determined, $w = u - v$, and ψ_R being defined in the proof of Theorem 2.2. Hence w satisfies the following problem,

$$\begin{cases} \lambda w + \nu\Delta w + \nabla p = f - \lambda v - \nu \Delta v, & \text{in } \Omega, \\ \text{div } w = 0, & \text{in } \Omega, \\ w(y) = 0, & \text{on } \partial\Omega. \end{cases}$$

It follows from Proposition 2.1 and the estimates (5.30) and (5.31) that

$$\begin{aligned} |\lambda|\|w\|_{L^p(\Omega)} &\leq C\left[\|f\|_{L^p(\Omega)} + |\lambda|\|v\|_{L^p(\Omega)} + \|\Delta v\|_{L^p(\Omega)}\right] \\ &\leq C\|f\|_{L^p(\Omega)} + C\left[|\lambda|^{\frac{1}{2}}\mu_1^{\frac{3}{p}} + |\lambda|^{\frac{1}{2}}\mu_2^{\frac{3}{p}} + |\lambda|^{\frac{1}{2}}\mu_2^{-1+\frac{3}{p}}\right] \cdot \|f\|_{L^2(\mathbb{R}^3)} \\ &\quad + C\left[|\lambda|^{-\frac{1}{2}}\mu_1^{-2+\frac{3}{p}} + |\lambda|^{-\frac{1}{2}}\mu_2^{-1+\frac{3}{p}} + |\lambda|^{-\frac{1}{2}}\mu_2^{-3+\frac{3}{p}}\right] \cdot \|f\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.32}$$

Setting $\mu_1 = |\lambda|^{-\frac{1}{2}}$, and $\mu_2 = |\lambda|^{-1}$, then one gets

$$|\lambda|\|w\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} + C\left[|\lambda|^{\frac{1}{2}-\frac{3}{2p}} + |\lambda|^{\frac{1}{2}-\frac{3}{p}}\right]\|f\|_{L^2(\mathbb{R}^3)}.$$

Since $p \geq 6$ and $|\lambda| \leq \frac{1}{2}$, combining the estimates (5.27), (5.28) and (5.32), one gets

$$\begin{aligned} |\lambda|\|u\|_{L^p(\mathbb{R}^3)} &\leq |\lambda|\|w\|_{L^p(\Omega)} + |\lambda|\|v\|_{L^p(\Omega)} + C|\lambda|[|V_u| + |\omega_u|] \\ &\leq C\left[\|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\right]. \end{aligned} \tag{5.33}$$

On the other hand, when $|\lambda| \geq \frac{1}{2}$, let

$$v = \text{curl } \left[\frac{1}{2} V_u \times y\psi_1(y)\right] - \text{curl } \left[\frac{1}{2}\omega_u |y|^2\psi_1(y)\right],$$

and $w = u - v$. It's easy to verify that

$$\begin{cases} \lambda w + \nu\Delta w + \nabla p = f - \lambda v - \nu \Delta v, & \text{in } \Omega, \\ \text{div } w = 0, & \text{in } \Omega, \\ w(y) = 0, & \text{on } \partial\Omega. \end{cases}$$

By virtue of Proposition 2.1′,

$$\begin{aligned} |\lambda|\|w\|_{L^p(\Omega)} &\leq C\left[\|f\|_{L^p(\Omega)} + |\lambda| \cdot (|V_u| + |\omega_u|) + |V_u| + |\omega_u|\right] \\ &\leq C\left[\|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}\right]. \end{aligned} \tag{5.34}$$

Collecting the estimates (5.27), (5.28) and (5.34), we conclude that

$$\begin{aligned} |\lambda| \|u\|_{L^p(\mathbb{R}^3)} &\le |\lambda| \left[\|w\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\right] + C|\lambda| \left[|V_u| + |\omega_u|\right] \\ &\le C \left[\|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}\right]. \end{aligned} \tag{5.35}$$

Therefore, (5.25) follows, which completes the proof.

Remark 5.1 In the case of two-dimensional motion, we just need to take the function $v(y)$ as $\nabla^\perp \left[V_u \cdot y^\perp \psi_{|\lambda|^{-1/2}}(y) - \omega_u \ln|y| \cdot \psi_{|\lambda|^{-1}}(y)\right]$.

