THE BOLTZMANN EQUATION WITH SOFT POTENTIALS NEAR A LOCAL MAXWELLIAN

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Abstract. In this paper, we consider the Boltzmann equation with soft potentials and prove the stability of a class of non-trivial profiles defined as some given local Maxwellians. The method consists of the analytic techniques for viscous conservation laws, properties of Burnett functions and energy method through the micro-macro decomposition of the Boltzmann equation. In particular, one of the key observations is a detailed analysis of the Burnett functions so that the energy estimates can be obtained in a clear way. As an application of the main results in this paper, we prove the large time nonlinear asymptotic stability of rarefaction waves to the Boltzmann equation with soft potentials.

1. Introduction

Consider the Boltzmann equation with slab symmetry in one space variable

$$f_t + \xi_1 f_x = Q(f,f), \qquad f(0,x,\xi) = f_0(x,\xi), \tag{1.1}$$

where $f(t,x,\xi)$ is the particle distribution function at time $t \geq 0$, position $x \in \mathbf{R}$ with velocity $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3$. Here, the collision operator is given by

$$Q(f,g)(\xi) = \frac{1}{2}\int_{\mathbf{R}^3\times\mathbf{S}^2} B(|\xi-\xi_*|,\vartheta)\{f(\xi')g(\xi_*')+f(\xi_*')g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\}d\xi_* d\omega$$

$$\equiv Q^1_{gain}(f,g) + Q^2_{gain}(f,g) + Q^1_{loss}(f,g) + Q^2_{loss}(f,g), \tag{1.2}$$

where $f(\xi) = f(t,x,\xi)$, $\omega \in \mathbf{S}^2$ with $\mathbf{S}^2$ denoting the unit sphere in $\mathbf{R}^3$, and

$$\xi' = \xi - [(\xi-\xi_*)\cdot\omega]\omega, \quad \xi_*' = \xi_* + [(\xi-\xi_*)\cdot\omega]\omega, \tag{1.3}$$

which give the relations between velocities of particles before and after an elastic collision.

For the interaction potential satisfying the inverse power law and under the Grad's angular cutoff assumption, the cross-section $B(|\xi-\xi_*|,\vartheta)$ takes the form

$$B(|\xi-\xi_*|,\vartheta) = B(\vartheta)|\xi-\xi_*|^\gamma, \quad \cos\vartheta = ((\xi-\xi_*)\cdot\omega)/|\xi-\xi_*|, \quad -3<\gamma\leq 1,$$

where $B(\vartheta)$ satisfies that $0 < B(\vartheta) \leq \text{const.}|\cos\vartheta|$. Throughout this paper, we will consider the case with soft potentials, that is, the case when $-3<\gamma<0$.

We decompose the Boltzmann equation and its solution with respect to the local Maxwellian [16] as:

$$f(t,x,\xi) = M(t,x,\xi) + G(t,x,\xi),$$

where the local Maxwellian M and G represent the macroscopic and microscopic component in the solution respectively. Precisely, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(t,x)$, momentum $m(t,x) = \rho(t,x)u(t,x)$, and energy density $(E(t,x) + \frac{1}{2}|u(t,x)|^2)$:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m_i(t,x) \equiv \int_{\mathbf{R}^3} \psi_i(\xi) f(t,x,\xi)d\xi, \quad \text{for} \quad i = 1,2,3, \\ \left[\rho(E(t,x) + \frac{1}{2}|u(t,x)|^2)\right] \equiv \int_{\mathbf{R}^3} \psi_4(\xi) f(t,x,\xi)d\xi, \end{cases} \tag{1.4}$$

as

$$M \equiv M_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\Big(- \frac{|\xi - u(t,x)|^2}{2R\theta(t,x)} \Big). \tag{1.5}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy E by $E = \frac{3}{2}R\theta$ with R being the gas constant, and $u(t,x)$ is the fluid velocity. It is well known that the collision invariants $\psi_\alpha(\xi)$ used above are given by

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i, \quad \text{for} \quad i = 1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.6}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_i(\xi) Q(f,g) d\xi = 0, \quad \text{for} \quad i = 0,1,2,3,4.$$

From now on, the inner product of two functions h and g in $L^2_\xi(\mathbf{R}^3)$ with respect to a given Maxwellian $\widetilde{M}$ is defined by:

$$\langle h, g \rangle_{\widetilde{M}} \equiv \int_{\mathbf{R}^3} \frac{1}{\widetilde{M}} h(\xi) g(\xi) d\xi,$$

when the integral is well defined. If $\widetilde{M}$ is the local Maxwellian M, corresponding to this inner product, the macroscopic space is spanned by the following five pair-wise orthonormal functions

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} M, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} M, \quad \text{for} \quad i = 1,2,3, \\ \chi_4(\xi) \equiv \frac{1}{6\rho} \Big(\frac{|\xi - u|^2}{R\theta} - 3 \Big) M, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i,j = 0,1,2,3,4. \end{cases} \tag{1.7}$$

In terms of these five basic functions, the macroscopic projection P_0 and microscopic projection P_1 are given by

$$\begin{cases} P_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j \rangle \chi_j, \\ P_1 h \equiv h - P_0 h. \end{cases}$$

Obviously, they satisfy

$$P_0 P_0 = P_0, \quad P_1 P_1 = P_1, \quad P_1 P_0 = P_0 P_1 = 0.$$

And, a function $h(\xi)$ is called microscopic if

$$\int_{\mathbf{R}^3} h(\xi)\psi_i(\xi) d\xi = 0, \quad \text{for} \quad i = 0, 1, 2, 3, 4.$$

Based on this decomposition, the solution $f(t, x, \xi)$ of the Boltzmann equation satisfies

$$P_0 f = M, \qquad P_1 f = G,$$

and the Boltzmann equation becomes

$$(M + G)_t + \xi_1 (M + G)_x = Q(G, M) + Q(M, G) + Q(G, G),$$

which is equivalent to the following fluid-type system for the macroscopic components:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\int_{\mathbf{R}^3} \xi_1^2 G_x d\xi, \\ (\rho u_i)_t + (\rho u_i^2)_x = -\int_{\mathbf{R}^3} \xi_1 \xi_i G_x d\xi, \qquad i = 2, 3, \\ \left(\rho\left(e + \frac{|u|^2}{2}\right)\right)_t + \left(\rho u_1 \left(e + \frac{|u|^2}{2} + p u_1\right)\right)_x = -\int_{\mathbf{R}^3} \xi_1 |\xi|^2 G_x d\xi, \end{cases} \tag{1.8}$$

together with the equation for the microscopic component G:

$$G_t + P_1(\xi_1 G_x) + P_1(\xi_1 M_x) = L_M G + Q(G, G), \tag{1.9}$$

where

$$G = L_M^{-1}(P_1(\xi_1 M_x)) + L_M^{-1}\Theta, \tag{1.10}$$

and

$$\Theta = G_t + P_1(\xi_1 G_x) - Q(G, G). \tag{1.11}$$

Here, L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(h, M) + Q(M, h),$$

and the null space $\mathcal{N}$ of L_M is spanned by the macroscopic variables, χ_j, $j = 0, 1, 2, 3, 4$. Moreover, the linearized operator takes the form

$$(L_M h)(\xi) = -\nu_M(\xi) h(\xi) + K_M h(\xi)$$

where the collisional frequency $\nu_M(\xi)$ is defined by

$$\nu_M(\xi)=\int_{\mathbf{R}^3\times\mathbf{S}^2}|\xi-\xi_*|^\gamma M(\xi_*)B(\vartheta)d\xi_*d\omega=c\int_{\mathbf{R}^3}|\xi-\xi_*|^\gamma M(\xi_*)d\xi_*, \tag{1.12}$$

for some constant $c>0$. And $K_M=K_{2M}-K_{1M}$ is given by

$$K_{1M}g=\int_{\mathbf{R}^3\times\mathbf{S}^2}|\xi-\xi_*|^\gamma M(\xi)g(\xi_*)B(\vartheta)d\xi_*d\omega, \tag{1.13}$$

$$K_{2M}g=\int_{\mathbf{R}^3\times\mathbf{S}^2}|\xi-\xi_*|^\gamma\{M(\xi')g(\xi'_*)+g(\xi')M(\xi'_*)\}B(\vartheta)d\xi_*d\omega. \tag{1.14}$$

Furthermore, it is known that there exists a positive constant $\sigma_0(\rho,u,\theta)>0$ such that for any function $h(\xi)\in\mathcal{N}^\perp$, cf. [7],

$$\langle h,L_Mh\rangle\le-\sigma_0\langle\nu_M(\xi)h,h\rangle. \tag{1.15}$$

Notice that for the soft potentials with angular cutoff, the collision frequency $\nu_M(\xi)$ has the following property

$$\nu_0(1+|\xi-u|^2)^{\gamma/2}\le\nu_M(\xi)\le\nu_1(1+|\xi-u|^2)^{\gamma/2}, \tag{1.16}$$

for some positive constants ν_0 and ν_1.

To have a clear representation for the macroscopic variables, we plug (1.10) into (1.8) to obtain

$$\begin{cases}\rho_t+(\rho u_1)_x=0,\\ (\rho u_1)_t+(\rho u_1^2+p)_x=-\Big(\int_{\mathbf{R}^3}\xi_1^2L_M^{-1}(\xi_1M_x)d\xi\Big)_x-\Big(\int_{\mathbf{R}^3}\xi_1^2L_M^{-1}\Theta d\xi\Big)_x,\\ (\rho u_i)_t+(\rho u_i^2)_x=-\Big(\int_{\mathbf{R}^3}\xi_1\xi_iL_M^{-1}(\xi_1M_x)d\xi\Big)_x-\Big(\int_{\mathbf{R}^3}\xi_1\xi_iL_M^{-1}\Theta d\xi\Big)_x,\qquad i=2,3,\\ \Big(\rho\Big(e+\frac{|u|^2}{2}\Big)\Big)_t+\Big(\rho u_1\Big(e+\frac{|u|^2}{2}+pu_1\Big)\Big)_x\\ \qquad=-\Big(\int_{\mathbf{R}^3}\xi_1|\xi|^2L_M^{-1}(\xi_1M_x)d\xi\Big)_x-\Big(\int_{\mathbf{R}^3}\xi_1|\xi|^2L_M^{-1}\Theta d\xi\Big)_x.\end{cases} \tag{1.17}$$

To present the main results in this paper, we will use the following notations. Let α and β be a non-negative integer and a multi-index $\beta=[\beta_1,\beta_2,\beta_3]$, respectively. Denote

$$\partial_\beta^\alpha\equiv\partial_x^\alpha\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3}.$$

If each component of β is not greater than the corresponding one of $\overline{\beta}$, we use the standard notation $\beta\le\overline{\beta}$. And $\beta<\overline{\beta}$ means that $\beta\le\overline{\beta}$ and $|\beta|<|\overline{\beta}|$. $C_\beta^{\overline{\beta}}$ is the usual binomial coefficient. We shall use $\|\cdot\|$ to denote the L^2 norms in $\mathbf{R}_x$ or $\mathbf{R}_x\times\mathbf{R}_\xi^3$ with the weight function $\frac{1}{M_-}$ and $\|\cdot\|_\nu$ to denote the L^2 norm in $\mathbf{R}_x\times\mathbf{R}_\xi^3$ with the weight function $\frac{\nu(\xi)}{M_-}$ where $M_-=M_{[\rho_-,u_-,\theta_-]}$ is a given global Maxwellian. And $\nu(\xi)$ and L_{M_-} denote the collision frequency and linearized collision operator corresponding to the global Maxwellian M_-. In addition, we introduce a weight function of ξ as $w(\xi)=(1+|\xi-u_-|^2)^{\gamma/2}$. And C denotes a generic positive constant which may vary from line to line.

In this paper, we consider the stability of a given local Maxwellian

$$M_{[\overline{\rho},\overline{u},\overline{\theta}]}=\frac{\overline{\rho}(t,x)}{\sqrt{(2\pi R\overline{\theta}(t,x))^3}}\exp\Big(-\frac{|\xi-\overline{u}(t,x))|^2}{2R\overline{\theta}(t,x)}\Big), \tag{1.18}$$

when its fluid variables $(\bar{\rho}(t,x),\bar{u}(t,x),\bar{\theta}(t,x))$ satisfy some assumptions given later. We define $(\widetilde{\rho},\widetilde{u},\widetilde{\theta})$ and $\widetilde{G}$ as

$$\begin{aligned}
\widetilde{\rho}(t,x)&=\rho(t,x)-\overline{\rho}(t,x),\\
\widetilde{u}(t,x)&=u(t,x)-\overline{u}(t,x),\\
\widetilde{\theta}(t,x)&=\theta(t,x)-\overline{\theta}(t,x),\\
\widetilde{G}(t,x,\xi)&=G(t,x,\xi)-\overline{G}(t,x,\xi),
\end{aligned}$$

where

$$\overline{G}(t,x,\xi)=L_M^{-1}P_1\xi_1M\Big\{\frac{|\xi-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(\xi-u)\cdot\overline{u}_x}{R\theta}\Big\}. \tag{1.19}$$

To show the stability of a given local Maxwellian (1.18), a key step is to establish some suitable uniform energy estimates. In fact, the following instant energy functional $\mathcal{E}(t)$ will be used:

$$\begin{aligned}
\mathcal{E}(t)=&\|(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\rho,u,\theta)(t)\|^2\\
&+\sum_{1\le|\alpha|,|\alpha|+|\beta|\le N}\|w^{|\beta|}\partial^\alpha_\beta G(t)\|^2+\sum_{|\beta|\le N}\|w^{|\beta|}\partial_\beta\widetilde{G}(t)\|^2.
\end{aligned} \tag{1.20}$$

As usual, the instant energy functional $\mathcal{E}(t)$ is assumed to be small enough a priorily. And this will be closed by the energy estimate in the end.

Correspondingly, the dissipation rate $\mathcal{D}(t)$ is given by

$$\begin{aligned}
\mathcal{D}(t)=&\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\rho,u,\theta)(t)\|^2\\
&+\sum_{1\le|\alpha|,|\alpha|+|\beta|\le N}\|w^{|\beta|}\partial^\alpha_\beta G(t)\|^2_\nu+\sum_{|\beta|\le N}\|w^{|\beta|}\partial_\beta\widetilde{G}(t)\|^2_\nu.
\end{aligned} \tag{1.21}$$

As in [16, 17], the following macroscopic entropy S will be estimated for the lower order energy estimates. Set

$$-\frac{3}{2}\rho S\equiv\int_{\mathbf{R}^3}M\ln M d\xi. \tag{1.22}$$

Direct calculation yields

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}(\rho u_1 S)_x+\Big(\int_{\mathbf{R}^3}(\xi_1\ln M)Gd\xi\Big)_x=\int_{\mathbf{R}^3}\frac{G\xi_1M_x}{M}d\xi \tag{1.23}$$

and

$$S=-\frac{2}{3}\ln\rho+\ln(2\pi R\theta)+1,\quad p=\frac{2}{3}\rho\theta=k\rho^{\frac{5}{3}}\exp(S),\quad e=\theta,\quad R=\frac{2}{3}. \tag{1.24}$$

Rewrite the conservation laws (1.8) by

$$m_t+n_x=-\begin{pmatrix}0\\ \int_{\mathbf{R}^3}\xi_1^2Gd\xi\\ \int_{\mathbf{R}^3}\xi_1\xi_2Gd\xi\\ \int_{\mathbf{R}^3}\xi_1\xi_3Gd\xi\\ \frac{1}{2}\int_{\mathbf{R}^3}\xi_1|\xi|^2Gd\xi\end{pmatrix}_x.$$

Here

$$m=(m_0,m_1,m_2,m_3,m_4)^t=(\rho,\rho u_1,\rho u_2,\rho u_3,\rho(\frac{1}{2}|u|^2+\theta))^t,$$

$$m=(n_0,n_1,n_2,n_3,n_4)^t=(\rho u_1,\rho u_1^2+\frac{2}{3}\rho\theta,\rho u_1u_2,\rho u_1u_3,\rho u_1(\frac{1}{2}|u|^2+\frac{5}{3}\theta))^t.$$

Then define an entropy-entropy flux pair (η,q) around a Maxwellian $\overline{M}=M_{[\overline{\rho},\overline{u},\overline{\theta}]}$ $(\overline{u}_i=0,\ i=2,3)$ as

$$\eta=\overline{\theta}\{-\frac{3}{2}\rho S+\frac{3}{2}\overline{\rho}\overline{S}+\frac{3}{2}\nabla_m(\rho S)|_{m=\overline{m}}(m-\overline{m})\},$$

$$q=\overline{\theta}\{-\frac{3}{2}\rho u_1S+\frac{3}{2}\overline{\rho u_1}\overline{S}+\frac{3}{2}\nabla_m(\rho S)|_{m=\overline{m}}(n-\overline{n})\}. \tag{1.25}$$

Since

$$(\rho S)_{m_0}=S+\frac{|u|^2}{2\theta}-\frac{5}{3},\quad (\rho S)_{m_i}=-\frac{u_i}{\theta},\quad i=1,2,3,\quad (\rho S)_{m_4}=\frac{1}{\theta},$$

it holds that

$$\eta=\frac{3}{2}\{\rho\theta-\overline{\theta}\rho S+\rho[(\overline{S}-\frac{5}{3})\overline{\theta}+\frac{|u-\overline{u}|^2}{2}]+\frac{2}{3}\overline{\rho}\overline{\theta}\},$$

$$q=u_1\eta+(u_1-\overline{u}_1)(\rho\theta-\overline{\rho}\overline{\theta}). \tag{1.26}$$

Note that for m in any closed bounded region in $\sum=\{m:\rho>0,\quad \theta>0\}$, there exists constant $C>1$ such that

$$C^{-1}|m-\overline{m}|^2\le\eta\le C|m-\overline{m}|^2.$$

We are now ready to state the assumptions on the macro components $(\overline{\rho},\overline{u},\overline{\theta})(t,x)$:

H1. $(\overline{\rho},\overline{u}_1,\overline{\theta})(t,x)$ solves the Euler equations

$$\begin{cases}\overline{\rho}_t+(\overline{\rho u}_1)_x=0,\\ (\overline{\rho u}_1)_t+(\overline{\rho u}_1^2+\overline{p})_x=0,\\ \Big(\overline{\rho}\Big(\overline{e}+\frac{|\overline{u}_1|^2}{2}\Big)\Big)_t+\Big(\overline{\rho u}_1\Big(\overline{e}+\frac{|\overline{u}_1|^2}{2}+\overline{pu}_1\Big)\Big)_x=0,\end{cases} \tag{1.27}$$

where $\overline{p}=R\overline{\rho}\overline{\theta}$ and $\overline{e}=\frac{3}{2}R\overline{\theta}$.

H2. $\overline{u}_2 = \overline{u}_3 = 0$, $\overline{u}_{1x} \geq 0$ for any $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$. And there exist positive constants $\rho_- > 0$, $\theta_- > 0$, and $\eta_0 > 0$, $u_- \in \mathbf{R}^3$ such that for all (t,x),

$$\frac{1}{2} \sup_{(t,x)\in\mathbf{R}_+\times\mathbf{R}} \overline{\theta}(t,x) < \theta_- < \inf_{(t,x)\in\mathbf{R}_+\times\mathbf{R}} \overline{\theta}(t,x), \tag{1.28}$$

$$|\overline{\rho}(t,x) - \rho_-| + |\overline{u}(t,x) - u_-| + |\overline{\theta}(t,x) - \theta_-| < \eta_0. \tag{1.29}$$

H3. For any p $(1 \leq p \leq \infty)$, there exists a constant $C(p) > 0$ depending on p such that for some sufficiently large constant $t_0 > 0$,

$$\|(\overline{\rho}, \overline{u}, \overline{\theta})_x(t,x)\|_{L^p} \leq C(p)(t+t_0)^{-1+\frac{1}{p}},$$

$$\|\frac{\partial^j}{\partial x^j}(\overline{\rho}, \overline{u}, \overline{\theta})(t,x)\|_{L^p} \leq C(p)(t+t_0)^{-1}, \quad j \geq 2.$$

H4. The following estimate on the entropy-entropy flux (η, q) pair in (1.25) holds:

$$\int_{\mathbf{R}} \Big[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta \cdot (\overline{\rho}, \overline{u}, \overline{S})_t + \nabla_{(\overline{\rho},\overline{u},\overline{S})} q \cdot (\overline{\rho}, \overline{u}, \overline{S})_x\Big] dx \leq g(t)\|\sqrt{\eta(t)}\|^2, \tag{1.30}$$

where the function $g(t) \geq 0$ satisfies $\int_0^\infty g(t)dt \leq C_g \epsilon$ for some small constant $\epsilon > 0$.

As will be shown in the last section, our assumptions are valid for rarefaction wave profiles. Obviously, they also hold true for the global Maxwellian $M_{[\overline{\rho},\overline{u},\overline{\theta}]}$ with $\overline{\rho} = \rho_- + \epsilon_1$, $\overline{u}_1 = u_{-1} + \epsilon_1$ and $\overline{\theta} = \theta_- + \epsilon_1$ for some small constant $\epsilon_1 > 0$.

For $(\overline{\rho}, \overline{u}, \overline{\theta})(t,x)$ satisfying the assumptions **H1-H4**, we define $\mathcal{I}(\epsilon_0, \epsilon, \eta_0; \overline{\rho}, \overline{u}, \overline{\theta})$ to be the set of initial data $f_0(x,\xi)$ satisfying

$$\sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial^\alpha_\beta(f_0(x,\xi) - M_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]})\|^2 \leq \epsilon_0, \tag{1.31}$$

for any $N \geq 6$ and a global Maxwellian M_- satisfying (1.28) and (1.29).

With the above preparation, the main result of this paper can be stated as follows.

Theorem 1.1. Let ϵ and ϵ_0 be suitably small positive constants. Then for each $f_0(x,\xi) \in \mathcal{I}(\epsilon_0, \epsilon, \eta_0; \overline{\rho}, \overline{u}, \overline{\theta})$, the Cauchy problem for the Boltzmann equation (1.1) with initial data $f_0(x,\xi)$ has a unique global solution $f(t,x,\xi)$ satisfying, for some small positive constant $\delta_0 > 0$ and any $t > 0$,

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)ds \leq C\delta_0, \tag{1.32}$$

and

$$\sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial^\alpha_\beta(f(t,x,\xi) - M_{[\overline{\rho},\overline{u},\overline{\theta}]})\| \leq C\delta_0. \tag{1.33}$$

As an important application of Theorem 1.1, we can prove the nonlinear large time asymptotic stability of rarefaction waves to the Boltzmann equation with soft potentials as stated in the following theorem.

