On 3D Lagrangian Navier-Stokes α Model with a Class of Vorticity-Slip Boundary Conditions

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Abstract. This paper concerns the 3-dimensional Lagrangian Navier-Stokes α model and the limiting Navier-Stokes system on smooth bounded domains with a class of vorticity-slip boundary conditions and the Navier-slip boundary conditions. It establishes the spectrum properties and regularity estimates of the associated Stokes operators, the local well-posedness of the strong solution and global existence of weak solutions for initial boundary value problems for such systems. Furthermore, the vanishing α limit to a weak solution of the corresponding initial-boundary value problem of the Navier-Stokes system is proved and a rate of convergence is shown for the strong solution.

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1. Introduction

The Lagragian Navier-Stokes α model (LNS-α) as a regularization system of the Navier-Stokes equations (NS) is given by

$$\partial_t v - \Delta v + T_\alpha v \cdot \nabla v + \nabla (T_\alpha v)^T \cdot v + \nabla p = 0 \tag{1.1}$$

$$\nabla \cdot v = 0 \tag{1.2}$$

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which describes large scale fluid motions in the turbulence theory, where $T_\alpha v = u$ is a filtered version of the velocity v determined usually by

$$u - \alpha\Delta u = v \tag{1.3}$$

$$\nabla \cdot u = 0 \tag{1.4}$$

with $\alpha > 0$ being a constant. This filter u is also called the averaged velocity. The system can be regarded as a system for this filter, and is also called the Lagrangian averaged Navier-Stokes equations (LANS). The ideal case, called the Lagrangian averaged Euler equations (LAE) or Camass-Holm equations, was first introduced in [14, 27]. The viscosity was added in [15, 16, 28] yielding the LANS which is sometime called viscous Camass-Holm equations.

The global well-posedness for the LANS was first obtained in [21] for periodic boundary conditions. The convergence of its solutions to that of the NS equations and the continuity of attractors when $\alpha \to 0$ are also considered there.

For bounded domains, the situation becomes more complicated since the LANS is a 4th odder system for the filter u, and only the no-slip boundary condition $u = 0$ on the boundary was considered by [37] under the assumption that $Au = -P\Delta u = 0$ on the boundary with P being the Leray projection operator. The boundary effects related to such a boundary condition were analyzed in [29]. We also refer [21, 24, 29, 37] for more details along this line.

On the other hand, the LNS-α model emphasizes the system (1.1)-(1.4) as equations for the physical velocity v, which is a regularized system of the NS equations by filtering some part of the nonlinearity through a global quantity which is then called filtered velocity (see [24] and the references therein). There are many filtered formulations, which thus lead to many α models(see [12, 25] for instance). It is also mentioned in [18] in the stochastic Lagrangian derivation of (1.1), (1.2) that any translation-invariant filter $u = T_\alpha v$ may be adaptable.

Although, there is no any serious difference between the two aspects for the equations (1.1), (1.2) filtered by (1.3), (1.4) in domains without boundary, the situation may be different for domains with boundaries. To our knowledge, very little is known to the LNS-α models in domains with boundaries from this point of view.

In this paper, we investigate the initial boundary value problem for the LNS-α model (1.1), (1.2) in the following equivalent form

$$\partial_t v - \Delta v + \nabla \times v \times T_\alpha v + \nabla p = 0 \text{ in } \Omega \tag{1.5}$$

$$\nabla \cdot v = 0 \text{ in } \Omega \tag{1.6}$$

in a smooth bounded domain with the property that both Ω and $\partial\Omega$ have only finite many simply connected components, where $\nabla\cdot$ and $\nabla\times$ denote the div and curl operator, respectively.

Once the filter mapping T_α is given, equations (1.5) and (1.6) become a Navier-Stokes type system for v, and for which, some boundary conditions are needed. Here we consider the following vorticity-slip boundary condition (VSB):

$$v \cdot n = 0, \ n \times \nabla \times v = \beta v \text{ on } \partial\Omega \tag{1.7}$$

Since there is a boundary, the filter $u = T_\alpha v$ can not be determined by solving (1.3) and (1.4). Some boundary conditions are also needed. We propose that the filter $u = T_\alpha v$ be determined by solving the following Stokes boundary value problem

$$u - \alpha\Delta u + \nabla\tilde{p} = v \text{ in } \Omega \tag{1.8}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{1.9}$$

with the VSB:

$$u \cdot n = 0, \ n \times \nabla \times u = \beta u \text{ on } \partial\Omega \tag{1.10}$$

We also consider the associated boundary value problem for the Navier-Stokes equations

$$\partial_t v^0 - \nu\Delta v^0 + (\nabla \times v^0) \times v^0 + \nabla p = 0 \tag{1.11}$$

$$\nabla \cdot v^0 = 0 \tag{1.12}$$

with the corresponding boundary conditions (1.7) as a singular limit problem by passing to the vanishing α limit in (1.5)-(1.10).

The equivalence between (1.1) and (1.5) follows from the vector formula

$$\nabla(u \cdot v) = u \cdot \nabla v + \nabla u^T \cdot v - \nabla \times v \times u \tag{1.13}$$

for any divergence free vectors u and v.

There have been extensive studies of the Navier-Stokes systems on bounded domains with various boundary conditions, such as the well known no-slip condition and various slip boundary conditions. In particular, substantial understanding has been achieved for the well-posedness of initial boundary value problems for the Navier-Stokes system with these boundary conditions and problems of vanishing viscosity limit and boundary layers, see [3, 4, 5, 7, 8, 13, 17, 20, 23, 33, 34, 35, 45, 48] and the references therein. Note that the no-slip boundary condition corresponds to our VSB with $\beta = \infty$. Yet one of the main motivations for the proposed VSB is its relation to the well known Navier-slip boundary condition (see [1, 3, 5, 6, 30, 33, 41, 48] and the references therein). Indeed, the Navier-slip boundary condition (NSB) says that the fluid at the boundary is allowed to slip and the slip velocity is proportional to the shear stress (see [39]), i.e.,

$$v \cdot n = 0, \ 2((S(v)n))_\tau = -\gamma v_\tau \text{ on } \partial\Omega \tag{1.14}$$

where $2S(v) = (\nabla v + (\nabla v)^T)$ is the stress tensor. Note that

$$(2(S(v)n) - (\nabla \times v) \times n)_\tau = GD(v)_\tau \text{ on } \partial\Omega \tag{1.15}$$

where $GD(v) = -2S(n)v$ is the lower order term due to the geometry of the boundary, see lemma 3.10. below. In the special case that the boundary $\partial\Omega$ is flat, one has $GD(v) = 0$. Thus the VSB (1.7) coincides with NSB (1.14). It should be mentioned that as far as we know, all the previous physical and numerical studies concerning the NSB deal with only the case of flat boundaries [1, 6, 31, 32, 42, 45]. Another main motivation for the proposed VSB (1.7) and (1.10) is that the vorticity formulations of the fluid equations have played important roles in analyzing fluid motions, and suitable boundary conditions on the vorticity should be important for such formulations, see [2, 7, 8, 13, 19, 36] and the references therein. For example, the equivalent vorticity form of the NSB conditions are crucial in the studies of the corresponding boundary value problems in [10, 17], and the VSB (with $\beta = 0$) was found very useful to understand the vanishing viscosity limit problem of the Navier-Stokes equations in [3, 35, 46, 47]. It is hoped that the VSB conditions proposed here can share light on understanding the fluid motions in bounded domains.

The rest of the paper is organized as follows: First, as a preparation, we present in the next section a L^2 version of the general Hodge decomposition theory that was stated in [11] for smooth vector spaces, which will be used to study the Stokes problems associated with various slip boundary conditions. Then we give general and systematic results on well-posedness and spectrum properties of the Stokes operators associated with various VSB and NSB conditions in section 3. Our results apply to domains with general topology. It should be mentioned that all the previous analysis deals with only the NSB conditions in some special cases. Based on the properties of the Stokes operators, in section 4, we can formulate the initial boundary value problem of the LNS-α model, (1.5)-(1.10), together with the limit problem of the NS equations, (1.11),(1.12),(1.7), to be a series of abstract equations in a Hilbert space for the parameter $\alpha \in [0, \infty)$. In section 5, we study the well-posedness of the weak solutions for the LNS-α equations with the VSB conditions for each $\alpha > 0$, by the Galerkin method. The local well-posedness, theorem 5.1., is obtained by direct estimates on the velocity v, while the global theory, theorem 5.2., is proved by combining energy estimates on both the velocity field v and the filter u. Note that our approach is somewhat different from [21, 37] in emphasizing the velocity v but not the filter u. In section 6, we investigate the vanishing α limit of solutions of the initial boundary value problem of the LNS-α equations with VSB condition to the corresponding solutions of the NS equations. The global in time convergence of weak solutions is obtained in theorem 6.1. similar to periodic case in [21], while local in time convergence of strong solutions is given in theorem 6.2.. The existence of the global weak solutions and local unique strong solution for the NS equations with corresponding VSB condition are then followed. Furthermore, some estimates on convergence rates are given in theorem 6.3.. Finally, we present some generalizations in section 7. In particular, a parallel theory holds for the NSB condition.

2. Preparations and Hodge decompositions

The Hodge decomposition theory plays an important role in the analysis of vector spaces in a 3D bounded smooth domain, our analysis on the boundary conditions will be based on this theory. To be self content, we give a simple L^2 version below. For more details, we refer [11, 40] and the references therein.

Let $\Omega \subset R^3$ be a bounded smooth domain, $H^s(\Omega)$ denote the standard Hilbert space with $H^0(\Omega) = L^2(\Omega)$. Then the following estimate is well known.

$$\|v\|_s \leq c(\|\nabla \times v\|_{s-1} + \|\nabla \cdot v\|_{s-1} + |n \cdot v|_{s-\frac{1}{2}} + \|v\|_{s-1}) \tag{2.1}$$

for all $v \in H^s(\Omega),\ s \geq 1$ (see [9, 22]).

Let $u \in L^2(\Omega)$. Set

$$u = v + \nabla\varphi_g$$

Note that $\nabla \cdot u \in H^{-1}(\Omega)$. Let φ_g solve

$$\Delta\varphi_g = \nabla \cdot u \text{ in } \Omega \tag{2.2}$$

$$\varphi_g = 0 \text{ on } \partial\Omega \tag{2.3}$$

It follows that

$$\nabla \cdot v = 0 \text{ in } \Omega \tag{2.4}$$

Set

$$DF = \{u \in L^2(\Omega); \nabla \cdot u = 0\}$$

$$GG = \{u \in L^2(\Omega); u = \nabla\varphi, \varphi \in H_0^1(\Omega)\}$$

Note that

$$(u, \nabla\varphi) = 0,\ \forall u \in DF, \varphi \in H_0^1(\Omega) \tag{2.5}$$

One has

Lemma 2.1. *The following decomposition holds:*

$$L^2(\Omega) = DF \oplus GG \tag{2.6}$$

Let $u \in DF$. Then $u \cdot n$ is well-defined on $\partial\Omega$ (see [23]) and

$$\int_{\partial\Omega} u \cdot n = \int_{\Omega} \nabla \cdot u = 0 \tag{2.7}$$

Let φ solve

$$\Delta\varphi = 0 \text{ in } \Omega \tag{2.8}$$

$$\partial_n\varphi = u \cdot n \text{ on } \partial\Omega \tag{2.9}$$

Set

$$v = u - \nabla\varphi$$

and

$$H = \{u \in L^2(\Omega); \nabla \cdot u = 0, \text{in } \Omega;\ u \cdot n = 0;\ \text{on } \partial\Omega\}$$

$$DFG = \{u \in L^2(\Omega); u = \nabla\varphi, \nabla \cdot u = 0, \int_{\partial\Omega} u \cdot n = 0\}$$

Hence $v \in H$. Note that

$$(u, v) = 0,\ \forall u \in H, v \in DFG \tag{2.10}$$

It then follows that

Lemma 2.2. *DF has the following decomposition:*

$$DF = H \oplus DFG \tag{2.11}$$

Note that $u = \nabla\varphi \in DFG$ may not belong to the range of curl, and the range of curl $\nabla \times H^1(\Omega)$ is closed in $L^2(\Omega)$. DFG can be further decomposed to

$$DFG = CG \oplus HG \tag{2.12}$$

where

$$CG = DFG \cap (\nabla \times H^1(\Omega)),\ HG = DFG \cap (\nabla \times H^1(\Omega))^\perp$$

Let $u = \nabla\varphi \in HG$. Since

$$0 = ((\nabla \times v), \nabla\varphi) = \int_{\partial\Omega} (n \times v) \cdot \nabla\varphi \tag{2.13}$$

for all $v \in H^1(\Omega)$, thus $\partial_\tau\varphi = 0,$ on $\partial\Omega$ with τ being any tangential direction on $\partial\Omega$ which implies φ is a constant on each component Γ_i of $\partial\Omega$. So

$$HG = \{\nabla\varphi; \Delta\varphi = 0, \varphi = c_i \ on \ \ \Gamma_i\}$$

consists only smooth vectors, and is finite dimensional, which is called the harmonic gradient space.

Remark 2.3. CG can also be expressed as

$$CG = \{u \in L^2(\Omega);\ u = \nabla\varphi, \nabla \cdot u = 0,\ \int_{\Gamma_i} u \cdot n = 0\}$$

Since $CG \subset \nabla \times H^1(\Omega)$, we will call it curl type gradient space.

Note that $H \cap Ker(\nabla\times)$ is compact in $L^2(\Omega)$ due to (2.1). Set

$$HH = H \cap Ker(\nabla\times)$$

Then

$$HH = \{u \in L^2(\Omega); \nabla \cdot u = 0,\ \nabla \times u = 0 \text{ in } \Omega,\ u \cdot n = 0;\ \text{on } \partial\Omega\}$$

This is called the harmonic knots space, which consists only smooth functions and is finite dimensional (see [11]). Now H can be decomposed to

$$H = FH \oplus HH \tag{2.14}$$

where

$$FH = H \cap (Ker(\nabla \times))^{\perp}$$

In conclusion, we have

Lemma 2.4. *The following decomposition holds:*

$$L^2(\Omega) = FH \oplus HH \oplus CG \oplus HG \oplus GG \tag{2.15}$$

Then for any $u \in L^2(\Omega)$, it is uniquely written to

$$u = P_{HH}u + P_{FH}u + P_{CG}u + P_{HG}u + P_{GG}u \tag{2.16}$$

where P_X denotes the projection on the corresponding subspace.