6 L^q-L^r Estimates

In this section, we give some $L^q - L^r$ estimates associated with the semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$, which will be the key for the proof of the local well-posedness of the problem. In section 6 and section 7, we will write A for $A_{\frac{6}{5}\cap p}$ for simplicity .

Proposition 6.1 Assume that $2 \le p < \infty$, and q satisfies that

$$\begin{cases} q \in [p, \infty], & \text{if } 2 \le p < 3, \\ q \in [p, \infty), & \text{if } p = 3, \\ q \in [p, \infty], & \text{if } p > 3. \end{cases}$$

Then there exist some positive constants $C_1(\Omega, p)$ and $C_2(\Omega, p, q)$ such that, for any $u_0 \in H_1^{\frac{6}{5}} \cap H_1^p$ and $t > 0$,

$$\left\|\nabla e^{-tA} u_0\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \le C_1(\Omega, p)(1 + t^{-\frac{1}{2}}) \|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}, \tag{6.36}$$

and

$$\left\|e^{-tA} u_0\right\|_{L^q(\mathbb{R}^3)} \le C_2(\Omega, p, q) \left(1 + t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q})}\right) \|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}. \tag{6.37}$$

Proof Let $u \in D(A)$ be given. First, we derive an estimate on $\|\nabla u\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}$ in terms of $\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}$ and $\|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}$. Suppose

$$-u - Au = f,$$

and

$$u(y) = V_u + \omega_u \times y, \quad \text{in } O.$$

As in section 4, there exists some $p \in \mathcal{D}'$ such that (u, p) satisfies the system

$$\begin{cases} -u + \nu \Delta u + \nabla p = f, \quad \text{in } \Omega, \\ \operatorname{div} u = 0, \quad \text{in } \Omega, \\ u(y) = V_u + \omega_u \times y, \quad \text{on } \partial\Omega. \end{cases}$$

Following the proof of Theorem 2.2, we choose some R large enough such that $O \subset B_R(0)$. Let

$$v = \text{curl } \left[\frac{1}{2}V_u \times y\psi_R(y)\right] - \text{curl } \left[\frac{1}{2}\omega_u|y|^2\psi_R(y)\right],$$

and $w = u - v$. Then w is the solution to the problem

$$\begin{cases} -w + \nu\Delta w + \nabla p = f + v - \nu\Delta v, \quad \text{in } \Omega, \\ \text{div } w = 0, \quad \text{in } \Omega, \\ w(y) = 0, \quad \text{on } \partial\Omega. \end{cases}$$

By virtue of Proposition $2.1'$, one has

$$\begin{aligned} & \|w\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} \\ \leq \; & C\left[\|f\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + \|v\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} + \|w\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)}\right] \\ \leq \; & C\left[\|f\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + |V_u| + |\omega_u| + \|u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)}\right] \\ \leq \; & C\left[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\right] \\ \leq \; & C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}, \end{aligned}$$

where the last inequality comes from Theorem 2.2 by letting $\lambda = -1$.

Then

$$\begin{aligned} \|u\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} & \leq \|w\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} + \|v\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} \\ & \leq C\left[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + |V_u| + |\omega_u|\right] \\ & \leq C\left[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\right] \\ & \leq C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}. \end{aligned}$$

By the interpolation inequality,

$$\begin{aligned} & \|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \\ \leq \; & C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \|u\|^{\frac{1}{2}}_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} \\ \leq \; & C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \|f\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\ \leq \; & C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \left[\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\right]^{\frac{1}{2}} \end{aligned}$$

Consequently,

$$
\begin{aligned}
& \|\nabla u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\leq\ & \|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + \|\nabla u\|_{L^{\frac{6}{5}}(O)\cap L^p(O)} \\
\leq\ & \|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + C|\omega_u| \\
\leq\ & C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \cdot \left[\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\right]^{\frac{1}{2}} \\
& + C\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\leq\ & C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \|Au\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + C\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
\end{aligned}
\tag{6.38}
$$