Theorem 1.2. Let $(\overline{\rho}, \overline{u}, \overline{\theta})(t, x)$ be the smooth rarefaction wave constructed in (4.6) with (4.8) and (4.9) in section 4. If the initial data $f_0(x, \xi)$ satisfies

$$\sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial^\alpha_\beta(f_0(x,\xi) - M_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]})\|^2 \leq \epsilon_0$$

for some global Maxwellian $M_- = M_{[\rho_-, u_-, \theta_-]}$, then when ϵ_0 is small enough, the problem (1.1) admits a unique global solution $f(t, x, \xi)$ satisfying for some small positive constant $\delta_0 > 0$ and for all t,

$$\sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial^\alpha_\beta(f(t,x,\xi) - M_{[\overline{\rho},\overline{u},\overline{\theta}]})\| \leq \delta_0.$$

Moreover, the solution tends to the local equilibrium rarefaction wave profile time asymptotically in the following sense

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} \sum_{|\alpha|+|\beta|\leq N-1} \|w^{|\beta|}\partial^\alpha_\beta(f(t,x,\xi) - M_{[\rho^R,u^R,\theta^R]})\|_{L^2_\xi} = 0.$$

Here, the global Maxwellian M_- satisfies that for all (t, x), $\frac{1}{2}\theta(t, x) < \theta_- < \theta(t, x)$, and $|\rho(t, x) - \rho_-| + |u(t, x) - u_-| + |\theta(t, x) - \theta_-| < \eta_0$.

In the rest of the introduction, we will first review some previous works related to this paper. Although there is extensive literature on the mathematical theory for the Boltzmann equation for hard potentials with angular cutoff, less is known for the soft potentials. For the study on the wave patterns, around 1980, under the angular cutoff condition, Nicolaenko, Thurber and Caflisch constructed the shock profile solutions of the Boltzmann equation in [24] and [4] for $\gamma \in [0, 1]$. Recently, Liu and Yu [18, 31] established the positivity and hydrodynamic limit of shock profile solutions of the Boltzmann equation for hard sphere model, that is, when $\gamma = 1$. On the other hand, the nonlinear stability of rarefaction waves to the Boltzmann equation was considered in [17, 30] with different boundary conditions for $\gamma \in [0, 1]$. And Huang, Xin and Yang studied the stability of contact waves with general perturbations for $\gamma \in [0, 1]$ in [8]. In addition, the hydrodynamic limits of contact discontinuities and rarefaction waves were also obtained in [10] and [29], respectively. Note that the above results are obtained under the assumption when $\gamma \geq 0$ so that it is has been an interesting problem to consider the corresponding problems when $-3 < \gamma < 0$.

For the Boltzmann equation with soft potentials, there are results on the perturbation of vacuum and a global Maxwellian. Precisely, global existence of the renormalized solution with large initial data was constructed in [6] for all $\gamma > -3$. This result was partially generalized to the case without angular cutoff in [1]. Caflish [3] and Ukai-Asano [26] obtained global classical solution near a global Maxwellian for $\gamma > -1$. Guo [7, 25] constructed global solutions near a global Maxwellian for all $\gamma > -3$ in a torus. The results in the torus were generalized to the case in the whole space by Hsiao and Yu in [9].

In this paper, we show that the Boltzmann equation with soft potential admits a unique global solution when the initial data is a small perturbation of a given local Maxwellian. In particular, this yields the stability of the rarefaction waves for the Boltzmann equation for soft potentials. Here, we would like to mention that for the Navier-Stokes equations, the stability of rarefaction waves with or without boundary effects has been extensively studied in [14, 15, 21, 22, 23]. Moreover, the case for the Broadwell model of discrete velocity gas was studied in [19, 28]. The problem we considered in this paper corresponds to the Navier-Stokes equations with ideal gas law when $p = R\rho\theta$ with $\theta = 3Re/2$. For more works to the compressible Navier-Stokes equations, see [12, 15, 23] and the references therein.

Finally we would like to make some comments on the analysis of this paper. Since the solution is a uniform small perturbation of a local Maxwellian, the structure and properties of the underlying local Maxwellian should play a key role in our analysis. Thus it is natural to use the general framework based on the deeper analytical understanding of the compressible Navier-Stokes equations and the macro-micro decomposition of the solutions to the Boltzmann equation with respect to the local Maxwellian in [18, 16] as in the case of hard potentials. However, for the case of soft potentials, the collision operator has a strong singularities for $\gamma < 0$, thus elaborated analytical techniques are needed even in the case of perturbation of a global Maxwellian as Guo has done successfully in [7]. These difficulties are more pronounced in the case of perturbations of a local Maxwellian. Indeed, one of the key elements in our energy estimates is to control the bounds on $\widetilde{G}$ defined in (1.19) to deal with the difficulties that $\|(\overline{u}_x, \overline{\theta}_x)\|^2$ is not integrable to time t. However, in contrast to the hard potentials [17], now the inverse of the linearized operator L_M^{-1} is an unbounded operator in $L^2(\mathbf{R}^3)$, which leads to considerable difficulties in our analysis, in particular, in the lower order estimates in the terms of entropy-entropy flux pairs, see section 3.2. One of the key observations in this paper is that by making use of the Burnett functions and their integrability and decay properties, we are able to identify a set of functions on which L_M^{-1} is an bounded operator in suitable weighted L^2 spaces, see Lemma 2.5. Furthermore, this makes it possible to calculate some microscopic terms more precisely and thus to complete the energy estimates in a systematic and clear way. On the other hand, to overcome the difficulties caused by the strong singularity in the kernels for the soft potentials, we also employ some useful techniques and ideas developed by Guo in [7] for a global Maxwellian.

The rest of the paper will be organized as follows. In the next section, we will give some basic estimates on the collision operator and the Burnett functions. The global existence of the solution will be proved in the third section. Finally, in the last section, we will apply our main theorem to the nonlinear stability of rarefaction waves for the Boltzmann equation.

2. Basic estimates

In this section, we will prove some basic estimates on the collision operator and the Burnett functions for the Boltzmann equation with soft potentials.

By the translation invariant of the collision operator Q, it is known that

$$\partial^\alpha_\beta Q(f,g) = \sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g) \equiv \sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta [Q^1_{gain}(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g) + Q^2_{gain}(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g) - Q^1_{loss}(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g) - Q^2_{loss}(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g)]. \quad (2.1)$$

The weighted estimate on the collision operator with derivatives will be given in the following lemmas. In the proof of the following lemma, we use some ideas due to Guo in [7].

Lemma 2.1. Suppose that M is defined by (1.5), M_- satisfies (1.28)-(1.29) and the following inequalities hold for some sufficiently small $\epsilon_0 \in (0, \eta_0)$,

$$\frac{1}{2}\theta(t,x) < \theta_- < \theta(t,x),$$

$$|\rho(t,x) - \rho_-| + |u(t,x) - u_-| + |\theta(t,x) - \theta_-| < \epsilon_0.$$

Let $M_* = M_{[\rho_*, u_*, \theta_*]}$ be either M or M_-. Assume $\beta_1 + \beta_2 = \beta$, $\alpha_1 + \alpha_2 = \alpha$, and $|\beta| \le \theta$. If $|\alpha_1| + |\beta_1| \le N/2$, then

$$\Big| \int_{\mathbf{R}^3} w^{2\theta}(\xi) Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} g) \frac{h(\xi)}{M_*(\xi)} d\xi \Big| \le C \sum_{|\alpha_1|+|\beta'|\le N-1} \Big\{ \int_{\mathbf{R}^3} w^{2|\beta'|}(\xi) \frac{|\partial^{\alpha_1}_{\beta'} f(\xi)|^2}{M_*(\xi)} d\xi \Big\}^{1/2} \times \Big\{ \int_{\mathbf{R}^3} w^{2|\beta_2|}(\xi)\nu(\xi) \frac{|\partial^{\alpha_2}_{\beta_2} g(\xi)|^2}{M_*(\xi)} d\xi \Big\}^{1/2} \Big\{ \int_{\mathbf{R}^3} w^{2\theta}(\xi)\nu(\xi) \frac{|h(\xi)|^2}{M_*(\xi)} d\xi \Big\}^{1/2}. \quad (2.2)$$

Similarly, if $|\alpha_2|+|\beta_2|\le N/2$, then

$$\Big|\int_{\mathbf{R}^3} w^{2\theta}(\xi)Q(\partial^{\alpha_1}_{\beta_1}f,\partial^{\alpha_2}_{\beta_2}g)\frac{h(\xi)}{M_*(\xi)}d\xi\Big|\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi)\frac{|\partial^{\alpha_2}_{\beta'}g(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}$$

$$\times\Big\{\int_{\mathbf{R}^3} w^{2|\beta_1|}(\xi)\nu(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3} w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}. \tag{2.3}$$

Proof. Consider the first loss term Q^1_{loss} in (2.1). If $|\alpha_2|+|\beta_2|\le N/2$, it follows that

$$\int_{\mathbf{R}^3\times\mathbf{S}^2}|\xi-\xi_*|^\gamma B(\vartheta)|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|d\xi_*d\omega$$

$$\le C\Big\{\int_{\mathbf{R}^3}|\xi-\xi_*|^\gamma M_*^{1/2}(\xi_*)d\xi_*\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}|\xi-\xi_*|^\gamma M_*^{1/2}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}$$

$$\le C\sup_{\xi_*}\Big\{M_*^{1/8}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|}{\sqrt{M_*(\xi_*)}}\Big\}\int_{\mathbf{R}^3}|\xi-\xi_*|^\gamma M_*^{1/4}(\xi_*)d\xi_*$$

$$\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta'}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}\times[1+|\xi-u_*|]^\gamma.$$

Here the following inequality has been used:

$$\sup_{\xi}\Big\{M_*^{1/8}(\xi')\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\Big\}\le C\sum_{|\alpha_1|+|\beta'|\le N-1}\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi)\frac{|\partial^{\alpha_1}_{\beta'}f(\xi)|^2}{M_*(\xi)}d\xi. \tag{2.4}$$

Since $M_*=M_{[\rho_*,u_*,\theta_*]}$ is M or M_-, it follows from the assumptions that

$$(1+|\xi-u_*|^2)^{\gamma/2}\le C(1+|\xi-u_-|^2)^{\gamma/2}\le\nu(\xi).$$

This leads to

$$\Big|\int_{\mathbf{R}^3} w^{2\theta}(\xi)Q^1_{loss}(\partial^{\alpha_1}_{\beta_1}f,\partial^{\alpha_2}_{\beta_2}g)\frac{h(\xi)}{M_*(\xi)}d\xi\Big|$$

$$=\Big|\int_{\mathbf{R}^3\times\mathbf{S}^2} w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\partial^{\alpha_1}_{\beta_1}f(\xi)\partial^{\alpha_2}_{\beta_2}g(\xi_*)\frac{h(\xi)}{M_*(\xi)}d\xi d\xi_*d\omega\Big|$$

$$\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta'}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}$$

$$\times\int_{\mathbf{R}^3}[1+|\xi-u_*|]^\gamma w^{2\theta}(\xi)|\partial^{\alpha_1}_{\beta_1}f(\xi)|\frac{|h(\xi)|}{M_*(\xi)}d\xi$$

$$\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta'}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}$$

$$\times\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}, \tag{2.5}$$

which is bounded by the right hand side of (2.3).

Now turn to the case of $|\alpha_1|+|\beta_1|\le N/2$. For this, the (ξ,ξ_*) integration domain is split into two parts

$$\{|\xi_*-u_*|\ge|\xi-u_*|/2\}\cup\{|\xi_*-u_*|\le|\xi-u_*|/2\}\equiv\mathcal{A}\cup\mathcal{B}.$$

For the first region $\mathcal{A}=\{|\xi_*-u_*|\ge|\xi-u_*|/2\}$,

$$\sqrt{M_*(\xi_*)}\le CM_*^{1/4}(\xi_*)M_*^{1/16}(\xi).$$

The term Q^1_{loss} over the domain $\mathcal{A}$ can be bounded by

$$\begin{aligned}
&\Big|\int_{\mathcal{A}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\partial^{\alpha_1}_{\beta_1}f(\xi)\partial^{\alpha_2}_{\beta_2}g(\xi_*)\frac{h(\xi)}{M_*(\xi)}d\xi d\xi_* d\omega\Big|\\
&=\Big|\int_{\mathcal{A}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\sqrt{M_*(\xi_*)}\frac{\partial^{\alpha_1}_{\beta_1}f(\xi)}{\sqrt{M_*(\xi)}}\frac{\partial^{\alpha_2}_{\beta_2}g(\xi_*)}{\sqrt{M_*(\xi_*)}}\frac{h(\xi)}{\sqrt{M_*(\xi)}}d\xi d\xi_* d\omega\Big|\\
&\le C\Big\{\int_{\mathbf{R}^3\times\mathbf{R}^3}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma M_*^{1/4}(\xi_*)M_*^{1/16}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\frac{|h(\xi)|^2}{M_*(\xi)}d\xi d\xi_*\Big\}^{1/2}\\
&\quad\times\Big\{\int_{\mathbf{R}^3\times\mathbf{R}^3}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma M_*^{1/4}(\xi_*)M_*^{1/16}(\xi)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi d\xi_*\Big\}^{1/2}\\
&\le C\sup_\xi\Big\{M_*^{1/32}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|}{\sqrt{M_*(\xi)}}\Big\}\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\\
&\quad\times\Big\{\int_{\mathbf{R}^3}M_*^{1/4}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi d\xi_*\Big\}^{1/2}.
\end{aligned}$$

Applying (2.4) to the first term of the last line concludes the first part $\mathcal{A}$.

Next consider the term Q^1_{loss} over the domain $\mathcal{B}$. We assume further that $|\xi-u_*|\le 1$. Since $\gamma<0$, $|\xi-\xi_*|^\gamma\le C|\xi-u_*|^\gamma$, thus

$$\begin{aligned}
&\Big|\int_{\mathcal{B}\cap\{|\xi-u_*|\le 1\}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\sqrt{M_*(\xi_*)}\frac{\partial^{\alpha_1}_{\beta_1}f(\xi)}{\sqrt{M_*(\xi)}}\frac{\partial^{\alpha_2}_{\beta_2}g(\xi_*)}{\sqrt{M_*(\xi_*)}}\frac{h(\xi)}{\sqrt{M_*(\xi)}}d\xi d\xi_* d\omega\Big|\\
&\le C\int_{|\xi-u_*|\le1}\Big\{\int_{\mathbf{R}^3}|\xi-\xi_*|^{\gamma/2}\sqrt{M_*(\xi_*)}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|}{\sqrt{M_*(\xi_*)}}d\xi_*\Big\}|\xi-u_*|^{\gamma/2}\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|}{\sqrt{M_*(\xi)}}\frac{|h(\xi)|}{\sqrt{M_*(\xi)}}d\xi\\
&\quad\le C\Big\{\int_{|\xi-u_*|\le1}|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\Big\{\int_{|\xi-u_*|\le1}\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3}M_*^{1/4}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}\\
&\le\sum_{|\alpha_1|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial^{\alpha_1}_{\beta'}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\\
&\quad\times\Big\{\int_{\mathbf{R}^3}w^{2|\beta_2|}(\xi_*)\nu(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2},
\end{aligned}$$

where one has used the inequality

$$
\begin{aligned}
\Big\{\int_{|\xi-u_*|\le 1}|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}&\le C\sup_{|\xi-u_*|\le 1}\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|}{\sqrt{M_*(\xi)}}\\
&\le C\sum_{|\alpha_1|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial^{\alpha_1}_{\beta'}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}. \qquad (2.6)
\end{aligned}
$$

For the last part $\mathcal{B}\cap\{|\xi-u_*|\ge 1\}$, it holds that

$$
\begin{aligned}
&\Big|\int_{\mathcal{B}\cap\{|\xi-u_*|\ge 1\}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\sqrt{M_*(\xi_*)}\frac{\partial^{\alpha_1}_{\beta_1}f(\xi)}{\sqrt{M_*(\xi)}}\frac{\partial^{\alpha_2}_{\beta_2}g(\xi_*)}{\sqrt{M_*(\xi_*)}}\frac{h(\xi)}{\sqrt{M_*(\xi)}}d\xi d\xi_* d\omega\Big|\\
&\le C\int_{\mathbf{R}^3}\sqrt{M_*(\xi_*)}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|}{\sqrt{M_*(\xi_*)}}d\xi_*\times\int_{|\xi-u_*|\ge 1}w^{2\theta}(\xi)|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|}{\sqrt{M_*(\xi)}}\frac{|h(\xi)|}{\sqrt{M_*(\xi)}}d\xi\\
&\le C\Big\{\int_{\mathbf{R}^3}w^{2|\beta_1|}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}w^{2|\beta_2|}(\xi_*)\nu(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3}w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2},
\end{aligned}
$$

which is bounded by (2.2). Similarly for the second loss term, one can obtain the same estimates.

In what follows we will estimate the first gain term in (2.1). We use the partition $\mathcal{A}$ and $\mathcal{B}$ again. In the region $\mathcal{A}=\{|\xi_*-u_*|\ge|\xi-u_*|/2\}$,

$$
\sqrt{M_*(\xi_*)}\le CM_*^{1/4}(\xi_*)M_*^{1/16}(\xi),
$$

and we have

$$
\begin{aligned}
&\int_{\mathcal{A}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\partial^{\alpha_1}_{\beta_1}f(\xi')\partial^{\alpha_2}_{\beta_2}g(\xi'_*)\frac{h(\xi)}{M_*(\xi)}d\xi_* d\omega d\xi\\
&=\int_{\mathcal{A}}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\sqrt{M_*(\xi_*)}\frac{\partial^{\alpha_1}_{\beta_1}f(\xi')}{\sqrt{M_*(\xi')}}\frac{\partial^{\alpha_2}_{\beta_2}g(\xi'_*)}{\sqrt{M_*(\xi'_*)}}\frac{h(\xi)}{\sqrt{M_*(\xi)}}d\xi_* d\omega d\xi\\
&\le C\Big\{\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}|\xi-\xi_*|^\gamma B(\vartheta)M_*^{1/4}(\xi_*)M^{1/16}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_* d\omega d\xi\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3\times\mathbf{R}^3}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma M_*^{1/4}(\xi_*)M_*^{1/16}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi_* d\xi\Big\}^{1/2}\\
&\le C\Big\{\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}|\xi'-\xi'_*|^\gamma B(\vartheta)M_*^{1/16}(\xi'_*)M_*^{1/16}(\xi')\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi'_* d\omega d\xi'\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}, \qquad (2.7)
\end{aligned}
$$

where one has used the facts $|\xi|^2+|\xi_*|^2=|\xi'|^2+|\xi_*'|^2$ and $|\xi-\xi_*|=|\xi'-\xi_*'|$. Assuming that $|\alpha_1|+|\beta_1|\le N/2$ and using a similar inequality as in (2.4), one can bound the integral in (2.7) by

$$\begin{aligned}
&\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}|\xi'-\xi_*'|^\gamma B(\vartheta)M_*^{1/16}(\xi_*')M_*^{1/16}(\xi')\frac{|\partial_{\beta_1}^{\alpha_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial_{\beta_2}^{\alpha_2}g(\xi_*')|^2}{M_*(\xi_*')}d\xi_*'d\omega d\xi'\\
&\le C\sum_{|\alpha_1|+|\beta'|\le N-1}\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial_{\beta'}^{\alpha_1}f(\xi)|^2}{M_*(\xi)}d\xi\\
&\qquad\times\int_{\mathbf{R}^3\times\mathbf{R}^3}|\xi'-\xi_*'|^\gamma M_*^{1/32}(\xi_*')M_*^{1/32}(\xi')\frac{|\partial_{\beta_2}^{\alpha_2}g(\xi_*')|^2}{M_*(\xi_*')}d\xi_*'d\xi'\\
&\le C\sum_{|\alpha_1|+|\beta'|\le N-1}\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial_{\beta'}^{\alpha_1}f(\xi)|^2}{M_*(\xi)}d\xi\int_{\mathbf{R}^3}M_*^{1/32}(\xi_*')\frac{|\partial_{\beta_2}^{\alpha_2}g(\xi_*')|^2}{M_*(\xi_*')}d\xi_*'.
\end{aligned}$$

Therefore for $|\alpha_1|+|\beta_1|\le N/2$, it holds that

$$\begin{aligned}
&\Big|\int_{\mathcal{A}}w^{2\theta}(\xi)Q_{gain}^1(\partial_{\beta_1}^{\alpha_1}f,\partial_{\beta_2}^{\alpha_2}g)\frac{h(\xi)}{M_*(\xi)}d\xi\Big|\\
&\le C\sum_{|\alpha_1|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial_{\beta'}^{\alpha_1}f(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}M_*^{1/32}(\xi_*')\frac{|\partial_{\beta_2}^{\alpha_2}g(\xi_*')|^2}{M_*(\xi_*')}d\xi_*'\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2},
\end{aligned}\tag{2.8}$$

which is controlled by the right hand side of (2.2) because $M_*^{1/32}(\xi_*')\le C\nu(\xi_*')w^{2|\beta_2|}(\xi_*')$.

Similarly, if $|\alpha_2|+|\beta_2|\le N/2$, one has

$$\begin{aligned}
&\Big|\int_{\mathcal{A}}w^{2\theta}(\xi)Q_{gain}^1(\partial_{\beta_1}^{\alpha_1}f,\partial_{\beta_2}^{\alpha_2}g)\frac{h(\xi)}{M_*(\xi)}d\xi\Big|\\
&\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\Big\{\int_{\mathbf{R}^3}w^{2|\beta'|}(\xi)\frac{|\partial_{\beta'}^{\alpha_2}g(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}M^{1/32}(\xi')\frac{|\partial_{\beta_1}^{\alpha_1}f(\xi')|^2}{M_*(\xi')}d\xi'\Big\}^{1/2}\\
&\qquad\times\Big\{\int_{\mathbf{R}^3}\nu(\xi)w^{2\theta}(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}.
\end{aligned}\tag{2.9}$$

Since the last inequality in (2.7) is symmetric about ξ' and ξ_*', the second gain term over such a region admits the same bound as the first one.

