Analytical Solutions to the Compressible Navier-Stokes Equations with Density-dependent Viscosity Coefficients and Free Boundaries

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Abstract

In this paper, we study a class of analytical solutions to the compressible Navier-Stokes equations with density-dependent viscosity coefficients, which describe compressible fluids moving into outer vacuum. For suitable viscous polytropic fluids, we construct a class of radial symmetric and self-similar analytical solutions in $\mathbb{R}^N (N \geq 2)$ with both continuous density condition and the stress free condition across the free boundaries separating the fluid from vacuum. Such solutions exhibit interesting new information such as the formation of vacuum at the center of the symmetry as time tends to infinity and explicit regularities and large time decay estimates of the velocity field.

1 Introduction

The compressible Navier-Stokes equations with density-dependent viscosity coefficients can be written as

$$\rho_t + \mathrm{div}(\rho \mathbf{U}) = 0, \tag{1.1}$$

$$(\rho \mathbf{U})_t + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \mathrm{div}(h(\rho) D(\mathbf{U})) - \nabla (g(\rho) \mathrm{div} \mathbf{U}) + \nabla P(\rho) = 0, \tag{1.2}$$

where $t \in (0, +\infty)$ is the time and $\mathbf{x} \in \mathbb{R}^N, N = 2, 3$ is the spatial coordinate, while $\rho(\mathbf{x}, t), \mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ stand for the fluid density, velocity and pressure, respectively. And

$$D(\mathbf{U}) = \frac{\nabla \mathbf{U} +^t \nabla \mathbf{U}}{2}$$

is the strain tensor, $h(\rho)$ and $g(\rho)$ are the Lamé viscosity coefficients satisfying

$$h(\rho) > 0, h(\rho) + Ng(\rho) \geq 0. \tag{1.3}$$

In the last several decades, significant progress on the system (1.1)-(1.2) has been achieved by many authors since it was introduced by Liu, Xin and Yang in [22]. Meanwhile, in geophysical flows, many mathematical models correspond to (1.1)-(1.2) (see [21]). In particular, the viscous Saint-Venant system for shallow water is expressed exactly as (1.1)-(1.2) with $N = 2, h(\rho) = \rho, g(\rho) = 0$ and $\gamma = 2$. Local smooth solutions or global smooth solutions for data close to equilibrium were established in [30] and related topics have been extensively studied by many authors. For one-dimensional compressible Navier-Stokes equations (1.1) and (1.2) with $h(\rho) = \rho^\alpha, g(\rho) = 0(\alpha \in (0, \frac{3}{2}))$ and free boundary conditions, there are many literatures on the well-posedness theory of the solutions (see[7], [9], [14], [15], [22], [27], [32], [39], [40], [41], [20] and references therein). However, few results are available for multi-dimensional problems. The first multi-dimensional result is due to Mellet and Vasseur [26], where they had proved the L^1 stability of weak solutions to the system (1.1)-(1.2) based on a new entropy estimate established in [1],[2] in a priori way, extending the corresponding L^1 stability results of [2] and [1]. However, although L^1 stability is considered as one of the main steps to prove existence of weak solutions, the global existence of weak solutions of Korteweg's system (see [2]) and the compressible Navier-Stokes equations with density-dependent viscosity (1.1)-(1.2) remains open in the multi-dimensional cases. The first successful example due to Guo, Jiu and Xin in [8] with spherically symmetric initial data and fixed boundary conditions and later Guo, Li, and Xin extended it to the free boundary conditions with discontinuously symmetric initial data ([5]). A local existing result was given by Chen and Zhang in [3] with spherically symmetric initial data between a solid core and a free boundary connected to a surrounding vacuum state. Recently, we have discussed the Lagrangian structure and large time behavior of solutions to compressible spherically symmetric Navier-Stokes equation with density-dependent viscosity coefficients both under fixed boundary conditions and free boundary conditions of discontinuous initial data(see [5],[6]).

In particular in [5], we had discussed the global existence, the Lagrangian structure and large time behavior of solutions to the compressible spherically symmetric Navier-Stokes equations (1.1)-(1.2) with the following stress free boundary conditions

$$\rho^\gamma = \rho u_r, \text{ on } r = a(t),$$

with $h(\rho) = \rho, g(\rho) = 0$ and $\gamma \in [2, \frac{N}{N-2})$. Notice that our results in [5] can be also generalized to multi-dimensional compressible spherically symmetric Navier-Stokes equations (1.1)-(1.2) with general viscosity coefficients $h(\rho) = \rho^\theta$ and $g(\rho) = (\theta-1)\rho^\theta$ with $\frac{N-1}{N} < \theta \leq 1$ with $N \geq 2$ the space dimension or supplying with the following free boundary

$$\rho^\gamma = \theta\rho^\theta(u_r + \frac{2u}{r}), \text{ on } r = a(t).$$

On the other hand, there are many references considering the analytical solutions or blowup solutions to the Navier-Stokes equations, Navier-Stokes-Poisson equations or Euler-Poisson equations (see [4], [35]-[38]). It is worthy mentioning that, L. H.

Yeung and M. W. Yuen in [38], considered (1.1)-(1.2) with $h(\rho) = 0, g(\rho) = \rho^\theta$ in radial symmetry and $\mathbb{R}^N$, they looked for a family of solutions of the form

$$\rho(r,t) = \frac{y(\frac{r}{a(t)})}{a(t)^N}, u(r,t) = \frac{a'(t)}{a(t)}r, \tag{1.4}$$

and derived that $a(t), y(z) \in C^1$ are two functions satisfying some ordinary differential equations. Yet the solvability of such two ordinary differential equations is not discussed.

In the present paper, we will construct a class of global analytical solutions to the three dimensional compressible Navier-Stokes equations (1.1)-(1.2) with spherically symmetric initial data and free boundaries separating the fluid from the vacuum. Here $h(\rho) = \rho^\theta, g(\rho) = (\theta-1)\rho^\theta$ in which $h(\rho)$ and $g(\rho)$ satisfying $g(\rho) = \rho h'(\rho) - h(\rho)$ and $\gamma = \theta > 1$ or $\gamma = 2\theta - \frac{1}{3}, \gamma > 1$. We assume that there exists a curve which separate the gas and the vacuum, and that the fluid moves continuously across this curve into vacuum or jump into vacuum through this curve but keep stress-free (see (2.10) and (2.11) below). Motivited by the construction in [38], we will also look for solutions of the form (1.4) which satisfy (1.1)-(1.2) and (2.10) or (2.11). Indeed, we can show that $a(t)$ and $y(z) \in C^1$ exist globally satisfying two ordinary differential equations, which yield the desired solution given in the form of (1.4). It should be emphasized that such a class of analytical solutions are new and exhibit a great deal of information, such as the formation of vacuum at the center of the symmetry as time approaches infinity and explicit regularity and decay estimates on the velocity field, etc.., see Remark 2.2. Our results also imply that the domain, where fluid is located on, expands outwards into vacuum at an algebraic rate as the time goes large due to the dispersion effect of the total pressure.

