Global classical solutions to the two-dimensional compressible Navier-Stokes equations in $\mathbb{R}^2$

Quansen Jiu,[1,3*] **Yi Wang**[2,3†] and **Zhouping Xin**[3‡]

[1] School of Mathematical Sciences, Capital Normal University, Beijing 100048, P. R. China

[2] Institute of Applied Mathematics, AMSS, CAS, Beijing 100190, P. R. China

[3] The Institute of Mathematical Sciences, Chinese University of HongKong, HongKong

Abstract: In this paper, we prove the global well-posedness of the classical solution to the 2D Cauchy problem of the compressible Navier-Stokes equations with arbitrarily large initial data when the shear viscosity μ is a positive constant and the bulk viscosity $\lambda(\rho) = \rho^\beta$ with $\beta > \frac{4}{3}$. Here the initial density keeps a non-vacuum states $\bar{\rho} > 0$ at far fields and our results generalize the ones by Vaigant-Kazhikhov [41] for the periodic problem and by Jiu-Wang-Xin [26] and Huang-Li [18] for the Cauchy problem with vacuum states $\bar{\rho} = 0$ at far fields. It shows that the solution will not develop the vacuum states in any finite time provided the initial density is uniformly away from vacuum. And the results also hold true when the initial data contains vacuum states in a subset of $\mathbb{R}^2$ and the natural compatibility conditions are satisfied. Some new weighted estimates are obtained to establish the upper bound of the density.

Key Words: compressible Navier-Stokes equations, Cauchy problem, global well-posedness, large data, vacuum

1 Introduction

We consider the following compressible and isentropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + \nabla((\mu + \lambda(\rho))\operatorname{div} u), \quad x \in \mathbb{R}^2, t > 0, \end{cases} \tag{1.1}$$

where $\rho(t,x) \geq 0$, $u(t,x) = (u_1, u_2)(t,x)$ represent the density and the velocity of the fluid, respectively. And $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \in [0,T]$ for any fixed $T > 0$. Here it is assumed that

*The research is partially supported by National Natural Sciences Foundation of China (No. 11171229) and Project of Beijing Education Committee. E-mail: jiuqs@mail.cnu.edu.cn

†The research is partially supported by National Natural Sciences Foundation of China (No. 11171326) and by the National Center for Mathematics and Interdisciplinary Sciences, CAS. E-mail: wangyi@amss.ac.cn.

‡The research is partially supported by Zheng Ge Ru Funds, Hong Kong RGC Earmarked Research Grant CUHK4042/08P and CUHK4041/11P, and a grant from the Croucher Foundation. Email: zpxin@ims.cuhk.edu.hk

the shear viscosity $\mu > 0$ is a positive constant and the bulk viscosity

$$\lambda(\rho) = \rho^\beta \tag{1.2}$$

with $\beta > 0$ in general such that the operator

$$\mathcal{L}_\rho u \equiv \mu\Delta u + \nabla((\mu + \lambda(\rho))\mathrm{div}u)$$

is strictly elliptic. The pressure function is given by $P(\rho) = A\rho^\gamma$, where $\gamma > 1$ denotes the adiabatic exponent and $A > 0$ is a constant which is normalized to be 1 for simplicity. We impose the initial values as

$$(\rho, u)(t = 0, x) = (\rho_0, u_0)(x) \to (\bar{\rho}, 0), \quad \text{as} \quad |x| \to +\infty, \tag{1.3}$$

where $\bar{\rho} > 0$ is a given positive constant.

In the case that both the shear and bulk viscosities are positive constants, there are a large number of literatures on the well-posedness theories of the compressible Navier-Stokes equations. In particular, the one-dimensional theory is rather satisfactory, see [16, 32, 28, 29] and the references therein. In multi-dimensional case, the local well-posedness theory of classical solutions was established in the absence of vacuum (see [37], [20] and [40]) and the global well-posedness theory of classical solutions was obtained for initial data close to a non-vacuum steady state (see [35], [14], [8], [3] and references therein). The local well-posedness of classical solutions containing vacuum was studied by Cho-Kim [6] and Luo [34] and the global well-posedness of classical solutions to the 3D isentropic compressible Navier-Stokes equations with small energy was proved by Huang-Li-Xin [19]. For the large initial data permitting vacuums, the global existence of weak solutions was investigated in [31], [11], [22]. It should be noted that if the initial data are arbitrarily large and the vacuums are permitted, the solution will also contain possible vacuums and one could not expect the global well-posedness in general, see [43] [39] and [44] for blow-up results of classical solutions.

The case that both the shear and bulk viscosities depend on the density has also received a lot attention recently, see [1, 2, 8, 10, 13, 21, 22, 23, 24, 27, 30, 33, 36, 45, 46, 47] and the references therein. When deriving by Chapman-Enskog expansions from the Boltzmann equation, the viscosity of the compressible Navier-Stokes equations depends on the temperature and thus on the density for isentropic flows. Moreover, in geophysical flows, the viscous Saint-Venant system for the shallow water corresponds exactly to a kind of compressible Navier-Stokes equations with density-dependent viscosities. However, except for the one-dimensional problems, few results are available for the multi-dimensional problems and even the short time well-posedness of classical solutions in the presence of vacuum remains open.

The system (1.1) was first proposed and studied by Vaigant-Kazhikhov in [41]. For the periodic problem on the torus $\mathbb{T}^2$ and under assumptions that the initial density is uniformly away from vacuum and $\beta > 3$ in (1.2), Vaigant-Kazhikhov established the global well-posedness of the classical solution to (1.1) in [41] and the global existence and large time behavior of weak solutions was stuided by Perepelitsa in [38]. Recently, Jiu-Wang-Xin [25] improved the result in [41] and obtained the global well-posedness of the classical solution to the periodic problem with large initial data permitting vacuum. Later on, Huang-Li relaxed the index β to be $\beta > \frac{4}{3}$ and studied the large time behavior of the solutions in [17]. However, all the above results are concerned with the 2D periodic problems. For the 2D Cauchy problems with vacuum states at far fields, Jiu-Wang-Xin [26] and Huang-Li [18] independently considered the global well-posedness of classical solution in different weighted spaces.

In the present paper, we study the global well-posedness of the classical solution to the Cauchy problem (1.1)-(1.3) with large data which keeps a non-vacuum states $\bar\rho > 0$ at far fields. In particular, our results show that the solution will not develop the vacuum states in any finite time provided the initial density is uniformly away from vacuum. The results of this paper generalize the ones by Vaigant-Kazhikhov in [41] to the Cauchy problem and the index β is relaxed to be $\beta > \frac{4}{3}$. The results also improve ones by Jiu-Wang-Xin [26] and Huang-Li [18] for the Cauchy problem with vacuum states $\bar\rho = 0$ at far fields. Moreover, the results hold true if the initial data contains vacuum states in a subset of $\mathbb{R}^2$ under appropriate compatibility conditions (see (1.9) in Theorem 1.2).

To study the global well-posedness of the classical solution of the compressible Navier-Stokes equations, it is crucial to obtain the uniformly upper bound of the density. To do that, similar to [41], [25] and [26], we first obtain any $L^p(2 \le p < \infty)$ estimates of the density $\rho - \bar\rho$ and then obtain the estimates of the first order derivative of the velocity. A new transport equation (3.36) is derived by means of the effective viscous flux $F = (2\mu + \lambda(\rho))\text{div}u - (P(\rho) - P(\bar\rho))$ and two new functions ξ and η satisfying the elliptic problems

$$-\Delta\xi = \text{div}(\rho u), \qquad -\Delta\eta = \text{div}[\text{div}(\rho u \otimes u)], \tag{1.4}$$

respectively, which was introduced in [41]. Comparing with the periodic problem and the Cauchy problem with vacuum at far fields, new difficulties will be encountered. Since no integrability is expected for the density itself, we will decompose the first elliptic problem in (1.4) into the following two parts:

$$-\Delta\xi_1 = \text{div}(\sqrt{\rho}u(\sqrt{\rho} - \sqrt{\bar\rho})), \tag{1.5}$$

$$-\Delta\xi_2 = \sqrt{\bar\rho}\ \text{div}(\sqrt{\rho}u). \tag{1.6}$$

For the elliptic problem (1.5), one can make use of the similar properties as the periodic case and the Cauchy problem with vanishing density at the far fields thanks to the expected integrability of $\sqrt{\rho} - \sqrt{\bar\rho}$. For the second elliptic problem (1.6), since it is expected that $\rho \in L^\infty$ and $\sqrt{\rho}u \in L^\infty([0,T]; L^2(\mathbb{R}^2))$ by the elementary energy estimate, it follows from (1.6) that $\nabla\xi_2 \in D^1(\mathbb{R}^2)$ which is a homogeneous and critical Sobolev space. Therefore, the integrability of ξ_2 can not be derived in a direct way. However, the integrability of ξ_2 is crucial to obtain the $L^p(2 \le p < \infty)$ estimates of $\rho - \bar\rho$ and the upper bound of the density ρ. In order to circumvent this difficulty, some new weighted estimates are needed and the integrability of the velocity and ξ_2 is proved by using Cafferelli-Kohn-Nirenberg type inequality [4, 5]. It should be remarked that these weighted estimates are motivated by our previous work [26] and in comparison with the uniform constant in [26], the weight power α here depends on the ratio $\frac{\lambda(\bar\rho)}{\mu}$. At the same time, if $\bar\rho = 0$, then the weight α is exactly same as the one in our previous work [26]. Moreover, when deriving the first-order derivative estimates of the velocity, since L^p-integrability $(2 \le p < \infty)$ is not available, it would be required to use the L^∞-norm of the density ρ in a priori way which is motivated by the work [38]. In this way, a log-type inequality of the first-order derivative of the velocity can be obtained (see Lemma 3.6). Finally, with help of a higher energy estimate in Lemma 3.7, one can get a upper bound of the density under the restriction $\beta > \frac{4}{3}$ (see [17, 18]).

Denote the potential energy by

$$\Psi(\rho, \bar\rho) = \frac{1}{\gamma - 1}\big[\rho^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(\rho - \bar\rho)\big].$$

Our main results can be stated as follows.

Theorem 1.1 *Let $\beta > \frac{4}{3}$ and $1 < \gamma \leq 2\beta$. Suppose that the initial values $(\rho_0, u_0)(x)$ satisfy*

$$
\begin{aligned}
&0 < c \leq \rho_0 \leq C, \quad (\rho_0 - \bar{\rho}, P(\rho_0) - P(\bar{\rho})) \in W^{2,q}(\mathbb{R}^2) \times W^{2,q}(\mathbb{R}^2), \quad u_0(x) \in H^2(\mathbb{R}^2),\\
&\Psi(\rho_0, \bar{\rho})(1 + |x|^\alpha) \in L^1(\mathbb{R}^2), \quad \sqrt{\rho_0} u_0 |x|^{\frac{\alpha}{2}} \in L^2(\mathbb{R}^2),
\end{aligned}
$$

where q, c, C and α are positive constants satisfying $q > 2$, $0 < c < C$ and $0 < \alpha^2 < \frac{4(\sqrt{2+\frac{\lambda(\bar{\rho})}{\mu}}-1)}{1+\frac{\lambda(\bar{\rho})}{\mu}}$ respectively. Then, for any $T > 0$, there exists a unique global classical solution $(\rho, u)(t, x)$ to the Cauchy problem (1.1)-(1.3) satisfying

$$0 < c_1 \leq \rho \leq C_1$$

for some positive constants c_1 and C_1. Moreover, one has

$$
\begin{aligned}
&(\rho - \bar{\rho}, P(\rho) - P(\bar{\rho}))(t, x) \in C([0, T]; W^{2,q}(\mathbb{R}^2)),\\
&\Psi(\rho, \bar{\rho})(1 + |x|^\alpha) \in C([0, T]; L^1(\mathbb{R}^2)), \quad \sqrt{\rho} u |x|^{\frac{\alpha}{2}} \in C([0, T]; L^2(\mathbb{R}^2)),\\
&u \in C([0, T]; H^2(\mathbb{R}^2)) \cap L^2(0, T; H^3(\mathbb{R}^2)), \quad \sqrt{t} u \in L^\infty(0, T; H^3(\mathbb{R}^2)),\\
&t u \in L^\infty(0, T; W^{3,q}(\mathbb{R}^2)), \quad u_t \in L^2(0, T; H^1(\mathbb{R}^2)),\\
&\sqrt{t} u_t \in L^2(0, T; H^2(\mathbb{R}^2)) \cap L^\infty(0, T; H^1(\mathbb{R}^2)), \quad t u_t \in L^\infty(0, T; H^2(\mathbb{R}^2)),\\
&\sqrt{t}\sqrt{\rho} u_{tt} \in L^2(0, T; L^2(\mathbb{R}^2)), \quad t\sqrt{\rho} u_{tt} \in L^\infty(0, T; L^2(\mathbb{R}^2)), \quad t\nabla u_{tt} \in L^2(0, T; L^2(\mathbb{R}^2)).
\end{aligned}
\tag{1.7}
$$

If the initial values contain vacuum states in a subset of $\mathbb{R}^2$, then the following results can be obtained.

