SUBSONIC FLOWS IN A MULTI-DIMENSIONAL NOZZLE*

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Abstract. In this paper, we study the global subsonic irrotational flows in a multi-dimensional ($n \geq 2$) infinitely long nozzle with variable cross sections. The flow is described by the inviscid potential equation, which is a second order quasilinear elliptic equation when the flow is subsonic. First, we prove the existence of the global uniformly subsonic flow in a general infinitely long nozzle for arbitrary dimension for sufficiently small incoming mass flux and obtain the uniqueness of the global uniformly subsonic flow. Furthermore, we show that there exists a critical value of the incoming mass flux such that a global uniformly subsonic flow exists uniquely, provided that the incoming mass flux is less than the critical value. This gives a positive answer to the problem of Bers on global subsonic irrotational flows in infinitely long nozzles for arbitrary dimension [5]. Finally, under suitable asymptotic assumptions of the nozzle, we obtain the asymptotic behavior of the subsonic flow in far fields by a blow-up argument. The main ingredients of our analysis are methods of calculus of variations, the Moser iteration techniques for the potential equation and a blow-up argument for infinitely long nozzles.

1. Introduction

This paper is devoted to the existence and the uniqueness of global subsonic flows for the Euler equations for steady irrotational compressible fluids. Our focus is on the global subsonic flows in a general multi-dimensional infinite nozzle, which is an important subject in gas dynamics (see [4] [5] [8][12][22]).

Consider the steady isentropic compressible Euler equations

$$\begin{cases} \operatorname{div}(\rho u) = 0, & \text{in } \Omega, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, \ldots, u_n)$, p represent the density, velocity, and the pressure of the fluid, respectively. Moreover, the pressure $p = p(\rho)$ is a smooth function of ρ and $p'(\rho) > 0$, $p''(\rho) > 0$ for $\rho > 0$.

It is easy to derive the following so-call Bernoulli's law [8]

$$u \cdot \nabla \left(\frac{1}{2}|u|^2 + h(\rho) \right) = 0, \tag{1.2}$$

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where $h(\rho)$ is the enthalpy, defined by $h(\rho) = \int_1^\rho \frac{p'(s)}{s} ds$. The relation (1.2) implies that the quantity $B(\rho, |u|^2) = \frac{1}{2}|u|^2 + h(\rho)$, named Bernoulli's function, remains constant along the stream line in a steady isentropic flow.

If, in addition, the flow is assumed to be irrotational, ie. the vorticity of the flow velocity

$$\nabla \times u = 0, \qquad \text{in } \Omega,$$

then there exists a velocity potential function φ, at least locally, such that

$$u(x) = \nabla\varphi(x).$$

In this case, the relation (1.2) simplifies to the following strong version of the Bernoulli's law

$$\nabla B(\rho, |\varphi|^2) = \nabla\left(\frac{1}{2}|\nabla\varphi|^2 + h(\rho)\right) = 0. \tag{1.3}$$

This yields a density-speed relation for steady irrotational flows. Therefore, the density ρ can be determined by the speed $|\nabla\varphi|$, denoted by $\rho\left(|\nabla\varphi|^2\right)$. Then the steady Euler equations (1.1) are reduced to the following well-known scaler potential equation

$$\operatorname{div}\left(\rho(|\nabla\varphi|^2)\nabla\varphi\right) = 0, \qquad \text{in } \Omega. \tag{1.4}$$

One of the most important parameters to the fluid dynamics is the *Mach number*, which is defined as a non-dimensional ratio of the fluid velocity to local sound speed,

$$M = \frac{|u|}{c(\rho)},$$

where $c(\rho) = \sqrt{p'(\rho)}$ is the local sound speed. Mathematically, the second-order nonlinear equation (1.4) is elliptic in the subsonic region, ie. $M < 1$ and hyperbolic in the supersonic region where $M > 1$.

Subsonic flows are those in which the local velocity speed is smaller than sonic speed everywhere, i.e. the Mach number of the flow is less than 1. Since the corresponding equations of subsonic flows possess some elliptic properties, problems related to subsonic flows are, in general, have extra-smoothness to those related to transonic flows or supersonic flows. There are many literatures in this field in the past decades. The first result is due to Frankl and Keldysh [15]. They studied the subsonic flows around a 2D finite body (or airfoil) and proved the existence and the uniqueness for small data by the method of successive approximations. Later on, Bers [1][2] proved the existence of subsonic flows with arbitrarily high local subsonic speed for the Chaplygin gas (minimal surface). By a variational method, Shiffman [25][26] proved that, if the infinite free stream flow speed u_∞ is less than some critical speed, there exists a unique subsonic potential flow around a given profile with finite energy. Shortly afterwards, Bers [3] improved the uniqueness results of Shiffman. Finn and Gilbarg [13] proved the uniqueness of the 2D potential subsonic flow about a bounded obstacle with given circulation and velocity at infinity. All above the results are related to two dimensional problems. For three (or higher) dimensional case, Finn and Gilbarg [14] proved the existence, uniqueness and the asymptotic behavior with implicit restriction on Mach number M. Payne and Weinberger [23] improved their results soon after. Later, Dong [9] extended the results of Finn and Gilbarg [14] to any Mach number $M < 1$ and to arbitrary dimensions. Furthermore, in [10], Dong and Ou extended the results of Shiffman to higher dimensions by the direct method of calculus of variations and the standard Hilbert space method.

All results as above (including [16]-[20]) are related to the subsonic flows past a profile. Another important problem is the study of subsonic flows is the theory of global subsonic flow in a variable nozzles as formulated by Bers in [5]:

Problem 1. *Find φ such that,*

$$\begin{cases} div\Big(\rho(|\nabla\varphi|^2)\nabla\varphi\Big) = 0, & in \ \ \Omega, \\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & on \ \ \partial\Omega, \\ \displaystyle\int_{S_0} \rho\left(|\nabla\varphi|^2\right)\frac{\partial\varphi}{\partial\vec{l}}dS = m_0 > 0, & \\ |\nabla\varphi| < c(\rho), & in \ \ \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is an infinitely long nozzle, $m_0 > 0$ is the mass flux passing through the nozzle, S_0 is an arbitrary cross section of the nozzle, $\vec{n}$ and $\vec{l}$ are the unit outer normal of the domain Ω and S_0, respectively (Please see Fig. 1).

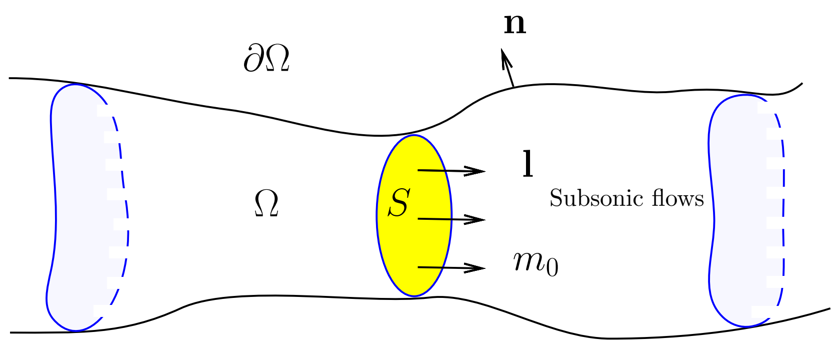

FIGURE 1. Subsonic flow in a nozzle

In the famous survey [5], Bers claimed without proof the unique solvability of sufficiently slow subsonic irrotational flows in two dimensional channel. The rigorous proof of this fact was achieved mathematically recently by Xie and Xin [27]. They established a very complete, satisfactory and systematic theory for the two dimensional subsonic flows in an infinitely long nozzle for potential flows, which not only solves the Problem 1 in this case, but also yields the existence of subsonic-sonic flows in the nozzle as limits of subsonic flows. One of the key ideas in [27], is to use the stream function to formulate the problem to a quasilinear elliptic problem with Dirichlet boundary conditions. The benefit of the stream function formation of the problem is that, the stream function ψ has a priori L^∞ bound, and the flow region of two dimensional nozzle, though infinitely long, has finite "width". So one can obtain the boundary L^∞ estimate of the gradient of the stream function, $\nabla\psi$, by constructing proper barrier functions and the standard comparison principle for subsolution to second order elliptic equation. Similar approach has been applied in 3D axis-asymmetric nozzles by Xie and Xin in [28]. Furthermore, these ideas are also useful to study the physically more important case, subsonic Euler flows, by Xie and Xin in [29] (see also the generalization in [11]). However, it seems difficult to apply the method in [27] and [28] in general multi-dimensional ($n \geq 3$) nozzles, since the stream function formulation can not work in this case. Thus, we have to consider a different approach from that in [27] to treat the subsonic problem in multi-dimension case.

On the other hand, since the domain of an infinitely long nozzle is differentiable homeomorphism to an infinitely long cylinder which is unbounded, the nozzle flow problems are different to the airfoil problems in which the domains are exterior domains. The main advantage of the exterior domain is that it can be transformed to a bounded domain through a Kelvin-like transformation. Then the airfoil problem can be transformed (explicitly or implicitly) to a scalar quasilinear elliptic problem with a bounded domain. This feature of the exterior domain plays an essential role in the previous airfoil results. For instance, in [24], a Hardy-type inequality in the exterior domain is essential. But there is no similar Hardy-type inequality for the domain of nozzle flows, which is the one of the main difficulties in our case. For more detailed discussions, we refer to [30].

The main purpose of this paper is to study subsonic flows in general multi-dimensional ($n \geq 2$) infinitely long nozzles. First, we formulate a subsonic truncated problem, which is a uniformly elliptic equation in a bounded domain. Moreover, we prove the existence of the weak solution to the truncated problem by a variational method, and use the approximated variational problems in bounded domains to approximate the original Problem 1. To realize this procedure, some uniform estimates are needed to show that the approximated solutions converge to the ones of the original Problem 1. However, one can not expect to get the uniform boundary gradient estimate of φ by the classical barrier function argument, since the potential function φ is essentially unbounded, which is another main difficulty in this paper. The key observation here is that, though the potential function φ is unbounded, the L^2 average of $\nabla\varphi$ is uniformly bounded (see the estimate (3.6)). Using this fact and the uniform ellipticity, we prove the "local average estimate" which states that the average estimate implies the local average of the gradient $\nabla\varphi$ is uniformly bounded (see (3.23) for details). That is, $\nabla\varphi$ is locally L^2 bounded. Then, it is easy to get the L^∞ bound of $\nabla\varphi$ by the standard Moser iteration. With this key estimate of uniformly L^∞ bound of $\nabla\varphi$, we establish the existence of the subsonic flows in an infinitely long nozzle for arbitrary dimensions for suitable small incoming mass flux, including the two dimensional case in [27]. Next, we show that the global uniformly subsonic flow is unique. The proof is based on considering the linear equation satisfied be the difference of two solutions of the nonlinear potential equation. Moreover, we prove the existence of the critical incoming mass flux for subsonic flows. Finally, with the additional asymptotic assumptions on the nozzle at the far field, we obtain some asymptotic behaviors of the subsonic flow at the far field by a blow-up argument.

Before stating the main results in this paper, we first give the following assumptions on the nozzle.

Basic assumptions on Ω. There exists an invertible $C^{2,\alpha}$ map $T: \overline{\Omega} \to \overline{\mathbf{C}} : x \mapsto y$ satisfying

$$\begin{cases} T(\partial\Omega) = \partial\mathbf{C}, \\ \text{For any } k \in \mathbb{R},\ T(\Omega \cap \{x_n = k\}) = B(0,1) \times \{y_n = k\}, \\ \|T\|_{C^{2,\alpha}},\ \|T^{-1}\|_{C^{2,\alpha}} \leq K, \end{cases} \tag{1.6}$$

where K is a uniform constant, $\mathbf{C} = B(0,1) \times (-\infty, \infty)$ is a unit cylinder in $\mathbb{R}^n$, $B(0,1)$ is unit ball in $\mathbb{R}^{n-1}$ centered at the origin, x_n is the longitudinal coordinate.

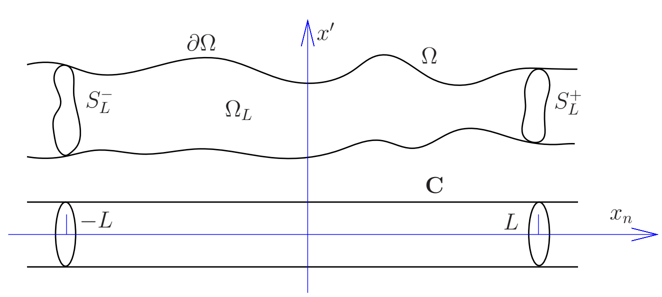

FIGURE 2. Basic assumptions on Ω

Asymptotic assumptions on Ω. Suppose that the nozzle approaches to a cylinder in the far fields, ie.

$$\Omega \cap \{x_n = k\} \to S_\pm, \qquad \text{as} \quad k \to \pm\infty, \tag{1.7}$$

respectively, where $S_\pm$ are $n-1$ dimensional, simply connected, $C^{2,\alpha}$ domains.

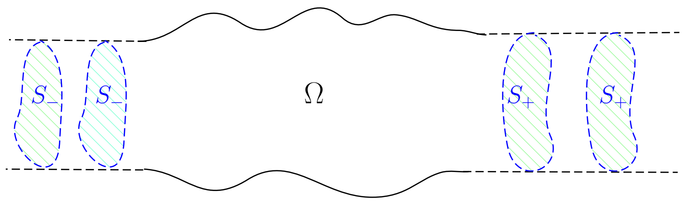

FIGURE 3. Asymptotic assumptions on Ω

Nondimensionalization of the quantities. It follows from Bernoulli's Law (1.2) that in a potential flow the density is a given function of speed. Applying the fact ([5], [8]) that there exists a critical speed q_{cr} such that the flow is subsonic if the speed is less than q_{cr}, we can introduce the nondimensional velocity and density as

$$\hat{u} = \frac{u}{q_{cr}}, \quad \hat{v} = \frac{v}{q_{cr}}, \quad \hat{\rho} = \frac{\rho}{\rho(q_{cr}^2)}.$$

With an abuse of the notation, we still denote the nondimesional quantities by u, v, ρ. Then it is easy to check that $\rho q \le 1$ for $q \ge 0$ and that the flow is subsonic provided that $q < 1$ or $\rho > 1$.

Our main results in this paper are stated as follows.

Theorem 1.1. *Suppose that the nozzle Ω satisfies the basic assumptions (1.6). Then*

(i) there exists a positive number M_0 depending only on Ω, such that if $m_0 \le M_0$, then there exists a uniformly subsonic flow through the nozzle, ie., the Problem 1 has a smooth solution $\varphi \in C^\infty(\Omega)$. Moreover,

$$\|\nabla\varphi(x)\|_{C^{1,\alpha}(\Omega)} \le C m_0,$$

where $C > 0$ is a uniform constant independent of M_0, m_0, and φ.

(ii) There exists a critical mass flux $M_c \le 1$, which depends only on Ω, such that if $0 \le m_0 < M_c$, then there exists a unique uniformly subsonic flow through the nozzle with the following properties

$$Q(m_0) = \sup_{x\in\bar{\Omega}} |\nabla\varphi| < 1, \tag{1.8}$$

and $Q(m_0)$ ranges over $[0, 1)$ as m_0 varies in $[0, M_c)$.

(iii) Furthermore, assume that the nozzle satisfies the asymptotic assumption (1.7), then the flow approaches the uniform flows at the far fields, ie.

$$\nabla\varphi = (0, \cdots, q_\pm), \qquad \text{as} \quad x_n \to \pm\infty, \tag{1.9}$$

respectively, with $q_\pm$ being constants determined uniquely by

$$\rho(q_\pm^2)q_\pm = \frac{m_0}{|S_\pm|},$$

here $|S_\pm|$ represents the measure of the domain $S_\pm$, respectively.

Remark 1.1. In the first statement of Theorem 1.1, it follows from the proof in Section 3 that one can derive an explicit form of M_0, which depends only on the nozzle Ω. In particular, it does not depend on the equation of the states. On the other hand, in the second statement of the Theorem 1.1, we just give the existence of the critical mass flux M_c for a given infinite long nozzle. Clearly, M_0 is a lower bound of M_c.

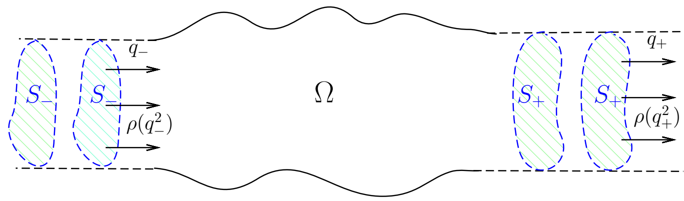

FIGURE 4. Asymptotic behaviors of the subsonic flows at the far fields

Remark 1.2. In the proof of the uniqueness of uniformly subsonic flows, it is not necessary to require the asymptotic assumption (1.7) on the nozzle. It is quite different from the strategy in [27] for the 2D case, in which the proof of the uniqueness depends on the asymptotic behaviors of the uniformly subsonic flow in the far fields. However, in this paper, the uniqueness of the uniformly subsonic flow is obtained in arbitrary dimensional nozzle without the asymptotic assumption (1.7).

This paper is organized as follows. In the next section, we introduce some necessary preliminaries. In Section 3, we prove the first statement of the Theorem 1.1. Our strategy for the existence of subsonic flows with small incoming mass flux can be divided into six steps: Step 1, truncate the coefficients of the potential equation to guarantee the strong ellipticity and truncate the unbounded nozzle to a series of bounded domains Ω_L, to formulate the approximated strong elliptic problems in bounded domains. Step 2, solve the approximate truncated problems by a direct variational method. Step 3, improve the regularity of the variational solutions to give the H^2 regularity. Step 4, prove the L^2 local average estimates to the gradient of the solutions. Step 5, obtain the classical $C^{1,\alpha}$ estimate of the approximate solutions. Step 6, based on these key estimates, the existence of the subsonic solution to the nozzle problem for suitable small incoming mass flux is proved. The uniqueness of the uniformly subsonic solution is given in Section 4, while the existence of the critical value for incoming mass flux is obtained in Section 5. In the last section, we prove that the subsonic nozzle flows approach to the uniform flows at the far fields when the nozzle satisfies the asymptotic assumption (1.7).

In this paper, x, y always denote the variables in Ω and $\mathbf{C}$ respectively, φ denotes the function defined in Ω and $\tilde{\varphi} = \varphi \circ T^{-1}$ denotes the corresponding function defined in $\mathbf{C}$. ∂, and ∇ denote the derivatives with respect to x in Ω, while $\tilde{\partial}$, and $\tilde{\nabla}$ denote the derivatives with respect to y in $\mathbf{C}$. $A \sim B$ means

$$\frac{1}{C} A \le B \le CA,$$

with C a positive constant.

2. Preliminaries

In this section, we give some basic notations, definitions and facts to be used in this paper.