Now we consider the first gain term over $\mathcal{B}=\{|\xi_*-u_*|\le|\xi-u_*|/2\}$. Assume further $|\xi-u_*|\le 1$. Then $|\xi_*-u_*|\le 1/2$ and the gain term is bounded by

$$\begin{aligned}
&\Big|\int_{\mathcal{B},|\xi-u_*|\le1}w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\partial_{\beta_1}^{\alpha_1}f(\xi')\partial_{\beta_2}^{\alpha_2}g(\xi_*')\frac{h(\xi)}{M_*(\xi)}d\xi_*d\omega d\xi\Big|\\
&\le C\Big\{\int_{\mathcal{B},|\xi-u_*|\le1}w^{2\theta}(\xi)B(\vartheta)|\xi-u_*|^\gamma\frac{|\partial_{\beta_1}^{\alpha_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial_{\beta_2}^{\alpha_2}g(\xi_*')|^2}{M_*(\xi_*')}d\xi d\xi_*d\omega\Big\}^{1/2}
\end{aligned}$$

$$\times\Big\{\int_{\mathcal{B},|\xi-u_*|\leq 1} w^{2\theta}(\xi)|\xi-\xi_*|^\gamma M_*(\xi_*)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi_* d\xi\Big\}^{1/2}$$

$$\leq C\Big\{\int_{\mathcal{B},|\xi-u_*|\leq 1} w^{2\theta}(\xi)B(\vartheta)|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi d\xi_* d\omega\Big\}^{1/2}$$

$$\times\Big\{\int_{\mathbf{R}^3} w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2},$$

where one has used the fact that $|\xi-\xi_*|^\gamma\leq 2^{-\gamma}|\xi-u_*|^\gamma$. We now estimate the first factor above. Since $|\xi_*-u_*|\leq|\xi-u_*|/2$, it follows from (1.3) that

$$|\xi'-u_*|+|\xi'_*-u_*|\leq C[|\xi-u_*|+|\xi_*-u_*|]\leq C|\xi-u_*|. \tag{2.10}$$

Since $\gamma<0$, this implies

$$|\xi-u_*|^\gamma\leq C|\xi'-u_*|^\gamma,\quad |\xi-u_*|^\gamma\leq C|\xi'_*-u_*|^\gamma, \tag{2.11}$$

and $w^{2\theta}(\xi)\leq C\min\{w^{2\theta}(\xi'),w^{2\theta}(\xi'_*)\}$ for $\theta\geq 0$. Thus, we have

$$\Big|\int_{\mathcal{B},|\xi-u_*|\leq 1} w^{2\theta}(\xi)B(\vartheta)|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi d\xi_* d\omega\Big|$$

$$\leq C\int_{|\xi'-u_*|\leq C,|\xi'_*-u_*|\leq C}\min\{w^{2\theta}(\xi'),w^{2\theta}(\xi'_*)\}B(\vartheta)$$

$$\times\min\{|\xi'-u_*|^\gamma,|\xi'_*-u_*|^\gamma\}\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi d\xi_* d\omega.$$

Now changing variables $(\xi',\xi'_*)\to(\xi,\xi_*)$, one can rewrite the right hand side of the above inequality as

$$C\int_{|\xi-u_*|\leq C,|\xi_*-u_*|\leq C}\min\{w^{2\theta}(\xi),w^{2\theta}(\xi_*)\}B(\vartheta)\min\{|\xi-u_*|^\gamma,|\xi_*-u_*|^\gamma\}$$

$$\times\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi d\xi_* d\omega. \tag{2.12}$$

For $|\alpha_1|+|\beta_1|\leq N/2$, (2.12) is bounded by

$$C\int_{|\xi-u_*|\leq C} w^{2\theta}(\xi)|\xi-u_*|^\gamma\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\int_{|\xi_*-u_*|\leq C}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*$$

$$\leq C\sup_{|\xi-u_*|\leq C}\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\int_{|\xi_*-u_*|\leq C}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*$$

$$\leq C\sum_{|\alpha_1|+|\beta'|\leq N-1}\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi)\frac{|\partial^{\alpha_1}_{\beta'}f(\xi)|^2}{M_*(\xi)}d\xi\int_{\mathbf{R}^3} w^{2\theta}(\xi_*)\nu(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*, \tag{2.13}$$

which is bounded by the right hand side of (2.2).

Similarly, if $|\alpha_2|+|\beta_2|\le N/2$, we have

$$
\begin{aligned}
&\Big|\int_{\mathcal{B},|\xi-u_*|\le 1} w^{2\theta}(\xi)Q^1_{gain}(\partial^{\alpha_1}_{\beta_1}f,\partial^{\alpha_2}_{\beta_2}g)\frac{h(\xi)}{M_*(\xi)}d\xi\Big|\\
&\le C\sum_{|\alpha_2|+|\beta'|\le N-1}\int_{\mathbf{R}^3} w^{2|\beta'|}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta'}g(\xi_*)|^2}{M_*(\xi_*)}d\xi\int_{\mathbf{R}^3} w^{2\theta}(\xi)\nu(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi_*\\
&\qquad\times\Big\{\int_{\mathbf{R}^3} w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2}.
\end{aligned}\tag{2.14}
$$

Since (2.12) is symmetric about ξ and ξ_*, the second gain term over such a region has the same bound as the first one.

It remains to estimate the gain terms over the region $\mathcal{B}\cap\{|\xi-u_*|\ge 1\}$. The first gain term will be controlled by

$$
\begin{aligned}
&\Big|\int_{\mathcal{B},|\xi-u_*|\ge 1} w^{2\theta}(\xi)|\xi-\xi_*|^\gamma B(\vartheta)\partial^{\alpha_1}_{\beta_1}f(\xi')\partial^{\alpha_2}_{\beta_2}g(\xi'_*)\frac{h(\xi)}{M_*(\xi)}d\xi_*d\omega d\xi\Big|\\
&\le C\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma\sqrt{M_*(\xi_*)}B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|}{\sqrt{M_*(\xi')}}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|}{\sqrt{M_*(\xi'_*)}}\frac{|h(\xi)|}{\sqrt{M_*(\xi)}}d\xi_*d\omega d\xi\\
&\le C\Big\{\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi\Big\}^{1/2}\\
&\qquad\times\Big\{\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma M_*(\xi_*)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi_*d\xi\Big\}^{1/2}\\
&\le C\Big\{\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi\Big\}^{1/2}\\
&\qquad\times\Big\{\int w^{2\theta}(\xi)\nu(\xi)\frac{|h(\xi)|^2}{M_*(\xi)}d\xi\Big\}^{1/2},
\end{aligned}\tag{2.15}
$$

due to $|\xi-\xi_*|^\gamma\le 4^{-\gamma}(1+|\xi-u_*|)^\gamma$. Next we estimate the first factor in the last inequality. It follows from (2.10) and (2.11) that

$$
\begin{aligned}
(1+|\xi-u_*|)^\gamma&\le C\min\{(1+|\xi'-u_*|)^\gamma,(1+|\xi'_*-u_*|)^\gamma\},\\
w^{2\theta}(\xi)&\le w^{2|\beta|}(\xi)\le Cw^{2|\beta_1|}(\xi')w^{2|\beta_2|}(\xi'_*).
\end{aligned}\tag{2.16}
$$

Assume that $|\alpha_1|+|\beta_1|\le N/2$. Using these estimates and the change of variable $(\xi',\xi'_*)\to(\xi,\xi_*)$ gives

$$
\begin{aligned}
&\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi\\
&\le C\int w^{2|\beta_1|}(\xi')w^{2|\beta_2|}(\xi'_*)(1+|\xi'_*-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi\\
&=C\int w^{2|\beta_1|}(\xi)w^{2|\beta_2|}(\xi_*)(1+|\xi_*-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*d\omega d\xi
\end{aligned}
$$

$$\leq C\int w^{2|\beta_1|}(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\times\int w^{2|\beta_2|}(\xi_*)\nu(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*.$$

Similarly, if $|\alpha_2|+|\beta_2|\leq N/2$, we have from (2.16) that

$$\int w^{2\theta}(\xi)(1+|\xi-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi$$

$$\leq C\int w^{2|\beta_1|}(\xi')w^{2|\beta_2|}(\xi'_*)(1+|\xi'-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi')|^2}{M_*(\xi')}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi'_*)|^2}{M_*(\xi'_*)}d\xi_*d\omega d\xi$$

$$=C\int w^{2|\beta_1|}(\xi)w^{2|\beta_2|}(\xi_*)(1+|\xi-u_*|)^\gamma B(\vartheta)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*d\omega d\xi$$

$$\leq C\int w^{2|\beta_1|}(\xi)\nu(\xi)\frac{|\partial^{\alpha_1}_{\beta_1}f(\xi)|^2}{M_*(\xi)}d\xi\times\int w^{2|\beta_2|}(\xi_*)\frac{|\partial^{\alpha_2}_{\beta_2}g(\xi_*)|^2}{M_*(\xi_*)}d\xi_*.$$

It follows from the above two inequalities and (2.15) that the first gain term in the last case can bounded by (2.2) and (2.3). Similarly, the second gain term has the same bound as the first one. This concludes the proof of the lemma.

Similar arguments for Lemma 2.1 yield the following lemma.

Lemma 2.2. Under the assumptions of Lemma 2.1, it holds that

$$\Big|\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)Q(f,\partial_\beta(M-M_-))h(\xi)}{M_-}d\xi\Big|$$

$$\leq C\eta_0\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|f(\xi)|^2}{M_-}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|h(\xi)|^2}{M_-}d\xi\Big\}^{1/2}. \tag{2.17}$$

Moreover, if $|\alpha|\geq 1$, then

$$\Big|\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)Q(f,\partial^\alpha_\beta(M-M_-))h(\xi)}{M_-}d\xi\Big|$$

$$\leq C\Big(|\partial^\alpha(\rho,u,\theta)|+\sum_{1\leq|\alpha'|\leq|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)||\partial^{\alpha'}(\rho,u,\theta)|+...+|(\rho_x,u_x,\theta_x)|^{|\alpha|}\Big)$$

$$\times\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|f(\xi)|^2}{M_-}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|h(\xi)|^2}{M_-}d\xi\Big\}^{1/2}, \tag{2.18}$$

and

$$\Big|\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)Q(f,\partial_\beta M)h(\xi)}{M_*}d\xi\Big| \tag{2.19}$$

$$\leq C(\rho_-,u_-,\theta_-)\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|f(\xi)|^2}{M_*}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{w^{2\theta}(\xi)\nu(\xi)|h(\xi)|^2}{M_*}d\xi\Big\}^{1/2},$$

where M_* is either M or M_-.

This lemma together with the dissipative property of the linearized collision operator yields immediately the following lemma.

Lemma 2.3. Under the assumptions in Lemma 2.1, for $h(\xi) \in \mathcal{N}^{\perp}$, there exists a constant $\sigma_1 = \sigma_1(\rho_-, u_-, \theta_-) > 0$ such that

$$-\int_{\mathbf{R}^3} \frac{hL_M h}{M_-} d\xi \geq \sigma_1 \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{M_-} d\xi. \tag{2.20}$$

Proof. Due to (1.15) and Lemma 2.2, one has

$$\begin{aligned}
-\int_{\mathbf{R}^3} \frac{hL_M h}{M_-} d\xi &= -\int_{\mathbf{R}^3} \frac{hL_{M_-} h}{M_-} d\xi + 2\int_{\mathbf{R}^3} \frac{hQ(h, M_- - M)}{M_-} d\xi \\
&\geq \sigma_0 \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{M_-} d\xi - C\eta_0 \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{M_-} d\xi. \qquad (2.21)
\end{aligned}$$

Choose $\eta_0 > 0$ small enough so that $\sigma_1 = \sigma_0 - C\eta_0 > 0$. Then the lemma follows.

The Hölder inequality and Lemma 2.3 imply that for $h(\xi) \in \mathcal{N}^{\perp}$, it holds that

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)|L_M^{-1} h|^2}{M_-} d\xi \leq C \int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)h^2}{M_-} d\xi. \tag{2.22}$$

Now recall the Burnett functions, cf. [2, 5, 13, 27], defined as :

$$\hat{A}_j(\xi) = \frac{|\xi|^2 - 5}{2}\xi_j \quad \text{and} \quad \hat{B}_{ij}(\xi) = \xi_i \xi_j - \frac{1}{3}\delta_{ij}|\xi|^2 \quad \text{for } i, j = 1, 2, 3. \tag{2.23}$$

Because $\hat{A}_j M$ and $\hat{B}_{ij} M$ are in $L^2(\mathbf{R}^3)$ with the weight function $\frac{\nu^{-1}(\xi)}{M_-}$, there exist functions $A_j(\xi)$ and $B_{ij}(\xi)$ in $L^2(\mathbf{R}^3)$ with the weight function $\frac{\nu(\xi)}{M_-}$ such that $P_0 A_j = 0$, $P_0 B_{ij} = 0$, and

$$A_j\Big(\frac{\xi - u}{\sqrt{R\theta}}\Big) = L_M^{-1}\Big(\hat{A}_j(\frac{\xi - u}{\sqrt{R\theta}})M\Big) \quad \text{and} \quad B_{ij}\Big(\frac{\xi - u}{\sqrt{R\theta}}\Big) = L_M^{-1}\Big(\hat{B}_{ij}(\frac{\xi - u}{\sqrt{R\theta}})M\Big).$$

Before going further, we list some elementary but important properties of the Burnett functions summarized in the following lemma, cf. [5, 13].

Lemma 2.4. The Burnett functions have the following properties:

- $-\langle \hat{A}_i M, A_i \rangle_M$ is positive and independent of i;
- $\langle \hat{A}_i M, A_j \rangle_M = 0$ for any $i \neq j$;
- $\langle \hat{A}_i M, B_{jk} \rangle_M = 0$ for any i, j, k;
- $\langle \hat{B}_{ij} M, B_{kl} \rangle_M = \langle \hat{B}_{kl} M, B_{ij} \rangle_M = \langle \hat{B}_{ji} M, B_{kl} \rangle_M$, which is independent of i, j for fixed k, l;
- $-\langle \hat{B}_{ij} M, B_{ij} \rangle_M$, is positive and independent of i, j when $i \neq j$;
- $-\langle \hat{B}_{ii} M, B_{jj} \rangle_M$, is positive and independent of i, j when $i \neq j$;
- $-\langle \hat{B}_{ii} M, B_{ii} \rangle_M$, is positive and independent of i;
- $\langle \hat{B}_{ij} M, B_{kl} \rangle_M = 0$ unless either $(i, j) = (k, l)$ or (l, k), or $i = j$ and $k = l$;

- $\langle \hat{B}_{ii}M, B_{ii}\rangle_M - \langle \hat{B}_{ii}M, B_{jj}\rangle_M = 2\langle \hat{B}_{ij}M, B_{ij}\rangle_M$ holds for any $i \neq j$.

In terms of Burnett functions, the viscosity coefficient $\mu(\theta)$ and the heat conductivity coefficient $\kappa(\theta)$ can be represented by

$$\mu(\theta) = -R\theta \int_{\mathbf{R}^3} B_{ij}(\frac{\xi - u}{\sqrt{R\theta}})\hat{B}_{ij}(\frac{\xi - u}{\sqrt{R\theta}})d\xi > 0, \quad i \neq j, \tag{2.24}$$

$$\kappa(\theta) = -R^2\theta \int_{\mathbf{R}^3} A_j(\frac{\xi - u}{\sqrt{R\theta}})\hat{A}_j(\frac{\xi - u}{\sqrt{R\theta}})d\xi > 0. \tag{2.25}$$

Notice that these coefficients are independent of the density function ρ and they are positive smooth functions of the temperature θ.

To study the decay and regularity of the Burnett functions, we prove the following lemma which will be used frequently in the later energy estimates.

Lemma 2.5. Under the assumptions in Lemma 2.1, for any $|\beta| \geq 0$ and $\sigma > 0$

$$\int_{\mathbf{R}^3} (1 + |\xi - u_-|)^\sigma \frac{|\partial_\beta A_j(\frac{\xi-u}{\sqrt{R\theta}})|^2}{M_-}d\xi + \int_{\mathbf{R}^3} (1 + |\xi - u_-|)^\sigma \frac{|\partial_\beta B_{ij}(\frac{\xi-u}{\sqrt{R\theta}})|^2}{M_-}d\xi < \infty.$$

Proof. By using (2.22), one obtains that

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)|A_j(\frac{\xi-u}{\sqrt{R\theta}})|^2}{M_-}d\xi = \int_{\mathbf{R}^3} \frac{\nu(\xi)|L_M^{-1}(\hat{A}_j(\frac{\xi-u}{\sqrt{R\theta}})M)|^2}{M_-}d\xi$$

$$\leq C\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)|\hat{A}_j(\frac{\xi-u}{\sqrt{R\theta}})M|^2}{M_-}d\xi \leq C_1(\rho_-, u_-, \theta_-). \tag{2.26}$$

For any $|\beta_1| = 1$, it holds that

$$\partial_{\beta_1} A_j = \partial_{\beta_1} L_M^{-1}(\hat{A}_j M) = L_M^{-1}\partial_{\beta_1}(\hat{A}_j M) - 2L_M^{-1}Q(A_j, \partial_{\beta_1} M).$$

Due to (2.22), we obtain

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)|L_M^{-1}\partial_{\beta_1}(\hat{A}_j M)|^2}{M_-}d\xi \leq C\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)|\partial_{\beta_1}(\hat{A}_j M)|^2}{M_-}d\xi \leq C_1.$$

By (2.22), (2.26) and Lemma 2.2, one has

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)|L_M^{-1}Q(A_j, \partial_{\beta_1} M)|^2}{M_-}d\xi \leq C\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)|Q(A_j, \partial_{\beta_1} M)|^2}{M_-}d\xi$$

$$\leq C\int_{\mathbf{R}^3} \frac{\nu(\xi)|A_j|^2}{M_-}d\xi \leq C_2. \tag{2.27}$$

Thus

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_{\beta_1} A_j|^2}{M_-}d\xi \leq C_3. \tag{2.28}$$

By repeating the above procedure inductively, we can obtain for any $|\beta|\in N$ that

$$\int_{\mathbf{R}^3}\frac{(1+|\xi-u_-|^2)^{\gamma/2}|\partial_\beta A_j|^2}{M_-}d\xi\le C\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_\beta A_j|^2}{M_-}d\xi\le C_{|\beta|}(\rho_-,u_-,\theta_-).\quad(2.29)$$

We claim that for any $\sigma>0$, it holds that

$$\int_{\mathbf{R}^3}\frac{(1+|\xi-u_-|^2)^{\sigma}|\partial_\beta A_j|^2}{M_-}d\xi\le C_\sigma.\quad(2.30)$$

In fact, for any $m\in N$ and $|\overline{\beta}|\in N$, (2.29) implies that

$$\begin{aligned}&\int_{\mathbf{R}^3}(a_0+a_1|\xi-u_-|^2+...+|\xi-u_-|^{2m})e^{|\xi-u_-|^2/2R\theta_-}(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\overline{\beta}}A_j|^2d\xi\\
&=\int_{\mathbf{R}^3}(\mathbf{I}+\Delta_\xi)^m e^{|\xi-u_-|^2/2R\theta_-}(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\overline{\beta}}A_j|^2d\xi\\
&=\int_{\mathbf{R}^3}e^{|\xi-u_-|^2/2R\theta_-}(\mathbf{I}+\Delta_\xi)^m\{(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\overline{\beta}}A_j|^2\}d\xi\\
&=\sum_{\beta_1,\beta_2,\beta_3}C_{\beta_1\beta_2\beta_3}\int_{\mathbf{R}^3}e^{|\xi-u_-|^2/2R\theta_-}\partial_{\beta_1}(1+|\xi-u_-|^2)^{\gamma/2}\partial_{\beta_2}A_j\partial_{\beta_3}A_jd\xi,\end{aligned}$$

where a_i $(i=0,1,..m)$ are some positive numbers, β_j $(j=1,2,3)$ are some multi-indices. It is clear that

$$|\partial_{\beta_1}(1+|\xi-u_-|^2)^{\gamma/2}|\le C(1+|\xi-u_-|^2)^{\gamma/2}.$$

Hence, for any $\sigma>0$, there exists $m\in N$ such that

$$\begin{aligned}&\int_{\mathbf{R}^3}\frac{(1+|\xi-u_-|^2)^{\sigma}|\partial_\beta A_j|^2}{M_-}d\xi\\
&\le\int_{\mathbf{R}^3}(a_0+a_1|\xi-u_-|^2+...+|\xi-u_-|^{2m})e^{|\xi-u_-|^2/2R\theta_-}(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\overline{\beta}}A_j|^2d\xi\\
&\le C\sum\Big\{\int_{\mathbf{R}^3}e^{|\xi-u_-|^2/2R\theta_-}(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\beta_2}A_j|^2d\xi\Big\}^{1/2}\\
&\times\Big\{\int_{\mathbf{R}^3}e^{|\xi-u_-|^2/2R\theta_-}(1+|\xi-u_-|^2)^{\gamma/2}|\partial_{\beta_3}A_j|^2d\xi\Big\}^{1/2}<\infty.\end{aligned}$$

Therefore, the claim (2.30) has been proved. Since $B_{ij}(\xi)$ shares the similar properties as $A_j(\xi)$, the above argument works also for $B_{ij}(\xi)$, and this completes the proof of the lemma.

By the similar arguments as Lemma 2.2 and Lemma 2.5, we can obtain

Corollary 2.6. Under the assumptions in Lemma 2.1, for any $|\beta|\ge 0$ and $\sigma>0$,

$$\int_{\mathbf{R}^3}(1+|\xi-u_-|)^\sigma\frac{|\partial_\beta A_j(\frac{\xi-u}{\sqrt{R\theta}})|^2}{M}d\xi+\int_{\mathbf{R}^3}(1+|\xi-u_-|)^\sigma\frac{|\partial_\beta B_{ij}(\frac{\xi-u}{\sqrt{R\theta}})|^2}{M}d\xi<\infty,\quad(2.31)$$

$$\Big|\int_{\mathbf{R}^3}\frac{w^{2l}(\xi)Q(f,\partial_\beta A_i)h(\xi)}{M_*}d\xi\Big|+\Big|\int_{\mathbf{R}^3}\frac{w^{2l}(\xi)Q(f,\partial_\beta B_{ij})h(\xi)}{M_*}d\xi\Big|$$
$$\le C\Big\{\int_{\mathbf{R}^3}\frac{w^{2l}(\xi)\nu(\xi)|f(\xi)|^2}{M_*}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{w^{2l}(\xi)\nu(\xi)|h(\xi)|^2}{M_*}d\xi\Big\}^{1/2}, \tag{2.32}$$
where M_* is either M or M_-.