For any $u_0 \in H_1^{\frac{6}{5}} \cap H_1^p$, applying (6.38) to $e^{-tA}u_0$ yields

$$
\begin{aligned}
& \|\nabla e^{-tA}u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\leq\ & C\|e^{-tA}u_0\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \|Ae^{-tA}u_0\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + C\|e^{-tA}u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
\end{aligned}
$$

Note that

$$
\|e^{-tA}u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq M_1\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)},
$$

and

$$
\|Ae^{-tA}u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq \frac{M_1}{t}\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
$$

Hence,

$$
\begin{aligned}
& \|\nabla e^{-tA}u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\leq\ & C_1(p,\Omega)\left(1+t^{-\frac{1}{2}}\right)\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
\end{aligned}
\tag{6.39}
$$

Let $u(t) = e^{-tA}u_0$, and $u(t) = l_u(t) + \omega_u(t) \times y$ in O. When $2 \leq p < 3$, and $q \in [p, \infty]$, using the Sobolev embedding inequality, one can get

$$
\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^3)} & \leq \|u(t)\|_{L^q(\Omega)} + \|u(t)\|_{L^q(O)} \\
& \leq C\|u(t)\|^{\theta}_{L^p(\Omega)} \cdot \|u(t)\|^{1-\theta}_{W^{2,p}(\Omega)} + C\left[|l_u(t)| + |\omega_u(t)|\right] \\
& \leq C\|u(t)\|^{\theta}_{L^p(\Omega)} \cdot \left[\|u(t)\|_{L^p(\Omega)} + \|Au(t)\|_{L^p(\Omega)}\right]^{1-\theta} + C\|u(t)\|_{L^p(O)} \\
& \leq C\|u(t)\|^{\theta}_{L^p(\mathbb{R}^3)} \cdot \left[\|u(t)\|_{L^p(\mathbb{R}^3)} + \|Au(t)\|_{L^p(\mathbb{R}^3)}\right]^{1-\theta} + C\|u(t)\|_{L^p(\mathbb{R}^3)} \\
& \leq CM_1\|u_0\|^{\theta}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \cdot \left[\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + t^{-1}\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\right]^{1-\theta} \\
& \quad + CM_1\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
& \leq C(p,q,\Omega)\left[1 + t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right]\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}
\end{aligned}
$$

where θ satisfies $\frac{1}{q} = \frac{\theta}{p} + \left(\frac{1}{p} - \frac{2}{3}\right)(1-\theta)$.

When $p = 3$, $q \in [p, \infty)$, or $p > 3$, $q \in [p, \infty]$, using the Sobolev embedding inequality, we have

$$\begin{aligned}\|u(t)\|_{L^q(\mathbb{R}^3)} &\leq C\|u(t)\|^{\theta}_{L^p(\mathbb{R}^3)} \cdot \|\nabla u(t)\|^{1-\theta}_{L^p(\mathbb{R}^3)} \\ &\leq C(p,q,\Omega)\left(1 + t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\right)\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)},\end{aligned}$$

where θ satisfies $\frac{1}{q} = \frac{\theta}{p} + \left(\frac{1}{p} - \frac{1}{3}\right)(1-\theta)$.

Therefore, we complete the proof of Proposition 6.1.

Remark 6.1 Comparing to the estimates of the classical Stokes semigroup in [11], we were not able to get the corresponding decay estimates of $\nabla e^{-tA}u_0$. In section 7 we will see that Proposition 6.1 is the key estimate to guarantee the local existence of a strong solution. However, without decay estimates on $\nabla e^{-tA}u_0$, we can not get any global strong solution even when the initial data is small.

Remark 6.2 When O is a ball in $\mathbb{R}^3$, applying Theorem 2.3 instead of Theorem 2.2, we can prove the corresponding result for the case $e^{-tA_{2\cap p}}$, $p \geq 6$.

7 Local Existence of Strong Solutions

Assume that O is a unit ball in $\mathbb{R}^3$. We treat this particular case as an example to investigate the local existence of strong solutions to the system (1.2).