2.1. Morrey theorem.

Definition 2.1. Let Ω be bounded region in $\mathbb{R}^n$. Ω is said to be of *A-type*, if there exists a positive number A such that, for any $x \in \Omega$ and $0 < r < \operatorname{diam} \Omega$,

$$|\Omega \cap B(x, r)| \ge A r^n.$$

Now, we state the following Morrey theorem (see, for instance, [7]):

Theorem 2.1. *Assume that Ω is of A-type, $u \in W^{1,p}(\Omega)$, $p > 1$ and there exist constants $K > 0$, $0 < \alpha < 1$ such that, for any B_R,*

$$\int_{\Omega \cap B_R} |\nabla u(x)|^p dx \le K R^{n-p+\alpha p}$$

holds. Then $u \in C^{\alpha}(\overline{\Omega})$ *and*

$$osc_{B_R} u \leq CKR^{\alpha},$$

where C *depends on* n, α, p *and* A.

2.2. Uniform Poincaré inequality.

Here, we prove a useful lemma of Poincaré type inequality. Assume that S is a bounded domain in $\mathbb{R}^n$, and there is a constant $C(n,p,U)$ such that the following classical Poincaré inequality holds:

$$\Big(\int_U |u(y)|^p dy\Big)^{\frac{1}{p}} \leq C(n,p,U)\Big(\int_U |\nabla_y u(y)|^p dy\Big)^{\frac{1}{p}},$$

with $\int_U u(y)dy = 0$, where $C(n,p,U)$ depending only on n,p,U, not on u.

Define a class $\mathbb{U}_K$ by

$$\mathbb{U}_K = \big\{\Omega \big| \ \exists \text{ an invertible smooth mapping } T:\Omega \to U, \text{such that} \\ \|T,\ T^{-1}\|_{C^{2,\alpha}} \leq K < \infty\big\}.$$

Then, the following useful uniform Poincaré type inequality holds:

Proposition 2.2. *For any* $1 \leq p < \infty$*, there exists a constant* $C(n,p,U,K)$ *depending only on* n,p,U,K*, such that, for any* $\Omega \in \mathbb{U}_K$*,*

$$\int_\Omega |u(x)|^p dx \leq C(n,p,U,K)\int_\Omega |\nabla u(x)|^p dx$$

or

$$\|u\|_{L^p(\Omega)} \leq C(n,p,U,K)\|\nabla u\|_{L^p(\Omega)} \tag{2.1}$$

holds, provided that $\int_\Omega u(x)dx = 0$.

Proof. Set $\alpha = \frac{1}{|U|}\int_U u\circ T^{-1}(z)dz$ and $J = \frac{\partial x}{\partial y}$. Since

$$\int_U u\circ T^{-1}(y)Jdy = \int_\Omega u(x)dx = 0,$$

one has

$$\alpha|\Omega| = \int_U \big(\alpha - u\circ T^{-1}(y)\big)\, Jdy.$$

Then, by the classical Poincaré inequality for $p=1$ on U , one gets

$$\begin{aligned}
|\alpha||\Omega| &= \Big|\int_U \big(u\circ T^{-1}(y) - \alpha\big)\, Jdy\Big| \\
&\leq \|J\|_{L^\infty}\int_U \Big|u\circ T^{-1}(y) - \alpha\Big| dy \\
&\leq \|J\|_{L^\infty} C(n,1,U)\int_U \big|\nabla_y(u\circ T^{-1}(y))\big|\, dy \\
&\leq \|J\|_{L^\infty} C(n,1,U)\,\big\|\nabla T^{-1}\big\|_{L^\infty}\int_U \big|\nabla_x u\circ T^{-1}(y)\big|\, dy \\
&\leq C(n,1,U)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\Big(\int_U \big|\nabla_x u\circ T^{-1}(y)\big|^p\, dy\Big)^{\frac{1}{p}}|U|^{1-\frac{1}{p}} \\
&\leq C(n,1,U)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\Big\|\frac{1}{J}\Big\|_{L^\infty}^{\frac{1}{p}}\Big(\int_\Omega |\nabla_x u(x)|^p dx\Big)^{\frac{1}{p}}|U|^{1-\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}|\alpha||U|^{\frac{1}{p}} &\le C(n,1,U)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)}\frac{|U|}{|\Omega|}\\ &\le C(n,1,U)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{1+\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)}.\end{aligned} \tag{2.2}$$

On the other hand, by the classical Poincaré inequality for p on U, one has

$$\begin{aligned}\left(\int_\Omega |u(x)|^p\frac{1}{J}dx\right)^{\frac{1}{p}} &= \left(\int_U |u\circ T^{-1}(y)|^p dy\right)^{\frac{1}{p}}\\ &\le \left(\int_U |u\circ T^{-1}(y)-\alpha|^p dy\right)^{\frac{1}{p}} + |\alpha||U|^{\frac{1}{p}}\\ &\le C(n,p,U)\left(\int_U |\nabla_y(u\circ T^{-1}(y))|^p dy\right)^{\frac{1}{p}} + |\alpha||U|^{\frac{1}{p}}\\ &\le C(n,p,U)\|\nabla T^{-1}\|_{L^\infty}\left(\int_\Omega |\nabla_x u(x)|^p\frac{1}{J}dx\right)^{\frac{1}{p}} + |\alpha||U|^{\frac{1}{p}}\\ &\le C(n,p,U)\|\nabla T^{-1}\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)} + |\alpha||U|^{\frac{1}{p}}\\ &\le C(n,p,U)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{1+\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)} + |\alpha||U|^{\frac{1}{p}},\end{aligned}$$

which, together with (2.2) shows

$$\begin{aligned}\left(\int_\Omega |u(x)|^p\frac{1}{J}dx\right)^{\frac{1}{p}} &\le C(n,p,U)\|\nabla T^{-1}\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)} + |\alpha||U|^{\frac{1}{p}}\\ &\le \Big(C(n,p,U)+C(n,1,U)\Big)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}\left\|\frac{1}{J}\right\|_{L^\infty}^{1+\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u(x)\|_{L^p(\Omega)} &= \left(\int_\Omega |u(x)|^p\frac{1}{J}Jdx\right)^{\frac{1}{p}}\\ &\le \|J\|_{L^\infty}^{\frac{1}{p}}\left(\int_\Omega |u(x)|^p\frac{1}{J}dx\right)^{\frac{1}{p}}\\ &\le \Big(C(n,p,U)+C(n,1,U)\Big)\|\nabla T^{-1}\|_{L^\infty}\|J\|_{L^\infty}^{1+\frac{1}{p}}\left\|\frac{1}{J}\right\|_{L^\infty}^{1+\frac{1}{p}}\|\nabla_x u(x)\|_{L^p(\Omega)},\end{aligned}$$

which implies the inequality (2.1). □

Theorem 2.3. *(Uniform Poincaré Inequality) For any $a\in\mathbb{R}$, $1\le p<\infty$, one has*

$$\left\|f(x)-⨍_{\Omega_{a,a+1}} f(x)dx\right\|_{L^p(\Omega_{a,a+1})} \le C\|\nabla f(x)\|_{L^p(\Omega_{a,a+1})}. \tag{2.3}$$

Here

$$\Omega_{a,b} = \{x=(x_1,\cdots,x_n)\in\Omega | a<x_n<b\},$$
$$⨍_{\Omega_{a,a+1}} f(x)dx = \frac{1}{|\Omega_{a,a+1}|}\int_{\Omega_{a,a+1}} f(x)dx,$$

C is a positive constant depending only on n, p, Ω, independent of f, a.

Proof. This can be easily deduced from Proposition 2.2. □

2.3. Basic properties of Ω.

Lemma 2.4. *Under the assumption (1.6), for any $0 \le k \le 2$, if $\varphi \in C^{k,\alpha}(\Omega)$, $\tilde{\varphi} = \varphi \circ T^{-1} \in C^{k,\alpha}(\mathbf{C})$ and vise versa, and*

$$\|\varphi\|_{C^{k,\alpha}(\overline{\Omega})} \sim \|\tilde{\varphi}\|_{C^{k,\alpha}(\overline{\mathbf{C}})}.$$

Similar equivalence holds for H^s norms, $s = 0, 1, 2$.

Proof. The proof follows from simple calculations, and is omitted. □

According to Lemma 2.4, we may abuse a bit of the notations by simply denoting $\|\varphi\|_{C^{k,\alpha}(\Omega)}$ and $\|\tilde{\varphi}\|_{C^{k,\alpha}(\mathbf{C})}$ by $\|\varphi\|_{C^{k,\alpha}}$ or $\|\tilde{\varphi}\|_{C^{k,\alpha}}$, $\|\varphi\|_{H^s(\Omega)}$ and $\|\tilde{\varphi}\|_{H^s(\mathbf{C})}$ by $\|\varphi\|_s$ or $\|\tilde{\varphi}\|_s$ respectively.

Lemma 2.5. *Assume that Ω satisfies (1.6). Then for any $x_0 \in \partial\Omega$, there exists an invertible $C^{2,\alpha}$ map $T_{x_0} : U_{x_0} \to B_{\delta_0} : x \mapsto y$ satisfying the following properties*

$$\begin{cases} T_{x_0}(U_{x_0} \cap \Omega) = B^+_{\delta_0},\ T_{x_0}(U_{x_0} \cap \partial\Omega) = B_{\delta_0} \cap \{y_n = 0\}, & (2.4\text{-}1)\\ \sigma_{ij}\sigma_{in}(x) = \sigma_{ij}\sigma_{in}(y) = 0,\ for\ x \in \partial\Omega\ (i.e.\ y_n = 0), \quad 1 \le j \le n-1, & (2.4\text{-}2)\\ \left\|T_{x_0}, T^{-1}_{x_0}\right\|_{C^{2,\alpha}} \le K, & (2.4\text{-}3)\\ |\sigma_{ij}(x)\xi_j|,\ |\sigma_{ij}(y)\xi_j| \sim |\xi|, \quad \forall x \in U_{x_0}, \forall\ y \in B^+_{\delta_0},\ \forall\ \xi \in \mathbb{R}^n. & (2.4\text{-}4) \end{cases}$$

where U_{x_0} is a neighbourhood of x_0 in $\mathbb{R}^n$, B_{δ_0} is a ball centered at the origin with radius δ_0, $B^+_{\delta_0} = B_{\delta_0} \cap \{y_n > 0\}$, $\sigma_{ij} = \dfrac{\partial y_j}{\partial x_i}$, δ_0 and C are positive numbers independent on $x_0 \in \partial\Omega$.

Proof. By assumption (1.6), $T(x_0) = \tilde{x}_0 \in \partial\mathbf{C}$. Set

$$V_{x_0} = T^{-1}\left(B_{1/4}(\tilde{x}_0) \cap \mathbf{C}\right), \quad S_{x_0} = \overline{V_{x_0}} \cap \partial\Omega, \quad \tilde{S}_{\tilde{x}_0} = \overline{B_{1/4}(\tilde{x}_0)} \cap \partial\mathbf{C},$$

then

$$S_{x_0} = T^{-1}(\tilde{S}_{\tilde{x}_0}).$$

Suppose that $\tilde{x}(y_1, ..., y_{n-1}) = \tilde{x}(y')$ is the standard surface parameter of $\tilde{S}_{\tilde{x}_0}$(and then, of S_{x_0}), $\vec{N}_{x_0}(y')$ is the unit inner normal vector on S_{x_0}. Let

$$\lambda_i(y') = \vec{e}_i(y') \cdot \vec{N}_{x_0}(y'), \qquad 1 \le i \le n,$$

where $\vec{e}_i$, $(1 \le i \le n)$ are the unit coordinate vectors. Then

$$\vec{N}_{x_0}(y') = (\lambda_1(y'), ..., \lambda_n(y')), \qquad \lambda_i(y') \in C^{1,\alpha}.$$

Define $y = T_{x_0}(x)$ by

$$x_i = x_i(y') + y_n^{2-n} \int_{y_1}^{y_1+y_n} \int_{y_2}^{y_2+y_n} \cdots \int_{y_{n-1}}^{y_{n-1}+y_n} \lambda_i(s_1, ..., s_{n-1}) ds_1 ds_2 \cdots ds_{n-1}, \tag{2.5}$$

for $1 \le i \le n$. Since $\|\lambda_i(y')\|_{C^{1,\alpha}} \le C\|T,\ T^{-1}\|_{C^{2,\alpha}}$, there exists a $\delta_0 > 0$ independent of x_0 such that $T^{-1}_{x_0}(y)$ is well defined on B_{δ_0} and $\|T_{x_0},\ T^{-1}_{x_0}\|_{C^{2,\alpha}} \le K$. Then, define $U_{x_0} = T^{-1}(B_{\delta_0})$. Clearly T_{x_0} satisfies (2.4-1) and (2.4-3).

Denote the matrix $(\sigma_{ij}(x))$ by $A(x)$. For any $\xi \in \mathbb{R}^n$,

$$|A(x)\xi| \le |A(x)||\xi| \le C|\xi|, \quad |A^{-1}(x)\xi| \le |A^{-1}(x)||\xi| \le C|\xi|.$$

Then (2.4-4) follows immediately.

To prove (2.4-2), we differentiate (2.5) with respect to y_j and note that $y = (y', 0)$ for $x \in \partial\Omega$,

$$\frac{\partial x_i}{\partial y_j}(y', 0) = \frac{\partial x_i(y')}{\partial y_j} + \lambda_i(y')\delta_{jn}.$$

Since

$$A^{-1}(y',0) = (\sigma_{ij})^{-1}(y',0) = \left(\frac{\partial y_i}{\partial x_j}(y',0)\right)^{-1} = \left(\frac{\partial x_i}{\partial y_j}(y',0)\right),$$

hence

$$\left(\sigma_{ji} \cdot \frac{\partial x_i}{\partial y_n}\right)(y',0) = \sigma_{ji}(y',0) \cdot \lambda_i(y',0) = 0, \quad 1 \le j \le n-1. \tag{2.6}$$

On the other hand, since

$$\sigma_{ni}(y',0) \cdot \frac{\partial x_i}{\partial y_j}(y',0) = 0, \quad 1 \le j \le n-1,$$

$(\sigma_{n1}(y',0), ..., \sigma_{nn}(y',0))$ is the normal direction of S_{x_0} at $x \in S_{x_0}$, that is, $(\sigma_{n1}(y',0), ..., \sigma_{nn}(y',0))$ is parallel to the inner normal $\vec{N}_{x_0}(x) = (\lambda_1(y',0), ..., \lambda_n(y',0))$. Comparing with (2.6) yields

$$\sigma_{ji}(y',0)\sigma_{ni}(y',0) = 0, \quad 1 \le j \le n-1.$$

□

Remark 2.1. Hypothesis (1.6) is stronger than the $C^{2,\alpha}$-regularity hypothesis on Ω. If one only assumes that $\Omega \in C^{2,\alpha}$, then C and δ_0 in lemma 2.5, in general, may depend on $x_0 \in \partial\Omega$.

Lemma 2.6. *There exists a $\delta_1 > 0$ such that*

$$\delta_0 \sim \delta_1, \quad \Omega = \left(\bigcup_{x_0 \in \partial\Omega} T_{x_0}^{-1}\left(B^+_{\frac{\delta_0}{2}}\right)\right) \bigcup \left(\bigcup_{B_{2\delta_1} \subset \Omega} B_{\delta_1}\right).$$

where T_{x_0} and δ_0 are the same as in Lemma 2.5.

Proof. Since $\|T,\ T^{-1}\|_{C^{2,\alpha}} \le C$, $|x_1 - x_2| \sim |T(x_1) - T(x_2)|$, there exists a constant $\delta \sim \delta_0$ such that

$$B_\delta(x_0) \cap \Omega \subset T_{x_0}^{-1}\left(B^+_{\frac{\delta_0}{2}}\right), \quad \forall\ x_0 \in \partial\Omega.$$

Taking $\delta_1 = \dfrac{\delta}{2}$ yields the Lemma. □

3. The existence of subsonic flow for small incoming mass flux

There are two major obstacles to solve the Problem 1. First, the ellipticity of the equation (1.5) is not guaranteed beforehand, since there is no a priori L^∞ bound for $\nabla\varphi$, the gradient of the solution to the Problem 1. Second, the nozzle region is unbounded, and can not be transformed to a bounded domain by Kelvin-like transformations. In order to overcome these difficulties, we first truncate the coefficients of the equation in (1.5) to ensure the strong ellipticity, and then, truncate the domain Ω to a series of bounded domains Ω_L, with additional boundary conditions. Therefore, to solve the Problem 1 becomes to study a series of approximate strong elliptic problems in bounded domains and their uniform estimates, which ensure to pass the limit of the approximate solutions to the Problem 1.

3.1. **A subsonic truncation and approximate solutions.**

3.1.1. *A subsonic truncation.* By normalizing the equation if necessary [5], [27], one can assume that the critical sound speed of the flow is one. Thus, the density-speed relation (1.3), $\rho = \rho(q^2)$, is positive, sufficiently smooth and nonincreasing in $q = |\nabla\varphi| \in [0,1]$. However, the potential equation is not uniformly elliptic as q approaches to 1. To guaranteed the uniformly ellipticity, we truncate the coefficients as follows.

Define two functions $\Theta(s^2)$ and $F(q^2)$ as follows

$$\Theta(s^2) = \begin{cases} \rho(s^2), & \text{if } s^2 < 1 - 2\tilde{\delta}_0, \\ \text{monotone and smooth}, & \text{if } 1 - 2\tilde{\delta}_0 \le s^2 \le 1 - \tilde{\delta}_0, \\ \rho(1-\tilde{\delta}_0), & \text{if } s^2 > 1 - \tilde{\delta}_0, \end{cases} \tag{3.1}$$

and

$$F(q^2) = \frac{1}{2}\int_0^{q^2} \Theta(s^2)ds^2, \tag{3.2}$$

where $\tilde{\delta}_0 > 0$. Moreover, $\Theta(s^2)$ is a smooth non-increasing functions and $F(q^2)$ is a smooth increasing function. Set

$$a_{ij}(\nabla\varphi) = \Theta(|\nabla\varphi|^2)\delta_{ij} + 2\Theta'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi.$$

It is easy to check the following facts,

$$F\left(q^2\right) \sim q^2, \quad \frac{1}{C(\tilde{\delta}_0)} < \Theta(s^2), \quad \Theta(s^2) + 2\Theta'(s^2)s^2 < C(\tilde{\delta}_0), \tag{3.3}$$

and there exist two positive constants λ and Λ, such that

$$\lambda|\xi|^2 < a_{ij}(\nabla\varphi)\xi_i\xi_j < \Lambda|\xi|^2, \tag{3.4}$$

where $C(\tilde{\delta}_0)$, λ and Λ depend only on the subsonic truncation parameter $\tilde{\delta}_0$. Note that a solution of the potential equation derived from the new density-speed relation $\Theta(q^2)$ is also a solution of the actual potential equation provided that $|\nabla\varphi|^2 \le 1 - 2\tilde{\delta}_0$. Therefore, in the end of this section, we will show that the solution of the truncated problem satisfies $|\nabla\varphi|^2 \le 1 - 2\tilde{\delta}_0$, as long as the incoming mass flux m_0 is suitable small. Consequently, the subsonic truncation can be removed.