Now we explain more on the motivations of this paper. First, in our previous work [5], we had obtained the global existence of weak solutions to the system (1.1)-(1.2) with spherically symmetric discontinuous initial data and $\frac{N}{N-1} < \theta \leq 1$. But it seems difficult to apply the method in [5] for either the case $\theta > 1$ or more importantly the case of continuous initial density (degenerate boundary conditions) due to the high degeneracy at vacuum. The present paper provides a concrete analytic solution to the system (1.1)-(1.2) for both cases. Furthermore, these analytic solutions yield much more information on the structure of solutions than the ones given in [5] for the case of spherically symmetric discontinuous initial data and $\frac{N}{N-1} < \theta \leq 1$.

Second, note that in [38], since the viscosity coefficients are given by $h(\rho) = 0, g(\rho) = \rho^\theta$. Thus an entropy estimate, such as (2.16) in our case, is not expected. This may explain the reason that there was no existence of global C^1 of $a(t)$ in [38]. However, as discussed as [1],[2], that the choice of viscosity coefficients, $h(\rho) = \rho^\theta, g(\rho) = (\theta - 1)\rho^\theta$ is physically more reasonable , which yields the desired entropy estimate (2.16). Based on this, the global existence of solution to the free boundary problem (2.7)-(2.11) can be established as shown in this paper. More importantly, we obtain the large time behavior of the analytical solution and the free boundary $a(t)$ to the free boundary problem (2.7)-(2.11) .

The plan of this paper is as follows. In Section 2 we give the main results of this paper. In Section 3 we will first give the self-similar solution to the equation (1.1), and then Theorem 2.1 and 2.2 will be proved in the following Section 3.1 and 3.2, respectively. In Section 4, we will give some examples of blow-up solutions.

2 Notations and main results

Set $h(\rho) = \rho^\theta$ and $g(\rho) = (\theta - 1)\rho^\theta, N = 3$ in (1.1)-(1.2). Then the isentropic compressible Navier-Stokes system with density-dependent viscosity coefficients become

$$\rho_t + \mathrm{div}(\rho\mathbf{U}) = 0, \tag{2.1}$$
$$(\rho\mathbf{U})_t + \mathrm{div}(\rho\mathbf{U}\otimes\mathbf{U}) - \mathrm{div}(\rho^\theta D(\mathbf{U})) - (\theta-1)\nabla(\rho^\theta \mathrm{div}\mathbf{U}) + \nabla P(\rho) = 0 \tag{2.2}$$

for $t \in (0, +\infty)$ and $\mathbf{x} \in \mathbb{R}^3$. Here $\rho(\mathbf{x}, t), \mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ are the same as in (1.1)-(1.2). The initial conditions of (2.1)-(2.2) are imposed as:

$$(\rho, \rho\mathbf{U})|_{t=0}(\mathbf{x}) = (\rho_0, \mathbf{m}_0)(\mathbf{x}), 0 \le |\mathbf{x}| \le a_0. \tag{2.3}$$

For simplicity, we will take $D(\mathbf{U}) = \nabla\mathbf{U}$ in (2.2), though the full strain tensor could be considered without any additional difficulty. The boundary conditions are imposed as either the stress free condition,

$$\rho^\theta\nabla\mathbf{U} + (\theta-1)\rho^\theta \mathrm{div}\mathbf{U} = P(\rho)I, \text{ on } \mathbf{x} = \partial\Omega_t, \tag{2.4}$$

or the continuous density condition

$$\rho = 0, \text{ on } \mathbf{x} = \partial\Omega_t, \tag{2.5}$$

where $\partial\Omega_t = \psi(\partial\Omega_0, t)$ is a free boundary separating fluid from vacuum. Here, $\partial\Omega_0 = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| = a_0\}$ is the initial free boundary and ψ is the flow of $\mathbf{U}$:

$$\begin{cases} \partial_t\psi(\mathbf{x}, t) = \mathbf{U}(\psi(\mathbf{x}, t), t), \mathbf{x} \in \mathbb{R}^3, \\ \psi(\mathbf{x}, 0) = \mathbf{x}. \end{cases}$$

In this paper, we are concerned with the following spherically symmetric solutions to the initial-boundary value problem (2.1)-(2.4). To this end, we denote

$$|\mathbf{x}| = r, \rho(\mathbf{x}, t) = \rho(r, t), \mathbf{U}(\mathbf{x}, t) = u(r, t)\frac{\mathbf{x}}{r}. \tag{2.6}$$

This leads to the following system of equations for $r > 0$,

$$\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \tag{2.7}$$
$$(\rho u)_t + (\rho u^2 + \rho^\gamma)_r + \frac{2\rho u^2}{r} - \theta\big(\rho^\theta(u_r + \frac{2u}{r})\big)_r + (\rho^\theta)_r(\frac{2u}{r}) = 0, \tag{2.8}$$

with the initial condition

$$(\rho, u)|_{t=0} = (\rho_0(r), u_0(r)), 0 \le r \le a_0, \tag{2.9}$$

where $a_0 > 0$ is a constant, and the free boundary condition

$$\rho(a(t), t) = 0, \tag{2.10}$$

or

$$\rho^\gamma = \theta\rho^\theta(u_r + \frac{2u}{r}), \text{ on } a(t), \tag{2.11}$$

where

$$a'(t) = u(a(t), t), a(0) = a_0, \forall t \geq 0. \tag{2.12}$$

It is easy to get the following usual a priori energy estimate for smooth solutions to (2.7), (2.8) and the boundary condition (2.10) or (2.11):

$$\begin{aligned}\tfrac{d}{dt}\textstyle\int_0^{a(t)}(\tfrac{1}{2}\rho u^2 + \tfrac{1}{\gamma-1}\rho^\gamma)r^2dr + (\theta-1)\int_0^{a(t)}\rho^\theta(u_r r + 2u)^2dr \\ + \textstyle\int_0^{a(t)}\rho^\theta(u_r^2r^2 + 2u^2)dr \leq 0, \forall\theta \geq 1.\end{aligned} \tag{2.13}$$

However, the system (2.1)-(2.2) with the boundary conditions $\rho\mathbf{U} = 0$ admits an additional a priori estimate, as observed by Bresch, Desjardins and Lin [2], which reads as follows