Lemma 2.7. Assume that $1\le|\alpha|\le N$. Under the assumptions in Lemma 2.1 and the assumption **H3**, it holds that for any $\lambda>0$
$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\partial^\alpha Q(G,M-M_-)\frac{h}{M_-}d\xi dx\le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4$$
$$+C(\lambda+\eta_0)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx. \tag{2.33}$$

Proof. Since $1\le|\alpha|\le N$, one has
$$\partial^\alpha Q(G,M-M_-)=Q(\widetilde{G},\partial^\alpha(M-M_-))+Q(\overline{G},\partial^\alpha(M-M_-))+Q(\partial^\alpha G,(M-M_-))$$
$$+\sum_{1\le|\alpha_1|<|\alpha|}C_\alpha^{\alpha_1}Q(\partial^{\alpha_1}G,\partial^{\alpha-\alpha_1}(M-M_-)). \tag{2.34}$$
For the first term on the right hand side of (2.34), Lemma 2.2 implies that
$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}Q(\widetilde{G},\partial^\alpha(M-M_-))\frac{h}{M_-}d\xi dx\le\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx+C_\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\widetilde{G}|^2}{M_-}d\xi$$
$$\times\Big(|\partial^\alpha(\rho,u,\theta)|^2+\sum_{1\le|\alpha'|\le|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+...+|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\Big)dx. \tag{2.35}$$

To estimate the last term in (2.35), we will consider the first and the last products as the other products can be treated similarly.

First, since for any function $g(x)\in H^1(\mathbf{R})\subset L^\infty(\mathbf{R})$,
$$\sup_x|g(x)|\le C\|g(x)\|^{1/2}\|g_x(x)\|^{1/2}. \tag{2.36}$$
Hence
$$\int_{\mathbf{R}}|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\widetilde{G}|^2}{M_-}d\xi dx\le C\|(\rho_x,u_x,\theta_x)\|^{|\alpha|}\|(\rho_{xx},u_{xx},\theta_{xx})\|^{|\alpha|}$$
$$\times\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\widetilde{G}|^2}{M_-}d\xi dx\le C\mathcal{E}^{|\alpha|}(t)\mathcal{D}(t)\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.37}$$
Here, one has used the fact that the a priori assumption that $\mathcal{E}(t)$ is sufficiently small.

To handle the first product, one has that for $1 \leq |\alpha| \leq N/2$,

$$\int_{\mathbf{R}} |\partial^\alpha(\rho,u,\theta)|^2 \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx \leq C \sup_x |\partial^\alpha(\rho,u,\theta)|^2 \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx$$

$$\leq C\|\partial^\alpha(\rho,u,\theta)\| \cdot \|\partial^\alpha(\rho_x,u_x,\theta_x)\| \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx, \tag{2.38}$$

which is bounded by $C\mathcal{E}(t)\mathcal{D}(t)$. On the other hand, if $|\alpha| \geq N/2$, it holds that

$$\int_{\mathbf{R}} |\partial^\alpha(\rho,u,\theta)|^2 \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx \leq \sup_x \Big(\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi\Big) \|\partial^\alpha(\rho,u,\theta)\|^2$$

$$\leq C\|\partial^\alpha(\rho,u,\theta)\|^2 \Big\{\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx\Big\}^{1/2} \Big\{\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}_x|^2}{M_-} d\xi dx\Big\}^{1/2}. \tag{2.39}$$

It is clear that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}_x|^2}{M_-} d\xi dx \leq C \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|G_x|^2}{M_-} d\xi dx + C \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\overline{G}_x|^2}{M_-} d\xi dx.$$

Moreover, it follows from (1.19) that

$$\overline{G}(t,x,\xi) = \frac{\sqrt{R}\theta_x}{\sqrt{\theta}} A_1\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big) + \overline{u}_{1x} B_{11}\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big),$$

which implies that

$$\begin{aligned}
\overline{G}_x = {}& \frac{\sqrt{R}\theta_{xx}}{\sqrt{\theta}} A_1\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big) - \frac{\sqrt{R}\theta_x\theta_x}{2\sqrt{\theta^3}} A_1\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big) \\
& - \frac{\sqrt{R}\theta_x}{\sqrt{\theta}} A_1'\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big)\frac{u_x}{\sqrt{R\theta}} - \frac{\sqrt{R}\theta_x\theta_x}{\sqrt{\theta}} A_1'\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big)\frac{\xi-u}{2\sqrt{R\theta^3}} \\
& + \overline{u}_{1xx} B_{11}\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big) - \frac{\overline{u}_{1x}u_x}{\sqrt{R\theta}} B_{11}'\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big) - \frac{\overline{u}_{1x}\theta_x}{2\sqrt{R\theta^3}} B_{11}'\Big(\frac{\xi-u}{\sqrt{R\theta}}\Big).
\end{aligned} \tag{2.40}$$

Here, we have represented the term $\overline{G}_x$ precisely by using the Burnett functions so that its estimates can be calculated by using the properties of Burnett functions.

It follows from Lemma 2.5 that

$$\Big\{\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\overline{G}_x|^2}{M_-} d\xi dx\Big\}^{1/2} \leq C(\|(\overline{u}_{1xx},\overline{\theta}_{xx})\| + \|(\overline{u}_{1x},\overline{\theta}_x)\cdot(u_x,\theta_x)\|). \tag{2.41}$$

Thus, we obtain

$$\int_{\mathbf{R}} |\partial^\alpha(\rho,u,\theta)|^2 \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx \leq C(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))\mathcal{D}(t) \leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

where the assumption **H3** has been used for some constant $t_0 > 0$ large enough.

Then

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\widetilde{G}, \partial^\alpha(M-M_-))\frac{h}{M_-} d\xi dx \leq \lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

By Lemma 2.2, one has

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{Q(\overline{G},\partial^\alpha(M-M_-))h}{M_-}d\xi dx$$

$$\leq \lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx + C_\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\overline{G}|^2}{M_-}d\xi\Big(|\partial^\alpha(\rho,u,\theta)|^2$$

$$+\sum_{1\leq|\alpha'|\leq|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+\ldots+|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\Big)dx. \quad (2.42)$$

As before, for the last term in (2.42), it suffices to estimate the first and last products. For the last product, one has from the assumption **H3** and Lemma 2.5 that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\overline{G}|^2}{M_-}d\xi|(\rho_x,u_x,\theta_x)|^{2|\alpha|}dx \leq C\sum_{|\alpha|=1}^{N}\int_{\mathbf{R}}|(\overline{u}_{1x},\overline{\theta}_x)|^2|(\rho_x,u_x,\theta_x)|^{2|\alpha|}dx$$

$$\leq C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4 + C\epsilon\mathcal{D}(t) + C\sum_{|\alpha|=2}^{N}\|(\rho_x,u_x,\theta_x)\|^{|\alpha|}\|(\rho_{xx},u_{xx},\theta_{xx})\|^{|\alpha|}\|(\overline{u}_{1x},\overline{\theta}_x)\|^2$$

$$\leq C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4 + C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t). \quad (2.43)$$

Here, one has used the assumption **H3** for some constant $t_0>0$ large enough.

For the first product in second term on the right hand side of (2.42), one has from the assumption **H3** and Lemma 2.5 that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\overline{G}|^2}{M_-}d\xi|\partial^\alpha(\rho,u,\theta)|^2dx \leq C\sum_{|\alpha|=1}^{N}\int_{\mathbf{R}}|(\overline{u}_{1x},\overline{\theta}_x)|^2|\partial^\alpha(\rho,u,\theta)|^2dx$$

$$\leq C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4 + C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t). \quad (2.44)$$

Finally, we estimate the fourth term on the right hand side of (2.34). It follows from (2.38), (2.39) and Lemma 2.2 that

$$\sum_{1\leq|\alpha_1|<|\alpha|}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{Q(\partial^{\alpha_1}G,\partial^{\alpha-\alpha_1}(M-M_-))h}{M_-}d\xi dx$$

$$\leq \lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx + C_\lambda\sum_{1\leq|\alpha_1|<|\alpha|}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\alpha_1}G|^2}{M_-}d\xi\Big(|\partial^{\alpha-\alpha_1}(\rho,u,\theta)|^2$$

$$+\sum_{1\leq|\alpha'|\leq|\alpha-\alpha_1|}|\partial^{\alpha-\alpha_1-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+\ldots+|(\rho_x,u_x,\theta_x)|^{2|\alpha-\alpha_1|}\Big)dx$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t) + \lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx.$$

Notice that the third term on the right hand side of (2.34) can be estimated directly by using Lemma 2.2. This completes the proof of the lemma.

The following lemma gives an estimate for the term studied in the above lemma with differentiations and the weight in the velocity variables.

Lemma 2.8. Let $|\alpha|+|\beta|\le N$ with $|\alpha|\ge 1$ and $|\beta|\ge 1$. Under the assumptions in Lemma 2.7, it holds that for any $\lambda>0$,

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}\partial^\alpha_\beta Q(G,M-M_-)\frac{h}{M_-}d\xi dx \le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|^4_{L^4}$$
$$+C(\lambda+\eta_0)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

Proof. It is clear that

$$\partial^\alpha_\beta Q(G,M-M_-)=\sum_{\beta_1\le\beta}C^{\beta_1}_\beta[Q(\partial_{\beta_1}\widetilde{G},\partial^\alpha_{\beta-\beta_1}(M-M_-))+Q(\partial_{\beta_1}\overline{G},\partial^\alpha_{\beta-\beta_1}(M-M_-))$$
$$+Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}(M-M_-))+\sum_{1\le|\alpha_1|<|\alpha|}C^{\alpha_1}_\alpha Q(\partial^{\alpha_1}_{\beta_1}G,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(M-M_-))]. \quad (2.45)$$

For the first term on the right hand side of (2.45), one can use Lemma 2.2 and a similar argument as for (2.37), (2.38) and (2.39) to get

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\widetilde{G},\partial^\alpha_{\beta-\beta_1}(M-M_-))\frac{h}{M_-}d\xi dx$$
$$\le\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx+C\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial_{\beta_1}\widetilde{G}|^2}{M_-}d\xi\Big(|\partial^\alpha(\rho,u,\theta)|^2$$
$$+\sum_{1\le|\alpha'|\le|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+...+|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\Big)dx$$
$$\le C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

For the second term on the right hand side of (2.45), it follows from Lemma 2.2, Lemma 2.5 and a similar argument for (2.43) and (2.44) that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\overline{G},\partial^\alpha_{\beta-\beta_1}(M-M_-))\frac{h}{M_*}d\xi dx$$
$$\le\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx+C\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial_{\beta_1}\overline{G}|^2}{M_-}d\xi\Big(|\partial^\alpha(\rho,u,\theta)|^2$$
$$+\sum_{1\le|\alpha'|\le|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+...+|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\Big)dx$$
$$\le C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|^4_{L^4}+C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t).$$

For the third term on the right hand side of (2.45), one can apply Lemma 2.2 to get

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}(M-M_-))\frac{h}{M_-}d\xi dx$$

$$\leq C\eta_0\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial^\alpha_{\beta_1}G|^2}{M_-}d\xi dx+C\eta_0\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

By Lemma 2.2, finally, the last term on the right hand side of (2.45) is bounded by

$$\sum_{1\leq|\alpha_1|<|\alpha|}\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial^{\alpha_1}_{\beta_1}G,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(M-M_-))\frac{h}{M_-}d\xi dx$$

$$\leq \lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx+C\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial^{\alpha_1}_{\beta_1}G|^2}{M_-}d\xi\Big(|\partial^\alpha(\rho,u,\theta)|^2$$

$$+\sum_{1\leq|\alpha'|\leq|\alpha|}|\partial^{\alpha-\alpha'}(\rho,u,\theta)|^2|\partial^{\alpha'}(\rho,u,\theta)|^2+...+|(\rho_x,u_x,\theta_x)|^{2|\alpha|}\Big)dx$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\epsilon)\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

And this completes the proof of the lemma.

The following four lemmas give the corresponding estimates of the above lemmas when $Q(G,M-M_-)$ is replaced by $Q(G,G)$.

Lemma 2.9. Let M_* be either M or M_-. Under assumptions of Lemma 2.7, it holds that for any $\lambda>0$,

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(G,G)\frac{h}{M_*}d\xi dx\leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|^4_{L^4}$$

$$+C(\lambda+\epsilon)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx+C(\lambda+\epsilon)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\widetilde{G}|^2}{M_-}d\xi dx.$$

Proof. Recalling that $G=\overline{G}+\widetilde{G}$, one has

$$Q(G,G)=Q(\overline{G},\overline{G})+2Q(\overline{G},\widetilde{G})+Q(\widetilde{G},\widetilde{G}).\tag{2.46}$$

It follows from Lemma 2.1 that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\overline{G},\overline{G})\frac{h}{M_*}d\xi dx\leq C\sum_{|\beta'|\leq N-1}\int_{\mathbf{R}}\Big\{\int_{\mathbf{R}^3}\frac{|\partial_{\beta'}\overline{G}|^2}{M_*}d\xi\Big\}^{1/2}$$

$$\times\Big\{\int_{\mathbf{R}^3}\frac{\nu(\xi)|\overline{G}|^2}{M_*}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_*}d\xi\Big\}^{1/2}dx.\tag{2.47}$$

By (1.19), Lemma 2.5 and Corollary 2.6, we can obtain that for any $|\beta| \geq 0$ and $\sigma > 0$,

$$\int_{\mathbf{R}^3} \frac{(1+|\xi-u_-|)^\sigma |\partial_\beta \overline{G}|^2}{M_*} d\xi \leq C|(\overline{\rho}_x, \overline{u}_{1x}, \overline{\theta}_x)|^2. \tag{2.48}$$

Due to (2.47) and (2.48), one can obtain that

$$\int_{\mathbf{R}} \int_{\mathbf{R}^3} Q(\overline{G}, \overline{G}) \frac{h}{M_*} d\xi dx \leq C_\lambda \|(\overline{\rho}_x, \overline{u}_{1x}, \overline{\theta}_x)\|_{L^4}^4 + \lambda \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx.$$

By (2.31) and (2.32) we have

$$\begin{aligned}
\int_{\mathbf{R}} \int_{\mathbf{R}^3} Q(\overline{G}, \widetilde{G}) \frac{h}{M_*} d\xi dx &\leq C \sup_x |(\overline{\theta}_x, \overline{u}_{1x})| \Big(\int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx + \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx \Big), \\
&\leq C\epsilon \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx + C\epsilon \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx.
\end{aligned}$$

Thanks to (2.2), one has that

$$\begin{aligned}
&\int_{\mathbf{R}} \int_{\mathbf{R}^3} Q(\widetilde{G}, \widetilde{G}) \frac{h}{M_*} d\xi dx \\
&\leq C_\lambda \sum_{|\beta'| \leq N-1} \int_{\mathbf{R}} \Big\{ \int_{\mathbf{R}^3} \frac{w^{2|\beta'|} |\partial_{\beta'} \widetilde{G}|^2}{M_*} d\xi \Big\} \Big\{ \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_*} d\xi \Big\} dx + \lambda \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_*} d\xi dx \\
&\leq C_\lambda \sum_{|\beta'| \leq N-1} \Big\{ \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{w^{2|\beta'|} |\partial_{\beta'} \widetilde{G}|^2}{M_-} d\xi dx \Big\}^{1/2} \Big\{ \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{w^{2|\beta'|} |\partial_{\beta'} \widetilde{G}_x|^2}{M_-} d\xi dx \Big\}^{1/2} \\
&\qquad \times \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx + \lambda \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi \\
&\leq C_\lambda (\sqrt{\mathcal{E}(t)} + \mathcal{E}(t)) \mathcal{D}(t) + \lambda \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx.
\end{aligned} \tag{2.49}$$

Here one has used (2.40) and (2.41). Combining the above inequalities completes the proof of the lemma.

Lemma 2.10. Let $1 \leq |\alpha| \leq N$. Under assumptions of Lemma 2.7, it holds that

$$\begin{aligned}
\int_{\mathbf{R}} \int_{\mathbf{R}^3} \partial^\alpha Q(G, G) \frac{h}{M_*} d\xi dx &\leq C_\lambda \sqrt{\mathcal{E}(t)} \mathcal{D}(t) \\
&+ C(\lambda + \epsilon) \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|h|^2}{M_-} d\xi dx + C\epsilon \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\alpha G|^2}{M_-} d\xi dx.
\end{aligned}$$

Proof. Since $1 \leq |\alpha| \leq N$, it holds that

$$\partial^\alpha Q(G, G) = 2Q(G, \partial^\alpha G) + \sum_{1 \leq |\alpha_1| < |\alpha|} C_\alpha^{\alpha_1} Q(\partial^{\alpha_1} G, \partial^{\alpha - \alpha_1} G). \tag{2.50}$$

By Corollary 2.6 and the expression of $\overline{G}$, one has

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\overline{G},\partial^\alpha G)\frac{h}{M_*}d\xi dx \le C\sup_x|(\overline{\theta}_x,\overline{u}_{1x})|\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\alpha G|^2}{M_*}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_*}d\xi dx\Big)$$

$$\le C\epsilon\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\alpha G|^2}{M_-}d\xi dx + C\epsilon\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx.$$

It follows from Lemma 2.1 and a similar argument as for (2.38) that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\widetilde{G},\partial^\alpha G)\frac{h}{M_*}d\xi dx \le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx.$$

Similarly, Lemma 2.1 and a similar argument as for (2.38) and (2.39) lead to

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\partial^{\alpha_1}G,\partial^{\alpha-\alpha_1}G)\frac{h}{M_*}d\xi dx \le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|h|^2}{M_-}d\xi dx.$$

In summary, the above inequalities yield the estimate given in the lemma.

Lemma 2.11. Let $1\le|\beta|\le N$. Under assumptions of Lemma 2.7, it holds that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}\partial_\beta Q(G,G)\frac{h}{M_*}d\xi dx \le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C_\lambda\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4$$

$$+C(\lambda+\epsilon)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx+C\epsilon\sum_{\beta_1\le\beta}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial_{\beta_1}\widetilde{G}|^2}{M_-}d\xi dx.$$

Proof. Since $1\le|\beta|\le N$, one has

$$\partial_\beta Q(G,G)=\sum_{\beta_1\le\beta}C_\beta^{\beta_1}Q(\partial_{\beta_1}G,\partial_{\beta-\beta_1}G)=\sum_{\beta_1\le\beta}C_\beta^{\beta_1}(Q(\partial_{\beta_1}\overline{G},\partial_{\beta-\beta_1}\overline{G})$$

$$+Q(\partial_{\beta_1}\widetilde{G},\partial_{\beta-\beta_1}\overline{G})+Q(\partial_{\beta_1}\overline{G},\partial_{\beta-\beta_1}\widetilde{G})+Q(\partial_{\beta_1}\widetilde{G},\partial_{\beta-\beta_1}\widetilde{G})).$$

Due to Lemma 2.1, Lemma 2.5, Corollary 2.6 and the expression of $\overline{G}$, it is clear that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\overline{G},\partial_{\beta-\beta_1}\overline{G})\frac{h}{M_*}d\xi dx \le C_\lambda\|(\overline{\theta}_x,\overline{u}_{1x})\|_{L^4}^4+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

By (2.42), (2.43) and Lemma 2.5, Corollary 2.6, one can obtain

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\widetilde{G},\partial_{\beta-\beta_1}\overline{G})\frac{h}{M_*}d\xi dx$$

$$\le C\sup_x|(\overline{\theta}_x,\overline{u}_{1x})|\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial_{\beta_1}\widetilde{G}|^2}{M_-}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx\Big),$$

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\overline{G},\partial_{\beta-\beta_1}\widetilde{G})\frac{h}{M_*}d\xi dx$$

$$\le C\sup_x|(\overline{\theta}_x,\overline{u}_{1x})|\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta-\beta_1|}|\partial_{\beta-\beta_1}\widetilde{G}|^2}{M_-}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx\Big).$$

Then finally, by using Lemma 2.1 and a similar argument as for (2.38) and (2.39), one has

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial_{\beta_1}\widetilde{G},\partial_{\beta-\beta_1}\widetilde{G})\frac{h}{M_-}d\xi dx \le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

Collecting all the inequalities above completes the proof of the lemma.

Lemma 2.12. Let $|\alpha|+|\beta|\le N-1$ with $|\alpha|\ge 1$ and $|\beta|\ge 1$. Under assumptions of Lemma 2.8, it holds that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}\partial^\alpha_\beta Q(G,G)\frac{h}{M_*}d\xi dx \le C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C_\lambda\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|^4_{L^4}$$

$$+C(\lambda+\epsilon)\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx + C\epsilon\sum_{\beta_1\le\beta}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial^\alpha_{\beta_1}G|^2}{M_-}d\xi dx.$$

Proof.

$$\partial^\alpha_\beta Q(G,G)=\sum_{\beta_1\le\beta}C^{\beta_1}_\beta\Big(Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}\overline{G})+Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}\widetilde{G})$$

$$+Q(\partial_{\beta_1}\overline{G},\partial^\alpha_{\beta-\beta_1}G)+Q(\partial_{\beta_1}\widetilde{G},\partial^\alpha_{\beta-\beta_1}G)+\sum_{0<\alpha_1<\alpha}C^{\alpha_1}_\alpha Q(\partial^{\alpha_1}_{\beta_1}G,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}G)\Big). \qquad (2.51)$$

By Lemma 2.5, Corollary 2.6 and the expression of $\overline{G}$, it is straightforward to show that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}\overline{G})\frac{h}{M_*}d\xi dx$$

$$\le C\sup_x|(\overline{\theta}_x,\overline{u}_{1x})|\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta_1|}|\partial^\alpha_{\beta_1}G|^2}{M_-}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx\Big).$$

By using Lemma 2.1 and a similar argument as for (2.38) and (2.39), one has

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial^\alpha_{\beta_1}G,\partial_{\beta-\beta_1}\widetilde{G})\frac{h}{M_*}d\xi dx$$

$$\le C_\lambda(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx,$$

$$\sum_{0<\alpha_1<\alpha}\int_{\mathbf{R}}\int_{\mathbf{R}^3} w^{2|\beta|}Q(\partial^{\alpha_1}_{\beta_1}G,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}G)\frac{h}{M_*}d\xi dx \le C_\epsilon\mathcal{E}(t)\mathcal{D}(t)+\epsilon\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)w^{2|\beta|}|h|^2}{M_-}d\xi dx.$$

Since the third and fourth terms on the right hand side of (2.51) can be treated similarly, the proof of the lemma is completed.