3.1.2. *Domain truncation.* Our strategy to deal with the unbounded domain here is to construct a series of truncated problems to approximate the Problem 1 with subsonic truncation.

Let $L > 0$ be sufficiently large. Define

$$\Omega_L = \left\{x \in \Omega \;\middle|\; |x_n| < L\right\}, \quad S_L^{\pm} = \Omega \cap \{x_n = \pm L\}, \quad S_L = S_L^- \cup S_L^+.$$

Consider the following truncated problem with $m_0 > 0$.

Problem 2. *Find a φ such that,*

$$\begin{cases} div(\Theta(|\nabla\varphi|^2)\nabla\varphi) = 0, & x \in \Omega_L, \\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & \partial\Omega \cap \partial\Omega_L, \\ \Theta(|\nabla\varphi|^2)\dfrac{\partial\varphi}{\partial x_n} = \dfrac{m_0}{|S_L^+|}, & on \;\; S_L^+ \\ \varphi = 0, & on \;\; S_L^-. \end{cases} \tag{3.5}$$

The additional boundary condition on S_L^+ implies the mass flux of the flow remains m_0.

Clearly, the truncated problem 2 is a strong quasilinear elliptic problem in a bounded domain. From now on, instead of the original Problem 1, we consider a series of the truncated Problem 2 for any fixed sufficiently large L. With some uniform estimates of the approximate solutions, we can conclude that the solution of the truncated problem 2 converges to the original Problem 1.

3.2. **Truncated variational problem.**

In this subsection, we solve the truncated problem 2 by a variational method. Define

$$H_L = \left\{\varphi \in H^1(\Omega_L) : \ \varphi\big|_{S_L^-} = 0\right\}.$$

Then, H_L is a Hilbert space under H^1-norm. The additional boundary condition on S_L^- is understood in the sense of traces. Define a functional $J(\psi)$ on H_L as

$$J(\psi) = \int_{\Omega_L} F(|\nabla\psi|^2)dx - \frac{m_0}{|S_L^+|}\int_{S_L^+} \psi dx',$$

where $F(q^2)$ is defined by (3.2) and $x' = (x_1, x_2, \dots, x_{n-1})$. The existence of solution to problem 2 is equivalent to the following variational problem:

Problem 3. *Find a minimizer $\varphi \in H_L$ such that*

$$J(\varphi) = \min_{\psi\in H_L} J(\psi).$$

Theorem 3.1. *Problem 3 has a nonnegative minimizer $\varphi \in H_L$. Moreover,*

$$\frac{1}{|\Omega_L|}\int_{\Omega_L} |\nabla\varphi|^2 dx \le Cm_0^2, \tag{3.6}$$

where the constant C does not depend on L.

Proof. Step 1. $J(\psi)$ is coercive on H_L. In fact, by Lemma 2.4, for any $\psi \in H_L$,

$$\begin{aligned} \left|\int_{S_L^+} \psi dx'\right| &\le C\left|\int_{B(0,1)} \tilde\psi dy'\right| \le C\left|\int_{B(0,1)}\int_{-L}^{L} \tilde\partial_n\tilde\psi dy_n dy'\right| \\ &\le C\int_{\mathbf{C}_L} |\tilde\nabla\tilde\psi| dy \le C\int_{\Omega_L} |\nabla\psi| dx \\ &\le C|\Omega_L|^{\frac12}\|\nabla\psi\|_{L^2} \end{aligned} \tag{3.7}$$

Therefore, applying (3.7) and Cauchy inequality yields

$$\begin{aligned} J(\psi) &= \int_{\Omega_L} F(|\nabla\psi|^2)dx - \frac{m_0}{|S_L^+|}\int_{S_L^+}\psi dx' \\ &\ge \lambda\int_{\Omega_L} |\nabla\psi|^2 dx - C(m_0, |S_L^+|, |\Omega_L|)\|\nabla\psi\|_{L^2} \\ &\ge \frac{\lambda}{2}\|\nabla\psi\|_{L^2}^2 - \frac{1}{\lambda}C(m_0, |S_L^+|, |\Omega_L|), \end{aligned}$$

which implies $J(\psi)$ is coercive.

Step 2. The existence of the minimizer $\varphi \in H_L$. Since $J(\psi)$ is coercive in H_L, there is a minimizer sequence $\{\varphi_n\} \subset H_L$ such that

$$J(\varphi_n) \to \alpha = \inf_{\psi\in H_L} J(\psi) > -\infty.$$

Then,

$$\begin{aligned}\|\nabla\varphi_n\|_{L^2}^2 &\le \frac{2}{\lambda}J(\varphi_n)+\frac{2}{\lambda^2}C(m_0,|S_L^+|,|\Omega_L|)\\ &\le \frac{2}{\lambda}J(0)+\frac{2}{\lambda^2}C(m_0,|S_L^+|,|\Omega_L|)\\ &= \frac{1}{\lambda^2}C(m_0,|S_L^+|,|\Omega_L|).\end{aligned}$$

Therefore, there exists a subsequence, denoted by $\{\varphi_n\}$ converges weakly to some $\varphi\in H_L$ and

$$\|\nabla\varphi\|_{L^2}^2 \le \frac{1}{\lambda^2}C(m_0,|S_L^+|,|\Omega_L|).$$

By Fatou's Lemma, it is easy to check that

$$\int_{\Omega_L} F(|\nabla\varphi|^2)dx \le \liminf_{n\to\infty}\int_{\Omega_L} F(|\nabla\varphi_n|^2)dx. \tag{3.8}$$

On the other hand,

$$\begin{aligned}\int_{S_L^+}(\varphi_n-\varphi)^2dx' &\le C\int_{B(0,1)}(\tilde\varphi_n-\tilde\varphi)^2dy'\\ &\le C\Big|\int_{B(0,1)}\int_{-L}^{L}(\tilde\varphi_n-\tilde\varphi)\tilde\partial_n(\tilde\varphi_n-\tilde\varphi)dy_ndy'\Big|\\ &\le C\int_{\mathbf{C}}|\tilde\varphi_n-\tilde\varphi||\tilde\nabla\tilde\varphi_n-\tilde\nabla\tilde\varphi|dy\\ &\le C\int_{\Omega_L}|\varphi_n-\varphi||\nabla\varphi_n-\nabla\varphi|dx\\ &\le C\Big(\int_{\Omega_L}|\varphi_n-\varphi|^2dx\Big)^{\frac12}\Big(\int_{\Omega_L}|\nabla\varphi_n-\nabla\varphi|^2dx\Big)^{\frac12}\to 0,\end{aligned}$$

as $n\to\infty$. Then,

$$\int_{S_L^+}|\varphi_n-\varphi|dx'\to 0,\qquad \text{as } n\to\infty. \tag{3.9}$$

Therefore, it follows from (3.8) and (3.9) that

$$J(\varphi)\le\liminf_{n\to\infty}J(\varphi_n)=\alpha.$$

i.e.

$$J(\varphi)=\min_{\psi\in H_L}J(\psi)=\alpha.$$

Step 3. $\varphi^+=\max\{\varphi,0\}$ is a nonnegative minimizer in H_L. Indeed, since $\varphi\in H_L$, $\varphi^+\in H_L$, and

$$\begin{aligned}&|\nabla\varphi^+|^2\le|\nabla\varphi|^2,\qquad F(|\nabla\varphi^+|^2)\le F(|\nabla\varphi|^2),\\ &\frac{m_0}{|S_L^+|}\int_{S_L^+}\varphi^+dx'\ge\frac{m_0}{|S_L^+|}\int_{S_L^+}\varphi dx'.\end{aligned}$$

Hence,

$$J(\varphi^+)\le J(\varphi).$$

Since φ is a minimizer, $J(\varphi^+)=J(\varphi)$, which implies that $\varphi^+\ge0$ is also a minimizer.

Step 4. By direct computations,

$$\begin{aligned}\int_{\Omega_L}F(|\nabla\varphi|^2)dx &= J(\varphi)+\frac{m_0}{|S_L^+|}\int_{S_L^+}\varphi dx'\le J(0)+\frac{m_0}{|S_L^+|}\int_{S_L^+}\varphi dx'\\ &\le C\frac{m_0}{|S_L^+|}|\Omega_L|^{\frac12}\|\nabla\varphi\|_{L^2}.\end{aligned}$$

It follows from (3.3) and (3.4) that

$$\|\nabla\varphi\|_{L^2}^2 \le \frac{1}{\lambda}\int_{\Omega_L} F(|\nabla\varphi|^2)dx \le C\frac{1}{\lambda}\frac{m_0}{|S_L^+|}|\Omega_L|^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}.$$

That is

$$\|\nabla\varphi\|_{L^2}^2 \le C\frac{m_0^2}{\lambda^2|S_L^+|^2}|\Omega_L|,$$

i.e.

$$\frac{1}{|\Omega_L|}\int_{\Omega_L}|\nabla\varphi|^2dx \le C\frac{m_0^2}{\lambda^2|S_L^+|^2} \le C\frac{m_0^2}{\lambda^2 S_{min}^2},$$

where S_{min} denotes the minimal of $|S_L^+|$. □

Remark 3.1. The estimate (3.6) is the key estimate for the existence of the classical solution to Problem 2. Indeed, the potential φ is essentially unbounded, one can not expect to get uniform bounds on $\|\nabla\varphi\|_{L^\infty}$ through $\|\varphi\|_{L^\infty}$ as in the standard elliptic theory.

Proposition 3.2. *$\varphi \in H_L$ is a weak solution to the equations in (3.5) in the following sense:*

$$\int_{\Omega_L}\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx - \frac{m_0}{|S_L^+|}\int_{S_L^+}\psi dx' = 0, \qquad \forall\ \psi \in H_L \tag{3.10}$$

Proof. This is a standard variation problem. In fact, for any $t \in \mathbb{R},\ t > 0$ and any $\psi \in H_L$, $\varphi + t\psi \in H_L$. Then,

$$0 \le J(\varphi + t\psi) - J(\varphi) = \int_{\Omega_L} F(|\nabla\varphi + t\nabla\psi|^2) - F(|\nabla\varphi|^2)dx - \frac{m_0 t}{|S_L^+|}\int_{S_L^+}\psi dx'. \tag{3.11}$$

Mean value theorem yields that

$$\begin{aligned}
&\int_{\Omega_L} F(|\nabla\varphi + t\nabla\psi|^2) - F(|\nabla\varphi|^2)dx \\
&= \int_{\Omega_L}\int_0^1 F'(\theta|\nabla\varphi + t\nabla\psi|^2 + |\nabla\varphi|^2(1-\theta))d\theta(|\nabla\varphi + t\nabla\psi|^2 - |\nabla\varphi|^2)dx \qquad (3.12)\\
&= \int_{\Omega_L}\int_0^1 F'(|\nabla\varphi|^2 + \theta(t^2|\nabla\psi|^2 + 2t\nabla\psi\cdot\nabla\varphi))d\theta(t^2|\nabla\psi|^2 + 2t\nabla\psi\cdot\nabla\varphi)dx.
\end{aligned}$$

Since $|F'(\cdot)| \le C$, $\nabla\varphi,\ \nabla\psi \in L^2(\Omega_L)$, substituting (3.12) into (3.11) shows that

$$\begin{aligned}
0 &\le \liminf_{t\to 0^+}\frac{1}{t}(J(\varphi + t\psi) - J(\varphi)) \\
&= \liminf_{t\to 0^+}\int_{\Omega_L}\int_0^1 F'(|\nabla\varphi|^2 + \theta(t^2|\nabla\psi|^2 + 2t\nabla\psi\cdot\nabla\varphi))d\theta(2\nabla\psi\cdot\nabla\varphi)dx \\
&\quad - \int_{S_L^+}\frac{m_0}{|S_L^+|}\psi dx' \\
&= \int_{\Omega_L}\int_0^1 F'(|\nabla\varphi|^2)d\theta(2\nabla\psi\cdot\nabla\varphi))dx - \frac{m_0}{|S_L^+|}\int_{S_L^+}\psi dx' \\
&\quad \text{(by Lebesgue's theorem)} \\
&= \int_{\Omega_L}\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx - \frac{m_0}{|S_L^+|}\int_{S_L^+}\psi dx'.
\end{aligned}$$

Therefore, for any $\psi \in H_L$,

$$\int_{\Omega_L}\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx - \frac{m_0}{|S_L^+|}\int_{S_L^+}\psi dx' = 0.$$

□

3.3. H^2 **regularity of the weak solution.** We are now ready to improve the regularity of the minimizer φ. Indeed, one has

Proposition 3.3. *$\varphi \in H^2\left(\Omega_{L/2}\right)$. Moreover,*

$$\tilde{\partial}_n \tilde{\varphi}(y)\Big|_{y_n=0} = \frac{\partial \tilde{\varphi}(y)}{\partial y_n}\Big|_{y_n=0} = 0. \tag{3.13}$$

To prove this, one needs the following estimates in Lemma 3.4, 3.5.

Lemma 3.4. *(Interior Estimate) For any $B_{2R}(x_0) \subset \Omega_L,\ R \le \delta_1$,*

$$\nabla^2 \varphi \in L^2(B_R).$$

Here δ_1 is the same number in the Lemma 2.6.

Proof. For any $B_{2R}(x_0) \subset \Omega_L$, $v \in H_0^1\left(B_{\frac{3}{2}R}\right)$, $h < \frac{1}{2}R$, one has

$$0 = \int_{B_{2R}} \Theta(|\nabla\varphi|^2)\nabla\varphi \cdot \nabla(\delta_{-h} v)dx = -\int_{B_{2R}} \delta_h(\Theta(|\nabla\varphi|^2)\nabla\varphi) \cdot \nabla v dx, \tag{3.14}$$

where $\delta_h v(x) \overset{\text{def}}{=} \frac{1}{h}(v(x+h\vec{e}_k) - v(x))$ is the k-th difference quotient, $k = 1, 2, \cdots, n$.

Set

$$\tilde{q} = t\nabla\varphi^h + (1-t)\nabla\varphi, \quad \varphi^h(x) = \varphi(x + h\vec{e}_k),$$

$$a_{ij}(\tilde{q}, t) = \Theta(\tilde{q}^2)\delta_{ij} + 2\Theta'(\tilde{q}^2)\tilde{q}_i\tilde{q}_j, \quad a_{ij} = a_{ij}(\tilde{q}) = \int_0^1 a_{ij}(\tilde{q}, t)dt.$$

Then, direct calculations give

$$\delta_h(\Theta(|\nabla\varphi|^2)\nabla\varphi) = \int_0^1 a_{ij}(\tilde{q}, t)dt\partial_j(\delta_h\varphi) = a_{ij}\partial_j(\delta_h\varphi). \tag{3.15}$$

Therefore, substituting (3.15) into (3.14), one has

$$\int_{B_{2R}} a_{ij}\partial_j(\delta_h\varphi)\partial_j v dx = 0. \tag{3.16}$$

Take $v = \eta^2\delta_h\varphi$ in (3.16), where $\eta \in C_0^\infty\left(B_{\frac{3}{2}R}\right)$, $\eta \equiv 1$ in B_R, $|D\eta| \le \frac{C}{R}$. Then,

$$\begin{aligned} 0 &= \int_{B_{2R}} a_{ij}\partial_j(\delta_h\varphi)\partial_j(\eta^2\delta_h\varphi)dx \\ &= \int_{B_{2R}} \eta^2 a_{ij}\partial_j(\delta_h\varphi)\partial_j(\delta_h\varphi)dx + 2\int_{B_{2R}} a_{ij}\partial_j(\delta_h\varphi)\eta\partial_j\eta\delta_h\varphi dx. \end{aligned} \tag{3.17}$$

It follows from Hölder inequality and (3.17) that

$$\begin{aligned} &\int_{B_{2R}} \eta^2 a_{ij}\partial_j(\delta_h\varphi)\partial_j(\delta_h\varphi)dx = -2\int_{B_{2R}} a_{ij}\partial_j(\delta_h\varphi)\eta\partial_j\eta\delta_h\varphi dx \\ \le\ & 2\left(\int_{B_{2R}} \eta^2 a_{ij}\partial_i(\delta_h\varphi)\partial_j(\delta_h\varphi)dx\right)^{\frac{1}{2}}\left(\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta(\delta_h\varphi)^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

namely,

$$\int_{B_{2R}} \eta^2 a_{ij}\partial_i(\delta_h\varphi)\partial_j(\delta_h\varphi)dx \le 4\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta(\delta_h\varphi)^2 dx.$$

Consequently, by the strong ellipticity of a_{ij}, one gets

$$\begin{aligned}\lambda\int_{B_R}|\nabla(\delta_h\varphi)|^2dx &\le \int_{B_{2R}} a_{ij}\partial_i(\delta_h\varphi)\partial_j(\delta_h\varphi)dx\\ &\le 4\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta(\delta_h\varphi)^2dx\\ &\le C\frac{\Lambda}{R^2}\int_{B_{\frac{3}{2}R}}(\delta_h\varphi)^2dx\\ &\le C\frac{\Lambda}{R^2}\int_{B_{\frac{3}{2}R}}|\nabla\varphi|^2dx,\end{aligned}$$

and then

$$\int_{B_R}|\nabla(\delta_h\varphi)|^2dx \le C\frac{\Lambda}{\lambda R^2}\int_{B_{2R}}|\nabla\varphi|^2dx, \quad \forall\ h<R. \tag{3.18}$$

Therefore, according to (3.18) and H^1 regularity of minimizer φ, we can conclude that $\nabla^2\varphi\in L^2(B_R)$. □

Next, we derive the boundary estimate of the minimizer φ.

Lemma 3.5. *(Boundary Estimate) For any $x_0\in\partial\Omega_{L/2}$,*

$$\nabla^2\varphi\in L^2\left(B_{\frac{\delta_0}{2}}(x_0)\cap\Omega_L\right). \tag{3.19}$$

Proof. Set $U_{x_0,\delta_0}=B_{\delta_0}(x_0)\cap\Omega_L$, and

$$T_{x_0}:U_{x_0,\delta_0}\to B^+_{\delta_0}:x\mapsto y,\ y=T_{x_0}(x),\ \sigma_{ij}(y)=\frac{\partial y_j}{\partial x_i}(y),\ J(y)=\frac{\partial x}{\partial y}.$$

For simplification, we write U_{x_0,δ_0} and $B^+_{\delta_0}$ as U and B^+ respectively in the remaining of the proof.