Lemma 2.1. *(see [26]) Assume that $h(\rho)$ and $g(\rho)$ are two C^2 functions such that*

$$g(\rho) = \rho h'(\rho) - h(\rho).$$

Then, the following inequality holds for any smooth solutions of (2.1)-(2.2) with the fixed boundary condition $\rho\mathbf{U} = 0$ and $\rho > 0$:

$$\frac{d}{dt}\int_\Omega(\frac{1}{2}\rho|\mathbf{U} + \nabla\varphi(\rho)|^2 + \frac{1}{\gamma-1}\rho^\gamma)dx + \int_\Omega\nabla\varphi(\rho)\cdot\nabla\rho^\gamma dx \leq 0, \tag{2.14}$$

with φ being an enthalpy such that

$$\varphi'(\rho) = \frac{h'(\rho)}{\rho}.$$

In particular, for three-dimensional spherically symmetric solutions to (2.1)-(2.2) with the continuous density condition, (2.10), one has

$$\frac{d}{dt}\int_0^{a(t)}(\frac{1}{2}\rho|u + \theta\rho^{\theta-2}\rho_r|^2 + \frac{1}{\gamma-1}\rho^\gamma)r^2dr + \int_0^{a(t)}\gamma\theta\rho^{\gamma+\theta-3}\rho_r^2r^2dr \leq 0. \tag{2.15}$$

While, for the stress free condition (2.11), Lemma 2.1 is no longer true. Fortunately, we can obtain the desired entropy estimate for three-dimensional spherically symmetric solutions (2.1)-(2.2) with (2.11) in a similar way as Lemma 2.1, and the proof can be found in [6]. More precisely

Lemma 2.2. *If (ρ, u) is a smooth solution to (2.1)-(2.2) with (2.11) and $\rho > 0$, then the following inequality holds:*

$$\frac{d}{dt}\int_0^{a(t)}\frac{1}{2}\rho|u + \theta\rho^{\theta-2}\rho_r|^2r^2dr + \int_0^{a(t)}\gamma\theta\rho^{\gamma+\theta-3}\rho_r^2r^2dr \leq 0. \tag{2.16}$$

i.e.,

$$\begin{aligned}\tfrac{d}{dt}\textstyle\int_0^{a(t)}\{\tfrac{1}{2}\rho u^2 + (\rho^\theta)_r u + \tfrac{\theta^2}{2(\theta-\frac{1}{2})^2}|(\rho^{\theta-\frac{1}{2}})_r|^2\}r^2dr \\ + \textstyle\int_0^{a(t)}\tfrac{4\gamma\theta}{(\gamma+\theta-1)^2}((\rho^{\frac{\gamma+\theta-1}{2}})_r r)^2dr \leq 0.\end{aligned} \tag{2.17}$$

Our main result is the following theorem:

Theorem 2.1. *Assume that* $\gamma = \theta > 1$. *For the radial symmetry compressible Navier-Stokes equations* (2.7)-(2.8)*, with the continuous density condition* (2.10)*, there exists a solution of the form*

$$\rho(r,t) = \frac{[\frac{1}{2}(\theta-1)(\frac{r^2}{a^2(t)}-1)]^{\frac{1}{\theta-1}}}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)}r. \tag{2.18}$$

Where $a(t) \in C^1([0,\infty))$ *is the free boundary satisfying* (2.12) *and exists for all* $t \geq 0$. *Furthermore,* $a(t)$ *tends to* $+\infty$ *as* $t \to +\infty$ *with the following rates:*

$$C_1(1+t)^{\frac{2-\alpha}{3(\theta-1)}} \leq a(t) \leq C_2(1+t)^{\frac{9\theta-8}{3\theta-3}}, \forall t \geq 0. \tag{2.19}$$

Where C_1 *and* C_2 *are constants and*

$$\alpha = \begin{cases} 1, & \text{if } \theta > \frac{4}{3}, \\ 1+\sigma, \sigma > 0 \text{ small }, & \text{if } \theta = \frac{4}{3}, \\ 5-3\theta, & \text{if } 1 < \theta < \frac{4}{3}. \end{cases} \tag{2.20}$$

Remark 2.1. *The solution* (2.18) *constructed in Theorem 2.1 satisfies the following properties:*

$$\rho(0,t) \to 0, \text{ as } t \to +\infty, \tag{2.21}$$

$$|\Omega(t)| = \frac{4}{3}\pi a^3(t) \to +\infty, \text{ as } t \to +\infty, \tag{2.22}$$

where the domain of the fluid is defined by

$$\Omega(t) = \{(r,t) \in \mathbb{R}^3 \times [0,\infty) | 0 \leq r \leq a(t), t \geq 0\}.$$

On the other hand, for the stress free boundary condition (2.11), we have:

Theorem 2.2. *Assume that* $\gamma = 2\theta - \frac{1}{3}, \gamma > \frac{5}{3}$. *Then the free boundary value problem for the radial symmetry compressible Navier-Stokes system,* (2.7)-(2.8)*, with the stress free condition* (2.11)*, has a unique solution with the free boundary* $r = a(t)$ *given by*

$$a(t) = f^{\frac{1}{3}}(1)\big[\rho_0^{\theta-\gamma}(a_0) + \frac{\gamma-\theta}{\theta}t\big]^{\frac{1}{3(\gamma-\theta)}}, \tag{2.23}$$

and

$$\rho(r,t) = \frac{f(\frac{r}{a(t)})}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)}r = \frac{r}{3\theta\rho_0^{\theta-\gamma}(a_0) + 3(\gamma-\theta)t}. \tag{2.24}$$

Where $f(z) > 0, f(z) \in C([0,1]) \cap C^1((0,1])$ *satisfies*

$$\frac{6(\gamma-1)}{(3\gamma+1)^2}z - \gamma f(1)^{\frac{1}{3}-\gamma} f^{\gamma-2}(z)f'(z) + \frac{\gamma-1}{2}f(1)^{\frac{1}{6}-\frac{\gamma}{2}} f^{\frac{3\gamma-11}{6}}(z)f'(z) = 0. \tag{2.25}$$

Remark 2.2. *It can be verified easily that the solution,* (2.23) *in Theorem 2.2, satisfies the following properties:*

$$\rho(0,t), \rho(a(t),t) \to 0, \ \text{as } t \to +\infty, \tag{2.26}$$

$$|\Omega(t)| = \frac{4}{3}\pi a^3(t) \to +\infty, \ \text{as } t \to +\infty. \tag{2.27}$$

Moreover, for $0 \le r \le a(t)$, *we have*

$$|u(r,t)| = \frac{|a'(t)|}{a(t)} r \le |a'(t)| \le C(1+t)^{-\frac{\gamma-1}{\gamma-\frac{1}{3}}} \to 0, \ \text{as } t \to +\infty, \tag{2.28}$$

$$|u_r(r,t)| = \frac{1}{3\theta\rho_0^{\theta-\gamma}(a_0) + 3(\gamma-\theta)t} \to 0, \ \text{as } t \to +\infty. \tag{2.29}$$

3 The proofs of the main theorems

First, we will give a self-similar solution to the continuity equation (2.7), which was derived in [35]:

Lemma 3.1. *For any two* C^1 *functions* $f(z) \ge 0$ *and* $a(t) > 0$, *define*

$$\rho(r,t) = \frac{f(\frac{r}{a(t)})}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)} r. \tag{3.1}$$

Then $(\rho,u)(r,t)$ *solves the continuity equation* (2.7), *i. e.,*

$$\rho_t + \rho_r u + \rho u_r + \frac{2\rho u}{r} = 0. \tag{3.2}$$

Here, we can choose $a(t)$ as a free boundary which satisfying the condition(2.10) or (2.11). So in the following we will determine the form of the function $f(x)$ and then prove the global existence of the free boundary $a(t)$.

