

LOW MACH NUMBER LIMIT FOR THE FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH GENERAL INITIAL DATA

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ABSTRACT. The low Mach number limit for the full compressible magnetohydrodynamic equations with general initial data is rigorously justified in the whole space \mathbb{R}^3 . The uniform estimates of the solutions in Sobolev space are obtained on a time interval independent of the Mach number. The limits are proved by using a theorem of G. Métiver & S. Schochet that established the decay of energy of the acoustic equations.

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1. INTRODUCTION

In this paper we study the low Mach number limit of local smooth solutions to the following full compressible magnetohydrodynamic (MHD) equations with general initial data in the whole space \mathbb{R}^3 (see [20, 27, 36, 42]):

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \frac{1}{4\pi}(\operatorname{curl} \mathbf{H}) \times \mathbf{H} + \operatorname{div} \Psi(\mathbf{u}), \tag{1.2}$$

$$\partial_t \mathbf{H} - \operatorname{curl}(\mathbf{u} \times \mathbf{H}) = -\operatorname{curl}(\nu \operatorname{curl} \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0, \tag{1.3}$$

$$\partial_t \mathcal{E} + \operatorname{div}(\mathbf{u}(\mathcal{E}' + P)) = \frac{1}{4\pi} \operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H})$$

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$$+ \operatorname{div} \left(\frac{\nu}{4\pi} \mathbf{H} \times (\operatorname{curl} \mathbf{H}) + \mathbf{u} \Psi(\mathbf{u}) + \kappa \nabla \theta \right). \quad (1.4)$$

Here the unknowns ρ , $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, $\mathbf{H} = (H_1, H_2, H_3) \in \mathbb{R}^3$, and θ denote the density, velocity, magnetic field, and temperature, respectively; $\Psi(\mathbf{u})$ is the viscous stress tensor given by

$$\Psi(\mathbf{u}) = 2\mu \mathbb{D}(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u} \mathbf{I}_3$$

with $\mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)/2$, \mathbf{I}_3 the 3×3 identity matrix, and $\nabla \mathbf{u}^\top$ the transpose of the matrix $\nabla \mathbf{u}$; \mathcal{E} is the total energy given by $\mathcal{E} = \mathcal{E}' + |\mathbf{H}|^2/(8\pi)$ and $\mathcal{E}' = \rho(e + |\mathbf{u}|^2/2)$ with e being the internal energy, $\rho|\mathbf{u}|^2/2$ the kinetic energy, and $|\mathbf{H}|^2/(8\pi)$ the magnetic energy. The viscosity coefficients λ and μ of the flow satisfy $\mu > 0$ and $2\mu + 3\lambda > 0$. The parameter $\nu > 0$ is the magnetic diffusion coefficient of the magnetic field and $\kappa > 0$ the heat conductivity. For simplicity, we assume that μ, λ, ν and κ are constants. The equations of state $P = P(\rho, \theta)$ and $e = e(\rho, \theta)$ relate the pressure P and the internal energy e to the density ρ and the temperature θ of the flow.

Multiplying (1.2) by \mathbf{u} and (1.3) by $\mathbf{H}/(4\pi)$ and summing over, one finds that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{8\pi} |\mathbf{H}|^2 \right) + \frac{1}{2} \operatorname{div} (\rho |\mathbf{u}|^2 \mathbf{u}) + \nabla P \cdot \mathbf{u} \\ &= \operatorname{div} \Psi \cdot \mathbf{u} + \frac{1}{4\pi} (\operatorname{curl} \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \frac{1}{4\pi} \operatorname{curl} (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H} \\ & \quad - \frac{\nu}{4\pi} \operatorname{curl} (\operatorname{curl} \mathbf{H}) \cdot \mathbf{H}. \end{aligned} \quad (1.5)$$

Due to the identities

$$\begin{aligned} \operatorname{div}(\mathbf{H} \times (\operatorname{curl} \mathbf{H})) &= |\operatorname{curl} \mathbf{H}|^2 - \operatorname{curl} (\operatorname{curl} \mathbf{H}) \cdot \mathbf{H}, \\ \operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H}) &= (\operatorname{curl} \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \operatorname{curl} (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H}, \end{aligned} \quad (1.6)$$

one can subtract (1.5) from (1.4) to rewrite the energy equation (1.4) in terms of the internal energy as

$$\partial_t(\rho e) + \operatorname{div}(\rho \mathbf{u} e) + (\operatorname{div} \mathbf{u})P = \frac{\nu}{4\pi} |\operatorname{curl} \mathbf{H}|^2 + \Psi(\mathbf{u}) : \nabla \mathbf{u} + \kappa \Delta \theta, \quad (1.7)$$

where $\Psi(\mathbf{u}) : \nabla \mathbf{u}$ denotes the scalar product of two matrices:

$$\Psi(\mathbf{u}) : \nabla \mathbf{u} = \sum_{i,j=1}^3 \frac{\mu}{2} \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right)^2 + \lambda |\operatorname{div} \mathbf{u}|^2 = 2\mu |\mathbb{D}(\mathbf{u})|^2 + \lambda |\operatorname{tr} \mathbb{D}(\mathbf{u})|^2.$$

To establish the low Mach number limit for the system (1.1)–(1.3) and (1.7), in this paper we shall focus on the ionized fluids obeying the perfect gas relations

$$P = \mathfrak{R} \rho \theta, \quad e = c_V \theta, \quad (1.8)$$

where the parameters $\mathfrak{R} > 0$ and $c_V > 0$ are the gas constant and the heat capacity at constant volume, respectively, which will be assumed to be one for simplicity of the presentation. We also ignore the coefficient $1/(4\pi)$ in the magnetic field.

Let ϵ be the Mach number, which is a dimensionless number. Consider the system (1.1)–(1.3), (1.7) in the physical regime:

$$P \sim P_0 + O(\epsilon), \quad \mathbf{u} \sim O(\epsilon), \quad \mathbf{H} \sim O(\epsilon), \quad \nabla \theta \sim O(1),$$

where $P_0 > 0$ is a certain given constant which is normalized to be $P_0 = 1$. Thus we consider the case when the pressure P is a small perturbation of the given state 1

while the temperature θ has a finite variation. As in [2], we introduce the following transformation to ensure the positivity of P and θ

$$P(x, t) = e^{\epsilon p^\epsilon(x, \epsilon t)}, \quad \theta(x, t) = e^{\theta^\epsilon(x, \epsilon t)}, \quad (1.9)$$

where a longer time scale $t = \tau/\epsilon$ (still denote τ by t later for simplicity) is introduced in order to seize the evolution of the fluctuations. Note that (1.8) and (1.9) imply that $\rho(x, t) = e^{\epsilon p^\epsilon(x, \epsilon t) - \theta^\epsilon(x, \epsilon t)}$ since $\mathfrak{R} \equiv c_V \equiv 1$. Set

$$\mathbf{H}(x, t) = \epsilon \mathbf{H}^\epsilon(x, \epsilon t), \quad \mathbf{u}(x, t) = \epsilon \mathbf{u}^\epsilon(x, \epsilon t), \quad (1.10)$$

and

$$\mu = \epsilon \mu^\epsilon, \quad \lambda = \epsilon \lambda^\epsilon, \quad \nu = \epsilon \nu^\epsilon, \quad \kappa = \epsilon \kappa^\epsilon. \quad (1.11)$$

Under these changes of variables and coefficients, the system, (1.1)–(1.3), (1.7) with (1.8), takes the following equivalent form:

$$\begin{aligned} \partial_t p^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) p^\epsilon + \frac{1}{\epsilon} \operatorname{div}(2\mathbf{u}^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon) \\ = \epsilon e^{-\epsilon p^\epsilon} [\nu^\epsilon |\operatorname{curl} \mathbf{H}^\epsilon|^2 + \Psi(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla p^\epsilon \cdot \nabla \theta^\epsilon, \end{aligned} \quad (1.12)$$

$$e^{-\theta^\epsilon} [\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon] + \frac{\nabla p^\epsilon}{\epsilon} = e^{-\epsilon p^\epsilon} [(\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon + \operatorname{div} \Psi^\epsilon(\mathbf{u}^\epsilon)], \quad (1.13)$$

$$\partial_t \mathbf{H}^\epsilon - \operatorname{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) - \nu^\epsilon \Delta \mathbf{H}^\epsilon = 0, \quad \operatorname{div} \mathbf{H}^\epsilon = 0, \quad (1.14)$$

$$\begin{aligned} \partial_t \theta^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \theta^\epsilon + \operatorname{div} \mathbf{u}^\epsilon \\ = \epsilon^2 e^{-\epsilon p^\epsilon} [\nu^\epsilon |\operatorname{curl} \mathbf{H}^\epsilon|^2 + \Psi^\epsilon(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon} \operatorname{div}(e^{\theta^\epsilon} \nabla \theta^\epsilon), \end{aligned} \quad (1.15)$$

where $\Psi^\epsilon(\mathbf{u}^\epsilon) = 2\mu^\epsilon \mathbb{D}(\mathbf{u}^\epsilon) + \lambda^\epsilon \operatorname{div} \mathbf{u}^\epsilon \mathbf{I}_3$, and the identity $\operatorname{curl}(\operatorname{curl} \mathbf{H}^\epsilon) = \nabla \operatorname{div} \mathbf{H}^\epsilon - \Delta \mathbf{H}^\epsilon$ and the constraint $\operatorname{div} \mathbf{H}^\epsilon = 0$ are used.

We shall study the limit as $\epsilon \rightarrow 0$ of solutions to (1.12)–(1.15). Formally, as ϵ goes to zero, if the sequence $(\mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ converges strongly to a limit $(\mathbf{w}, \mathbf{B}, \vartheta)$ in some sense, and $(\mu^\epsilon, \lambda^\epsilon, \nu^\epsilon, \kappa^\epsilon)$ converges to a constant vector $(\bar{\mu}, \bar{\lambda}, \bar{\nu}, \bar{\kappa})$, then taking the limit to (1.12)–(1.15), we have

$$\operatorname{div}(2\mathbf{w} - \bar{\kappa} e^{\vartheta} \nabla \vartheta) = 0, \quad (1.16)$$

$$e^{-\vartheta} [\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w}] + \nabla \pi = (\operatorname{curl} \mathbf{B}) \times \mathbf{B} + \operatorname{div} \Phi(\mathbf{w}), \quad (1.17)$$

$$\partial_t \mathbf{B} - \operatorname{curl}(\mathbf{w} \times \mathbf{B}) - \bar{\nu} \Delta \mathbf{B} = 0, \quad \operatorname{div} \mathbf{B} = 0, \quad (1.18)$$

$$\partial_t \vartheta + (\mathbf{w} \cdot \nabla) \vartheta + \operatorname{div} \mathbf{w} = \bar{\kappa} \operatorname{div}(e^{\vartheta} \nabla \vartheta), \quad (1.19)$$

with some function π , where $\Phi(\mathbf{w})$ is defined by

$$\Phi(\mathbf{w}) = 2\bar{\mu} \mathbb{D}(\mathbf{w}) + \bar{\lambda} \operatorname{div} \mathbf{w} \mathbf{I}_3. \quad (1.20)$$

The purpose of this paper is to establish the above limit process rigorously. For this purpose, we supplement the system (1.12)–(1.15) with the following initial conditions

$$(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)|_{t=0} = (p_{\text{in}}^\epsilon(x), \mathbf{u}_{\text{in}}^\epsilon(x), \mathbf{H}_{\text{in}}^\epsilon(x), \theta_{\text{in}}^\epsilon(x)), \quad x \in \mathbb{R}^3. \quad (1.21)$$

For simplicity of presentation, assume that $\mu^\epsilon \equiv \bar{\mu} > 0$, $\nu^\epsilon \equiv \bar{\nu} > 0$, $\kappa^\epsilon \equiv \bar{\kappa} > 0$, and $\lambda^\epsilon \equiv \bar{\lambda}$. The general case $\mu^\epsilon \rightarrow \bar{\mu} > 0$, $\nu^\epsilon \rightarrow \bar{\nu} > 0$, $\kappa^\epsilon \rightarrow \bar{\kappa} > 0$ and $\lambda^\epsilon \rightarrow \bar{\lambda}$ as $\epsilon \rightarrow 0$ can be treated by modifying slightly the arguments presented here.

As in [2], we will use the notation $\|v\|_{H_\eta^\sigma} := \|v\|_{H^{\sigma-1}} + \eta\|v\|_{H^\sigma}$ for any $\sigma \in \mathbb{R}$ and $\eta \geq 0$. For each $\epsilon > 0$, $t \geq 0$ and $s \geq 0$, we will also use the following norm:

$$\begin{aligned} & \|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(t)\|_{s,\epsilon} \\ & := \sup_{\tau \in [0,t]} \left\{ \|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(\tau)\|_{H^s} + \|(\epsilon p^\epsilon, \epsilon \mathbf{u}^\epsilon, \epsilon \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(\tau)\|_{H_\epsilon^{s+2}} \right\} \\ & \quad + \left\{ \int_0^t [\|\nabla(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)\|_{H^s}^2 + \|\nabla(\epsilon \mathbf{u}^\epsilon, \epsilon H^\epsilon, \theta^\epsilon)\|_{H_\epsilon^{s+2}}^2](\tau) d\tau \right\}^{1/2}. \end{aligned} \quad (1.22)$$

Then, the main result of this paper reads as follows.

Theorem 1.1. *Let $s \geq 4$. Assume that the initial data $(p_{\text{in}}^\epsilon, \mathbf{u}_{\text{in}}^\epsilon, \mathbf{H}_{\text{in}}^\epsilon, \theta_{\text{in}}^\epsilon)$ satisfy*

$$\|(p_{\text{in}}^\epsilon, \mathbf{u}_{\text{in}}^\epsilon, \mathbf{H}_{\text{in}}^\epsilon, \theta_{\text{in}}^\epsilon - \bar{\theta})(t)\|_{s,\epsilon} \leq L_0 \quad (1.23)$$

for all $\epsilon \in (0, 1]$ and two given positive constants $\bar{\theta}$ and L_0 . Then there exist positive constants T_0 and $\epsilon_0 < 1$, depending only on L_0 and $\bar{\theta}$, such that the Cauchy problem (1.12)–(1.15), (1.21) has a unique solution $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ satisfying

$$\|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(t)\|_{s,\epsilon} \leq L, \quad \forall t \in [0, T_0], \quad \forall \epsilon \in (0, \epsilon_0], \quad (1.24)$$

where L depends only on L_0 , $\bar{\theta}$ and T_0 . Moreover, assume further that the initial data satisfy the following conditions

$$|\theta_0^\epsilon(x) - \bar{\theta}| \leq N_0|x|^{-1-\zeta}, \quad |\nabla\theta_0^\epsilon(x)| \leq N_0|x|^{-2-\zeta}, \quad \forall \epsilon \in (0, 1], \quad (1.25)$$

$$(p_{\text{in}}^\epsilon, \text{curl}(e^{-\theta_{\text{in}}^\epsilon} \mathbf{u}_{\text{in}}^\epsilon), \mathbf{H}_{\text{in}}^\epsilon, \theta_{\text{in}}^\epsilon) \rightarrow (0, \mathbf{w}_0, \mathbf{B}_0, \vartheta_0) \text{ in } H^s(\mathbb{R}^3) \text{ as } \epsilon \rightarrow 0, \quad (1.26)$$

where N_0 and ζ are fixed positive constants. Then the solution sequence $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ converges weakly in $L^\infty(0, T_0; H^s(\mathbb{R}^3))$ and strongly in $L^2(0, T_0; H_{\text{loc}}^{s_2}(\mathbb{R}^3))$ for all $0 \leq s_2 < s$ to the limit $(0, \mathbf{w}, \mathbf{B}, \vartheta)$, where $(\mathbf{w}, \mathbf{B}, \vartheta)$ satisfies the system (1.16)–(1.19) with initial data $(\mathbf{w}, \mathbf{B}, \vartheta)|_{t=0} = (\mathbf{w}_0, \mathbf{B}_0, \vartheta_0)$.