Then for any $\psi\in H^1_0(U)$, $\tilde\psi=\psi\circ T^{-1}_{x_0}$,

$$0=\int_U\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx=\int_{B^+}\Theta(|\sigma_{\alpha\beta}\tilde\partial_\beta\tilde\varphi|^2)\sigma_{ij}\tilde\partial_j\tilde\varphi\sigma_{il}\tilde\partial_l\tilde\psi Jdy,$$

where $\tilde\varphi=\varphi\circ T^{-1}_{x_0}$. Taking $\tilde\psi$ as the k-th difference quotient

$$\tilde\delta_{-h}\tilde\psi\overset{\text{def}}{=}\frac{1}{h}\left(\tilde\psi(y)-\tilde\psi(y-h\vec e_k)\right)\ \text{for } k=1,2,\cdots,n-1,$$

we may get from the property and the "integrate by parts" formula for difference quotient that for suitable small $h>0$

$$\begin{aligned}0&=\int_{B^+}\tilde\delta_h\left(\Theta(|\sigma_{\alpha\beta}\tilde\partial_\beta\tilde\varphi|^2)\sigma_{ij}\sigma_{il}J\tilde\partial_j\tilde\varphi\right)\tilde\partial_l\tilde\psi dy\\ &=\int_{B^+}\tilde\delta_h\left(\Theta(|\sigma_{\alpha\beta}\tilde\partial_\beta\tilde\varphi|^2)\sigma_{ij}\tilde\partial_j\tilde\varphi\right)\sigma_{il}\tilde\partial_l\tilde\psi Jdy\\ &\quad+\int_{B^+}\left(\Theta(|\sigma_{\alpha\beta}\tilde\partial_\beta\tilde\varphi|^2)\sigma_{ij}\tilde\partial_j\tilde\varphi\right)^h\tilde\delta_h(\sigma_{il}J)\tilde\partial_l\tilde\psi dy\\ &=I+II.\end{aligned}$$

Set

$$\tilde A_{ij}=\int_0^1\tilde A_{ij}(t)dt,\quad \tilde A_{ij}(t)=\Theta(|q(t)|^2)\delta_{ij}+2\Theta'(|q(t)|^2)q_i(t)q_j(t),$$
$$q(t)=tq^h(y)+(1-t)q(y)=(q_1(t),q_2(t),\cdots,q_n(t)),$$
$$q^h(y)=\sigma_{\alpha\beta}(y+h\vec e_k)\tilde\partial_\beta\tilde\varphi(y+h\vec e_k),\ q(y)=\sigma_{\alpha\beta}(y)\tilde\partial_\beta\tilde\varphi(y).$$

Now, the term I can be rewritten as

$$\begin{aligned} I &= \int_{B^+} \tilde{\delta}_h \left(\Theta(|\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}|^2)\sigma_{ij}\tilde{\partial}_j\tilde{\varphi} \right) \sigma_{il}\tilde{\partial}_l\tilde{\psi} J dy \\ &= \int_{B^+} \tilde{A}_{ij}\frac{1}{h}\left(q_j^h - q_j\right)\sigma_{il}\tilde{\partial}_l\tilde{\psi} J dy \\ &= \int_{B^+} \tilde{A}_{ij}\sigma_{js}\tilde{\partial}_s(\tilde{\delta}_h\tilde{\varphi})\sigma_{il}\tilde{\partial}_l\tilde{\psi} J dy + \int_{B^+} \tilde{A}_{ij}\left(\tilde{\partial}_s\tilde{\varphi}\right)^h \tilde{\delta}_h(\sigma_{js})\sigma_{il}\tilde{\partial}_l\tilde{\psi} J dy \\ &= I_1 + I_2. \end{aligned}$$

Set

$$\tilde{\psi} = \tilde{\eta}^2\tilde{u}_h, \quad \tilde{u}_h = \tilde{\delta}_h\tilde{\varphi},$$

$$\tilde{\eta} \in C_0^\infty(B^+), \quad \tilde{\eta} \equiv 1 \ \text{ in } \ \tilde{B}^+ = T_{x_0}\left(U_{x_0,\frac{\delta_0}{2}} \cap \Omega_L\right), \quad |\tilde{\nabla}\tilde{\eta}| \le 2$$

and

$$\psi = \tilde{\psi} \circ T_{x_0} = \eta^2 u_h, \quad u_h = \tilde{u}_h \circ T_{x_0},$$

$$\eta = \tilde{\eta} \circ T_{x_0} \in C_0^\infty(U \cap \Omega_L), \quad \eta \equiv 1 \ \text{ in } \ U_{x_0,\frac{\delta_0}{2}} \cap \Omega_L, \quad |\nabla\eta| \le 2.$$

Then

$$\begin{aligned} I_1 &= \int_{B^+} \tilde{A}_{ij}\sigma_{js}\tilde{\partial}_s\tilde{u}_h\sigma_{il}\tilde{\partial}_l\tilde{\psi} J dy \\ &= \int_U \tilde{A}_{ij}\partial_i\psi\partial_j u_h dx \\ &= \int_U \tilde{A}_{ij}\eta^2\partial_i u_h\partial_j u_h dx + 2\int_U \tilde{A}_{ij}\eta u_h\partial_j u_h\partial_i\eta dx \\ &= I_{11} + I_{12}, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{B^+} \tilde{A}_{ij}(\tilde{\partial}_s\tilde{\varphi})^h\tilde{\delta}_h(\sigma_{js})\sigma_{il}\tilde{\partial}_l\tilde{u}_h\tilde{\eta}^2 J dy + 2\int_{B^+} \tilde{A}_{ij}(\tilde{\partial}_s\tilde{\varphi})^h\tilde{\delta}_h(\sigma_{js})\sigma_{il}\tilde{u}_h\tilde{\eta}\tilde{\partial}_l\tilde{\eta} J dy \\ &= I_{21} + I_{22}. \end{aligned}$$

Due to the strong ellipticity,

$$I_{11} = \int_U \tilde{A}_{ij}\eta^2\partial_i u_h\partial_j u_h dx \ge \lambda\|\eta\nabla u_h\|^2_{L^2(U)}. \tag{3.20}$$

To estimate the term I_{11}, we will deal with I_{12}, I_{21}, I_{22} and II first.

By Hölder inequality and the strong ellipticity of $\tilde{A}_{ij}$, we have

$$\begin{aligned} |I_{12}| &= \left| 2\int_U \tilde{A}_{ij}\eta u_h\partial_j u_h\partial_i\eta dx \right| \\ &\le 2\left(\int_U \tilde{A}_{ij}\eta^2\partial_i u_h\partial_j u_h dx\right)^{\frac{1}{2}}\left(\int_U \tilde{A}_{ij}u_h^2\partial_j\eta\partial_i\eta dx\right)^{\frac{1}{2}} \\ &\le C I_{11}^{\frac{1}{2}}\left(\Lambda\|\nabla\eta\|^2_{L^\infty}\cdot\|u_h\|^2_{L^2(U)}\right)^{\frac{1}{2}} \\ &\le C I_{11}^{\frac{1}{2}}\left(\Lambda\|\nabla\varphi\|^2_{L^2(U)}\right)^{\frac{1}{2}} \\ &\le \frac{1}{4}I_{11} + C\Lambda\|\nabla\varphi\|^2_{L^2(U)}, \end{aligned}$$

and

$$
\begin{aligned}
|I_{21}| &= \left| \int_{B^+} \tilde{A}_{ij}(\tilde{\partial}_s\tilde{\varphi})^h \tilde{\delta}_h(\sigma_{js})\sigma_{il}\tilde{\partial}_l\tilde{u}_h\tilde{\eta}^2 J dy \right| \\
&\le \left(\int_{B^+} \tilde{A}_{ij}\sigma_{il}\tilde{\partial}_l\tilde{u}_h\sigma_{js}\tilde{\partial}_s\tilde{u}_h\tilde{\eta}^2 J dy \right)^{\frac{1}{2}} \left(\int_{B^+} \tilde{A}_{ij}\tilde{\eta}^2\tilde{\delta}_h(\sigma_{js})(\tilde{\partial}_s\tilde{\varphi})^h\tilde{\delta}_h(\sigma_{il})(\tilde{\partial}_l\tilde{\varphi})^h J dy \right)^{\frac{1}{2}} \\
&\le C I_{11}^{\frac{1}{2}} \left(\Lambda |K|^2 \|\nabla\varphi\|^2_{L^2(U)} \right)^{\frac{1}{2}} \\
&\le \frac{1}{4} I_{11} + C\left(\Lambda |K|^2 \|\nabla\varphi\|^2_{L^2(U)} \right),
\end{aligned}
$$

where K is $C^{2,\alpha}$ norm of the boundary (See assumption (1.6)).

$$
I_{22} = 2\int_{B^+} \tilde{A}_{ij}(\tilde{\partial}_s\tilde{\varphi})^h\tilde{\delta}_h(\sigma_{js})\sigma_{il}\tilde{u}_h\tilde{\eta}\tilde{\partial}_l\tilde{\eta} J dy \le C\Lambda K \|\nabla\varphi\|^2_{L^2(U)}.
$$

Next, we estimate II.

$$
\begin{aligned}
II &= \int_{B^+} \left(\Theta(|\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}|^2)\sigma_{ij}\tilde{\partial}_j\tilde{\varphi} \right)^h \tilde{\delta}_h(\sigma_{il}J)\tilde{\partial}_l(\eta^2\tilde{u}_h)dy \\
&= \int_{B^+} \tilde{\eta}^2 \left(\Theta(|\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}|^2)\sigma_{ij}\tilde{\partial}_j\tilde{\varphi} \right)^h \tilde{\delta}_h(\sigma_{il}J)\tilde{\partial}_l\tilde{u}_h dy \\
&\quad + 2\int_{B^+} \tilde{\eta} \left(\Theta(|\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}|^2)\sigma_{ij}\tilde{\partial}_j\tilde{\varphi} \right)^h \tilde{\delta}_h(\sigma_{il}J)\tilde{u}_h\tilde{\partial}_l\tilde{\eta} dy \\
&= II_1 + II_2.
\end{aligned}
$$

Then direct computations yield that

$$
\begin{aligned}
II_1 &= \int_{B^+} \tilde{\eta}^2 \left(\Theta(|\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}|^2)\sigma_{ij}\tilde{\partial}_j\tilde{\varphi} \right)^h \tilde{\delta}_h(\sigma_{il}J)\tilde{\partial}_l\tilde{u}_h dy \\
&\le C\Lambda K \int_{B^+} \tilde{\eta}^2 |\tilde{\nabla}\tilde{\varphi}||\tilde{\nabla}\tilde{u}_h| dy \\
&\le C\Lambda K \int_U \eta^2 |\nabla\varphi||\nabla u_h| dx \\
&\le \frac{\lambda}{4}\|\eta\nabla u_h\|^2_{L^2(U)} + \frac{C}{\lambda}\Lambda^2 K^2 \|\nabla\varphi\|^2_{L^2(U)},
\end{aligned}
$$

and

$$
II_2 \le 2\Lambda K \int_{B^+} |\tilde{\nabla}\tilde{\varphi}||\tilde{\nabla}\tilde{\eta}||\tilde{u}_h| dy \le C\Lambda K \|\nabla\varphi\|^2_{L^2(U)}.
$$

Therefore, noticing that

$$
0 = I_1 + I_2 + II_1 + II_2 = I_{11} + I_{12} + I_{21} + I_{22} + II_1 + II_2,
$$

and applying the estimates as above, we get

$$
\begin{aligned}
I_{11} &\le C\Lambda(K^2+1)\|\nabla\varphi\|^2_{L^2(U)} + \frac{\lambda}{2}\|\eta\nabla u_h\|^2_{L^2(U)} + \frac{C}{\lambda}\Lambda^2 K^2 \|\nabla\varphi\|^2_{L^2(U)} \\
&\le \frac{\lambda}{2}\|\eta\nabla u_h\|^2_{L^2(U)} + C(K^2+1)\|\nabla\varphi\|^2_{L^2(U)}.
\end{aligned}
$$

Then, combining with (3.20), we obtain the gradient estimates for the kth difference quotient $\tilde{u}_h$ $(k = 1, 2, \cdots, n-1)$,

$$
\|\tilde{\nabla}\tilde{u}_h\|^2_{L^2(\tilde{B}^+)} \le C(K^2+1)\|\nabla\varphi\|^2_{L^2(U)}.
$$

Furthermore, the following derivatives estimates hold,

$$
\sum_{k=1}^{n-1} \|\tilde{\nabla}(\tilde{D}_k\tilde{\varphi})\|^2_{L^2\left(\tilde{B}^+\right)} \le C(K^2+1)\|\nabla\varphi\|^2_{L^2(U)}. \tag{3.21}
$$

For the $\tilde{D}^2_{nn}\tilde{\varphi}$, by the potential equation, and the estimates for $\tilde{D}^2_{kj},\ 1 \le k \le n-1,\ 1 \le j \le n$,

$$\|\tilde{D}^2_{nn}\tilde{\varphi}\|^2_{L^2(\tilde{B}^+)} \le C(K^2+1)\|\nabla\varphi\|^2_{L^2(U)}. \tag{3.22}$$

Combining the estimates (3.21), (3.22) and the H^1-estimate (3.6) yields (3.19). □

Proof of Proposition 3.3: It follows from Lemma 3.4, Lemma 3.5 and a finite cover argument. □

3.4. **Local average estimate.** Set

$$\Omega_{x_0,r} = \Omega \cap \{x = (x', x_n):\ |x_n - x_{0,n}| < r\},\ \text{where } x_0 = (x'_0, x_{0,n}) \in \Omega.$$

Proposition 3.6. *(Local average estimate). For any $x_0 \in \Omega$ with $|x_{0,n}| < \frac{1}{2}L$, one has*

$$\frac{1}{|\Omega_{x_0,1}|}\int_{\Omega_{x_0,1}} |\nabla\varphi|^2 dx \le Cm_0^2, \tag{3.23}$$

where C does not depend on x_0, L.

Proof. For any $-\frac{L}{2} < a-1 < a < b < b+1 < \frac{L}{2}$, define $\eta \in C^\infty(\Omega_L)$, $0 \le \eta \le 1$, $|\nabla\eta| \le 2$ by

$$\eta(x) = \begin{cases} 0, & x_n \le a-1, \\ 1, & a \le x_n \le b, \\ 0, & x_n \ge b+1. \end{cases}$$

For any constants k_1, k_2, set

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) - k_1, & x_n \le a, \\ \varphi(x) - k_1 - \dfrac{k_2-k_1}{b-a}(x_n - a), & a \le x_n \le b, \\ \varphi(x) - k_2, & x_n \ge b. \end{cases}$$

Then $\eta^2\hat{\varphi} \in H^1(\Omega_L)$ and $\left(\eta^2\hat{\varphi}\right)|_{x_n=\pm L} = 0$. Therefore $\eta^2\hat{\varphi} \in H_L$ and

$$\int_{\Omega_L} \Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla(\eta^2\hat{\varphi})dx = 0.$$

Thus,

$$\int_{\Omega_{a-1,b+1}} \eta^2\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\hat{\varphi}dx = -2\int_{\Omega_{a-1,b+1}} \eta\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\eta\hat{\varphi}dx,$$

where $\nabla\hat{\varphi} = \nabla\varphi - \frac{k_2-k_1}{b-a}\chi_{a,b}(x)\vec{e}_n$, $\vec{e}_n = (0, ..., 0, 1)$,

$$\Omega_{a,b} = \{x = (x_1, x_2, \cdots, x_n) \in \Omega | a < x_n < b\}$$

and $\chi_{a,b}(x)$ is the characteristic function of $\Omega_{a,b}$. Then,

$$\begin{aligned} &\int_{\Omega_{a-1,b+1}} \eta^2\Theta(|\nabla\varphi|^2)|\nabla\varphi|^2 dx + \int_{\Omega_{a,b}} \eta^2\Theta(|\nabla\varphi|^2)\frac{\partial\varphi}{\partial x_n}\left(-\frac{k_2-k_1}{b-a}\right)dx \\ =& -2\int_{\Omega_{a-1.b+1}} \eta\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\eta\hat{\varphi}dx. \end{aligned}$$

Since $\eta = 1$ on $\Omega_{a,b}$ and $\int_{S_{x_n}} \Theta(|\nabla\varphi|^2)\frac{\partial\varphi}{\partial x_n}dx' = m_0$,

$$\int_{\Omega_{a-1,b+1}} \eta^2\Theta(|\nabla\varphi|^2)|\nabla\varphi|^2 dx = -2\int_{\Omega_{a-1.b+1}} \eta\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\eta\hat{\varphi}dx + (k_2-k_1)m_0.$$

Consequently,

$$\begin{aligned}\lambda\int_{\Omega_{a,b}}|\nabla\varphi|^2dx &\le \int_{\Omega_{a-1,b+1}}\eta^2\Theta(|\nabla\varphi|^2)|\nabla\varphi|^2dx\\ &\le 2\left|\int_{\Omega_{a-1,a}}+\int_{\Omega_{b,b+1}}\eta\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\eta\hat{\varphi}dx\right|+|k_2-k_1|m_0\\ &\le 4\Lambda\left|\int_{\Omega_{a-1,a}}+\int_{\Omega_{b,b+1}}|\nabla\varphi||\hat{\varphi}|dx\right|+|k_2-k_1|m_0\\ &\le 4\Lambda\Big[\Big(\int_{\Omega_{a-1,a}}|\nabla\varphi|^2dx\Big)^{\frac12}\Big(\int_{\Omega_{a-1,a}}|\varphi-k_1|^2dx\Big)^{\frac12}\\ &\quad+\Big(\int_{\Omega_{b,b+1}}|\nabla\varphi|^2dx\Big)^{\frac12}\Big(\int_{\Omega_{b,b+1}}|\varphi-k_2|^2dx\Big)^{\frac12}\Big]+|k_2-k_1|m_0.\end{aligned}\tag{3.24}$$

Set

$$k_1=\fint_{\Omega_{a-1,a}}\varphi dx,\qquad k_2=\fint_{\Omega_{b,b+1}}\varphi dx.$$

It follows the uniform Poincaré Inequality that

$$\int_{\Omega_{a-1,a}}|\varphi-k_1|^2dx\le C\int_{\Omega_{a-1,a}}|\nabla\varphi|^2dx,\quad \int_{\Omega_{b,b+1}}|\varphi-k_2|^2dx\le C\int_{\Omega_{b,b+1}}|\nabla\varphi|^2dx,\tag{3.25}$$

where C does not depend on a,b.