As in [7], a similar argument gives

Lemma 2.13. Let $|\beta| > 0$ and $\theta \geq 0$. For any $\eta > 0$, there exists $C_\eta > 0$ such that

$$-\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{w^{2|\beta|}\partial_\beta \widetilde{G} \partial_\beta L_{M_-} \widetilde{G}}{M_-} d\xi dx \geq \|w^{|\beta|}\partial_\beta \widetilde{G}\|_\nu^2 - \eta \sum_{|\beta_1|\leq|\beta|} \|w^{|\beta_1|}\partial_{\beta_1}\widetilde{G}\|_\nu^2 - C_\eta \|\widetilde{G}\|_\nu^2.$$

Here, we omit its proof for brevity.

3. Global existence

For the local existence of solutions to the Boltzmann equation (1.1) around the local Maxwellian $\overline{M} = M_{[\overline{\rho},\overline{u},\overline{\theta}]}$, we set $f = F + \overline{M}$. Then equation (1.1) becomes

$$\begin{aligned} F_t + \xi_1 F_x &= L_{\overline{M}} F + Q(F,F) - (\overline{M}_t + \xi_1 \overline{M}_x) \\ &= L_{M_-} F + Q(F,F) + 2Q(F, \overline{M} - M_-) - (\overline{M}_t + \xi_1 \overline{M}_x), \end{aligned} \tag{3.1}$$

with $F(0,x,\xi) = F_0(x,\xi)$. Here the global Maxwellian M_- satisfies (1.28) and (1.29). Without the last two terms of right hand side in (3.1), the local in time existence was constructed by Guo [7]. By using the conditions **H2** and **H3** on $\overline{M}$ and the properties of the nonlinear collision operator given in the previous section, a slightly modified argument used in [7] gives the following local existence theorem. Here, we omit the details of its proof for brevity.

Theorem 3.1. For any sufficiently small $\epsilon_0 > 0$, there exists $T^* > 0$ such that if

$$\sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial_\beta^\alpha F_0\|^2 < \epsilon_0, \tag{3.2}$$

then there is a unique classical solution $F(t,x,\xi)$ to (3.1) in $[0,T^*) \times \mathbf{R} \times \mathbf{R}^3$ such that

$$\sup_{0\leq t\leq T^*} \sum_{|\alpha|+|\beta|\leq N} \|w^{|\beta|}\partial_\beta^\alpha F(t)\|^2 \leq C\epsilon_0.$$

Moreover, if $f_0(x,\xi) = M_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]} + F_0(x,\xi) \geq 0$, then

$$f(t,x,\xi) = M_{[\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)]} + F_0(t,x,\xi) \geq 0.$$

In the following, we will perform the energy analysis and establish a uniform energy estimate by using the the assumptions **H1-H4**. Then the standard continuity argument combining the local existence theorem with the uniform energy estimates gives the statement in Theorem 1.1.

3.1. **Basic estimates.** To obtain the lower order estimates, we first estimate $\|\widetilde{\rho}_x\|^2$ and $\|(\widetilde{\rho}_t, \widetilde{u}_t, \widetilde{\theta}_t)\|^2$. For this, we will use some techniques from [11, 16]. First, we prove the following lemma.

Lemma 3.2. Under assumptions of **H1-H3** and Lemma 2.1, there exists a constant $C > 0$ such that

$$\frac{1}{C}\|\widetilde{\rho}_x\|^2 \le -\frac{d}{dt}\int_{\mathbf{R}} \widetilde{u}_1\widetilde{\rho}_x dx + \|(\widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|G_x\|_\nu^2 + \epsilon(t+t_0)^{-4/3} + \sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

and

$$\frac{1}{C}\|(\widetilde{\rho}_t, \widetilde{u}_t, \widetilde{\theta}_t)\|^2 \le \epsilon(t+t_0)^{-4/3} + \|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|G_x\|_\nu^2 + \sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Proof. It follows from (1.8) and (1.27) that $(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(t,x)$ satisfies

$$\widetilde{\rho}_t + (\rho\widetilde{u}_1)_x = -H_1, \tag{3.3}$$

$$\widetilde{u}_{1t} + \widetilde{u}_1\widetilde{u}_{1x} + \frac{2}{3}\widetilde{\theta}_x + \frac{2\theta}{3\rho}\widetilde{\rho}_x = -\int_{\mathbf{R}^3} \frac{\xi_1^2 G_x}{\rho} d\xi - H_2, \tag{3.4}$$

$$\widetilde{u}_{it} + \widetilde{u}_1\widetilde{u}_{ix} = -\int_{\mathbf{R}^3} \frac{\xi_1\xi_i G_x}{\rho} d\xi - \overline{u}_1\widetilde{u}_{ix}, \quad i = 2, 3, \tag{3.5}$$

$$\widetilde{\theta}_t + \frac{2}{3}\widetilde{\theta}\widetilde{u}_{1x} + \widetilde{u}_1\widetilde{\theta}_x = -\int_{\mathbf{R}^3} \frac{\xi_1(\xi\cdot u - \frac{1}{2}|\xi|^2)G_x}{\rho} d\xi - H_3, \tag{3.6}$$

where

$$H_1 = (\overline{\rho}\widetilde{u}_1 + \overline{u}_1\widetilde{\rho})_x, \tag{3.7}$$

$$H_2 = \widetilde{u}_1\overline{u}_{1x} + \overline{u}_1\widetilde{u}_{1x} + \frac{\overline{\rho}\widetilde{\theta} - \frac{2}{3}\widetilde{\rho}\overline{\theta}}{\rho\overline{\rho}}\overline{\rho}_x, \tag{3.8}$$

$$H_3 = \frac{2}{3}(\widetilde{\theta}\overline{u}_{1x} + \overline{\theta}\widetilde{u}_{1x}) + (\widetilde{u}_1\overline{\theta}_x + \overline{u}_1\widetilde{\theta}_x). \tag{3.9}$$

Multiplying (3.4) by $\widetilde{\rho}_x$ and integrating with respect to x yield

$$\int_{\mathbf{R}} \frac{2\theta}{3\rho}|\widetilde{\rho}_x|^2 dx = -\int_{\mathbf{R}} \widetilde{u}_{1t}\widetilde{\rho}_x dx - \int_{\mathbf{R}} \widetilde{u}_1\widetilde{u}_{1x}\widetilde{\rho}_x dx - \int_{\mathbf{R}} \frac{2}{3}\widetilde{\theta}_x\widetilde{\rho}_x dx - \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\xi_1^2 G_x\widetilde{\rho}_x}{\rho} d\xi dx - \int_{\mathbf{R}} H_2\widetilde{\rho}_x dx. \tag{3.10}$$

The Hölder inequality and assumption **H3** give

$$\Big|\int_{\mathbf{R}} \frac{2}{3}\widetilde{\theta}_x\widetilde{\rho}_x dx\Big| \le \lambda\|\widetilde{\rho}_x\|^2 + C_\lambda\|\widetilde{\theta}_x\|^2,$$

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\xi_1^2 G_x\widetilde{\rho}_x}{\rho} d\xi dx\Big| \le C_\lambda\|G_x\|_\nu^2 + \lambda\|\widetilde{\rho}_x\|^2,$$

$$\Big|\int_{\mathbf{R}} \widetilde{u}_1\widetilde{u}_{1x}\widetilde{\rho}_x dx\Big| \le \lambda\|\widetilde{\rho}_x\|^2 + C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

$$\Big|\int_{\mathbf{R}} H_2\widetilde{\rho}_x dx\Big| \leq \lambda\|\widetilde{\rho}_x\|^2 + C_\lambda\|\widetilde{u}_x\|^2 + \epsilon(t+t_0)^{-4/3} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

For the first term of the right hand side of (3.10), one uses (3.3) and the integration by part to get

$$-\int_{\mathbf{R}} \widetilde{u}_{1t}\widetilde{\rho}_x dx = -\frac{d}{dt}\int_{\mathbf{R}} \widetilde{u}_1\widetilde{\rho}_x dx - \int_{\mathbf{R}} \widetilde{u}_{1x}\widetilde{\rho}_t dx$$

$$\leq -\frac{d}{dt}\int_{\mathbf{R}} \widetilde{u}_1\widetilde{\rho}_x dx + \lambda\|\widetilde{\rho}_x\|^2 + C_\lambda\|\widetilde{u}_x\|^2 + \epsilon(t+t_0)^{-4/3} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \quad (3.11)$$

Combining (3.10) and (3.11) yields the first estimate in Lemma 3.2.

By using equations (3.3)-(3.6) and the assumptions **H1-H3**, a similar argument gives the second estimate. And this completes the proof of the lemma.

To obtain the high order estimates, we need to estimate $\|\partial^\alpha\rho_x\|^2$ and $\|\partial^\alpha(\rho_t, u_t, \theta_t)\|^2$ as follows.

Lemma 3.3. Under assumptions **H1-H3** and Lemma 2.1, there exists some constant $C > 0$ that

$$\frac{1}{C}\|\partial^\alpha\rho_x\|^2 \leq -\frac{d}{dt}\int_{\mathbf{R}} \partial^\alpha u_1\partial^\alpha\rho_x dx + \|\partial^\alpha(u_x, \theta_x)\|^2$$
$$+\|\partial^\alpha G_x\|_\nu^2 + \epsilon(t+t_0)^{-4/3} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

and

$$\frac{1}{C}\|\partial^\alpha(\rho_t, u_t, \theta_t)\|^2 \leq \|\partial^\alpha(\rho_x, u_x, \theta_x)\|^2 + \|\partial^\alpha G_x\|_\nu^2 + \epsilon(t+t_0)^{-4/3} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t),$$

where $1 \leq |\alpha| \leq N-1$.

Remark 3.4. We will use Lemma 3.2 and 3.3 to close the energy estimate. Furthermore, we will not use the temporal derivatives in the subsequent energy analysis with the help of these two lemmas and integration by part about time t so that we do not impose the temporal derivatives on the initial data and thus improve that in [16, 17].

Proof of Lemma 3.3. It follows from (1.8) that

$$\rho_t + (\rho u_1)_x = 0, \quad (3.12)$$

$$u_{1t} + u_1u_{1x} + \frac{2}{3}\theta_x + \frac{2\theta}{3\rho}\rho_x = -\int_{\mathbf{R}^3} \frac{\xi_1^2 G_x}{\rho} d\xi, \quad (3.13)$$

$$u_{it} + u_1u_{ix} = -\int_{\mathbf{R}^3} \frac{\xi_1\xi_i G_x}{\rho} d\xi, \quad i = 2, 3, \quad (3.14)$$

$$\theta_t + \frac{2}{3}\theta u_{1x} + u_1\theta_x = -\int_{\mathbf{R}^3} \frac{\xi_1(\xi\cdot u - \frac{1}{2}|\xi|^2)G_x}{\rho} d\xi. \quad (3.15)$$

Taking ∂^α $(1 \le |\alpha| \le N-1)$ of (3.13) in x variable and multiplying the resulting equation by $\partial^\alpha \rho_x$ lead to

$$\int_{\mathbf{R}} \frac{2\theta}{3\rho}|\partial^\alpha \rho_x|^2 dx = -\int_{\mathbf{R}} \partial^\alpha u_{1t}\partial^\alpha \rho_x dx - \int_{\mathbf{R}} \partial^\alpha (u_1 u_{1x})\partial^\alpha \rho_x dx - \int_{\mathbf{R}} \frac{2}{3}\partial^\alpha \theta_x \partial^\alpha \rho_x dx$$
$$-\sum_{|\alpha'|<|\alpha|} C_\alpha^{\alpha'} \int_{\mathbf{R}} \partial^{\alpha-\alpha'}(\frac{2\theta}{3\rho})\partial^{\alpha'}\rho_x \partial^\alpha \rho_x dx - \int_{\mathbf{R}}\int_{\mathbf{R}^3} \partial^\alpha \rho_x \partial^\alpha (\frac{\xi_1^2 G_x}{\rho}) d\xi dx$$
$$\le -\frac{d}{dt}\int_{\mathbf{R}} \partial^\alpha u_1 \partial^\alpha \rho_x dx + \lambda \|\partial^\alpha \rho_x\|^2 + C_\lambda \|\partial^\alpha (u_x, \theta_x)\|^2$$
$$+\|\partial^\alpha G_x\|_\nu^2 + C\epsilon(t+t_0)^{-4/3} + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Taking $\lambda > 0$ small enough gives the first estimate of the lemma.

Applying ∂^α $(1 \le |\alpha| \le N-1)$ to (3.12)-(3.15) in x variable and then taking the inner product in L^2, one gets the second estimate of the lemma by straightforward calculations. And this completes the proof of the lemma.

3.2. **Lower Order Estimates.** The lower order estimates can be obtained by the entropy-entropy flux analysis as done in [16, 17]. Here for soft potentials, the differences comes from the calculations of some microscopic terms very precisely by using the Burnett functions and their properties obtained in the previous section so that the estimates can be derived in a clear and systematic way, which is one of the main observations in this paper.

Due to (1.25), one can obtain

$$\eta_t + q_x = \Big\{\nabla_{(\bar\rho,\bar u,\bar S)}\eta \cdot (\bar\rho,\bar u,\bar S)_t + \nabla_{(\bar\rho,\bar u,\bar S)} q \cdot (\bar\rho,\bar u,\bar S)_x\Big\}$$
$$+\int_{\mathbf{R}^3} \Big[\xi_1(\bar\theta \ln M)_x - \frac{3}{2}\bar u_{1x}\xi_1^2\Big] G d\xi + \Big\{\int_{\mathbf{R}^3}\Big(\xi_1\bar\theta \ln M - \frac{1}{2}\xi_1|\xi|^2 - \frac{3}{2}\xi_1^2 \bar u_1\Big) G d\xi\Big\}_x. \tag{3.16}$$

Integrating the above equation over $\mathbf{R}$ yields

$$\frac{d}{dt}\int_{\mathbf{R}} \eta(t) dx = \int_{\mathbf{R}} \Big[\nabla_{(\bar\rho,\bar u,\bar S)}\eta \cdot (\bar\rho,\bar u,\bar S)_t + \nabla_{(\bar\rho,\bar u,\bar S)} q \cdot (\bar\rho,\bar u,\bar S)_x\Big] dx$$
$$+\int_{\mathbf{R}}\int_{\mathbf{R}^3} \Big[\xi_1(\bar\theta \ln M)_x - \frac{3}{2}\bar u_{1x}\xi_1^2\Big] G d\xi dx. \tag{3.17}$$

First, we estimate the second term on the right hand side of (3.17). By using (1.10) and the self-adjoint property of L_M^{-1}, one has

$$\int_{\mathbf{R}^3} \Big[\xi_1(\bar\theta \ln M)_x - \frac{3}{2}\bar u_{1x}\xi_1^2\Big] G d\xi = \int_{\mathbf{R}^3} L_M^{-1} P_1[\xi_1(\bar\theta \ln M)_x M - \frac{3}{2}\bar u_{1x}\xi_1^2 M]\frac{(P_1(\xi_1 M_x) + \Theta)}{M} d\xi.$$

Noticing that

$$M_x = \Big(\frac{\rho}{(2\pi R\theta)^{3/2}} e^{-\frac{|\xi-u|^2}{2R\theta}}\Big)_x = M\Big(\frac{\rho_x}{\rho} - \frac{3\theta_x}{2\theta} + \frac{(\xi-u)^2\theta_x}{2R\theta^2} + \sum_{i=1}^3 \frac{u_{ix}(\xi_i - u_i)}{R\theta}\Big), \tag{3.18}$$

one can show that

$$P_1\xi_1 M_x = P_1\xi_1 M\Big\{\frac{|\xi-u|^2\theta_x}{2R\theta^2}+\frac{(\xi-u)\cdot u_x}{R\theta}\Big\} \tag{3.19}$$

$$= P_1\xi_1 M\Big\{\frac{|\xi-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(\xi-u)\cdot \overline{u}_x}{R\theta}\Big\}+P_1\xi_1 M\Big\{\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{(\xi-u)\cdot \widetilde{u}_x}{R\theta}\Big\}$$

$$=\frac{\sqrt{R}\overline{\theta}_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M+\frac{\partial\overline{u}_1}{\partial x}\hat{B}_{11}(\frac{\xi-u}{\sqrt{R\theta}})M+\frac{\sqrt{R}\widetilde{\theta}_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M+\sum_{j=1}^{3}\frac{\partial\widetilde{u}_j}{\partial x}\hat{B}_{1j}(\frac{\xi-u}{\sqrt{R\theta}})M.$$

A direct calculation yields

$$P_1\xi_1(\overline{\theta}\ln M)_x M-\frac{3}{2}P_1\overline{u}_{1x}\xi_1^2M = P_1\xi_1 M\overline{\theta}_x\ln M+P_1\xi_1 M\overline{\theta}(\ln M)_x-\frac{3}{2}P_1\overline{u}_{1x}\xi_1^2M$$

$$=-\overline{\theta}_xP_1\xi_1M\frac{|\xi-u|^2}{2R\theta}+\overline{\theta}P_1\xi_1M\Big\{\frac{|\xi-u|^2\theta_x}{2R\theta^2}+\frac{(\xi-u)\cdot u_x}{R\theta}\Big\}-\frac{3}{2}P_1\overline{u}_{1x}\xi_1^2M$$

$$=-\overline{\theta}_xP_1\xi_1M\frac{|\xi-u|^2}{2R\theta}-\frac{3}{2}P_1\overline{u}_{1x}\xi_1^2M+P_1\xi_1M\Big\{\frac{|\xi-u|^2\overline{\theta}_x}{2R\theta}+\frac{(\xi-u)\cdot\overline{u}_x}{R}\Big\}$$

$$+P_1\xi_1M\Big\{\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta}+\frac{(\xi-u)\cdot\widetilde{u}_x}{R}\Big\}-\widetilde{\theta}P_1\xi_1M\Big\{\frac{|\xi-u|^2\theta_x}{2R\theta^2}+\frac{(\xi-u)\cdot u_x}{R\theta}\Big\}$$

$$=\sqrt{R\theta}\widetilde{\theta}_x\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M+\theta\sum_{j=1}^{3}\frac{\partial\widetilde{u}_j}{\partial x}\hat{B}_{1j}(\frac{\xi-u}{\sqrt{R\theta}})M-\frac{\sqrt{R}\widetilde{\theta}\theta_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M-\widetilde{\theta}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x}\hat{B}_{1j}(\frac{\xi-u}{\sqrt{R\theta}})M.$$

Note that

$$L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_x-\frac{3}{2}\overline{u}_{1x}\xi_1^2]M$$

$$=\sqrt{R\theta}\widetilde{\theta}_xA_1(\frac{\xi-u}{\sqrt{R\theta}})+\theta\sum_{j=1}^{3}\frac{\partial\widetilde{u}_j}{\partial x}B_{1j}(\frac{\xi-u}{\sqrt{R\theta}})-\frac{\sqrt{R}\widetilde{\theta}\theta_x}{\sqrt{\theta}}A_1(\frac{\xi-u}{\sqrt{R\theta}})-\widetilde{\theta}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x}B_{1j}(\frac{\xi-u}{\sqrt{R\theta}}).$$

It follows from the two equalities above and Lemma 2.4 that

$$\int_{\mathbf{R}^3}L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{P_1(\xi_1M_x)}{M}d\xi$$

$$=-\frac{\kappa(\theta)}{R\theta}\overline{\theta}_x\widetilde{\theta}_x+\frac{\kappa(\theta)}{R\theta^2}\overline{\theta}_x\theta_x\widetilde{\theta}-\frac{\kappa(\theta)}{R\theta}\widetilde{\theta}_x^2+\frac{\kappa(\theta)}{R\theta^2}\theta_x\widetilde{\theta}\widetilde{\theta}_x-\frac{\mu(\theta)}{3R}|\frac{\partial\widetilde{u}_1}{\partial x}|^2-\frac{\mu(\theta)}{R}|\widetilde{u}_x|^2$$

$$+\frac{\widetilde{\theta}\mu(\theta)}{3R\theta}\frac{\partial\widetilde{u}_1}{\partial x}\frac{\partial u_1}{\partial x}+\frac{\mu(\theta)\widetilde{\theta}}{R}\sum_{j=1}^{3}\frac{\partial\widetilde{u}_j}{\partial x}\frac{\partial u_j}{\partial x}-\frac{4\mu(\theta)}{3R}\frac{\partial\overline{u}_1}{\partial x}\frac{\partial\widetilde{u}_1}{\partial x}+\frac{4\mu(\theta)\widetilde{\theta}}{3R\theta}\frac{\partial\overline{u}_1}{\partial x}\frac{\partial u_1}{\partial x}.$$

Here, one has used the expressions (2.24) and (2.25) of $\mu(\theta)$ and $\kappa(\theta)$. Since $\mu(\theta)$ and $\kappa(\theta)$ are smooth functions of θ, by the assumption of Lemma 2.1, there exist positive constants κ_1 and κ_2 such that $\mu(\theta)$, $\kappa(\theta)\in[\kappa_1,\kappa_2]$. Thus,

$$-\int_{\mathbf{R}}[\frac{\kappa(\theta)}{R\theta}\widetilde{\theta}_x^2+\frac{\mu(\theta)}{3R}|\frac{\partial\widetilde{u}_1}{\partial x}|^2+\frac{\mu(\theta)}{R}|\widetilde{u}_x|^2]dx\le-\kappa_1\int_{\mathbf{R}}(|\widetilde{u}_x|^2+|\widetilde{\theta}_x|^2)dx.$$

Setting $q(\theta)=\frac{\kappa(\theta)}{R\theta^3}$, one gets

$$
\begin{aligned}
-\int_{\mathbf{R}}\frac{\kappa(\theta)}{R\theta^3}\overline{\theta}_x\widetilde{\theta}_x dx &= \int_{\mathbf{R}} q'(\theta)\overline{\theta}_x^2\widetilde{\theta}dx+\int_{\mathbf{R}} q'(\theta)\widetilde{\theta}_x\overline{\theta}_x\widetilde{\theta}dx+\int_{\mathbf{R}} q(\theta)\overline{\theta}_{xx}\widetilde{\theta}dx\\
&\le C\|\widetilde{\theta}\|^{1/2}\|\widetilde{\theta}_x\|^{1/2}(\|\overline{\theta}_x\|^2+\|\overline{\theta}_{xx}\|_{L^1})+C\|\overline{\theta}_x\|^{1/2}\|\overline{\theta}_{xx}\|^{1/2}\|\widetilde{\theta}\|\cdot\|\widetilde{\theta}_x\|\\
&\le C_\epsilon\|\widetilde{\theta}\|^2\|\widetilde{\theta}_x\|^2+\epsilon(t+t_0)^{-4/3}. \qquad (3.20)
\end{aligned}
$$