It should be noticed that the space FH has the following expressions (see [9, 20, 36, 47]).

Lemma 2.5. *The space FH can be expressed as*

$$FH = \{u \in L^2(\Omega); \nabla \cdot u = 0, u \cdot n = 0 \ on \ \partial\Omega, F(u) = 0\} \tag{2.17}$$

$$FH = \{u; u = \nabla \times v, \ v \in H^1(\Omega), \ \nabla \cdot v = 0, \ n \times v = 0 \ on \ \partial\Omega\} \tag{2.18}$$

where $F(u) = 0$ means

$$\int_{\Sigma} u \cdot n = 0$$

for any smooth cross section Σ of Ω.

It follow from (2.1) and the fact that $HH \subset \nabla \times (FH \cap H^1(\Omega))$ (see [11]) respectively that

Proposition 2.6.

$$L^2(\Omega) = \nabla \times (FH \cap H^1(\Omega)) \oplus HG \oplus GG \tag{2.19}$$

Similarly, in general, it holds that

Proposition 2.7.

$$H^s(\Omega) = \nabla \times (FH \cap H^{s+1}(\Omega)) \oplus (HG \cap H^s(\Omega)) \oplus (GG \cap H^s(\Omega)) \tag{2.20}$$

for $s \geq 0$.

It follows from (2.1),(2.2),(2.3),(2.8),(2.9) and the fact that HH, HG are finite dimensional that

Proposition 2.8. *$C^\infty(\Omega) \cap X$ is dense in $H^s(\Omega) \cap X$, $s \geq 0$ for*

$$X = FH, HH, CG, HG, GG$$

3. The Stokes operators

In this section, we apply the Hodge decomposition theory to the Stokes problems with both the VSB and NSB conditions. We first consider a special Stokes problem with the VSB (3.1)-(3.3) and prove theorem 3.1.. Next, since the topology of the domain is assumed to be general, to avoid the uniqueness of the solutions for the general Stokes problems, we consider the perturbed Stokes problem associated with VSB (3.25)-(3.27). Based on theorem 3.1., by using the Hodge decomposition theory, we prove the associated Stokes operator is a self-adjoint extension of the associated positive definite bilinear form (see theorem 3.5.). The proof of theorem 3.5. is constructive, and the techniques can also be used to prove the well-posedness of the non-homogeneous problem (3.55)-(3.57)(see theorem 3.7.). More generally, we can prove well-posedness of the boundary value problem (3.63)-(3.65) (see theorem 3.9.) by construction a contraction map. Finally, we identify the relationship between the NSB and VSB, and establish a similar theory for the Stokes problem associated with NSB (3.67)-(3.69).

3.1. A special Stokes problem

Let us start by considering the following special Stokes problem with homogenous VSB condition

$$-\Delta u = f \text{ in } \Omega \tag{3.1}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{3.2}$$

$$u \cdot n = 0, \ n \times \nabla \times u = 0 \text{ on } \partial\Omega \tag{3.3}$$

with $f \in FH$. Set

$$W = \{u \in H^2(\Omega);\ n \times (\nabla \times u) = 0 \text{ on}, \partial\Omega\}$$

Then we have

Theorem 3.1. *The Stokes operator $A_F = -\Delta$ with the domain $D(A_F) = W \cap FH$ is self-adjoint in the Hilbert space FH.*

Proof. It is clear that $A_F = -\Delta$ with the domain $W \cap FH$ is symmetric. Since $C_0^\infty(\Omega) \cap H$ is dense in H, it follows that A_F is densely defined due to the orthogonality of FH and HH and the compactness of HH. Let $u \in W$. Since $n \times (\nabla \times u) = 0$ on $\partial\Omega$, then $-\Delta u = \nabla \times (\nabla \times u) \in FH$ by lemma 2.5., thus A_F maps $W \cap FH$ to FH. Now, for any $f \in FH$, it follows from lemma 2.5. that there is a $\Phi \in H^1(\Omega)$ satisfying

$$\nabla \times \Phi = f \text{ in } \Omega \tag{3.4}$$

$$\nabla \cdot \Phi = 0 \text{ in } \Omega \tag{3.5}$$

$$\Phi \times n = 0 \text{ on } \partial\Omega \tag{3.6}$$

Due to proposition 2.7. and lemma 2.1., there is a $v \in FH \cap H^2(\Omega)$ so that

$$\Phi = \nabla \times v + P_{HG}\Phi \tag{3.7}$$

Note that $P_{HG}\Phi \times n = 0$ on $\partial\Omega$. It follows that

$$n \times (\nabla \times v) = 0 \text{ } on \text{ } \partial\Omega \tag{3.8}$$

Then $\nabla \times (P_{HG}\Phi) = 0$ and (3.7) imply that

$$-\Delta v = f \text{ in } \Omega \tag{3.9}$$

Thus $A_F : W \cap FH \to FH$ is surjective. If $f = 0$, then integration by part shows

$$\|\nabla \times v\| = 0 \tag{3.10}$$

It follows that $u = 0$ due to the orthogonality of FH and HH and then $A_F : W \cap FH \to FH$ is one to one.

Noting that W and FH are closed in $H^2(\Omega)$ and $L^2(\Omega)$, and

$$\|\Delta v\| \leq \|v\|_2 \tag{3.11}$$

we obtain from the Banach inverse operator theorem that

$$\|v\|_2 \leq c\|\Delta v\| \tag{3.12}$$

The theorem was proved. □

Equivalently, we have shown the problem (3.1)-(3.3) has a unique solution $u \in H^2(\Omega)$ for any $f \in FH$.

It follows from the proof of theorem 3.1. that

$$\nabla \times : H^1_n(\Omega) \mapsto HF$$

is also one to one and onto, where

$$H^1_n(\Omega) = \{u \in H^1(\Omega); \nabla \cdot u = 0; \quad n \times u = 0 \text{ } on \text{ } \partial\Omega; (u, \varphi) = 0, \forall \varphi \in HG\}$$

It follows from the trace theorem and the continuity of the divergence operator that $H^1_n(\Omega)$ is closed in $H^1(\Omega)$. Then

$$\|u\|_1 \leq c\|\nabla \times u\| \tag{3.13}$$

follows from

$$\|\nabla \times u\| \leq \|u\|_1 \tag{3.14}$$

for any $u \in H^1_n(\Omega)$. This yields immediately that

Lemma 3.2. *Let $u \in H^1_n(\Omega)$. Then the following Poincaré type inequality holds*

$$\|u\| \leq c\|\nabla \times u\| \tag{3.15}$$

Let $v \in FH \cap H^1(\Omega)$. Then there is $u \in H^1_n(\Omega)$ such that $\nabla \times u = v$ and

$$(v, v) = (\nabla \times u, v) = (u, \nabla \times v) \leq \|u\|\|\nabla \times v\| \tag{3.16}$$

This, together with (3.15), shows

$$(v, v) \leq c\|\nabla \times v\|\|\nabla \times u\| \tag{3.17}$$

Thus, one gets

Lemma 3.3. *Let* $u \in FH \cap H^1(\Omega)$. *Then the following Poincaré type inequality*

$$\|u\| \leq c\|\nabla \times u\| \tag{3.18}$$

holds.

As a consequence, we can obtain

Corollary 3.4. *The operator* A_F *in theorem 3.1. is the self adjoint extension of the following bilinear form*

$$a(u, \phi) = (\nabla \times u, \nabla \times \phi),\ \mathrm{D}(a) = V_F = FH \cap H^1(\Omega) \tag{3.19}$$

in FH.

Proof. From proposition 2.8., $a(u, \phi)$ with $D(a) = FH \cap H^1(\Omega)$ is densely defined. Due to (2.1) and lemma 3.3., $a(u, \phi)$ is closed and positive. It follows that there is a self-adjoint operator A with domain $D(A) \subset D(a)$ such that

$$a(u, \phi) = (Au, \phi), \forall \phi \in FH \cap H^1(\Omega) \tag{3.20}$$

for any $u \in D(A)$. It is clear that $D(A_F) = W \cap FH \subset D(A)$ and $Au = -\Delta u$ for any $u \in W \cap FH$. Let $u \in D(A)$ and $f = Au$. It then follows that $f \in FH$. It follows from theorem 3.1. that there is a $v \in D(A_F)$ such that (3.1)-(3.3) are valid (with u replaced by v) and

$$a(v, \phi) = (f, \phi) \tag{3.21}$$

for all $\phi \in V_F$. On the other hand

$$a(u, \phi) = (Au, \phi) = (f, \phi) \tag{3.22}$$

for all $\phi \in V_F$, hence

$$a(u - v, \phi) = (\nabla \times (u - v), \nabla \times \phi) = 0 \tag{3.23}$$

for all $\phi \in V_F$. Taking $\phi = u - v$ shows that $\nabla \times (u - v) = 0$. Thus $u = v$ due to (2.14). Thus $D(A) = D(A_F)$ and $A = A_F$. □

Denote by V_F' the dual space of V_F respect to the L^2 inner product. Then the notation of weak solutions can be extended for $f \in V_F'$: u is called a weak solution to (3.1)-(3.3) for $f \in V_F'$ if

$$a(u, \phi) = (f, \phi),\ \forall\ \phi \in V_F \tag{3.24}$$

3.2. The Stokes problem with VSB condition

Next, we consider the Stokes problem with general VSB condition. Since the domain is allowed to have general topologe, the kernel of $-\Delta$ may be not empty. To avoid it, we consider the following boundary value problem instead:

$$(I - \Delta)u + \nabla p = f \text{ in } \Omega \tag{3.25}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{3.26}$$

$$u \cdot n = 0,\ n \times (\nabla \times u) = \beta u \text{ on } \partial\Omega \tag{3.27}$$

where β is a nonnegative smooth function.
Define

$$V = H^1(\Omega) \cap H$$

$$W_\beta = \{u \in H^2(\Omega);\ n \times (\nabla \times u) = \beta u \text{ on}, \partial\Omega\}$$

Define a bilinear form as

$$\tilde{a}_\beta(u, \phi) = (u, \phi) + a_\beta(u, \phi)$$

where

$$a_\beta(u, \phi) = \int_{\partial\Omega} \beta u \cdot \phi + \int_\Omega (\nabla \times u) \cdot (\nabla \times \phi) \tag{3.28}$$

with the domain $D(\tilde{a}_\beta) = V$. $u \in V$ is said to be a weak solution to the boundary value problem (3.25)-(3.27) on H for $f \in V'$ if

$$\tilde{a}_\beta(u, \phi) = (f, \phi),\ \forall\ \phi \text{ in } V \tag{3.29}$$

where V' is the dual space of V. Based on theorem 3.1., we can prove

Theorem 3.5. *The self-adjoint extension of the bilinear form $\tilde{a}_\beta(u, \phi)$ with the domain $\mathrm{D}(\tilde{a}_\beta) = V$ is the Stokes operator $A_\beta = I + P(-\Delta)$ with $\mathrm{D}(A_\beta) = W_\beta \cap H$, and A_β is an isomorphism between $\mathrm{D}(A_\beta)$ and H with compact inverse on H. Consequently, the eigenvalues of the Stokes operator A_β can be listed as*

$$1 \leq 1 + \lambda_1 \leq 1 + \lambda_2 \cdots \to \infty$$

with the corresponding eigenvectors $\{e_j\} \subset W_\beta$, i.e.,

$$A_\beta e_j = (1 + \lambda_j) e_j \tag{3.30}$$

which form a complete orthogonal basis in H. Furthermore, it holds that

$$(1 + \lambda_1)\|u\|^2 \leq \tilde{a}_\beta(u, u) \leq \frac{1}{1 + \lambda_1}\|A_\beta u\|^2,\ \forall\ u \in \mathrm{D}(A_\beta) \tag{3.31}$$

Proof. It is clear that $\tilde{a}_\beta(u, \phi)$ with the domain $D(\tilde{a}_\beta) = V$ is a positive densely defined closed bilinear form. Let A_β be the self-adjoint extension of $\tilde{a}_\beta(u, \phi)$. It follows that $W_\beta \cap H \subset D(A_\beta)$ and $A_\beta u = u + P(-\Delta u)$, for any $u \in W_\beta \cap H$ by integrating by part. It remains to show that $D(A_\beta) \subset W_\beta \cap H$. Let $u \in D(A_\beta)$ and $f = A_\beta u$. Since $D(A_\beta) \subset D(\tilde{a}_\beta) = V$, it follows from (3.29) that

$$\|u\|_1 \leq c\|f\| \tag{3.32}$$

Let $n(x)$ and $\beta(x)$ be internal smooth extensions of the normal vector β respectively. Then $\beta(x) u \times n(x) \in H^1(\Omega)$. Proposition 2.7. yields

$$\beta(x) u \times n(x) = \nabla \times v + \nabla h + \nabla g \tag{3.33}$$

with $\nabla h = P_{HG}(\beta(x) u \times n(x))$, $\nabla g = P_{GG}(\beta(x) u \times n(x))$ and $v \in FH \cap H^2(\Omega)$. It follows that

$$\|\nabla g\|_1 \leq c\|u\|_1 \tag{3.34}$$

since g satisfies

$$\Delta g = \nabla \cdot (\beta(x) u \times n(x)) \text{ in } \Omega \tag{3.35}$$

$$g = 0 \text{ on } \partial\Omega \tag{3.36}$$

Since HG is finite dimensional, so

$$\|P_{HG}(\beta(x) u \times n(x))\|_1 \leq c\|P_{HG}(\beta(x) u \times n(x))\| \leq c\|u\| \tag{3.37}$$