Theorem 1.2 *Suppose that the initial values $(\rho_0, u_0)(x)$ satisfy*

$$
\begin{aligned}
&\rho_0 \geq 0, \quad (\rho_0 - \bar{\rho}, P(\rho_0) - P(\bar{\rho})) \in W^{2,q}(\mathbb{R}^2) \times W^{2,q}(\mathbb{R}^2), \quad u_0(x) \in H^2(\mathbb{R}^2),\\
&\Psi(\rho_0, \bar{\rho})(1 + |x|^\alpha) \in L^1(\mathbb{R}^2), \quad \sqrt{\rho_0} u_0 |x|^{\frac{\alpha}{2}} \in L^2(\mathbb{R}^2),
\end{aligned}
\tag{1.8}
$$

with q, α, γ and β being the same as in Theorem 1.1. Suppose that the compatibility conditions

$$\mathcal{L}_{\rho_0} u_0 - \nabla P(\rho_0) = \sqrt{\rho_0} g(x) \tag{1.9}$$

are satisfied for some $g \in L^2(\mathbb{R}^2)$. Then, for any $T > 0$, there exists a unique global classical solution $(\rho, u)(t, x)$ to the Cauchy problem (1.1)-(1.3) satisfying $0 \leq \rho \leq C_2$ for some positive constant C_2 and (1.7) in Theorem 1.1.

Remark 1.1 *If $\lambda(\bar{\rho}) < 7\mu$, one has $\frac{4(\sqrt{2+\frac{\lambda(\bar{\rho})}{\mu}}-1)}{1+\frac{\lambda(\bar{\rho})}{\mu}} > 1$. Then one can choose a weight α satisfying $1 < \alpha^2 < \frac{4(\sqrt{2+\frac{\lambda(\bar{\rho})}{\mu}}-1)}{1+\frac{\lambda(\bar{\rho})}{\mu}}$. In this case, the condition $\gamma \leq 2\beta$ in Theorems 1.1 and 1.2 can be removed and both theorems hold true for any $\gamma > 1$ and $\beta > \frac{4}{3}$ (see [26] for more details).*

Remark 1.2 *If $\bar{\rho} = 0$, then $\frac{4(\sqrt{2+\frac{\lambda(\bar{\rho})}{\mu}}-1)}{1+\frac{\lambda(\bar{\rho})}{\mu}} = 4(\sqrt{2} - 1)$. This is exactly same as our previous work [26] for the Cauchy problem with the vanishing density at the far fields.*

The rest of the paper is organized as follows. In Section 2, we present some elementary facts which will be used later. In Section 3, we derive a priori estimates which are needed to extend the local solution to a global one. The sketch of proof of our main results is given in Section 4.

Notations. Throughout this paper, positive generic constants are denoted by c and C, which are independent of δ, m and $t \in [0,T]$, without confusion, and $C(\cdot)$ stands for some generic constant(s) depending only on the quantity listed in the parenthesis. For functional spaces, $L^p(\mathbb{R}^2), 1 \le p \le \infty$, denote the usual Lebesgue spaces on $\mathbb{R}^2$ and $\|\cdot\|_p$ denotes its L^p norm. $W^{k,p}(\mathbb{R}^2)$ denotes the standard k^{th} order Sobolev space and $H^k(\mathbb{R}^2) := W^{k,2}(\mathbb{R}^2)$. For $1 < p < \infty$, the homogenous Sobolev space $D^{k,p}(\mathbb{R}^2)$ is defined by $D^{k,p}(\mathbb{R}^2) = \{u \in L^1_{loc}(\mathbb{R}^2) | \|\nabla^k u\|_p < +\infty\}$ with $\|u\|_{D^{k,p}} := \|\nabla^k u\|_p$ and $D^k(\mathbb{R}^2) := D^{k,2}(\mathbb{R}^2)$.

2 Preliminaries

Motivated by [41], we introduce the following variables. First denote the effective viscous flux by

$$F = (2\mu + \lambda(\rho))\mathrm{div}u - (P(\rho) - P(\bar{\rho})), \tag{2.1}$$

and the vorticity by

$$\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

Also, we define that

$$H = \frac{1}{\rho}(\mu\omega_{x_1} + F_{x_2}), \qquad L = \frac{1}{\rho}(-\mu\omega_{x_2} + F_{x_1}).$$

Then the momentum equation $(1.1)_2$ can be rewritten as

$$\begin{cases} \dot{u}_1 = u_{1t} + u \cdot \nabla u_1 = \frac{1}{\rho}(-\mu\omega_{x_2} + F_{x_1}) = L, \\ \dot{u}_2 = u_{2t} + u \cdot \nabla u_2 = \frac{1}{\rho}(\mu\omega_{x_1} + F_{x_2}) = H, \end{cases}$$

that is,

$$\dot{u} = (\dot{u}_1, \dot{u}_2)^t = (L, H)^t.$$

Then the effective viscous flux F and the vorticity ω solve the following system:

$$\begin{cases} \omega_t + u \cdot \nabla\omega + \omega\mathrm{div}u = H_{x_1} - L_{x_2}, \\ (\frac{F+P(\rho)-P(\bar{\rho})}{2\mu+\lambda(\rho)})_t + u \cdot \nabla(\frac{F+P(\rho)-P(\bar{\rho})}{2\mu+\lambda(\rho)}) + (u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2 = H_{x_2} + L_{x_1}. \end{cases}$$

Due to the continuity equation $(1.1)_1$, it holds that

$$\begin{cases} \omega_t + u \cdot \nabla\omega + \omega\mathrm{div}u = H_{x_1} - L_{x_2}, \\ F_t + u \cdot \nabla F - \rho(2\mu + \lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)-P(\bar{\rho})}{2\mu+\lambda(\rho)})']\mathrm{div}u \\ \qquad +(2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2] = (2\mu + \lambda(\rho))(H_{x_2} + L_{x_1}). \end{cases} \tag{2.2}$$

Furthermore, the system for (H, L) can be derived as

$$\begin{cases} \rho H_t + \rho u \cdot \nabla H - \rho H \mathrm{div} u + u_{x_2} \cdot \nabla F + \mu u_{x_1} \cdot \nabla \omega + \mu(\omega \mathrm{div} u)_{x_1} \\ \qquad -\left\{\rho(2\mu + \lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)})']\mathrm{div} u\right\}_{x_2} \\ \qquad +\left\{(2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2]\right\}_{x_2} \\ \qquad = [(2\mu + \lambda(\rho))(H_{x_2} + L_{x_1})]_{x_2} + \mu(H_{x_1} - L_{x_2})_{x_1}, \\ \rho L_t + \rho u \cdot \nabla L - \rho L \mathrm{div} u + u_{x_1} \cdot \nabla F - \mu u_{x_2} \cdot \nabla \omega - \mu(\omega \mathrm{div} u)_{x_2} \\ \qquad -\left\{\rho(2\mu + \lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)})']\mathrm{div} u\right\}_{x_1} \\ \qquad +\left\{(2\mu + \lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2]\right\}_{x_1} \\ \qquad = [(2\mu + \lambda(\rho))(H_{x_2} + L_{x_1})]_{x_1} - \mu(H_{x_1} - L_{x_2})_{x_2}. \end{cases}$$

In the following, we will utilize the above systems in different steps. Note that these systems are equivalent to each other for the smooth solution to the original system (1.1).

Several elementary Lemmas are needed later. The first one is the various Gagliardo-Nirenberg inequalities.

Lemma 2.1 *(1) $\forall h \in W^{1,m}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$, it holds that*

$$\|h\|_q \le C\|\nabla h\|_m^\theta \|h\|_r^{1-\theta},$$

where $\theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{m} + \frac{1}{2})^{-1}$, and if $m < 2$, then q is between r and $\frac{2m}{2-m}$, that is, $q \in [r, \frac{2m}{2-m}]$ if $r < \frac{2m}{2-m}$, $q \in [\frac{2m}{2-m}, r]$ if $r \ge \frac{2m}{2-m}$, if $m = 2$, then $q \in [r, +\infty)$, if $m > 2$, then $q \in [r, +\infty]$.

(2) (Best constant for the Gagliardo-Nirenberg inequality)
$\forall h \in \mathbb{D}^m(\mathbb{R}^2) \doteq \left\{h \in L^{m+1}(\mathbb{R}^2) \middle| \nabla h \in L^2(\mathbb{R}^2), h \in L^{2m}(\mathbb{R}^2)\right\}$ with $m > 1$, it holds that

$$\|h\|_{2m} \le A_m \|\nabla h\|_2^\theta \|u\|_{m+1}^{1-\theta},$$

where $\theta = \frac{1}{2} - \frac{1}{2m}$ and

$$A_m = \Big(\frac{m+1}{2\pi}\Big)^{\frac{\theta}{2}} \Big(\frac{2}{m+1}\Big)^{\frac{1}{2m}} \le C m^{\frac{1}{4}}$$

with the positive constant C independent of m, and A_m is the optimal constant.

(3) $\forall h \in W^{1,m}(\mathbb{R}^2)$ with $1 \le m < 2$, then

$$\|h\|_{\frac{2m}{2-m}} \le C(2-m)^{-\frac{1}{2}} \|\nabla h\|_m,$$

where the positive constant C is independent of m.

Proof: The proof of (1) can be found in [41] while the proof of (2) can be found in [7]. The proof of (3) can be found in [12].

The following Lemma is the Caffarelli-Kokn-Nirenberg weighted inequalities, which is crucial to the weighted estimates in the two-dimensional Cauchy problem.

Lemma 2.2 *(1) $\forall h\in C_0^\infty(\mathbb{R}^2)$, it holds that*

$$\||x|^\kappa h\|_r\le C\||x|^\alpha|\nabla h|\|_p^\theta\ \||x|^\beta h\|_q^{1-\theta}$$

where $1\le p,q<\infty, 0<r<\infty, 0\le\theta\le 1, \frac{1}{p}+\frac{\alpha}{2}>0, \frac{1}{q}+\frac{\beta}{2}>0, \frac{1}{r}+\frac{\kappa}{2}>0$ and satisfying

$$\frac{1}{r}+\frac{\kappa}{2}=\theta(\frac{1}{p}+\frac{\alpha-1}{2})+(1-\theta)(\frac{1}{q}+\frac{\beta}{2}),$$

and

$$\kappa=\theta\sigma+(1-\theta)\beta,$$

with $0\le\alpha-\sigma$ if $\theta>0$ and $0\le\alpha-\sigma\le 1$ if $\theta>0$ and $\frac{1}{p}+\frac{\alpha-1}{2}=\frac{1}{r}+\frac{\kappa}{2}$.

(2) (Best constant for Caffarelli-Kohn-Nirenberg inequality)
$\forall h\in C_0^\infty(\mathbb{R}^2)$, it holds that

$$\||x|^b h\|_p\le C_{a,b}\||x|^a\nabla h\|_2 \tag{2.3}$$

where $a>0, a-1\le b\le a$ and $p=\frac{2}{a-b}$. If $b=a-1$, then $p=2$ and the best constant in the inequality (2.3) is

$$C_{a,b}=C_{a,a-1}=a.$$

Proof: The proof of (1) can be found in [4] while the proof of (2) can be found in [5].

The proof of the following Lemma is referred to [26].