A similar argument as for (3.20) implies that

$$
\begin{aligned}
&\int_{\mathbf{R}^3} L_M^{-1}P_1\Big[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{P_1(\xi_1M_x)}{M}d\xi\\
&\le -\kappa_1\int_{\mathbf{R}}(|\widetilde{u}_x|^2+|\widetilde{\theta}_x|^2)dx+C\mathcal{E}(t)\mathcal{D}(t)+\epsilon(t+t_0)^{-4/3}. \qquad (3.21)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\overline{G}_t=&\frac{\sqrt{R\theta}_{xt}}{\sqrt{\theta}}A_1(\frac{\xi-u}{\sqrt{R\theta}})-\frac{\sqrt{R\theta}_x\theta_t}{2\sqrt{\theta^3}}A_1(\frac{\xi-u}{\sqrt{R\theta}})\\
&-\frac{\sqrt{R\theta}_x}{\sqrt{\theta}}A_1'(\frac{\xi-u}{\sqrt{R\theta}})\frac{u_t}{\sqrt{R\theta}}-\frac{\sqrt{R\theta}_x\theta_t}{\sqrt{\theta}}A_1'(\frac{\xi-u}{\sqrt{R\theta}})\frac{\xi-u}{2\sqrt{R\theta^3}}\\
&+\overline{u}_{1xt}B_{11}(\frac{\xi-u}{\sqrt{R\theta}})-\frac{\overline{u}_{1x}u_t}{\sqrt{R\theta}}B_{11}'(\frac{\xi-u}{\sqrt{R\theta}})-\frac{\overline{u}_{1x}\theta_t}{2\sqrt{R\theta^3}}B_{11}'(\frac{\xi-u}{\sqrt{R\theta}}).
\end{aligned}
$$

One can apply assumption **H3**, Lemma 3.2 and Corollary 2.6 to obtain

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\overline{G}_t|^2}{M}d\xi dx\le C\|(\overline{u}_{1xt},\overline{\theta}_{xt})\|^2+C\|(\overline{u}_{1x},\overline{\theta}_x)\cdot(u_t,\theta_t)\|^2\\
&\le C\epsilon(t+t_0)^{-4/3}+C\epsilon\|(\widetilde{\rho}_t,\widetilde{u}_t,\widetilde{\theta}_t)\|^2+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)\\
&\le C\epsilon(t+t_0)^{-4/3}+C\epsilon\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C\epsilon\|G_x\|_\nu^2+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\Big|L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]\Big|^2}{M}d\xi dx\le C(\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\widetilde{\theta}\cdot(u_x,\theta_x)\|^2)\\
&\le C\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C\epsilon(t+t_0)^{-4/3}+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t).
\end{aligned}
$$

Thus, one obtains

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]\frac{\overline{G}_t}{M}d\xi dx\\
&\le\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\Big|L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]\Big|^2}{M}d\xi dx+C_\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\overline{G}_t|^2}{M}d\xi dx\\
&\le C\epsilon(t+t_0)^{-4/3}+C(\epsilon+\lambda)(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|G_x\|_\nu^2)+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t). \qquad (3.22)
\end{aligned}
$$

Moreover, integration by parts over t gives

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]\frac{\widetilde{G}_t}{M}d\xi dx$$

$$=\frac{d}{dt}\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]\frac{\widetilde{G}}{M}d\xi\Big)$$
$$-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\Big(\frac{L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]}{M}\Big)_t\widetilde{G}d\xi dx.$$

For the second term on the right hand side of the above equation, it follows from Lemma 3.2 and 3.3 that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\Big(\frac{L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]}{M}\Big)_t\widetilde{G}d\xi dx\Big|$$
$$\leq C_\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\Big|\Big(\frac{L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M]}{M}\Big)_t\Big|^2Md\xi dx+\lambda\|\widetilde{G}\|_\nu^2$$
$$\leq C_\lambda\|(\widetilde{u}_{xt},\widetilde{\theta}_{xt})\|^2+C_\lambda\|(\widetilde{u}_x,\widetilde{\theta}_x)\cdot(\rho_t,u_t,\theta_t)\|^2+C_\lambda\|(\widetilde{\theta}_t,\widetilde{u}_t)\cdot(u_x,\theta_x)\|^2$$
$$+C_\lambda\|\widetilde{\theta}\cdot(\rho_t,u_t,\theta_t)\cdot(u_x,\theta_x)\|^2+C_\lambda\|\widetilde{\theta}\cdot(u_{xt},\theta_{xt})\|^2+\lambda\|\widetilde{G}\|_\nu^2$$
$$\leq C_\lambda(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|G_{xx}\|_\nu^2)+\lambda\|\widetilde{G}\|_\nu^2+C\epsilon(t+t_0)^{-4/3}+C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Finally, one can show that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{G_t}{M}d\xi dx\Big|$$
$$\leq\frac{d}{dt}\Big(\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{\widetilde{G}}{M}d\xi dx\Big)+C_\lambda(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|G_{xx}\|_\nu^2)$$
$$+C\epsilon(t+t_0)^{-4/3}+C(\epsilon+\lambda)(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|G_x\|_\nu^2+\|\widetilde{G}\|_\nu^2)+C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t).\quad(3.23)$$

Similarly, it holds that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{\xi_1G_x}{M}d\xi dx\Big|$$
$$\leq C\lambda\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C_\lambda\|G_x\|_\nu^2.$$

For the term involving the collision term, if follows from **H3** and Lemma 2.9 that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{Q(G,G)}{M}d\xi dx\Big|$$
$$\leq C\epsilon(t+t_0)^{-4/3}+C(\lambda+\epsilon)(\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\widetilde{G}\|_\nu)+C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).\quad(3.24)$$

In addition, by the assumption **H4**, one has

$$\int_{\mathbf{R}}\Big[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta\cdot(\overline{\rho},\overline{u},\overline{S})_t+\nabla_{(\overline{\rho},\overline{u},\overline{S})}q\cdot(\overline{\rho},\overline{u},\overline{S})_x\Big]dx\leq g(t)\|\sqrt{\eta(t)}\|^2.\quad(3.25)$$

Collecting all the estimates (3.21), (3.23), (3.24) and (3.25) and taking $\lambda > 0$ and $\epsilon > 0$ small enough, we have from (3.17) and Lemma 3.2 that

$$\frac{d}{dt}\Big(\int_{\mathbf{R}}\eta(t)dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{\widetilde{G}}{M}d\xi dx+C(\lambda+\epsilon)\int_{\mathbf{R}}\widetilde{u}_1\widetilde{\rho}_xdx\Big)$$
$$+d_3\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2\leq g(t)\|\sqrt{\eta(t)}\|^2+C(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|G_x\|_\nu^2+\|G_{xx}\|_\nu^2)$$
$$+C(\lambda+\epsilon)\|\widetilde{G}\|_\nu^2+C\epsilon(t+t_0)^{-4/3}+C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \quad (3.26)$$

To complete the lower estimates, one needs to estimate the microscopic component G. Since $\|G\|^2$ is not integrable with respect to the time t. Thus, we will first derive the lower order estimates from (1.9) for the microscopic part $\widetilde{G}$. Note that $\widetilde{G}$ solves

$$\widetilde{G}_t-L_{M_-}\widetilde{G}=2Q(M-M_-,\widetilde{G})-P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]$$
$$-P_1(\xi_1G_x)+Q(G,G)-\overline{G}_t. \quad (3.27)$$

Multiplying (3.27) by $\widetilde{G}/M_-$ and integrating its product with over $\mathbf{R}\times\mathbf{R}^3$ yield

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{G}\|^2+C_1\|\widetilde{G}\|_\nu^2\leq C\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}Q(M-M_-,\widetilde{G})}{M_-}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}\partial^\alpha Q(G,G)}{M_-}d\xi dx$$
$$-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}P_1(\xi_1G_x)}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}\overline{G}_t}{M_-}d\xi dx. \quad (3.28)$$

By Lemma 2.2, the first term on the right hand side of (3.28) is bounded by

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}Q(M-M_-,\widetilde{G})}{M_-}d\xi dx\leq C\eta_0\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\widetilde{G}|^2}{M_-}d\xi dx.$$

Noticing that

$$P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]=\frac{\sqrt{R}\widetilde{\theta}_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{R\theta})M+\sum_{j=1}^3\widetilde{u}_{jx}\hat{B}_{1j}(\frac{\xi-u}{R\theta})M,$$

one can get

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]}{M_-}d\xi dx\leq C_\lambda\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C\lambda\|\widetilde{G}\|_\nu^2.$$

It follows from **H3** and Lemma 2.9 that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\widetilde{G}Q(G,G)}{M_-}d\xi dx\leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C(\lambda+\epsilon)\|\widetilde{G}\|_\nu^2.$$

Since

$$P_1(\xi_1 G_x) = \xi_1 \widetilde{G}_x + \xi_1 \overline{G}_x - \sum_{j=0}^{4} \langle \xi_1 G_x, \chi_j \rangle \chi_j,$$

one can obtain

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\widetilde{G}\xi_1 \overline{G}_x}{M_-} d\xi dx \le \lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx + C_\lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)|\xi_1|^2|\overline{G}_x|^2}{M_-} d\xi dx,$$

and

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\widetilde{G}\langle \xi_1 G_x, \chi_j\rangle \chi_j}{M_-} d\xi dx \le \lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\widetilde{G}|^2}{M_-} d\xi dx + C_\lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|G_x|^2}{M_-} d\xi dx.$$

By Lemma 2.5 and the Sobolev imbedding, one gets

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)(1+|\xi_1|^2)|\overline{G}_x|^2}{M_-} d\xi dx \le C(\|(\overline{u}_{1xx}, \overline{\theta}_{xx}\|^2 + \|(\overline{u}_{1x}, \overline{\theta}_x)\cdot(u_x, \theta_x)\|^2)$$

$$\le C\mathcal{E}(t)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3}.$$

Then it follows that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\widetilde{G}\overline{G}_t}{M_-} d\xi dx \le \lambda \|\widetilde{G}\|_\nu^2 + C_\lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu^{-1}(\xi)|\overline{G}_t|^2}{M_-} d\xi dx$$

$$\le \lambda\|\widetilde{G}\|_\nu^2 + C\epsilon(t+t_0)^{-4/3} + C\epsilon\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + C\epsilon\|G_x\|_\nu^2 + C(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))\mathcal{D}(t).$$

Combining the estimates above and choosing ϵ, η_0 and λ small enough, we have

$$\frac{d}{dt}\|\widetilde{G}\|^2 + C_1\|\widetilde{G}\|_\nu^2 \le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)$$

$$+C\epsilon(t+t_0)^{-4/3} + C(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|G_x\|_\nu^2). \tag{3.30}$$

In summary, a suitable linear combination of (3.26) and (3.30) gives the following lower order estimates

$$\frac{d}{dt}\Big\{\|\widetilde{G}\|^2 + \widetilde{C}_1\Big(\int_{\mathbf{R}} \eta(t)dx - \int_{\mathbf{R}}\int_{\mathbf{R}^3} L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_x M - \frac{3}{2}\overline{u}_{1x}\xi_1^2 M\Big]\frac{\widetilde{G}}{M} d\xi dx$$

$$+C(\lambda+\epsilon)\int_{\mathbf{R}} \widetilde{u}_1\widetilde{\rho}_x dx\Big)\Big\} + d_4\Big(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|\widetilde{G}\|_\nu^2\Big)$$

$$\le Cg(t)\|\sqrt{\eta(t)}\|^2 + C(\|(\rho_{xx}, u_{xx}, \theta_{xx})\|^2 + \|G_x\|_\nu^2 + \|G_{xx}\|_\nu^2)$$

$$+C\epsilon(t+t_0)^{-4/3} + C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t), \tag{3.31}$$

where $\widetilde{C}_1 > 0$ is a constant that can be large.

3.3. **High order estimates on spatial derivatives.** In what follows we will estimate $\partial^\alpha G$ with $1\le|\alpha|\le N-1$. To this end, we apply ∂^α to (1.9) to have

$$\partial^\alpha G_t+\partial^\alpha P_1(\xi_1 G_x)+\partial^\alpha P_1(\xi_1 M_x)$$
$$=L_{M_-}\partial^\alpha G+2\partial^\alpha Q(G,M-M_-)+\partial^\alpha Q(G,G). \quad (3.32)$$

Multiplying (3.32) by $\partial^\alpha G/M_-$ and integrating the resulting equality over $\mathbf{R}\times\mathbf{R}^3$ yield

$$\frac{d}{dt}\|\partial^\alpha G\|^2+C\|\partial^\alpha G\|_\nu^2\le C\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha Q(G,M-M_-)\partial^\alpha G+\partial^\alpha Q(G,G)\partial^\alpha G}{M_-}d\xi dx$$
$$-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha P_1(\xi_1 G_x)\partial^\alpha G}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha P_1(\xi_1 M_x)\partial^\alpha G}{M_-}d\xi dx. \quad (3.33)$$

Then the right hand side of (3.33) can be estimated term by term as follows. First, by using Lemma 2.7, one can obtain

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha Q(G,M-M_-)\partial^\alpha G}{M_-}d\xi dx\Big|$$
$$\le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|_{L^4}^4+C(\lambda+\eta_0)\|\partial^\alpha G\|_\nu^2$$
$$\le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C(\lambda+\eta_0)\|\partial^\alpha G\|_\nu^2.$$

For the second term of (3.33), it follows from **H3** and Lemma 2.10 that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha Q(G,G)\partial^\alpha G}{M_-}d\xi dx\Big|\le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C(\lambda+\epsilon)\|\partial^\alpha G\|_\nu^2.$$

For the third term of (3.33), since

$$\partial^\alpha P_1(\xi_1 G_x)=\xi_1\partial^\alpha G_x-\sum_{j=0}^{4}\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}C_\alpha^{\alpha_1\alpha_2\alpha_3}\partial^{\alpha_1}\chi_j\int_{\mathbf{R}^3}\xi_1\partial^{\alpha_2}G_x\partial^{\alpha_3}(\frac{\chi_j}{M})d\xi,$$

it suffices to consider the second term on the right hand side of the above equation because the first term vanishes after integration. For this, there are the following three cases to be investigated:

Case 1. When $[\alpha_1,\alpha_2,\alpha_3]=[0,\alpha,0]$, it is straightforward to have

$$\int_{\mathbf{R}}\Big(\int_{\mathbf{R}^3}\frac{\chi_j\partial^\alpha G}{M_-}d\xi\Big)\Big(\int_{\mathbf{R}^3}\xi_1\partial^\alpha G_x(\frac{\chi_j}{M})d\xi\Big)dx\le\lambda\|\partial^\alpha G\|_\nu^2+C_\lambda\|\partial^\alpha G_x\|_\nu^2.$$

Case 2. When $[\alpha_1,\alpha_2,\alpha_3]=[1,\alpha_2,0]$ or $[\alpha_1,\alpha_2,\alpha_3]=[0,\alpha_2,1]$, one has

$$\int_{\mathbf{R}}\Big(\int_{\mathbf{R}^3}\frac{\partial^{\alpha_1}\chi_j\partial^\alpha G}{M_-}d\xi\Big)\Big(\int_{\mathbf{R}^3}\xi_1\partial^{\alpha_2}G_x(\frac{\chi_j}{M})d\xi\Big)+\Big(\int_{\mathbf{R}^3}\frac{\chi_j\partial^\alpha G}{M_-}d\xi\Big)\Big(\int_{\mathbf{R}^3}\xi_1\partial^{\alpha_2}G_x\partial^{\alpha_3}(\frac{\chi_j}{M})d\xi\Big)dx$$
$$\le\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\alpha G|^2}{M_-}d\xi dx+C_\lambda\int_{\mathbf{R}}|(\rho_x,u_x,\theta_x)|^2\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\alpha_2}G_x|^2}{M_-}d\xi dx$$
$$\le\lambda\|\partial^\alpha G\|_\nu^2+C_\lambda(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t).$$

Case 3. When $[\alpha_1,\alpha_2,\alpha_3]=[1,\alpha_2,1]$, it holds that

$$\begin{aligned}
&\int_{\mathbf{R}}\Big(\int_{\mathbf{R}^3}\frac{\partial^{\alpha_1}\chi_j\partial^\alpha G}{M_-}d\xi\Big)\Big(\int_{\mathbf{R}^3}\xi_1\partial^{\alpha_2}G_x\partial^{\alpha_3}(\frac{\chi_j}{M})d\xi\Big)dx\\
&\le C\int_{\mathbf{R}}|(\rho_x,u_x,\theta_x)|^2\Big\{\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\alpha G|^2}{M_-}d\xi\Big\}^{1/2}\Big\{\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\alpha_2}G_x|^2}{M_-}d\xi\Big\}^{1/2}dx\\
&\le C\|(\rho_x,u_x,\theta_x)\|\|(\rho_{xx},u_{xx},\theta_{xx})\|\|\partial^\alpha G\|_\nu^2\|\partial^{\alpha_2}G_x\|_\nu^2\\
&\le C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t).
\end{aligned}$$

Since the other cases can be discussed similarly, we thus have

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha P_1(\xi_1G_x)\partial^\alpha G}{M_-}d\xi dx\le C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C_\lambda\|\partial^\alpha G_x\|_\nu^2+C\lambda\|\partial^\alpha G\|_\nu^2.$$

For the fourth term on the right hand side of (3.33), since

$$P_1\xi_1M_x=\frac{\sqrt{R}\theta_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M+\sum_{i=1}^3u_{ix}\hat{B}_{1i}(\frac{\xi-u}{\sqrt{R\theta}})M,$$

then

$$\begin{aligned}
\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha P_1(\xi_1M_x)\partial^\alpha G}{M_-}d\xi dx\le{}&\lambda\|\partial^\alpha G\|_\nu^2+C\|\partial^\alpha(\rho_x,u_x,\theta_x)\|^2\\
&+C_\lambda\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}.
\end{aligned}$$

By combining the above estimates and choosing λ, η_0 and ϵ small enough, one gets

$$\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha G\|^2+C\|\partial^\alpha G\|_\nu^2\le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)\\
&\qquad+C\epsilon(t+t_0)^{-4/3}+C\|\partial^\alpha(\rho_x,u_x,\theta_x)\|^2+C\|\partial^\alpha G_x\|_\nu^2.
\end{aligned}\tag{3.34}$$

For any $1\le|\alpha|\le N-1$, the summation of (3.34) over $|\alpha|$ through a suitable linear combination gives

$$\begin{aligned}
&\sum_{1\le|\alpha|\le N-1}\Big[\frac{d}{dt}\|\partial^\alpha G\|^2+C\|\partial^\alpha G\|_\nu^2\Big]\le C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)\\
&+C\epsilon(t+t_0)^{-4/3}+C\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha(\rho_x,u_x,\theta_x)\|^2+C\sum_{|\alpha|=N}\|\partial^\alpha G\|_\nu^2.
\end{aligned}\tag{3.35}$$

To have the dissipative estimate on the N-order derivative of the microscopic component G, we consider the original equation for $f(t,x,\xi)$. Applying ∂^α $(2\le|\alpha|\le N)$ to (1.1) and integrating its product with $\partial^\alpha f/M_-$ over $\mathbf{R}\times\mathbf{R}^3$ lead to

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha GL_{M_-}\partial^\alpha G}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha M\partial^\alpha L_MG}{M_-}d\xi dx$$

$$= 2\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha Q(G, M-M_-)\partial^\alpha G}{M_-} d\xi dx + \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha Q(G,G)\partial^\alpha f}{M_-} d\xi dx. \tag{3.36}$$

We now estimate (3.36) term by term as follows. (1.15) implies that

$$-\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha G L_{M_-}\partial^\alpha G}{M_-} d\xi dx \geq \sigma_1 \|\partial^\alpha G\|_\nu^2.$$

By Lemma 2.7 and 2.10, one can get

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha Q(G, M-M_-)\partial^\alpha f}{M_-} d\xi dx \leq C(\sqrt{\mathcal{E}(t)} + \eta_0 + \epsilon)\mathcal{D}(t)$$
$$+ C\epsilon(t+t_0)^{-4/3} + C(\lambda + \eta_0)(\|\partial^\alpha(\rho,u,\theta)\|^2 + \|\partial^\alpha G\|_\nu^2),$$

and

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha Q(G,G)\partial^\alpha f}{M_-} d\xi dx \leq C(\sqrt{\mathcal{E}(t)} + \eta_0 + \epsilon)\mathcal{D}(t)$$
$$+ C\epsilon(t+t_0)^{-4/3} + C(\lambda + \epsilon)(\|\partial^\alpha(\rho,u,\theta)\|^2 + \|\partial^\alpha G\|_\nu^2), \tag{3.37}$$

where one has used the fact that

$$\|\partial^\alpha f\|_\nu^2 \leq C\|\partial^\alpha G\|_\nu^2 + C\|\partial^\alpha(\rho,u,\theta)\|^2 + C\epsilon(t+t_0)^{-4/3} + C(\sqrt{\mathcal{E}(t)} + \eta_0 + \epsilon)\mathcal{D}(t).$$

For the third term of the left hand side of (3.36), noting that

$$\partial^\alpha L_M G = L_M \partial^\alpha G + 2\sum_{\alpha_1<\alpha} C_\alpha^{\alpha_1} Q(\partial^{\alpha_1} G, \partial^{\alpha-\alpha_1}(M-M_-)),$$

and $P_1(\partial^\alpha M)$ does not contain $\partial^\alpha(\rho,u,\theta)$, one gets from (2.19) that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\partial^\alpha M L_M \partial^\alpha G}{M} d\xi dx = \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{P_1(\partial^\alpha M) L_M \partial^\alpha G}{M} d\xi dx$$
$$\leq C(\sqrt{\mathcal{E}(t)} + \eta_0 + \epsilon)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3} + \lambda\|\partial^\alpha G\|_\nu^2.$$

Thus, to estimate the third term on the left hand side of (3.36), one only needs to estimate $\int_{\mathbf{R}}\int_{\mathbf{R}^3} \partial^\alpha M L_M \partial^\alpha G\Big(\frac{1}{M_-} - \frac{1}{M}\Big) d\xi dx$. For this, it follows from (2.19) that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} \partial^\alpha M L_M \partial^\alpha G\Big(\frac{1}{M_-} - \frac{1}{M}\Big) d\xi dx$$
$$\leq C\lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\alpha G|^2}{M_-} d\xi dx + C_\lambda \int_{\mathbf{R}}\int_{\mathbf{R}^3} \nu(\xi)|\partial^\alpha M|^2 M_- \Big|\frac{1}{M_-} - \frac{1}{M}\Big|^2 d\xi dx.$$

For the second term on the right hand side of the above inequality, one has

$$\int_{\mathbf{R}}\int_{|\xi|\geq R} \nu(\xi)|\partial^\alpha M|^2 M_- \Big|\frac{1}{M_-} - \frac{1}{M}\Big|^2 d\xi dx = \int_{\mathbf{R}}\int_{|\xi|\geq R} \frac{\nu(\xi)|\partial^\alpha M|^2}{M_-}\Big|1 - \frac{M_-}{M}\Big|^2 d\xi dx.$$

Taking $R > 0$ large enough and $\eta_0 > 0$ small enough, when $\frac{\theta}{2} < \theta_- < \theta$ and

$$|\rho - \rho_-| + |u - u_-| + |\theta - \theta_-| < \eta_0,$$

one has

$$\int_{\mathbf{R}}\int_{|\xi|\geq R}\frac{\nu(\xi)|\partial^\alpha M|^2}{M_-}\Big|1-\frac{M_-}{M}\Big|^2 d\xi dx$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\lambda\|\partial^\alpha(\rho,u,\theta)\|^2+C\epsilon(t+t_0)^{-4/3}.$$

On the other hand, it holds that

$$\int_{\mathbf{R}}\int_{|\xi|\leq R}\frac{\nu(\xi)|\partial^\alpha M|^2 M_-}{M^2}\Big|1-\frac{M}{M_-}\Big|^2 d\xi dx$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\lambda\|\partial^\alpha(\rho,u,\theta)\|^2+C\epsilon(t+t_0)^{-4/3}.$$

By (2.18) and a similar argument as for Lemma 2.8, one can obtain

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3} Q(\partial^{\alpha_1}G,\partial^{\alpha-\alpha_1}(M-M_-))\frac{\partial^\alpha G}{M_-}d\xi dx$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C(\lambda+\eta_0)\|\partial^\alpha G\|_\nu^2.$$

Combining all the estimates above and choosing λ, η_0 and ϵ small enough, one has

$$\frac{d}{dt}\|\partial^\alpha f\|^2+C\|\partial^\alpha G\|_\nu^2$$

$$\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C(\lambda+\eta_0+\epsilon)\|\partial^\alpha(\rho,u,\theta)\|^2. \quad (3.38)$$

For any $2\leq|\alpha|\leq N$, the summation of (3.38) over $|\alpha|$ through a suitable linear combination gives

$$\sum_{2\leq|\alpha|\leq N}\Big[\frac{d}{dt}\|\partial^\alpha f\|^2+C\|\partial^\alpha G\|_\nu^2\Big]\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)$$

$$+C\epsilon(t+t_0)^{-4/3}+C(\lambda+\eta_0+\epsilon)\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\rho,u,\theta)\|^2. \quad (3.39)$$

We now turn to the estimate on the macroscopic components, that is, the fluid variables. Note that in the following estimates, Burnett functions play an important role.