It then follows from (2.1) and lemma 3.3 that

$$\|v\|_2 \leq c\|\nabla \times v\|_1 \leq c\|u\|_1 \leq c\|f\| \tag{3.38}$$

Integrating by part and noting that $n \times \nabla h = 0$, $n \times \nabla g = 0$ on the boundary, we have

$$\int_\Omega (\nabla \times v) \cdot (\nabla \times \phi) + \int_{\partial\Omega} \beta n \times (u \times n) \cdot \phi = (-\Delta v, \phi) \tag{3.39}$$

for all $\phi \in H^1(\Omega)$. It follows from $n \times (u \times n) = u$ on the boundary and the definition of the weak solution that

$$\int_\Omega (\nabla \times (u - v)) \cdot (\nabla \times \phi) = (P_{FH}(f - u + \Delta v), \phi) \tag{3.40}$$

for all $\phi \in H^1(\Omega) \cap FH$. Note that $\nabla \times u = \nabla \times P_{FH}(u)$ and $P_{FH}(u) \in H^1(\Omega) \cap FH$. It follows that

$$a(P_{FH}(u) - v, \phi) = (P_{FH}(f - u + \Delta v), \phi), \ \forall \phi \ \in \ H^1(\Omega) \cap FH \tag{3.41}$$

It follows from theorem 3.1. that $P_{FH}(u) - v \in W$ and

$$\|P_{FH}(u) - v\|_2 \leq c(\|f\| + \|\Delta v\| + \|u\|) \tag{3.42}$$

Since HH is finite dimensional, it holds that

$$\|P_{HH}(u)\|_2 \leq c\|u\| \tag{3.43}$$

One gets from (3.32),(3.38),(3.42) and (3.43) that

$$\|u\|_2 \leq c\|f\| \tag{3.44}$$

Since $P_{FH}(u) - v \in W$, it holds that

$$n \times \nabla \times u = n \times \nabla \times P_{FH}(u) = n \times \nabla \times v = \beta u \text{ on } \partial\Omega \tag{3.45}$$

Thus we have shown $u \in W_\beta \cap H$. Integrating by part in (3.29) yields

$$(u - \Delta u - f, \phi) = 0 \tag{3.46}$$

for all $\phi \in V$, which implies

$$u - \Delta u + \nabla p = f \text{ in } \Omega \tag{3.47}$$

with p given by

$$-\Delta p = 0 \text{ in } \Omega \tag{3.48}$$

$$(\nabla p) \cdot n = \Delta u \cdot n \text{ on } \partial\Omega \tag{3.49}$$

It is noted that $\|A_\beta u\|$ is an equivalent norm of $H^2(\Omega)$ on $W_\beta \cap H$ due to (3.44) and

$$\|A_\beta u\| \leq \|u\| + \|\Delta u\| \leq c\|u\|_2 \tag{3.50}$$

for all $u \in W_\beta \cap H$. The theorem was proved. □

Let V' be the dual space of V with respect to the L^2 inner product. $u \in V$ is called a weak solution to (3.1)-(3.3) for $f \in V'$ if

$$\tilde{a}_\beta(u, \phi) = (f, \phi), \ \forall \ \phi \in V \tag{3.51}$$

By using a standard density argument, one can show

Corollary 3.6. *For any $f \in V'$, the boundary value problem (3.25)-(3.27) has a unique weak solution $u \in V$*

Now, let $b \in H^{\frac{1}{2}}(\partial\Omega)$ and $b \cdot n = 0$ on $\partial\Omega$. From the extension theorem, it has an extension denoted by $b(x) \in H^1(\Omega)$. Similar to the proof of theorem 3.5., one can show that there exists a $\Phi \in H^2(\Omega) \cap FH$ such that

$$n \times (\nabla \times \Phi) = b \text{ on } \partial\Omega$$

It follows that Φ solves the following problem:

$$u - \Delta u + \nabla p = f \text{ in } \Omega \tag{3.52}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{3.53}$$

$$u \cdot n = 0, \ n \times (\nabla \times u) = b \text{ on } \partial\Omega \tag{3.54}$$

with $f = u + P(-\Delta\Phi)$ and $\nabla p = \Delta\Phi - P(\Delta\Phi)$. This fact and theorem 3.5. for $\beta = 0$ yield

Theorem 3.7. *Let $b \in H^{\frac{1}{2}}(\partial\Omega)$, $b \cdot n = 0$ and $\lambda > 0$. Then the following problem*

$$\lambda u - \Delta u + \nabla p = f \text{ in } \Omega \tag{3.55}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{3.56}$$

$$u \cdot n = 0, \ n \times (\nabla \times u) = b \text{ on } \partial\Omega \tag{3.57}$$

has a unique solution $u \in H^2(\Omega)$ for any $f \in H$.

The boundary value problem (3.55)-(3.57) also have a weak formulation

$$\lambda(u, \phi) + \int_\Omega (\nabla \times u) \cdot (\nabla \times \phi) + \int_{\partial\Omega} b \cdot \phi = (f, \phi), \ \forall \ \phi \text{ in } V \tag{3.58}$$

Similar to corollary 3.4., one has

Corollary 3.8. *Let $b \in H^{-\frac{1}{2}}(\partial\Omega)$, $b \cdot n = 0$. Then for any $f \in V'$, the boundary value problem (3.55)-(3.57) has a unique weak solution $u \in V$ in the sense of (3.58).*

We omit the details of the proof here, and refer to [26] for the definition of the weak tangential trace $H^{-\frac{1}{2}}(\partial\Omega)$.

For any given smooth and nonnegative function β, we define the map

$$T : H^{\frac{1}{2}}(\Omega) \cap H \mapsto V \subset H^{\frac{1}{2}}(\Omega) \cap H$$

by $u = Tv$ determined by (3.58) with b replaced by $\beta v + b$ and $f = 0$. Let $v_i \in H^{\frac{1}{2}}(\Omega) \cap H$ and $u_i = Tv_i$, $i = 1, 2$. It then follows from (3.58) that

$$\lambda\|u_1 - u_2\|^2 + \|\nabla \times (u_1 - u_2)\|^2 + \int_{\partial\Omega} \beta(u_1 - u_2) \cdot (v_1 - v_2) = 0 \quad (3.59)$$

Note that

$$|\int_{\partial\Omega} \beta(u_1 - u_2) \cdot (v_1 - v_2)| \le c\|u_1 - u_2\|_{H^{\frac{1}{2}}(\Omega)} \|v_1 - v_2\|_{H^{\frac{1}{2}}(\Omega)} \quad (3.60)$$

and

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\Omega)} \le c\|\varphi\|\|\varphi\|_{H^1(\Omega)} \le c\|\varphi\|\|\nabla \times \varphi\|, \ \forall \ \varphi \in V \quad (3.61)$$

It follows that

$$\|u_1 - u_2\|^2_{H^{\frac{1}{2}}(\Omega)} \le c\lambda^{-\frac{1}{2}} \|v_1 - v_2\|^2_{H^{\frac{1}{2}}(\Omega)} \quad (3.62)$$

for $\lambda \ge 1$. Take λ large enough such that T becomes a contraction map on $H^{\frac{1}{2}}(\Omega)$. It follows that

$$Tv = v$$

has a unique solution Ψ on $H^{\frac{1}{2}}(\Omega)$ and $\Psi = T\Psi \in H^1(\Omega)$.

For any $\tilde{f} \in V'$, let v be the weak solution of (3.25)-(3.27) with $f = \tilde{f} - (1-\lambda)\Psi$. It is clear that $u = v + \Psi \in V$ is a weak solution of the following problem:

$$u - \Delta u + \nabla p = \tilde{f} \text{ in } \Omega \quad (3.63)$$

$$\nabla \cdot u = 0 \text{ in } \Omega \quad (3.64)$$

$$u \cdot n = 0, \ n \times (\nabla \times u) = \beta u + b \text{ on } \partial\Omega \quad (3.65)$$

in the sense that

$$(u, \phi) + \int_{\Omega} (\nabla \times u) \cdot (\nabla \times \phi) + \int_{\partial\Omega} (\beta u + b) \cdot \phi = (\tilde{f}, \phi), \ \forall \ \phi \text{ in } V \quad (3.66)$$

The uniqueness can be proved in the same way as for theorem 3.1.. We conclude

Theorem 3.9. *Let $b \in H^{-\frac{1}{2}}(\partial\Omega)$, $b \cdot n = 0$. Then for any $\tilde{f} \in V'$, the boundary value problem (3.63)-(3.65) has a unique solution $u \in V$ in the sense of (3.66). Moreover, if $\tilde{f} \in H$ and $b \in H^{\frac{1}{2}}(\partial\Omega)$, then $u \in H^2(\Omega)$.*

3.3. The Stokes problem with the NSB condition

We can establish a similar theory for the Stokes problem with the NSB just as with VSB. For completeness, we sketch it here. Consider the following Stokes problem with the NSB condition.

$$(I - \Delta)u + \nabla p = f \text{ in } \Omega \quad (3.67)$$

$$\nabla \cdot u = 0 \text{ in } \Omega \quad (3.68)$$

$$u \cdot n = 0, \ 2(S(u)n)_\tau = -\gamma u_\tau u \text{ on } \partial\Omega \quad (3.69)$$

where γ is a nonnegative smooth function.
Define

$$\tilde{W}_\gamma = \{u \in H^2(\Omega);\ 2(S(u)n)_\tau = -\gamma u_\tau \text{ on}, \partial\Omega\}$$

and a bilinear form

$$\tilde{a}_\gamma(u, \phi) = (u, \phi) + a_\gamma(u, \phi),\ D(\tilde{a}_\gamma) = V$$

where

$$a_\gamma(u, \phi) = \int_{\partial\Omega} \gamma u \cdot \phi + 2\int_\Omega S(u) \cdot S(\phi) \tag{3.70}$$

and $S(u) \cdot S(\phi)$ denotes the trace of the product of the two matrices.

u is said to be a weak solution to the boundary value problem (3.67)-(3.69) on H for $f \in V'$ if

$$\tilde{a}_\gamma(u, \phi) = (f, \phi),\ \forall\ \phi \text{ in } V \tag{3.71}$$

where V' is the dual space of V.

To compare it with the VSB case, we first calculate that

Lemma 3.10. *Let $u \in H^2(\Omega)$ and $u \cdot n = 0$ on the boundary. It holds that*

$$(2(S(u)n) - \omega \times n)_\tau = GD(u)_\tau \tag{3.72}$$

with $GD(u) = -2S(n)u$.

Proof. Note that

$$\partial_n u = \frac{1}{2}\omega \times n + S(u)n \tag{3.73}$$

and

$$\partial_\tau u = \frac{1}{2}\omega \times \tau + S(u)\tau \tag{3.74}$$

It follows that

$$2(S(u)n) \cdot \tau = \partial_\tau u \cdot n + \partial_n u \cdot \tau \tag{3.75}$$

$$(n \times \omega)\tau = \partial_\tau u \cdot n - \partial_n u \cdot \tau \tag{3.76}$$

and

$$2(S(u)n) \cdot \tau + (n \times \omega)\tau = 2\partial_\tau u \cdot n \tag{3.77}$$

Note that $u \cdot n = 0$ on the boundary. It follows that

$$\partial_\tau u \cdot n = -u \cdot \partial_\tau n \tag{3.78}$$

We conclude that

$$(2S(u)n - \omega \times n) \cdot \tau = -2u \cdot \partial_\tau n \tag{3.79}$$

Note that

$$\partial_\tau n = \frac{1}{2}(\nabla \times n) \times \tau + S(n)\tau \tag{3.80}$$

thus

$$(2S(u)n - \omega \times n) \cdot \tau = ((\nabla \times n) \times u) \cdot \tau - 2S(n)u \cdot \tau \tag{3.81}$$

Note that

$$u \times \tau = \lambda n \tag{3.82}$$

and

$$(\nabla \times n) \cdot n = 0 \tag{3.83}$$

on the boundary. It follows that

$$(2S(u)n - \omega \times n) \cdot \tau = -2S(n)u \cdot \tau \tag{3.84}$$

Set

$$GD(u) = -2S(n)u$$

The lemma is proved. □

It follows from a simple calculation and by using the density method that

Lemma 3.11. *Let $u \in H^1(\Omega) \cap H$. Then*

$$2\int_\Omega S(u) \cdot S(\phi) = \int_\Omega (\nabla \times u) \cdot (\nabla \times \phi) + \int_{\partial\Omega} GD(\phi) \cdot u \tag{3.85}$$

$$\int_{\partial\Omega} GD(\phi) \cdot u = \int_{\partial\Omega} GD(u) \cdot \phi \tag{3.86}$$

As a counterpart of theorem 3.5., we can obtain

Theorem 3.12. *The self-adjoint extension of the bilinear form $\tilde{a}_\gamma(u, \phi)$ with domain $\mathrm{D}(\tilde{a}_\gamma) = V$ is the Stokes operator $A_\gamma = I + P(-\Delta)$ with $\mathrm{D}(A_\gamma) = \tilde{W}_\gamma \cap H$, and A_γ is an isomorphism between $\mathrm{D}(A_\gamma)$ and H with a compact inverse on H. Consequently, the eigenvalues of the Stokes operator A_γ can be listed as*

$$1 \le 1 + \lambda_1 \le 1 + \lambda_2 \cdots \to \infty$$

with the corresponding eigenvectors $\{e_j\} \subset \tilde{W}_\gamma$, i.e.,

$$A_\gamma e_j = (1 + \lambda_j)e_j \tag{3.87}$$

which form a complete orthogonal basis in H. Furthermore, it holds that

$$(1 + \lambda_1)\|u\|^2 \le \tilde{a}_\beta(u, u) \le \frac{1}{1 + \lambda_1}\|A_\gamma u\|^2, \ \forall u \in \mathrm{D}(A_\gamma) \tag{3.88}$$