Lemma 2.3 *(1) It holds that for $1<p<\infty$ and $u\in C_0^\infty(\mathbb{R}^2)$,*

$$\|\nabla u\|_p\le C(\|\mathrm{div}u\|_p+\|\omega\|_p);$$

(2) It holds that for $1<p<\infty$, $-2<\alpha<2(p-1)$ and $u\in C_0^\infty(\mathbb{R}^2)$,

$$\||x|^{\frac{\alpha}{p}}|\nabla u|\|_p\le C(\||x|^{\frac{\alpha}{p}}\mathrm{div}u\|_p+\||x|^{\frac{\alpha}{p}}\omega\|_p).$$

3 A priori estimates

In this section, we first prove Theorem 1.1. Various a priori estimates and upper and lower bound of the density will be obtained.

<u>Step 1. Elementary energy estimates:</u>

Lemma 3.1 *There exists a positive constant C depending on (ρ_0,u_0), such that*

$$\sup_{t\in[0,T]}\big(\|\sqrt{\rho}u\|_2^2+\|\Psi(\rho,\bar\rho)\|_1\big)+\int_0^T\big(\|\nabla u\|_2^2+\|\omega\|_2^2+\|(2\mu+\lambda(\rho))^{\frac{1}{2}}\mathrm{div}u\|_2^2\big)dt\le C.$$

Proof: Multiplying the equation $(1.1)_2$ by u, the continuity equation $(1.1)_1$ by $\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$, then summing the resulting equations, and using the continuity equation $(1.1)_1$, yield that

$$\begin{aligned}&\big[\rho\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)\big]_t+\mathrm{div}\big[\rho u\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)u+(P(\rho)-P(\bar\rho))u\big]\\&=\mathrm{div}\big[\mu\nabla\frac{|u|^2}{2}+(\mu+\lambda(\rho))(\mathrm{div}u)u\big]-\mu|\nabla u|^2-(\mu+\lambda(\rho))(\mathrm{div}u)^2.\end{aligned}\tag{3.1}$$

Therefore, integrating the above equality over $[0,t]\times\mathbb{R}^2$ with respect to t and x and noting that

$$\int\big[\mu|\nabla u|^2+(\mu+\lambda(\rho))(\text{div}u)^2\big]dx=\int\big[\mu\omega^2+(2\mu+\lambda(\rho))(\text{div}u)^2\big]dx,$$

complete the proof of Lemma 3.1. $\square$

Step 2. Weighted energy estimates:

The following weighted energy estimates are fundamental and crucial in our analysis.

Lemma 3.2 *For $\alpha>0$ satisfying $\alpha^2<\frac{4(\sqrt{2+\frac{\lambda(\bar\rho)}{\mu}}-1)}{1+\frac{\lambda(\bar\rho)}{\mu}}$ and $\gamma\le 2\beta$, it holds that for sufficiently large $m>1$ and $\forall t\in[0,T]$,*

$$\begin{aligned}&\int_{\mathbb{R}^2}|x|^\alpha\big[\rho|u|^2+\Psi(\rho,\bar\rho)\big](t,x)dx+\int_0^t\big[\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2(s)+\||x|^{\frac{\alpha}{2}}\text{div}u\|_2^2(s)+\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\text{div}u\|_2^2(s)\big]ds\\&\le C_\alpha\Big[1+\int_0^t(\|\rho-\bar\rho\|_{2m\beta+1}^\beta(s)+1)(\|\nabla u\|_2^2(s)+1)ds\Big],\end{aligned}\tag{3.2}$$

where the positive constant C_α may depend on α but is independent of m.

Proof: Multiplying the equality (3.1) by $|x|^\alpha$ yields that

$$\begin{aligned}&\big[|x|^\alpha(\rho\frac{|u|^2}{2}+\Psi(\rho,\bar\rho))\big]_t+\big[\mu|\nabla u|^2+(\mu+\lambda(\rho))(\text{div}u)^2\big]|x|^\alpha\\&=-\text{div}\big[|x|^\alpha\big(\rho u\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)u+(P(\rho)-P(\bar\rho))u\big)\big]+\text{div}\big[\big(\mu\nabla\frac{|u|^2}{2}+(\mu+\lambda(\rho))(\text{div}u)u\big)|x|^\alpha\big]\\&\quad+\big[\rho u\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)u+(P(\rho)-P(\bar\rho))\big]u\cdot\nabla(|x|^\alpha)-\big[\mu\nabla\frac{|u|^2}{2}+(\mu+\lambda(\rho))(\text{div}u)u\big]\cdot\nabla(|x|^\alpha).\end{aligned}\tag{3.3}$$

Integrating the above equation (3.3) with respect to x over $\mathbb{R}^2$ yields that

$$\begin{aligned}&\frac{d}{dt}\int|x|^\alpha\big[\rho\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)\big](t,x)dx+\Big[\mu\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+\mu\||x|^{\frac{\alpha}{2}}\text{div}u\|_2^2+\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\text{div}u\|_2^2\Big](t)\\&=\int\big[\rho u\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)u+(P(\rho)-P(\bar\rho))\big]u\cdot\nabla(|x|^\alpha)dx\\&\qquad\qquad-\int\big[\mu\nabla\frac{|u|^2}{2}+(\mu+\lambda(\rho))(\text{div}u)u\big]\cdot\nabla(|x|^\alpha)dx.\end{aligned}\tag{3.4}$$

Now we estimate the terms on the right hand side of (3.4). First, it holds that

$$\begin{aligned}&|\int\rho\frac{|u|^2}{2}u\cdot\nabla(|x|^\alpha)dx|=|\int\frac{|u|^2}{2}\big((\sqrt{\rho}-\sqrt{\bar\rho})+\sqrt{\bar\rho}\big)\sqrt{\rho}u\cdot\nabla(|x|^\alpha)dx|\\&\quad\le|\int\frac{|u|^2}{2}\big(\sqrt{\rho}-\sqrt{\bar\rho}\big)\sqrt{\rho}u\cdot\nabla(|x|^\alpha)dx|+\sqrt{\bar\rho}\,|\int\frac{|u|^2}{2}\sqrt{\rho}u\cdot\nabla(|x|^\alpha)dx|:=I_{11}+I_{12}.\end{aligned}\tag{3.5}$$

Then, it follows that

$$\begin{aligned}
I_{11} &= |\int \frac{|u|^2}{2}\big(\sqrt{\rho}-\sqrt{\bar\rho}\big)\big(\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}+\mathbf{1}|_{\{\rho>2\bar\rho\}}\big)\sqrt{\rho}u\cdot\nabla(|x|^\alpha)dx| \\
&\le C\|\sqrt{\rho}u\|_2\Big[\|(\sqrt{\rho}-\sqrt{\bar\rho})\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_{p_1}\||x|^{\alpha-1}|u|^2\|_{q_1}+\|(\sqrt{\rho}-\sqrt{\bar\rho})\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{2\gamma}\||x|^{\alpha-1}|u|^2\|_{\frac{2\gamma}{\gamma-1}}\Big] \\
&\le C\big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_{p_1}\||x|^{\frac{\alpha-1}{2}}u\|_{2q_1}^2+\|\Psi(\rho,\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_1^{\frac{1}{2\gamma}}\||x|^{\frac{\alpha-1}{2}}u\|_{\frac{4\gamma}{\gamma-1}}^2\big] \\
&\le C\big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_{p_1}\|\nabla u\|_2^{2\theta_1}\||x|^{\frac{\alpha}{2}}\nabla u\|_2^{2(1-\theta_1)}+\|\nabla u\|_2^{\frac{2}{\alpha\gamma}}\||x|^{\frac{\alpha}{2}}\nabla u\|_2^{2(1-\frac{1}{\alpha\gamma})}\big] \\
&\le \sigma\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+C_\sigma\big[1+\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_{p_1}^{\frac{1}{\theta_1}}\big]\|\nabla u\|_2^2,
\end{aligned} \tag{3.6}$$

where and in the sequel $\sigma>0$ is a small constant to be determined, C_σ is a positive constant depending on σ. By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.2 (1), the positive constants $p_1>2, q_1>2, \theta_1\in(0,1]$ in the above inequality (3.8) satisfying

$$\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{2},$$

and

$$\frac{1}{2q_1}+\frac{\frac{\alpha-1}{2}}{2}=\theta_1(\frac{1}{2}+\frac{0-1}{2})+(1-\theta_1)(\frac{1}{2}+\frac{\frac{\alpha}{2}-1}{2})=\frac{\alpha}{4}(1-\theta_1).$$

The combination of the above two equalities yields that

$$p_1=\frac{2}{\alpha\theta_1}, \tag{3.7}$$

with $\alpha>0$, $\theta_1\in(0,1)$ and $p_1>2$. Therefore, it holds that

$$\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_{p_1}^{\frac{1}{\theta_1}}\le C\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_2^{\frac{1}{\theta_1}}\le C\|\Psi(\rho,\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_1^{\frac{1}{2\theta_1}}\le C,$$

which together with (3.6) gives that

$$I_{11}\le\sigma\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+C_\sigma\|\nabla u\|_2^2. \tag{3.8}$$

Then, one can obtain

$$\begin{aligned}
I_{12} &\le \frac{\alpha}{2}\sqrt{\bar\rho}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2\||x|^{\frac{\frac{\alpha}{2}-1}{2}}|u|\|_4^2 \\
&\le C\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2\|\nabla u\|_2\||x|^{\frac{\alpha}{2}}\nabla u\|_2 \\
&\le \sigma\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+C_\sigma\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2\|\nabla u\|_2^2.
\end{aligned} \tag{3.9}$$

Then it holds that

$$\begin{aligned}
&|\int\big[\Psi(\rho,\bar\rho)+(P(\rho)-P(\bar\rho))\big]u\cdot\nabla(|x|^\alpha)dx| \\
&=|\int\big[\Psi(\rho,\bar\rho)+(P(\rho)-P(\bar\rho))\big]\big(\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}+\mathbf{1}|_{\{\rho>2\bar\rho\}}\big)u\cdot\nabla(|x|^\alpha)dx| \\
&\le C|\int\big[|\rho-\bar\rho|\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}+|\rho-\bar\rho|^\gamma\mathbf{1}|_{\{\rho>2\bar\rho\}}\big]u\cdot\nabla(|x|^\alpha)dx| \\
&\le C\big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_2\||x|^{\alpha-1}u\|_2+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\gamma p_2}^\gamma\||x|^{\alpha-1}u\|_{q_2}\big] \\
&\le C\big[\|\Psi(\rho,\bar\rho)\mathbf{1}|_{\{0\le\rho\le 2\bar\rho\}}\|_1^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u|x|^{\frac{\alpha}{2}}\|_2^{\frac{1}{2}}+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\gamma p_2}^\gamma\|\nabla u\|_2^{\theta_2}\||x|^{\frac{\alpha}{2}}\nabla u\|_2^{1-\theta_2}\big] \\
&\le \sigma\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+C_\sigma\big[1+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\gamma p_2}^{\frac{2\gamma}{1+\theta_2}}\big](\|\nabla u\|_2^2+1),
\end{aligned} \tag{3.10}$$

By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.2 (1), the positive constants $p_2 > 1, q_2 > 1, \theta_2 \in (0, 1]$ in the above inequality (3.10) satisfying

$$\frac{1}{p_2} + \frac{1}{q_2} = 1,$$

$$\frac{1}{q_2} + \frac{\alpha - 1}{2} = \theta_2(\frac{1}{2} + \frac{0-1}{2}) + (1-\theta_2)(\frac{1}{2} + \frac{\frac{\alpha}{2} - 1}{2}) = \frac{\alpha}{4}(1-\theta_2).$$

The combination of the above three equalities yields that

$$p_2 = \frac{4}{2 + \alpha(1+\theta_2)}, \tag{3.11}$$

with the parameters $\alpha > 0$, $\theta_2 \in (0, 1)$ and $p_2 > 1$. Note that $p_2 > 1$ is equivalent to the condition that

$$\frac{\alpha}{2}(1+\theta_2) < 1. \tag{3.12}$$

Then one can compute that

$$\begin{aligned} &|-\int \mu\nabla\frac{|u|^2}{2}\cdot\nabla(|x|^\alpha)dx| = \mu\alpha|\int u\cdot\nabla u\cdot x|x|^{\alpha-2}dx| \\ &\le \mu\alpha\||x|^{\frac{\alpha}{2}}\nabla u\|_2\||x|^{\frac{\alpha}{2}-1}u\|_2 \le \frac{\mu\alpha^2}{2}\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2, \end{aligned} \tag{3.13}$$

where in the last inequality one has used the best constant $\frac{\alpha}{2}$ for the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.2 (2). Similarly, it holds that

$$\begin{aligned} &|-\int \mu(\mathrm{div}u)u\cdot\nabla(|x|^\alpha)dx| = \mu\alpha|\int(\mathrm{div}u)|x|^{\alpha-2}u\cdot xdx| \\ &\le \mu\alpha\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\||x|^{\frac{\alpha}{2}-1}u\|_2 \le \frac{\mu\alpha^2}{2}\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\||x|^{\frac{\alpha}{2}}\nabla u\|_2. \end{aligned} \tag{3.14}$$