To this end, denoting $z = \frac{r}{a(t)}$, one can obtain from (2.8) that

$$f(z)\frac{a''(t)}{a^4(t)} r + \gamma f^{\gamma-1}(z) f'(z) \frac{1}{a^{3\gamma+1}(t)} + (2-3\theta)\theta f^{\theta-1}(z) f'(z) \frac{a'(t)}{a^{3\theta+2}(t)} = 0. \tag{3.3}$$

Next, we will solve (3.3) according to the free boundary conditions (2.10) or (2.11) respectively.

3.1 The continuous boundary condition

Assume that $\gamma = \theta > 1$. We require

$$z = f^{\theta-2}(z) f'(z). \tag{3.4}$$

Then it follows from (3.3)-(3.4) that

$$f(z) = [f^{\theta-1}(1) + \frac{1}{2}(\theta-1)(z^2-1)]^{\frac{1}{\theta-1}}, \tag{3.5}$$

$$a''(t) + \theta a^{2-3\theta}(t) + (2-3\theta)\theta a'(t) a^{1-3\theta}(t) = 0. \tag{3.6}$$

Using the boundary condition (2.10) yields $f(z) = [\frac{1}{2}(\theta-1)(z^2-1)]^{\frac{1}{\theta-1}}$ and then

$$\rho(r,t) = \frac{[\frac{1}{2}(\theta-1)(\frac{r^2}{a^2(t)}-1)]^{\frac{1}{\theta-1}}}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)}r. \tag{3.7}$$

Clearly, if $a(t) > 0$ is a free boundary satisfying the condition (2.10), then it is straightforward to check that (ρ, u) defined by (3.7) is a solution of (2.7)-(2.8), where $a(t)$ can be determined by

$$\begin{cases} a'(t) = a_1 + \theta a_0^{2-3\theta} - \theta a^{2-3\theta}(t) - \theta \int_0^t a^{2-3\theta}(s)ds, \\ a(0) = a_0, a'(0) = a_1, \end{cases} \tag{3.8}$$

with $a_0 > 0$ and a_1 be the initial location and slope of the free boundary.

Thus, it remains to solve the boundary value problem (3.8). We start with estimates on solutions of (3.8).

Lemma 3.2. *Let $a(t) \in C^1[0,1]$ be a solution to (3.8) for $T > 0$. Then there exist two uniform positive constants C_1 and $C_2 > 0$, independent of T, such that*

$$C_1(1+t)^{\frac{2-\alpha}{3(\theta-1)}} \le a(t) \le C_2(1+t)^{\frac{9\theta-8}{3\theta-3}}, \quad \text{for all } t \in [0,T], \tag{3.9}$$

where

$$\alpha = \begin{cases} 1, & \text{if } \theta > \frac{4}{3}, \\ 1+\sigma, \sigma > 0 \text{ small }, & \text{if } \theta = \frac{4}{3}, \\ 5-3\theta, & \text{if } 1 < \theta < \frac{4}{3}. \end{cases} \tag{3.10}$$

Proof. We first verify the fact that $a(t) \ge C_1(1+t)^{\frac{2-\alpha}{3(\theta-1)}}$ for all $t \in [0,T]$. Note that (3.7) implies

$$\rho(a(t),t) = 0, u(0,t) = 0. \tag{3.11}$$

We introduce

$$\begin{aligned} H(t) &= \int_0^{a(t)} (r-(1+t)u)^2 \rho r^2 dr + \frac{2}{\gamma-1}(1+t)^2 \int_0^{a(t)} \rho^\gamma r^2 dr \\ &= \int_0^{a(t)} \rho r^4 dr - 2(1+t)\int_0^{a(t)} \rho u r^3 dr + (1+t)^2 \int_0^{a(t)} \rho u^2 r^2 dr \\ &\quad + \frac{2}{\gamma-1}(1+t)^2 \int_0^{a(t)} \rho^\gamma r^2 dr. \end{aligned} \tag{3.12}$$

Due to $a'(t) = u(a(t),t)$ and (3.11), a direct computation gives

$$\begin{aligned} H'(t) &= \int_0^{a(t)} (\rho_t r^4 - 2\rho u r^3)dr + (1+t)^2 \int_0^{a(t)} \Big((\rho u^2)_t + \frac{2}{\gamma-1}(\rho^\gamma)_t\Big) r^2 dr \\ &\quad +2(1+t)\int_0^{a(t)} \Big(\rho u^2 r^2 - (\rho u)_t r^3 + \frac{2}{\gamma-1}\rho^\gamma r^2\Big)dr \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.13}$$

(2.7) and (3.11) yield

$$I_1 = -\int_0^{a(t)} \big(r^2(r^2\rho u)_r + 2\rho u r^3\big)dr = -\int_0^{a(t)} (\rho u r^4)_r dr = 0.$$

Similarly, one has

$$\begin{aligned}
I_2 &= (1+t)^2 \int_0^{a(t)} \big(\rho_t u^2 + 2\rho u u_t + \frac{2\gamma}{\gamma-1}\rho^{\gamma-1}\rho_t\big)r^2 dr \\
&= (1+t)^2 \int_0^{a(t)} \big(-(r^2\rho u)_r u^2 - 2\rho u^2 u_r r^2 - 2ur^2(\rho^\gamma)_r + 2\gamma u r^2[\rho^\gamma(u_r + \frac{2u}{r})]_r \\
&\quad -4(\rho^\gamma)_r u^2 r - \frac{2\gamma\rho^{\gamma-1}}{\gamma-1}(r^2\rho u)_r\big)dr \\
&= (1+t)^2 \int_0^{a(t)} \big\{[2\gamma\rho^\gamma(uu_r r + 2u^2 r) - 4\rho^\gamma u^2 r]_r - (r^2\rho u^3)_r - (\frac{2\gamma\rho^\gamma u r^2}{\gamma-1})_r \\
&\quad -2\gamma\rho^\gamma u_r^2 r^2 + (4-8\gamma)\rho^\gamma u^2 + (8-8\gamma)\rho^\gamma u u_r r\big\}dr \\
&= -4(\gamma-1)(1+t)^2 \int_0^{a(t)} \rho^\gamma(u_r r + u)^2 dr + (4-6\gamma)(1+t)^2 \int_0^{a(t)} \rho^\gamma(u_r^2 r^2 + 2u^2)dr \\
&\le (4-6\gamma)(1+t)^2 \int_0^{a(t)} \rho^\gamma(u_r^2 r^2 + 2u^2)dr.
\end{aligned}$$