Proof. It suffices to show that $D(A_\gamma) \subset \tilde{W}_\gamma \cap H$ since the rest is similar to the proof of theorem 3.5.. Let $u \in D(A_\gamma)$ and $f = A_\gamma u$. Since $D(A_\gamma) \subset D(\tilde{a}_\beta) = H^1(\Omega) \cap H$, it follows from (3.71) that

$$\|u\|_1^2 \le c\|f\|^2 \tag{3.89}$$

Let $n(x)$ and $\gamma(x)$ be internal smooth extensions of the normal vector n and γ. Then $(\gamma(x)u + GD(u)) \times n(x) \in H^1(\Omega)$. Due to proposition 2.7., one has

$$(\gamma(x)u + GD(u)) \times n(x) = \nabla \times v + \nabla h + \nabla g \tag{3.90}$$

with $v \in H^2(\Omega) \cap FH$, $\nabla h = P_{HG}((\gamma(x)u + GD(u)) \times n(x))$ and $\nabla g = P_{GG}((\gamma(x)u + GD(u)) \times n(x))$. Similar to the proof of theorem 3.1., one can get

$$\|v\|_2 \le c\|\nabla \times v\|_1 \le c\|u\|_1 \tag{3.91}$$

Note that $n \times (\nabla h) = 0$ and $n \times (\nabla g) = 0$. Thus

$$\int_\Omega (\nabla \times v) \cdot (\nabla \times \phi) + \int_{\partial\Omega} (\gamma u + GD(u)) \cdot \phi = (-\Delta v, \phi), \ \forall \phi \in V \tag{3.92}$$

Then the definition of the weak solution and lemma 3.11. imply

$$\int_\Omega (\nabla \times u) \cdot (\nabla \times \phi) + \int_{\partial\Omega} \gamma u \cdot \phi + \int_{\partial\Omega} GD(\phi) \cdot u = (f - u, \phi), \ \forall \phi \in V \tag{3.93}$$

Combine them and note (3.86) to get

$$\int_\Omega (\nabla \times (u - v)) \cdot (\nabla \times \phi) = (P_{FH}(f - u + \Delta v), \phi), \ \forall \phi \in V \tag{3.94}$$

Note that $\nabla \times u = \nabla \times P_{FH}(u)$ and $P_F(u) \in H^1(\Omega) \cap FH$. It follows that

$$a(P_{FH}(u) - v, \phi) = (P_{FH}(f - u + \Delta v), \phi), \ \forall \phi \ \in \ H^1(\Omega) \cap FH \tag{3.95}$$

Since $P_{FH}(f - u + \Delta v) \in FH$, so $P_{FH}(u) - v \in W$, and

$$\|P_{FH}(u) - v\|_2 \le c(\|f\| + \|u\|_1) \tag{3.96}$$

Since HH is a finite dimensional, so

$$\|P_{HH}(u)\|_2 \le c\|u\| \tag{3.97}$$

It follows from (3.89),(3.91),(3.96) and (3.97) that

$$\|u\|_2 \le c\|f\| \tag{3.98}$$

Note that

$$(\nabla \times u) \times n = (\nabla \times P_{FH}(u)) \times n = (\nabla \times v) \times n = -\gamma u - GD(u) \tag{3.99}$$

It follows that

$$2(S(u)n)_\tau = ((\nabla \times u) \times n + GD(u))_\tau = -\gamma u_\tau \tag{3.100}$$

The theorem was proved. □

Similar to the discussion for VSB, we have

Theorem 3.13. *Let $b \in H^{-\frac{1}{2}}(\partial\Omega)$, $b \cdot n = 0$, γ be a nonnegative smooth function on the boundary. Then for any $f \in V'$, the following boundary value problem*

$$u - \Delta u + \nabla p = f \text{ in } \Omega \tag{3.101}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{3.102}$$

$$u \cdot n = 0, \ 2(S(u)n)_\tau = -\gamma u_\tau + b \text{ on } \partial\Omega \tag{3.103}$$

has a unique solution $u \in V$ in the sense that

$$(u, \phi) + \int_{\partial\Omega} (\gamma u + b) \cdot \phi + 2\int_\Omega S(u) \cdot S(\phi) = (f, \phi), \forall \phi \text{ in } V \tag{3.104}$$

Moreover, if $f \in H$ and $b \in H^{\frac{1}{2}}(\partial\Omega)$, then $u \in H^2(\Omega)$.

4. Functional setting of the LNS-α equation

In this section, we formulate the following boundary value problem for the LNS-α system:

$$\partial_t v - \Delta v + \nabla \times v \times u + \nabla p = 0 \text{ in } \Omega \tag{4.1}$$

$$\nabla \cdot v = 0 \text{ in } \Omega \tag{4.2}$$

$$u - \alpha\Delta u + \nabla\tilde{p} = v \text{ in } \Omega \tag{4.3}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{4.4}$$

with the VBS conditions

$$v \cdot n = 0, \ n \times \nabla \times v = \beta v \text{ on } \partial\Omega \tag{4.5}$$

$$u \cdot n = 0, \ n \times \nabla \times u = \beta u \text{ on } \partial\Omega \tag{4.6}$$

Due to theorem 3.5., $A_\alpha = I - \alpha P\Delta$ is also a positive definite self-adjoint operator with domain $D(A_\alpha) = W_\beta \cap H$ for any $\alpha > 0$. We have

Proposition 4.1. *The linear operator $T_\alpha = A_\alpha^{-1} : H \mapsto W_\beta \cap H$ is well defined with $u = T_\alpha v \in W_\beta \cap H$ given by the Stokes boundary value problem*

$$u - \alpha\Delta u + \nabla\tilde{p} = v \text{ in } \Omega \tag{4.7}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{4.8}$$

$$u \cdot n = 0, \ n \times (\nabla \times u) = \beta u \text{ on } \partial\Omega \tag{4.9}$$

and is bounded, i.e.

$$\|u\|_2 \le c_\alpha \|v\| \tag{4.10}$$

for some constant c_α depending on α.

We now estimate the nonlinearity. Let $v \in V \subset H$ so that $T_\alpha v$ is defined. Set

$$B(v,u) = P(\nabla \times v \times u), \quad \forall\, u \in W_\beta, \quad v \in V \tag{4.11}$$

$$B_\alpha(v) = B(v, T_\alpha v), \quad v \in V \tag{4.12}$$

for $\alpha > 0$. Then we have

Lemma 4.2. *The nonlinearity $B_\alpha(v) : V \mapsto H$ is locally Lipshitz for $\alpha > 0$.*

Proof. Clearly, B_α is well-defined due to (4.10). For any v_1, $v_2 \in V$,

$$\|B_\alpha(v_1) - B_\alpha(v_2)\| \le \|\nabla \times (v_1 - v_2) \times T_\alpha v_1 - \nabla \times v_2 \times T_\alpha(v_1 - v_2)\| \tag{4.13}$$

Note that for all $\phi \in V$, $\psi \in L^\infty(\Omega)$,

$$\|\nabla \times (\phi) \times \psi\| \le c\|\phi\|_1 \|\psi\|_{L^\infty(\Omega)} \tag{4.14}$$

and

$$\|w\|^2_{L^\infty(\Omega)} \le c\|w\|_1 \|w\|_2 \tag{4.15}$$

It follows that

$$\|B_\alpha(v_1) - B_\alpha(v_2)\| \le c(\|v_1\|_1 + \|v_2\|_1)\|(v_1 - v_2)\|_1 \tag{4.16}$$

which implies the lemma. □

We now can formulate the initial boundary problem of the LNS-α equations (4.1)-(4.6) as an abstract equation

$$v' - P\Delta v + B(v,u) = 0 \tag{4.17}$$

$$u = T_\alpha v \tag{4.18}$$

on H, with a parameter $\alpha \in (0,\infty)$.

The weak solutions of the initial boundary problem can be defined as below.

Definition 4.3. (v,u) is a weak solution of (4.1)-(4.6) with $\alpha > 0$ for LNS-α equations with initial data $v_0 \in H$ on the time interval $[0,T)$ if $v \in L^2(0,T;V) \cap C_w([0,T];H)$, $v' \in L^1(0,T;V')$, $u \in L^2(0,T;V \cap H^3(\Omega)) \cap C_w([0,T];W_\beta)$, $u' \in L^1(0,T;V)$ such that

$$(v',w) + a_\beta(v,w) + (B(v,u),w) = 0,\ \ a.e.\ t \in [0,T) \tag{4.19}$$

$$u = T_\alpha v,\ \ a.e.\ t \in [0,T) \tag{4.20}$$

for all $w \in V$.

For the special case $\alpha = 0$, we define also the corresponding weak solutions for the NS as follows

Definition 4.4. (v,u) is a weak solution of (4.1)-(4.6) with $\alpha = 0$ (NS equations) initial data $v_0 \in H$ on the time interval $[0,T)$ if $v,u \in L^2(0,T;V) \cap C_w([0,T];H)$ and $v',v' \in L^1(0,T;V')$ such that

$$(v',w) + a_\beta(v,w) + (B(v,u),w) = 0,\ \ a.e.\ t \in [0,T) \tag{4.21}$$

$$u = v,\ \ a.e.\ t \in [0,T) \tag{4.22}$$

for all $w \in V$.

For later use, one can also define the fractional powers of the operator $A_\beta = I - P\Delta$ in theorem 3.5., $A_\beta^s : D(A_\beta^s) \mapsto H$ for $s \ge 0$ by

$$A_\beta^s\, u = \sum_{j=1}^{\infty} (1+\lambda_j)^s\, u_j\, e_j \tag{4.23}$$

for $u \in \sum_{j=1}^{\infty} u_j\, e_j \in D(A_\beta^s)$, where

$$D(A_\beta^s) = \left\{ u = \sum_{j=1}^{\infty} u_j\, e_j ; \sum_{j=1}^{\infty} (1+\lambda_j)^{2s}\, |u_j|^2 < \infty \right\} \tag{4.24}$$

equipped with the graph norm

$$||u||^2_{D(A_\beta^s)} = (A_\beta^s\, u, A_\beta^s\, u).$$

It can be checked easily that $A_\beta^s : D(A_\beta^{s+t}) \mapsto D(A_\beta^t)$ is an isomorphism for all $s,\ t \geq 0$, $D(A_\beta^1) = D(A_\beta) = H^2(\Omega) \cap W_\beta$, and $D(A_\beta^{\frac{1}{2}}) = V$ with equivalent norms $||u||_{D(A_\beta^{\frac{1}{2}})}$ and $H^1(\Omega)$-norm. Denote by $D(A_\beta^{-s})$ the dual space of $D(A_\beta^s)$ for any $s \geq 0$. Then the operator A_β^s can be extended to an operator: $H \mapsto D(A_\beta^{-s})$ by

$$(A_\beta^s u, v) = (u, A_\beta^s v), \qquad \forall u \in H, \quad v \in D(A_\beta^s) \tag{4.25}$$

It follows from the definition that

$$||u||^2_{D(A_\beta^{-s})} = ||A_\beta^{-s} u||^2, \qquad \forall u \in D(A_\beta^{-s}) \tag{4.26}$$

and $A^s : D(A_\beta^{s+t}) \mapsto D(A_\beta^t)$ is an isomorphism for $s,\ t \in \mathbb{R}$, and furthermore,

$$||A_\beta^{\frac{s+t}{2}} u||^2 = (A_\beta^s u, A_\beta^t u) \leq ||A_\beta^s u||\, ||A_\beta^t u||, \forall u \in D(A_\beta^s) \cap D(A_\beta^t) \tag{4.27}$$

holds true for all $s,\ t \in \mathbb{R}$.

5. Well-Posedness of the LNS-α Equations

In this section, we investigate the well-posedness of the initial boundary value problem of the LNS-α equations (4.1)-(4.6) by a Gelerkin approximation based on the orthogonal basis given in theorem 3.5..

5.1. Local well-posedness

We start with the following local well-posedness result.

Theorem 5.1. *Let $v_0 \in H$ and $\alpha > 0$. Then there is a time $T^* = T^*(v_0) > 0$ such that the problem (4.1)-(4.8) has a unique weak solution of (v, u) with initial data v_0 on the interval $[0, T^*)$ in the sense of definition 4.1 for any $T \in (0, T^*)$, which satisfies the energy equation*

$$\frac{d}{dt}\|v\|^2 + 2a_\beta(v, v) + (B(v, u), v) = 0, \text{ on } [0, T] \tag{5.1}$$

in the sense of distribution. Furthermore, if $v_0 \in V$, then

$$v \in L^2(0, T; W_\beta \cap H) \cap C([0, T); V) \tag{5.2}$$

$$v' \in L^2(0, T; H) \tag{5.3}$$

and the energy equation

$$\frac{d}{dt}a_\beta(v, v) + 2\|P\Delta v\|^2 + 2(B(v), -\Delta v) = 0 \tag{5.4}$$

is valid.

Proof. Let $v_0 \in H$. Consider the following system of ordinary differential equations

$$v_j'(t) + \lambda_j v_j(t) + g_j(\mathcal{V}) = 0 \tag{5.5}$$

$$v_j(0) = (u_0, e_j) \tag{5.6}$$

$j = 1, \cdots m$, where $\mathcal{V} = (v_j)$ and

$$g_i(\mathcal{V}) = (B(\Sigma_1^m v_j e_j, u_m), e_i) \tag{5.7}$$

$$u_m = T_\alpha(\Sigma_1^m v_j e_j) \tag{5.8}$$

Note that all norms are equivalent in a finite dimensional linear space. It follows from lemma 4.2. that $(g_j(\mathcal{V}))$ is locally Lipshitz in $\mathcal{V}$ and thus the systems is locally well posed and equivalent to the following partial differential equations

$$v_m'(t,x) - P\Delta v_m(t,x) + P_m B(v_m, u_m)(t,x) = 0 \tag{5.9}$$

$$u_m = T_\alpha(\Sigma_1^m v_j e_j) \tag{5.10}$$

$$v_m(0) = P_m(v_0) \tag{5.11}$$

where $v_m(t,x) = \Sigma_1^m v_j(t) e_j(x)$, and P_m is the orthogonal projection of H onto the space $spin\{e_j\}_1^m$.