Then it follows that

$$\begin{aligned} &|-\int\lambda(\rho)(\mathrm{div}u)u\cdot\nabla(|x|^\alpha)dx| = \alpha|\int\sqrt{\lambda(\rho)}(\mathrm{div}u)\big[(\sqrt{\lambda(\rho)}-\sqrt{\lambda(\bar\rho)})+\sqrt{\lambda(\bar\rho)}\big]|x|^{\alpha-2}u\cdot xdx| \\ &\le \alpha|\int\sqrt{\lambda(\rho)}(\mathrm{div}u)\big(\sqrt{\lambda(\rho)}-\sqrt{\lambda(\bar\rho)}\big)|x|^{\alpha-2}u\cdot xdx| + \sqrt{\lambda(\bar\rho)}\alpha|\int\sqrt{\lambda(\rho)}(\mathrm{div}u)|x|^{\alpha-2}u\cdot xdx| \\ &:= I_{21} + I_{22}. \end{aligned} \tag{3.15}$$

It holds that

$$\begin{aligned} &I_{21} \le \alpha\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\|\sqrt{\lambda(\rho)}-\sqrt{\lambda(\bar\rho)}\|_{p_3}\||x|^{\frac{\alpha}{2}-1}u\|_{q_3} \\ &\le C\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_{p_3} + \|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{2}}\big]\|\nabla u\|_2^{\theta_3}\||x|^{\frac{\alpha}{2}}\nabla u\|_2^{1-\theta_3} \\ &\le \sigma\big[\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2 + \||x|^{\frac{\alpha}{2}}\nabla u\|_2^2\big] \\ &\qquad + C_\sigma\big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_{p_3}^{\frac{2}{\theta_3}} + \|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}}\big]\|\nabla u\|_2^2. \end{aligned} \tag{3.16}$$

By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.2 (1), the positive constants $p_3 > 2, q_3 > 2, \theta_3 \in (0, 1]$ in the above inequality (3.18) satisfying

$$\frac{1}{p_3} + \frac{1}{q_3} = \frac{1}{2},$$

$$\frac{1}{q_3}+\frac{\frac{\alpha}{2}-1}{2}=\theta_3(\frac{1}{2}+\frac{0-1}{2})+(1-\theta_3)(\frac{1}{2}+\frac{\frac{\alpha}{2}-1}{2})=\frac{\alpha}{4}(1-\theta_3).$$

The combination of the above three equalities yields that

$$p_3=\frac{4}{\alpha\theta_3}. \tag{3.17}$$

with $\alpha>0$, $\theta_3\in(0,1)$ and $p_3>2$. Therefore, it holds that

$$\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_{p_3}^{\frac{2}{\theta_3}}\le C\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_2^{\frac{2}{\theta_3}}\le C\|\Psi(\rho,\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_1^{\frac{1}{\theta_3}}\le C,$$

which together with (3.16) gives that

$$I_{21}\le\sigma\big[\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2+\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2\big]+C_\sigma\big[1+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}}\big]\|\nabla u\|_2^2. \tag{3.18}$$

Meanwhile, it holds that

$$\begin{aligned}I_{22}&\le\sqrt{\lambda(\bar\rho)}\alpha\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\||x|^{\frac{\alpha}{2}-1}u\|_2\\&\le\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{2}\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2\||x|^{\frac{\alpha}{2}}\nabla u\|_2.\end{aligned} \tag{3.19}$$

Substituting (3.8) and (3.9) into (3.5), (3.18) and (3.19) into (3.15) and then substituting the resulting (3.5), (3.15) and (3.10), (3.13) and (3.14) into (3.4) yield that

$$\begin{aligned}&\frac{d}{dt}\int|x|^\alpha\big[\rho\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)\big](t,x)dx+J(t)\le\sigma\big[\|\sqrt{\lambda(\rho)}|x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2+4\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2\big]\\&\quad+C_\sigma\big[1+\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{p_2\gamma}^{\frac{2\gamma}{1+\theta_2}}+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}}\big](\|\nabla u\|_2^2+1),\end{aligned} \tag{3.20}$$

where $\theta_i\in(0,1]$, p_i $(i=1,2,3)$ are given in (3.7), (3.11) and (3.17), respectively, and $p_1,p_3>2$ and $p_2>1$ and

$$\begin{aligned}J(t)=&\mu(1-\frac{\alpha^2}{2})\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2-\frac{\mu\alpha^2}{2}\||x|^{\frac{\alpha}{2}}\nabla u\|_2\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2+\mu\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2\\&+\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\mathrm{div}u\|_2^2(t)-\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{2}\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\mathrm{div}u\|_2\||x|^{\frac{\alpha}{2}}\nabla u\|_2.\end{aligned} \tag{3.21}$$

The corresponding matrix of the above quadratic term (3.21) is

$$A=\begin{pmatrix}\mu(1-\frac{\alpha^2}{2}) & -\frac{\mu\alpha^2}{4} & -\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{4}\\ -\frac{\mu\alpha^2}{4} & \mu & 0\\ -\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{4} & 0 & 1\end{pmatrix}.$$

The matrix A is positively definite if and only if all the principal minor determinant of A is positive, that is,

$$\mu(1-\frac{\alpha^2}{2})>0,\qquad\begin{vmatrix}\mu(1-\frac{\alpha^2}{2}) & -\frac{\mu\alpha^2}{4}\\ -\frac{\mu\alpha^2}{4} & \mu\end{vmatrix}>0,\quad\text{and}\quad\begin{vmatrix}\mu(1-\frac{\alpha^2}{2}) & -\frac{\mu\alpha^2}{4} & -\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{4}\\ -\frac{\mu\alpha^2}{4} & \mu & 0\\ -\frac{\alpha^2\sqrt{\lambda(\bar\rho)}}{4} & 0 & 1\end{vmatrix}>0.$$

Therefore, if the weight α satisfies

$$0<\alpha^2<\frac{4(\sqrt{2+\frac{\lambda(\bar\rho)}{\mu}}-1)}{1+\frac{\lambda(\bar\rho)}{\mu}}, \tag{3.22}$$

then the matrix A is positively definite, and then there exists a positive constant C_α such that

$$J(t)\geq C_\alpha^{-1}\Big[\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2(t)+\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2(t)+\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\mathrm{div}u\|_2^2(t)\Big]. \tag{3.23}$$

Consequently, if the weight α satisfies (3.22), then substituting (3.23) into (3.20) and choosing σ suitably small yield that

$$\begin{aligned}&\frac{d}{dt}\int|x|^\alpha\big[\rho\frac{|u|^2}{2}+\Psi(\rho,\bar\rho)\big](t,x)dx+\frac{\mu}{2}C_\alpha^{-1}\Big[\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2+\||x|^{\frac{\alpha}{2}}\mathrm{div}u\|_2^2+\||x|^{\frac{\alpha}{2}}\sqrt{\lambda(\rho)}\mathrm{div}u\|_2^2(t)\Big]\\&\leq C\big[1+\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2++\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{p_2\gamma}^{\frac{2\gamma}{1+\theta_2}}+\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}}\big](\|\nabla u\|_2^2+1).\end{aligned} \tag{3.24}$$

Now choose $m>1$ sufficiently large such that

$$2m\beta+1\geq\max\Big\{p_2\gamma,\frac{\beta p_3}{2}\Big\}.$$

Then, it holds that

$$\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{p_2\gamma}^{\frac{2\gamma}{1+\theta_2}}\leq\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\gamma}^{\frac{2a_2\gamma}{1+\theta_2}}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{2(1-a_2)\gamma}{1+\theta_2}}\leq C\|\Psi(\rho,\bar\rho)\|_1^{\frac{2a_2}{1+\theta_2}}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{2\gamma(1-a_2)}{1+\theta_2}}, \tag{3.25}$$

with $a_2\in(0,1)$ satisfying

$$\frac{a_2}{\gamma}+\frac{1-a_2}{2m\beta+1}=\frac{1}{p_2\gamma}=\frac{2+\alpha(1+\theta_2)}{4\gamma},$$

which implies that

$$a_2=\frac{(2+\alpha(1+\theta_2))(2m\beta+1)-4\gamma}{4(2m\beta+1-\gamma)}\to\frac{2+\alpha(1+\theta_2)}{4},\quad \text{as } m\to+\infty.$$

The following restriction should be imposed to (3.25)

$$\frac{2\gamma(1-a_2)}{1+\theta_2}\leq\beta,$$

which is satisfied provided

$$(1+\theta_2)(\frac{\beta}{\gamma}+\frac{\alpha}{2})>1 \tag{3.26}$$

and $m\gg1$. Since $\gamma\leq2\beta$, then $\frac{\beta}{\gamma}\geq\frac12$, thus one can choose $\theta_2\in(0,1)$ such that $(1+\theta_2)\frac{\beta}{\gamma}\geq1$ and $\frac{\alpha}{2}(1+\theta_2)<1$, and thus satisfies the restrictions (3.26) and (3.12) if $m\gg1$. If $\lambda(\bar\rho)<7\mu$, then $\frac{4(\sqrt{2+\frac{\lambda(\bar\rho)}{\mu}}-1)}{1+\frac{\lambda(\bar\rho)}{\mu}}>1$. Thus we can choose the weight $\alpha>0$ satisfying $1<\alpha^2<\frac{4(\sqrt{2+\frac{\lambda(\bar\rho)}{\mu}}-1)}{1+\frac{\lambda(\bar\rho)}{\mu}}$. In this case, one can choose $\theta_2\in(0,1)$ satisfying the restrictions (3.12) and (3.26) for any fixed

$\gamma, \beta > 1$, that is, the condition $\gamma \le 2\beta$ in the Theorem 1.2 can be removed as in Remark 2. Then it follows from (3.25) that

$$\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{p_2\gamma}^{\frac{2\gamma}{1+\theta_2}} \le C(\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}+1) \tag{3.27}$$

with the positive constant C independent of m.

Similarly, one has

$$\begin{aligned} \|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}} &\le \|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{1}^{\frac{\beta}{\theta_3}a_3}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{\theta_3}(1-a_3)} \\ &\le \|\Psi(\rho,\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{1}^{\frac{\beta}{\theta_3}a_3}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{\theta_3}(1-a_3)}, \end{aligned} \tag{3.28}$$

with with $a_3 \in (0,1)$ satisfying

$$\frac{a_3}{1}+\frac{1-a_3}{2m\beta+1}=\frac{2}{p_3\beta}=\frac{\alpha\theta_3}{2\beta},$$

which implies that

$$a_3=\frac{\alpha\theta_3(2m\beta+1)-2\beta}{4m\beta^2}\to\frac{\alpha\theta_3}{2\beta},\quad \text{as } m\to+\infty.$$

The following restriction should be imposed to (3.28)

$$\frac{1-a_3}{\theta_3}\le 1,$$

which is satisfied provided we choose $m \gg 1$ and $\theta_3 \in (0,1)$ such that

$$\theta_3(1+\frac{\alpha}{2\beta})>1.$$

Then it follows from (3.28) that

$$\|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_{\frac{\beta p_3}{2}}^{\frac{\beta}{\theta_3}} \le C(\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}+1) \tag{3.29}$$

with the positive constant C independent of m. Substituting (3.27) and (3.29) into (3.24), then integrating the resulting inequality over $[0,t]$ with $t\in[0,T]$ and using Gronwall inequality yield the estimate (3.2) in Lemma 3.2. □

Step 3. Density estimates:

Applying the operator *div* to the momentum equation $(1.1)_2$, it holds that

$$[\operatorname{div}(\rho u)]_t+\operatorname{div}[\operatorname{div}(\rho u\otimes u)]=\Delta F. \tag{3.30}$$

Consider the following three elliptic problems on the whole space $\mathbb{R}^2$:

$$-\Delta\xi_1=\operatorname{div}(\sqrt{\rho}u(\sqrt{\rho}-\sqrt{\bar\rho})), \tag{3.31}$$

$$-\Delta\xi_2=\sqrt{\bar\rho}\,\operatorname{div}(\sqrt{\rho}u), \tag{3.32}$$

$$-\Delta\eta=\operatorname{div}[\operatorname{div}(\rho u\otimes u)], \tag{3.33}$$

all with the boundary conditions $\xi_1,\xi_2,\eta\to 0$ as $|x|\to\infty$.