We now give some comments on the proof of Theorem 1.1. The key point in the proof is to establish the uniform estimates in Sobolev norms for the acoustic components of solutions, which are propagated by wave equations whose coefficients are functions of the temperature. Our main strategy is to bound the norm of $(\nabla p^\epsilon, \text{div} \mathbf{u}^\epsilon)$ in terms of the norm of $(\epsilon \partial_t)(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$ and that of $(\epsilon p^\epsilon, \epsilon \mathbf{u}^\epsilon, \epsilon \mathbf{H}^\epsilon, \theta^\epsilon)$ through the density and the momentum equations. This approach is motivated by the previous works due to Alazard in [1, 2], and Levermore, Sun and Trivisa [37]. It should be pointed out that the analysis for (1.12)–(1.15) is complicated and difficult due to the strong coupling of the hydrodynamic motion and the magnetic fields. Moreover, it is observed that the terms $(\text{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon$ in the momentum equations, $\text{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon)$ in the magnetic field equation, and $|\nabla \times \mathbf{H}^\epsilon|^2$ in the temperature equation change basically the structure of the system. More efforts should be paid on the estimates involving these terms, in particular on the higher order spatial derivatives. We shall exploit the special structure of the system to obtain the tamed estimate on higher order derivatives, so that we can enclose our arguments on the uniform boundedness of the solutions. Once the uniform bounds of the solutions are obtained, the convergence result in Theorem 1.1 can be proved by applying the compactness arguments and the dispersive estimates on the acoustic wave equations in the whole space developed in [39].

Remark 1.1. The positivity of the coefficients μ , ν and κ plays an fundamental role in the proof of Theorem 1.1. The arguments given in this paper can not be applied

to the case when one of them disappears. We shall discuss this situation in another forthcoming paper.

We point out that the low Mach number limit is an interesting topic in fluid dynamics and applied mathematics. Now we briefly review some related results on the Euler, Navier-Stokes and MHD equations. In [47], Schochet obtained the convergence of the non-isentropic compressible Euler equations to the incompressible non-isentropic Euler equations in a bounded domain for local smooth solutions and well-prepared initial data. As mentioned above, in [39] Métivier and Schochet proved rigorously the incompressible limit of the compressible non-isentropic Euler equations in the whole space with general initial data, see also [1, 2, 37] for further extensions. In [40] Métivier and Schochet showed the incompressible limit of the one-dimensional non-isentropic Euler equations in a periodic domain with general data. For compressible heat-conducting flows, Hagstrom and Lorenz established in [18] the low Mach number limit under the assumption that the variations density and temperature are small. In the case of without heat conductivity, Kim and Lee [32] investigated the incompressible limit to the non-isentropic Navier-Stokes equations in a periodic domain with well-prepared data, while Jiang and Ou [31] investigated the incompressible limit in three-dimensional bounded domains, also for well-prepared data. The justification of the low Mach number limit to non-isentropic Euler or Navier-Stokes equations with general initial data in bounded domains or multi-dimensional periodic domains is still open. We refer the interested reader to [6] for formal computations for viscous polytropic gases, and to [5, 40] for the study on the acoustic waves of the non-isentropic Euler equations in periodic domains. Compared with the non-isentropic case, the description of the propagation of oscillations in the isentropic case is simpler and there are many articles on this topic in the literature, see, for example, Ukai [49], Asano [3], Desjardins and Grenier [11] in the whole space case; Isozaki [25, 26] in the case of exterior domains; Iguchi [24] in the half space case; Schochet [46] and Gallagher [16] in the case of periodic domains; and Lions and Masmoudi [43], and Desjardins, et al. [12] in the case of bounded domains.

For the compressible isentropic MHD equations, the justification of the low Mach number limit has been established in several aspects. In [33] Klainerman and Majda studied the low Mach number limit to the compressible isentropic MHD equations in the spatially periodic case with well-prepared initial data. Recently, the low Mach number limit to the compressible isentropic viscous (including both viscosity and magnetic diffusivity) MHD equations with general data was studied in [23, 28, 29]. In [23] Hu and Wang obtained the convergence of weak solutions to the compressible viscous MHD equations in bounded domains, periodic domains and the whole space. In [28] Jiang, Ju and Li employed the modulated energy method to verify the limit of weak solutions of the compressible MHD equations in the torus to the strong solution of the incompressible viscous or partially viscous MHD equations (zero shear viscosity but with magnetic diffusion), while in [29] the convergence of weak solutions of the viscous compressible MHD equations to the strong solution of the ideal incompressible MHD equations in the whole space was established by using the dispersion property of the wave equation, as both shear viscosity and magnetic diffusion coefficients go to zero. For the full compressible MHD equations, the incompressible limit in the framework of the so-called variational solutions was established in [34, 35, 41]. Recently, the low Mach number limit for the ideal and full

non-isentropic MHD equations with small entropy or temperature variations was justified rigorously in [30]. Besides the references mentioned above, the interested reader can refer to the monograph [14] and the survey papers [9, 44, 48] for more related results on the low Mach number limit to fluid models.

We also mention that there are a lot of articles in the literatures on the other topics related to the compressible MHD equations due to their physical importance, complexity, rich phenomena, and mathematical challenges, see, for example, [4, 7, 8, 10, 13, 15, 19–22, 38, 42, 51] and the references cited therein.

This paper is arranged as follows. In Section 2, we describe some notations, recall basic facts and present commutators estimates. In Section 3 we first establish a priori estimates on $(\mathbf{H}^\epsilon, \theta^\epsilon)$, $(\epsilon p^\epsilon, \epsilon \mathbf{u}^\epsilon, \epsilon \mathbf{H}^\epsilon, \theta^\epsilon)$ and on $(p^\epsilon, \mathbf{u}^\epsilon)$. Then, with the help of these estimates we establish the uniform boundness of the solutions and prove the existence part of Theorem 1.1. Finally, in Section 4 we study the decay of the local energy to the acoustic wave equations and prove the convergence part of Theorem 1.1.

2. PRELIMINARY

In this section, we give some notations and recall basic facts which will be used frequently throughout the paper. We also present some commutators estimates introduced in [37] and state the results on local solutions to the Cauchy problem (1.12)–(1.15), (1.21).

We denote $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(\mathbb{R}^3)$ with norm $\langle f, f \rangle = \|f\|_{L^2}^2$ and H^k the standard Sobolev space $W^{k,2}$ with norm $\|\cdot\|_{H^k}$. The notation $\|(A_1, \dots, A_k)\|_{L^2}$ means the summation of $\|A_i\|_{L^2}$, $i = 1, \dots, k$, and it also applies to other norms. For the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$. We will omit the spatial domain \mathbb{R}^3 in integrals for convenience. We use $l_i > 0$ ($i \in \mathbb{N}$) to denote given constants. We also use the symbol K or C_0 to denote generic positive constants, and $C(\cdot)$ to denote a smooth function which may vary from line to line.

For a scalar function f , vector functions \mathbf{a} and \mathbf{b} , we have the following basic vector identities:

$$\operatorname{div}(f\mathbf{a}) = f \operatorname{div} \mathbf{a} + \nabla f \cdot \mathbf{a}, \quad (2.1)$$

$$\operatorname{curl}(f\mathbf{a}) = f \cdot \operatorname{curl} \mathbf{a} - \nabla f \times \mathbf{a}, \quad (2.2)$$

$$\operatorname{div}(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \operatorname{curl} \mathbf{a} - \mathbf{a} \cdot \operatorname{curl} \mathbf{b}, \quad (2.3)$$

$$\operatorname{curl}(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} + \mathbf{a}(\operatorname{div} \mathbf{b}) - \mathbf{b}(\operatorname{div} \mathbf{a}), \quad (2.4)$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\operatorname{curl} \mathbf{b}) + \mathbf{b} \times (\operatorname{curl} \mathbf{a}). \quad (2.5)$$

For simplicity of presentation, we will use

Definition 2.1 ([2]). For $s \in \mathbb{R}^1$ and $g \in H^{s+1}(\mathbb{R}^3)$, we define the weighted Sobolev norm $\|\cdot\|_{H_\epsilon^{s+1}}$ as

$$\|g\|_{H_\epsilon^{s+1}} = \|g\|_{H^s} + \epsilon \|\nabla g\|_{H^s}.$$

Below we recall some results on commutators estimates.

Lemma 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index such that $|\alpha| = k$. Then, for any $\sigma \geq 0$, there exists a positive constant C_0 , such that for all $f, g \in H^{k+\sigma}(\mathbb{R}^3)$,*

$$\|[f, \partial^\alpha]g\|_{H^\sigma} \leq C_0(\|f\|_{W^{1,\infty}} \|g\|_{H^{\sigma+k-1}} + \|f\|_{H^{\sigma+k}} \|g\|_{L^\infty}). \quad (2.6)$$

Lemma 2.2. *Let $s > 5/2$. Then there exists a positive constant C_0 , such that for all $\epsilon \in (0, 1]$, $T > 0$ and multi-index $\beta = (\beta_1, \beta_2, \beta_3)$ satisfying $0 \leq |\beta| \leq s - 1$, and any $f, g \in C^\infty([0, T], H^s(\mathbb{R}^3))$, it holds that*

$$\| [f, \partial^\beta(\epsilon \partial_t)]g \|_{L^2} \leq \epsilon C_0 (\|f\|_{H^{s-1}} \|\partial_t g\|_{H^{s-2}} + \|\partial_t f\|_{H^{s-1}} \|g\|_{H^{s-1}}). \quad (2.7)$$

Since the system (1.1)–(1.3), (1.7), (1.8) is hyperbolic-parabolic, so the classical result of Vol’pert and Khudiaev [50] implies that

Proposition 2.3. *Let $s \geq 4$. Assume that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{H}_0, \theta_0)$ satisfy*

$$\|(\rho_0 - \underline{\rho}, \mathbf{u}_0, \mathbf{H}_0, \theta_0 - \underline{\theta})\|_{H^s} \leq C_0$$

for some positive constants $\underline{\rho}$, $\underline{\theta}$ and C_0 . Then there exists a $\tilde{T} > 0$, such that the system (1.1)–(1.3), (1.7), and (1.8) with these initial data has a unique classical solution $(\rho, \mathbf{u}, \mathbf{H}, \theta)$ enjoying $\rho \in C([0, \tilde{T}], H^s(\mathbb{R}^3))$, $(\mathbf{u}, \mathbf{H}, \theta) \in C([0, \tilde{T}], H^s(\mathbb{R}^3)) \cap L^2(0, \tilde{T}; H^{s+1}(\mathbb{R}^3))$, and

$$\begin{aligned} \sup_{0 \leq t \leq \tilde{T}} \|(\rho, \mathbf{u}, \mathbf{H}, \theta)\|_{H^s}^2 + \int_0^{\tilde{T}} \left\{ \mu \|\mathbb{D}(\mathbf{u})\|_{H^s}^2 + \lambda \|\operatorname{div} \mathbf{u}\|_{H^s}^2 \right. \\ \left. + \nu \|\nabla \mathbf{H}\|_{H^s}^2 + \kappa \|\nabla \theta\|_{H^s}^2 \right\}(\tau) d\tau \leq 4C_0^2. \end{aligned}$$

It follows from Proposition 2.3 and the transforms (1.9), (1.10) that there exists a $T_\epsilon > 0$, depending on ϵ and L_0 , such that for each fixed ϵ and any initial data (1.21) satisfying (1.23), the Cauchy problem (1.12)–(1.15), (1.21) has a unique solution $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ satisfying $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon) \in C([0, T_\epsilon], H^s(\mathbb{R}^3))$ and $(\mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon) \in L^2(0, T_\epsilon; H^{s+1}(\mathbb{R}^3))$. Moreover, let T_ϵ^* be the maximal time of existence of such smooth solution, then if T_ϵ^* is finite, one has

$$\limsup_{t \rightarrow T_\epsilon^*} \{ \| (p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)(t) \|_{W^{1,\infty}} + \| (\mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)(t) \|_{W^{2,\infty}} \} = \infty.$$

Therefore, the existence part of Theorem 1.1 is a consequence of the above assertion and the following a priori estimates which can be shown in the same manner as in [39].

Proposition 2.4. *For any given $s \geq 4$ and fixed $\epsilon > 0$, let $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ be the classical solution to the Cauchy problem (1.12)–(1.15) and (1.21). Denote*

$$\mathcal{O}(T) := \|(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(T)\|_{s,\epsilon}, \quad (2.8)$$

$$\mathcal{O}_0 := \|(p_{\text{in}}^\epsilon, \mathbf{u}_{\text{in}}^\epsilon, \mathbf{H}_{\text{in}}^\epsilon, \theta_{\text{in}}^\epsilon)\|_{H^s} + \|\epsilon(p_{\text{in}}^\epsilon, \mathbf{u}_{\text{in}}^\epsilon, \mathbf{H}_{\text{in}}^\epsilon, \theta_{\text{in}}^\epsilon - \bar{\theta})\|_{H_\epsilon^{s+2}}. \quad (2.9)$$

Then there exist positive constants \hat{T}_0 and $\epsilon_0 < 1$, and an increasing positive function $C(\cdot)$, such that for all $T \in [0, \hat{T}_0]$ and $\epsilon \in (0, \epsilon_0]$,

$$\mathcal{O}(T) \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}.$$

3. UNIFORM ESTIMATES

In this section we shall establish the uniform bounds of the solutions to the Cauchy problem (1.12)–(1.15) and (1.21) by modifying the approaches developed in [2, 37, 39] and making careful use of the special structure of the system (1.12)–(1.15). In the rest of this section, we will drop the superscripts ϵ of the variables in the Cauchy problem and denote

$$\Psi(\mathbf{u}) = 2\bar{\mu}\mathbb{D}(\mathbf{u}) + \bar{\lambda}\operatorname{div} \mathbf{I}_3.$$

Recall that it has been assumed that $\mu^\epsilon \equiv \bar{\mu} > 0$, $\nu^\epsilon \equiv \bar{\nu} > 0$, $\kappa^\epsilon \equiv \bar{\kappa} > 0$, and $\lambda^\epsilon \equiv \bar{\lambda}$ independent of ϵ .