Taking the inner product of (5.9) with v_m and noting that

$$(P_m B(v_m, u_m), v_m) = \int_\Omega \nabla \times v_m \times u_m \cdot (v_m) dx \tag{5.12}$$

one can get

$$\frac{d}{dt}\|v_m\|^2 + 2a_\beta(v_m, v_m) + (B(v_m, u_m), v_m) = 0 \tag{5.13}$$

It follows from the definition of T_α that

$$|(B(v_m, u_m), v_m)| \le c\|v_m\|_1 \|v_m\| \|u_m\|_{L^\infty(\Omega)} \le c\|v_m\|_1 \|v_m\|^2 \tag{5.14}$$

Note that

$$\|\phi\|^2 \le c\|\phi\|_1^2 \le c(\|\phi\|^2 + a_\beta(\phi, \phi)) \tag{5.15}$$

for all $\phi \in V$. It follows that

$$\frac{d}{dt}\|v_m\|^2 + a_\beta(v_m, v_m) \le c(\|v_m\|^2 + 1)\|v_m\|^2 \tag{5.16}$$

Hence, there is a time $T > 0$ such that

$$\{v_m\} \ \ \textit{is bounded in } L^\infty(0,T;H)$$

$$\{v_m\} \ \ \textit{is bounded in } L^2(0,T;V)$$

Note that for $\phi \in V$,

$$|(A_\beta v_m, \phi)| \le |(v_m, \phi)| + |a_\beta(v_m, \phi)| \tag{5.17}$$

which implies that

$$\{A_\beta v_m\} \textit{ is bounded in } \text{Ł}^2(0,T;V') \tag{5.18}$$

Since

$$\|u_m\|_{L^\infty(\Omega)} = c\|T_\alpha v_m\|_2 \le c\|v_m\| \tag{5.19}$$

it follows that

$$|(P_m B(v_m, u_m), \phi)| = |(\nabla \times v_m \times u_m, P_m \phi)| \le C\|v_m\|_1 \|v_m\| \|\phi\|_1 \tag{5.20}$$

for all $\phi \in V$, which implies that

$$\{P_m B(v_m, u_m)\} \text{ is bounded in } Ł^2(0,T;V') \tag{5.21}$$

Hence

$$\{v_m'\} \text{ is bounded in } Ł^2(0,T;V') \tag{5.22}$$

By using a similar argument in [20], it shows that there is a subsequence also denoted by v_m and a $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$ such that

$$v_m \to v \text{ in } L^\infty(0,T;H) \text{ weak} - \text{star} \tag{5.23}$$

$$v_m \to v \text{ in } L^2(0,T;V) \text{ weakly} \tag{5.24}$$

$$v_m \to v \text{ in } L^2(0,T;H) \text{ strongly} \tag{5.25}$$

Consequently, $u_m = T_\alpha v_m$ has the property:

$$u_m \to T_\alpha v \text{ in } L^\infty(0,T;W_\beta \cap H) \text{ weak} - \text{star} \tag{5.26}$$

$$u_m \to T_\alpha v \text{ in } L^2(0,T;V \cap H^3(\Omega)) \text{ weakly} \tag{5.27}$$

$$u_m \to T_\alpha v \text{ in } L^2(0,T;W_\beta \cap H) \text{ strongly} \tag{5.28}$$

Passing to the limit of a subsequence, it is showed that (v, u) is a weak solution in the sense of definition 4.3.. It also follows that the energy equation

$$\frac{d}{dt}\|v\|^2 + 2a_\beta(v,v) + (B_\alpha(v), v) = 0 \tag{5.29}$$

is valid on the interval $[0, T]$ in the sense of distribution.

Let v_1 and v_2 be any two solutions. Then $w = v_1 - v_2$ satisfies the following equation

$$w' - P\Delta w + P(B_\alpha(v_1) - B_\alpha(v_2)) = 0 \tag{5.30}$$

$$w(0) = 0 \tag{5.31}$$

and the energy equation

$$\frac{d}{dt}\|w\|^2 + 2a_\beta(w,w) + (B_\alpha(v_1) - B_\alpha(v_2), w) = 0 \tag{5.32}$$

It follows from the local Lipshitz continuity stated in lemma 4.2. and the Gronwall inquality that

$$\|w\|^2 \le c(T)\|w(0)\|^2 \quad \text{on}[0,T] \tag{5.33}$$

which implies the uniqueness of the solution. Consequently, the convergence of the whole sequence follows.

By the standard continuation method, there is a $T^* > 0$ such that the weak solution does exist on $[0, T]$ for all $T < T^*$, and if $T^* < \infty$ then

$$\|v(t)\| \to \infty, as\ t \to T^*$$

Let $v_0 \in V$. Taking the inner product of (5.9) with $-P\Delta v_m$ and noting that

$$(P_m B(v_m, u_m), -P\Delta v_m) = (B(v_m, u_m), -P\Delta v_m) \tag{5.34}$$

one gets

$$\frac{d}{dt}a_\beta(v_m, v_m) + 2\| - P\Delta v_m\|^2 + (B_\alpha(v_m), -P\Delta v_m) = 0 \tag{5.35}$$

It follows that

$$\frac{d}{dt}a_\beta(v_m, v_m) + \| - P\Delta v_m\|^2 \leq \|v_m\|_1^2 \|u_m\|_{L^\infty(\Omega)}^2 \tag{5.36}$$

Due to

$$\|u_m\|_{L^\infty(\Omega)} \leq c_\alpha \|v_m\| \tag{5.37}$$

$v_0 \in V$, the bounds of v_m in $L^2(0, T; V)$, and the Gronwall's inequality, one has

$$\{v_m\} \;\; \textit{is bounded in } L^\infty(0, T; V)$$
$$\{v_m\} \;\; \textit{is bounded in } L^2(0, T; W_\beta \cap H)$$

which, together with the uniqueness, implies that the whole sequence indeed converges in the sense

$$v_m \to v \textit{ in } L^\infty(0, T; V) \textit{ weak} - \textit{star} \tag{5.38}$$
$$v_m \to v \textit{ in } L^2(0, T; W_\beta \cap H) \textit{ weakly} \tag{5.39}$$
$$v_m \to v \textit{ in } L^2(0, T; V) \textit{ strongly} \tag{5.40}$$

This completes the proof of theorem 5.1.. □

5.2. Global well-posedness

Now, we prove the following global well-posedness result.

Theorem 5.2. *If $v_0 \in H, \alpha > 0$, then the solution v obtained in theorem 5.1. is global, i.e., $T^* = T^*(v_0) = \infty$.*

Proof. Let v be the weak solution on the interval $[0, T]$. Then,

$$u = T_\alpha v \in L^\infty(0, T; W_\beta \cap H) \tag{5.41}$$

Taking u as a test function yields

$$(v', u) + a_\beta(v, u) + (B(v, u), u) = 0 \tag{5.42}$$

Since $v = (I - \alpha P\Delta)u$, then

$$2(v', u) = \frac{d}{dt}(\|u\|^2 + \alpha a_\beta(u, u)) \tag{5.43}$$

in the sense of distribution on $[0, T]$. Note that

$$(B(v, u), u) = \int_\Omega (\nabla \times v) \times u \cdot u = 0 \tag{5.44}$$

It follows that

$$\frac{d}{dt}(\|u\|^2 + \alpha a_\beta(u, u)) + 2(\int_{\partial\Omega} \beta u \cdot v + \int_\Omega (\nabla \times v) \cdot (\nabla \times u)) = 0 \tag{5.45}$$

Due to the smoothness and the boundary condition for u, it holds that

$$\int_\Omega (\nabla \times v) \cdot (\nabla \times u) = -\int_{\partial\Omega} \beta u \cdot v + \int_\Omega (-\Delta u) \cdot v \tag{5.46}$$

Consequently

$$\frac{d}{dt}(\|u\|^2 + \alpha a_\beta(u,u)) + 2(a_\beta(u,u) + \alpha\|P\Delta u\|^2) = 0 \tag{5.47}$$

It follows that

$$(\|u\|^2 + \alpha a_\beta(u,u)) \le (\|u_0\|^2 + \alpha a_\beta(u_0,u_0)) \tag{5.48}$$

and

$$\int_0^t (a_\beta(u,u) + \alpha\|P\Delta u\|^2)d\tau \le (\|u_0\|^2 + \alpha a_\beta(u_0,u_0)) \tag{5.49}$$

On the other hand, it follows from the energy equation (5.1) and a similar argument as for (5.17) that

$$\frac{d}{dt}\|v\|^2 + a_\beta(v,v) \le c\|v\|^4 + 1 \tag{5.50}$$

Noting that

$$\|v\|^2 \le c(\|u\|^2 + \alpha^2\|P\Delta u\|^2) \tag{5.51}$$

it follows that

$$\|v\|^2 + \int_0^t a_\beta(v,v) \le c \tag{5.52}$$

for some constant c depending only on v_0 and α. Thus $T^* = \infty$. The theorem is proved. □

6. Vanishing α Limit and the NS Equations

In this section, we investigate the vanishing α limit of the solutions of the LNS-α equations ($\alpha \to 0$) to that of the NS equations. We will prove both weak and strong convergence results. Then, the global existence of weak solutions and the local unique strong solution to the NS equations with the VSB condition are followed.

6.1. Weak Convergence and Global Weak Solutions of the NS

We first prove

Theorem 6.1. *Let $v_0 \in H$, and (v^α, u^α) be the global weak solution stated in theorem 5.2. corresponding to the parameter $\alpha > 0$. Then for any given $T > 0$ there is a subsequence u^{α_j} of u^α and a (v^0, u^0) satisfying*

$$v^0 \in L^2(0,T;V) \cap C_w([0,T];H) \tag{6.1}$$

$$(v^0)' \in L^{\frac{4}{3}}(0,T;V') \tag{6.2}$$

such that

$$v^{\alpha_j} \to v^0 \textit{ in } L^2(0,T;H) \textit{ weakly} \tag{6.3}$$

$$v^{\alpha_j} \to v^0 \textit{ in } L^2(0,T;D(A_\beta^{-\frac{1}{4}})) \textit{ strongly} \tag{6.4}$$

$$u^{\alpha_j} \to v^0 \textit{ in } L^2(0,T;V_\beta) \textit{ weakly} \tag{6.5}$$

$$u^{\alpha_j} \to v^0 \textit{ in } L^2(0,T;D(A_\beta^{-\frac{1}{4}})) \textit{ strongly} \tag{6.6}$$

Moreover (v^0, v^0) is a weak solution of the initial boundary problem of the NS equations (4.1)-(4.6) with $\alpha = 0$ and satisfies the energy inequality

$$\frac{d}{dt}\|v^0\|^2 + 2a_\beta(v^0, v^0) \le 0 \tag{6.7}$$

Proof. Let $v_0 \in H$, $T > 0$, and (v^α, u^α) be the global weak solution to (4.1)-(4.6) corresponding to $1 \ge \alpha > 0$. It follows from (5.47) that

$$\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha, u^\alpha) + \int_0^t (a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2)d\tau \le c \tag{6.8}$$

for some constant c independent of α. For any $\phi \in W_\beta \cap H$, we have

$$(B(v^\alpha, u^\alpha), \phi) = \int_\Omega (\nabla \times v^\alpha \times u^\alpha)\phi dx = I + II \tag{6.9}$$

where

$$I = \int_{\partial\Omega} (n \times v^\alpha) \cdot (u^\alpha \times \phi) dS \tag{6.10}$$

$$II = \int_\Omega v^\alpha \cdot (-u^\alpha \cdot \nabla\phi - \phi \cdot \nabla u^\alpha) dx \tag{6.11}$$

Since $u \cdot n = 0$ and $\phi \cdot n = 0$ on the boundary so

$$u^\alpha \times \phi = \lambda n \text{ on } \partial\Omega \tag{6.12}$$

Hence

$$I = 0 \tag{6.13}$$

To estimate II, we note that

$$|\int_\Omega v^\alpha \cdot (u^\alpha \cdot \nabla\phi)dx| \le c(\|u^\alpha\| + \alpha\|P\Delta u^\alpha\|)\|u^\alpha\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)} \tag{6.14}$$

$$\|u^\alpha\|^2_{L^3(\Omega)} \le c\|u^\alpha\|\|u^\alpha\|_1 \le c\|u^\alpha\|^{\frac{3}{2}}(\|u^\alpha\| + \|P\Delta u^\alpha\|)^{\frac{1}{2}} \tag{6.15}$$

$$\|\nabla\phi\|_{L^6(\Omega)} \le c\|A_\beta\phi\| \tag{6.16}$$

Then, due to (6.8), it holds that

$$|\int_\Omega v^\alpha \cdot (u^\alpha \cdot \nabla\phi)dx| \le c((a_\beta(u^\alpha, u^\alpha))^{\frac{1}{2}} + \alpha||P\Delta u^\alpha|| + \alpha||P\Delta u^\alpha\|^{\frac{5}{4}})\|A_\beta\phi\| \tag{6.17}$$

Next,

$$|\int_\Omega v^\alpha \cdot (\phi \cdot \nabla u^\alpha)dx| \le c(\|u^\alpha\| + \alpha\|P\Delta u^\alpha\|)\|u^\alpha\|_1\|\phi\|_{L^\infty(\Omega)} \tag{6.18}$$

which implies that

$$|\int_\Omega v^\alpha \cdot (\phi \cdot \nabla u^\alpha)dx| \le c((a_\beta(u^\alpha, u^\alpha))^{\frac{1}{2}} + \alpha||P\Delta u^\alpha|| + \alpha||P\Delta u^\alpha\|^{\frac{3}{2}})\|A_\beta\phi\| \tag{6.19}$$

Then for $\alpha < 1$,

$$|(B(v^\alpha, u^\alpha), \phi)| \le c(1 + (a_\beta(u^\alpha, u^\alpha))^{\frac{1}{2}} + \alpha^{\frac{3}{4}}\|P\Delta u^\alpha\|^{\frac{3}{2}})\|A_\beta\phi\| \tag{6.20}$$

It follows from (6.8) and (6.20) that $B(v^\alpha, u^\alpha)$ and then $\frac{d}{dt}(v^\alpha)$ are uniformly bounded in $L^{\frac{4}{3}}(0,T; D(A_\beta^{-1}))$. It follows from (4.7)-(4.9) that

$$(1-\alpha)u_t^\alpha + \alpha A_\beta(u_t^\alpha) = v_t^\alpha,$$

which yields immediately

$$(1-\alpha)||A_\beta^{-1}\, u_t^\alpha||^2 + \alpha||A_\beta^{-1}\, u_t^\alpha||^2 = ||A_\beta^{-1}\, v_t^\alpha||^2.$$