The proof of Theorem 2.4 is in spirit similar to those given in [14]. In fact, it was proved in [23], the system (1.2) can be rewritten in the abstract form

$$\partial_t v + Av + \mathbb{P}(v \cdot \nabla v) - \mathbb{P}(l_v \cdot \nabla v) = 0,$$

with the initial data

$$v(y,0) = v_0(y) = \begin{cases} a(y), & y \in \Omega, \\ b + c \times y, & y \in O. \end{cases}$$

Here $\mathbb{P}$ is the projection operator mentioned in section 2, and l_v is associated with v such that $v = l_v + \omega_v \times y$ in O.

The above equation can be converted into the integral equation

$$v(y,t) = e^{-tA}v_0 - \int_0^t e^{-(t-s)A}[\mathbb{P}(\widehat{v \cdot \nabla v}) - \mathbb{P}(\widehat{l_v \cdot \nabla v})](s)ds,$$

where $\hat{f}$ denotes the restriction of f on the domain Ω, i.e.,

$$\hat{f}(y) = \begin{cases} f(y), & y \in \Omega, \\ 0, & y \in O. \end{cases}$$

Suppose that $\|v_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} = K$, and set

$$X_{T_0} = \left\{ u(y,t): \begin{array}{l} \text{div } u = 0 \ \text{ in } \mathbb{R}^3, \ D(u) = 0 \text{ in } O, \ \ \|u\|_{L^\infty(0,T_0;L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \leq NK, \\ \text{and } \ \|t^{\frac{1}{2}}\nabla u\|_{L^\infty(0,T_0;L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \leq NK. \end{array} \right\}$$

where $N \geq 4\max\{M_1, C_1\}$ and T_0 is to be determined later. Set

$$\|u\|_{X_{T_0}} = \max\left\{ \|u(t)\|_{L^\infty(0,T_0;L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))}, \ \|t^{\frac{1}{2}}\nabla u(t)\|_{L^\infty(0,T_0;L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \right\}.$$

For any $u \in X_{T_0}$, and $u = l_u + \omega_u \times y$ in O, define the map $\mathcal{L}$,

$$\mathcal{L}u = e^{-tA}v_0 - \int_0^t e^{-(t-s)A}[\mathbb{P}(\widehat{u\cdot\nabla u}) - \mathbb{P}(\widehat{l_u\cdot\nabla u})](s)ds.$$

We will show that, for suitable T_0, $\mathcal{L}$ maps X_{T_0} int X_{T_0} and $\mathcal{L}$ is a contraction mapping.

$\mathcal{L}u$ can be estimated as the sum of three parts. Thanks to the estimates (2.7) and (6.36), one has

$$\|e^{-tA}v_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq M_1 K,$$

and

$$\|\nabla e^{-tA}v_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C_1(1+t^{-\frac{1}{2}})K.$$

Since $\mathbb{P}$ is a bounded operator from $L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ to $H_1^{\frac{6}{5}} \cap H_1^p$, then it follows from the definition of X_{T_0} and Sobolev's inequality that

$$\begin{aligned} & \left\| \int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\ \leq \ & \int_0^t M_1\|\mathbb{P}(\widehat{u\cdot\nabla u})(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds \\ \leq \ & \int_0^t CM_1\|u(s)\|_{L^\infty(\mathbb{R}^3)}\cdot\|\nabla u(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds \\ \leq \ & \int_0^t CM_1 s^{-\frac{3}{2p}}(NK)^2 s^{-\frac{1}{2}}ds \\ \leq \ & \frac{2p}{p-3}CM_1(NK)^2 t^{\frac{1}{2}-\frac{3}{2p}} \\ = \ & C_3(NK)^2 t^{\frac{1}{2}-\frac{3}{2p}}, \end{aligned}$$

where $C_3 = C_3(p,\Omega)$ depends only on p and Ω.