Therefore, substituting (3.25) into (3.24) yields

$$\lambda\int_{\Omega_{a,b}}|\nabla\varphi|^2dx\le C\Lambda\int_{\Omega_{a-1,b+1}\setminus\Omega_{a,b}}|\nabla\varphi|^2dx+|k_2-k_1|m_0.$$

We now claim that

$$|k_2-k_1|\le C\int_{\Omega_{a-1,b+1}}|\nabla\varphi|dx.\tag{3.26}$$

Assuming (3.26) for a moment, one gets

$$\int_{\Omega_{a,b}}|\nabla\varphi|^2dx\le C\frac{\Lambda}{\lambda}\int_{\Omega_{a-1,b+1}\setminus\Omega_{a,b}}|\nabla\varphi|^2dx+\frac{C}{\lambda}m_0\int_{\Omega_{a-1,b+1}}|\nabla\varphi|dx.\tag{3.27}$$

Set $S_{\max}=\max_{x_n}|S_{x_n}|$, $S_{\min}=\min_{x_n}|S_{x_n}|$. On another hand,

$$\begin{aligned}m_0\int_{\Omega_{a-1,b+1}}|\nabla\varphi|dx&\le m_0S_{max}^{\frac12}\Big(\int_{\Omega_{a-1,b+1}}|\nabla\varphi|^2dx\Big)^{\frac12}(b-a+2)^{\frac12}\\ &\le\varepsilon\int_{\Omega_{a-1,b+1}}|\nabla\varphi|^2dx+\frac{S_{max}}{4\varepsilon}(b-a+2)m_0^2.\end{aligned}\tag{3.28}$$

Combining (3.27) and (3.28) leads to

$$\begin{aligned}\int_{\Omega_{a,b}}|\nabla\varphi|^2dx\le C\frac{\Lambda}{\lambda}\int_{\Omega_{a-1,b+1}\setminus\Omega_{a,b}}|\nabla\varphi|^2dx+\frac{C}{\lambda}\varepsilon\int_{\Omega_{a-1,b+1}}|\nabla\varphi|^2dx\\ +\frac{C'}{\varepsilon\lambda}(b-a+2)m_0^2.\end{aligned}$$

Taking $\dfrac{C\varepsilon}{\lambda}=\dfrac12$ yields

$$\frac12\int_{\Omega_{a,b}}|\nabla\varphi|^2dx\le\Big(C\frac{\Lambda}{\lambda}+\frac12\Big)\int_{\Omega_{a-1,b+1}\setminus\Omega_{a,b}}|\nabla\varphi|^2dx+\frac{C'}{\lambda^2}(b-a+2)m_0^2.$$

Therefore, one has

$$\int_{\Omega_{a,b}} |\nabla\varphi|^2 dx \leq \left(C\frac{\Lambda}{\lambda}+1\right)\int_{\Omega_{a-1,b+1}\setminus\Omega_{a,b}} |\nabla\varphi|^2 dx + \frac{C'}{\lambda^2}(b-a+2)m_0^2,$$

ie.

$$\left(C\frac{\Lambda}{\lambda}+2\right)\int_{\Omega_{a,b}} |\nabla\varphi|^2 dx \leq \left(C\frac{\Lambda}{\lambda}+1\right)\int_{\Omega_{a-1,b+1}} |\nabla\varphi|^2 dx + \frac{C'}{\lambda^2}(b-a+2)m_0^2.$$

Set $\theta_0 = \dfrac{C\frac{\Lambda}{\lambda}+1}{C\frac{\Lambda}{\lambda}+2}$. Then $0<\theta_0<1$ and

$$\int_{\Omega_{a,b}} |\nabla\varphi|^2 dx \leq \theta_0\int_{\Omega_{a-1,b+1}} |\nabla\varphi|^2 dx + \frac{C'}{\lambda(C\Lambda+2\lambda)}(b-a+2)m_0^2. \tag{3.29}$$

Set

$$A_{a,b} = \frac{1}{b-a}\int_{\Omega_{a,b}} |\nabla\varphi|^2 dx.$$

It follows from (3.29) that

$$A_{a,b} \leq \theta_0\frac{b-a+2}{b-a}A_{a-1,b+1} + \frac{C'}{\lambda(C\Lambda+2\lambda)}\frac{b-a+2}{b-a}m_0^2.$$

Taking $\theta_0' = \dfrac{1+\theta_0}{2}$ and a positive constant $k(\theta_0)\geq 2$ such that, if $b-a\geq k(\theta_0)$, one has

$$\frac{b-a+2}{b-a}\leq 2, \qquad \theta_0\frac{b-a+2}{b-a}\leq\theta_0'<1,$$

$$A_{a,b}\leq\theta_0' A_{a-1,b+1} + \frac{C'}{\lambda(C\Lambda+2\lambda)}m_0^2 \qquad \text{for } \forall b-a\geq k(\theta_0).$$

Then,

$$\begin{aligned} A_{a,b} &\leq (\theta_0')^N A_{a-N,b+N} + \frac{Cm_0^2}{\lambda(C\Lambda+2\lambda)}\sum_{i=0}^{N-1}(\theta_0')^i \\ &\leq (\theta_0')^N A_{a-N,b+N} + \frac{C'm_0^2}{\lambda(C\Lambda+2\lambda)(1-\theta_0')}. \end{aligned}$$

Applying (3.6), one has

$$\begin{aligned} A_{a,b} &\leq (\theta_0')^N\frac{|\Omega_{a-N,b+N}|}{b-a+2N}Cm_0^2 + \frac{Cm_0^2}{\lambda(C\Lambda+2\lambda)(1-\theta_0')} \\ &\leq C(\theta_0')^N S_{max}m_0^2 + \frac{C'm_0^2}{\lambda(C\Lambda+2\lambda)(1-\theta_0')}. \end{aligned}$$

Therefore, for any $-\frac{2}{3}L<a<b<\frac{2}{3}L$ and $b-a\geq k(\theta_0)$, letting $N\to\infty$ yields

$$A_{a,b}\leq Cm_0^2,$$

where C does not depend on L.

Then, for any x_0, $|x_{0,n}| \leq \frac{L}{2}$,

$$\begin{aligned}\frac{1}{|\Omega_{x_0,1}|}\int_{\Omega_{x_0,1}}|\nabla\varphi|^2dx &\leq \frac{1}{|\Omega_{x_0,1}|}\int_{\Omega_{x_0,k(\theta_0)}}|\nabla\varphi|^2dx\\ &\leq \frac{2k(\theta_0)S_{\max}}{|\Omega_{x_0,1}|}\cdot\frac{1}{2k(\theta_0)S_{\max}}\int_{\Omega_{x_0,k(\theta_0)}}|\nabla\varphi|^2dx\\ &\leq \frac{2k(\theta_0)S_{\max}}{|\Omega_{x_0,1}|}Cm_0^2\\ &\leq C\frac{2k(\theta_0)S_{\max}}{S_{\min}}m_0^2,\end{aligned}$$

which yields (3.23).

Now, it remains to prove the claim (3.26). For any $a \in \mathbb{R}$, we define

$$\alpha_i = ⨍_{\Omega_{a+i-1,a+i}}\varphi dx, \qquad \alpha_{i+\frac{1}{2}} = ⨍_{\Omega_{a+i-1,a+i+1}}\varphi dx, \quad i = 1, 2, \cdots, n.$$

By the uniform Poincaré inequality (2.3), one has

$$\int_{\Omega_{a-1,a}}|\varphi-\alpha_0|dx \leq C\int_{\Omega_{a-1,a}}|\nabla\varphi|dx \leq C\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx, \tag{3.30}$$

$$\int_{\Omega_{a-1,a}}\left|\varphi-\alpha_{\frac{1}{2}}\right|dx \leq \int_{\Omega_{a-1,a+1}}\left|\varphi-\alpha_{\frac{1}{2}}\right|dx \leq C\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx. \tag{3.31}$$

Then, it follows (3.30) and (3.31) that

$$\int_{\Omega_{a-1,a}}\left|\alpha_{\frac{1}{2}}-\alpha_0\right|dx \leq C\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx.$$

Consequently,

$$\left|\alpha_{\frac{1}{2}}-\alpha_0\right| \leq \frac{C}{|\Omega_{a-1,a}|}\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx \leq \frac{C}{S_{\min}}\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx.$$

Similarly,

$$\left|\alpha_1-\alpha_{\frac{1}{2}}\right| \leq \frac{C}{S_{\min}}\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx.$$

Hence,

$$|\alpha_1-\alpha_0| \leq \left|\alpha_1-\alpha_{\frac{1}{2}}\right| + \left|\alpha_{\frac{1}{2}}-\alpha_0\right| \leq \frac{C}{S_{\min}}\int_{\Omega_{a-1,a+1}}|\nabla\varphi|dx.$$

In a similar way, one gets

$$|\alpha_2-\alpha_1| \leq \frac{C}{S_{\min}}\int_{\Omega_{a,a+2}}|\nabla\varphi|dx, \quad \ldots, \quad |\alpha_n-\alpha_{n-1}| \leq \frac{C}{S_{\min}}\int_{\Omega_{a+n-2,a+n}}|\nabla\varphi|dx.$$

Therefore, by induction, it holds that

$$|\alpha_n-\alpha_0| \leq \frac{C}{S_{\min}}\int_{\Omega_{a-1,a+n}}|\nabla\varphi|dx,$$

which proves the claim (3.26). □

3.5. $C^{1,\alpha}$ **regularity of the weak solution.**

Lemma 3.7. *(Gradient estimate). It holds that*

$$\|\nabla\varphi\|_{L^\infty(\Omega_{L/2})} \le Cm_0. \tag{3.32}$$

where C does not depend on L.

Proof. The proof is based on Moser's iteration technique.
Step 1. Interior estimate: It follows from the definition of weak solutions that for any $B_{2R} \subset \Omega_L$

$$\int_{B_{2R}} \Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx = 0, \quad \forall\psi\in C_0^\infty(B_{2R}).$$

Regarding $\partial_s\psi$ as a test function, $s = 1, 2, \cdots, n$, one gets

$$\begin{aligned}
0 &= \int_{B_{2R}} \Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla(\partial_s\psi)dx = -\int_{B_{2R}} \partial_s(\Theta(|\nabla\varphi|^2)\nabla\varphi)\cdot\nabla\psi dx \\
&= -\int_{B_{2R}} \Big(\Theta(|\nabla\varphi|^2)\delta_{ij} + 2\Theta'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi\Big)\partial_i(\partial_s\varphi)\partial_j\psi dx \\
&= -\int_{B_{2R}} a_{ij}\partial_i w_s\partial_j\psi dx,
\end{aligned}$$

where

$$a_{ij} = \Theta(|\nabla\varphi|^2)\delta_{ij} + 2\Theta'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi \in L^\infty(B_{2R}), \quad w_s = \partial_s\varphi \in L^2(B_{2R}).$$

Therefore

$$\int_{B_{2R}} a_{ij}\partial_i w_s\partial_j\psi dx = 0, \qquad \forall\psi\in H_0^1(B_{2R}). \tag{3.33}$$

Taking

$$\psi = \eta^2 w_s^{p-1}, \quad \eta\in C_0^\infty(B_{2R}), \quad \eta\equiv 1 \text{ in } B_R, \quad p\ge 2.$$

in (3.33), one has

$$\begin{aligned}
0 &= \int_{B_{2R}} a_{ij}\partial_i w_s\partial_j\left(\eta^2 w_s^{p-1}\right)dx \\
&= (p-1)\int_{B_{2R}} \eta^2 a_{ij}\partial_i w_s\partial_j w_s w_s^{p-2}dx + 2\int_{B_{2R}} \eta a_{ij}\partial_i w_s\partial_j\eta w_s^{p-1}dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
(p-1)\int_{B_{2R}} \eta^2 a_{ij}\partial_i w_s\partial_j w_s w_s^{p-2}dx &\le 2\left|\int_{B_{2R}} \eta a_{ij}\partial_i w_s\partial_j\eta w_s^{p-1}dx\right| \\
&\le 2\left(\int_{B_{2R}} \eta^2 a_{ij}\partial_i w_s\partial_j w_s w_s^{p-2}dx\right)^{\frac12}\left(\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta w_s^p dx\right)^{\frac12} \\
&\le \frac{p-1}{2}\int_{B_{2R}} \eta^2 a_{ij}\partial_i w_s\partial_j w_s w_s^{p-2}dx + \frac{2}{p-1}\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta w_s^p dx,
\end{aligned}$$

ie.

$$\int_{B_{2R}} \eta^2 a_{ij}\partial_i w_s\partial_j w_s w_s^{p-2}dx \le \frac{4}{(p-1)^2}\int_{B_{2R}} a_{ij}\partial_i\eta\partial_j\eta w_s^p dx.$$

Due to (3.4), we have

$$\int_{B_{2R}} \eta^2|\nabla w_s|^2 w_s^{p-2}dx \le \frac{4\Lambda}{(p-1)^2\lambda}\int_{B_{2R}} |\nabla\eta|^2 w_s^p dx. \tag{3.34}$$

Since

$$\eta^2|\nabla w_s|^2 w_s^{p-2} = \frac{4}{p^2}\left|\nabla\left(\eta w_s^{\frac p2}\right) - w_s^{\frac p2}\nabla\eta\right|^2 \ge \frac{2}{p^2}\left|\nabla\left(\eta w_s^{\frac p2}\right)\right|^2 - \frac{4}{p^2}|\nabla\eta|^2 w_s^p, \tag{3.35}$$

Combining (3.34) with (3.35) yields that

$$\frac{2}{p^2}\int_{B_{2R}}\left|\nabla\left(\eta w_s^{\frac{p}{2}}\right)\right|^2 dx \leq \frac{4}{p^2}\int_{B_{2R}}|\nabla\eta|^2 w_s^p dx + \frac{4\Lambda}{(p-1)^2\lambda}\int_{B_{2R}}|\nabla\eta|^2 w_s^p dx,$$

namely,

$$\int_{B_{2R}}\left|\nabla\left(\eta w_s^{\frac{p}{2}}\right)\right|^2 dx \leq 2\left(\frac{p^2\Lambda}{(p-1)^2\lambda}+1\right)\int_{B_{2R}}|\nabla\eta|^2 w_s^p dx.$$

Then the Sobolev's inequality implies that

$$\begin{aligned}\left(\int_{B_{2R}}\left(\eta w_s^{\frac{p}{2}}\right)^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} &\leq C\int_{B_{2R}}\left|\nabla\left(\eta w_s^{\frac{p}{2}}\right)\right|^2 dx \\ &\leq C\left(\frac{p^2\Lambda}{(p-1)^2\lambda}+1\right)\int_{B_{2R}}|\nabla\eta|^2 w_s^p dx.\end{aligned} \tag{3.36}$$

for $s = 1, 2, \cdots, n$.

Set

$$p_k = p\left(\frac{n}{n-2}\right)^k, \quad R_k = R\left(1+\frac{1}{2^k}\right),$$

$$\eta_k \in C_0^\infty(B_{R_k}), \quad \eta_k \equiv 1 \text{ in } B_{R_{k+1}}, \quad |\nabla\eta_k|^2 \leq \frac{C}{R_k - R_{k+1}} = \frac{C2^{k+1}}{R}.$$

Note that, $\{p_k\}$ is a strictly increasing sequence and tends to infinity as $k \to +\infty$, and $\{R_k\}$ is strictly decreasing sequence and tends to R as k goes to infinity. The following is the standard Moser's iteration process.

Taking $p = p_k$, $\eta = \eta_k$ in (3.36) yields that

$$\begin{aligned}\left(\int_{B_{R_{k+1}}} w_s^{p_{k+1}} dx\right)^{\frac{n-2}{n}} &\leq C\left(\frac{p_k^2\Lambda}{(p_k-1)^2\lambda}+1\right)\int_{B_{R_k}}|\nabla\eta_k|^2 w_s^{p_k} dx \\ &\leq C\frac{\Lambda 2^k}{\lambda R}\int_{B_{R_k}} w_s^{p_k} dx,\end{aligned}$$

and so,

$$\|w_s\|_{L^{p_{k+1}}\left(B_{R_{k+1}}\right)} \leq \left[C\frac{\Lambda 2^k}{\lambda R}\right]^{\frac{1}{p_k}}\|w_s\|_{L^{p_k}\left(B_{R_k}\right)}.$$

Let $M_k = \|w_s\|_{L^{p_k}\left(B_{R_k}\right)}$ and $D_k = \left[C\frac{\Lambda 2^k}{\lambda R}\right]^{\frac{1}{p_k}}$. Then by induction

$$M_{k+1} \leq D_k M_k \leq \cdots \leq D_k D_{k-1}\cdots D_0 M_0 = M_0\prod_{j=0}^{k} D_j. \tag{3.37}$$

Due to

$$\prod_{j=0}^{\infty} D_j = \prod_{j=1}^{\infty}\left[C\frac{\Lambda 2^j}{\lambda R}\right]^{\frac{1}{p_j}} = \left[C\frac{\Lambda}{\lambda R}\right]^{\sum_{j=0}^{\infty}\frac{1}{p_j}} 2^{\sum_{j=0}^{\infty}\frac{j}{p_j}} = C\left(C\frac{\Lambda}{\lambda R}\right)^{\frac{n}{2p}},$$

so taking $k \to \infty$ in (3.37) yields

$$\sup_{B_R} w_s \leq C\left(C\frac{\Lambda}{\lambda R}\right)^{\frac{n}{2p}}\|w_s\|_{L^p(B_{2R})}, \quad \text{for any } s = 1, 2, \cdots, n. \tag{3.38}$$

Step 2. Boundary estimate: For any $x_0 \in \partial\Omega_{L/2}$, according to Lemma 2.5, there exists a neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ and an invertible $C^{2,\alpha}$ map

$$T_{x_0} : U_{x_0} \cap \Omega_{L/2} \to B_{\delta_0}^+ : x \mapsto y$$

satisfying (2.4), where B_{δ_0} is independent of x_0. Define

$$\sigma_{ij} = \frac{\partial y_j}{\partial x_i}.$$

Then (2.4) implies

$$\sigma_{ij}(y)\sigma_{in}(y) = 0, \quad \text{on } \ y_n = 0, \quad j = 1, 2, \cdots, n-1.$$

For any $0 < R \le \delta_0$,

$$\int_{B_R^+} \Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}\tilde{\partial}_l\tilde{\psi}J dy = 0, \qquad \forall\tilde{\psi} \in H_0^1\left(B_{\delta_0}^+\right). \tag{3.39}$$

Taking $(s = 1, 2, \cdots, n)$

$$\tilde{\psi} = \tilde{\partial}_s\left(\tilde{\eta}^2\left(\tilde{\partial}_s\tilde{\varphi}\right)^{p-1}\right), \quad \tilde{\eta} \in C_0^\infty\left(B_{\delta_0}^+\right) \quad \text{and} \quad \tilde{\eta} \equiv 1 \ \text{ in } B_{\frac{\delta_0}{2}}^+$$

in (3.39) and integrating by parts show that

$$\begin{aligned} 0 &= \int_{B_R^+} \Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}\tilde{\partial}_l\left(\tilde{\partial}_s(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1})\right) J dy \\ &= -\int_{B_R^+} \tilde{\partial}_s\left(\Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J\right)\tilde{\partial}_l\left(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1}\right) dy \\ &\quad -\int_{\partial B_R^+\cap\{y_n=0\}} \left(\Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}\right)\tilde{\partial}_l\left(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1}\right) J\delta_{sn} dy' \\ &= -\int_{B_R^+} \tilde{\partial}_s\left(\Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J\right)\tilde{\partial}_l\left(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1}\right) dy, \end{aligned} \tag{3.40}$$

where the boundary terms vanish according to (2.4-2) and (3.13).