3.1. H^s estimates on (\mathbf{H}, θ) and $(\epsilon p, \epsilon \mathbf{u})$. To prove the Proposition 2.4, we first give some estimates derived directly from the system (1.12)–(1.15). Denote

$$\mathcal{Q} := \|(p, \mathbf{u}, \mathbf{H}, \theta - \bar{\theta})\|_{H^s} + \|(\epsilon p, \epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta - \bar{\theta})\|_{H_\epsilon^{s+2}}, \quad (3.1)$$

$$\mathcal{S} := \|(\nabla \mathbf{u}, \nabla p, \nabla \mathbf{H})\|_{H^s} + \|\nabla(\epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta)\|_{H_\epsilon^{s+2}}. \quad (3.2)$$

One has

Lemma 3.1. *There exists an increasing function $C(\cdot)$ such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \{ \|(\mathbf{H}, \theta)\|_{H^s} + \|\epsilon \mathbf{H}\|_{H_\epsilon^{s+1}} \} + \|\nabla(\mathbf{H}, \theta)\|_{L^2(0, T; H^s)} + \|\nabla(\epsilon \mathbf{H})\|_{L^2(0, T; H_\epsilon^{s+1})} \\ & \leq C(\mathcal{O}_0) \exp\{TC(\mathcal{O})\}. \end{aligned} \quad (3.3)$$

Proof. For any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ satisfying $|\alpha| \leq s$, let $\mathbf{H}_\alpha = \partial^\alpha \mathbf{H}$. Then

$$\partial_t \mathbf{H}_\alpha + (\mathbf{u} \cdot \nabla) \mathbf{H}_\alpha - \bar{\nu} \Delta \mathbf{H}_\alpha = -[\partial^\alpha, \mathbf{u}] \cdot \nabla \mathbf{H} - \partial^\alpha(\mathbf{H} \operatorname{div} \mathbf{u}) + \partial^\alpha((\mathbf{H} \cdot \nabla) \mathbf{u}).$$

Taking inner product of the above equations with \mathbf{H}_α and integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{H}_\alpha\|_{L^2}^2 + \bar{\nu} \|\nabla \mathbf{H}_\alpha\|_{L^2}^2 &= -\langle (\mathbf{u} \cdot \nabla) \mathbf{H}_\alpha, \mathbf{H}_\alpha \rangle - \langle [\partial^\alpha, \mathbf{u}] \cdot \nabla \mathbf{H}, \mathbf{H}_\alpha \rangle \\ &\quad - \langle \partial^\alpha(\mathbf{H} \operatorname{div} \mathbf{u}), \mathbf{H}_\alpha \rangle + \langle \partial^\alpha((\mathbf{H} \cdot \nabla) \mathbf{u}), \mathbf{H}_\alpha \rangle. \end{aligned}$$

An integration by parts gives

$$-\langle (\mathbf{u} \cdot \nabla) \mathbf{H}_\alpha, \mathbf{H}_\alpha \rangle = \frac{1}{2} \int \operatorname{div} \mathbf{u} |\mathbf{H}_\alpha|^2 dx \leq C(\mathcal{O}) \|\mathbf{H}_\alpha\|_{L^2}^2.$$

It follows from the commutator inequality (2.6) that

$$\|[\partial^\alpha, \mathbf{u}] \cdot \nabla \mathbf{H}\|_{L^2} \leq C_0(\|\mathbf{u}\|_{W^{1, \infty}} \|\nabla \mathbf{H}\|_{H^{s-1}} + \|\mathbf{u}\|_{H^s} \|\nabla \mathbf{H}\|_{L^\infty}) \leq C(\mathcal{O}).$$

By Sobolev's inequality, one gets

$$-\langle \partial^\alpha(\mathbf{H} \operatorname{div} \mathbf{u}), \mathbf{H}_\alpha \rangle + \langle \partial^\alpha(\mathbf{H} \cdot \nabla \mathbf{u}), \mathbf{H}_\alpha \rangle \leq C_0 \|\mathbf{H}\|_{H^s}^2 \|\mathbf{u}\|_{H^{s+1}} \leq C(\mathcal{O}) \mathcal{S}.$$

Thus, we conclude that

$$\begin{aligned} \|\mathbf{H}\|_{H^s} + \bar{\nu} \|\nabla \mathbf{H}\|_{L^2(0, T; H^s)} &\leq C(\mathcal{O}_0) + C(\mathcal{O})T + C(\mathcal{O}) \int_0^T \mathcal{S}(\tau) d\tau \\ &\leq C(\mathcal{O}_0) + C(\mathcal{O})T + C(\mathcal{O})\sqrt{T} \\ &\leq C(\mathcal{O}_0) + C(\mathcal{O})\sqrt{T} \\ &\leq C(\mathcal{O}_0) e^{\sqrt{T}C(\mathcal{O})}. \end{aligned}$$

Now denote $\hat{\mathbf{H}} = \epsilon \mathbf{H}$ and $\hat{\mathbf{H}}_\alpha = \partial^\alpha(\epsilon \mathbf{H})$ for $|\alpha| = s + 1$. Then, $\hat{\mathbf{H}}_\alpha$ satisfies

$$\partial_t \hat{\mathbf{H}}_\alpha + (\mathbf{u} \cdot \nabla) \hat{\mathbf{H}}_\alpha - \bar{\nu} \Delta \hat{\mathbf{H}}_\alpha = -\epsilon [\partial^\alpha, \mathbf{u}] \cdot \nabla \mathbf{H} - \epsilon \partial^\alpha(\mathbf{H} \operatorname{div} \mathbf{u}) + \epsilon \partial^\alpha((\mathbf{H} \cdot \nabla) \mathbf{u}). \quad (3.4)$$

The commutator inequality (2.6) implies that

$$\|-\epsilon [\partial^\alpha, \mathbf{u}] \cdot \nabla \mathbf{H}\|_{L^2} \leq C_0(\|\mathbf{u}\|_{W^{1, \infty}} \|\nabla \hat{\mathbf{H}}\|_{H^s} + \|\epsilon \mathbf{u}\|_{H^{s+1}} \|\nabla \mathbf{H}\|_{L^\infty}) \leq C(\mathcal{O}),$$

while an integration by parts and Sobolev's inequality lead to

$$\begin{aligned} -\langle \epsilon \partial^\alpha (\mathbf{H} \operatorname{div} \mathbf{u}), \hat{\mathbf{H}}_\alpha \rangle + \langle \epsilon \partial^\alpha ((\mathbf{H} \cdot \nabla) \mathbf{u}), \hat{\mathbf{H}}_\alpha \rangle &\leq \frac{\bar{\nu}}{2} \|\nabla \hat{\mathbf{H}}_\alpha\|_{L^2}^2 + C_0 \|\mathbf{H}\|_{H^s}^2 \|\epsilon \mathbf{u}\|_{H^{s+1}}^2 \\ &\leq \frac{\bar{\nu}}{2} \|\nabla \hat{\mathbf{H}}_\alpha\|_{L^2}^2 + C(\mathcal{O}). \end{aligned}$$

Hence, we obtain

$$\|\epsilon \mathbf{H}\|_{H^{s+1}} + \bar{\nu} \|\nabla(\epsilon \mathbf{H})\|_{L^2(0,T;H^{s+1})} \leq C(\mathcal{O}_0) + C(\mathcal{O})T.$$

Similarly, the terms $\epsilon^2 \|\mathbf{H}\|_{H^{s+2}}$ and $\epsilon^2 \|\nabla \mathbf{H}\|_{L^2(0,T;H^{s+1})}$ can be estimated.

Next, we estimate θ . Using Sobolev's inequality, one finds that

$$\begin{aligned} \|\partial^s (\epsilon^2 e^{-\epsilon p} [\bar{\nu} |\operatorname{curl} \mathbf{H}|^2 + \Psi(\mathbf{u}) : \nabla \mathbf{u}])\|_{L^2} \\ \leq C_0 \|\epsilon p\|_{H^s} (\|\epsilon \mathbf{H}\|_{H^{s+1}} + \|\epsilon \mathbf{u}\|_{H^s}) \leq C(\mathcal{O}). \end{aligned}$$

Employing arguments similar to those used for \mathbf{H} , we can obtain

$$\|\theta\|_{H^s} + \|\nabla \theta\|_{L^2(0,T;H^s)} \leq C(\mathcal{O}_0) \exp\{TC(\mathcal{O})\}.$$

Thus, the lemma is proved. \square

Lemma 3.2. *There exists an increasing function $C(\cdot)$ such that*

$$\sup_{t \in [0,T]} \|(\epsilon p, \epsilon \mathbf{u})\|_{H^s} + \|\nabla(\epsilon \mathbf{u})\|_{L^2(0,T;H^s)} \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O})\}. \quad (3.5)$$

Proof. Let $\check{p} = \epsilon p$, and $\check{p}_\alpha = \partial^\alpha(\epsilon p)$ for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ satisfying $|\alpha| \leq s$. Then

$$\begin{aligned} \partial_t \check{p}_\alpha + (\mathbf{u} \cdot \nabla) \check{p}_\alpha &= -[\partial^\alpha, \mathbf{u}] \cdot (\nabla \check{p}) - \partial^\alpha [\operatorname{div}(2\mathbf{u} - \kappa a(\epsilon p) b(\theta) \nabla \theta)] \\ &\quad + \partial^\alpha \{a(\epsilon p) [\bar{\nu} |\operatorname{curl}(\epsilon \mathbf{H})|^2 + \Psi(\epsilon \mathbf{u}) : \nabla(\epsilon \mathbf{u})]\} \\ &\quad + \kappa \partial^\alpha \{a(\epsilon p) b(\theta) \nabla(\epsilon p) \cdot \nabla \theta\} \\ &:= h_1 + h_2 + h_3 + h_4, \end{aligned} \quad (3.6)$$

where, for simplicity of presentation, we have set

$$a(\epsilon p) := e^{-\epsilon p}, \quad b(\theta) := e^\theta.$$

It is easy to see that the energy estimate for (3.6) gives

$$\frac{1}{2} \frac{d}{dt} \|\check{p}_\alpha\|_{L^2}^2 = -\langle (\mathbf{u} \cdot \nabla) \check{p}_\alpha, \check{p}_\alpha \rangle + \langle h_1 + h_2 + h_3 + h_4, \check{p}_\alpha \rangle, \quad (3.7)$$

where we have to estimate each term on the right-hand side of (3.7). First, an integration by parts yields

$$-\langle (\mathbf{u} \cdot \nabla) \check{p}_\alpha, \check{p}_\alpha \rangle = \frac{1}{2} \int \operatorname{div} \mathbf{u} |\check{p}_\alpha|^2 dx \leq C(\mathcal{O}) \|\check{p}_\alpha\|_{L^2}^2,$$

while the commutator inequality leads to

$$\|h_1\| \leq C_0 (\|\mathbf{u}\|_{W^{1,\infty}} \|\nabla \check{p}\|_{H^{s-1}} + \|\mathbf{u}\|_{H^s} \|\nabla \check{p}\|_{L^\infty}) \leq C(\mathcal{O}).$$

Consequently,

$$\langle h_1, \check{p}_\alpha \rangle \leq \|\check{p}_\alpha\|_{L^2} \|h_1\|_{L^2} \leq C(\mathcal{O}).$$

From Sobolev's inequality one gets

$$\|h_2\| \leq C_0 \|\mathbf{u}\|_{H^{s+1}} + \|\theta\|_{H^s} \|\epsilon p\|_{H^s} \|\theta\|_{H^{s+2}} \leq C(\mathcal{S})(1 + C(\mathcal{O})),$$

whence,

$$\langle h_2, \check{p}_\alpha \rangle \leq C(\mathcal{O})C(\mathcal{S}).$$

Similarly, one can prove that

$$\langle (h_3 + h_4), \check{p}_\alpha \rangle \leq C(\mathcal{O}).$$

Hence, we conclude that

$$\|\epsilon p\|_{H^s} \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O})\}.$$

In a similar way, we can obtain estimates on \mathbf{u} . Thus the proof of the lemma is completed. \square

Next, we control the term $\|(\mathbf{u}, p)\|_{H^s}$. The idea is to bound the norm of $(\operatorname{div} \mathbf{u}, \nabla p)$ in terms of the suitable norm of $(\epsilon \mathbf{u}, \epsilon p, \epsilon \mathbf{H}, \theta)$ and $\epsilon(\partial_t \mathbf{u}, \partial_t p)$ by making use of the structure of the system. To this end, we first estimate $\|(\epsilon \mathbf{u}, \epsilon p, \theta)\|_{H^{s+1}}$.

3.2. H^{s+1} estimates on $(\epsilon \mathbf{u}, \epsilon p, \epsilon \mathbf{H}, \theta)$. Following [2], we set

$$(\hat{p}, \hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{\theta}) := (\epsilon p - \theta, \epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta - \bar{\theta}).$$

A straightforward calculation results in that $(\hat{p}, \hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{\theta})$ solves the following system:

$$\partial_t \hat{p} + (\mathbf{u} \cdot \nabla) \hat{p} + \frac{1}{\epsilon} \operatorname{div} \hat{\mathbf{u}} = 0, \quad (3.8)$$

$$\begin{aligned} b(-\theta)[\partial_t \hat{\mathbf{u}} + (\mathbf{u} \cdot \nabla) \hat{\mathbf{u}}] + \frac{1}{\epsilon} (\nabla \hat{p} + \nabla \hat{\theta}) \\ = a(\epsilon p)[(\operatorname{curl} \mathbf{H}) \times \hat{\mathbf{H}} + \operatorname{div} \Psi(\hat{\mathbf{u}})], \end{aligned} \quad (3.9)$$

$$\partial_t \hat{\mathbf{H}} + \mathbf{u} \cdot \nabla \hat{\mathbf{H}} + \mathbf{H} \operatorname{div} \hat{\mathbf{u}} - \mathbf{H} \cdot \nabla \hat{\mathbf{u}} - \bar{\nu} \Delta \hat{\mathbf{H}} = 0, \quad \operatorname{div} \hat{\mathbf{H}} = 0, \quad (3.10)$$

$$\begin{aligned} \partial_t \hat{\theta} + (\mathbf{u} \cdot \nabla) \hat{\theta} + \frac{1}{\epsilon} \operatorname{div} \hat{\mathbf{u}} = \epsilon a(\epsilon p)[\bar{\nu} \operatorname{curl} \mathbf{H} : \operatorname{curl} \hat{\mathbf{H}} + \epsilon a(\epsilon p) \Psi(\mathbf{u}) : \nabla \hat{\mathbf{u}}] \\ + \bar{\kappa} a(\epsilon p) \operatorname{div}(b(\theta) \nabla \hat{\theta}). \end{aligned} \quad (3.11)$$

One has

Lemma 3.3. *Let $s \geq 4$ and $(p, \mathbf{u}, \mathbf{H}, \theta)$ be a solution to the problem (1.12)–(1.15), (1.21) on $[0, T_1]$. Then there exists a constant $l_1 > 0$, such that for any $\epsilon \in (0, 1]$ and $t \in [0, T]$, $T = \min\{T_1, 1\}$, it holds that*

$$\begin{aligned} \sup_{\tau \in [0, T]} \|(\epsilon q, \epsilon \mathbf{u}, \theta - \bar{\theta})(\tau)\|_{H^{s+1}} + l_1 \left\{ \int_0^T \|\nabla(\epsilon \mathbf{u}, \theta)\|_{H^{s+1}}^2(\tau) d\tau \right\}^{1/2} \\ \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O}(T))\}, \end{aligned} \quad (3.12)$$

where $\mathcal{O}(T)$ and \mathcal{O}_0 are defined by (2.8) and (2.9), respectively.