$$
\begin{aligned}
&\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\\
\leq\ & \int_0^t \left\|\nabla e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} ds\\
\leq\ & \int_0^t CC_1[1+(t-s)^{-\frac{1}{2}}]\|(u\cdot\nabla u)(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t C[1+(t-s)^{-\frac{1}{2}}]\|u(s)\|_{L^\infty(\mathbb{R}^3)}\cdot\|\nabla u(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t C[1+(t-s)^{-\frac{1}{2}}]s^{-\frac{1}{2}}s^{-\frac{3}{2p}}(NK)^2ds\\
\leq\ & C_4(p,\Omega)(NK)^2\left[t^{\frac{1}{2}-\frac{3}{2p}}+t^{-\frac{3}{2p}}\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\|\int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{l_u\cdot\nabla u})(s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\\
\leq\ & \int_0^t CM_1|l_u(s)|\cdot\|\nabla u(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t CM_1\|u(s)\|_{L^\infty(\mathbb{R}^3)}\cdot\|\nabla u(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t CM_1 s^{-\frac{3}{2p}}(NK)^2 s^{-\frac{1}{2}}ds\\
\leq\ & \frac{2p}{p-3}CM_1(NK)^2t^{\frac{1}{2}-\frac{3}{2p}}\\
=\ & C_5(p,\Omega)(NK)^2t^{\frac{1}{2}-\frac{3}{2p}},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{l_u\cdot\nabla u})(s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\\
\leq\ & \int_0^t \left\|\nabla e^{-(t-s)A}\mathbb{P}(\widehat{l_u\cdot\nabla u})(s)\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} ds\\
\leq\ & \int_0^t CC_1[1+(t-s)^{-\frac{1}{2}}]\|(l_u\cdot\nabla u)(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t C[1+(t-s)^{-\frac{1}{2}}]\|u(s)\|_{L^\infty(\mathbb{R}^3)}\cdot\|\nabla u(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds\\
\leq\ & \int_0^t C[1+(t-s)^{-\frac{1}{2}}]s^{-\frac{1}{2}}s^{-\frac{3}{2p}}(NK)^2ds\\
\leq\ & C_6(p,\Omega)(NK)^2\left[t^{-\frac{3}{2p}}+t^{\frac{1}{2}-\frac{3}{2p}}\right].
\end{aligned}
$$

Hence

$$
\|\mathcal{L}u(t)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\leq M_1K+(C_3+C_5)(NK)^2t^{\frac{1}{2}-\frac{3}{2p}},
$$

and

$$\left\|t^{\frac{1}{2}}\nabla\mathcal{L}u(t)\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C_1(1+t^{\frac{1}{2}})K + (C_4+C_6)(NK)^2\left[t^{1-\frac{3}{2p}} + t^{\frac{1}{2}-\frac{3}{2p}}\right].$$

If T_0 is chosen to be sufficiently small such that

$$T_0 \leq T_1 = \min\left\{[(C_3+C_5)NK]^{-\frac{2p}{p-3}},\ [(C_4+C_6)NK]^{-\frac{2p}{p-3}},\ 1\right\},$$

then $\mathcal{L}$ maps X_{T_0} to X_{T_0}.

Furthermore, for any $u, \tilde{u} \in X_{T_0}$,

$$\begin{aligned}
\mathcal{L}u - \mathcal{L}\tilde{u} &= -\int_0^t e^{-(t-s)A}\left[\mathbb{P}(\widehat{u\cdot\nabla u}) - \mathbb{P}(\widehat{l_u\cdot\nabla u})\right](s)ds \\
&\quad + \int_0^t e^{-(t-s)A}\left[\mathbb{P}(\widehat{\tilde{u}\cdot\nabla\tilde{u}}) - \mathbb{P}(\widehat{l_{\tilde{u}}\cdot\nabla\tilde{u}})\right](s)ds \\
&= \int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{\tilde{u}-u\cdot\nabla})\tilde{u}\right](s)ds + \int_0^t e^{-(t-s)A}\mathbb{P}\left[(u\cdot\widehat{\nabla)(\tilde{u}}-u)\right](s)ds \\
&\quad + \int_0^t e^{-(t-s)A}\mathbb{P}\left[(l_u\widehat{-l_{\tilde{u}}\cdot}\nabla)u\right](s)ds + \int_0^t e^{-(t-s)A}\mathbb{P}\left[(l_{\tilde{u}}\cdot\widehat{\nabla)(u}-\tilde{u})\right](s)ds
\end{aligned}$$