Then

$$||A_\beta^{-1}\, u_t^\alpha||^2 \leq 2||A_\beta^{-1}\, v_t^\alpha||^2$$

for $0 < \alpha \leq \frac{1}{2}$. This shows that $\partial_t\, u^\alpha$ are uniformly bounded in $L^{\frac{4}{3}}(0,T; D(A_\beta^{-1}))$ as $\partial_t\, v^\alpha$ are. Note that (6.8) also implies that (u^α) are uniformly bounded in $L^2(0,T;V)$ and the duality between $V = D(A_\beta^{\frac{1}{2}})$ and $D(A_\beta^{-1})$ with respect to the inner product of $D(A_\beta^{-\frac{1}{4}})$, i.e.,

$$(A_\beta^{-\frac{1}{4}} u, A_\beta^{-\frac{1}{4}} \phi) = (A_\beta^{\frac{1}{2}} u, A_\beta^{-1}\phi)$$

By using the standard compactness argument (see [21, 20]), one can show that there exist a subsequence u^{α_j} of u^α and a v^0 such that

$$u^{\alpha_j} \to v^0 \ \textit{in}\ L^2(0,T;V)\ \textit{weakly} \tag{6.21}$$

$$u^{\alpha_j} \to v^0 \ \textit{in}\ L^2(0,T; D(A_\beta^{-\frac{1}{4}}))\ \textit{strongly} \tag{6.22}$$

Note that

$$|(B(v^\alpha, u^\alpha) - B(v^0, v^0), \phi)| \leq I + II \tag{6.23}$$

where

$$I = |(B(u^\alpha - v^0, u^\alpha) + B(v^0, u^\alpha - v^0), \phi)| \tag{6.24}$$

$$II = \alpha|(B(P\Delta u^\alpha, u^\alpha), \phi)| \tag{6.25}$$

Similar to (6.9) and (6.13), integrating by part yields

$$|(B(u^\alpha - v^0, u^\alpha), \phi)| = |\int_\Omega (u^\alpha - v^0)\cdot(u^\alpha \cdot \nabla\phi + \phi\cdot\nabla u^\alpha)| \tag{6.26}$$

Note that

$$|\int_\Omega (u^\alpha - v^0)\cdot(u^\alpha\cdot\nabla\phi)| \leq c\|u^\alpha - v^0\|\|u^\alpha\|^{\frac{1}{2}}\|u^\alpha\|^{\frac{1}{2}}_{L^6(\Omega)}\|\nabla\phi\|_{L^6(\Omega)} \tag{6.27}$$

and

$$\|u^\alpha - v^0\|^2 \leq c\|u^\alpha - v^0\|_{D(A_\beta^{-\frac{1}{4}})}\|u^\alpha - v^0\|_{D(A_\beta^{\frac{1}{4}})} \tag{6.28}$$

$$\|u^\alpha - v^0\|^2_{D(A_\beta^{\frac{1}{4}})} \leq c\|u^\alpha - v^0\|\|u^\alpha - v^0\|_1 \tag{6.29}$$

This, together with (6.8), shows that

$$|\int_\Omega (u^\alpha - v^0)|\cdot(u^\alpha\cdot\nabla\phi) \leq c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}\|u^\alpha - v_0\|_1^{\frac{1}{4}}\|u^\alpha\|_1^{\frac{1}{2}}\|A_\beta\phi\| \tag{6.30}$$

Hence

$$|\int_\Omega (u^\alpha - v^0)|\cdot(u^\alpha\cdot\nabla\phi) \leq c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}(\|u^\alpha\|_1^{\frac{3}{4}} + \|v^0\|_1^{\frac{3}{4}})\|A_\beta\phi\| \tag{6.31}$$

While

$$|\int_\Omega (u^\alpha - v^0)\cdot(\phi\cdot\nabla u^\alpha)| \le \|u^\alpha - v^0\|\|u^\alpha\|_1\|\phi\|_{L^\infty(\Omega)} \tag{6.32}$$

It follows that

$$|\int_\Omega (u^\alpha - v^0)|\cdot(u^\alpha\cdot\nabla\phi) \le c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}(\|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}})\|A_\beta\phi\| \tag{6.33}$$

Then

$$|(B(u^\alpha - v^0, u^\alpha),\phi)| \le c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}})\|A_\beta\phi\| \tag{6.34}$$

Similarly, one can obtain

$$|(B(v^0, u^\alpha - v^0),\phi)| \le c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}})\|A_\beta\phi\| \tag{6.35}$$

It follows that

$$I \le c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{-\frac{1}{4}})}(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}})\|A_\beta\phi\| \tag{6.36}$$

Similarly,

$$|(B(P\Delta u^\alpha, u^\alpha),\phi)| = |\int_\Omega (P\Delta u^\alpha)\cdot(u^\alpha\cdot\nabla\phi + \phi\cdot\nabla u^\alpha)| \tag{6.37}$$

Then

$$|(B(P\Delta u^\alpha, u^\alpha),\phi)| \le c\|P\Delta u^\alpha\|\|u^\alpha\|_1\|A_\beta\phi\| \tag{6.38}$$

It follows that

$$II \le c\alpha^{\frac{1}{2}}(\alpha\|P\Delta u^\alpha\|^2 + \|u^\alpha\|_1^2)\|\phi\|_2 \tag{6.39}$$

It follows from (6.23),(6.36),(6.39), (6.8) and (6.21) that

$$B(v^{\alpha_j}, u^{\alpha_j}) \to B(v^0, v^0) \ in \ L^1(0,T; D(A_\beta^{-1})) \ strongly \tag{6.40}$$

which enables us to pass the limit in (4.19)-(4.20) to show that v^0 satisfies

$$((v^0)',\phi) + a_\beta(v^0,\phi) + ((\nabla\times v^0)\times v^0,\phi) = 0, \ a.e. \ t \tag{6.41}$$

for all $\phi \in C^\infty(\Omega)\cap V$ in the sense of distribution on $[0,T]$. Note that $v^0 \in L^2(0,T;V)$ implies $(v^0)' \in L^{\frac{4}{3}}(0,T;V')$. Thus (6.41) is also valid for all $\phi \in V$. Due to (5.47), it holds that

$$\frac{d}{dt}(\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha,u^\alpha)) + 2a_\beta(u^\alpha,u^\alpha) \le 0 \tag{6.42}$$

Passing to the limit and noting the weak lower semi-continuity of the norm, one gets

$$\frac{d}{dt}\|v^0\|^2 + 2a_\beta(v^0,v^0) \le 0 \tag{6.43}$$

Note that

$$(v^\alpha - v^0,\phi) = (u^\alpha - v^0,\phi) + \alpha((A_\beta^{\frac{3}{4}}u^\alpha, A_\beta^{\frac{1}{4}}\phi) - (u^\alpha,\phi)) \tag{6.44}$$

for $\phi \in D(A_\beta^{-\frac{1}{4}})$. Then

$$\|v^\alpha - v^0\|^2_{D(A_\beta^{-\frac{1}{4}})} \leq \|v^\alpha - v^0\|^2_{D(A_\beta^{-\frac{1}{4}})} + c\alpha^{\frac{1}{2}}(\alpha\|P\Delta u^\alpha\|^2 + \|u^\alpha\|_1^2)$$

It follows that

$$v^{\alpha_j} \to v^0 \ in \ L^2(0,T; D(A_\beta^{-\frac{1}{4}})) \ strongly \tag{6.45}$$

Note that

$$(v^\alpha - v^0, \phi) = (u^\alpha - v^0, \phi) - \alpha(P\Delta u^\alpha, \phi) \tag{6.46}$$

It follows from (6.8) and (6.21) that

$$v^{\alpha_j} \to v^0 \ in \ L^2(0,T; H) \ weakly \tag{6.47}$$

Hence, the theorem is proved. □

6.2. Strong Convergence and the Strong Solutions of the NS

We now turn to the strong convergence of the strong solutions of the LNS-α to that of the NS equations, and prove

Theorem 6.2. *Let $v_0 \in V$ and (v^α, u^α) be the strong solution stated in theorem 5.1. corresponding to the parameter $\alpha > 0$. Then there is a $T > 0$ and a v^0 in $L^\infty(0,T;V) \cap L^2(0,T; W_\beta \cap H)$ such that*

$$v^\alpha \to v^0 \ in \ L^2(0,T; W_\beta \cap H) \ weakly \tag{6.48}$$

$$v^\alpha \to v^0 \ in \ L^2(0,T; V) \ strongly \tag{6.49}$$

$$u^\alpha \to v^0 \ in \ L^2(0,T; W_\beta \cap H) \ weakly \tag{6.50}$$

$$u^\alpha \to v^0 \ in \ L^2(0,T; V) \ strongly \tag{6.51}$$

with v^0 being a weak solution to the initial boundary problem of the NS equation (4.1)-(4.6) with $\alpha = 0$ which is unique and thus called the strong solution. Consequently, it can be extended to the maximal existence time interval $[0, T^*)$ such that if $T^* < \infty$ then

$$\|v^0\|_1 \to \infty, \ as \ t \to T^*$$

Moreover, the following energy equation holds:

$$\frac{d}{dt}a_\beta(v^0, v^0) + 2\|P\Delta v^0\|^2 - 2(B(v^0, v^0), P\Delta v^0) = 0 \tag{6.52}$$

Proof. It follows from the energy equation (5.5) that

$$\frac{d}{dt}a_\beta(v^\alpha, v^\alpha) + \|P\Delta v^\alpha\|^2 \leq c\|B(v^\alpha, u^\alpha)\|^2 \tag{6.53}$$

Note that

$$\|B(v^\alpha, u^\alpha)\|^2 \leq c\int_\Omega |\nabla \times v^\alpha|^2 |u^\alpha|^2 dx \leq c\|\nabla \times v^\alpha\|^2_{L^3(\Omega)} \|u^\alpha\|^2_{L^6(\Omega)} \tag{6.54}$$

$$\|\nabla \times v^\alpha\|^2_{L^3(\Omega)} \leq c(\|v^\alpha\| + \|P\Delta v^\alpha\|)\|v^\alpha\|_1 \tag{6.55}$$

$$\|u^\alpha\|_{L^6(\Omega)} \leq c\|u^\alpha\|_1 \tag{6.56}$$

and

$$||u^\alpha||_1 \le c||v^\alpha||_1 \tag{6.57}$$

which follows from the fact that

$$\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha, u^\alpha) = (v^\alpha, u^\alpha) \tag{6.58}$$

$$a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2 = a_\beta(u^\alpha, v^\alpha) \tag{6.59}$$

Consequently,

$$\frac{d}{dt}a_\beta(v^\alpha, v^\alpha) + \frac{1}{2}\|P\Delta v^\alpha\|^2 \le c(1 + \|v^\alpha\|_1^2)||v^\alpha||_1^4 \tag{6.60}$$

Combining this with similar estimates for (5.17) yields

$$\frac{d}{dt}(\|v^\alpha\|^2 + a_\beta(v^\alpha, v^\alpha)) + \frac{1}{2}(\|v^\alpha\|^2 + \|P\Delta v^\alpha\|^2) \le c(1 + \tilde{a}_\beta(v^\alpha, v^\alpha))^3 \tag{6.61}$$

Comparing it with the following ordinary differential equation

$$\frac{d}{dt}y = c(1+y)^3$$

with $y(0) = \tilde{a}_\beta(v_0, v_0)$ shows that there is a time T such that

$$v^\alpha \textit{ is uniform bounded in } L^\infty(0, T; V) \cap L^2(0, T; W_\beta \cap H)$$

It follows from this, (6.55)-(6.60), and (4.19) that

$$(v^\alpha)' \textit{ is uniform bounded in } L^2(0, T; H).$$

Hence, by using the standard compactness argument, we find a subsequence v^{α_j} of v^α and a v^0 such that

$$v^{\alpha_j} \to v^0 \textit{ in } L^2(0, T; W_\beta \cap H) \textit{ weakly} \tag{6.62}$$

$$v^{\alpha_j} \to v^0 \textit{ in } L^2(0, T; V) \textit{ strongly} \tag{6.63}$$

which enables one to pass to the limit to find $v^0 \in C([0, T]; V) \cap L^2(0, T; W_\beta \cap H)$ such that $(v^0, v^0)$is a (strong) solution of the NS equations.
Let v_1^0 and v_2^0 be two strong solutions to the Navier-Stokes equations with same initial data. Set $w = v_1^0 - v_2^0$. Then

$$\frac{d}{dt}\|w\|^2 + 2a_\beta(w, w) + 2(B(v_1^0, v_1^0) - B_0(v_2^0, v_2^0), w) = 0 \tag{6.64}$$

Note that

$$\begin{aligned} |(B(v_1^0, v_1^0) - B(v_2^0, v_2^0), w)| \le \; & |(B(w, v_1^0), w)| + |(B(v_2^0, w), w)| \\ \le \; & \tilde{a}_\beta(w, w) + c(||v_1^0||_{L_\infty}(t) + ||\nabla \times v_2^0||^4)||w||^2 \end{aligned}$$

which, together with (5.53) and Gronwall's inequality, yields $||w|| = 0$. Thus we have obtained the uniqueness of the strong solution to the initial boundary value problem for the Navier-Stokes equations. By the standard continuation method, the strong solution can be extended to the maximum existent time interval $[0, T^*) \supset [0, T]$, and the energy equation follows from the smoothness of the solution. Consequently, the convergence of the whole sequence of v^α follows.