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index such that $|\alpha| = s + 1$. Set

$$(\hat{p}_\alpha, \hat{\mathbf{u}}_\alpha, \hat{\mathbf{H}}_\alpha, \hat{\theta}_\alpha) := (\partial^\alpha(\epsilon p - \theta), \partial^\alpha(\epsilon \mathbf{u}), \partial^\alpha(\epsilon \mathbf{H}), \partial^\alpha(\theta - \bar{\theta})).$$

Then, $\hat{\mathbf{H}}_\alpha$ satisfies (3.4) and $(\hat{p}_\alpha, \hat{\mathbf{u}}_\alpha, \hat{\theta}_\alpha)$ solves

$$\partial_t \hat{p}_\alpha + (\mathbf{u} \cdot \nabla) \hat{p}_\alpha + \frac{1}{\epsilon} \operatorname{div} \hat{\mathbf{u}}_\alpha = g_1, \quad (3.13)$$

$$\begin{aligned} b(-\theta)[\partial_t \hat{\mathbf{u}}_\alpha + (\mathbf{u} \cdot \nabla) \hat{\mathbf{u}}_\alpha] + \frac{1}{\epsilon} (\nabla \hat{p}_\alpha + \nabla \hat{\theta}_\alpha) \\ = a(\epsilon p)(\operatorname{curl} \mathbf{H}) \times \hat{\mathbf{H}}_\alpha + a(\epsilon p) \operatorname{div} \Psi(\hat{\mathbf{u}}_\alpha) + g_2, \end{aligned} \quad (3.14)$$

$$\begin{aligned} \partial_t \hat{\theta}_\alpha + (\mathbf{u} \cdot \nabla) \hat{\theta}_\alpha + \frac{1}{\epsilon} \operatorname{div} \hat{\mathbf{u}}_\alpha &= \epsilon a(\epsilon p) [\bar{\nu} \operatorname{curl} \mathbf{H} : \operatorname{curl} \hat{\mathbf{H}}_\alpha + \Psi(\mathbf{u}) : \nabla \hat{\mathbf{u}}_\alpha] \\ &+ \bar{\kappa} a(\epsilon p) \operatorname{div}(b(\theta) \nabla \hat{\theta}_\alpha) + g_3 \end{aligned} \quad (3.15)$$

with initial data

$$\begin{aligned} (\hat{p}_\alpha, \hat{\mathbf{u}}_\alpha, \hat{\mathbf{H}}_\alpha, \hat{\theta}_\alpha)|_{t=0} &:= (\partial^\alpha(\epsilon p_{\text{in}}(x) - \theta_{\text{in}}(x)), \partial^\alpha(\epsilon \mathbf{u}_{\text{in}}(x)), \\ &\partial^\alpha \mathbf{H}_{\text{in}}(x), \partial^\alpha(\theta_{\text{in}}(x) - \bar{\theta})), \end{aligned} \quad (3.16)$$

where

$$g_1 := -[\partial^\alpha, \mathbf{u}] \cdot \nabla(\epsilon p - \theta), \quad (3.17)$$

$$\begin{aligned} g_2 &:= -[\partial^\alpha, b(-\theta)] \partial_t(\epsilon \mathbf{u}) - [\partial^\alpha, b(-\theta) \mathbf{u}] \cdot \nabla(\epsilon \mathbf{u}) \\ &+ [\partial^\alpha, a(\epsilon p) \operatorname{curl}(\epsilon \mathbf{H})] \times \mathbf{H} + [\partial^\alpha, a(\epsilon p)] \operatorname{div} \Psi(\epsilon \mathbf{u}), \end{aligned} \quad (3.18)$$

$$\begin{aligned} g_3 &:= -[\partial^\alpha, \mathbf{u}] \cdot \nabla \theta + \bar{\nu} [\partial^\alpha, a(\epsilon p) \operatorname{curl}(\epsilon \mathbf{H})] : \operatorname{curl}(\epsilon \mathbf{H}) \\ &+ \epsilon [\partial^\alpha, a(\epsilon p) \Psi(\mathbf{u})] : \nabla(\epsilon \mathbf{u}) + \bar{\kappa} \partial^\alpha(a(\epsilon p) \operatorname{div}(b(\theta) \nabla \theta)) \\ &- \bar{\kappa} a(\epsilon p) \operatorname{div}(b(\theta) \nabla \hat{\theta}_\alpha). \end{aligned} \quad (3.19)$$

It follows from Proposition 2.3 and the positivity of $a(\cdot)$ and $b(\cdot)$ that $a(\cdot)$ and $b(\cdot)$ are bounded away from 0 uniformly with respect to ϵ , i.e.

$$a(\epsilon p) \geq \underline{a} > 0, \quad b(-\theta) \geq \underline{b} > 0. \quad (3.20)$$

The standard L^2 -energy estimates for (3.13), (3.14) and (3.15) yield that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\hat{p}_\alpha\|_{L^2}^2 + \langle b(-\theta) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle + \|\hat{\theta}_\alpha\|_{L^2}^2) \\ &\leq \frac{1}{2} \langle b_t(\theta) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle - \langle (\mathbf{u} \cdot \nabla) \hat{p}_\alpha, \hat{p}_\alpha \rangle - \langle b(-\theta) (\mathbf{u} \cdot \nabla) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle - \langle (\mathbf{u} \cdot \nabla) \hat{\theta}_\alpha, \hat{\theta}_\alpha \rangle \\ &+ \langle a(\epsilon p) (\operatorname{curl} \mathbf{H}) \times \hat{\mathbf{H}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle + \langle a(\epsilon p) \operatorname{div} \Psi(\hat{\mathbf{u}}_\alpha), \hat{\mathbf{u}}_\alpha \rangle \\ &\quad \langle \epsilon a(\epsilon p) [\bar{\nu} \operatorname{curl} \mathbf{H} : \operatorname{curl} \hat{\mathbf{H}}_\alpha + \Psi(\mathbf{u}) : \nabla \hat{\mathbf{u}}_\alpha], \theta_\alpha \rangle + \langle \bar{\kappa} a(\epsilon p) \operatorname{div}(b(\theta) \nabla \theta_\alpha), \hat{\theta}_\alpha \rangle \\ &+ \langle g_1, \hat{p}_\alpha \rangle + \langle g_2, \hat{\mathbf{u}}_\alpha \rangle + \langle g_3, \hat{\theta}_\alpha \rangle. \end{aligned} \quad (3.21)$$

It follows from equation (1.15) and the definition of \mathcal{Q} and \mathcal{S} that

$$\|b_t(\theta)\|_{L^\infty} \leq \|b(\theta)\|_{H^s} \|\theta_t\|_{H^s} \leq C(\mathcal{Q})(1 + \mathcal{S}).$$

Therefore,

$$\frac{1}{2} \langle b_t(\theta) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle \leq C(\mathcal{Q})(1 + \mathcal{S}).$$

On the other hand, it is easy to see that

$$-\langle (\mathbf{u} \cdot \nabla) \hat{p}_\alpha, \hat{p}_\alpha \rangle - \langle b(-\theta) (\mathbf{u} \cdot \nabla) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle - \langle (\mathbf{u} \cdot \nabla) \hat{\theta}_\alpha, \hat{\theta}_\alpha \rangle \leq C(\mathcal{Q})$$

and

$$\langle a(\epsilon p) (\operatorname{curl} \mathbf{H}) \times \hat{\mathbf{H}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle \leq C(\mathcal{Q}).$$

A partial integration shows that

$$\begin{aligned} -\langle a(\epsilon p_0) \operatorname{div} \Psi(\hat{\mathbf{u}}), \hat{\mathbf{u}} \rangle &= \int \mu a(\epsilon p_0) [|\nabla \hat{\mathbf{u}}_\alpha|^2 + (\mu + \lambda) |\operatorname{div} \hat{\mathbf{u}}_\alpha|^2] dx \\ &+ \mu \langle (\nabla a(\epsilon p) \cdot \nabla) \hat{\mathbf{u}}_\alpha, \hat{\mathbf{u}}_\alpha \rangle \\ &+ (\mu + \lambda) \langle (\nabla a(\epsilon p) \operatorname{div} \hat{\mathbf{u}}_\alpha), \hat{\mathbf{u}}_\alpha \rangle dx \end{aligned}$$

$$\equiv d_1 + d_2 + d_3. \quad (3.22)$$

Thanks to the assumption that $\bar{\mu} > 0$ and $2\bar{\mu} + 3\bar{\lambda} > 0$, there exists a positive constant ξ_1 , such that

$$d_1 \geq \underline{a}\xi \int |\nabla \hat{\mathbf{u}}_\alpha|^2 dx, \quad (3.23)$$

while Cauchy-Schwarz's inequality implies

$$|d_2| + |d_3| \leq C(\mathcal{Q})\mathcal{S}. \quad (3.24)$$

Similarly, one can obtain

$$-\langle \bar{\kappa}a(\epsilon p) \operatorname{div}(b(\theta)\nabla \hat{\theta}_\alpha), \hat{\theta}_\alpha \rangle \geq \bar{\kappa}\underline{a}b \|\nabla \hat{\theta}_\alpha\|_{L^2}^2 - C(\mathcal{Q})\mathcal{S}. \quad (3.25)$$

Easily, one has

$$|\langle \epsilon a(\epsilon p) [\bar{\nu} \operatorname{curl} \mathbf{H} : \operatorname{curl} \hat{\mathbf{H}}_\alpha + \Psi(\mathbf{u}) : \nabla \hat{\mathbf{u}}_\alpha], \hat{\theta}_\alpha \rangle| \leq C(\mathcal{Q})(1 + \mathcal{S}).$$

It remains to estimate $\langle g_1, \hat{p}_\alpha \rangle$, $\langle g_2, \hat{u}_\alpha \rangle$ and $\langle g_3, \hat{\theta}_\alpha \rangle$ in (3.21). First, an application of Hölder's inequality gives

$$|\langle \hat{p}_\alpha, g_1 \rangle| \leq C_0 \|\hat{p}_\alpha\|_{L^2} \|g_1\|_{L^2},$$

where $\|g_1\|_{L^2}$ can be bounded, by using (2.6), as follows

$$\begin{aligned} \|g_1\|_{L^2} &= \|[\partial^\alpha, \mathbf{u}] \cdot \nabla(\epsilon p - \theta)\|_{L^2} \\ &\leq C_0 (\|\mathbf{u}\|_{W^{1,\infty}} \|\nabla(\epsilon p - \theta)\|_{H^s} + \|\mathbf{u}\|_{H^{s+1}} \|\nabla(\epsilon p - \theta)\|_{L^\infty}). \end{aligned}$$

It follows from the definition of \mathcal{Q} and Sobolev's inequalities that

$$\|\nabla(\epsilon p, \theta)\|_{H^s} \leq \mathcal{Q}, \quad \|\nabla(\epsilon p, \theta)\|_{L^\infty} \leq \mathcal{Q}.$$

Therefore, we obtain $\|g_1\|_{L^2} \leq C(\mathcal{Q})(1 + \mathcal{S})$, and

$$|\langle p_\alpha, g_1 \rangle| \leq C(\mathcal{Q})(1 + \mathcal{S}). \quad (3.26)$$

Next, we turn to the term $|\langle \mathbf{u}_\alpha, g_2 \rangle|$. Due to the equation (1.13), one has

$$\begin{aligned} -[\partial^\alpha, b(-\theta)] \partial_t(\epsilon \mathbf{u}) &= [\partial^\alpha, b(-\theta)]((\mathbf{u} \cdot \nabla)(\epsilon \mathbf{u})) + [\partial^\alpha, b(-\theta)](b(\theta)\nabla p) \\ &\quad - [\partial^\alpha, b(-\theta)](b(\theta)a(\epsilon p)(\operatorname{curl} \mathbf{H} \times (\epsilon \mathbf{H}))) \\ &\quad - [\partial^\alpha, b(-\theta)](b(\theta)a(\epsilon p)\operatorname{div} \Psi(\epsilon \mathbf{u})). \end{aligned} \quad (3.27)$$

The inequality (2.6) implies that

$$\begin{aligned} &|\langle \mathbf{u}_\alpha, [\partial^\alpha, b(-\theta)]((\mathbf{u} \cdot \nabla)(\epsilon \mathbf{u})) \rangle| \\ &\leq C_0 \|\mathbf{u}_\alpha\|_{L^2} \|[\partial^\alpha, b(-\theta)](\mathbf{u} \cdot \nabla(\epsilon \mathbf{u}))\|_{L^2} \\ &\leq C(\mathcal{Q}) (\|b(-\theta)\|_{W^{1,\infty}} \|\mathbf{u} \cdot \nabla(\epsilon \mathbf{u})\|_{H^s} + \|b(-\theta)\|_{H^{s+1}} \|\mathbf{u} \cdot \nabla(\epsilon \mathbf{u})\|_{L^\infty}) \\ &\leq C(\mathcal{Q}), \end{aligned}$$

and

$$\begin{aligned} &|\langle \mathbf{u}_\alpha, [\partial^\alpha, b(-\theta)](b(\theta)\nabla p) \rangle| \\ &\leq C_0 \|\mathbf{u}_\alpha\|_{L^2} \|[\partial^\alpha, b(-\theta)](b(\theta)\nabla p)\|_{L^2} \\ &\leq C(\mathcal{Q}) (\|b(-\theta)\|_{W^{1,\infty}} \|b(\theta)\nabla p\|_{H^s} + \|b(-\theta)\|_{H^{s+1}} \|b(\theta)\nabla p\|_{L^\infty}) \\ &\leq C(\mathcal{Q})(1 + \mathcal{S}). \end{aligned}$$

The third term on the right-hand side of (3.27) can be treated in a similar manner, and we obtain

$$|\langle \mathbf{u}_\alpha, [\partial^\alpha, b(-\theta)](b(\theta)a(\epsilon p)(\operatorname{curl} \mathbf{H} \times (\epsilon \mathbf{H})) \rangle| \leq C(\mathcal{Q})(1 + \mathcal{S}).$$

To bound the last term on the right-hand side of (3.27), we use (2.6) to deduce that

$$\begin{aligned} & \langle \mathbf{u}_\alpha, [\partial^\alpha, b(-\theta)](b(\theta)a(\epsilon p)\operatorname{div} \Psi(\epsilon \mathbf{u})) \rangle \\ & \leq C_0 \|\mathbf{u}_\alpha\|_{L^2} \|[\partial^\alpha, b(-\theta)](b(\theta)a(\epsilon p)\operatorname{div} \Psi(\epsilon \mathbf{u}))\|_{L^2} \\ & \leq C(\mathcal{Q})(\|b(-\theta)\|_{W^{1,\infty}} \|b(\theta)a(\epsilon p)\operatorname{div} \Psi(\epsilon \mathbf{u})\|_{H^s} \\ & \quad + \|b(-\theta)\|_{H^{s+1}} \|b(\theta)a(\epsilon p)\operatorname{div} \Psi(\epsilon \mathbf{u})\|_{L^\infty}) \\ & \leq C(\mathcal{Q})(1 + \mathcal{S}). \end{aligned}$$

Hence, it holds that

$$|\langle \mathbf{u}_\alpha, g_2 \rangle| \leq C(\mathcal{Q})(1 + \mathcal{S}). \quad (3.28)$$

Since g_3 is similar to g_1 in structure, we easily get

$$|\langle \hat{\theta}_\alpha, g_3 \rangle| \leq C(\mathcal{Q})(1 + \mathcal{S}). \quad (3.29)$$

Therefore, it follows from (3.26), (3.28)–(3.29), the positivity of $b(-\theta)$, and the definition of \mathcal{O} , \mathcal{O}_0 , \mathcal{Q} and \mathcal{S} , that there exists a constant $l_1 > 0$, such that for $t \in [0, T]$ and $T = \min\{T_1, 1\}$,

$$\begin{aligned} & \sup_{\tau \in [0, t]} \|(\hat{p}_\alpha, \hat{\mathbf{u}}_\alpha, \hat{\theta}_\alpha)(\tau)\|_{H^{s+1}}^2 + l_1 \int_0^t \|\nabla(\hat{\mathbf{u}}_\alpha, \hat{\theta}_\alpha)\|_{H^{s+1}}(\tau) d\tau \\ & \leq C(\mathcal{O}_0) + C(\mathcal{O}(t))t + C(\mathcal{O}(t)) \int_0^t \mathcal{S}(\tau) d\tau \\ & \leq C(\mathcal{O}_0) + C(\mathcal{O}(t))\sqrt{t} \\ & \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O}(T))\}. \end{aligned}$$

Summing up the above estimates for all α with $0 \leq |\alpha| \leq s + 1$ leads to the desired inequality (3.12). \square

In a way similar to the proof of Lemma 3.3, we can show that

Lemma 3.4. *Let $s \geq 4$ and $(p, \mathbf{u}, \mathbf{H}, \theta)$ be a solution to (1.12)–(1.15), (1.21) on $[0, T_1]$. Then there exists a constant $l_2 > 0$, such that for any $\epsilon \in (0, 1]$ and $t \in [0, T]$, $T = \min\{T_1, 1\}$, it holds that*

$$\begin{aligned} & \sup_{\tau \in [0, t]} \|(\epsilon^2 q, \epsilon^2 \mathbf{u}, \epsilon(\theta - \bar{\theta}))(\tau)\|_{H^{s+2}} + l_2 \left\{ \int_0^t \|\nabla(\epsilon^2 \mathbf{u}, \epsilon \theta)\|_{H^{s+2}}^2(\tau) d\tau \right\}^{1/2} \\ & \leq C(\mathcal{O}_0) \exp\{(\sqrt{T})C(\mathcal{O}(T))\}, \end{aligned} \quad (3.30)$$

where $\mathcal{O}(T)$ and \mathcal{O}_0 are defined by (2.8) and (2.9), respectively.