By the elliptic estimates and Hölder inequality, it holds that

Lemma 3.3 *(1)* $\|\nabla\xi_1\|_{2m} \le Cm\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{2mk}$, *for any* $k>1, m\ge 1$;

(2) $\|\nabla\xi_2\|_{2m} \le Cm\Big[\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{2mk} + \sqrt{\bar\rho}\|u\|_{2m}\Big]$, *for any* $k>1, m\ge 1$;

(3) $\|\nabla\xi_2|x|^{\frac{\alpha}{2}}\|_2 \le C\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2$, *for* α *satisfying* (3.22)*;*

(4) $\|\eta\|_{2m} \le Cm\big[\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{4mk}^2 + \bar\rho\|u\|_{4m}^2\big]$, *for any* $k>1, m\ge 1$;

where C *are positive constants independent of* m, k *and* r*.*

Proof: By the elliptic estimates to the equations (3.31), (3.32), respectively, and then using the Hölder inequality, one has for any $k>1, m\ge 1$,

$$\begin{aligned}\|\nabla\xi_1\|_{2m} &\le Cm\|\sqrt{\rho}u(\sqrt{\rho}-\sqrt{\bar\rho})\|_{2m} = Cm\|u(\rho-\bar\rho)\tfrac{\sqrt{\rho}}{\sqrt{\rho}+\sqrt{\bar\rho}}\|_{2m}\\ &\le Cm\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|\frac{\sqrt{\rho}}{\sqrt{\rho}+\sqrt{\bar\rho}}\|_\infty\|u\|_{2mk} \le Cm\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{2mk},\end{aligned}$$

and

$$\begin{aligned}\|\nabla\xi_2\|_{2m} &\le Cm\|\sqrt{\rho}u\|_{2m} \le Cm\big[\|(\sqrt{\rho}-\sqrt{\bar\rho})u\|_{2m} + \sqrt{\bar\rho}\|u\|_{2m}\big]\\ &\le Cm\big[\|\sqrt{\rho}-\sqrt{\bar\rho}\|_{\frac{2mk}{k-1}}\|u\|_{2mk} + \sqrt{\bar\rho}\|u\|_{2m}\big] \le Cm\big[\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{2mk} + \sqrt{\bar\rho}\|u\|_{2m}\big].\end{aligned}$$

Thus the proofs of (1) and (2) are completed.

By the similar proof as in Lemma 2.3 (2) in [26], the statements (3) can be proved.

Now we prove (4). By the elliptic estimates to the equation (3.33) and then using the Hölder inequality, one has for any $k>1, m\ge 1$,

$$\begin{aligned}\|\eta\|_{2m} &\le Cm\|\rho|u|^2\|_{2m} = Cm\big[\|(\rho-\bar\rho)|u|^2\|_{2m} + \bar\rho\||u|^2\|_{2m}\big]\\ &\le Cm\big[\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}\|u\|_{4mk}^2 + \bar\rho\|u\|_{4m}^2\big].\end{aligned}$$

Thus Lemma 3.3 is proved. □

Based on Lemmas 2.1-2.3 and Lemma 3.3, it holds that

Lemma 3.4 *(1)* $\|\xi_1\|_{2m} \le Cm^{\frac12}\|\rho-\bar\rho\|_{2m}$, *for any* $m\ge 2$;

(2) $\|\xi_2\|_{2m} \le Cm^{\frac12}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^{\frac{2}{m\alpha}}$, *for any* $m+1\ge\frac{4}{\alpha}$ *and* α *satisfying* (3.22)*;*

(3) $\|u\|_{2m} \le Cm^{\frac12}\big[\|\nabla u\|_2+1\big]$, *for any* $m\ge 1$;

(4) $\|\nabla\xi_1\|_{2m} \le Cm^{\frac32}k^{\frac12}\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}(\|\nabla u\|_2+1)$, *for any* $k>1, m\ge 1$;

(5) $\|\nabla\xi_2\|_{2m} \le Cm^{\frac32}\big[k^{\frac12}\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}+1\big](\|\nabla u\|_2+1)$, *for any* $k>1, m\ge 1$;

(6) $\|\eta\|_{2m} \le Cm^2\big[k\|\rho-\bar\rho\|_{\frac{2mk}{k-1}}+1\big]\big(\|\nabla u\|_2^2+1\big)$, *for any* $k>1, m\ge 1$;

where C *are positive constants independent of* m, k*.*

Proof: (1) By Lemma 2.2, it holds that

$$\|\xi_1\|_{2m} \le Cm^{\frac{1}{2}}\|\nabla\xi_1\|_{\frac{2m}{m+1}} \le Cm^{\frac{1}{2}}\|\sqrt{\rho}u\|_2\|\sqrt{\rho}-\sqrt{\bar\rho}\|_{2m} \le Cm^{\frac{1}{2}}\|\rho-\bar\rho\|_{2m},$$

where in the last inequality one has used the elementary energy estimates.

(2). If $m+1 > \frac{4}{\alpha}$, then by interpolation inequality, Caffarelli-Kohn-Nirenberg inequality and Lemma 3.3 (2), it holds that

$$\|\xi_2\|_{m+1} \le \|\xi_2\|_{2m}^{\theta}\|\xi_2\|_{\frac{4}{\alpha}}^{1-\theta} \le C\|\xi_2\|_{2m}^{\theta}\||x|^{\frac{\alpha}{2}}\nabla\xi_2\|_2^{1-\theta} \le C\|\xi_2\|_{2m}^{\theta}\||x|^{\frac{\alpha}{2}}\sqrt{\rho}u\|_2^{1-\theta} \tag{3.34}$$

where

$$\theta = \frac{\frac{1}{m+1}-\frac{\alpha}{4}}{\frac{1}{2m}-\frac{\alpha}{4}}.$$

Then it follows from Lemma 2.1 (2) and (3.34) that

$$\begin{aligned}\|\xi_2\|_{2m} &\le Cm^{\frac{1}{4}}\|\nabla\xi_2\|_2^{\frac{1}{2}-\frac{1}{2m}}\|\xi_2\|_{m+1}^{\frac{1}{2}+\frac{1}{2m}} \le Cm^{\frac{1}{4}}\|\sqrt{\rho}u\|_2^{\frac{1}{2}-\frac{1}{2m}}\|\xi_2\|_{2m}^{(\frac{1}{2}+\frac{1}{2m})\theta}\||x|^{\frac{\alpha}{2}}\sqrt{\rho}u\|_2^{(\frac{1}{2}+\frac{1}{2m})(1-\theta)}\\ &\le Cm^{\frac{1}{4}}\|\xi_2\|_{2m}^{(\frac{1}{2}+\frac{1}{2m})\theta}\||x|^{\frac{\alpha}{2}}\sqrt{\rho}u\|_2^{(\frac{1}{2}+\frac{1}{2m})(1-\theta)},\end{aligned}$$

which implies Lemma 3.4 (2) immediately.

Now we prove (3). First,

$$\begin{aligned}\bar\rho\int|u|^2dx &= \int(\bar\rho-\rho)|u|^2dx + \int\rho|u|^2dx\\ &= \int(\bar\rho-\rho)\big(\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}+\mathbf{1}|_{\{\rho>2\bar\rho\}}\big)|u|^2dx + \int\rho|u|^2dx\\ &\le \|(\bar\rho-\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_2\|u\|_4^2 + \|(\bar\rho-\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_\gamma\|u\|_{\frac{2\gamma}{\gamma-1}}^2 + C\\ &\le \|\Psi(\rho,\bar\rho)\|_1^{\frac{1}{2}}\|u\|_2\|\nabla u\|_2 + \|\Psi(\rho,\bar\rho)\|_1^{\frac{1}{\gamma}}\|u\|_2^{2-\frac{2}{\gamma}}\|\nabla u\|_2^{\frac{2}{\gamma}} + C\\ &\le \sigma\|u\|_2^2 + C_\sigma\|\nabla u\|_2^2 + C.\end{aligned}$$

Choosing $\sigma = \frac{\bar\rho}{2}$ in the above inequality yields that

$$\|u\|_2^2 \le C\big(\|\nabla u\|_2^2+1\big). \tag{3.35}$$

By Lemma 2.1 and the interpolation inequality, it holds that

$$\|u\|_{2m} \le Cm^{\frac{1}{4}}\|\nabla u\|_2^{\frac{1}{2}-\frac{1}{2m}}\|u\|_{m+1}^{\frac{1}{2}+\frac{1}{2m}} \le Cm^{\frac{1}{4}}\|\nabla u\|_2^{\frac{1}{2}-\frac{1}{2m}}\big(\|u\|_2^{\frac{1}{m+1}}\|u\|_{2m}^{\frac{m}{m+1}}\big)^{\frac{1}{2}+\frac{1}{2m}},$$

thus one has

$$\|u\|_{2m} \le Cm^{\frac{1}{2}}\|\nabla u\|_2^{1-\frac{1}{m}}\|u\|_2^{\frac{1}{m}} \le Cm^{\frac{1}{2}}(\|\nabla u\|_2+1),$$

where in the last inequality we have used (3.35). The statement (3) is proved.

The assertions (3), (4) and (5) in Lemma 3.4 are the direct consequences of Lemma 3.4 (2) and Lemma 3.3 (1), (2), (4), respectively. Thus the proof of Lemma 3.4 is completed. □

Substituting (3.31), (3.32) and (3.33) into (3.30) yields that

$$-\Delta\Big(\xi_{1t}+\xi_{2t}+\eta+F\Big) = 0,$$

which implies that

$$\xi_{1t}+\xi_{2t}+\eta+F=0.$$

It follows from the definition (2.1) of the effective viscous flux F that

$$\xi_{1t}+\xi_{2t}+(2\mu+\lambda(\rho))\mathrm{div}u-(P(\rho)-P(\bar\rho))+\eta=0.$$

Then the continuity equation $(1.1)_1$ yields that

$$\xi_{1t}+\xi_{2t}-\frac{2\mu+\lambda(\rho)}{\rho}(\rho_t+u\cdot\nabla\rho)-(P(\rho)-P(\bar\rho))+\eta=0.$$

Define

$$\Lambda(\rho)=\int_{\bar\rho}^{\rho}\frac{2\mu+\lambda(s)}{s}ds=2\mu\ln\frac{\rho}{\bar\rho}+\frac{1}{\beta}(\rho^\beta-\bar\rho^\beta).$$

Then we obtain a new transport equation

$$(\Lambda(\rho)-\xi_1-\xi_2)_t+u\cdot\nabla(\Lambda(\rho)-\xi_1-\xi_2)+(P(\rho)-P(\bar\rho))+u\cdot\nabla(\xi_1+\xi_2)-\eta=0, \tag{3.36}$$

which is crucial in the following Lemma for the higher integrability of the density function.