Finally, we prove the convergence of u^α. It follows from (4.3) that

$$\nabla \times u^\alpha - \alpha\Delta(\nabla \times u^\alpha) = \nabla \times v^\alpha, \text{ in } \Omega \tag{6.65}$$

Taking the inner product of above equality with $-\Delta(\nabla \times u^\alpha)$ and integrating by part, we can get

$$\|\Delta u^\alpha\|^2 + \alpha\|(\nabla\times)^3 u^\alpha\| = (\Delta u^\alpha, \Delta v^\alpha) + \int_{\partial\Omega} \Delta u^\alpha \cdot \beta(v^\alpha - u^\alpha) \tag{6.66}$$

To handle the last term on the right hand side above, we use the fact $v^\alpha - u^\alpha = n \times ((v^\alpha - u^\alpha) \times n)$ on $\partial\Omega$ and the Stokes formula to get

$$\begin{aligned}
&\int_{\partial\Omega} \Delta u^\alpha \cdot (\beta(v^\alpha - u^\alpha)) = \int_{\partial\Omega} \Delta u^\alpha \cdot (n \times (\beta(v^\alpha - u^\alpha) \times n)) \\
= &\int_{\partial\Omega} (n \times \Delta u^\alpha) \cdot (\beta(u^\alpha - v^\alpha) \times n) \\
= &\int_{\Omega} (\nabla \times (\Delta u^\alpha)) \cdot (\beta(u^\alpha - v^\alpha) \times n) - \int_{\Omega} \Delta u^\alpha \cdot \nabla \times (\beta(u^\alpha - v^\alpha) \times n)
\end{aligned} \tag{6.67}$$

where we have extended β and n smoothly to $\bar{\Omega}$. It follows from (4.7) that

$$||v^\alpha - u^\alpha||^2 = (-\alpha\Delta u^\alpha, v^\alpha - u^\alpha) \leq \alpha||\Delta u^\alpha||\,||v^\alpha - u^\alpha||$$

which yields

$$||v^\alpha - u^\alpha|| \leq \alpha||\Delta u^\alpha|| \tag{6.68}$$

It follows from (6.57) and (6.68) that

$$\left|\int_{\Omega} \Delta u^\alpha \cdot \nabla \times (\beta(u^\alpha - v^\alpha) \times n)\right| \leq \frac{1}{4}||\Delta u^\alpha||^2 + c||v^\alpha||_1^2 \tag{6.69}$$

for suitably small α. Using (6.68) again gives

$$\begin{aligned}
&\left|\int_\Omega (\nabla \times \Delta u^\alpha) \cdot (\beta(u^\alpha - v^\alpha) \times n)\right| \\
\leq\ &\frac{1}{2}\alpha \int_{\Omega} |\nabla \times (\Delta u^\alpha)|^2 + \alpha^{-1} c||u^\alpha - v^\alpha||^2 \\
\leq\ &\frac{1}{2}\alpha \int_{\Omega} |\nabla \times (\Delta u^\alpha)|^2 + c\alpha||\Delta u^\alpha||^2
\end{aligned} \tag{6.70}$$

Collecting (6.66),(6.67), (6.69) and (6.70) leads to

$$||\Delta u^\alpha||^2 + \alpha||(\nabla\times)^3\, u^\alpha||^2 \leq c||v^\alpha||_2^2 \tag{6.71}$$

for suitably small α. This, together with the bound of $\partial_t\, u^\alpha$ in H, implies the desired convergence in (6.50),(6.51). Thus the theorem is proved. □

6.3. Estimates on Convergence Rates

Finally, we study the rates of convergence in the case of strong solutions. We start with the case that the limiting Navier-Stokes system has a strong solution.

Theorem 6.3. *Let $v_0 \in V$ and v^0 be the strong solution to the Navier-Stokes equation with initial data v_0 on any given finite interval $[0,T]$ with $T > 0$. Then there exists a $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0]$, the LNS-α with the initial data v_0 has a unique strong solution (v^α, u^α) on the same interval $[0,T]$ satisfying*

$$\sup_{0\le t\le T} ||(v^\alpha, u^\alpha) - (v^0, v^0)||^2 + \int_0^T ||(v^\alpha, u^\alpha) - (v^0, v^0)||_1^2(t)dt \le c\alpha \quad (6.72)$$

$$\sup_{0\le t\le T} ||v^\alpha - v^0||_1^2 + \int_0^T ||v^\alpha - v^0||_2^2 dt \le c\alpha^{\frac{1}{2}} \quad (6.73)$$

with c being a positive constant depending on v^0.

Proof. Thanks to the local well-posedness of the strong solution to the initial-boundary value problem for the LNS-α and the standard continuation arguments, theorem 6.3. will follow immediately from the following a priori estimates. □

Proposition 6.4. *Let $T_1 \in (0,T]$ and (u^α, v^α) be the strong solution to the LANS-α with the initial data v^0 on the interval $[0,T_1]$ with the property that*

$$||v^\alpha||_1^2(t) + \int_0^t ||v^\alpha||_2^2(\tau)d\tau \le c_0 \qquad \text{for } t \in [0,T_1] \quad (6.74)$$

with a positive constant c_0 depending only on T and $v^0 \in L^\infty(0,T:V) \cap L^\tau(0,T:W_\beta)$. Then there exist uniform constants α_1 and c with the same dependence as c_0 such that

$$\sup_{0\le t\le T_1} ||(v^\alpha, u^\alpha) - (v^0, v^0)||^2 + \int_0^{T_1} ||(v^\alpha, u^\alpha) - (v^0, v^0)||_1^2(t)dt \le c\alpha \quad (6.75)$$

$$\sup_{0\le t\le T_1} ||(v^\alpha - v^0)||_1^2(t) + \int_0^{T_1} ||v^\alpha - v^0||_2^2(t)dt \le c\alpha^{\frac{1}{2}} \quad (6.76)$$

for $\alpha \in (0, \alpha_1]$.

Remark 6.5. Assuming Proposition 6.4. for a moment, one can verify the a priori assumption (6.74) by choosing

$$c_0 = 1 + 4 \sup_{0\le t\le T} ||v^0||_1^2 + \int_0^T ||v^0||_2^2 dt$$

and using (6.75) and (6.76) to choose α_0. Thus (6.75),(6.76) hold for $T_1 = T$. This yields theorem 6.3. immediately. It remains to verify proposition 6.4.

Proof of Proposition 6.4. Set $w = v^\alpha - v^0$. Then it holds that for $t \in [0,T_1]$,

$$\frac{d}{dt}\|w\|^2(t) + 2a_\beta(w,w)(t) + 2(B(v^\alpha, u^\alpha) - B(v^0, u^0), w)(t) = 0 \quad (6.77)$$

Note that

$$2(B(v^\alpha, u^\alpha) - B(v^0, v^0), w) = 2(B(v^\alpha, v^\alpha) - B(v^0, v^0), w) + 2\alpha((\nabla\times v^\alpha)\times P(\Delta u^\alpha), w)$$

$$|(B(v^\alpha, v^\alpha) - B(v^0, v^0), w)| = |(B(w, v^0), w)| \le c||v^0||_2\,||w||_1\,||w||$$
$$|(\nabla \times v^\alpha) \times P(\Delta u^\alpha), w| \le c||P(\Delta u^\alpha)||\,||v^\alpha||_2\,||w||_1$$

It follows that for all $t \in [0, T_1]$

$$\frac{d}{dt}\|w\|^2(t) + a_\beta(w,w)(t) \le c(1 + \|v^0\|_2^2)\|w\|^2 + c\alpha^2\|P(\Delta u^\alpha)\|^2\,\|v^\alpha\|_2^2 \tag{6.78}$$

Due to (6.59), (6.60) and (6.74), one has

$$\alpha||P(\Delta u^\alpha)||^2 \le cc_0 \qquad \text{for all } t \in [0, T_1] \tag{6.79}$$

which, together with (6.78), yields

$$\frac{d}{dt}\|w\|^2(t) + a_\beta(w,w)(t) \le c(1 + \|v^0\|_2^2)\,\|w\|^2(t) + cc_0\alpha\|v^\alpha\|_2^2(t) \tag{6.80}$$

Since $w(0) = 0$, so Gronwall's inequality leads to

$$\begin{aligned} &\sup\nolimits_{0\le t\le T_1} ||w(t)||^2 + \textstyle\int_0^{T_1} ||w||_1^2(t)dt \\ \le\ & (c\alpha c_0\, e^{c\int_0^{T_1}(1+||v^0||_2^2(t))dt}) \int_0^{T_1} ||v^\alpha||_2^2(t)dt \\ \le\ & c\alpha\, c_0^2 e^{cc_1} \ \equiv\ c_2\alpha \end{aligned} \tag{6.81}$$

with c_1 depending only T and the $L^2(0, T : H^2)$-norm of v^0.

To prove (6.75) for u^α, we note that (4.3) implies that

$$||u^\alpha - v^0||^2 + \alpha a_\beta(u^\alpha, u^\alpha - v^0) = (v^\alpha - v^0, u^\alpha - v^0) \tag{6.82}$$

$$a_\beta(u^\alpha - v^0, u^\alpha - v^0) + \alpha(P(\Delta u^\alpha), \Delta(u^\alpha - v^0)) = a_\beta(v^\alpha - v^0, u^\alpha - v^0) \tag{6.83}$$

It follows from (6.82), (6.83), and (6.74) that

$$\begin{aligned} ||u^\alpha - v^0||^2 + a_\beta(u^\alpha - v^0, u^\alpha - v^0) \ &\le\ ||v^\alpha - v^0||^2 + \alpha a_\beta(v^0, v^0) \\ &\le\ c_2\alpha + cc_0\alpha \end{aligned} \tag{6.84}$$

Since

$$\begin{aligned} \alpha(P(\Delta u^\alpha), \Delta(u^\alpha - u^0)) \ &=\ \alpha||P(\Delta u^\alpha)||^2 - \alpha(P(\Delta u^\alpha), \Delta v^0) \\ &\ge\ \frac{1}{2}\alpha||P(\Delta u^\alpha)||^2 - \frac{\alpha}{2}||\Delta v^0||^2 \end{aligned}$$

This, together with (6.84), shows that

$$a_\beta(u^\alpha - v^0, u^\alpha - v^0) + \alpha||P(\Delta u^\alpha)||^2 \le a_\beta(v^\alpha - v^0, v^\alpha - v^0) + \alpha||\Delta v^0||^2 \tag{6.85}$$

Hence, one obtains from (6.82), (6.84), and (6.85) that

$$\sup_{0\le t\le T_1} ||u^\alpha - v^0||^2 + \int_0^{T_1} ||u^\alpha - v^0||_1^2\, dt \le c_3\,\alpha$$

This completes the verification of (6.75). It remains to prove (6.76).

By the definition of strong solutions, one has that for a.e. $t \in [0, T_1]$,

$$\frac{d}{dt}a_\beta(w,w) + 2||P\Delta w||^2 + 2(B(v^\alpha, u^\alpha) - B(v^0, v^0), P\Delta w) = 0 \tag{6.86}$$

Rewrite the last term on the left hand side above as

$$2(B(v^\alpha, u^\alpha) - B(v^0, v^0), -P(\Delta w)) = I + II$$

One can estimate each term as

$$|I_1| = |2(B(v^\alpha, v^\alpha) - B(v^0, v^0), -P(\Delta w))| \le c(||v^\alpha||_2 + ||v||_2)||w||_1\, ||P\Delta w||$$

$$|II_2| = |2\alpha((\nabla\times v^\alpha)\times(P\Delta u^\alpha), P\Delta w)| \le c^\alpha ||P\Delta u^\alpha||_1\, ||v^\alpha||_1^{\frac{1}{2}}\, ||v^\alpha||_2^{\frac{1}{2}}\, ||P\Delta w||$$

It follows that for a.e. $t \in [0, T_1]$,

$$\begin{aligned} & \frac{d}{dt}\tilde{a}_\beta(w,w)(t) + ||P\Delta w||^2(t) \\ \le\ & c(1 + ||v^0||_2^2 + ||v^\alpha||_2^2)\tilde{a}_\beta(w,w) \\ & + c\alpha^2 ||P\Delta u^\alpha||_1^2\, ||v^\alpha||_1\, ||v^\alpha||_2 + c\alpha^2 ||P\Delta u^\alpha||^2\, ||v^\alpha||_2^2 \end{aligned} \tag{6.87}$$

As a consequence of (6.57), (6.74) and

$$\alpha||P(\Delta u^\alpha)||_1 = ||u^\alpha - v^\alpha||_1$$

one has

$$\begin{aligned} & c\alpha^2\, ||P\Delta u^\alpha||_1\, ||v^\alpha||_1\, ||v^\alpha||_2 \\ \le\ & c||v^\alpha - u^\alpha||_1^2\, ||v^\alpha||_1\, ||v^\alpha||_2 \\ \le\ & c||v^\alpha||_1\, ||v||_2^\alpha\, \tilde{a}_\beta(w,w) + c||u^\alpha - v^0||_1^2\, ||v^\alpha||_1\, ||v^\alpha||_2 \end{aligned} \tag{6.88}$$

It follows from (6.87), (6.88), and (6.79) that

$$\begin{aligned} & \frac{d}{dt}\tilde{a}_\beta(w,w) + \nu||A_\beta w||^2 \\ \le\ & c(1 + ||v^0||_2^2 + ||v^\alpha||_2^2)\tilde{a}_\beta(w,w) \\ & + cc_0^{\frac{1}{2}}\, ||u^\alpha - v^0||_1^2\, ||v^\alpha||_2 + cc_0\alpha||v^\alpha||_2^2 \end{aligned} \tag{6.89}$$

Consequently, we can get

$$\begin{aligned} & \sup_{0\le t\le T_1} ||w(t)||_1^2 + \int_0^{T_1} ||w(t)||_2^2 dt \\ \le\ & \int_0^{T_1} e^{c\int_0^{T_1}(1+||v_0||_2^2+||v^\alpha||_2^2)dt}\, [cc_0\alpha||v^\alpha(t)||_2^2 + cc_0^{\frac{1}{2}}||u^\alpha - v^0||_1^2\, ||v^\alpha||_2]\, dt \\ \le\ & c_1\,\alpha + c_1\int_0^{T_1} ||u^\alpha - v^0||_1^2\, ||v^\alpha||_2\, dt \\ \le\ & c_1\,\alpha + c_1\left(\int_0^{T_1} ||u^\alpha - v^0||_1^4 dt\right)^{\frac{1}{2}} \left(\int_0^{T_1} ||v^\alpha||_2^2 dt\right)^{\frac{1}{2}} \\ \le\ & c_1\,\alpha + c_1(||v^\alpha||_1 + ||v^0||_1^2)\left(\int_0^{T_1} ||u^\alpha - v^0||_1^2 dt\right)^{\frac{1}{2}} \le c_1\,\alpha^{\frac{1}{2}} \end{aligned}$$

where we have used (6.75). Thus (6.76) holds, and the proposition is proved. □

Remark 6.6. It is not clear to us whether the stronger estimate as (6.73) holds for u^α under the assumptions in theorem 6.3.. However, under the additional assumption that the strong solution $v^0 \in L^\infty([0,T],H^2)$, there holds also

$$\sup_{0\le t\le T} ||u^\alpha - v^0||_1^2 + \int_0^T ||u^\alpha - v^0||_2^2\,dt \le c\alpha^{\frac{1}{2}} \tag{6.90}$$

This follows from

$$\begin{aligned} & a_\beta(u^\alpha - v^0, u^\alpha - v^0) + \alpha(P\Delta(u^\alpha - v^0), P\Delta(u^\alpha - v^0)) \\ = \; & a_\beta(v^\alpha - v^0, u^\alpha - v^0) + \alpha(-P\Delta v^0, P\Delta(u^\alpha - v^0)) \end{aligned}$$

(due to (6.83)) and (6.73).