Recalling Lemma 2.2 and the definitions of \mathcal{Q} and \mathcal{S} , one finds that

$$\|\partial_t(\epsilon p, \epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta)\|_{H^{s-1}} \leq C(\mathcal{Q}), \quad \|\partial_t(\epsilon p, \epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta)\|_{H^s} \leq C(\mathcal{Q})(1 + \mathcal{S}), \quad (3.31)$$

$$\epsilon \|\partial_t(\epsilon p, \epsilon \mathbf{u}, \epsilon \mathbf{H}, \theta)\|_{H^s} \leq C(\mathcal{Q}). \quad (3.32)$$

Moreover, it follows easily from Lemmas 3.1–3.4 and the equation (1.15) that for some constant $l_3 > 0$, one has

$$\sup_{\tau \in [0, T]} \|\epsilon \partial_t \theta\|_{H^s}^2 + l_3 \int_0^T \|\nabla((\epsilon \partial_t) \theta)\|_{H^s}^2(\tau) d\tau \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O}(T))\}. \quad (3.33)$$

3.3. H^{s-1} estimates on $(\operatorname{div} \mathbf{u}, \nabla p)$. To establish the estimates for p and the acoustic part of \mathbf{u} , we first control the term $(\epsilon \partial_t)(p, \mathbf{u})$. To this end, we start with a L^2 -estimate for the linearized system. For a given state $(p_0, \mathbf{u}_0, \mathbf{H}_0, \theta_0)$, consider the following linearized system of (1.12)–(1.15):

$$\begin{aligned} \partial_t p + (\mathbf{u}_0 \cdot \nabla) p + \frac{1}{\epsilon} \operatorname{div}(2\mathbf{u} - \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \theta) &= \epsilon a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \mathbf{H}] \\ &+ \epsilon a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \mathbf{u} + \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla p_0 \cdot \nabla \theta + f_1, \end{aligned} \quad (3.34)$$

$$b(-\theta_0) [\partial_t \mathbf{u} + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}] + \frac{\nabla p}{\epsilon} = a(\epsilon p_0) [(\operatorname{curl} \mathbf{H}_0) \times \mathbf{H} + \operatorname{div} \Psi(\mathbf{u})] + f_2, \quad (3.35)$$

$$\partial_t \mathbf{H} - \operatorname{curl}(\mathbf{u}_0 \times \mathbf{H}) - \bar{\nu} \Delta \mathbf{H} = f_3, \quad \operatorname{div} \mathbf{H} = 0, \quad (3.36)$$

$$\begin{aligned} \partial_t \theta + (\mathbf{u}_0 \cdot \nabla) \theta + \operatorname{div} \mathbf{u} &= \epsilon^2 a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \mathbf{H} + \Psi(\mathbf{u}_0) : \nabla \mathbf{u}] \\ &+ \bar{\kappa} a(\epsilon p_0) \operatorname{div}(b(\theta_0) \nabla \theta) + f_4, \end{aligned} \quad (3.37)$$

where we have added the source terms f_i ($1 \leq i \leq 4$) on the right-hands sides of (3.34)–(3.37) for latter use, and used the following notations:

$$a(\epsilon p_0) := e^{-\epsilon p_0}, \quad b(\theta_0) := e^{\theta_0}.$$

The system (3.34)–(3.37) is supplemented with initial data

$$(p, \mathbf{u}, \mathbf{H}, \theta)|_{t=0} = (p_{\text{in}}(x), \mathbf{u}_{\text{in}}(x), \mathbf{H}_{\text{in}}(x), \theta_{\text{in}}(x)), \quad x \in \mathbb{R}^3. \quad (3.38)$$

Lemma 3.5. *Let $(p, \mathbf{u}, \mathbf{H}, \theta)$ be a solution to the Cauchy problem (3.34)–(3.38) on $[0, T]$. Then there exist a constant $l_4 > 0$ and an increasing positive function $C(\cdot)$, such that*

$$\begin{aligned} &\sup_{\tau \in [0, T]} \|(p, \mathbf{u}, \mathbf{H})(\tau)\|_{L^2}^2 + l_4 \int_0^T \|\nabla(\mathbf{u}, \mathbf{H})\|_{L^2}^2(\tau) d\tau \\ &\leq e^{TC(R_0)} \|(p, \mathbf{u}, \mathbf{H})(0)\|_{L^2}^2 + C(R_0) e^{TC(R_0)} \sup_{\tau \in [0, T]} \|\nabla \theta(\tau)\|_{L^2}^2 \\ &\quad + C(R_0) \int_0^T \|\nabla(\epsilon \mathbf{u}, \epsilon \mathbf{H})\|_{L^2}^2 d\tau + C(R_0) \int_0^T \|\nabla \theta\|_{H^1}^2(\tau) d\tau \\ &\quad + C(R_0) \int_0^T \{\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 + \|f_3\|_{L^2}^2 + \|\nabla f_4\|_{L^2}^2\}(\tau) d\tau, \end{aligned} \quad (3.39)$$

where

$$R_0 = \sup_{\tau \in [0, T]} \{\|\partial_t \theta_0(\tau)\|_{L^\infty}, \|(p_0, \mathbf{u}_0, \mathbf{H}_0, \theta_0)(\tau)\|_{W^{1, \infty}}\}. \quad (3.40)$$

Proof. Set

$$(\tilde{p}, \tilde{\mathbf{u}}, \tilde{\mathbf{H}}, \tilde{\theta}) = (p, 2\mathbf{u} - \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \theta, \mathbf{H}, \theta). \quad (3.41)$$

Then \tilde{p} and $\tilde{\mathbf{H}}$ satisfy

$$\partial_t \tilde{p} + (\mathbf{u}_0 \cdot \nabla) \tilde{p} + \frac{1}{\epsilon} \operatorname{div} \tilde{\mathbf{u}} = \epsilon a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}] + \frac{\epsilon}{2} a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}}$$

$$+ \frac{\epsilon}{2} a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) + \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla p_0 \cdot \nabla \tilde{\theta} + f_1 \quad (3.42)$$

and

$$\partial_t \tilde{\mathbf{H}} - \operatorname{curl}(\mathbf{u}_0 \times \tilde{\mathbf{H}}) - \bar{\nu} \Delta \tilde{\mathbf{H}} = f_3, \quad \operatorname{div} \tilde{\mathbf{H}} = 0, \quad (3.43)$$

respectively. One can derive the equation for $\tilde{\mathbf{u}}$ by applying the operator ∇ to (3.37) to obtain

$$\begin{aligned} & \partial_t \nabla \tilde{\theta} + (\mathbf{u}_0 \cdot \nabla) \nabla \tilde{\theta} + \frac{1}{2} \nabla \operatorname{div} \tilde{\mathbf{u}} + \frac{1}{2} \nabla \operatorname{div}(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \\ &= \nabla \left\{ \epsilon^2 a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}] \right\} + \frac{1}{2} \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}} \right\} \\ & \quad + \frac{1}{2} \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \right\} \\ & \quad + \nabla \left\{ \bar{\kappa} a(\epsilon p_0) \operatorname{div}(b(\theta_0) \nabla \tilde{\theta}) \right\} + [\nabla, \mathbf{u}_0] \cdot \nabla \tilde{\theta} + \nabla f_4. \end{aligned} \quad (3.44)$$

If we multiply (3.44) with $\frac{1}{2} \bar{\kappa} a(\epsilon p_0)$, we get

$$\begin{aligned} & \frac{1}{2} b(-\theta_0) \left\{ \partial_t (\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) + (\mathbf{u}_0 \cdot \nabla) [\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}] \right\} \\ &= \frac{\bar{\kappa}}{2} b(-\theta_0) \partial_t \{ a(\epsilon p_0) b(\theta_0) \} \nabla \tilde{\theta} + \frac{\bar{\kappa}}{2} b(-\theta_0) \left\{ \mathbf{u}_0 \cdot \nabla [a(\epsilon p_0) b(\theta_0)] \nabla \tilde{\theta} \right\} \\ & \quad - \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \operatorname{div} \tilde{\mathbf{u}} - \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \operatorname{div}(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \\ & \quad + \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}] \right\} + \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}} \right\} \\ & \quad + \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \right\} \\ & \quad + \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \bar{\kappa} a(\epsilon p_0) \operatorname{div}(b(\theta_0) \nabla \tilde{\theta}) \right\} \\ & \quad + \frac{1}{2} \bar{\kappa} a(\epsilon p_0) [\nabla, \mathbf{u}_0] \cdot \nabla \tilde{\theta} + \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla f_4. \end{aligned} \quad (3.45)$$

Subtracting (3.45) from (3.35) yields

$$\begin{aligned} & \frac{1}{2} b(-\theta_0) [\partial_t \tilde{\mathbf{u}} + \mathbf{u}_0 \cdot \nabla \tilde{\mathbf{u}}] + \frac{\nabla \tilde{p}}{\epsilon} \\ &= - \frac{\bar{\kappa}}{2} b(-\theta_0) \partial_t \{ a(\epsilon p_0) b(\theta_0) \} \nabla \tilde{\theta} - \frac{\bar{\kappa}}{2} b(-\theta_0) \left\{ \mathbf{u}_0 \cdot \nabla [a(\epsilon p_0) b(\theta_0)] \nabla \tilde{\theta} \right\} \\ & \quad + \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \operatorname{div} \tilde{\mathbf{u}} + \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \operatorname{div}(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \\ & \quad - \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}] \right\} \\ & \quad - \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}} \right\} \\ & \quad - \frac{1}{4} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \epsilon^2 a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta}) \right\} \\ & \quad - \frac{1}{2} \bar{\kappa} a(\epsilon p_0) \nabla \left\{ \bar{\kappa} a(\epsilon p_0) \operatorname{div}(b(\theta_0) \nabla \tilde{\theta}) \right\} + a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}] \\ & \quad + \frac{1}{2} a(\epsilon p_0) [\Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}}] + \frac{1}{2} a(\epsilon p_0) [\Psi(\mathbf{u}_0) : \nabla(\bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla \tilde{\theta})] \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\bar{\kappa}a(\epsilon p_0)[\nabla, \mathbf{u}_0] \cdot \nabla \tilde{\theta} + a(\epsilon p_0)[(\operatorname{curl} \mathbf{H}_0) \times \tilde{\mathbf{H}}] \\
& + \frac{1}{2}a(\epsilon p_0)\operatorname{div}\Psi(\bar{\kappa}a(\epsilon p_0)b(\theta_0)\nabla\tilde{\theta}) + \frac{1}{2}a(\epsilon p_0)\operatorname{div}\Psi(\tilde{\mathbf{u}}) - \frac{1}{2}\bar{\kappa}a(\epsilon p_0)\nabla f_4 + f_2 \\
& := \sum_{i=1}^{14} h_i + \frac{1}{2}a(\epsilon p_0)\operatorname{div}\Psi(\tilde{\mathbf{u}}) - \frac{1}{2}\bar{\kappa}a(\epsilon p_0)\nabla f_4 + f_2. \tag{3.46}
\end{aligned}$$

Multiplying (3.42) by \tilde{p} , (3.43) by $\tilde{\mathbf{H}}$, and (3.46) by $\tilde{\mathbf{u}}$, respectively, integrating the results over \mathbb{R}^3 , and summing them together, we deduce that

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{1}{2} \langle \tilde{p}, \tilde{p} \rangle + \frac{1}{4} \langle b(-\theta_0) \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle + \frac{1}{2} \langle \tilde{\mathbf{H}}, \mathbf{H} \rangle \right\} + \bar{\nu} \|\nabla \tilde{\mathbf{H}}\|_{L^2}^2 \\
& = - \langle (\mathbf{u}_0 \cdot \nabla) \tilde{p}, \tilde{p} \rangle + \frac{1}{4} \langle \partial_t b(-\theta_0) \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle - \frac{1}{2} \langle b(-\theta_0) (\mathbf{u}_0 \cdot \nabla) \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \\
& \quad + \langle \epsilon a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}], \tilde{p} \rangle + \frac{\epsilon}{2} \langle a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}}, \tilde{p} \rangle \\
& \quad + \frac{\epsilon}{2} \langle a^2(\epsilon p_0) b(\theta_0) \Psi(\mathbf{u}_0) : \nabla(\nabla \tilde{\theta}), \tilde{p} \rangle + \langle \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla p_0 \cdot \nabla \tilde{\theta}, \tilde{p} \rangle \\
& \quad + \sum_{i=1}^{14} \langle h_i, \tilde{\mathbf{u}} \rangle + \frac{1}{2} \langle a(\epsilon p_0) \operatorname{div} \Psi(\tilde{\mathbf{u}}), \tilde{\mathbf{u}} \rangle \\
& \quad - \frac{1}{2} \langle \bar{\kappa} a(\epsilon p_0) \nabla f_4, \tilde{\mathbf{u}} \rangle + \langle f_2, \tilde{\mathbf{u}} \rangle + \langle f_3, \tilde{\mathbf{H}} \rangle + \langle f_1, \tilde{p} \rangle, \tag{3.47}
\end{aligned}$$

where the singular terms have been canceled out.

Now, the terms on the right-hand side of (3.47) can be estimated as follows. First, it follows from the regularity of $(p_0, \mathbf{u}_0, \mathbf{H}_0, \theta_0)$, a partial integration and Cauchy-Schwarz's inequality that

$$\frac{1}{4} |\langle \partial_t b(-\theta_0) \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle| \leq \frac{1}{4} \|\partial_t b(-\theta_0)\|_{L^\infty} \|\tilde{\mathbf{u}}\|_{L^2}^2 \leq C(R_0) \|\tilde{\mathbf{u}}\|_{L^2}^2, \tag{3.48}$$

$$|\langle (\mathbf{u}_0 \cdot \nabla) \tilde{p}, \tilde{p} \rangle| = \frac{1}{2} \left| \int (\operatorname{div} \mathbf{u}_0) |\tilde{p}|^2 dx \right| \leq C(R_0) \|\tilde{p}\|_{L^2}^2, \tag{3.49}$$

$$\frac{1}{2} |\langle b(-\theta_0) (\mathbf{u}_0 \cdot \nabla) \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle| \leq C(R_0) \|\tilde{\mathbf{u}}\|_{L^2}^2, \tag{3.50}$$

$$|\langle \epsilon a(\epsilon p_0) [\bar{\nu} \operatorname{curl} \mathbf{H}_0 : \operatorname{curl} \tilde{\mathbf{H}}], \tilde{p} \rangle| \leq C(R_0) (\|\epsilon \nabla \tilde{\mathbf{H}}\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2), \tag{3.51}$$

$$\frac{\epsilon}{2} |\langle a(\epsilon p_0) \Psi(\mathbf{u}_0) : \nabla \tilde{\mathbf{u}}, \tilde{p} \rangle| \leq C(R_0) (\|\epsilon \nabla \tilde{\mathbf{u}}\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2), \tag{3.52}$$

$$\begin{aligned}
\frac{\epsilon}{2} |\langle a^2(\epsilon p_0) b(\theta_0) \Psi(\mathbf{u}_0) : \nabla(\nabla \tilde{\theta}), \tilde{p} \rangle| & \leq C(R_0) \|\tilde{p}\|_{L^2}^2 \\
& + G_1(\epsilon p_0, \theta_0) \sum_{|\alpha|=2} \|\partial^\alpha(\epsilon \tilde{\theta})\|_{L^2}^2, \tag{3.53}
\end{aligned}$$

$$|\langle \bar{\kappa} a(\epsilon p_0) b(\theta_0) \nabla p_0 \cdot \nabla \tilde{\theta}, \tilde{p} \rangle| \leq C(R_0) (\|\nabla \tilde{\theta}\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2), \tag{3.54}$$

where $G_1(\cdot, \cdot)$ is a smooth function. Similarly, one can bound the terms involving h_i as follows.

$$\begin{aligned}
\sum_{i=1}^{14} |\langle h_i, \tilde{\mathbf{u}} \rangle| & \leq \frac{\bar{\nu}}{8} \|\nabla \tilde{\mathbf{H}}\|_{L^2}^2 + \frac{a\bar{\mu}}{8} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 + \frac{a\bar{\nu}}{8} \|\operatorname{div} \tilde{\mathbf{u}}\|_{L^2}^2 \\
& + C(R_0) \|\tilde{\mathbf{u}}\|_{L^2}^2 + C(R_0) \|\nabla(\epsilon \tilde{\mathbf{u}}, \tilde{\theta})\|_{L^2}^2 + G_2(\epsilon p_0, \theta_0) \|\Delta \tilde{\theta}\|_{L^2}^2, \tag{3.55}
\end{aligned}$$

where $G_2(\cdot, \cdot)$ is a smooth function.