Detailed calculations show that

$$\begin{aligned} \tilde{\partial}_s(\Theta\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J) &= \Theta\sigma_{ij}\tilde{\partial}_j(\tilde{\partial}_s\tilde{\varphi})\sigma_{il}J + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma}\tilde{\partial}_\gamma\tilde{\varphi})\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J \\ &\quad + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J) \\ &= \Theta\sigma_{ij}\tilde{\partial}_j(\tilde{\partial}_s\tilde{\varphi})\sigma_{il}J + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\sigma_{\alpha\gamma}\tilde{\partial}_\gamma(\tilde{\partial}_s\tilde{\varphi})\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J \\ &\quad + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma})\tilde{\partial}_\gamma\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J) \\ &= \left(\Theta\sigma_{ij}\tilde{\partial}_j(\tilde{\partial}_s\tilde{\varphi}) + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\sigma_{\alpha\gamma}\tilde{\partial}_\gamma(\tilde{\partial}_s\tilde{\varphi})\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\right)\sigma_{il}J \\ &\quad + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma})\tilde{\partial}_\gamma\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J) \\ &= \left(\Theta\delta_{i\alpha} + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\right)\sigma_{\alpha\gamma}\tilde{\partial}_\gamma(\tilde{\partial}_s\tilde{\varphi})\sigma_{il}J \\ &\quad + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma})\tilde{\partial}_\gamma\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J) \\ &= \left(\Theta\delta_{ij} + 2\Theta'\sigma_{i\alpha}\tilde{\partial}_\alpha\tilde{\varphi}\sigma_{j\beta}\tilde{\partial}_\beta\tilde{\varphi}\right)\sigma_{i\gamma}\tilde{\partial}_\gamma(\tilde{\partial}_s\tilde{\varphi})\sigma_{jl}J \\ &\quad + 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma})\tilde{\partial}_\gamma\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J). \end{aligned}$$

Set

$$\begin{aligned} \tilde{A}_{l\gamma} &= \left(\Theta\delta_{ij} + 2\Theta'\sigma_{i\alpha}\tilde{\partial}_\alpha\tilde{\varphi}\sigma_{j\beta}\tilde{\partial}_\beta\tilde{\varphi}\right)\sigma_{i\gamma}\sigma_{jl}J, \\ \tilde{B}_{ls} &= 2\Theta'\sigma_{\alpha\beta}\tilde{\partial}_\beta\tilde{\varphi}\tilde{\partial}_s(\sigma_{\alpha\gamma})\tilde{\partial}_\gamma\tilde{\varphi}\sigma_{ij}\tilde{\partial}_j\tilde{\varphi}\sigma_{il}J + \Theta\tilde{\partial}_j\tilde{\varphi}\tilde{\partial}_s(\sigma_{ij}\sigma_{il}J). \end{aligned}$$

Then

$$\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_\gamma(\tilde{\partial}_s\tilde{\varphi})\tilde{\partial}_l(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1}) + \tilde{B}_{ls}\tilde{\partial}_l(\tilde{\eta}^2(\tilde{\partial}_s\tilde{\varphi})^{p-1})dy = 0. \tag{3.41}$$

Denoting $\tilde{\partial}_s\tilde{\varphi}$ by $\tilde{w}_s$, we have

$$\int_{B_R^+}\tilde{A}_{l\gamma}\tilde{\partial}_\gamma\tilde{w}_s\tilde{\partial}_l(\tilde{\eta}^2\tilde{w}_s^{p-1})+\tilde{B}_{ls}\tilde{\partial}_l(\tilde{\eta}^2\tilde{w}_s^{p-1})dy=0, \tag{3.42}$$

which can be rewritten as,

$$\begin{aligned}0&=\int_{B_R^+}\Big(\tilde{A}_{l\gamma}\tilde{\partial}_\gamma\tilde{w}_s+\tilde{B}_{ls}\Big)\Big((p-1)\tilde{\eta}^2\tilde{w}_s^{p-2}\tilde{\partial}_l\tilde{w}_s+2\tilde{\eta}\tilde{\partial}_l\tilde{\eta}\tilde{w}_s^{p-1}\Big)dy\\&=(p-1)\int_{B_R^+}\tilde{A}_{l\gamma}\tilde{\partial}_\gamma\tilde{w}_s\tilde{\eta}^2\tilde{w}_s^{p-2}\tilde{\partial}_l\tilde{w}_sdy+2\int_{B_R^+}\tilde{A}_{l\gamma}\tilde{\partial}_\gamma\tilde{w}_s\tilde{\eta}\tilde{\partial}_l\tilde{\eta}\tilde{w}_s^{p-1}dy\\&\quad+(p-1)\int_{B_R^+}\tilde{B}_{ls}\tilde{\eta}^2\tilde{w}_s^{p-2}\tilde{\partial}_l\tilde{w}_sdy+2\int_{B_R^+}\tilde{B}_{ls}\tilde{\eta}\tilde{\partial}_l\tilde{\eta}\tilde{w}_s^{p-1}dy\\&=I_1+I_2+I_3+I_4.\end{aligned} \tag{3.43}$$

Note that,

$$\begin{aligned}|I_2|&\le 2\Big|\int_{B_R^+}\tilde{\eta}\tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{\eta}\tilde{\partial}_\gamma\tilde{w}_s\tilde{w}_s^{p-1}dy\Big|\\&\le\frac{p-1}{2}\int_{B_R^+}\tilde{\eta}^2\tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{w}_s\tilde{\partial}_\gamma\tilde{w}_s\tilde{w}_s^{p-2}dy+\frac{2}{p-1}\int_{B_R^+}\tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{\eta}\tilde{\partial}_\gamma\tilde{\eta}\tilde{w}_s^pdy,\end{aligned} \tag{3.44}$$

$$\begin{aligned}|I_3|&\le(p-1)\Big|\int_{B_R^+}\tilde{\eta}^2\tilde{B}_{ls}\tilde{\partial}_l\tilde{w}_s\tilde{w}_s^{p-2}dy\Big|\\&\le(p-1)\frac{\lambda}{4}\int_{B_R^+}\tilde{\eta}^2|\tilde{\nabla}\tilde{w}_s|^2\tilde{w}_s^{p-2}dy+\frac{p-1}{\lambda}\int_{B_R^+}\tilde{\eta}^2|\tilde{B}_{ls}|^2\tilde{w}_s^{p-2}dy,\end{aligned} \tag{3.45}$$

and

$$|I_4|\le 2\Big|\int_{B_R^+}\tilde{\eta}\tilde{B}_{ls}\tilde{\partial}_l\tilde{\eta}\tilde{w}_s^{p-1}dy\Big|\le 2\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{B}_{ls}|\tilde{w}_s^{p-1}dy. \tag{3.46}$$

Therefore, substituting (3.44), (3.45) and (3.46) into (3.43) yields that

$$\begin{aligned}\frac{I_1}{2}&=\frac{(p-1)}{2}\int_{B_R^+}\tilde{\eta}^2\tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{w}_s\tilde{\partial}_\gamma\tilde{w}_s\tilde{w}_s^{p-2}dy\\&\le\frac{2}{p-1}\int_{B_R^+}\tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{\eta}\tilde{\partial}_\gamma\tilde{\eta}\tilde{w}_s^pdy+(p-1)\frac{\lambda}{4}\int_{B_R^+}\tilde{\eta}^2|\tilde{\nabla}\tilde{w}_s|^2\tilde{w}_s^{p-2}dy\\&\quad+\frac{p-1}{\lambda}\int_{B_R^+}\tilde{\eta}^2|\tilde{B}_{ls}|^2\tilde{w}_s^{p-2}dy+2\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{B}_{ls}|\tilde{w}_s^{p-1}dy.\end{aligned}$$

Then the uniform ellipticity yields

$$\begin{aligned}\int_{B_R^+}\tilde{\eta}^2|\tilde{\nabla}\tilde{w}_s|^2\tilde{w}_s^{p-2}dy\le\frac{8\Lambda}{\lambda(p-1)^2}\int_{B_R^+}|\tilde{\nabla}\tilde{\eta}|^2\tilde{w}_s^pdy+\frac{4}{\lambda^2}\int_{B_R^+}\tilde{\eta}^2|\tilde{B}_{ls}|^2\tilde{w}_s^{p-2}dy\\+\frac{8}{\lambda(p-1)}\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{B}_{ls}|\tilde{w}_s^{p-1}dy.\end{aligned}$$

Note that (3.3) implies that

$$|\tilde{B}_{ls}|\le C(1+\Lambda)|\tilde{w}|\le C\Lambda|\tilde{w}|,$$

where $\tilde{w}=(\tilde{w}_1,\tilde{w}_2,\cdots,\tilde{w}_n)=(\tilde{\partial}_1\tilde{\varphi},\tilde{\partial}_2\tilde{\varphi},\cdots,\tilde{\partial}_n\tilde{\varphi})$ and $|\tilde{w}|=\sum_{s=1}^n|\tilde{w}_s|$. Then,

$$\begin{aligned}\int_{B_R^+}\tilde{\eta}^2|\tilde{\nabla}\tilde{w}_s|^2\tilde{w}_s^{p-2}dy\le\frac{8\Lambda}{\lambda(p-1)^2}\int_{B_R^+}|\tilde{\nabla}\tilde{\eta}|^2\tilde{w}_s^pdy+\frac{C\Lambda^2}{\lambda^2}\int_{B_R^+}\tilde{\eta}^2|\tilde{w}|^2\tilde{w}_s^{p-2}dy\\+\frac{C\Lambda}{\lambda(p-1)}\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{w}|\tilde{w}_s^{p-1}dy\end{aligned}$$

which implies

$$\begin{aligned}\int_{B_R^+}\tilde{\eta}^2\left|\tilde{\nabla}\left(\tilde{w}_s^{\frac{p}{2}}\right)\right|^2dy\le\frac{2\Lambda p^2}{\lambda(p-1)^2}\int_{B_R^+}|\tilde{\nabla}\tilde{\eta}|^2\tilde{w}_s^pdy+\frac{C\Lambda^2}{\lambda^2}p^2\int_{B_R^+}\tilde{\eta}^2|\tilde{w}|^2\tilde{w}_s^{p-2}dy\\+\frac{C\Lambda p^2}{\lambda(p-1)}\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{w}|\tilde{w}_s^{p-1}dy.\end{aligned}\tag{3.47}$$

It follows from (3.47) that

$$\begin{aligned}&\int_{B_R^+}\left|\tilde{\nabla}\left(\tilde{\eta}\tilde{w}_s^{\frac{p}{2}}\right)\right|^2dy\\&\le\left(\frac{2\Lambda p^2}{\lambda(p-1)^2}+1\right)\int_{B_R^+}|\tilde{\nabla}\tilde{\eta}|^2\tilde{w}_s^pdy+\frac{C\Lambda^2}{\lambda^2}p^2\int_{B_R^+}\tilde{\eta}^2|\tilde{w}|^2\tilde{w}_s^{p-2}dy\\&\quad+\frac{C\Lambda p^2}{\lambda(p-1)}\int_{B_R^+}\tilde{\eta}|\tilde{\nabla}\tilde{\eta}||\tilde{w}|\tilde{w}_s^{p-1}dy,\end{aligned}\tag{3.48}$$

for $s=1,2,\cdots,n$.

Let

$$R_k=\delta_0\left(\theta+\frac{1-\theta}{2^k}\right),\qquad p_k=p\left(\frac{n}{n-2}\right)^k,\qquad k=0,1,2,\dots,$$

$$\tilde{\eta}_k\in C_0^\infty(B_{R_k}),\quad \tilde{\eta}_k\equiv1\text{ in }B_{R_{k+1}}\quad\text{and}\quad|\tilde{\nabla}\tilde{\eta}_k|\le\frac{2}{R_k-R_{k+1}}=\frac{2^{k+1}}{(1-\theta)\delta_0}.$$

Taking $p=p_k$, $\tilde{\eta}=\tilde{\eta}_k$ in (3.48) and using Sobolev embedding Theorem, we obtain

$$\begin{aligned}&\left(\int_{B_{R_{k+1}}^+}\tilde{w}_s^{p_{k+1}}dy\right)^{\frac{n-2}{n}}\\&\le\left(\int_{B_{R_k}^+}\left(\tilde{\eta}_k\tilde{w}_s^{\frac{p_k}{2}}\right)^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\\&\le C\int_{B_{R_k}^+}\left|\tilde{\nabla}\left(\tilde{\eta}_k\tilde{w}_s^{\frac{p_k}{2}}\right)\right|^2dy\\&\le C\left(\frac{2\Lambda {p_k}^2}{\lambda(p_k-1)^2}+1\right)\int_{B_{R_k}^+}|\tilde{\nabla}\tilde{\eta}_k|^2\tilde{w}_s^{p_k}dy+\frac{C\Lambda^2}{\lambda^2}{p_k}^2\int_{B_{R_k}^+}\tilde{\eta}_k^2|\tilde{w}|^2\tilde{w}_s^{p_k-2}dy\\&\quad+\frac{C\Lambda {p_k}^2}{\lambda(p_k-1)}\int_{B_{R_k}^+}\tilde{\eta}_k|\tilde{\nabla}\tilde{\eta}_k||\tilde{w}|\tilde{w}_s^{p_k-1}dy\\&\le C\left(\frac{2\Lambda {p_k}^2}{\lambda(p_k-1)^2}+1\right)\int_{B_{R_k}^+}\frac{4^{k+1}}{(1-\theta)^2\delta_0^2}\tilde{w}_s^{p_k}dy+\frac{C\Lambda^2}{\lambda^2}{p_k}^2\int_{B_{R_k}^+}|\tilde{w}|^2\tilde{w}_s^{p_k-2}dy\\&\quad+\frac{C\Lambda {p_k}^2}{\lambda(p_k-1)}\int_{B_{R_k}^+}\frac{2^{k+1}}{(1-\theta)\delta_0}|\tilde{w}|\tilde{w}_s^{p_k-1}dy.\end{aligned}$$

Therefore,

$$\|\tilde{w}_s\|_{L^{p_{k+1}}\left(B^+_{R_{k+1}}\right)} \le \left(A_k \int_{B^+_{R_k}} \tilde{w}_s^{p_k} dy + B_k \int_{B^+_{R_k}} |\tilde{w}|^2 \tilde{w}_s^{p_k-2} dy + C_k \int_{B^+_{R_k}} |\tilde{w}| \tilde{w}_s^{p_k-1} dy\right)^{\frac{1}{p_k}},$$

with

$$A_k = C\left(\frac{2\Lambda {p_k}^2}{\lambda(p_k-1)^2} + 1\right)\frac{4^{k+1}}{(1-\theta)^2\delta_0^2}, \quad B_k = \frac{C\Lambda^2}{\lambda^2}{p_k}^2,$$

and

$$C_k = \frac{C\Lambda {p_k}^2}{\lambda(p_k-1)} \cdot \frac{2^{k+1}}{(1-\theta)\delta_0}.$$

Note that

$$\int_{B^+_{R_k}} \tilde{w}_s^{p_k-2}|\tilde{w}|^2 dy \le \left(\int_{B^+_{R_k}} \tilde{w}_s^{p_k} dy\right)^{\frac{p_k-2}{p_k}} \left(\int_{B^+_{R_k}} |\tilde{w}|^{p_k} dy\right)^{\frac{2}{p_k}},$$

$$\int_{B^+_{R_k}} \tilde{w}_s^{p_k-1}|\tilde{w}| dy \le \left(\int_{B^+_{R_k}} \tilde{w}_s^{p_k} dy\right)^{\frac{p_k-1}{p_k}} \left(\int_{B^+_{R_k}} |\tilde{w}|^{p_k} dy\right)^{\frac{1}{p_k}}.$$

Then

$$\begin{aligned}
\|\tilde{w}_s\|_{L^{p_{k+1}}\left(B^+_{R_{k+1}}\right)} \le& \Big[A_k \|\tilde{w}_s\|^{p_k}_{L^{p_k}\left(B^+_{R_k}\right)} + B_k \|\tilde{w}_s\|^{p_k-2}_{L^{p_k}\left(B^+_{R_k}\right)} \||\tilde{w}|\|^2_{L^{p_k}\left(B^+_{R_k}\right)} \\
&+ C_k \|\tilde{w}_s\|^{p_k-1}_{L^{p_k}\left(B^+_{R_k}\right)} \||\tilde{w}|\|_{L^{p_k}\left(B^+_{R_k}\right)}\Big]^{\frac{1}{p_k}}, \\
\le& \|\tilde{w}_s\|^{\frac{p_k-2}{p_k}}_{L^{p_k}(B^+_{R_k})} \Big[A_k \|\tilde{w}_s\|^2_{L^{p_k}(B^+_{R_k})} + B_k \||\tilde{w}|\|^2_{L^{p_k}(B^+_{R_k})} \\
&+ C_k \|\tilde{w}_s\|_{L^{p_k}(B^+_{R_k})} \||\tilde{w}|\|_{L^{p_k}(B^+_{R_k})}\Big]^{\frac{1}{p_k}}.
\end{aligned}$$

Hence,

$$\|\tilde{w}_s\|_{L^{p_{k+1}}\left(B^+_{R_{k+1}}\right)} \le \|\tilde{w}_s\|^{\frac{p_k-2}{p_k}}_{L^{p_k}\left(B^+_{R_k}\right)} \Big[A_k + B_k + C_k\Big]^{\frac{1}{p_k}} \||\tilde{w}|\|^{\frac{2}{p_k}}_{L^{p_k}\left(B^+_{R_k}\right)},$$

which implies that

$$\begin{aligned}
\sum_{s=1}^n \|\tilde{w}_s\|_{L^{p_{k+1}}\left(B^+_{R_{k+1}}\right)} \le& \Big[A_k + B_k + C_k\Big]^{\frac{1}{p_k}} \||\tilde{w}|\|^{\frac{2}{p_k}}_{L^{p_k}\left(B^+_{R_k}\right)} \sum_{s=1}^n \|\tilde{w}_s\|^{\frac{p_k-2}{p_k}}_{L^{p_k}\left(B^+_{R_k}\right)} \\
\le& \Big[A_k + B_k + C_k\Big]^{\frac{1}{p_k}} \||\tilde{w}|\|^{\frac{2}{p_k}}_{L^{p_k}\left(B^+_{R_k}\right)} \left(\sum_{s=1}^n \|\tilde{w}_s\|^{\frac{p_k-2}{p_k}\cdot\frac{p_k}{p_k-2}}_{L^{p_k}\left(B^+_{R_k}\right)}\right)^{\frac{p_k-2}{p_k}} n^{\frac{2}{p_k}} \\
\le& \Big[(A_k + B_k + C_k)n^2\Big]^{\frac{1}{p_k}} \left(\sum_{s=1}^n \|\tilde{w}_s\|_{L^{p_k}\left(B^+_{R_k}\right)}\right)^{\frac{2}{p_k}} \\
&\times \left(\sum_{s=1}^n \|\tilde{w}_s\|_{L^{p_k}\left(B^+_{R_k}\right)}\right)^{\frac{p_k-2}{p_k}}.
\end{aligned}$$

Therefore,

$$\sum_{s=1}^n \|\tilde{w}_s\|_{L^{p_{k+1}}\left(B^+_{R_{k+1}}\right)} \le \Big[(A_k + B_k + C_k)n^2\Big]^{\frac{1}{p_k}} \sum_{s=1}^n \|\tilde{w}_s\|_{L^{p_k}\left(B^+_{R_k}\right)}.$$

Define

$$M_k = \sum_{s=1}^{n} \|\tilde{w}_s\|_{L^{p_k}\left(B^+_{R_k}\right)}, \quad D_k = \left[(A_k + B_k + C_k)n^2\right]^{\frac{1}{p_k}}.$$

Then

$$M_{k+1} \le D_k M_k.$$

It is clear that $p_k = p\left(\frac{n}{n-2}\right)^k \le p4^k$, for $n \ge 3$, so

$$\begin{aligned} A_k + B_k + C_k &\le C\left(\frac{2\Lambda p_k^2}{\lambda(p_k-1)^2} + 1\right)\frac{4^{k+1}}{(1-\theta)^2\delta_0^2} + \frac{C\Lambda^2}{\lambda^2}p_k^2 + \frac{C\Lambda p_k^2}{\lambda(p_k-1)}\frac{2^{k+1}}{(1-\theta)\delta_0} \\ &\le C\left(\frac{\Lambda}{\lambda} + 1\right)\frac{4^k}{(1-\theta)^2\delta_0^2} + C\frac{\Lambda^2}{\lambda^2}p^2 16^k + C\frac{\Lambda}{\lambda}p\frac{8^k}{(1-\theta)\delta_0} \\ &\le T \cdot 16^k, \end{aligned}$$

where $T = C\left[\frac{1}{(1-\theta)^2\delta_0^2}\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}p^2 + \frac{\Lambda}{\lambda}\frac{p}{(1-\theta)\delta_0}\right]$, C does not depend on k.