To obtain the estimates on $\partial^\alpha M$, one may apply P_0 to (1.1) to yield

$$M_t+P_0(\xi_1 M_x)+P_0(\xi_1 G_x)=0. \quad (3.40)$$

Consider the case when $|\alpha|=1$ first. Applying ∂^α to (3.40) gives

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha M\|^2=-\int_{\mathbf{R}\times\mathbf{R}^3}\frac{\partial^\alpha P_0(\xi_1 M_x)\partial^\alpha M}{M_-}d\xi dx-\int_{\mathbf{R}\times\mathbf{R}^3}\frac{\partial^\alpha P_0(\xi_1 G_x)\partial^\alpha M}{M_-}d\xi dx. \quad (3.41)$$

The first term the right hand side of (3.41) can be estimated as follows. Since

$$\partial^\alpha P_0(\xi_1 M_x) = P_0(\xi_1\partial^\alpha M_x) + \sum_{j=0}^4 \langle \xi_1 M_x, \chi_j\rangle \partial^\alpha \chi_j + \sum_{j=0}^4 \int_{\mathbf{R}^3} \xi_1 M_x (\frac{\chi_j}{M})_x d\xi \chi_j, \quad (3.42)$$

it holds that

$$-\int_{\mathbf{R}\times\mathbf{R}^3} \frac{P_0(\xi_1\partial^\alpha M_x)\partial^\alpha M}{M} d\xi dx = \int_{\mathbf{R}\times\mathbf{R}^3} \frac{(\xi_1|\partial^\alpha M|^2)M_x}{2M^2} d\xi dx \le C\int_{\mathbf{R}} |(\rho_x, u_x, \theta_x)|^3 dx$$
$$\le C_\epsilon(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3}. \quad (3.43)$$

And it is straightforward to check that the other two terms on the right hand side of (3.42) have the same upper bound as the first one.

To take care of the difference in weights, that is, M and M_-, we claim that the following inequality holds

$$\int_{\mathbf{R}\times\mathbf{R}^3} \partial^\alpha P_0(\xi_1 M_x)\partial^\alpha M(\frac{1}{M} - \frac{1}{M_-}) d\xi dx \le C\epsilon(t+t_0)^{-4/3}$$
$$+C(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))\mathcal{D}(t) + C(\lambda+\epsilon)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|(\rho_{xx}, u_{xx}, \theta_{xx})\|^2). \quad (3.44)$$

First, note that

$$\partial^\alpha P_0(\xi_1 M_x) = \xi_1 \partial^\alpha M_x - \partial^\alpha P_1(\xi_1 M_x),$$
$$M_x = \Big(\frac{\rho}{(2\pi R\theta)^{3/2}} e^{-\frac{|\xi-u|^2}{2R\theta}}\Big)_x = M\Big(\frac{\rho_x}{\rho} - \frac{3\theta_x}{2\theta} + \frac{(\xi-u)^2\theta_x}{2R\theta^2} + \sum_{i=1}^3 \frac{u_{ix}(\xi_i - u_i)}{R\theta}\Big).$$

To prove (3.44), by (3.19) and the two equalities above, it suffices to estimate the following term as the other terms can be estimated similarly. Set

$$a(\rho,u,\theta) = \int_{\mathbf{R}\times\mathbf{R}^3} \frac{\sqrt{R}}{\rho\sqrt{\theta}} \hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}}) M^2 (\frac{1}{M} - \frac{1}{M_-}) d\xi.$$

Since $a(\rho,u,\theta)$ is bounded and differentiable under the assumption (1.24) and (1.25), one has

$$\int_{\mathbf{R}\times\mathbf{R}^3} \frac{\rho_x}{\rho}\frac{\sqrt{R}\theta_{xx}}{\sqrt{\theta}} \hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}}) M^2(\frac{1}{M} - \frac{1}{M_-}) d\xi dx = \int_{\mathbf{R}} a(\rho,u,\theta)\rho_x\theta_{xx} dx$$
$$= \int_{\mathbf{R}} a(\rho,u,\theta)(\widetilde{\rho}_x\theta_{xx} + \overline{\rho}_x\overline{\theta}_{xx} + \overline{\rho}_x\widetilde{\theta}_{xx}) dx = \int_{\mathbf{R}} a(\rho,u,\theta)(\widetilde{\rho}_x\theta_{xx} + \overline{\rho}_x\overline{\theta}_{xx}) dx$$
$$- \int_{\mathbf{R}} (a(\rho,u,\theta)\overline{\rho}_{xx}\widetilde{\theta}_x + a'(\rho,u,\theta)(\rho_x,u_x,\theta_x)\overline{\rho}_x\widetilde{\theta}_x dx$$
$$\le C(\lambda+\epsilon)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|(\rho_{xx}, u_{xx}, \theta_{xx})\|^2) + C\epsilon(t+t_0)^{-4/3} + C(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))\mathcal{D}(t),$$

where $\lambda > 0$ is small enough.

Thus, it follows from (3.43) and (3.44) that

$$-\int_{\mathbf{R}\times\mathbf{R}^3} \frac{\partial^\alpha P_0(\xi_1 M_x)\partial^\alpha M}{M_-} d\xi dx \le C(\lambda+\epsilon)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|(\rho_{xx}, u_{xx}, \theta_{xx})\|^2)$$

$$+C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}.$$

Next we consider the second term on the right hand side of (3.41). Due to (1.10), it holds that

$$P_0(\xi_1 G_x)=\sum_{j=0}^{4}\langle\xi_1 G_x,\chi_j\rangle\chi_j=\sum_{j=0}^{4}\langle\xi_1 G,\chi_j\rangle_x\chi_j-\sum_{j=0}^{4}\int_{\mathbf{R}^3}\xi_1 G(\frac{\chi_j}{M})_x d\xi\chi_j$$

$$=\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1 M_x)\rangle_x\chi_j+\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1\Theta\rangle_x\chi_j-\sum_{j=0}^{4}\int_{\mathbf{R}^3}\xi_1 G(\frac{\chi_j}{M})_x d\xi\chi_j.$$

By using the Burnett functions, one has for $j=1,2,3$,

$$L_M^{-1}P_1(\xi_1\chi_0)=0,\quad L_M^{-1}P_1(\xi_1\chi_j)=\frac{R\theta}{\sqrt{R\rho\theta}}B_{1j}(\frac{\xi-u}{\sqrt{R\theta}}),\quad L_M^{-1}P_1(\xi_1\chi_4)=\frac{2\sqrt{R\theta}}{\sqrt{6\rho}}A_1(\frac{\xi-u}{\sqrt{R\theta}}).$$

On the other hand, one also has

$$P_1\xi_1 M_x=P_1\xi_1 M\Big\{\frac{|\xi-u|^2\theta_x}{2R\theta^2}+\frac{(\xi-u)\cdot u_x}{R\theta}\Big\}=\frac{\sqrt{R}\theta_x}{\sqrt{\theta}}\hat{A}_1(\frac{\xi-u}{\sqrt{R\theta}})M+\sum_{j=1}^{3}\frac{\partial u_j}{\partial x}\hat{B}_{1j}(\frac{\xi-u}{\sqrt{R\theta}})M.$$

Then one can obtain from Lemma 2.5 that

$$\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1 M_x)\rangle_x\chi_j=-(\frac{2\kappa(\theta)\theta_x}{\sqrt{6\rho}R\theta})_x\chi_4-\frac{4}{3}(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}u_{1x})_x\chi_1-\sum_{j=2}^{3}(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}u_{jx})_x\chi_j. \tag{3.45}$$

It is straightforward to show that for $j=1,2,3$,

$$\langle\chi_j,M_x\rangle=\sqrt{\frac{\rho}{R\theta}}u_{jx},\qquad\langle\chi_4,M_x\rangle=\frac{\sqrt{6}\rho\theta_x}{2\theta},$$

$$\partial^\alpha P_0(\xi_1 G_x)=\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1 M_x)\rangle_{xx}\chi_j+\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1 M_x)\rangle_x\chi_{jx}$$

$$+\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1\Theta\rangle_{xx}\chi_j+\sum_{j=0}^{4}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1\Theta\rangle_x\chi_{jx}-\sum_{j=0}^{4}\Big(\int_{\mathbf{R}^3}\xi_1 G(\frac{\chi_j}{M})_x d\xi\chi_j\Big)_x.$$

According to the expressions above, the first term of $\langle\partial^\alpha P_0(\xi_1 G_x),M_x\rangle$ satisfies

$$\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_4),P_1(\xi_1 M_x)\rangle_{xx}\langle\chi_4,M_x\rangle dx=-\int_{\mathbf{R}}(\frac{2\kappa(\theta)\theta_x}{\sqrt{6\rho}R\theta})_{xx}\frac{\sqrt{6}\rho\theta_x}{2\theta}dx$$

$$=\int_{\mathbf{R}}\Big(\frac{\kappa(\theta)\theta_{xx}^2}{R\theta^2}+\frac{\kappa(\theta)\theta_{xx}\theta_x}{\sqrt{6\rho}R\theta}\Big(\frac{\sqrt{6\rho}}{\theta}\Big)_x+\Big(\frac{\kappa(\theta)}{\sqrt{6\rho}R\theta}\Big)_x\theta_x\frac{\sqrt{6\rho}}{R\theta}\theta_{xx}+\Big(\frac{\kappa(\theta)}{\sqrt{6\rho}R\theta^2}\Big)_x\theta_x^2\Big(\frac{\sqrt{6\rho}}{R\theta}\Big)_x\Big)dx,$$

and for $j=1,2,3$,

$$\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1 M_x)\rangle_{xx}\langle\chi_j,M_x\rangle dx=-\int_{\mathbf{R}}(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}u_{jx})_{xx}\sqrt{\frac{\rho}{R\theta}}u_{jx}dx$$

$$=\int_{\mathbf{R}}\Big(\frac{\mu(\theta)u_{jxx}^2}{R\theta}+\frac{\mu(\theta)u_{jxx}u_{jx}}{\sqrt{R\rho\theta}}\Big(\sqrt{\frac{\rho}{R\theta}}\Big)_x+\Big(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}\Big)_x u_{jx}\sqrt{\frac{\rho}{R\theta}}u_{jxx}+\Big(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}\Big)_x\Big(\frac{\mu(\theta)}{\sqrt{R\rho\theta}}\Big)_x u_{jx}^2\Big)dx.$$

Therefore, by assumption **H3** and Lemma 3.2, we obtain

$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1M_x)\rangle_{xx}\langle\chi_j,M_x\rangle dx\le -d_2\|(u_{xx},\theta_{xx})\|^2+C\|(\rho_x,u_x,\theta_x)\|_{L^4}^4$$

$$\le -d_2\|(u_{xx},\theta_{xx})\|^2+C\|(\rho_x,u_x,\theta_x)\|\|(\rho_{xx},u_{xx},\theta_{xx})\|(\|(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x)\|^2+\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2)$$

$$\le -d_2\|(u_{xx},\theta_{xx})\|^2+C\mathcal{E}(t)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}, \tag{3.46}$$

and

$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1M_x)\rangle_{xx}\int_{\mathbf{R}^3}\chi_jM_x(\frac{1}{M_-}-\frac{1}{M})d\xi dx$$

$$=\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1M_x)\rangle_{x}\Big(\int_{\mathbf{R}^3}\chi_jM_x(\frac{1}{M_-}-\frac{1}{M})d\xi\Big)_x dx$$

$$\le C(\eta_0+\lambda)\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+C\mathcal{E}(t)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}. \tag{3.47}$$

Combining (3.46) and (3.47) and taking λ and η_0 small enough give

$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1M_x)\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_jM_x}{M_-}d\xi dx$$

$$\le -d_3\|(u_{xx},\theta_{xx})\|^2+C(\eta_0+\lambda)\|\rho_{xx}\|^2+C\mathcal{E}(t)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}. \tag{3.48}$$

Thanks to (3.45), the second term in $\langle\partial^\alpha P_0(\xi_1G_x),M_x\rangle$ is bounded by

$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1(\xi_1M_x)\rangle_{x}\int_{\mathbf{R}^3}\frac{\chi_{jx}M_x}{M_-}d\xi dx$$

$$\le C\lambda\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+C_\lambda\mathcal{E}(t)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}. \tag{3.49}$$

By the expression (1.11) of Θ, Lemma 2.9, 2.10, 3.2, 3.3 and assumption **H3**, the first term involving Θ in $\langle\partial^\alpha P_0(\xi_1G_x),M_x\rangle$ can be estimated as follows:

$$\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),\xi_1G_x+Q(G,G)\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_jM_x}{M_-}d\xi dx$$

$$=-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),\xi_1G_x+Q(G,G)\rangle_{x}\Big(\int_{\mathbf{R}^3}\frac{\chi_jM_x}{M_-}d\xi\Big)_x dx$$

$$\le C(\eta_0+\epsilon+\lambda)(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|\widetilde{G}\|_\nu^2)$$

$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\|G_{xx}\|_\nu^2,$$

and

$$\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G_t\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_jM_x}{M_-}d\xi dx=\frac{d}{dt}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_jM_x}{M_-}d\xi dx$$

$$+\int_{\mathbf{R}}\Big(\int\Big(\frac{L_M^{-1}P_1(\xi_1\chi_j)}{M}\Big)_t G d\xi\Big)_x\Big(\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi\Big)_x dx-\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{xx}\Big(\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi\Big)_t dx$$

$$\le \frac{d}{dt}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi dx+C(\lambda+\epsilon)\|(\rho_{xx},u_{xx},\theta_{xx})\|^2$$
$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\|G_{xx}\|_\nu^2.$$

Therefore, we obtain

$$\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1\Theta\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi dx$$

$$\le\sum_{j=0}^{4}\frac{d}{dt}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi dx+C(\eta_0+\epsilon+\lambda)(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|\widetilde{G}\|_\nu^2)$$
$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\|G_{xx}\|_\nu^2. \tag{3.50}$$

Similarly, the second term involving Θ in $\langle\partial^\alpha P_0(\xi_1 G_x),M_x\rangle$ satisfies

$$\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),P_1\Theta\rangle_{x}\int_{\mathbf{R}^3}\frac{\chi_{jx} M_x}{M_-}d\xi dx$$

$$\le\sum_{j=0}^{4}\frac{d}{dt}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{x}\int_{\mathbf{R}^3}\frac{\chi_{jx} M_x}{M_-}d\xi dx+C(\eta_0+\epsilon+\lambda)(\|(\rho_{xx},u_{xx},\theta_{xx})\|^2+\|\widetilde{G}\|_\nu^2)$$
$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\|G_{xx}\|_\nu^2. \tag{3.51}$$

For the last term in $\langle\partial^\alpha P_0(\xi_1 G_x),M_x\rangle$, we have

$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\Big(\int_{\mathbf{R}^3}\xi_1 G(\frac{\chi_j}{M})_x d\xi\Big)_x\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi dx$$
$$\le C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\epsilon\|(\rho_{xx},u_{xx},\theta_{xx})\|^2.$$

Combining the above estimates and choosing $\eta_0>0$, $\lambda>0$ and $\epsilon>0$ small enough, one derives from (3.41) that

$$\frac{d}{dt}\Big\{\|\partial^\alpha M\|^2-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{xx}\int_{\mathbf{R}^3}\frac{\chi_j M_x}{M_-}d\xi dx$$
$$-\sum_{j=0}^{4}\int_{\mathbf{R}}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_{x}\int_{\mathbf{R}^3}\frac{\chi_{jx} M_x}{M_-}d\xi dx\Big\}+d_3\|(u_{xx},\theta_{xx})\|^2$$
$$\le C(\lambda+\epsilon+\eta_0)(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\widetilde{G}\|_\nu^2)+C(\eta_0+\epsilon+\lambda)\|\rho_{xx}\|^2$$
$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\|G_{xx}\|_\nu^2.$$

Similarly, for any $1 \le |\alpha| \le N-1$, it holds that

$$\sum_{1\le|\alpha|\le N-1} \frac{d}{dt}\Big\{\|\partial^\alpha M\|^2 - \sum_{j=0}^{4}\sum_{\alpha_1\le\alpha} C_\alpha^{\alpha_1}\int_{\mathbf{R}} \partial^{\alpha_1}\langle L_M^{-1}P_1(\xi_1\chi_j), G\rangle_x$$

$$\times \int_{\mathbf{R}^3} \frac{\partial^{\alpha-\alpha_1}\chi_j\partial^\alpha M}{M_-} d\xi dx\Big\} + d_4 \sum_{2\le|\alpha|\le N} \|(\partial^\alpha u, \partial^\alpha\theta)\|^2$$

$$\le C(\lambda+\epsilon+\eta_0)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|\widetilde{G}\|_\nu^2) + C(\eta_0+\epsilon+\lambda)\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha \rho_x\|^2$$

$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3} + C\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha G_x\|_\nu^2.$$

To recover the estimate on $\partial^\alpha \rho_x$, one may use Lemma 3.3 to deduce that

$$\sum_{1\le|\alpha|\le N-1} \frac{d}{dt}\Big\{\|\partial^\alpha M\|^2 - \sum_{j=0}^{4}\sum_{\alpha_1\le\alpha} C_\alpha^{\alpha_1}\int_{\mathbf{R}} \partial^{\alpha_1}\langle L_M^{-1}P_1(\xi_1\chi_j), G\rangle_x$$

$$\times \int_{\mathbf{R}^3} \frac{\partial^{\alpha-\alpha_1}\chi_j\partial^\alpha M}{M_-} d\xi dx + C\int_{\mathbf{R}}\partial^\alpha u_1 \partial^\alpha \rho_x dx\Big\} + d_5 \sum_{2\le|\alpha|\le N} \|\partial^\alpha(\rho, u, \theta)\|^2$$

$$\le C(\lambda+\epsilon+\eta_0)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|\widetilde{G}\|_\nu^2) + C\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha G_x\|_\nu^2$$

$$+C(\sqrt{\mathcal{E}(t)}+\eta_0)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3}. \tag{3.52}$$

By using (3.35), (3.39) and (3.52), a suitable linear combination gives the following high order estimate with spatial derivatives:

$$\frac{d}{dt}\Big\{\widetilde{C}_3\sum_{2\le|\alpha|\le N}\|\partial^\alpha f\|^2 + \sum_{1\le|\alpha|\le N-1}\|\partial^\alpha G\|^2 + \widetilde{C}_2\sum_{1\le|\alpha|\le N-1}\Big(\|\partial^\alpha M\|^2 + C\int_{\mathbf{R}}\partial^\alpha u_1\partial^\alpha\rho_x dx$$

$$-\sum_{j=0}^{4}\sum_{\alpha_1\le\alpha} C_\alpha^{\alpha_1}\int_{\mathbf{R}} \partial^{\alpha_1}\langle L_M^{-1}P_1(\xi_1\chi_j), G\rangle_x \int_{\mathbf{R}^3} \frac{\partial^{\alpha-\alpha_1}\chi_j\partial^\alpha M}{M_-} d\xi dx\Big)\Big\}$$

$$+d_6\Big(\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\rho,u,\theta)\|^2 + \sum_{1\le|\alpha|\le N}\|\partial^\alpha G\|_\nu^2\Big)$$

$$\le C(\lambda+\epsilon+\eta_0)(\|(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|\widetilde{G}\|_\nu^2) + C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3}, \tag{3.53}$$

where $\widetilde{C}_2 > 0$ and $\widetilde{C}_3 > 0$ are some large constants with $\widetilde{C}_3 > \widetilde{C}_2$.