For the dissipative term $\frac{1}{2}\langle a(\epsilon p_0)\text{div}\Psi(\tilde{\mathbf{u}}), \tilde{\mathbf{u}} \rangle$, we can employ arguments similar to those used in the estimate of the slow motion in (3.22)–(3.24) to obtain that

$$-\frac{1}{2}\langle a(\epsilon p_0)\text{div}\Psi(\hat{\mathbf{u}}), \hat{\mathbf{u}} \rangle \geq \frac{a\bar{\mu}}{4}(\|\nabla\hat{\mathbf{u}}\|_{L^2}^2 + \|\text{div}\hat{\mathbf{u}}\|_{L^2}^2) - C(R_0)\|\hat{\mathbf{u}}\|_{L^2}^2. \quad (3.56)$$

Finally, it follows from (3.47)–(3.56), and Cauchy-Schwarz's and Gronwall's inequalities that (3.39) holds. \square

In the next lemma we utilize Lemma 3.5 to estimate $((\epsilon\partial_t)p, (\epsilon\partial_t)\mathbf{u}, (\epsilon\partial_t)\mathbf{H})$.

Lemma 3.6. *Let $s \geq 4$ and $(p, \mathbf{u}, \mathbf{H}, \theta)$ be the solution to the Cauchy problem (1.12)–(1.15), (1.21) on $[0, T]$. Set*

$$(p_\beta, \mathbf{u}_\beta, \mathbf{H}_\beta, \theta_\beta) := \partial^\beta((\epsilon\partial_t)p, (\epsilon\partial_t)\mathbf{u}, (\epsilon\partial_t)\mathbf{H}, (\epsilon\partial_t)\theta),$$

where $1 \leq |\beta| \leq s-1$. Then there exist a constant $l_5 > 0$ and an increasing positive function $C(\cdot)$, such that

$$\begin{aligned} \sup_{\tau \in [0, T]} \|(p_\beta, \mathbf{u}_\beta, \mathbf{H}_\beta)(\tau)\|_{L^2}^2 + l_5 \int_0^T \|\nabla(\mathbf{u}_\beta, \mathbf{H}_\beta)\|_{L^2}^2(\tau) d\tau \\ \leq C(\mathcal{O}_0) \exp\{(\sqrt{T})C(\mathcal{O}(T))\}. \end{aligned} \quad (3.57)$$

Proof. An application of the operator $\partial^\beta(\epsilon\partial_t)$ to the system (1.12)–(1.15) leads to

$$\begin{aligned} \partial_t p_\beta + (\mathbf{u} \cdot \nabla)p_\beta + \frac{1}{\epsilon}\text{div}(2\mathbf{u}_\beta - \bar{\kappa}a(\epsilon p)b(\theta)\nabla\theta_\beta) = \epsilon a(\epsilon p)[\bar{\nu} \text{curl} \mathbf{H} : \text{curl} \mathbf{H}_\beta] \\ + \epsilon a(\epsilon p)\Psi(\mathbf{u}) : \nabla\mathbf{u}_\beta + \bar{\kappa}a(\epsilon p)b(\theta)\nabla p \cdot \nabla\theta_\beta + \tilde{g}_1, \end{aligned} \quad (3.58)$$

$$b(-\theta)[\partial_t \mathbf{u}_\beta + (\mathbf{u} \cdot \nabla)\mathbf{u}_\beta] + \frac{\nabla p_\beta}{\epsilon} = a(\epsilon p)[(\text{curl} \mathbf{H}) \times \mathbf{H}_\beta + \text{div}\Psi(\mathbf{u}_\beta)] + \tilde{g}_2, \quad (3.59)$$

$$\partial_t \mathbf{H}_\beta - \text{curl}(\mathbf{u} \times \mathbf{H}_\beta) - \bar{\nu}\Delta \mathbf{H}_\beta = \tilde{g}_3, \quad \text{div} \mathbf{H}_\beta = 0, \quad (3.60)$$

$$\begin{aligned} \partial_t \theta_\beta + (\mathbf{u} \cdot \nabla)\theta_\beta + \text{div}\mathbf{u}_\beta = \epsilon^2 a(\epsilon p)[\bar{\nu} \text{curl} \mathbf{H} : \text{curl} \mathbf{H}_\beta] \\ + \epsilon^2 a(\epsilon p)\Psi(\mathbf{u}) : \nabla\mathbf{u}_\beta + \bar{\kappa}a(\epsilon p)\text{div}(b(\theta)\nabla\theta_\beta) + \tilde{g}_4, \end{aligned} \quad (3.61)$$

where

$$\begin{aligned} \tilde{g}_1 := & -[\partial^\beta(\epsilon\partial_t), \mathbf{u}] \cdot \nabla p + \frac{1}{\epsilon}[\partial^\beta(\epsilon\partial_t), (\bar{\kappa}a(\epsilon p)b(\theta))]\Delta\theta \\ & + \frac{1}{\epsilon}[\partial^\beta(\epsilon\partial_t), \nabla(\bar{\kappa}a(\epsilon p)b(\theta))] \cdot \nabla\theta + \epsilon\bar{\nu}[\partial^\beta(\epsilon\partial_t), a(\epsilon p)\text{curl} \mathbf{H}] : \text{curl} \mathbf{H} \\ & + \epsilon[\partial^\beta(\epsilon\partial_t), a(\epsilon p)\Psi(\mathbf{u})] : \nabla\mathbf{u} + [\partial^\beta(\epsilon\partial_t), \bar{\kappa}a(\epsilon p)b(\theta)\nabla p] \cdot \nabla\theta, \end{aligned} \quad (3.62)$$

$$\begin{aligned} \tilde{g}_2 := & -[\partial^\beta(\epsilon\partial_t), b(-\theta)]\partial_t \mathbf{u} - [\partial^\beta(\epsilon\partial_t), b(-\theta)\mathbf{u}] \cdot \nabla\mathbf{u}, \\ & + [\partial^\beta(\epsilon\partial_t), a(\epsilon p)\text{curl} \mathbf{H}] \times \mathbf{H} + [\partial^\beta(\epsilon\partial_t), a(\epsilon p)]\text{div}\Psi(\mathbf{u}), \end{aligned} \quad (3.63)$$

$$\tilde{g}_3 := \partial^\beta(\epsilon\partial_t)(\text{curl}(\mathbf{u} \times \mathbf{H})) - \text{curl}(\mathbf{u} \times \mathbf{H}_\beta), \quad (3.64)$$

$$\begin{aligned} \tilde{g}_4 := & -[\partial^\beta(\epsilon\partial_t), \mathbf{u}] \cdot \nabla\theta + \epsilon^2\bar{\nu}[\partial^\beta(\epsilon\partial_t), a(\epsilon p)\text{curl} \mathbf{H}] : \text{curl} \mathbf{H} \\ & + \epsilon^2[\partial^\beta(\epsilon\partial_t), a(\epsilon p)\Psi(\mathbf{u})] : \nabla\mathbf{u} \\ & + \bar{\kappa}\partial^\beta(\epsilon\partial_t)(a(\epsilon p)\text{div}(b(\theta)\nabla\theta_\alpha)) - \bar{\kappa}a(\epsilon)\text{div}(b(\theta)\nabla\theta_\alpha). \end{aligned} \quad (3.65)$$

It follows from the linear estimate (3.39) that for some $l_4 > 0$,

$$\begin{aligned}
& \sup_{\tau \in [0, T]} \|(p_\beta, \mathbf{u}_\beta, \mathbf{H}_\beta)(\tau)\|_{L^2}^2 + l_4 \int_0^T \|\nabla(\mathbf{u}_\beta, \mathbf{H}_\beta)\|_{L^2}^2(\tau) d\tau \\
& \leq e^{TC(R)} \|(p_\beta, \tilde{\mathbf{u}}_\beta, \mathbf{H}_\beta)(0)\|_{L^2}^2 + C(R) e^{TC(R)} \sup_{\tau \in [0, T]} \|\nabla \theta_\beta(\tau)\|_{L^2}^2 \\
& \quad + TC(R) \sup_{\tau \in [0, T]} \|\nabla(\epsilon \mathbf{u}_\beta, \epsilon \mathbf{H}_\beta)(\tau)\|_{L^2}^2 + C(R) \int_0^T \|\nabla \theta_\beta\|_{H^1}^2(\tau) d\tau \\
& \quad + C(R) \int_0^T \{ \|\tilde{g}_1\|_{L^2}^2 + \|\tilde{g}_2\|_{L^2}^2 + \|\tilde{g}_3\|_{L^2}^2 + \|\nabla \tilde{g}_4\|_{L^2}^2 \}(\tau) d\tau, \tag{3.66}
\end{aligned}$$

where R is defined as R_0 in (3.40) with $(p_0, \mathbf{u}_0, \mathbf{H}_0, \theta_0)$ replaced with $(p, \mathbf{u}, \mathbf{H}, \theta)$.

It remains to control the terms $\|\tilde{g}_1\|_{L^2}^2$, $\|\tilde{g}_2\|_{L^2}^2$, $\|\tilde{g}_3\|_{L^2}^2$, and $\|\nabla \tilde{g}_4\|_{L^2}^2$. The first term of \tilde{g}_1 can be bounded as follows.

$$\begin{aligned}
\|[\partial^\beta(\epsilon \partial_t), \mathbf{u}] \cdot \nabla p\|_{L^2} & \leq \epsilon C_0 (\|\mathbf{u}\|_{H^{s-1}} \|(\epsilon \partial_t) \nabla p\|_{H^{s-2}} + \|(\epsilon \partial_t) \mathbf{u}\|_{H^{s-1}} \|\nabla p\|_{H^{s-1}}) \\
& \leq C(\mathcal{Q}). \tag{3.67}
\end{aligned}$$

Similarly, the second term of \tilde{g}_1 admits the following bound:

$$\begin{aligned}
& \frac{1}{\epsilon} \|[\partial^\beta(\epsilon \partial_t), (\bar{\kappa} a(\epsilon p) b(\theta))] \Delta \theta\| \\
& \leq C_0 (\|a(\epsilon p) b(\theta)\|_{H^{s-1}} \|\partial_t \Delta \theta\|_{H^{s-2}} + \|\partial_t (a(\epsilon p) b(\theta))\|_{H^{s-1}} \|\Delta \theta\|_{H^{s-1}}) \\
& \leq C(\mathcal{Q})(1 + \mathcal{S}). \tag{3.68}
\end{aligned}$$

The other four terms in \tilde{g}_1 can be treated similarly and hence can be bounded by $C(\mathcal{Q})(1 + \mathcal{S})$.

For the first term of \tilde{g}_2 , one has by the equation (1.13) that

$$\begin{aligned}
[\partial^\beta(\epsilon \partial_t), b(-\theta)] \partial_t \mathbf{u} & = [\partial^\beta(\epsilon \partial_t), b(-\theta)] \{(\mathbf{u} \cdot \nabla) \mathbf{u}\} \\
& \quad + \frac{1}{\epsilon} [\partial^\beta(\epsilon \partial_t), b(-\theta)] \{b^{-1}(-\theta_0) \nabla p\} \\
& \quad - [\partial^\beta(\epsilon \partial_t), b(-\theta)] \{b^{-1}(-\theta) a(\epsilon p) [(\operatorname{curl} \mathbf{H}_0) \times \mathbf{H}]\} \\
& \quad - [\partial^\beta(\epsilon \partial_t), b(-\theta)] \{b^{-1}(-\theta) a(\epsilon p) \operatorname{div} \Psi(\mathbf{u})\}. \tag{3.69}
\end{aligned}$$

Note that the terms on the right-hand side of (3.69) have similar structure as that of \tilde{g}_1 . Thus, we see that

$$\|[\partial^\beta(\epsilon \partial_t), b(-\theta)] \partial_t \mathbf{u}\|_{L^2} \leq C(\mathcal{Q})(1 + \mathcal{S}). \tag{3.70}$$

Similarly, the other four terms of \tilde{g}_2 can be bounded by $C(\mathcal{Q})(1 + \mathcal{S})$.

Next, by the identity (2.4), one can rewrite \tilde{g}_3 as

$$\tilde{g}_3 = -[\partial^\beta(\epsilon \partial_t), \operatorname{div} \mathbf{u}] \mathbf{H} - [\partial^\beta(\epsilon \partial_t), \mathbf{u}] \cdot \nabla \mathbf{H} + \sum_{i=1}^3 [\partial^\beta(\epsilon \partial_t), \nabla \mathbf{u}_i] \mathbf{H}. \tag{3.71}$$

Following a process similar to that in the estimate of \tilde{g}_1 , one gets

$$\|\tilde{g}_3\|_{L^2} \leq C(\mathcal{Q})(1 + \mathcal{S}). \tag{3.72}$$

And analogously,

$$\|\nabla \tilde{g}_4\|_{L^2} \leq C(\mathcal{Q})(1 + \mathcal{S}). \tag{3.73}$$

We proceed to control the other terms on the right-hand side of (3.66). It follows from (3.33) that

$$C(R)e^{TC(R)} \sup_{\tau \in [0, T]} \|\nabla \theta_\beta(\tau)\|_{L^2}^2 \leq C(\mathcal{O}) \exp\{\sqrt{T}C(\mathcal{O}(T))\}$$

and

$$\int_0^T \|\Delta \theta_\beta\|_{L^2}^2(\tau) d\tau \leq \int_0^T \|(\epsilon \partial_t) \theta\|_{H^{s+1}}^2(\tau) d\tau \leq C(\mathcal{O}_0) \exp\{\sqrt{T}C(\mathcal{O}(T))\}.$$

Thanks to (3.32), one has

$$\begin{aligned} TC(R) \sup_{\tau \in [0, T]} \|\nabla(\epsilon \mathbf{u}_\beta, \epsilon \mathbf{H}_\beta)(\tau)\|_{L^2}^2 &\leq TC(\mathcal{O}(T)) \sup_{\tau \in [0, T]} \|(\epsilon \partial_t)(\epsilon \mathbf{u}, \epsilon \mathbf{H})(\tau)\|_{H^s}^2 \\ &\leq TC(\mathcal{O}(T)). \end{aligned} \quad (3.74)$$

Then, the desired inequality (3.57) follows from the above estimates and the inequality (3.66). \square

Now we are in a position to estimate the Sobolev norm of $(\operatorname{div} \mathbf{u}, \nabla p)$ based on Lemma 3.6.