Then,

$$\begin{aligned} M_{k+1} &\le D_k M_k \le D_k \cdot D_{k-1} \cdots D_0 \cdot M_0 \\ &\le T^{\sum_{i=0}^{k}\frac{1}{p_i}} \cdot 16^{\sum_{i=0}^{k}\frac{i}{p_i}} \cdot M_0. \end{aligned}$$

Note that

$$\sum_{i=0}^{\infty}\frac{1}{p_i} = \frac{n}{2p}, \qquad \text{and} \qquad \sum_{i=0}^{k}\frac{i}{p_i} < \infty.$$

One has

$$M_{k+1} \le C T^{\frac{n}{2p}} M_0, \qquad \forall k > 0.$$

Letting $k \to \infty$ shows that

$$\sum_{s=1}^{n}\sup_{B_{\theta\delta_0}}|\tilde{w}_s| \le C T^{\frac{n}{2p}}\sum_{s=1}^{n}\|\tilde{w}_s\|_{L^p\left(B^+_{\delta_0}\right)}. \tag{3.49}$$

Combining the interior estimate (3.38) with the boundary estimate (3.49) yields the desired gradient estimate (3.32). □

Remark 3.2. It has been assumed that $w_s \ge 0$ and w_s is bounded in the above proof. The boundness assumption could be eliminated by a standard technique (see chapter 8 of [21]). If w_s is not positive, we can repeat the proof for w_s^+ and w_s^- respectively.

Remark 3.3. In the case that $n = 2$, choosing $p_1 = \infty$, one can obtain the estimate similarly to (3.49).

Lemma 3.8. *(Hölder estimate of gradient.)* $\varphi \in C^{1,\alpha}\left(\overline{\Omega}_{L/2}\right)$ *and*

$$\|\nabla\varphi\|_{C^{0,\alpha}\left(\Omega_{L/2}\right)} \le C m_0, \tag{3.50}$$

where C does not depend on L.

Proof. **Step 1. Interior Estimate.** For any $B_{2R} \subset \Omega$, $w_s = \partial_s\varphi$ $(s = 1, 2, \cdots, n)$ is a weak solution to

$$\partial_i(a_{ij}\partial_j w_s) = 0,$$

in the sense of (3.33), where $a_{ij} = \rho(|\nabla\varphi|^2)\delta_{ij} + 2\rho'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi$. Then, the desired interior Hölder estimate for w_s is just the standard interior Hölder estimate for the weak solutions to second order elliptic equation with bounded coefficients.

Step 2. Boundary Estimate. Similar to (3.41), one has for any $s = 1, 2, \cdots, n-1$,

$$\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_\gamma \tilde{w}_s \tilde{\partial}_l \tilde{\psi} + \tilde{B}_{ls}\tilde{\partial}_l \tilde{\psi} dy = 0, \quad \tilde{\psi} \in H_0^1(B_R^+). \tag{3.51}$$

where B_R^+, $\tilde{A}_{l\gamma}$ and $\tilde{B}_{ls}$ are same as in (3.41).

By an even symmetrizing procedure, $\hat{w}_s$, $\hat{A}_{l\gamma}$ and $\hat{B}_{ls}$ denote the even extensions of $\tilde{w}_s$, $\tilde{A}_{l\gamma}$ and $\tilde{B}_{ls}$, respectively. Then $\hat{w}_s$ satisfies

$$\int_{B_R} \hat{A}_{l\gamma}\tilde{\partial}_\gamma \hat{w}_s \tilde{\partial}_l \hat{\psi} + \hat{B}_{ls}\tilde{\partial}_l \hat{\psi} dy = 0, \quad \hat{\psi} \in H_0^1(B_R),\ 1 \le s \le n-1. \tag{3.52}$$

Since $\|\tilde{w}\|_{L^\infty} \le Cm_0$,

$$\|\tilde{A}_{l\gamma}, \tilde{B}_{ls}\|_{L^\infty} \le C, \quad \|\hat{A}_{l\gamma}, \hat{B}_{ls}\|_{L^\infty} \le C.$$

Therefore, for $1 \le s \le n-1$, the standard interior De Giorgi estimate gives

$$\|\tilde{w}_s\|_{C^\alpha\left(B_{R/2}^+\right)} \le C\left(\|\tilde{w}_s\|_{L^2(B_R^+)} + \frac{1}{\lambda}\|\tilde{B}_{ls}\|_{L^q(B_R^+)}\right) \le Cm_0, \quad q > n.$$

Now, we estimate $\tilde{w}_n$. For any $y_0 \in B_{R/2}^+$, $r \le \max\left\{\frac{1}{6}R, \frac{1}{2}\right\}$, taking $\tilde{\psi} = \tilde{\eta}^2(\tilde{w}_s - \bar{w}_s)$ in (3.52), $\tilde{\eta} \in C_0^\infty(B_{2r}(y_0))$, $\tilde{\eta} \equiv 1$ in $B_r(y_0)$, $|\tilde{\nabla}\tilde{\eta}| \le \frac{2}{r}$ and $\bar{w}_s = ⨍_{B_{2r}(y_0)} \tilde{w}_s dy$, one has

$$\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_\gamma \tilde{w}_s \tilde{\partial}_l(\tilde{\eta}^2(\tilde{w}_s - \bar{w}_s)) + \tilde{B}_{ls}\tilde{\partial}_l(\tilde{\eta}^2(\tilde{w}_s - \bar{w}_s))dy = 0.$$

Therefore,

$$\begin{aligned}
&\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_l \tilde{w}_s \tilde{\partial}_\gamma \tilde{w}_s \tilde{\eta}^2 dy \\
\le& \left|\int_{B_R^+} \tilde{B}_{ls}\tilde{\partial}_l \tilde{w}_s \tilde{\eta}^2 dy\right| + \left|2\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_\gamma \tilde{w}_s \tilde{\eta}\tilde{\partial}_l\tilde{\eta}(\tilde{w}_s - \bar{w}_s)dy\right| + \left|2\int_{B_R^+} \tilde{B}_{ls}\tilde{\eta}\tilde{\partial}_l\tilde{\eta}(\tilde{w}_s - \bar{w}_s)dy\right| \\
\le& \frac{1}{\lambda}\int_{B_R^+} |\tilde{B}_{ls}|^2\tilde{\eta}^2 dy + \frac{\lambda}{4}\int_{B_R^+} |\tilde{\nabla}\tilde{w}_s|^2\tilde{\eta}^2 dy + \frac{\lambda}{4\Lambda}\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_l \tilde{w}_s \tilde{\partial}_\gamma \tilde{w}_s \tilde{\eta}^2 dy \\
&+ \frac{4\Lambda}{\lambda}\int_{B_R^+} \tilde{A}_{l\gamma}\tilde{\partial}_l\tilde{\eta}\tilde{\partial}_\gamma\tilde{\eta}(\tilde{w}_s - \bar{w}_s)^2 dy + \int_{B_R^+} |\tilde{B}_{ls}|^2\tilde{\eta}^2 dy + \int_{B_R^+} \left|\tilde{\nabla}\tilde{\eta}\right|^2 (\tilde{w}_s - \bar{w}_s)^2 dy.
\end{aligned}$$

As a consequence,

$$\int_{B_R^+} \tilde{\eta}^2|\tilde{\nabla}\tilde{w}_s|^2 dy \le C\bigg(\int_{B_R^+\cap B_{2r}(y_0)} |\tilde{\nabla}\tilde{\eta}|^2|\tilde{w}_s - \bar{w}_s|^2 dy + \int_{B_R^+\cap B_{2r}(y_0)} \tilde{\eta}^2|\tilde{B}_{ls}|^2 dy\bigg).$$

Noting that $\|\tilde{w}_s\|_{C^\alpha\left(B_{R/2}^+\right)} \le Cm_0$ and $|\tilde{B}_{ls}| \le Cm_0$, one has that, for any $1 \le s \le n-1$,

$$\begin{aligned}
\int_{B_R^+\cap B_r(y_0)} |\tilde{\nabla}\tilde{w}_s|^2 &\le C\bigg(\int_{B_R^+\cap B_{2r}(y_0)} |\tilde{\nabla}\tilde{\eta}|^2|\tilde{w}_s - \bar{w}_s|^2 dy + \int_{B_R^+\cap B_{2r}(y_0)} \tilde{\eta}^2|\tilde{B}_{ls}|^2 dy\bigg) \\
&\le Cm_0^2\left(r^{n-2+2\alpha} + r^n\right) \\
&\le Cm_0^2 r^{n-2+2\alpha}.
\end{aligned} \tag{3.53}$$

According to the equation of $\tilde{\varphi}$, one has

$$\begin{aligned}\int_{B_R^+\cap B_r(y_0)}|\tilde{D}_{nn}^2\tilde{\varphi}|^2dy &\leq C\Big(\sum_{k=1}^{n-1}\int_{B_R^+\cap B_r(y_0)}|\tilde{\nabla}\tilde{w}_s|^2dy+\int_{B_R^+\cap B_r(y_0)}|\tilde{\nabla}\tilde{\varphi}|^2dy\Big)\\ &\leq Cm_0^2(r^{n-2+2\alpha}+r^n)\\ &\leq Cm_0^2r^{n-2+2\alpha}.\end{aligned} \tag{3.54}$$

Then, due to (3.53) and (3.54), one has by Theorem 2.1 that

$$\|w\|_{C^\alpha\left(B^+_{R/2}\right)}\leq Cm_0.$$

Now, the Hölder estimate of $\nabla\varphi$ follows from Step 1 and Step 2. □

3.6. **Proof of the existence of subsonic flows. Proof of the statement (i) of Theorem 1.** For any fixed suitably large L, according to previous subsections, one can get a H^1 function $\varphi_L(x)$ such that $(\varphi_L(x)-\varphi_L(0))\in H_L$ is a weak solution to problem 2. Set $\hat{\varphi}_L(x)=\varphi_L(x)-\varphi_L(0)$. Moreover, $\hat{\varphi}_L\in C^{1,\alpha}(\overline{\Omega}_{L/2})$ and

$$\|\nabla\hat{\varphi}_L\|_{C^{0,\alpha}\left(\Omega_{L/2}\right)}\leq Cm_0.$$

For any fixed $K\gg 1$, if $L>2K$,

$$\|\hat{\varphi}_L\|_{C^{1,\alpha}(\Omega_K)}\leq C,$$

where C does not depend on L, and $\hat{\varphi}_L$ satisfies

$$\int_{\Omega_K}\Theta\left(|\nabla\hat{\varphi}_L|^2\right)\nabla\hat{\varphi}_L\cdot\nabla\psi dx=0,\qquad\forall\psi\in C_0^\infty(\Omega_K).$$

Since $\hat{\varphi}_L\in H_L\cap C^{1,\alpha}(\Omega_K)$ satisfies the equation (3.10), one can check easily that

$$\int_{S_{x_0}}\Theta\left(|\nabla\hat{\varphi}_L|^2\right)\frac{\partial\hat{\varphi}_L}{\partial x_n}dx'=m_0,\quad\text{for any } x_0\in\Omega_K.$$

By a standard diagonal argument, there exists a $\varphi\in C^{1,\alpha}(\Omega)$ and a subsequence $\hat{\varphi}_{L_n}$ such that for any K,

$$\lim_{n\to\infty}\|\hat{\varphi}_{L_n}-\varphi\|_{C^{1,\alpha}(\Omega_K)}=0.$$

Therefore, one has

$$\int_\Omega\Theta(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx=0,\qquad\forall\psi\in C_0^\infty(\Omega),$$

and

$$\int_{S_{x_0}}\Theta(|\nabla\varphi|^2)\frac{\partial\varphi}{\partial x_n}dx'=m_0,\quad\text{for any } x_0\in\Omega.$$

It is clear that

$$\varphi\in C^{1,\alpha}(\Omega)\bigcap H^1_{loc}(\Omega)$$

and

$$\|\nabla\varphi\|_{C^{0,\alpha}(\Omega)}\leq Cm_0.$$

Similar to the previous subsections, one can prove that $\varphi\in H^2_{loc}(\Omega)$ and φ is a strong solution to

$$\begin{cases}\left(\Theta(|\nabla\varphi|^2)\delta_{ij}+2\Theta'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi\right)\partial^2_{ij}\varphi=0 & \text{in } \Omega,\\ \dfrac{\partial\varphi}{\partial\vec{n}}=0, & \text{on } \partial\Omega.\end{cases} \tag{3.55}$$

By the standard regularity theory for second order elliptic equations, one gets that $\varphi\in C^{2,\alpha}_{loc}(\overline{\Omega})$ is a solution to (3.55) with the property

$$\|\nabla\varphi\|_{C^{1,\alpha}(\Omega)}\leq Cm_0.$$

Choose m_0 small enough such that $Cm_0 \leq 1-2\hat{\delta}_0$. Then the subsonic truncation automatically disappears, so $\varphi \in C^{2,\alpha}_{loc}(\overline{\Omega})$ is a smooth solution to the original Problem 1. This proves the first part of Theorem 1.1.

Remark 3.4. In fact, we can conclude that $\varphi \in C^\infty(\Omega)$ by the standard bootstrap argument.

4. Uniqueness of the global subsonic flow

Theorem 4.1. *(Uniqueness) Suppose that Ω satisfies the assumptions (1.6), and φ_k $(k=1,2)$ are uniformly subsonic solutions to the following problem*

$$\begin{cases} div\left(\rho(|\nabla\varphi_k|^2)\,\nabla\varphi_k\right) = 0, & in \quad \Omega, \\ \dfrac{\partial\varphi_k}{\partial\vec{n}} = 0, & on \quad \partial\Omega, \end{cases}$$

associated with the same incoming mass flux m_0. Then

$$\nabla\varphi_1 = \nabla\varphi_2, \qquad in \quad \Omega.$$

Proof. Set $\varphi = \varphi_1 - \varphi_2$. Then φ satisfies

$$\begin{cases} \partial_i(A_{ij}\partial_j\varphi) = 0, & in \quad \Omega, \\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & on \quad \partial\Omega, \end{cases} \tag{4.1}$$

where

$$A_{ij} = \int_0^1 \rho(\bar{q}^2)\delta_{ij} + 2\rho'(\bar{q}^2)(s\partial_j\varphi_1 + (1-s)\partial_j\varphi_2)(s\partial_i\varphi_1 + (1-s)\partial_i\varphi_2)ds,$$

$$\bar{q}^2 = |s\nabla\varphi_1 + (1-s)\nabla\varphi_2|^2.$$

Moreover, there exist two positive constants $\lambda < \Lambda$, such that for any vector $\xi \in \mathbb{R}^n$

$$\lambda|\xi|^2 < A_{ij}\xi_i\xi_j < \Lambda|\xi|^2. \tag{4.2}$$

Let $\eta(x) = \eta(x_n)$ be a C_0^∞ function satisfying

$$\eta(x_n) \equiv 1 \ \text{ for } \ |x_n| \leq L; \qquad \eta(x_n) \equiv 0 \ \text{ for } \ |x_n| \geq L+1, \quad \textit{and} \ \ |\eta'(x_n)| \leq 2,$$

Denote $\Omega_{a,b} = \{x = (x', x_n) \in \Omega | a \leq x_n \leq b\}$ and for $L > 0$

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) - \varphi_L^-, & x \in \Omega \cap \{x_n \leq -L\}, \\ \varphi(x) - \varphi_L^- - \dfrac{\varphi_L^+ - \varphi_L^-}{2L}(x_n + L), & x \in \Omega \cap \{-L \leq x_n \leq L\}, \\ \varphi(x) - \varphi_L^+, & x \in \Omega \cap \{x_n \geq L\}, \end{cases}$$

where

$$\varphi_L^- = \frac{1}{|\Omega_{-L-1,-L}|}\int_{\Omega_{-L-1,-L}} \varphi(x)dx, \quad \varphi_L^+ = \frac{1}{|\Omega_{L,L+1}|}\int_{\Omega_{L,L+1}} \varphi(x)dx.$$

Note that $\nabla\hat{\varphi} = \nabla\varphi - \dfrac{\varphi_L^+ - \varphi_L^-}{2L}\chi_{-L,L}(x)\vec{e}_n$, $\vec{e}_n = (0,\cdots,0,1)$, $\chi_{-L,L}(x)$ is the characteristic function of $\Omega_{-L,L}$.

Multiplying on the both sides of the first equation in (4.1) by $\eta^2\hat{\varphi}$, and integrating it over Ω, one obtains

$$\int_{\Omega_{-L-1,L+1}} \eta^2 A_{ij}\partial_i\varphi\partial_j\varphi dx + \frac{\varphi_L^+ - \varphi_L^-}{2L}\int_{\Omega_{-L,L}} \eta^2\left(\rho(|\nabla\varphi_1|^2)\nabla\varphi_1 - \rho(|\nabla\varphi_2|^2)\nabla\varphi_2\right)\cdot\vec{e}_n dx$$
$$= -2\int_{\Omega_{-L-1,-L}} \eta(\varphi-\varphi_L^-)A_{ij}\partial_i\eta\partial_j\varphi dx - 2\int_{\Omega_{L,L+1}} \eta(\varphi-\varphi_L^+)A_{ij}\partial_i\eta\partial_j\varphi dx. \tag{4.3}$$

The second integral on the left hand side of (4.3) vanishes. Indeed,

$$\int_{\Omega_{-L,L}} \eta^2(\rho(|\nabla\varphi_1|^2)\nabla\varphi_1 - \rho(|\nabla\varphi_2|^2)\nabla\varphi_2)\cdot\vec{e}_n dx$$
$$= \int_{-L}^{L}\eta^2(t)\int_{S_t}\left(\rho(|\nabla\varphi_1|^2)\nabla\varphi_1\cdot\vec{e}_n - \rho(|\nabla\varphi_2|^2)\nabla\varphi_2\cdot\vec{e}_n\right)dx'dt = 0,$$

since the two solutions possess the same mass flux m_0, here $S_t = \Omega\cap\{x_n = t\}$ for $t\in[-L,L]$.