Lemma 3.5 *For any $k\ge 2$ and $\beta>1$, it holds that*

$$\sup_{t\in[0,T]}\|(\rho-\bar\rho)(t,\cdot)\|_k\le Ck^{\frac{2}{\beta-1}}. \tag{3.37}$$

Proof: Multiplying the equation (3.36) by $\rho[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}$ with m being sufficiently large integer, here and in what follows, the notation $(\cdots)_+$ denotes the positive part of $(\cdots)$, one can get that

$$\begin{aligned}&\frac{1}{2m}\frac{d}{dt}\int\rho[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m}dx+\int\rho(P(\rho)-P(\bar\rho))\mathbf{1}|_{\{\rho>2\bar\rho\}}[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx\\&=-\int\rho(P(\rho)-P(\bar\rho))\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx\\&\quad+\int\rho\eta[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx-\int\rho u\cdot\nabla(\xi_1+\xi_2)[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx.\end{aligned} \tag{3.38}$$

Denote

$$f(t)=\Big\{\int\rho[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m}dx\Big\}^{\frac{1}{2m}},\qquad t\in[0,T].$$

Now we estimate the three terms on the right hand side of (3.38). First, it holds that

$$\begin{aligned}&\Big|\int\rho(P(\rho)-P(\bar\rho))\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx\Big|\\&\le f(t)^{2m-1}\Big(\int\rho|P(\rho)-P(\bar\rho)|^{2m}\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}dx\Big)^{\frac{1}{2m}}\\&\le Cf(t)^{2m-1}\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_2\le Cf(t)^{2m-1}.\end{aligned} \tag{3.39}$$

Then, it follows that

$$
\begin{aligned}
&|\int \rho\eta[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx| \le \int \rho^{\frac{1}{2m}}|\eta|\big[\rho(\Lambda(\rho)-\xi_1-\xi_2)_+^{2m}\big]^{\frac{2m-1}{2m}}dx\\
&=\int \big[(\rho-\bar\rho)+\bar\rho\big]^{\frac{1}{2m}}|\eta|\big[\rho(\Lambda(\rho)-\xi_1-\xi_2)_+^{2m}\big]^{\frac{2m-1}{2m}}dx\\
&\le C\Big[\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{1}{2m}}\|\eta\|_{2m+\frac{1}{\beta}}+\|\eta\|_{2m}\Big]\|\rho(\Lambda(\rho)-\xi_1-\xi_2)_+^{2m}\|_1^{\frac{2m-1}{2m}}\\
&\le C\Big[\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{1}{2m}}(m+\frac{1}{2\beta})^2\big(k_1\|\rho-\bar\rho\|_{\frac{2(m+\frac{1}{2\beta})k_1}{k_1-1}}+1\big)\\
&\qquad\qquad +m^2\big(k_2\|\rho-\bar\rho\|_{\frac{2mk_2}{k_2-1}}+1\big)\Big]\big(\|\nabla u\|_2^2+1\big)f(t)^{2m-1}\\
&\le Cm^2\big[\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big]\big(\|\nabla u\|_2^2+1\big)f(t)^{2m-1},
\end{aligned}
\tag{3.40}
$$

where in the last inequality we have taken $k_1=\frac{\beta}{\beta-1}$ and $k_2=\frac{2m\beta+1}{2m(\beta-1)+1}$.

Next, for $\frac{1}{2m\beta+1}+\frac{1}{p_1}+\frac{1}{q_1}=1$ and $\frac{1}{p_2}+\frac{1}{q_2}=1$ with $p_i,q_i>1,(i=1,2)$, one has

$$
\begin{aligned}
&|-\int \rho u\cdot\nabla(\xi_1+\xi_2)[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx|\\
&\le \int \big[(\rho-\bar\rho)+\bar\rho\big]^{\frac{1}{2m}}|u||\nabla(\xi_1+\xi_2)|\big[\rho(\Lambda(\rho)-\xi_1-\xi_2)_+^{2m}\big]^{\frac{2m-1}{2m}}dx\\
&\le C\Big[\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{1}{2m}}\|u\|_{2mp_1}\|\nabla(\xi_1+\xi_2)\|_{2mq_1}+\|u\|_{2mp_2}\|\nabla(\xi_1+\xi_2)\|_{2mq_2}\Big]f(t)^{2m-1}\\
&\le C\Big[\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{1}{2m}}(mp_1)^{\frac12}(mq_1)^{\frac32}\big(k_1^{\frac12}\|\rho-\bar\rho\|_{\frac{2mq_1k_1}{k_1-1}}+1\big)\\
&\qquad +(mp_2)^{\frac12}(mq_2)^{\frac32}\big(k_2\|\rho-\bar\rho\|_{\frac{2mq_2k_2}{k_2-1}}+1\big)\Big]\big(\|\nabla u\|_2^2+1\big)f(t)^{2m-1}\\
&\le Cm^2\big[\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big]\big(\|\nabla u\|_2^2+1\big)f(t)^{2m-1},
\end{aligned}
\tag{3.41}
$$

where in the last inequality one has chosen $p_1=\frac{(2m\beta+1)(\beta+1)}{2m\beta(\beta-1)}, q_1=\frac{(\beta+1)(2m\beta+1)}{4m\beta}, k_1=\frac{2\beta}{\beta-1}$ and $p_2=\frac{2\beta}{\beta-1}, q_2=\frac{2\beta}{\beta+1}, k_2=\frac{(\beta+1)(2m\beta+1)}{2m\beta(\beta-1)+(\beta+1)}$.

Substituting (3.39), (3.40) and (3.41) into (3.38) yields that

$$
\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}(f^{2m}(t))+\int \rho(P(\rho)-P(\bar\rho))\mathbf{1}|_{\{\rho>2\bar\rho\}}[(\Lambda(\rho)-\xi_1-\xi_2)_+]^{2m-1}dx\\
&\le Cm^2\big[\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big]\big(\|\nabla u\|_2^2+1\big)f(t)^{2m-1}.
\end{aligned}
$$

Then it holds that

$$
\frac{d}{dt}f(t)\le Cm^2\big[\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big]\big(\|\nabla u\|_2^2+1\big).
$$

Integrating the above inequality over $[0,t]$ gives that

$$
f(t)\le f(0)+Cm^2\int_0^t\big[\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big]\big(\|\nabla u\|_2^2+1\big)d\tau. \tag{3.42}
$$

Now we calculate the quantity

$$
f(0)=\Big(\int \rho_0[(\Lambda(\rho_0)-\xi_{10}-\xi_{20})_+]^{2m}dx\Big)^{\frac{1}{2m}}.
$$

By Lemma 3.3 (1), (2) and Lemma 3.4 (1), (2) with $t=0$, we can easily get

$$\|\xi_{10}+\xi_{20}\|_{L^\infty}\le C.$$

Furthermore, by the definition of $\Lambda(\rho_0)=2\mu\ln\frac{\rho_0}{\bar\rho}+\frac{1}{\beta}((\rho_0)^\beta-\bar\rho^\beta)$, we have

$$\Lambda(\rho_0)-\xi_{10}-\xi_{20}\to-\infty,\quad \text{as}\quad \rho_0\to 0+.$$

Thus there exists a positive constant σ_0, such that if $0\le\rho_0\le\sigma_0$, then

$$(\Lambda(\rho_0)-\xi_{10}-\xi_{20})_+\equiv 0.$$

Now one has

$$\begin{aligned}
f(0)&=\Big[\Big(\int_{[0\le\rho_0\le\sigma_0]}+\int_{[\sigma_0\le\rho_0\le M]}\Big)\rho_0(\Lambda(\rho_0)-\xi_{10}-\xi_{20})_+^{2m}dx\Big]^{\frac{1}{2m}}\\
&=\Big[\int_{[\sigma_0\le\rho_0\le M]}\rho_0(\Lambda(\rho_0)-\xi_{10}-\xi_{20})_+^{2m}dx\Big]^{\frac{1}{2m}}\\
&\le C(\sigma_0,M)\Big[\|(\rho_0-\bar\rho)\mathbf{1}_{\sigma_0\le\rho_0\le M}\|_{2m}+\|\xi_{10}+\xi_{20}\|_{2m}\Big]\le C(\sigma_0,M)m^{\frac{3}{2}},
\end{aligned}\tag{3.43}$$

where the positive constant $C(\sigma_0,M)$ is independent of m and the lower bound of the density.

It follows from (3.42) and (3.43) that for $t\in[0,T]$,

$$f(t)\le Cm^2\Big[1+\int_0^t\big(\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}+1\big)\big(\|\nabla u\|_2^2+1\big)d\tau\Big].\tag{3.44}$$

For any $t\in[0,T]$, set $\Omega_1(t)=\{x\in\mathbb{R}^2|\rho(t,x)>2\bar\rho\}$ and $\Omega_2(t)=\{x\in\Omega_1(t)|(\Lambda(\rho)-\xi_1-\xi_2)(t,x)>0\}$. Then one has on $\Omega_1(t)$, $|\rho-\bar\rho|^\beta\le C\beta|\Lambda(\rho)|$ for some constant $C>0$, and on $\Omega_1(t)\setminus\Omega_2(t)$, $0<\Lambda(\rho)\le\xi_1+\xi_2$. Thus it holds that

$$\begin{aligned}
&\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}(t)=\Big(\int_{\Omega_1(t)}|\rho-\bar\rho|^{2m\beta+1}dx+\int_{\mathbb{R}^2\setminus\Omega_1(t)}|\rho-\bar\rho|^{2m\beta+1}dx\Big)^{\frac{\beta}{2m\beta+1}}\\
&\le\Big(\int_{\Omega_1(t)}|\rho-\bar\rho|^{2m\beta+1}dx+\bar\rho^{2m\beta-1}\int_{\mathbb{R}^2\setminus\Omega_1(t)}|\rho-\bar\rho|^2dx\Big)^{\frac{\beta}{2m\beta+1}}\\
&\le\Big[\int_{\Omega_1(t)}\big(|\rho-\bar\rho|^\beta\big)^{\frac{2m\beta+1}{\beta}}dx\Big]^{\frac{\beta}{2m\beta+1}}+C\le C\Big(\int_{\Omega_1(t)}|\beta\Lambda(\rho)|^{\frac{2m\beta+1}{\beta}}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\\
&\le C\Big(\int_{\Omega_1(t)}\Lambda(\rho)^{2m}\Lambda(\rho)^{\frac{1}{\beta}}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\le C\Big(\int_{\Omega_1(t)}\rho\Lambda(\rho)^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\\
&=C\Big(\int_{\Omega_2(t)}\rho|\Lambda(\rho)-\xi_1-\xi_2+(\xi_1+\xi_2)|^{2m}dx+\int_{\Omega_1(t)\setminus\Omega_2(t)}\rho|\Lambda(\rho)|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\\
&\le C\Big[\int_{\Omega_2(t)}\rho(\Lambda(\rho)-\xi_1-\xi_2)^{2m}dx+\int_{\Omega_2(t)}\rho|\xi_1+\xi_2|^{2m}dx+\int_{\Omega_1(t)\setminus\Omega_2(t)}\rho|\xi_1+\xi_2|^{2m}dx\Big]^{\frac{\beta}{2m\beta+1}}+C\\
&\le C\Big(f(t)^{2m}+\int_{\mathbb{R}^2}\rho|\xi_1+\xi_2|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\\
&\le C\Big[f(t)+\Big(\int_{\mathbb{R}^2}\rho|\xi_1|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+\Big(\int_{\mathbb{R}^2}\rho|\xi_2|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+1\Big].
\end{aligned}\tag{3.45}$$

Note that

$$\Big(\int_{\mathbb{R}^2}\rho|\xi_1|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}\le C\Big(\int_{\mathbb{R}^2}|\rho-\bar\rho||\xi_1|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}+C\Big(\int_{\mathbb{R}^2}|\xi_1|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}}:=K_{11}+K_{12}.\tag{3.46}$$

$$
\begin{aligned}
K_{11} &\le C\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\||\xi_1|^{2m}\|_{\frac{2m\beta+1}{2m\beta}}^{\frac{\beta}{2m\beta+1}} = C\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\|\xi_1\|_{2m+\frac{1}{\beta}}^{\frac{2m\beta}{2m\beta+1}} \\
&\le \|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\Big[C(m+\frac{1}{2\beta})^{\frac12}\|\rho-\bar\rho\|_{2m+\frac{1}{\beta}}\Big]^{\frac{2m\beta}{2m\beta+1}} \\
&\le Cm^{\frac12}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\Big[\|(\rho-\bar\rho)\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}\|_2^{\frac{2m\beta}{2m\beta+1}} \\
&\qquad\qquad + \|(\rho-\bar\rho)\mathbf{1}|_{\{\rho>2\bar\rho\}}\|_\gamma^{\frac{2\gamma m\beta(\beta-1)}{(2m\beta-\gamma+1)(2m\beta+1)}}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{2m\beta(2m\beta-\gamma\beta+1)}{(2m\beta-\gamma+1)(2m\beta+1)}}\Big] \\
&\le Cm^{\frac12}\big[\|\rho-\bar\rho\|_{2m\beta+1}+1\big],
\end{aligned}
\tag{3.47}
$$

and

$$
K_{12} = \|\xi_1\|_{2m}^{\frac{2m\beta}{2m\beta+1}} \le \Big(Cm^{\frac12}\|\rho-\bar\rho\|_{2m}\Big)^{\frac{2m\beta}{2m\beta+1}} \le Cm^{\frac12}\big[\|\rho-\bar\rho\|_{2m\beta+1}+1\big]. \tag{3.48}
$$