Lemma 3.7. *Let $s \geq 4$ and $(p, \mathbf{u}, \mathbf{H}, \theta)$ be the solution to the Cauchy problem (1.12)–(1.15), (1.21) on $[0, T]$. Then there exist a constant l_6 and an increasing function $C(\cdot)$, such that*

$$\begin{aligned} \sup_{\tau \in [0, T]} \{ \|p(\tau)\|_{H^s} + \|\operatorname{div} \mathbf{u}(\tau)\|_{H^{s-1}} \} + l_6 \int_0^T \{ \|\nabla p\|_{H^s}^2 + \|\nabla \operatorname{div} \mathbf{u}\|_{H^{s-1}}^2 \}(\tau) d\tau \\ \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \end{aligned} \quad (3.75)$$

Proof. Rewrite the equations (1.12) and (1.13) as

$$\begin{aligned} \operatorname{div} \mathbf{u} &= -\frac{1}{2}(\epsilon \partial_t) p - \frac{\epsilon}{2}(\mathbf{u} \cdot \nabla) p + \frac{1}{2} \operatorname{div}(\bar{\kappa} a(\epsilon p) b(\theta) \nabla \theta) + \frac{\epsilon^2 \bar{\nu}}{2} a(\epsilon p_0) |\operatorname{curl} \mathbf{H}|^2 \\ &\quad + \frac{\epsilon^2}{2} a(\epsilon p) \Psi(\mathbf{u}) : \nabla \mathbf{u} + \frac{\epsilon \bar{\kappa}}{2} a(\epsilon p) b(\theta) \nabla p \cdot \nabla \theta, \end{aligned} \quad (3.76)$$

$$\begin{aligned} \nabla p &= -b(-\theta)(\epsilon \partial_t) \mathbf{u} - \epsilon b(-\theta)(\mathbf{u} \cdot \nabla) \mathbf{u} \\ &\quad + \epsilon a(\epsilon p) [(\operatorname{curl} \mathbf{H}) \times \mathbf{H}] + \epsilon a(\epsilon p) \operatorname{div} \Psi(\mathbf{u}). \end{aligned} \quad (3.77)$$

Then,

$$\begin{aligned} \|\operatorname{div} \mathbf{u}\|_{H^{s-1}} &\leq C_0 \|(\epsilon \partial_t) p\|_{H^{s-1}} + C_0 \epsilon \|\mathbf{u}\|_{H^{s-1}} \|\nabla p\|_{H^{s-1}} \\ &\quad + C_0 \|\operatorname{div}(\bar{\kappa} a(\epsilon p) b(\theta) \nabla \theta)\|_{H^{s-1}} + C_0 \|a(\epsilon p_0)\|_{L^\infty} \|\epsilon \operatorname{curl} \mathbf{H}\|_{H^{s-1}}^2 \\ &\quad + C_0 \|a(\epsilon p)\|_{L^\infty} \|\Psi(\epsilon \mathbf{u}) : (\epsilon \nabla \mathbf{u})\|_{H^{s-1}} \\ &\quad + C_0 \|a(\epsilon p) b(\theta)\|_{L^\infty} \|(\epsilon \nabla p)\|_{H^{s-1}} \|\nabla \theta\|_{H^{s-1}}. \end{aligned} \quad (3.78)$$

It follows from Lemmas 3.2–3.4 and 3.6, and the inequalities (3.31)–(3.33) that

$$\begin{aligned} \|(\epsilon \partial_t) p\|_{H^{s-1}} &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}, \\ \epsilon \|\mathbf{u}\|_{H^{s-1}} \|\nabla p\|_{H^{s-1}} &\leq \epsilon C(\mathcal{O}), \\ \|\operatorname{div}(\bar{\kappa} a(\epsilon p) b(\theta) \nabla \theta)\|_{H^{s-1}} &\leq C_0 \|\Delta \theta\|_{H^{s-1}} + C_0 \|\nabla \theta\|_{H^{s-1}} \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}, \\ \|\epsilon \operatorname{curl} \mathbf{H}\|_{H^{s-1}} &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}, \end{aligned}$$

$$\begin{aligned} \|\Psi(\epsilon \mathbf{u}) : (\epsilon \nabla \mathbf{u})\|_{H^{s-1}} &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}, \\ \|(\epsilon \nabla p)\|_{H^{s-1}} \|\nabla \theta\|_{H^{s-1}} &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \end{aligned}$$

These bounds together with (3.78) imply that

$$\sup_{\tau \in [0, T]} \|\operatorname{div} \mathbf{u}\|(\tau)_{H^{s-1}} \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \quad (3.79)$$

Similar arguments applying to the equation (3.77) for ∇p yield

$$\sup_{\tau \in [0, T]} \|p\|_{H^s} + l_6 \int_0^T \|\nabla p\|_{H^s}^2(\tau) d\tau \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\} \quad (3.80)$$

for some positive constant $l_6 > 0$.

To obtain the desired inequality (3.75), we shall establish the following estimate

$$\int_0^T \|\nabla \operatorname{div} \mathbf{u}\|_{H^{s-1}}^2(\tau) d\tau \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \quad (3.81)$$

To this end, for any multi-index α satisfying $1 \leq |\alpha| \leq s$, one can apply ∂^α to (3.75) and then take the inner product with $\partial^\alpha \operatorname{div} \mathbf{u}$ to obtain

$$\begin{aligned} \int_0^T \|\partial^\alpha \operatorname{div} \mathbf{u}\|_{L^2}^2(\tau) d\tau &= -\frac{1}{2} \int_0^T \langle \partial^\alpha (\epsilon \partial_t) p, \partial^\alpha \operatorname{div} \mathbf{u} \rangle(\tau) d\tau \\ &\quad + \int_0^T \langle \Xi, \partial^\alpha \operatorname{div} \mathbf{u} \rangle(\tau) d\tau, \end{aligned} \quad (3.82)$$

where

$$\begin{aligned} \Xi &= -\frac{\epsilon}{2} (\mathbf{u} \cdot \nabla) p + \frac{1}{2} \operatorname{div}(\bar{\kappa} a(\epsilon p) b(\theta) \nabla \theta) + \frac{\epsilon^2 \bar{\nu}}{2} a(\epsilon p_0) |\operatorname{curl} \mathbf{H}|^2 \\ &\quad + \frac{\epsilon^2}{2} a(\epsilon p) \Psi(\mathbf{u}) : \nabla \mathbf{u} + \frac{\epsilon \bar{\kappa}}{2} a(\epsilon p) b(\theta) \nabla p \cdot \nabla \theta. \end{aligned}$$

It thus follows from (3.57) and similar arguments to those for (3.80), that for all $1 \leq |\alpha| \leq s$,

$$\int_0^T \|\partial^\alpha \Xi(\tau)\|_{L^2}^2 d\tau \leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\},$$

whence,

$$\begin{aligned} &\int_0^T |\langle \Xi, \partial^\alpha \operatorname{div} \mathbf{u} \rangle|(\tau) d\tau \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\} \left\{ \int_0^T \|\partial^\alpha \operatorname{div} \mathbf{u}\|_{L^2}^2(\tau) d\tau \right\}^{1/2} \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\} + \frac{1}{4} \int_0^T \|\partial^\alpha \operatorname{div} \mathbf{u}\|_{L^2}^2(\tau) d\tau. \end{aligned}$$

For the first term on the right-hand side of (3.82), one gets by integration by parts that

$$\begin{aligned} -\frac{1}{2} \int_0^T \langle \partial^\alpha (\epsilon \partial_t) p, \partial^\alpha \operatorname{div} \mathbf{u} \rangle(\tau) d\tau &= -\frac{1}{2} \langle \partial^\alpha p, \epsilon \partial^\alpha \operatorname{div}(\epsilon \mathbf{u}) \rangle \Big|_0^T \\ &\quad + \frac{1}{2} \int_0^T \langle \partial^\alpha \nabla p, \partial^\alpha (\epsilon \partial_t) \mathbf{u} \rangle(\tau) d\tau. \end{aligned}$$

By virtue of the estimate (3.12) on $(\epsilon q, \epsilon \mathbf{u}, \theta - \bar{\theta})$ and (3.80), we find that

$$\begin{aligned} \left| \frac{1}{2} \langle \partial^\alpha p, \epsilon \partial^\alpha \operatorname{div}(\epsilon \mathbf{u}) \rangle \Big|_0^T \right| &\leq \sup_{\tau \in [0, T]} \{ \|p\|_{H^s} \|\epsilon \mathbf{u}\|_{H^{s+1}} \} \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}, \\ \frac{1}{2} \left| \int_0^T \langle \partial^\alpha \nabla p, \partial^\alpha (\epsilon \partial_t \mathbf{u})(\tau) \rangle d\tau \right| &\leq \frac{1}{2} \left\{ \int_0^T \|\partial^\alpha \nabla p\|_{L^2}^2(\tau) d\tau \right\}^{1/2} \\ &\quad \cdot \left\{ \int_0^T \|\partial^\alpha (\epsilon \partial_t \mathbf{u})\|_{L^2}^2(\tau) d\tau \right\}^{1/2} \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \end{aligned}$$

These bounds, together with (3.82), yield the desired estimate (3.81). This completes the proof. \square

3.4. H^{s-1} estimate on $\operatorname{curl} \mathbf{u}$. The another key point to obtain a uniform bound for \mathbf{u} is the following estimate on $\operatorname{curl} \mathbf{u}$.

Lemma 3.8. *Let $s > 5/2$ be an integer and $(p, \mathbf{u}, \mathbf{H}, \theta)$ be the solution to the Cauchy problem (1.12)–(1.15), (1.21) on $[0, T]$. Then there exist a constant $l_6 > 0$ and an increasing function $C(\cdot)$, such that*

$$\begin{aligned} &\sup_{\tau \in [0, T]} \{ \|\operatorname{curl}(b(-\theta)\mathbf{u})(\tau)\|_{H^{s-1}}^2 + \|\operatorname{curl} \mathbf{H}(\tau)\|_{H^{s-1}}^2 \} \\ &+ l_6 \int_0^T \{ \|\nabla \operatorname{curl}(b(-\theta)\mathbf{u})\|_{H^{s-1}}^2 + \|\nabla \operatorname{curl} \mathbf{H}(\tau)\|_{H^{s-1}}^2 \}(\tau) d\tau \\ &\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}. \end{aligned} \quad (3.83)$$

Proof. Applying curl to the equations (1.13) and (1.14), using the identities (2.1) and (2.2), and the fact that $\operatorname{curl} \nabla = 0$, one infers

$$\begin{aligned} &\partial_t(\operatorname{curl}(b(-\theta)\mathbf{u})) + (\mathbf{u} \cdot \nabla)(\operatorname{curl}(b(-\theta)\mathbf{u})) \\ &= \operatorname{curl} \{ a(\epsilon p)(\operatorname{curl} \mathbf{H}) \times \mathbf{H} \} + \bar{\mu} \operatorname{div} \{ a(\epsilon p) b(\theta) \nabla(\operatorname{curl}(b(-\theta)\mathbf{u})) \} + \Upsilon_1, \end{aligned} \quad (3.84)$$

$$\partial_t(\operatorname{curl} \mathbf{H}) - \operatorname{curl}[\operatorname{curl}(\mathbf{u} \times \mathbf{H})] - \bar{\nu} \Delta(\operatorname{curl} \mathbf{H}) = 0, \quad (3.85)$$

where Υ_1 is defined by

$$\begin{aligned} \Upsilon_1 := &\bar{\mu} \operatorname{div}(a(\epsilon p)(\nabla b(\theta)) \otimes \operatorname{curl}(b(-\theta)\mathbf{u})) - \bar{\mu} \nabla a(\epsilon p) \cdot \nabla(b(\theta) \operatorname{curl}(b(-\theta)\mathbf{u})) \\ &- \bar{\mu} a(\epsilon p) \Delta((\nabla b(\theta)) \times (b(-\theta)\mathbf{u})) - \nabla a(\epsilon p) \times (\bar{\mu} \Delta \mathbf{u} + (\bar{\mu} + \bar{\lambda}) \nabla \operatorname{div} \mathbf{u}) \\ &+ \operatorname{curl}(b(-\theta)\mathbf{u} \partial_t \theta) + [\operatorname{curl}, \mathbf{u}] \cdot \nabla(b(-\theta)\mathbf{u}) + \operatorname{curl}(b(-\theta)\mathbf{u}(\mathbf{u} \cdot \nabla \theta)). \end{aligned}$$

For any multi-index α satisfying $0 \leq |\alpha| \leq s - 1$, we apply ∂^α to (3.84) and (3.85) to obtain

$$\begin{aligned} &\partial_t \partial^\alpha(\operatorname{curl}(b(-\theta)\mathbf{u})) + (\mathbf{u} \cdot \nabla)[\partial^\alpha(\operatorname{curl}(b(-\theta)\mathbf{u}))] \\ &= \partial^\alpha \operatorname{curl} \{ a(\epsilon p)(\operatorname{curl} \mathbf{H}) \times \mathbf{H} \} \\ &\quad + \bar{\mu} \operatorname{div} \{ a(\epsilon p) b(\theta) \nabla[\partial^\alpha(\operatorname{curl}(b(-\theta)\mathbf{u}))] \} + \partial^\alpha \Upsilon_1 + \Upsilon_2, \end{aligned} \quad (3.86)$$

$$\partial_t \partial^\alpha(\operatorname{curl} \mathbf{H}) - \partial^\alpha \operatorname{curl}[\operatorname{curl}(\mathbf{u} \times \mathbf{H})] - \bar{\nu} \Delta(\operatorname{curl} \mathbf{H}) = 0, \quad (3.87)$$

where

$$\Upsilon_2 := -[\partial^\alpha, \mathbf{u}] \cdot \nabla[\partial^\alpha(\operatorname{curl}(b(-\theta)\mathbf{u}))]$$

$$- [\partial^\alpha, \operatorname{div}(a(\epsilon p)b(\theta))] \nabla [\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))].$$

Multiplying (3.86) by $\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))$ and (3.87) by $\partial^\alpha (\operatorname{curl} \mathbf{H})$ respectively, summing up, and integrating over \mathbb{R}^3 , we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ \|\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))\|_{L^2}^2 + \|\partial^\alpha (\operatorname{curl} \mathbf{H})\|_{L^2}^2 \} + \bar{\nu} \|\partial^\alpha (\operatorname{curl} \mathbf{H})\|_{L^2}^2 \\ & + \langle a(\epsilon p)b(\theta) \nabla [\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))], \nabla [\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))] \rangle \\ & = - \langle (\mathbf{u} \cdot \nabla) [\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))], \partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u})) \rangle \\ & \quad - \langle \partial^\alpha \operatorname{curl} \{ a(\epsilon p) (\operatorname{curl} \mathbf{H}) \times \mathbf{H} \}, \partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u})) \rangle \\ & \quad + \langle \partial^\alpha \operatorname{curl} [\operatorname{curl}(\mathbf{u} \times \mathbf{H})], \partial^\alpha (\operatorname{curl} \mathbf{H}) \rangle + \langle \partial^\alpha \Upsilon_1 + \Upsilon_2, \partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u})) \rangle \\ & := \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4, \end{aligned} \tag{3.88}$$

where \mathcal{J}_i ($i = 1, \dots, 4$) will be bounded as follows.

An integration by parts leads to

$$|\mathcal{J}_1| \leq \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\nabla [\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))]\|_{L^2}^2.$$

By virtue of (2.3), the Cauchy-Schwarz's inequality and Moser-type inequality (see [33]), the term \mathcal{J}_2 can be bounded by

$$\begin{aligned} |\mathcal{J}_2| & \leq |\langle \partial^\alpha \{ a(\epsilon p) (\operatorname{curl} \mathbf{H}) \times \mathbf{H} \}, \partial^\alpha \operatorname{curl} (\operatorname{curl}(b(-\theta)\mathbf{u})) \rangle| \\ & \leq \|\partial^\alpha \{ a(\epsilon p) (\operatorname{curl} \mathbf{H}) \times \mathbf{H} \}\|_{L^2} \|\partial^\alpha \nabla (\operatorname{curl}(b(-\theta)\mathbf{u}))\|_{L^2} \\ & \leq \eta_2 \|\partial^\alpha \nabla (\operatorname{curl}(b(-\theta)\mathbf{u}))\|_{L^2}^2 \\ & \quad + C_\eta \{ \|\operatorname{curl} \mathbf{H}\|_{L^\infty}^2 \|a(\epsilon p) \mathbf{H}\|_{H^{s-1}}^2 + \|a(\epsilon p) \mathbf{H}\|_{L^\infty}^2 \|\operatorname{curl} \mathbf{H}\|_{H^{s-1}}^2 \}, \end{aligned}$$

where $\eta_2 > 0$ is a sufficiently small constant independent of ϵ .

If we integrate by parts, make use of (2.3), $\operatorname{curl} \operatorname{curl} \mathbf{a} = \nabla \operatorname{div} \mathbf{a} - \Delta \mathbf{a}$ and the fact that $\operatorname{div} \mathbf{H} = 0$, we see that the term \mathcal{J}_3 can be rewritten as

$$\mathcal{J}_3 = \langle \partial^\alpha \operatorname{curl}(\mathbf{u} \times \mathbf{H}), \partial^\alpha \Delta \mathbf{H} \rangle,$$

which, together with the Moser-type inequality, implies that

$$|\mathcal{J}_3| \leq C(\mathcal{S}) + \eta_3 \|\mathbf{H}^\epsilon(\tau)\|_{s+1}^2,$$

where $\eta_3 > 0$ is a sufficiently small constant independent of ϵ .