Next, I_3 can be treated as follows.

$$\begin{aligned}
I_3 &= 2(1+t)\int_0^{a(t)} \big\{3\rho u^2 r^2 + (\rho u^2)_r r^3 + (\rho^\gamma)_r r^3 + [(2-2\gamma)\rho^\gamma u r^2 - \gamma\rho^\gamma u_r r^3]_r \\
&\quad +(3\gamma-2)\rho^\gamma(u_r r^2 + 2ur) + \frac{2}{\gamma-1}\rho^\gamma r^2\big\}dr \\
&= 2(1+t)\int_0^{a(t)} \big[(3\gamma-2)\rho^\gamma(u_r r^2 + 2ur) + \frac{5-3\gamma}{\gamma-1}\rho^\gamma r^2\big]dr.
\end{aligned}$$

Thus, substituting above estimates into (3.13) and using the Cauchy-Schwarz's inequality, one may deduce

$$\begin{aligned}
H'(t) &= \frac{2(5-3\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma r^2 dr + 2(3\gamma-2)(1+t)\int_0^{a(t)} \rho^\gamma u_r r^2 dr \\
&\quad +4(3\gamma-2)(1+t)\int_0^{a(t)} \rho^\gamma u r dr - 4(3\gamma-2)(1+t)^2 \int_0^{a(t)} \rho^\gamma u^2 dr \\
&\quad -2(3\gamma-2)(1+t)^2 \int_0^{a(t)} \rho^\gamma u_r^2 r^2 dr \\
&\le \frac{2(5-3\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma r^2 dr + \frac{3(3\gamma-2)}{2}\int_0^{a(t)} \rho^\gamma r^2 dr, \qquad (3.14)
\end{aligned}$$

where one has used

$$\begin{aligned}
2(1+t)\int_0^{a(t)} \rho^\gamma u_r r^2 dr &\le 2(1+t)^2 \int_0^{a(t)} \rho^\gamma u_r^2 r^2 dr + \frac{1}{2}\int_0^{a(t)} \rho^\gamma r^2 dr, \\
4(1+t)\int_0^{a(t)} \rho^\gamma u r dr &\le 4(1+t)^2 \int_0^{a(t)} \rho^\gamma u^2 dr + \int_0^{a(t)} \rho^\gamma r^2 dr.
\end{aligned}$$

Note also that the conservation of total mass implies that

$$\int_0^{a(t)} \rho r^2 dr = \int_0^{a_0} \rho_0 r^2 dr = 1.$$

In the case $\gamma \geq \frac{5}{3}$, (3.14) yields

$$H'(t) \leq \frac{3(3\gamma-2)(\gamma-1)}{2} E_0, E_0 = \int_0^{a_0} (\frac{1}{2}\rho_0 u_0^2 + \frac{\rho_0^\gamma}{\gamma-1}) r^2 dr, \tag{3.15}$$

and so

$$H(t) \leq H(0) + \frac{3(3\gamma-2)(\gamma-1)}{2} E_0 t. \tag{3.16}$$

and consequently

$$\int_0^{a(t)} \rho^\gamma r^2 dr \leq C(1+t)^{-1}. \tag{3.17}$$

Thus, as a consequence of (3.17) and the conservation of mass, it holds that for any $t > 0$,

$$1 = \int_0^{a(t)} \rho r^2 dr \leq \left(\int_0^{a(t)} \rho^\gamma r^2 dr\right)^{\frac{1}{\gamma}} \left(\int_0^{a(t)} r^2 dr\right)^{\frac{\gamma-1}{\gamma}} \leq Ca(t)^{3-\frac{3}{\gamma}}(1+t)^{-\frac{1}{\gamma}},$$

which implies

$$a(t) \geq C(1+t)^{\frac{1}{3(\gamma-1)}}. \tag{3.18}$$

While for $1 < \gamma < \frac{5}{3}$, (3.14) gives

$$(H(t)(1+t)^{3\gamma-5})' \leq \frac{3(3\gamma-2)(\gamma-1)}{2} E_0 (1+t)^{3\gamma-5},$$

which yields

$$H(t) \leq C(1+t)^\alpha, \tag{3.19}$$

where

$$\alpha = \begin{cases} 1, & \text{if } \frac{4}{3} < \gamma < \frac{5}{3}, \\ 1+\sigma, \sigma > 0 \text{ small }, & \text{if } \gamma = \frac{4}{3}, \\ 5-3\gamma, & \text{if } 1 < \gamma < \frac{4}{3}. \end{cases} \tag{3.20}$$

As in (3.16)-(3.18), one can show that

$$a(t) \geq C_1(1+t)^{\frac{2-\alpha}{3(\gamma-1)}}. \tag{3.21}$$

Next, we derive a upper bound for $a(t)$. It follows from (3.8), (3.21) and $\alpha \in [1,2)$ that

$$\begin{aligned} a'(t) &\leq |a_1| + \theta a_0^{2-3\theta} + \theta C_1(1+t)^{\frac{3\theta-2}{3\theta-3}} + \theta C_1 \int_0^t (1+s)^{\frac{3\theta-2}{3\theta-3}} ds \\ &\leq C_2(1+t)^{\frac{6\theta-5}{3\theta-3}}. \end{aligned}$$

This yields (3.9) and completes the proof of lemma 2.2. $\square$

We are now ready to give the existence and uniqueness of the solution to the boundary value problem (3.8).

Lemma 3.3. *There exists a sufficiently small T such that* (3.8) *has a solution $a(t)$, which is unique in $C^1[0,T]$ and satisfies with $0 < \frac{1}{2}a_0 < a(t) < 2a_0$.*

Proof. The lemma can be proved by a fixed point argument. In fact, set

$$g(a(t)) = a_1 + \theta a_0^{2-3\theta} - \theta a^{2-3\theta}(t) - \theta \int_0^t a^{2-3\theta}(s)ds.$$

Then (3.8) can be rewrited as

$$\frac{da(t)}{dt} = g(a(t)), a(0) = a_0, g(a(0)) = a'(0) = a_1.$$

Let T_1 be a positive small constant to be determined. Define

$$X = \{a(t) \in C^1[0,T_1], 0 < \frac{1}{2}a_0 < a(t) < 2a_0, \forall t \in [0,T_1]\}.$$