Furthermore, it holds that

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^2}\rho|\xi_2|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}} &\le C\Big(\int_{\mathbb{R}^2}|\rho-\bar\rho||\xi_2|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}} + C\Big(\int_{\mathbb{R}^2}|\xi_2|^{2m}dx\Big)^{\frac{\beta}{2m\beta+1}} \\
&\le C\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\|\xi_2\|_{2m+\frac{1}{\beta}}^{\frac{2m\beta}{2m\beta+1}} + C\|\xi_2\|_{2m}^{\frac{2m\beta}{2m\beta+1}} \\
&\le Cm^{\frac12}\|\rho-\bar\rho\|_{2m\beta+1}^{\frac{\beta}{2m\beta+1}}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^{\frac{2}{(m+\frac{1}{2\beta})\alpha}\frac{2m\beta}{2m\beta+1}} + Cm^{\frac12}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^{\frac{2}{m\alpha}\frac{2m\beta}{2m\beta+1}} \\
&\le C\big[\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2 + m^{\frac12}\|\rho-\bar\rho\|_{2m\beta+1} + m^{\frac12}\big].
\end{aligned}
\tag{3.49}
$$

Substituting and (3.46), (3.47), (3.48) and (3.49) into (3.45) yields that

$$
\begin{aligned}
\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}(t) &\le C\Big[m^{\frac12} + f(t) + m^{\frac12}\|\rho-\bar\rho\|_{2m\beta+1}(t) + \|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2(t)\Big] \\
&\le \frac12\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}(t) + C\Big[f(t) + m^{\frac{\beta}{2(\beta-1)}} + \|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2(t)\Big].
\end{aligned}
\tag{3.50}
$$

Thus it follows from (3.44), (3.50) and the weighted estimates in Lemma 3.2 that

$$
\begin{aligned}
&\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}(t) \le C\Big[f(t) + m^{\frac{\beta}{2(\beta-1)}} + \|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^2(t)\Big] \\
&\le C\Big[m^2 + m^2\int_0^t\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}\big(\|\nabla u\|_2^2+1\big)d\tau + \int_0^t\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}\big(\|\nabla u\|_2^2+1\big)d\tau\Big].
\end{aligned}
$$

Applying Gronwall's inequality to the above inequality yields that

$$
\|\rho-\bar\rho\|_{2m\beta+1}^{\beta}(t) \le Cm^2\Big[1+\int_0^t\|\rho-\bar\rho\|_{2m\beta+1}^{1+\frac{1}{2m}}\big(\|\nabla u\|_2^2(\tau)+1\big)d\tau\Big].
$$

Denote

$$
y(t) = m^{-\frac{2}{\beta-1}}\|\rho-\bar\rho\|_{2m\beta+1}(t).
$$

Then it holds that

$$
y^\beta(t) \le C\Big[1+\int_0^t y(\tau)^{1+\frac{1}{2m}}\|\nabla u\|_2^2(\tau)d\tau\Big] \le C\Big[1+\int_0^t\big(y^\beta(\tau)+1\big)\|\nabla u\|_2^2(\tau)d\tau\Big].
$$

So applying the Gronwall's inequality to the above inequality yields that

$$
y(t) \le C, \quad \forall t\in[0,T],
$$

that is, for sufficiently large $m > 1$,

$$\|\rho - \bar{\rho}\|_{2m\beta+1}(t) \le Cm^{\frac{2}{\beta-1}}, \quad \forall t \in [0, T].$$

Equivalently, (3.37) holds for sufficiently large k. Now by the elementary energy estimate Lemma 3.1, if $\gamma \ge 2$, then

$$\|\rho - \bar{\rho}\|_2(t) \le C\|\Psi(\rho, \bar{\rho})\|_1^{\frac{1}{2}} \le C, \tag{3.51}$$

and if $1 < \gamma < 2$, then

$$\|(\rho - \bar{\rho})\mathbf{1}|_{\{0\le\rho\le2\bar{\rho}\}}\|_2(t) \le C\|\Psi(\rho, \bar{\rho})\mathbf{1}|_{\{0\le\rho\le2\bar{\rho}\}}\|_1^{\frac{1}{2}} \le C, \tag{3.52}$$

and

$$\|(\rho - \bar{\rho})\mathbf{1}|_{\{\rho>2\bar{\rho}\}}\|_\gamma(t) \le C\|\Psi(\rho, \bar{\rho})\mathbf{1}|_{\{\rho>2\bar{\rho}\}}\|_1^{\frac{1}{\gamma}} \le C.$$

Therefore, for $1 < \gamma < 2$, it holds that

$$\|(\rho - \bar{\rho})\mathbf{1}|_{\{\rho>2\bar{\rho}\}}\|_2(t) \le \|(\rho - \bar{\rho})\mathbf{1}|_{\{\rho>2\bar{\rho}\}}\|_\gamma^\theta\|\rho - \bar{\rho}\|_k^{1-\theta} \le C, \tag{3.53}$$

where k is sufficiently large such that (3.37) holds and $\theta \in (0, 1)$ satisfying $\frac{1}{2} = \frac{\theta}{\gamma} + \frac{1-\theta}{k}$. Thus by (3.51), (3.52) and (3.53), it holds that for any $\gamma > 1$ and $t \in [0, T]$,

$$\|\rho - \bar{\rho}\|_2(t) \le C.$$

Thus Lemma 3.5 is proved for any $k \ge 2$. □

Step 4: First-order derivative estimates of the velocity.

Set

$$Z^2(t) = \int (\mu\omega^2 + \frac{F^2}{2\mu + \lambda(\rho)})dx,$$

$$\varphi^2(t) = \int \rho(H^2 + L^2)dx = \int \rho|\dot{u}|^2 dx$$

and

$$\Phi_T = \sup_{t\in[0,T]} \|\rho(\cdot, t)\|_\infty + 1.$$

The following Lemma is motivated by [38].

Lemma 3.6 *For any $\varepsilon > 0$, there exists a positive constant C_ε, such that*

$$\sup_{t\in[0,T]} \log(e + Z^2(t)) + \int_0^T \frac{\varphi^2(t)}{e + Z^2(t)}dt \le C_\varepsilon \Phi_T^{1+\varepsilon\beta}.$$

Proof: Multiplying the equation $(2.2)_1$ by $\mu\omega$, the equation $(2.2)_2$ by $\frac{F}{2\mu+\lambda(\rho)}$, respectively, and then summing the resulted equations together, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (\mu\omega^2 + \frac{F^2}{2\mu + \lambda(\rho)})dx + \frac{\mu}{2}\int \omega^2 \mathrm{div}u dx - \frac{1}{2}\int \rho F^2(\frac{1}{2\mu + \lambda(\rho)})'\mathrm{div}u dx \\
&-\frac{1}{2}\int F^2\frac{\mathrm{div}u}{2\mu + \lambda(\rho)}dx - \int \rho F(\mathrm{div}u)(\frac{P(\rho) - P(\bar{\rho})}{2\mu + \lambda(\rho)})'dx + \int F[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2]dx \\
&= -\int \rho(H^2 + L^2)dx.
\end{aligned} \tag{3.54}$$

Notice that

$$
\begin{aligned}
&(u_{1x_1})^2+2u_{1x_2}u_{2x_1}+(u_{2x_2})^2=(u_{1x_1}+u_{2x_2})^2+2(u_{1x_2}u_{2x_1}-u_{1x_1}u_{2x_2})\\
&\quad=(\mathrm{div}u)^2+2(u_{1x_2}u_{2x_1}-u_{1x_1}u_{2x_2})\\
&\quad=(\mathrm{div}u)\left(\frac{F}{2\mu+\lambda(\rho)}+\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)}\right)+2(u_{1x_2}u_{2x_1}-u_{1x_1}u_{2x_2}),
\end{aligned}
$$

then one has

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int(\mu\omega^2+\frac{F^2}{2\mu+\lambda(\rho)})dx+\int\rho(H^2+L^2)dx\\
&\quad=-\frac{\mu}{2}\int\omega^2\mathrm{div}u dx+\frac12\int F^2(\mathrm{div}u)\Big[\rho(\frac{1}{2\mu+\lambda(\rho)})'-\frac{1}{2\mu+\lambda(\rho)}\Big]dx\\
&\quad+\int F(\mathrm{div}u)\Big[\rho(\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)})'-\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)}\Big]dx-\int 2F(u_{1x_2}u_{2x_1}-u_{1x_1}u_{2x_2})dx.
\end{aligned}
\tag{3.55}
$$

Then

$$
\begin{aligned}
&\|\nabla u\|_2+\|\omega\|_2+\|\mathrm{div}u\|_2+\|(2\mu+\lambda(\rho))^{\frac12}\mathrm{div}u\|_2\\
&\le C\Big[Z(t)+\Big(\int\frac{|P(\rho)-P(\bar\rho)|^2}{2\mu+\lambda(\rho)}dx\Big)^{\frac12}\Big]\le C(Z(t)+1).
\end{aligned}
\tag{3.56}
$$

Now we estimate the four terms on the right hand side of (3.55). First, by the interpolation inequality, Lemma 2.1 and (3.56), it holds that

$$
\begin{aligned}
|-\frac{\mu}{2}\int\omega^2\mathrm{div}u dx|&\le C\|\mathrm{div}u\|_2\|\omega\|_4^2\le C(Z(t)+1)\|\omega\|_2\|\nabla\omega\|_2\\
&\le C(Z(t)+1)\|\omega\|_2\|\rho\dot u\|_2\le C(Z(t)+1)\|\rho\|_\infty^{\frac12}\|\omega\|_2\|\sqrt{\rho}\dot u\|_2\\
&\le\sigma\|\sqrt{\rho}\dot u\|_2^2+C_\sigma(Z(t)^2+1)\|\rho\|_\infty\|\omega\|_2^2.
\end{aligned}
\tag{3.57}
$$

Next, one has

$$
\begin{aligned}
&|\frac12\int F^2\mathrm{div}u\Big[\rho(\frac{1}{2\mu+\lambda(\rho)})'-\frac{1}{2\mu+\lambda(\rho)}\Big]dx|\\
&\le C\int|F|^2\frac{|\mathrm{div}u|}{2\mu+\lambda(\rho)}dx\le\|\mathrm{div}u\|_2\|\frac{F^2}{2\mu+\lambda(\rho)}\|_2,
\end{aligned}
$$

while for any $\varepsilon>0$ suitably small,

$$
\begin{aligned}
&\|\frac{F^2}{2\mu+\lambda(\rho)}\|_2\le\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{1-\varepsilon}\|F\|_{\frac{2(1+\varepsilon)}{\varepsilon}}^{1+\varepsilon}\\
&\le C\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{1-\varepsilon}\big(\|F\|_2^{\frac{\varepsilon}{1+\varepsilon}}\|\nabla F\|_2^{\frac{1}{1+\varepsilon}}\big)^{1+\varepsilon}\\
&\le C\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{1-\varepsilon}\|F\|_2^{\varepsilon}\|\nabla F\|_2\le C\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\|\rho\|_\infty^{\frac{1+\beta\varepsilon}{2}}\|\sqrt{\rho}\dot u\|_2.
\end{aligned}
\tag{3.58}
$$

Then it holds that

$$
\begin{aligned}
&|\frac12\int F^2\mathrm{div}u\Big[\rho(\frac{1}{2\mu+\lambda(\rho)})'-\frac{1}{2\mu+\lambda(\rho)}\Big]dx|\\
&\le C\|\mathrm{div}u\|_2\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\|\rho\|_\infty^{\frac{1+\beta\varepsilon}{2}}\|\sqrt{\rho}\dot u\|_2\\
&\le\sigma\|\sqrt{\rho}\dot u\|_2^2+C_\sigma\|\rho\|_\infty^{1+\beta\varepsilon}\|\mathrm{div}u\|_2^2\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^2.
\end{aligned}
\tag{3.59}
$$