To handle \mathcal{J}_4 , we note that the leading order terms in Υ_1 are of third-order in θ and of second-order in \mathbf{u} , and the leading order terms in Υ_2 are of order $s+1$ in \mathbf{u} and of order $s+1$ in $(\epsilon p, \theta)$. Then it follows that

$$\begin{aligned} |\mathcal{J}_4| & \leq C_0 (\|\partial^\alpha \Upsilon_1\|_{L^2} + \|\Upsilon_2\|_{L^2}) \|\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))\|_{L^2} \\ & \leq C(\mathcal{S}) \|\partial^\alpha (\operatorname{curl}(b(-\theta)\mathbf{u}))\|_{L^2}. \end{aligned}$$

Putting the above estimates into the (3.88), choosing η_2 and η_3 sufficient small, summing over α for $0 \leq |\alpha| \leq s-1$, and then integrating the result on $[0, T]$, we conclude

$$\begin{aligned} & \sup_{\tau \in [0, T]} \{ \|\operatorname{curl}(b(-\theta)\mathbf{u})(\tau)\|_{H^{s-1}}^2 + \|\operatorname{curl} \mathbf{H}(\tau)\|_{H^{s-1}}^2 \} \\ & + l_1 \int_0^T \{ \|\nabla \operatorname{curl}(b(-\theta)\mathbf{u})\|_{H^{s-1}}^2 + \|\nabla \operatorname{curl} \mathbf{H}(\tau)\|_{H^{s-1}}^2 \} (\tau) d\tau \\ & \leq C_0 \{ \|\operatorname{curl}(b(-\theta)\mathbf{u})(0)\|_{H^{s-1}}^2 + \|\operatorname{curl} \mathbf{H}(0)\|_{H^{s-1}}^2 \} \\ & \quad + C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\} \end{aligned}$$

$$\leq C(\mathcal{O}_0) \exp\{(\sqrt{T} + \epsilon)C(\mathcal{O}(T))\}.$$

□

Proof of Proposition 2.4. By the definition of the norm $\|\cdot\|_{s,\epsilon}$ and the fact

$$\|\mathbf{v}\|_{H^{m+1}} \leq K(\|\operatorname{div}\mathbf{v}\|_{H^m} + \|\operatorname{curl}\mathbf{v}\|_{H^m} + \|\mathbf{v}\|_{H^m}), \quad \forall \mathbf{v} \in H^{m+1}(\mathbb{R}^3),$$

Proposition 2.4 follows directly from Lemmas 3.1, 3.3, 3.7 and 3.8. □

Once Proposition 2.4 is established, the existence part of Theorem 1.1 can be proved by applying the arguments in [2, 39] directly, and hence we omit the details here.

4. DECAY OF THE LOCAL ENERGY AND ZERO MACH NUMBER LIMIT

In this section, we shall prove the convergence part of Theorem 1.1 by modifying the arguments developed by Métivier and Schochet [39], see also some extensions in [1, 2, 37].

Proof of the convergence part of Theorem 1.1. The uniform estimate (1.24) implies that

$$\sup_{\tau \in [0, T_0]} \|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(\tau)\|_{H^s} + \sup_{\tau \in [0, T_0]} \|\theta^\epsilon - \bar{\theta}\|_{H^{s+1}} < +\infty.$$

Thus, after extracting a subsequence, one has

$$(p^\epsilon, \mathbf{u}^\epsilon) \rightharpoonup (\bar{p}, \mathbf{w}) \quad \text{weakly-}^* \text{ in } L^\infty(0, T_0; H^s(\mathbb{R}^3)), \quad (4.1)$$

$$\mathbf{H}^\epsilon \rightharpoonup \mathbf{B} \quad \text{weakly-}^* \text{ in } L^\infty(0, T_0; H^s(\mathbb{R}^3)), \quad (4.2)$$

$$\theta^\epsilon - \bar{\theta} \rightharpoonup \vartheta - \bar{\theta} \quad \text{weakly-}^* \text{ in } L^\infty(0, T_0; H^{s+1}(\mathbb{R}^3)). \quad (4.3)$$

It follows from the equations for \mathbf{H}^ϵ and θ^ϵ that

$$\partial_t \mathbf{H}^\epsilon, \partial_t \theta^\epsilon \in C([0, T_0], H^{s-2}(\mathbb{R}^3)).$$

Hence, after further extracting a subsequence, we obtain that for all $s' < s$,

$$\mathbf{H}^\epsilon \rightarrow \mathbf{B} \quad \text{strongly in } C([0, T_0], H_{\text{loc}}^{s'}(\mathbb{R}^3)), \quad (4.4)$$

$$\theta^\epsilon - \bar{\theta} \rightarrow \vartheta - \bar{\theta} \quad \text{strongly in } C([0, T_0], H_{\text{loc}}^{s'+1}(\mathbb{R}^3)), \quad (4.5)$$

where the limit $\mathbf{B} \in C([0, T_0], H_{\text{loc}}^{s'}(\mathbb{R}^3)) \cap L^\infty(0, T_0; H_{\text{loc}}^s(\mathbb{R}^3))$ and $(\vartheta - \bar{\theta}) \in C([0, T_0], H_{\text{loc}}^{s'+1}(\mathbb{R}^3)) \cap L^\infty(0, T_0; H_{\text{loc}}^{s+1}(\mathbb{R}^3))$.

Similarly, from (3.83) we get

$$\operatorname{curl}(e^{-\theta^\epsilon} \mathbf{u}^\epsilon) \rightarrow \operatorname{curl}(e^{-\vartheta} \mathbf{w}) \quad \text{strongly in } C([0, T_0], H_{\text{loc}}^{s'-1}(\mathbb{R}^3)) \quad (4.6)$$

for all $s' < s$.

In order to obtain the limit system, one needs to show that the limits in (4.1) hold in the strong topology of $L^2(0, T_0; H_{\text{loc}}^{s'}(\mathbb{R}^3))$ for all $s' < s$. To this end, we first show that $\bar{p} = 0$ and $\operatorname{div}(2\mathbf{w} - \bar{\kappa}e^\vartheta \nabla \vartheta) = 0$. In fact, the equations (1.12) and (1.13) can be rewritten as

$$\epsilon \partial_t p^\epsilon + \operatorname{div}(2\mathbf{u}^\epsilon - \bar{\kappa}e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon) = \epsilon f^\epsilon, \quad (4.7)$$

$$\epsilon e^{-\theta^\epsilon} \partial_t \mathbf{u}^\epsilon + \nabla p^\epsilon = \epsilon \mathbf{g}^\epsilon. \quad (4.8)$$

By virtue of (1.24), f^ϵ and \mathbf{g}^ϵ are uniformly bounded in $C([0, T_0], H^{s-1}(\mathbb{R}^3))$. Passing to the weak limit in (4.7) and (4.8), respectively, we see that $\nabla \bar{p} = 0$ and $\operatorname{div}(2\mathbf{w} - \bar{\kappa}e^\vartheta \nabla \vartheta) = 0$. Since $\bar{p} \in L^\infty(0, T_0; H^s(\mathbb{R}^3))$, we infer that $\bar{p} = 0$.

Notice that by virtue of (4.6), the strong compactness for the incompressible component of $e^{-\theta^\epsilon} \mathbf{u}^\epsilon$ holds. So, it is sufficient to prove the following proposition on the acoustic components in order to get the strong convergence of \mathbf{u}^ϵ .

Proposition 4.1. *Suppose that the assumptions in Theorem 1.1 hold. Then, p^ϵ converges to 0 strongly in $L^2(0, T_0; H_{\text{loc}}^{s'}(\mathbb{R}^3))$ and $\operatorname{div}(2\mathbf{u}^\epsilon - \bar{\kappa}e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon)$ converges to 0 strongly in $L^2(0, T_0; H_{\text{loc}}^{s'-1}(\mathbb{R}^3))$ for all $s' < s$.*

The proof of Proposition 4.1 is based on the following dispersive estimates on the wave equation obtained by Métivier and Schochet [39] and reformulated in [2].

Lemma 4.2. ([2, 39]) *Let $T > 0$ and v^ϵ be a bounded sequence in $C([0, T], H^2(\mathbb{R}^3))$, such that*

$$\epsilon^2 \partial_t (a^\epsilon \partial_t v^\epsilon) - \nabla \cdot (b^\epsilon \nabla v^\epsilon) = c^\epsilon,$$

where c^ϵ converges to 0 strongly in $L^2(0, T; L^2(\mathbb{R}^3))$. Assume further that for some $s > 3/2 + 1$, the coefficients (a^ϵ, b^ϵ) are uniformly bounded in $C([0, T], H^s(\mathbb{R}^3))$ and converge in $C([0, T], H_{\text{loc}}^s(\mathbb{R}^3))$ to a limit (a, b) satisfying the decay estimate

$$\begin{aligned} |a(x, t) - \hat{a}| &\leq C_0 |x|^{-1-\zeta}, & |\nabla_x a(x, t)| &\leq C_0 |x|^{-2-\zeta}, \\ |b(x, t) - \hat{b}| &\leq C_0 |x|^{-1-\zeta}, & |\nabla_x b(x, t)| &\leq C_0 |x|^{-2-\zeta}, \end{aligned}$$

for some positive constants \hat{a} , \hat{b} , C_0 and ζ . Then the sequence v^ϵ converges to 0 strongly in $L^2(0, T; L_{\text{loc}}^2(\mathbb{R}^3))$.

Proof of Proposition 4.1. We first show that p^ϵ converges to 0 strongly in $L^2(0, T_0; H_{\text{loc}}^{s'}(\mathbb{R}^3))$ for all $s' < s$. Applying $\epsilon^2 \partial_t$ to (1.12), we find that

$$\begin{aligned} &\epsilon^2 \partial_t \{ \partial_t p^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) p^\epsilon \} + \epsilon \partial_t \{ \operatorname{div}(2\mathbf{u}^\epsilon - \bar{\kappa}e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon) \} \\ &= \epsilon^3 \partial_t \{ e^{-\epsilon p^\epsilon} [\bar{\nu} |\operatorname{curl} \mathbf{H}^\epsilon|^2 + \Psi(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon] \} + \epsilon^2 \partial_t \{ \bar{\kappa} e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla p^\epsilon \cdot \nabla \theta^\epsilon \}. \end{aligned} \quad (4.9)$$

Dividing (1.13) by $e^{-\theta^\epsilon}$ and then applying the operator div to the resulting equation, one gets

$$\begin{aligned} \epsilon \partial_t \operatorname{div} \mathbf{u}^\epsilon + \operatorname{div}(e^{\theta^\epsilon} \nabla p^\epsilon) &= -\epsilon \operatorname{div} \{ (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon \} \\ &\quad + \epsilon \operatorname{div} \{ e^{-\epsilon p^\epsilon + \theta^\epsilon} [(\operatorname{curl} \mathbf{H}) \times \mathbf{H} + \operatorname{div} \Psi^\epsilon(\mathbf{u}^\epsilon)] \}, \end{aligned} \quad (4.10)$$

Subtracting (4.10) from (4.9), we have

$$\epsilon^2 \partial_t \left(\frac{1}{2} \partial_t p^\epsilon \right) - \operatorname{div}(e^{\theta^\epsilon} \nabla p^\epsilon) = \epsilon F^\epsilon(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon), \quad (4.11)$$

where $F^\epsilon(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ is a smooth function in its variables with $F(0) = 0$. By the uniform boundedness of $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ and the strong convergence of θ^ϵ , one infers that

$$F^\epsilon(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon) \rightarrow 0 \quad \text{strongly in } L^2(0, T_0; L^2(\mathbb{R}^3)),$$

and the coefficients in (4.11) satisfy the conditions in Lemma 4.2. Therefore, we can apply Lemma 4.2 to obtain

$$p^\epsilon \rightarrow 0 \quad \text{strongly in } L^2(0, T_0; L_{\text{loc}}^2(\mathbb{R}^3)).$$

Since p^ϵ is bounded uniformly in $C([0, T_0], H^s(\mathbb{R}^3))$, an interpolation argument gives

$$p^\epsilon \rightarrow 0 \quad \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'}(\mathbb{R}^3)) \quad \text{for all } s' < s.$$

Similarly, we can obtain the strong convergence of $\text{div}(2\mathbf{u}^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon)$. This completes the proof. \square

We continue our proof of Theorem 1.1. It follows from Proposition 4.1 and (4.5) that

$$\text{div } \mathbf{u}^\epsilon \rightarrow \text{div } \mathbf{w} \quad \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'-1}(\mathbb{R}^3)).$$

Thus, using (4.6), one obtains

$$\mathbf{u}^\epsilon \rightarrow \mathbf{w} \quad \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'}(\mathbb{R}^3)) \quad \text{for all } s' < s.$$

By (4.4), (4.5), and Proposition 4.1, it holds that

$$\begin{aligned} \nabla \mathbf{u}^\epsilon &\rightarrow \nabla \mathbf{w} && \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'-1}(\mathbb{R}^3)); \\ \nabla \mathbf{H}^\epsilon &\rightarrow \nabla \mathbf{B} && \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'-1}(\mathbb{R}^3)); \\ \nabla \theta^\epsilon &\rightarrow \nabla \vartheta && \text{strongly in } L^2(0, T_0; H_{\text{loc}}^{s'-1}(\mathbb{R}^3)). \end{aligned}$$

Passing to the limits in the equations for p^ϵ , \mathbf{H}^ϵ , and θ^ϵ , we see that the limit $(0, \mathbf{w}, \mathbf{B}, \vartheta)$ satisfies, in the sense of distributions, that

$$\text{div}(2\mathbf{w} - \bar{\kappa} e^\vartheta \nabla \vartheta) = 0, \quad (4.12)$$

$$\partial_t \mathbf{B} - \text{curl}(\mathbf{w} \times \mathbf{B}) - \bar{\nu} \Delta \mathbf{B} = 0, \quad \text{div } \mathbf{B} = 0, \quad (4.13)$$

$$\partial_t \vartheta + (\mathbf{w} \cdot \nabla) \vartheta + \text{div } \mathbf{w} = \bar{\kappa} \text{div}(e^\vartheta \nabla \vartheta). \quad (4.14)$$

On the other hand, applying *curl* to the momentum equations (1.13), using the equations (1.12) and (1.15) on p^ϵ and θ^ϵ , and then taking to the limit on the resulting equations, one sees that

$$\text{curl} \{ \partial_t (e^{-\vartheta} \mathbf{w}) + \text{div}(\mathbf{w} e^{-\vartheta} \otimes \mathbf{w}) - (\text{curl } \mathbf{B}) \times \mathbf{B} - \text{div} \Phi(\mathbf{w}) \} = 0$$

holds in the sense of distributions. Therefore it follows from (4.12)–(4.14) that

$$e^{-\vartheta} [\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w}] + \nabla \pi = (\text{curl } \mathbf{B}) \times \mathbf{B} + \text{div} \Phi(\mathbf{w}), \quad (4.15)$$

for some function π .

Following the same arguments as those in the proof of Theorem 1.5 in [39], we conclude that $(\mathbf{w}, \mathbf{B}, \vartheta)$ satisfies the initial condition

$$(\mathbf{w}, \mathbf{B}, \vartheta)|_{t=0} = (\mathbf{w}_0, \mathbf{B}_0, \vartheta_0). \quad (4.16)$$

Moreover, the standard iterative method shows that the system (4.12)–(4.15) with initial data (4.16) has a unique solution $(\mathbf{w}^*, \mathbf{B}^*, \vartheta^*) \in C([0, T_0], H^s(\mathbb{R}^3))$. Thus, the uniqueness of solutions to the limit system (4.12)–(4.15) implies that the above convergence holds for the full sequence of $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$. Therefore the proof is completed. \square

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