7. Concluding Remarks

We conclude this paper with a few remarks on related issues.

Remark 7.1. In exact same way, we can study the boundary value problem of LNS-α with NSB:

$$\partial_t v - \Delta v + \nabla\times v\times u + \nabla p = 0 \text{ in } \Omega \tag{7.1}$$
$$\nabla\cdot v = 0 \text{ in } \Omega \tag{7.2}$$
$$u - \alpha\Delta u + \nabla\tilde{p} = v \text{ in } \Omega \tag{7.3}$$
$$\nabla\cdot u = 0 \text{ in } \Omega \tag{7.4}$$
$$v\cdot n = 0,\ 2(S(v)n)_\tau = -\gamma v_\tau \text{ on } \partial\Omega \tag{7.5}$$
$$u\cdot n = 0,\ 2(S(u)n)_\tau = -\gamma u_\tau \text{ on } \partial\Omega \tag{7.6}$$

The functional setting is similar to that of (4.1)-(4.6), and all the results stated in section 4-6 are also valid.

Remark 7.2. The non-homogenous boundary value problems of LNS-α with VSB:

$$\partial_t v - \Delta v + \nabla\times v\times u + \nabla p = 0 \text{ in } \Omega \tag{7.7}$$
$$\nabla\cdot v = 0 \text{ in } \Omega \tag{7.8}$$
$$u - \alpha\Delta u + \nabla\tilde{p} = v \text{ in } \Omega \tag{7.9}$$
$$\nabla\cdot u = 0 \text{ in } \Omega \tag{7.10}$$
$$v\cdot n = 0,\ n\times\nabla\times v = \beta v + b \text{ on } \partial\Omega \tag{7.11}$$
$$u\cdot n = 0,\ n\times\nabla\times u = \beta v + b \text{ on } \partial\Omega \tag{7.12}$$

can also be considered by using a homogenous method to reduce it into

$$\partial_t v - \Delta v + \nabla\times v\times u + \nabla p = \xi \text{ in } \Omega \tag{7.13}$$
$$\nabla\cdot v = 0 \text{ in } \Omega \tag{7.14}$$
$$u - \alpha\Delta u + \nabla\tilde{p} = v + \eta_\alpha \text{ in } \Omega \tag{7.15}$$
$$\nabla\cdot u = 0 \text{ in } \Omega \tag{7.16}$$
$$v\cdot n = 0,\ n\times\nabla\times v = \beta v \text{ on } \partial\Omega \tag{7.17}$$
$$u\cdot n = 0,\ n\times\nabla\times u = \beta u \text{ on } \partial\Omega \tag{7.18}$$

for some ξ and η_α as was done for the steady homogenous case in Section 3. Similarly, the non-homogenous boundary value problems for LNS-α with NSB may be established too.

Remark 7.3. In the functional settings, the parameters associated with the velocity v and the filter u can be different, and different type boundary conditions, VSB or NSB, may be also allowed. However, in this case the analysis in the global existence and the vanishing α limit seems very difficult since (5.47) does not hold, there are some boundary terms arising, and the energy estimate in (5.16) depends on α. Yet the local well-posedness theory can be established by the method discussed in this paper.

Remark 7.4. Our approaches works also for other α models. For instance, one can consider the following Leray α model:

$$\partial_t v - \Delta v + u \cdot \nabla v + \nabla p = 0 \text{ in } \Omega \tag{7.19}$$

$$\nabla \cdot v = 0 \text{ in } \Omega \tag{7.20}$$

$$u - \alpha \Delta u + \nabla \tilde{p} = v \text{ in } \Omega \tag{7.21}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{7.22}$$

$$v \cdot n = 0, \; n \times (\nabla \times v) = \beta v \text{ on } \partial\Omega \tag{7.23}$$

$$u \cdot n = 0, \; 2(S(u)n)_\tau = -\gamma u_\tau \text{ on } \partial\Omega \tag{7.24}$$

which allowed different boundary conditions between the velocity v and the filter u. In fact, this model is easier to analyze than the LNS-α since it has the following energy equation

$$\frac{d}{dt}\|v\|^2 + 2a_\beta(v, v) = 0 \tag{7.25}$$

which yields the global existence directly, and the corresponding convergence result is better both in v^α and u^α than that in theorem 5.1..

References

[1] Achdou,Y., Pironneau,O. and Valentin,F., Effective boundary conditions for laminar flow over periodic rough boundaries, J.Comput. phys. 147(1998), 187-218.

[2] H. Beirão da Veiga, Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions, Adv. Differential Equations 9 (2004), no. 9-10, 1079-1114.

[3] H. Beirão da Veiga and F. Crispo, Sharp inviscid limit results under Navier type boundary conditions. An Lp theory, J. math. fluid mech. 12 (2010), 397-411.

[4] H. Beirão da Veiga and F. Crispo, Concerning the $W^{k,p}$-inviscid limit for 3D flows under a slip boundary condition, J. math. fluid mech. 13 (2011), 117-135.

[5] H. Beirão da Veiga and L.C. Berselli, Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary, J. Diff. Eqns. 246(2009), 597-628.

[6] Beavers,G.S. and Joseph,D.D., Boundary conditions at a naturally permeable wall, J.Fluid Mech. 30(1967), 197-207.

[7] Bellout, H., Neustupa, J. and Penel, P., On a ν continous family of strong solution to the Euler or Navier-Stokes equations with the Navier type boundary condition, Disc. Cont. Dyn. Sys. Vol. 27:4(2010) 1353-1373.

[8] Bellout, H., Neustupa, J. and Penel, P. On the Navier-Stokes equations with boundary conditions based on vorticity, Math. Nachr. 269/270(2004), 59C72.

[9] Bourguignon,J. P., Brezis, H. Remarks on the Euler equation, J. Funct. Anal 15(1974), 341-363.

[10] Busuioc, A. V. and Ratiu, T. S., Second grade fluid and averaged Euler equations with Navier slip boundary condition.Nonlinearity 16(2003), 1119-1149.

[11] Cantarella,J., De Turk,D and Gluck,H., Vector calculus and the topology of domains in 3-space, Amer. Math. monthly 109(2002), 257-286.

[12] Cao, Y. and Titi,E.S., On the rate of convergence of the two-dimensional $\alpha-$ models of the turbulence to the Navier-Stokes equations, Num. Func. Anal. Optim. 30(11-12)(2009), 1231-1271.

[13] Chen, G., Osborneb, D. and Qian, Z., The Navier-Stokes equations with the kinematic and vorticity boundary conditions on non-flat boundaries, Acta Math. Sci. Vol. 29(4)(2009), 919-948.

[14] Chen,S.Y.,Foias,C.,Holm,D.D,Olson,E.J., Titi, E.S. and Wynne, S., The Camassa-Holm equations as a closure model for turbulent channel and pipe flows. Phys. Rev. Lett. 81(1998), 5338-5341.

[15] Chen,S.Y.,Foias,C.,Holm,D.D,Olson,E.J., Titi, E.S. and Wynne, S., The Camassa-Holm equations and turbulence in pipes and channels. Phys. D 133(1999), 49-65.

[16] Chen,S.Y.,Foias,C.,Holm,D.D,Olson,E.J., Titi, E.S. and Wynne, S., The Camassa-Holm equations and turbulence in pipes and channels, Phys. Fluids 11(1999), 2343-2353.

[17] Clopeau,T.,Mikelić,A. and Robert,R., On the valishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions, Nonlinearity 11(1998),1625-1636.

[18] Constantin,P. and Iyer, G., A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations. Comm. Pure Appl. Math. 61 (2008), no. 3, 330–345.

[19] Constantin, P., An Eulerian-Lagrangian approach to the Navier-Stokes equations, Comm. Math. Phys. 216 (2001), no. 3, 663–686.

[20] Constantin,P. and Foias,C., *Navier Stokes equation*, Univ. of Chicago press IL(1988).

[21] Foias,C.,Holm,D.D. and Titi, E.S., The three-dimensional viscous Camassa-Holm equations and their relation to the Navier-Stokes equations and turbulence theory. J. Dyn. Diff. Eqns., Vol. 14.1,(2002) 1-35.

[22] Foias, C. and Temam R., Remarques sur les equations de Navier-Stokes stationaires et les phenomenes successifs de bifurcation, Ann. Sc. Norm. Syper. Pisa 5(1978) 29-63.

[23] Galdi,G.P., *An introduction to the mathematical theory of the Navier-Stokes equations: Vol.I: Linearized steady problems*, Springer Tracts in Natural Philosophy, Springer-verlag 38(1998).

[24] Gibbon,J.D. and Holm, D.D., Length-scale estimate for the LANS-α equations in terms of the Reynolds number. Phy. D 222(2)(2006), 69-78.

[25] Gibbon,J.D. and Holm, D.D., Estimates for the LANS-α, Leray-α and Bardina Models in Terms of a Navier-Stokes Reynolds Number, Indiana Univ. Math. J. 57 (2008), no. 6, 2761–2773.

[26] V. Oirault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Theory and Algorithms, Springer-Verlag, Berlin 1986.

[27] Holm, D. D., Marsden, J. E., Ratiu, T. S. (1998). Euler Poincare models of ideal fluids with nonlinear dispersion. Phys. Rev. Lett. 80, 4173-4176.

[28] Holm, D. D. (1999). Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion, Physica D 133, 215269.

[29] Holm,D.D.,Putkavadge, V.,Weiman, P.D. and Wingate, B.A. Boundary effects on exact solutions of the Lagrangian-Averaged Navier-Stokes α equations, J. Stat. Phys. Vol. 113.516(2003), 841-854.

[30] Itoh,S., Tanaka,N.and Tani,A., The initial value problem for the Navier-Stokes equations with general slip boundary condition in Hölder spaces, J. Math. Fluid Mech. 5(2003), 275-301.

[31] Jäger,W. and Mikelić,A., On the interface boundary condition of Beavers, Joseph, and Saffman, SIAM J. Appl.Math. 60(2000), 1111-1127.

[32] John,V., Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equation-numerical test and aspect of the implementation, J. Comput. Appl. Math. 147(2002), 287-300.

[33] Iftimie,D.S. and Sueur, F. Viscosity boundary layers for the Navier-Stokes equations with the Navier slip conditions, Arch. Ration. Mech. Anal. 199 (2011), no. 1, 145C175.

[34] Kato,T., *Remarks on zero viscosity limit for non-stationary Navier-Stokes flows with boundary. In: Seminar on PDE (S.S.Chen, eds)*, Springer, New York, (1984), 85-98.

[35] J.-L. Lions, *Quelques Méthodes de Résolution des Problemès aux Limites non Linéaires*, Dunod, Paris, 1969.

[36] Majda, A. J.; Bertozzi, A. L., *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[37] Marsden, J.E. and Shkoller, S. Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains, Phil. Trans. R.Soc. Lond. A 359(2001), 1449-1468.

[38] Morrey C.B., *Multiple integrals in the calculus of variations*, Berlin Heidelberg New York: Springer 1966.

[39] Navier, C.L.M.H., *Sur les lois de l'équilibre et du mouvement des corps élastiques*, Mem. Acad. R. Sci. Inst. France 6(1827), 369.

[40] Schwarz, G., *Hodge Decomposition-A Method for Solving Boundary Value Problems*, Lecture Notes in Math 1607, Spring-Verlag, Berlin (1995).

[41] Solonnikov,V.A. and Ščadilov,V.E., *A certain boundary value problem for the stationary system of Navier-Stokes equations*, Boundary Value Problem of Mathematical Physics, 8. Trudy Mat. Inst. Steklov. 125(1973), 196-210; translation in Proc. Steklov Inst. Math. 125(1973),186-199.

[42] Prizjev, N.V. and Troian, S.M. Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces, J. Fluid Mech. Vol. 554(2006), 25-46.

[43] Tang, C.L. and Xin, Z.P., Existence of solutions for three dimensional stationary incompressible Euler equations with non-vanishing vorticity, Chinese Ann. Math. 30B(6)(2009), 801-806.

[44] Temam, R., *Infinite dimensional dynamical systems in mechanics and physics*, second ed., Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1997.

[45] Thompson, P.A. and Troian, S.M., 1997 A general boundary condition for liquid flow at solid surface. Nature 389(1997), 360-362.

[46] Xiao, Y.L. and Xin, Z.P., On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, Comm. Pure Appl. Math. Vol. LX (2007) 1027-1055.

[47] Xiao, Y.L. and Xin, Z.P., Remarks on the vanishing viscosity limit for 3D Navier-Stokes equations with a slip boundary condition, Chinese Ann. Math., 32B(3)(2011), 321-332.

[48] Yoshida,Z. and Giga,Y., Remarks on spectra of operator rot, Math. Z. 204(1990), 235-245.

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