3.4. **High order estimates with velocity derivatives.** Next we estimate the velocity derivatives of $\widetilde{G}$. Taking ∂_β $(1 \leq |\beta| \leq N)$ over (3.27) and multiplying the resulting equation by $w^{2|\beta|}\partial_\beta\widetilde{G}/M_-$ and then integrating the resulting equality over $\mathbf{R}\times\mathbf{R}^3$, one can get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2 - \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta L_{M_-}\widetilde{G}}{M_-}d\xi dx + \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta P_1(\xi_1 G_x)}{M_-}d\xi dx\\
&\quad = \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta Q(\widetilde{G}, M-M_-)}{M_-}d\xi dx + \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta Q(G,G)}{M_-}d\xi dx\\
&\quad - \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]}{M_-}d\xi dx - \int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta\overline{G}_t}{M_-}d\xi dx.
\end{aligned}
\tag{3.54}
$$

We will estimate each term of (3.54). For the second term on the left hand side, it follows from Lemma 2.13 that

$$
-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta L_{M_-}\widetilde{G}}{M_-}d\xi dx \geq \|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2 - \lambda\sum_{|\beta_1|\leq|\beta|}\|w^{|\beta_1|}\partial_{\beta_1}\widetilde{G}\|_\nu^2 - C_\lambda\|\widetilde{G}\|_\nu^2.
$$

For the third term on the left hand side, since

$$
\partial_\beta P_1(\xi_1 G_x) = \xi_1\partial_\beta\widetilde{G}_x + \xi_1\partial_\beta\overline{G}_x + \sum_{|\beta_1|=1}\partial_{\beta_1}\xi_1\partial_{\beta-\beta_1}G_x - \sum_{j=0}^{4}\langle\xi_1 G_x,\chi_j\rangle\partial_\beta\chi_j, \tag{3.55}
$$

the inner product of the first term in (3.55) with $w^{2|\beta|}\partial_\beta\widetilde{G}$ with the weight M_- vanishes. And by Lemma 2.5 and (2.40), the inner product with the second term is bounded by

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\xi_1\partial_\beta\overline{G}_x\partial_\beta\widetilde{G}}{M_-}d\xi dx\Big|\\
&\leq \lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2 + C_\lambda\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}|\xi|^2\nu^{-1}(\xi)|\partial_\beta\overline{G}_x|^2}{M_-}d\xi dx\\
&\leq C\mathcal{E}(t)\mathcal{D}(t) + C\epsilon(t+t_0)^{-4/3} + \lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2.
\end{aligned}
$$

For the inner product with the third term in (3.55), one has

$$
\begin{aligned}
&\sum_{|\beta_1|=1}\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_{\beta_1}\xi_1\partial_{\beta-\beta_1}G_x\partial_\beta\widetilde{G}}{M_-}d\xi dx\Big|\\
&\leq C_\lambda\sum_{|\beta_1|=1}\|w^{|\beta|-\frac{1}{2}}\partial_{\beta-\beta_1}G_x\|^2 + \lambda\|w^{|\beta|+\frac{1}{2}}\partial_\beta\widetilde{G}\|^2\\
&\leq C_\lambda\sum_{|\beta_1|=1}\|w^{|\beta-\beta_1|}\partial_{\beta-\beta_1}G_x\|_\nu^2 + \lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2.
\end{aligned}
$$

Here, one has used (1.16) and the fact that $|\beta-\beta_1| = |\beta|-1$.

For the inner product with the last term in (3.55), one has

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\langle\xi_1 G_x,\chi_j\rangle\partial_\beta\chi_j\partial_\beta\widetilde{G}}{M_-}d\xi dx\Big|\leq C_\lambda\|G_x\|_\nu^2+\lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2.$$

Next, we handle the terms on the right hand side of (3.54). First, it follows from Lemma 2.2 that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta Q(\widetilde{G},M-M_-)}{M_-}d\xi dx\leq C\eta_0\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2+C\eta_0\sum_{|\beta_1|\leq|\beta|}\|w^{|\beta_1|}\partial_{\beta_1}\widetilde{G}\|_\nu^2.$$

Lemma 2.11 implies that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta Q(G,G)}{M_-}d\xi dx$$

$$\leq C_\lambda(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+\epsilon\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2+C(\lambda+\epsilon)\sum_{\beta_1\leq\beta}\|w^{|\beta_1|}\partial_{\beta_1}\widetilde{G}\|_\nu^2.$$

For the third term of the right hand side of (3.54), one has

$$\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta P_1\Big[\xi_1(\frac{|\xi-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{\xi\cdot\widetilde{u}_x}{R\theta})M\Big]}{M_-}d\xi dx\leq\lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2+C_\lambda\|(\widetilde{u}_x,\widetilde{\theta}_x)\|^2.$$

It follows from Lemma 2.5, Lemma 3.2 and Lemma 3.3 that

$$\Big|\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial_\beta\widetilde{G}\partial_\beta\overline{G}_t}{M_-}d\xi dx\Big|$$

$$\leq\lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2+C_\lambda(\|(\overline{u}_{1xt},\overline{\theta}_{xt})\|^2+\|(\overline{u}_{1x},\overline{\theta}_x)\cdot(u_t,\theta_t)\|^2)$$

$$\leq\lambda\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2+C_\lambda(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}+C\epsilon\|G_x\|_\nu^2+C\epsilon\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2.$$

Therefore, collecting all the estimates above and choosing λ, η_0 and ϵ small enough, one deduces that

$$\frac{d}{dt}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2+C\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2\leq C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}$$

$$+C\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C\sum_{|\beta_1|\leq N-1}\|w^{|\beta_1|}\partial_{\beta_1}G_x\|_\nu^2+C\sum_{\beta_1<\beta}\|w^{|\beta_1|}\partial_{\beta_1}\widetilde{G}\|_\nu^2. \tag{3.56}$$

Hence, for any $1\leq|\beta|\leq N$, the summation of (3.56) over $1\leq|\beta|\leq N$ through a suitable linear combination gives

$$\sum_{1\leq|\beta|\leq N}\Big[\frac{d}{dt}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2+C\|w^{|\beta|}\partial_\beta\widetilde{G}\|_\nu^2\Big]\leq C(\sqrt{\mathcal{E}(t)}+\mathcal{E}(t))\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}$$

$$+C\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C\sum_{|\beta_1|\leq N-1}\|w^{|\beta_1|}\partial_{\beta_1}G_x\|_\nu^2+C\|\widetilde{G}\|_\nu^2. \tag{3.57}$$

Next, we will estimate $\partial^\alpha_\beta G$ with $|\alpha|+|\beta|\leq N$ with $|\alpha|\geq 1$ and $|\beta|\geq 1$. Applying ∂^α_β to (1.9) gives

$$\partial^\alpha_\beta G_t+\partial^\alpha_\beta P_1(\xi_1 G_x)+\partial^\alpha_\beta P_1(\xi_1 M_x)=\partial^\alpha_\beta L_{M_-}G+2\partial^\alpha_\beta Q(G,M-M_-)+\partial^\alpha_\beta Q(G,G).$$

Multiplying the above equation by $\partial^\alpha_\beta G/M_-$ and then integrating over $\mathbf{R}\times\mathbf{R}^3$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}|\partial^\alpha_\beta G|^2}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial^\alpha_\beta L_{M_-}G\partial^\alpha_\beta G}{M_-}d\xi dx\\ =&\,2\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial^\alpha_\beta Q(G,M-M_-)\partial^\alpha_\beta G}{M_-}d\xi dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial^\alpha_\beta Q(G,G)\partial^\alpha_\beta G}{M_-}d\xi dx\\ &-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial^\alpha_\beta P_1(\xi_1 G_x)\partial^\alpha_\beta G}{M_-}d\xi dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{w^{2|\beta|}\partial^\alpha_\beta P_1(\xi_1 M_x)\partial^\alpha_\beta G}{M_-}d\xi dx. \quad (3.58)\end{aligned}$$

By Lemma 2.8, 2.12, 2.13 and a similar argument as for (3.56), one can obtain

$$\begin{aligned}&\frac{d}{dt}\|w^{|\beta|}\partial^\alpha_\beta G\|^2+C\|w^{|\beta|}\partial^\alpha_\beta G\|^2_\nu\leq C\epsilon(t+t_0)^{-4/3}+C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)\\ &\quad+C\|\partial^\alpha(\rho_x,u_x,\theta_x)\|^2+C\sum_{|\alpha_1|\leq|\alpha|}\|\partial^{\alpha_1}G_x\|^2_\nu+C\sum_{|\beta_1|<|\beta|}\|w^{|\beta_1|}\partial^\alpha_{\beta_1}G\|^2_\nu.\end{aligned}$$

Summing over $|\alpha|+|\beta|\leq N$ with $|\alpha|\geq 1$ and $|\beta|\geq 1$ through a suitable linear combination of the above inequality implies

$$\begin{aligned}&\sum_{(\alpha,\beta)\in\Lambda}\Big[\frac{d}{dt}\|w^{|\beta|}\partial^\alpha_\beta G\|^2+C\|w^{|\beta|}\partial^\alpha_\beta G\|^2_\nu\Big]\leq C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}\\ &\quad+C\sum_{1\leq|\alpha|\leq N-1}\|\partial^\alpha(\rho_x,u_x,\theta_x)\|^2+C\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha G\|^2_\nu, \quad (3.59)\end{aligned}$$

with the set of indices defined as $\Lambda=\{|\alpha|+|\beta|\leq N, |\alpha|\geq 1,\ |\beta|\geq 1\}$.

Then a suitable linear combination of (3.57) and (3.59) gives the following high order estimate:

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{1\leq|\beta|\leq N}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2+\widetilde{C}_4\sum_{(\alpha,\beta)\in\Lambda}\|w^{|\beta|}\partial^\alpha_\beta G\|^2\Big\}\\ &\qquad+C\Big(\sum_{1\leq|\beta|\leq N}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2_\nu+\sum_{(\alpha,\beta)\in\Lambda}\|w^{|\beta|}\partial^\alpha_\beta G\|^2\Big)\\ &\leq C(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\rho,u,\theta)\|^2+\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha G\|^2_\nu+\|\widetilde{G}\|^2_\nu)\\ &\qquad+C(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon)\mathcal{D}(t)+C\epsilon(t+t_0)^{-4/3}, \quad (3.60)\end{aligned}$$

where $\widetilde{C}_4>0$ is a large constant.

Finally, a suitable linear combination of (3.31), (3.53) and (3.60) implies that

$$\frac{d}{dt}\overline{\mathcal{E}}(t)+C_0\mathcal{D}(t)\leq C_1(\sqrt{\mathcal{E}(t)}+\eta_0+\epsilon+\lambda)\mathcal{D}(t)$$

$$+C_2\epsilon(t+t_0)^{-4/3}+C_2g(t)\|\sqrt{\eta(t)}\|^2, \qquad (3.61)$$

with $\mathcal{E}(t)$ and $\mathcal{D}(t)$ defined by (1.20) and (1.21) respectively, and $\overline{\mathcal{E}}(t)$ given by

$$\overline{\mathcal{E}}(t)=\|\widetilde{G}\|^2+\widetilde{C}_1\Big(\int_{\mathbf{R}}\eta(t)dx-\int_{\mathbf{R}}\int_{\mathbf{R}^3}L_M^{-1}P_1[\xi_1(\overline{\theta}\ln M)_xM-\frac{3}{2}\overline{u}_{1x}\xi_1^2M\Big]\frac{\widetilde{G}}{M}d\xi dx$$

$$+C(\lambda+\epsilon)\int_{\mathbf{R}}\widetilde{u}_1\widetilde{\rho}_xdx\Big)+\widetilde{C}_5\Big\{\widetilde{C}_3\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha f\|^2+\sum_{1\leq|\alpha|\leq N-1}\|\partial^\alpha G\|^2+\widetilde{C}_2\sum_{1\leq|\alpha|\leq N-1}\Big(\|\partial^\alpha M\|^2$$

$$+C\int_{\mathbf{R}}\partial^\alpha u_1\partial^\alpha\rho_xdx-\sum_{j=0}^{4}\sum_{\alpha_1\leq\alpha}C_\alpha^{\alpha_1}\int_{\mathbf{R}}\partial^{\alpha_1}\langle L_M^{-1}P_1(\xi_1\chi_j),G\rangle_x\int_{\mathbf{R}^3}\frac{\partial^{\alpha-\alpha_1}\chi_j\partial^\alpha M}{M_-}d\xi dx\Big)\Big\}$$

$$+\frac{1}{\widetilde{C}_6}\Big(\sum_{1\leq|\beta|\leq N}\|w^{|\beta|}\partial_\beta\widetilde{G}\|^2+\widetilde{C}_4\sum_{(\alpha,\beta)\in\Lambda}\|w^{|\beta|}\partial_\beta^\alpha G\|^2\Big).$$

Here, by choosing $\widetilde{C}_6>0$ and $\widetilde{C}_5>\widetilde{C}_1$ as some large constants, one can show easily that there exists a positive constant $\bar{C}>1$ such that $\bar{C}^{-1}(\mathcal{E}(t)-\zeta(t_0))<\overline{\mathcal{E}}<\bar{C}(\mathcal{E}(t)+\zeta(t_0))$ where $\zeta(t_0)\to 0$ as $t_0\to\infty$.

Assume a priorily that $\mathcal{E}(0)\leq a\epsilon_0$ and for some $T_1>0$,

$$\sup_{0\leq t\leq T_1}\mathcal{E}(t)<b(\epsilon_0+\epsilon),$$

with $b=\max\{3a\bar{C}^2,3\bar{C}C_2\}$. We can choose ϵ, η_0 and λ small enough such that for $0\leq t\leq T_1$

$$C_1(\mathcal{E}^{1/2}+\eta_0+\epsilon+\lambda)<C_0.$$

Then (3.61) and **H4** implies that

$$\sup_{0\leq t\leq T_1}\overline{\mathcal{E}}(t)<\overline{\mathcal{E}}(0)+C_2\epsilon+C_2C_g\epsilon\sup_{0\leq t\leq T_1}\overline{\mathcal{E}}(t).$$

Choosing $\epsilon>0$ small enough so that $C_2C_g\epsilon<\frac{1}{2}$, we can obtain

$$\sup_{0\leq t\leq T_1}\mathcal{E}(t)<\bar{C}\sup_{0\leq t\leq T_1}\overline{\mathcal{E}}(t)+\bar{C}\zeta(t_0)<2(\overline{\mathcal{E}}(0)+C_2\epsilon)+\bar{C}\zeta(t_0)$$

$$\leq 2\bar{C}(\bar{C}\mathcal{E}(0)+C_2\epsilon)+4\bar{C}\zeta(t_0)\leq 3\bar{C}(a\bar{C}\epsilon_0+C_2\epsilon)<b(\epsilon_0+\epsilon),$$

where t_0 is chosen to be large enough. This together with the local existence theorem yields the global existence of classical solutions for small perturbations.

4. Application to Nonlinear Stability of Rarefaction waves

Now we apply Theorem 1.1 to study the large time asymptotic behavior of the global solution $f(t,x,\xi)$ to the Boltzmann equation when the initial data is a small perturbation of a rarefaction wave profile. Let

$$f(t,x,\xi)|_{t=0} = f_0(x,\xi) \to \begin{cases} \frac{\rho_l}{\sqrt{(2\pi R\theta_l)^3}} \exp\left(-\frac{|\xi-u_l|^2}{2R\theta_l}\right), & x \to -\infty, \\ \frac{\rho_r}{\sqrt{(2\pi R\theta_r)^3}} \exp\left(-\frac{|\xi-u_r|^2}{2R\theta_r}\right), & x \to +\infty. \end{cases} \tag{4.1}$$

Here $\rho_r, \theta_r > 0$, $u_r = (u_{1r}, 0, 0)$ and $\rho_l, \theta_l > 0$, $u_l = (u_{1l}, 0, 0)$ are constants such that the Riemann problem of the compressible Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_i^2)_x = 0, \quad i = 2, 3, \\ \left(\rho\left(e + \frac{|u|^2}{2}\right)\right)_t + \left(\rho u_1\left(e + \frac{|u|^2}{2} + p u_1\right)\right)_x = 0, \end{cases} \tag{4.2}$$

$$(\rho, u, \theta)(t,x)|_{t=0} = (\rho_0^r, u_0^r, \theta_0^r)(x) = \begin{cases} (\rho_l, u_l, \theta_l), & x < 0, \\ (\rho_r, u_r, \theta_r), & x > 0, \end{cases} \tag{4.3}$$

admits a centered rarefaction wave solution of the third family. Here the equations of state are corresponding to the monatomic gas, $e = \theta$ and $p = \frac{2}{3}\rho\theta$ with the gas constant $R = \frac{2}{3}$. A centered rarefaction wave in the third family, denoted by $(\rho^R, u^R, \theta^R)(x/t)$ with $(\rho_r, u_r, \theta_r) \in R_3(\rho_l, u_l, \theta_l)$, satisfies

$$R_3(\rho_l, u_l, \theta_l) \equiv \Big\{(\rho, u, \theta) \ \Big| S = \overline{S}, \ u_1 - \frac{\sqrt{15k}}{3}\rho^{1/3}\exp\left(\frac{S}{2}\right) = u_{1r} - \sqrt{15k}\rho_r^{1/3}\exp\left(\frac{\overline{S}}{2}\right)$$

$$u_2 = u_3 = 0, \quad u_1 < u_{1r}, \quad \rho < \rho_r\Big\},$$

where

$$\begin{cases} S = -\frac{2}{3}\ln\rho + \ln(\frac{4}{3}\pi\theta) + 1 = -\frac{2}{3}\ln\rho_l + \ln(\frac{4}{3}\pi\theta_l) + 1 \\ \quad = -\frac{2}{3}\ln\rho_r + \ln(\frac{4}{3}\pi\theta_r) + 1 = \overline{S}, \\ k = \frac{1}{2\pi e}. \end{cases} \tag{4.4}$$

Note that along a given R_3 wave curve, the third characteristic speed, λ, satisfies the inviscid Burgers equation

$$\lambda_t + \lambda\lambda_x = 0.$$

Hence, a smooth rarefaction wave profile can be constructed as [20] with a parameter $\epsilon > 0$ as follows. Let λ satisfies

$$\lambda_t + \lambda\lambda_x = 0,$$
$$\lambda(0,x) = \frac{1}{2}(\lambda_r + \lambda_l) + \frac{1}{2}(\lambda_r - \lambda_l)\tanh(\epsilon x), \tag{4.5}$$

where

$$\lambda_l = \lambda_3(\rho_l, u_l, \theta_l), \quad \lambda_r = \lambda_3(\rho_r, u_r, \theta_r).$$

Then the smooth rarefaction profile $(\overline{\rho}, \overline{u}, \overline{\theta})(t, x)$ is defined by

$$(\overline{\rho}, \overline{u}, \overline{\theta})(t, x) = (\rho^A, u^A, \theta^A)(t + t_0, x), \tag{4.6}$$

where $t_0 = \frac{1}{d_1 \epsilon^2}$ with $d_1 > 0$, and $(\rho^A, u^A, \theta^A)(t, x)$ satisfies

$$\begin{aligned}
u_1^A(t,x) + \frac{\sqrt{15k}}{3}\Big(\rho^A(t,x)\Big)^{1/3} \exp\Big(\frac{\overline{S}}{2}\Big) &= \lambda(t,x), \quad i = 1, 3, \\
u_1^A - \sqrt{15k}\Big(\rho^A(t,x)\Big)^{1/3} \exp\Big(\frac{\overline{S}}{2}\Big) &= u_{1r} - \sqrt{15k}\rho_r^{1/3} \exp\Big(\frac{\overline{S}}{2}\Big), \\
\theta^A(t,x) = \frac{3}{2}k\Big(\rho^A(t,x)\Big)^{2/3}\overline{S}, \quad u_2^A &= u_3^A = 0.
\end{aligned} \tag{4.7}$$

In what follows we assume

$$|\rho_l - \rho_r| + |u_l - u_r| + |\theta_l - \theta_r| < \eta_0, \tag{4.8}$$

$$\frac{1}{2} \sup_{(t,x)\in \mathbf{R}_+ \times \mathbf{R}} \overline{\theta}(t,x) < \inf_{(t,x)\in \mathbf{R}_+ \times \mathbf{R}} \overline{\theta}(t,x). \tag{4.9}$$

Notice that ϵ appears in the initial data for (4.5) for the spreading of the wave, while η_0 represents the strength of the wave in condition (4.8).

We recall some properties of $(\overline{\rho}, \overline{u}, \overline{\theta})(t, x)$ from [20] in the following lemma.

Lemma 4.1. The smooth rarefaction wave $(\overline{\rho}, \overline{u}, \overline{\theta})(t, x)$ constructed in (4.6) has the following properties:

(i) $\overline{u}_{1x}(t, x) > 0$ for all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$.

(ii) For any p $(1 \le p \le \infty)$, there exists a constant $C(p) > 0$ depending on p such that

$$\|(\overline{\rho}, \overline{u}, \overline{\theta})_x(t,x)\|_{L^p} \le C(p)(t + t_0)^{-1+\frac{1}{p}},$$

$$\|\frac{\partial^j}{\partial x^j}(\overline{\rho}, \overline{u}, \overline{\theta})(t,x)\|_{L^p} \le C(p)(t + t_0)^{-1}, \quad j \ge 2.$$

(iii) $(\overline{\rho}, \overline{u}_1, \overline{\theta})(t, x)$ solves

$$\begin{cases}
\overline{\rho}_t + (\overline{\rho u}_1)_x = 0, \\
(\overline{\rho u}_1)_t + (\overline{\rho u}_1^2 + \overline{p})_x = 0, \\
\Big(\overline{\rho}\Big(\overline{e} + \frac{|\overline{u}_1|^2}{2}\Big)\Big)_t + \Big(\overline{\rho u}_1\Big(\overline{e} + \frac{|\overline{u}_1|^2}{2} + \overline{pu}_1\Big)\Big)_x = 0.
\end{cases} \tag{4.10}$$

(iv) $\quad \lim\limits_{t\to\infty} \sup\limits_{x\in\mathbf{R}} |(\overline{\rho}, \overline{u}, \overline{\theta})(t,x) - (\rho^R, u^R, \theta^R)(x/t)| = 0.$

It follows from Lemma 4.1 and a direct calculation that the smooth rarefaction profile (4.6) satisfies all the assumptions **H1-H4**. Hence, the nonlinear asymptotic stability theorem of the smooth rarefaction wave, Theorem 1.2, is a consequence of Theorem 1.1.

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