Then, for any $a_1(t)$ and $a_2(t) \in X$, since $\theta > 1$, we have

$$\begin{aligned}
&|g(a_1(t)) - g(a_2(t))| = |\theta a_2^{2-3\theta}(t) - \theta a_1^{2-3\theta}(t) + \theta \int_0^t (a_2^{2-3\theta}(s) - a_1^{2-3\theta}(s))ds| \\
&\le \theta|\frac{1}{a_2^{3\theta-2}(t)} - \frac{1}{a_1^{3\theta-2}(t)}| + \theta \int_0^t |\frac{1}{a_2^{3\theta-2}(s)} - \frac{1}{a_1^{3\theta-2}(s)}|ds \\
&\le \theta(\frac{1}{2}a_0)^{4-6\theta}|a_1 - a_2|^{3\theta-2}(t) + \theta(\frac{1}{2}a_0)^{4-6\theta}\int_0^t |a_1(s) - a_2(s)|^{3\theta-2}ds \\
&\le \theta(\frac{1}{2}a_0)^{4-6\theta}(1+T_1)\sup_{0\le t\le T_1}|a_1(t) - a_2(t)|^{3\theta-2} \le L\sup_{0\le t\le T_1}|a_1(t) - a_2(t)|,
\end{aligned}$$

where $L = \theta 3^{3(\theta-1)}(\frac{1}{2}a_0)^{1-3\theta}(1+T_1)$ is a constant.

We now define a mapping on X as

$$\mathbb{T}a(t) = a_0 + \int_0^t g(a(s))ds, \forall t \in [0,T_1].$$

Then $\mathbb{T}a(t) \in C^1[0,T_1]$, and for any $t < T_1$, one can deduce that

$$\mathbb{T}a(t) \le a_0 + t(|a_1| + \theta a_0^{2-3\theta}) \le 2a_0, \text{ if } t \le \frac{a_0}{|a_1| + \theta a_0^{2-3\theta}} = T_2,$$

and

$$\begin{aligned}
\mathbb{T}a(t) &\ge a_0 - |a_1|t - \theta \int_0^t a^{2-3\theta}(s)ds - \theta \int_0^t \int_0^s a^{2-3\theta}(\tau)d\tau ds \\
&\ge a_0 - |a_1|t - \theta(2a_0)^{2-3\theta}t - \frac{\theta}{2}(2a_0)^{2-3\theta}t^2 \ge \frac{1}{2}a_0,
\end{aligned}$$

$$\text{if } \frac{\theta}{2}(2a_0)^{2-3\theta}t^2 + (|a_1| + \theta(2a_0)^{2-3\theta})t \le \frac{1}{2}a_0 \Longleftrightarrow 0 < t \le T_3,$$

where $T_3 = \frac{\sqrt{[|a_1|+\theta(2a_0)^{2-3\theta}]^2+\theta a_0(2a_0)^{2-3\theta}}-(|a_1|+\theta(2a_0)^{2-3\theta})}{\theta(2a_0)^{2-3\theta}}$. Thus, if $T_1 \le \min\{T_2, T_3\}$, then $\mathbb{T}a(t) \in X$.

Furthermore, since

$$
\begin{aligned}
&|\mathbb{T}a_1(s) - \mathbb{T}a_2(s)| = |\int_0^t g(a_1(s))ds - \int_0^t g(a_2(s))ds| \\
&\le LT_1 \sup_{0\le t\le T_1} |a_1(t) - a_2(t)|,
\end{aligned}
$$

thus, $\mathbb{T}$ will be a contraction mapping if $(1+T_1) < \theta^{-1}(\frac{1}{2}a_0)^{3\theta-1}3^{3(1-\theta)}$, i.e., $T_1 < \sqrt{1+4\theta^{-1}(\frac{1}{2}a_0)^{3\theta-1}3^{3(1-\theta)}} - 1 = T_4$ (i.e. $LT_1 < 1$). The above argument shows that $\mathbb{T} : X \to X$ is a contraction with the sup-norm for any $T_1 = \min\{T_2, T_3, T_4\}$. By the contraction mapping theorem, there exists a unique $a(t) \in C^1[0, T_1]$ such that $\mathbb{T}a(t) = a(t)$ and then $a'(t) = g(a(t))$, which yields (3.8). This completes the proof of the lemma. $\square$

Now, Theorem 2.1 follows from Lemma 3.3, the a priori estimates, Lemma 3.2, and the stand continuity argument. This completes the proof of Theorem 2.1.

3.2 The stress free boundary condition

First, it follows from the free boundary (2.11) and the equation (2.7) that

$$\rho(a(t), t) = \big[\rho_0^{\theta-\gamma}(a_0) + \frac{\gamma-\theta}{\theta}t\big]^{\frac{1}{\theta-\gamma}}. \tag{3.22}$$

Using the ansatz in (3.1) shows that

$$a(t) = f^{\frac{1}{3}}(1)\big[\rho_0^{\theta-\gamma}(a_0) + \frac{\gamma-\theta}{\theta}t\big]^{\frac{1}{3(\gamma-\theta)}}. \tag{3.23}$$

Then (3.1) becomes

$$\rho(r,t) = \frac{f(\frac{r}{a(t)})}{a^3(t)}, u(r,t) = \frac{r}{3\theta\rho_0^{\theta-\gamma}(a_0) + 3(\gamma-\theta)t}, \tag{3.24}$$

Set $\alpha = f(1)$, and then (3.23) tells us that $\alpha > 0$. Recall the assumption that $\theta = \frac{1}{2}(\gamma + \frac{1}{3}), \gamma > \frac{5}{3}$. Then (3.3) becomes

$$\frac{6(\gamma-1)}{(3\gamma+1)^2}z - \gamma\alpha^{\frac{1}{3}-\gamma}f^{\gamma-2}(z)f'(z) + \frac{\gamma-1}{2}\alpha^{\frac{1}{6}-\frac{\gamma}{2}}f^{\frac{3\gamma-11}{6}}(z)f'(z) = 0. \tag{3.25}$$

Denoting $g(z) = (\frac{f(z)}{\alpha})^{\frac{3\gamma-5}{6}}$ for any $z \in [0,1]$, then (3.25) becomes

$$\begin{cases} g'(z)\{g^{\frac{3\gamma-1}{3\gamma-5}}(z) - \frac{\gamma-1}{2\gamma}\} = \frac{\alpha^{\frac{2}{3}}(\gamma-1)(3\gamma-5)z}{\gamma(3\gamma+1)^2}, \\ g(1) = 1. \end{cases} \tag{3.26}$$

Next, we will prove that the equation (3.26) can be solved on $[0,1]$. To this end, we start with the a-priori estimates and the uniqueness.

Lemma 3.4. *For any $\gamma > \frac{5}{3}$, let $g(z)$ be a solution to the system (3.26) in $C([0,1]) \cap C^1((0,1])$. Then*

$$(\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}} < g(z) \leq 1, \tag{3.27}$$

for all $z \in (0,1]$. Furthermore, such a solution is unique.