For each term on the right hand side, we have the following estimates,

$$\begin{aligned}
&\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{\tilde{u}-u\cdot\nabla})\tilde{u}\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\leq\ & \int_0^t CM_1\|\tilde{u}(s)-u(s)\|_{L^\infty(\mathbb{R}^3)}\cdot\|\nabla\tilde{u}(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds \\
\leq\ & \int_0^t CM_1 s^{-\frac{1}{2}}(NK)s^{-\frac{3}{2p}}ds\cdot\|u-\tilde{u}\|_{X_{T_0}} \\
\leq\ & C_7(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}}
\end{aligned}$$

Similarly,

$$\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[u\cdot\widehat{\nabla(\tilde{u}-u})\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C_8(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}},$$

Note that

$$|l_u - l_{\tilde{u}}| = \left|\frac{1}{m}\int_O(u-\tilde{u})dy\right| \leq C\|u-\tilde{u}\|_{L^\infty(\mathbb{R}^3)},$$

then

$$\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[(l_u\widehat{-l_{\tilde{u}}\cdot}\nabla)u\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C_9(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}},$$

and

$$\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[(l_{\tilde{u}}\cdot\widehat{\nabla)(u}-\tilde{u})\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leq C_{10}(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}}.$$

Hence, for every $t \in [0, T_0]$,

$$\|\mathcal{L}u(t) - \mathcal{L}\tilde{u}(t)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \le (C_7 + C_8 + C_9 + C_{10})NKt^{\frac{1}{2}-\frac{3}{2p}}\|u - \tilde{u}\|_{X_{T_0}}.$$

Furthermore,

$$\begin{aligned}
&\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{\tilde{u} - u \cdot \nabla})\tilde{u}\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\le\ & \int_0^t CC_1(t-s)^{-\frac{1}{2}}\|\tilde{u}(s) - u(s)\|_{L^\infty(\mathbb{R}^3)} \cdot \|\nabla\tilde{u}(s)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}ds \\
\le\ & \int_0^t CC_1(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}(NK)s^{-\frac{3}{2p}}ds \cdot \|u - \tilde{u}\|_{X_{T_0}} \\
\le\ & C_{11}(p,\Omega)(NK)\left[t^{\frac{1}{2}-\frac{3}{2p}} + t^{-\frac{3}{2p}}\right]\|u - \tilde{u}\|_{X_{T_0}}.
\end{aligned}$$

Similarly,

$$\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}\left[u \cdot \widehat{\nabla(\tilde{u} - u)}\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \le C_{12}(p,\Omega)(NK)t^{-\frac{3}{2p}}\|u - \tilde{u}\|_{X_{T_0}},$$

$$\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{l_u - l_{\tilde{u}} \cdot \nabla})u\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \le C_{13}(p,\Omega)(NK)t^{-\frac{3}{2p}}\|u - \tilde{u}\|_{X_{T_0}},$$

and

$$\begin{aligned}
&\left\|\nabla \int_0^t e^{-(t-s)A}\mathbb{P}\left[(l_{\tilde{u}} \cdot \widehat{\nabla)(u} - \tilde{u})\right](s)ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
\le\ & C_{14}(p,\Omega)(NK)\left[t^{\frac{1}{2}-\frac{3}{2p}} + t^{-\frac{3}{2p}}\right]\|u - \tilde{u}\|_{X_{T_0}}.
\end{aligned}$$

Let $T_2 = \min\left\{[(C_7 + C_8 + C_9 + C_{10})NK]^{-\frac{2p}{p-3}},\ [(C_{11} + C_{12} + C_{13} + C_{14})NK]^{-\frac{2p}{p-3}}\right\}$. Combining the above estimates, we obtain that when $T_0 \le \min\{T_1, T_2\}$, $\mathcal{L}$ is a contraction mapping on X_{T_0}. Therefore, there exists a fixed point $v \in X_{T_0}$ of $\mathcal{L}$, i.e., $\mathcal{L}v = v$. It is clear that the fixed point $v(y,t) \in X_{T_0}$ is a strong solution to the system (1.2). The uniqueness of the solution is implied in the proof of verifying the contraction property of $\mathcal{L}$.

Remark 7.1 Following almost the same proof of Theorem 2.4 and applying Theorem 2.3, we can also get a local strong solution starting from $H_1^2 \cap H_1^p$, $p \ge 6$.

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