It follows from (4.3) and (4.2) that

$$\lambda\int_{\Omega_{-L,L}}|\nabla\varphi|^2 dx$$
$$\leq 4\Lambda\int_{\Omega_{-L-1,-L}}|\varphi-\varphi_L^-||\nabla\varphi|dx + 4\Lambda\int_{\Omega_{L,L+1}}|\varphi-\varphi_L^+||\nabla\varphi|dx$$
$$\leq 2\Lambda\left(\int_{\Omega_{-L-1,-L}}|\varphi-\varphi_L^-|^2dx + \int_{\Omega_{L,L+1}}|\varphi-\varphi_L^+|^2dx + \int_{\Omega_{-L-1,-L}\cup\,\Omega_{L,L+1}}|\nabla\varphi|^2dx\right)$$

Due to the uniform Poincaré inequality, ie.

$$\int_{\Omega_{-L-1,-L}}|\varphi-\varphi_L^-|^2dx \leq C\int_{\Omega_{-L-1,-L}}|\nabla\varphi|^2dx,$$

and

$$\int_{\Omega_{L,L+1}}|\varphi-\varphi_L^+|^2dx \leq C\int_{\Omega_{L,L+1}}|\nabla\varphi|^2dx,$$

where C is independent of L, we have

$$\lambda\int_{\Omega_{-L,L}}|\nabla\varphi|^2dx \leq C\int_{\Omega_{-L-1,-L}\cup\,\Omega_{L,L+1}}|\nabla\varphi|^2dx. \tag{4.4}$$

By the estimate (3.6), one has

$$\int_{\Omega_{-L-1,-L}\cup\,\Omega_{L,L+1}}|\nabla\varphi_k|^2dx \leq Cm_0^2\left(|\Omega_{-L-1,-L}| + |\Omega_{L,L+1}|\right), \quad k=1,2,$$

which implies that

$$\int_{\Omega_{-L-1,-L}\cup\,\Omega_{L,L+1}}|\nabla\varphi|^2dx \leq C,$$

where C is independent of L.

Combining this with (4.4) shows

$$\int_{\Omega_{-L-1,-L}}|\nabla\varphi|^2dx \to 0 \quad\text{and}\quad \int_{\Omega_{L,L+1}}|\nabla\varphi|^2dx\to 0, \quad \text{as}\ \ L\to\infty.$$

Taking $L\to\infty$ in (4.4) yields

$$\nabla\varphi = 0 \quad in \ \ \Omega.$$

□

As a direct application of the uniqueness, we can obtain the explicit form of the subsonic solution $\varphi(x)$ to the Problem 1, provided that the nozzle is a cylinder.

Corollary 4.2. *(Cylinder case) Suppose that Ω is a cylinder, that is, $\Omega = S\times(-\infty,+\infty)$, S is a $n-1$ dimensional, simply connected, $C^{2,\alpha}$ domain. Then the unique solution to Problem 1 is given by*

$$\varphi = q_0 x_n + \varphi_0,$$

where φ_0 is an arbitrary constant, q_0 is a constant defined by

$$\rho(q_0^2)q_0 = \frac{m_0}{|S|}.$$

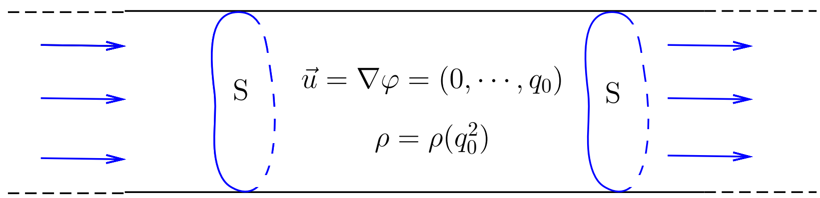

FIGURE 5. Subsonic flow in cylinder case

5. EXISTENCE OF THE CRITICAL INCOMING MASS FLUX

In the Section 3 and Section 4, we have obtained the existence of the uniformly subsonic flows associated with suitable small incoming mass flux m_0 and the uniqueness of the uniformly subsonic flow. In the following, it will be shown that there exists a critical mass flux M_c such that the flow is always uniformly subsonic, provided that the mass flux m_0 is less than M_c.

Theorem 5.1. *Suppose the nozzle satisfies the basic assumptions (1.6). Then there exists a positive constant $M_c \le 1$, which depends only on Ω, such that if $0\le m_0 < M_c$, then the following problem*

$$\begin{cases} \mathit{div}\big(\rho(|\nabla\varphi|^2)\nabla\varphi\big) = 0, & \mathit{in}\quad \Omega,\\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & \mathit{on}\quad \partial\Omega,\\ \displaystyle\int_S \rho\left(|\nabla\varphi|^2\right)\frac{\partial\varphi}{\partial\vec{l}}dS = m_0 & \end{cases}$$

has a unique uniformly subsonic solution $\varphi(x)$ up to a constant satisfying

$$Q(m_0) = \sup_{x\in\bar{\Omega}}|\nabla\varphi| < 1.$$

Moreover, $Q(m_0)$ ranges over $[0,1)$ as m_0 varies in $[0,M_c)$.

Proof. Choosing a strictly increasing sequence $\{q_n\}_{n=1}^{\infty}$ satisfying $\lim\limits_{n\to\infty} q_n = 1$. Consider the following truncated problem

$$\begin{cases} \mathrm{div}\big(\rho_n(|\nabla\varphi|^2)\nabla\varphi\big) = 0, & \text{in}\quad \Omega,\\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & \text{on}\quad \partial\Omega,\\ \displaystyle\int_S \rho_n\left(|\nabla\varphi|^2\right)\frac{\partial\varphi}{\partial\vec{l}}dS = m, & \end{cases} \tag{5.1}$$

where

$$\rho_n(s) = \begin{cases} \rho(s), & \text{if } 0 \le s \le q_n^2, \\ \text{smooth and decreasing} & \text{if } q_n^2 \le s \le \left(\dfrac{q_n+1}{2}\right)^2, \\ \rho\left(\left(\dfrac{q_n+1}{2}\right)^2\right), & \text{if } s > \left(\dfrac{q_n+1}{2}\right)^2, \end{cases}$$

satisfies $\rho_n(s) + 2s\rho_n'(s) > \lambda_n$ with some $\lambda_n > 0$ for all $s \ge 0$. Let $\varphi_n(\cdot; m)$ solve the problem (5.1) and set

$$Q_n(m) = \sup_{x \in \bar{\Omega}} |\nabla \varphi_n(\cdot; m)|.$$

We claim that $Q_n(m)$ is a continuous function of m.

In fact, we take a sequence $m_j \to m$, it suffices to prove

$$|\nabla \varphi_n(\cdot; m_j)| \to |\nabla \varphi_n(\cdot; m)|.$$

Without loss of generality, we assume that there exists a positive constant $\bar{M}$, such that $\sup_{j \ge 1} m_j < \bar{M}$.

It follows from Section 3 that the solution $\varphi_n(\cdot; m_j)$ to the problem (5.1) with the mass flux m_j satisfies the Hölder gradient estimate

$$\| \nabla \varphi_n(\cdot; m_j) \|_{C^{1,\alpha}(\Omega)} \le C(\bar{M}) \tag{5.2}$$

and

$$\| \varphi_n(\cdot; m_j) - \varphi_n(0; m_j) \|_{C^{2,\alpha}(\Omega_L)} \le C(\bar{M}, L) \qquad for \;\; any \;\; L > 0. \tag{5.3}$$

Therefore, by Arzela-Ascoli Lemma and a diagonal argument, there exists a subsequence $\varphi_n(\cdot; m_{j_k}) - \varphi_n(0; m_{j_k})$ such that for any $L > 0$ and $0 < \beta < \alpha$

$$(\varphi_n(\cdot; m_{j_k}) - \varphi_n(0; m_{j_k})) \to \varphi_n(\cdot) \quad \text{in} \quad C^{2,\beta}(\Omega_L) \quad \text{as} \quad m_{j_k} \to m.$$

And $\varphi_n(\cdot)$ solves the boundary value problem (5.1) and satisfies that

$$\| \nabla \varphi_n \|_{C^{1,\alpha}(\Omega)} \le C(\bar{M}).$$

On the other hand, it follows from the previous sections that there exists a $\varphi_n(\cdot; m)$ which solves (5.1). We can conclude that

$$\nabla \varphi_n(\cdot) = \nabla \varphi_n(\cdot; m)$$

by the uniqueness.

Hence, for any $L > 0$

$$\nabla \varphi_n(\cdot; m_j) \to \nabla \varphi_n(\cdot; m) \quad in \;\; C^{1,\beta}(\Omega_L), \quad as \;\; m_j \to m,$$

which proves the claim.

It follows from the claim that, there exists the largest $Q_n > 0$ and the smallest $S_n > 0$ such that

$$q_{n-1} < Q_n(m) < q_n, \qquad \text{for} \quad \text{any} \quad m \in (m_n, M_n).$$

Moreover, clearly $M_{n+1} \ge M_n$. Set $M_c = \lim_{n\to\infty} M_n$. It follows the definition of M_n that $M_n \le \rho(Q_n^2(M_n))Q_n(M_n) < 1$, hence $M_c \le 1$.

Then we can conclude that there exists a critical mass flux $M_c \le 1$, for any $m_0 < M_c$, there is M_n such that $M_n > m_0$, then

$$Q(m_0) = Q_n(m_0) < q_n < 1.$$

Moreover, for any normalized subsonic speed $Q \in (0,1)$, there exists some n, such that $Q \in (0, q_n)$, therefore, there exists a $m_0 \in (0, M_n)$, such that $Q(m_0) = Q_n(m_0) = Q$ by the continuity of $Q_n(m)$.

This completes the proof of Theorem 5.1. □

6. Properties of the subsonic flow

In this section, we consider the asymptotic behavior of the uniformly subsonic flows at the far fields under the asymptotic assumption (1.7).

Proposition 6.1. *Suppose that the nozzle satisfies the asymptotic assumption (1.7). Then the subsonic flow constructed before approaches to uniform flows at the far fields, ie.*

$$\nabla\varphi = (0, \cdots, q_\pm), \qquad as \quad x_n \to \pm\infty,$$

respectively, $q_\pm$ *are constants uniquely determined by*

$$\rho(q_\pm^2)q_\pm = \frac{m_0}{|S_\pm|},$$

respectively.

Proof. Assume that $\varphi(x)$ is a classical solution of

$$\begin{cases} \operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, & \text{in} \quad \Omega, \\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & \text{on} \quad \partial\Omega, \\ \displaystyle\int_S \rho\left(|\nabla\varphi|^2\right)\frac{\partial\varphi}{\partial\vec{l}}dS = m_0, \end{cases}$$

satisfying

$$\| \nabla\varphi \|_{C^{1,\alpha}(\Omega)} \leq Cm_0. \tag{6.1}$$

Step 1. A Special Case. Suppose that $\Omega \cap \{x_n \geq L_0\} = U_+ \times [L_0, +\infty)$ for some L_0. Define a sequence of functions as follows

$$\varphi_k(x', x_n) = \varphi(x', x_n + k)\chi_{\Omega_k},$$

here $\Omega_k = \{(x', x_n) | (x', x_n + k) \in \Omega, \ x_n + k > L_0 + 1\}$.

For any compact set $S \subset \overline{S}_+$ and k sufficiently large, it follows from the gradient estimate (6.1) that

$$\| \nabla\varphi_k \|_{C^{1,\alpha}(S\times[-k/2,k/2])} \leq C,$$

where C is independent of k. Set $\hat{\varphi}_k(x) = \varphi_k(x) - \varphi_k(0)$, for any fixed $L \geq 1$, if $k > 2L$, we have

$$\| \hat{\varphi}_k \|_{C^{2,\alpha}(S\times[-L,L])} \leq C,$$

with C independent of k. Therefore, by Ascoli-Arzela Lemma and a diagonal procedure, there exists a subsequence $\hat{\varphi}_{k_j}$, such that for any L

$$\hat{\varphi}_{k_j} \to \varphi_0, \qquad \text{in} \quad C^{2,\beta}(S \times [-L, L]) \quad \text{with} \quad \beta < \alpha,$$

for any compact set $S \subset \overline{S}_+$. Therefore φ_0 solves the following problem

$$\begin{cases} \mathrm{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, & \text{in} \quad E_+ = S_+ \times (-\infty, +\infty), \\ \dfrac{\partial\varphi}{\partial\vec{n}} = 0, & \text{in} \quad \partial S_+ \times (-\infty, +\infty), \\ \displaystyle\int_S \rho\left(|\nabla\varphi|^2\right) \frac{\partial\varphi}{\partial\vec{l}} dS = m_0, & \\ \varphi(x) = 0, & \text{on} \quad x_n = 0. \end{cases} \tag{6.2}$$

Moreover,

$$\nabla\varphi_k = \nabla\hat{\varphi}_k \to \nabla\varphi_0 \quad \text{in} \quad C^{1,\mu}(S \times [-L, L]) \quad \text{for} \quad \mu < \beta.$$

So, choosing $S = \overline{S}_+$ and $L = 2$, we have

$$\| \nabla\varphi_k - \nabla\varphi_0 \|_{C^\mu(\overline{S}_+ \times [-2,2])} \to 0 \qquad \text{as} \quad k \to +\infty.$$

By the definition of φ_k and Corollary 4.2, it follows that

$$\nabla\varphi \to \nabla\varphi_0 = (0, \cdots, q_+) \quad \text{as} \quad x_n \to +\infty.$$

This completes the proof of Proposition 6.1 in this special case.

Step 2. General Case. Suppose now that the nozzle satisfies (1.7). we can also define a sequence of functions as

$$\varphi_k(x', x_n) = \varphi(x', x_n + k)\chi_{\Omega_k},$$

here $\Omega_k = \{(x', x_n) | (x', x_n + k) \in \Omega, \ x_n + k > 1\}$. Then similar to the Step 1, we can show that

$$\nabla\varphi_{k_j} \to \nabla\varphi_0 \quad \text{in} \quad C^{1,\beta}(S \times [-L, L]) \tag{6.3}$$

for any compact set $S \subset S_+$ and any fixed L, here S may not reach the boundary ∂S_+, and φ_0 is still the solution of boundary value problem (6.2).

In particular, φ_0 satisfies the no-flow boundary condition on the nozzle wall. Indeed, for any given point $(y', y_n) \in \partial S_+ \times (-\infty, +\infty)$, $\vec{n} = (\vec{n}_1, 0)$ is the outer normal direction of the cylinder $S_+ \times (-\infty, +\infty)$ at (y', y_n). For any $\delta > 0$, there exists suitable large $K_0 > 0$, such that

$$(y' - \delta\vec{n}_1, y_n + k) \in S \times \{x_n = y_n + k\} \quad \text{for} \quad k > K_0,$$

where S is a compact set of $\Omega \cap \{x_n = y_n + k\}$.

There exists a sequence of $n-1$ dimensional vectors $\{\vec{z}_k\}_{k=1}^\infty$, such that $(y' - \delta\vec{n}_1 + \vec{z}_k, y_n + k) \in \partial\Omega$, and $|\vec{z}_k| = dist((y' - \delta\vec{n}_1, y_n + k), \partial\Omega)$. $\vec{n}_k$ is the out normal of the domain Ω at $(y' - \delta\vec{n}_1 + \vec{z}_k, y_n + k)$. Obviously,

$$\lim_{k\to+\infty} |\vec{z}_k| \to 0, \quad \text{and} \quad \lim_{k\to+\infty} \vec{n}_k \to \vec{n},$$

due to the assumption (1.7) on the nozzle at the far fields.

Therefore

$$\begin{aligned} \nabla\varphi_0(y' - \delta\vec{n}_1, y_n) \cdot \vec{n} &= \lim_{k\to+\infty} \nabla\varphi(y' - \delta\vec{n}_1, y_n + k) \cdot \vec{n} \\ &= \lim_{k\to+\infty} \left(\nabla\varphi(y' - \delta\vec{n}_1, y_n + k) - \nabla\varphi(y' - \delta\vec{n_1} + \vec{z}_k, y_n + k)\right) \cdot \vec{n} \\ &\quad + \lim_{k\to+\infty} \nabla\varphi(y' - \delta\vec{n}_1 + \vec{z}_k, y_n + k) \cdot (\vec{n} - \vec{n}_k) \\ &= 0. \end{aligned}$$

As a consequence, $\dfrac{\partial\varphi_0}{\partial\vec{n}}(y', y_n) = 0$.

Set

$$\Omega'(\delta) = \left\{ x' \in \overline{\Omega} \cap \{x_n = k\} \middle| \, dist(x', \partial\Omega) < \delta \right\}, \quad \Omega'_0(\delta) = (\Omega \cap \{x_n = k\}) \setminus \Omega'(\delta),$$

and

$$B_{i,\delta}(y'_i) = \left\{ x' \in \mathbb{R}^{n-1} \middle| \, |x' - y'_i| < 2\delta \right\}, \quad y'_i \in \partial\Omega \cap \{x_n = k\}, \quad i = 1, \cdots, N,$$

such that

$$\Omega'(\delta) \subset \bigcup_{i=1}^{N} B_{i,\delta}(y_i), \quad \text{for any} \quad \delta > 0.$$

For any fixed $\delta > 0$, there exists a sufficiently large $K_0 > 0$, such that $\Omega'_0(\delta) \subset S$ for $k > K_0$, S is a compact set in S_+.

Choosing $L = 2$, one has from (6.3) that

$$\| \nabla\varphi - \nabla\varphi_0 \|_{C^\mu\left(\Omega'_0(\delta)\times[k,k+2]\right)} \to 0 \quad as \quad k \to +\infty, \quad \text{for} \quad \mu < \beta. \tag{6.4}$$

Near the boundary $\partial\Omega \cap \{x_n = k\}$, φ possesses the following estimates

$$\| \nabla\varphi \|_{C^\eta\left(B^+_{i,\delta}(y'_i)\times(k,k+2)\right)} \leq C.$$

for $\eta > 0$, with $B^+_{i,\delta}(y'_i) = B_{i,\delta}(y'_i) \cap \left(\bar{\Omega} \cap \{x_n = k\}\right)$. Hence, for any $x', y' \in B^+_{i,\delta}(y'_i)$, one has

$$|\nabla\varphi(x', k) - \nabla\varphi(y', k)| < C\delta^\eta.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|\nabla\varphi(x', k) - \nabla\varphi(y', k)| < \frac{\varepsilon}{N+1}, \qquad \text{for any} \quad x', \; y' \in B^+_{i,\delta}(y'_i), \tag{6.5}$$

and $i = 1, 2, \cdots, N$.

On the other hand, it follows from (6.4) that there exists $K > 0$ such that

$$|\nabla\varphi - \nabla\varphi_0| \leq \frac{\varepsilon}{N+1}, \qquad \text{for any} \quad x \in \Omega'_0(\delta) \times (K, +\infty). \tag{6.6}$$

Then, combining that (6.5) and (6.6), one can conclude

$$|\nabla\varphi - (0, \cdots, q_+)| < \varepsilon, \qquad \text{for any} \quad x_n > K.$$

Similarly, one can get the asymptotic behavior as $x_n \to -\infty$. This completes the proof of Proposition 6.1. □

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