Proof. If $g^{\frac{3\gamma-1}{3\gamma-5}}(z) - \frac{\gamma-1}{2\gamma} = 0$, then (3.26) implies that z must be zero, i.e., $g^{\frac{3\gamma-1}{3\gamma-5}}(0) = \frac{\gamma-1}{2\gamma}$. Namely, if $z \neq 0$, then one has $g^{\frac{3\gamma-1}{3\gamma-5}}(z) \neq \frac{\gamma-1}{2\gamma}$.

If for any $z \in (0,1]$, $g^{\frac{3\gamma-1}{3\gamma-5}}(z)$ belongs to $[0, \frac{\gamma-1}{2\gamma})$, then (3.26) implies that $g'(z) \leq 0$ and thus

$$1 \leq g(z) < (\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}} < 1.$$

which is a contradiction. Thus we can deduce that $g^{\frac{3\gamma-1}{3\gamma-5}}(z) > \frac{\gamma-1}{2\gamma}$ for all $z \in (0,1]$ and together with (3.26) to get $g'(z) \geq 0$ and consequently

$$(\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}} < g(z) \leq 1, \forall z \in (0,1]. \tag{3.28}$$

It remains to prove the uniqueness.

To this end, let $\bar{g}(z) \in C([0,1]) \cap C^1((0,1])$ be another solution to (3.26) with $\bar{g}(1) = 1$ and $\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{3\gamma-5}{3\gamma-1}} < \bar{g}(z) \leq 1$ for all $z \in (0,1]$.

Define $w(z) = g(z) - \bar{g}(z)$. Then $w(z)$ solves the following problem:

$$\begin{cases} \frac{d}{dz}\{w(z) - \frac{\gamma(3\gamma-5)}{3(\gamma-1)^2}\{[w(z)+\bar{g}(z)]^{\frac{6(\gamma-1)}{3\gamma-5}} - (\bar{g}(z))^{\frac{6(\gamma-1)}{3\gamma-5}}\}\} = 0, \\ w(1) = 0, \bar{g}(1) = 1. \end{cases} \tag{3.29}$$

Set

$$I = \{z \in [0,1] | w(\xi) \equiv 0, z \leq \xi \leq 1\}.$$

Here $I \neq \emptyset$ because of $1 \in I$. Define $z_0 = \inf I$ and then $z_0 \in [0,1]$. Obviously, the uniqueness of solutions to the system (3.26) will be showed by proving that $z_0 \equiv 0$ and continuity argument.

If not, then $z_0 \in (0,1]$, and $w(z_0) = 0$. For any $z \in (0, z_0)$, (3.27) tells us that

$$(\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}} < \bar{g}(z) \leq 1, \forall z \in (0, z_0). \tag{3.30}$$

Integrating (3.29) over $[z, z_0]$ shows

$$w(z) - \frac{\gamma(3\gamma-5)}{3(\gamma-1)^2}\{[w(z)+\bar{g}(z)]^{\frac{6(\gamma-1)}{3\gamma-5}} - (\bar{g}(z))^{\frac{6(\gamma-1)}{3\gamma-5}}\} = 0. \tag{3.31}$$

Since for any $\gamma > \frac{5}{3}$ and then $\frac{6(\gamma-1)}{3\gamma-5} > 2$, Taylor expansion gives

$$\begin{aligned} &[w(z)+\bar{g}(z)]^{\frac{6(\gamma-1)}{3\gamma-5}} - (\bar{g}(z))^{\frac{6(\gamma-1)}{3\gamma-5}} \\ &= \frac{6(\gamma-1)}{3\gamma-5}[\bar{g}(z)]^{\frac{3\gamma-1}{3\gamma-5}} w(z) + O(1)w^2(z) \end{aligned} \tag{3.32}$$

for sufficiently small $w(z)$. Putting (3.32) into (3.31) and using the fact $w(z_0) = 0$, one has

$$\{1 - \frac{2\gamma}{\gamma - 1}[\bar{g}(z)]^{\frac{3\gamma-1}{3\gamma-5}}\}w(z) - \frac{\gamma(3\gamma - 5)}{3(\gamma - 1)^2}O(1)w^2(z) = 0, \tag{3.33}$$

for z close to z_0. Notices that

$$1 - \frac{2\gamma}{\gamma - 1}[\bar{g}(z)]^{\frac{3\gamma-1}{3\gamma-5}} < 0, \forall z \in (0, z_0)$$

by virtue of (3.30). Then one can easily deduce that $w(z) \equiv 0, \forall z \in (z_0 - \delta, z_0)$ for some $\delta > 0$. This contradicts to $z_0 = \inf I$. Thus $z_0 \equiv 0$ and the proof of Lemma 3.4 is completed.

□

Now we are ready to give an existence result to system (3.26).

Lemma 3.5. *For any $\gamma > \frac{5}{3}$, there is a positive function $y = g(z)$ in $C([0, 1]) \cap C^1((0, 1])$ satisfying* (3.26)*.*

Proof. We can rewrite (3.26) as follows:

$$\begin{cases} g'(z) = G(g(z), z) = \dfrac{\frac{\alpha^{\frac{2}{3}}(\gamma-1)(3\gamma-5)z}{\gamma(3\gamma+1)^2}}{g^{\frac{3\gamma-1}{3\gamma-5}}(z) - \frac{\gamma-1}{2\gamma}}, \\ g(1) = 1. \end{cases} \tag{3.34}$$

We look for a solution to (3.34) such that

$$g(z) \in C([0, 1]) \cap C^1((0, 1]), (\frac{\gamma - 1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}} < g(z) \leq 1, \forall z \in (0, 1]. \tag{3.35}$$

Set

$$\mathbb{R} = \{(z, g(z))|0 \leq 1 - z \leq a, 0 \leq 1 - g(z) \leq b\}$$

for small $a \in (0, 1), b \in (0, 1 - (\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}})$. Then we can easily deduce that

$$|G(g(z), z)| \leq M, \forall (z, g(z)) \in \mathbb{R},$$

where M is a positive constant only depends on γ, a, b.

Since $G(g(z), z)$ is continuous in $\mathbb{R}$, by choosing $h = \min\{a, \frac{b}{M}\}$ and one can show that the solution to the initial value problem (3.33) exists in the neighborhood $0 \leq 1 - z \leq h$.

Similarly, we can extend this solution from the left of the neighborhood $0 \leq 1 - z \leq h$ step by step. Let the maximum interval of the existence of solutions be $(\alpha, 1]$ for some $\alpha \geq 0$ and $g(z) \in C([\alpha, 1]) \cap C^1((\alpha, 1])$, we will show that $\alpha \equiv 0$.

If not, then $\alpha \in (0, 1)$. By Lemma 3.4, one has $g(\alpha) > (\frac{\gamma-1}{2\gamma})^{\frac{3\gamma-5}{3\gamma-1}}$ and $(3.34)_1$ is well defined in the small neighborhood of $z = \alpha$ for C^1 function $g(z)$. Thus the similar arguments as above show that the solution of this initial value problem (3.33) in the neighborhood $|\alpha - z| \leq h_0$ for small $h_0 > 0$ exists. This is to say, the solution $g(z)$ can be extended to the interval $[\alpha - h_0, 1]$ which contradicts to the fact that $(\alpha, 1]$ is the maximum interval of the existence of solutions. Then we have obtained a $C([0, 1]) \cap C^1((0, 1])$ solution $g(z)$ to the system (3.26). □

Finally, by virtue of (3.24) and Lemma 3.4-3.5, we have obtained the global existence of $y = f(z) \in C([0,1]) \cap C^1((0,1])$ to the equation (3.25) and the proof of Theorem 2.2 is complete. In according to Lemma 3.4, $f(0) > (\frac{\gamma-1}{2\gamma})^{\frac{6}{3\gamma-1}} f(1)$, and $f(z) = \rho(r,t)a^3(t)$, we can deduce from (3.22)-(3.23) that

Corollary 3.1. *The density at the origin of the center has the following estimate:*

$$(\frac{\gamma-1}{2\gamma})^{\frac{6}{3\gamma-1}} \big[\rho_0^{\frac{1-3\gamma}{6}}(a_0) + \frac{3\gamma-1}{3\gamma+1}t\big]^{\frac{-6}{3\gamma-1}} < \rho(0,t) = \frac{f(0)}{f(1)} \big[\rho_0^{\frac{1-3\gamma}{6}}(a_0) + \frac{3\gamma-1}{3\gamma+1}t\big]^{\frac{-6}{3\gamma-1}}$$

and then

$$\rho(0,t) \to 0, \ \text{as } t \to +\infty.$$

4 Examples of blow-up solutions

In this section, we will look for some examples of blow-up solutions to (2.7)-(2.8) without $a(t)$ to be the free boundary and the boundary conditions (2.10) or (2.11).

To this end, letting $\gamma = \theta = 1$, it can be deduced from (3.3) -(3.4) that

$$\rho(r,t) = \frac{e^{\frac{1}{2}(\frac{r^2}{a^2(t)}-1)}}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)}r, \tag{4.1}$$

solve the system with (2.7)-(2.8), with $a(t) \in C^2[0,T)$ satisfying the following ordinary differential equation:

$$a''(t) + a^{-1}(t) - a'(t)a^{-2}(t) = 0, a(0) = a_0 > 0, a'(0) = a_1. \tag{4.2}$$

As in Lemma 2.2, we can prove that $a(t)$ exists for small T_0 and $t \in [0,T_0]$. Let $[0,T)$ be the largest interval of existence of positive solutions to (4.2). The following lemma gives the condition on the initial data for T to be finite.

Lemma 4.1. *If the initial data in* (4.2) *satisfies*

$$a_1 + a_0^{-1} < 0,$$

then $T < +\infty$ and $a(T_-) = 0$.

Proof. First, integrating (4.2) over $[0,t]$ yields

$$a'(t) = a_1 + a_0^{-1} - \frac{1}{a(t)} - \int_0^t \frac{1}{a(s)}ds, \forall t \geq 0. \tag{4.3}$$

So, if $a_1 + a_0^{-1} < 0$, one has that $a'(t) < 0$ for all $t \geq 0$ and then

$$a(t) < a_0, a_1 + a_0^{-1} - \frac{1}{a(t)} - \int_0^t \frac{1}{a(s)}ds < 0, \forall t \geq 0.$$

That is to say

$$\frac{1}{a(t)} + \int_0^t \frac{1}{a(s)}ds - (a_1 + a_0^{-1}) > -a_1 - a_0^{-1} > 0. \tag{4.4}$$

Thus, for $t \geq 0$, one has

$$\begin{aligned} t &= \int_0^t ds = \int_{a_0}^{a(t)} \frac{1}{a_1 + a_0^{-1} - \frac{1}{a(t)} - \int_0^t \frac{1}{a(s)} ds} da(s) \\ &= \int_{a(t)}^{a_0} \frac{1}{\frac{1}{a(t)} + \int_0^t \frac{1}{a(s)} ds - (a_1 + a_0^{-1})} da(s) \\ &\leq \int_{a(t)}^{a_0} \frac{1}{-(a_1 + a_0^{-1})} da(s) = \frac{a(t) - a_0}{a_1 + a_0^{-1}} \leq \frac{a_0}{-a_1 - a_0^{-1}} < +\infty. \end{aligned}$$

This implies $\lim_{t \to T_-} a(t) = 0$ for a finite time T and completes the proof of lemma. □

As $\gamma = 1, \theta = \frac{2}{3}$, one can verify that

$$\rho(r,t) = \frac{1}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)} r, a(t) = mt + n, m \neq 0, n > 0, \tag{4.5}$$

is a family of solutions to (2.7)-(2.8). Obviously, If $m < 0$, $\lim_{t \to T_0} a(t) = 0, T_0 = -\frac{n}{m}$ and then $\rho(r,t)$ blows up. While as if $m > 0$, then the family solutions (4.5) exist globally with the following properties:

$$\rho(r,t) \to 0, u_r(r,t) \to 0, \forall r \geq 0,$$

as $t \to +\infty$.

Collecting all the conclusion above, we have

Theorem 4.1. *If $\gamma = \theta = 1$, there exists a blow-up solution to the system (2.7)-(2.8) of the form*

$$\rho(r,t) = \frac{e^{\frac{1}{2}(\frac{r^2}{a^2(t)} - 1)}}{a^3(t)}, u(r,t) = \frac{a'(t)}{a(t)} r, \tag{4.6}$$

with $a(t) \in C^2[0,T)$ solving the following problem:

$$a''(t) + a^{-1}(t) - a'(t) a^{-2}(t) = 0, a(0) = a_0 > 0, a'(0) = a_1 \tag{4.7}$$

for finite $T > 0$ and $\lim_{t \to T_-} a(t) = 0$ as if $a_1 + a_0^{-1} < 0$.

If $\gamma = 1, \theta = \frac{2}{3}$, then there exist a family of solutions to the system (2.7)-(2.8) of the form:

$$\rho(r,t) = \frac{1}{(mt+n)^3}, u(r,t) = \frac{mr}{mt+n}, m \neq 0, n > 0, \tag{4.8}$$

where if $m < 0, T_0 = -\frac{n}{m}$ and $\rho(r,t)$ blows up as t tends to T_0. While as if $m > 0$, then the family of solutions (4.8) exist globally with the following properties:

$$\rho(r,t) \to 0, u_r(r,t) \to 0, \forall r \geq 0,$$

as $t \to +\infty$.

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