On the other hand, it holds that

$$
\begin{aligned}
&|\int F(\mathrm{div}u)\Big[\rho(\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)})'-\frac{P(\rho)-P(\bar\rho)}{2\mu+\lambda(\rho)}\Big]dx|\\
&\le C\int|F||\mathrm{div}u|\frac{|P(\rho)-P(\bar\rho)|+1}{2\mu+\lambda(\rho)}dx\\
&\le C\|\mathrm{div}u\|_2\Big[\|F\|_{\frac{2(2+\varepsilon)}{\varepsilon}}\|P(\rho)-P(\bar\rho)\|_{2+\varepsilon}+\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\Big]\\
&\le C\|\mathrm{div}u\|_2\Big[\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac{\varepsilon}{2+\varepsilon}}\|\rho\|_\infty^{\frac{\beta\varepsilon}{2(2+\varepsilon)}}\|\nabla F\|_2^{\frac{2}{2+\varepsilon}}+\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\Big]\\
&\le C\|\mathrm{div}u\|_2\Big[\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac{\varepsilon}{2+\varepsilon}}\|\rho\|_\infty^{\frac12+\frac{\beta\varepsilon}{2(2+\varepsilon)}}\|\sqrt\rho\dot u\|_2^{\frac{2}{2+\varepsilon}}+\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\Big]\\
&\le\sigma\|\sqrt\rho\dot u\|_2^2+C_\sigma\|\rho\|_\infty^{1+\frac{\beta\varepsilon}{2+\varepsilon}}\|\mathrm{div}u\|_2^{\frac{2+\varepsilon}{1+\varepsilon}}\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac{\varepsilon}{1+\varepsilon}}+C\|\mathrm{div}u\|_2\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2\\
&\le\sigma\|\sqrt\rho\dot u\|_2^2+C_\sigma(1+\|\rho\|_\infty)^{1+\beta\varepsilon}\big(\|\mathrm{div}u\|_2^2+1\big)\big(\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^2+1\big).
\end{aligned}
\tag{3.60}
$$

Then due to [38], it holds that

$$
\begin{aligned}
&|-\int 2F(u_{1x_2}u_{2x_1}-u_{1x_1}u_{2x_2})dx|=|-\int 2F\nabla u_1\cdot\nabla^\perp u_2dx|\\
&\le C\|F\|_{\mathrm{BMO}}\|\nabla u_1\cdot\nabla^\perp u_2\|_{\mathcal H^1}\le C\|\nabla F\|_2\|\nabla u\|_2^2\le C\|\rho\|_\infty^{\frac12}\|\sqrt\rho\dot u\|_2\|\nabla u\|_2^2\\
&\le\sigma\|\sqrt\rho\dot u\|_2^2+C_\sigma\|\rho\|_\infty\|\nabla u\|_2^4\le\sigma\|\sqrt\rho\dot u\|_2^2+C_\sigma\|\rho\|_\infty\|\nabla u\|_2^2\Big[1+Z^2(t)\Big].
\end{aligned}
\tag{3.61}
$$

In summary, substituting (3.57), (3.59), (3.60) and (3.61) into (3.55), one can arrive at

$$
\frac{d}{dt}Z^2(t)+\varphi^2(t)\le C\Phi_T^{1+\beta\varepsilon}(1+\|\nabla u\|_2^2)(1+Z^2(t))
$$

Multiplying the above inequality by $\frac{1}{e+Z^2(t)}$ and then integrating over $[0,T]$ give the proof of Lemma 3.6 . □

Step 5: Upper and lower bound of the density:

The following Lemma comes from [17, 18]. With the following Lemma, the index β can be improved to $\beta>\frac43$ as in [17, 18].

Lemma 3.7 *There exists a positive constant C, such that*

$$
\sup_{t\in[0,T]}\int\rho|u|^{2+\nu}dx\le C,
$$

where $\nu=\frac{\mu^{\frac12}}{2(\mu+1)}\Phi_T^{-\frac{\beta}{2}}\in(0,\frac14]$.

Proof: Multiplying the momentum equation $(1.1)_2$ by $(2+\nu)u|u|^\nu$ and integrating over $\mathbb R^2$ with respect to x lead to

$$
\begin{aligned}
&\frac{d}{dt}\int\rho|u|^{2+\nu}dx+\mu(2+\nu)\int|\nabla u|^2|u|^\nu dx+(2+\nu)\int(\mu+\lambda(\rho))(\mathrm{div}u)^2|u|^\nu dx\\
&=(2+\nu)\int(P(\rho)-P(\bar\rho))\mathrm{div}(u|u|^\nu)dx-\mu(2+\nu)\int\nabla\frac{|u|^2}{2}\cdot\nabla|u|^\nu dx\\
&\quad-(2+\nu)\int(\mu+\lambda(\rho))(\mathrm{div}u)u\cdot\nabla|u|^\nu dx.
\end{aligned}
$$

Now we only estimate the first term on the right hand side of the above equality, since the other terms can be done similarly as in [18]. Then it holds that

$$
\begin{aligned}
&(2+\nu)|\int (P(\rho)-P(\bar\rho))\mathrm{div}(u|u|^\nu)dx| \le (2+\nu)(1+\nu)\int |P(\rho)-P(\bar\rho)||\nabla u||u|^\nu dx\\
&\le \sigma(2+\nu)\int |\nabla u|^2|u|^\nu dx + C_\sigma(2+\nu)(1+\nu)^2\int |P(\rho)-P(\bar\rho)|^2|u|^\nu dx\\
&\le \sigma(2+\nu)\int |\nabla u|^2|u|^\nu dx + C_\sigma(2+\nu)(1+\nu)^2\|P(\rho)-P(\bar\rho)\|_{2q_1}^2\|u\|_{q_2\nu}^\nu\\
&\le \sigma(2+\nu)\int |\nabla u|^2|u|^\nu dx + C_\sigma(2+\nu)(1+\nu)^2\big(\|\nabla u\|_2^2+1\big).
\end{aligned}
$$

where $q_1, q_2 > 1$ satisfying $\frac{1}{q_1}+\frac{1}{q_2}=1$. Thus Lemma 3.7 is proved. □

Now one can obtain the upper and lower bound of the density by using the transport equation (3.36).

Lemma 3.8 *There exists positive constants C_1 and c_1 such that*

$$
c_1 \le \rho(t,x) \le C_1, \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^2.
$$

Proof: First, for any $p>2$ and $q>1$ satisfying

$$
\frac{1}{p} = \frac{\frac{2}{p}}{2+\nu} + \frac{1-\frac{2}{p}}{q},
$$

it holds that

$$
\begin{aligned}
\|\rho u\|_p &\le \|\rho u\|_{2+\nu}^{\frac{2}{p}}\|\rho u\|_q^{1-\frac{2}{p}} \le C\Big(\|\rho^{\frac{1}{2+\nu}}u\|_{2+\nu}\|\rho\|_\infty^{\frac{1+\nu}{2+\nu}}\Big)^{\frac{2}{p}}\big(\|\rho\|_\infty\|u\|_q\big)^{1-\frac{2}{p}}\\
&\le C\|\rho\|_\infty^{1-\frac{2}{p}+\frac{2(1+\nu)}{p(2+\nu)}}\Big[q^{\frac{1}{2}}(\|\nabla u\|_2+1)\Big]^{1-\frac{2}{p}},
\end{aligned}
$$

where in the last inequality one has used Lemma 3.4 (3). It can be computed that

$$
q = (1+\frac{2}{\nu})(p-2) \le C_p\Phi_T^{\frac{\beta}{2}}.
$$

Therefore, one has

$$
\|\rho u\|_p \le C\|\rho\|_\infty^{1-\frac{2}{p(2+\nu)}}\Phi_T^{\frac{\beta}{4}(1-\frac{2}{p})}(\|\nabla u\|_2^{1-\frac{2}{p}}+1) \le C\Phi_T^{1+\frac{\beta}{4}}(\|\nabla u\|_2^{1-\frac{2}{p}}+1). \tag{3.62}
$$

Note that by the definition of ξ_i $(i=1,2)$ from (3.31) and (3.32)

$$
u\cdot\nabla(\xi_1+\xi_2)-\eta = [u, R_iR_j](\rho u), \tag{3.63}
$$

where $[\cdot,\cdot]$ is the usual commutator and R_i, R_j are the Riesz operators. Thus from (3.36), it holds that

$$
D_t\Lambda(\rho) - D_t(\xi_1+\xi_2) + (P(\rho)-P(\bar\rho)) + [u, R_iR_j](\rho u) = 0, \tag{3.64}
$$

where the material derivative $D_t := \partial_t + u\cdot\nabla$. Along the particle path $\vec X(\tau;t,x)$ through the point $(t,x)\in[0,T]\times\mathbb{R}^2$ defined by

$$
\begin{cases}
\dfrac{d\vec X(\tau;t,x)}{d\tau} = u(\tau, \vec X(\tau;t,x)),\\
\vec X(\tau;t,x)|_{\tau=t} = x,
\end{cases}
$$

from the equation (3.64), there holds the following ODE

$$\begin{aligned}&\frac{d}{d\tau}(\Lambda(\rho)-\xi_1-\xi_2)(\tau,\vec{X}(\tau;t,x))+(P(\rho)-P(\bar\rho))\mathbf{1}|_{\{\rho>2\bar\rho\}}(\tau,\vec{X}(\tau;t,x))\\&\quad=-\Big((P(\rho)-P(\bar\rho))\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}+[u,R_iR_j](\rho u)\Big)(\tau,\vec{X}(\tau;t,x)),\end{aligned}$$

and thus

$$\frac{d}{d\tau}(\Lambda(\rho)-\xi_1-\xi_2)(\tau,\vec{X}(\tau;t,x))\le-\Big((P(\rho)-P(\bar\rho))\mathbf{1}|_{\{0\le\rho\le2\bar\rho\}}+[u,R_iR_j](\rho u)\Big)(\tau,\vec{X}(\tau;t,x)).$$

Integrating the above inequality over $[0,t]$ yields that

$$\begin{aligned}&2\mu\ln\frac{\rho(t,x)}{\rho_0(\vec{X}_0)}+\frac{1}{\beta}\big(\rho^\beta(t,x)-\rho_0^\beta(\vec{X}_0)\big)-\big((\xi_1+\xi_2)(t,x)-(\xi_{10}+\xi_{20})(\vec{X}_0)\big)\\&\qquad\le C+\int_0^t\|[u,R_iR_j](\rho u)\|_\infty ds,\end{aligned}\tag{3.65}$$

with $\vec{X}_0=\vec{X}(\tau;t,x)|_{\tau=0}$.

Then for any sufficiently large $p>4$, by the commutator estimates for (3.63) and (3.62), it holds that

$$\begin{aligned}&\|[u,R_iR_j](\rho u)\|_\infty\le C\|[u,R_iR_j](\rho u)\|_p^{1-\frac4p}\|\nabla\big([u,R_iR_j](\rho u)\big)\|_{\frac{4p}{p+4}}^{\frac4p}\\&\le C\Big[\|u\|_{\mathrm{BMO}}\|\rho u\|_p\Big]^{1-\frac4p}\Big[\|\nabla u\|_4\|\rho u\|_p\Big]^{\frac4p}\\&\le C\|\nabla u\|_2^{1-\frac4p}\|\nabla u\|_4^{\frac4p}\|\rho u\|_p\le C\Phi_T^{1+\frac\beta4}\Big(\|\nabla u\|_2^{1-\frac2p}+1\Big)\|\nabla u\|_2^{1-\frac4p}\|\nabla u\|_4^{\frac4p},\end{aligned}$$

while

$$\begin{aligned}&\|\nabla u\|_4\le C\big(\|\mathrm{div}u\|_4+\|\omega\|_4\big)\le C\big(\|\frac{F+(P(\rho)-P(\bar\rho))}{2\mu+\lambda(\rho)}\|_4+\|\rho\|_\infty^{\frac14}\|\omega\|_2^{\frac12}\|\sqrt\rho\dot u\|_2^{\frac12}\big)\\&\le C\big(\|\frac{F^2}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac12}+1+\|\rho\|_\infty^{\frac14}\|\omega\|_2^{\frac12}\|\sqrt\rho\dot u\|_2^{\frac12}\big)\\&\le C\big(\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac{1-\varepsilon}{2}}\|\rho\|_\infty^{\frac{1+\beta\varepsilon}{4}}\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac\varepsilon2}\|\sqrt\rho\dot u\|_2^{\frac12}+1+\|\rho\|_\infty^{\frac14}\|\omega\|_2^{\frac12}\|\sqrt\rho\dot u\|_2^{\frac12}\big)\\&\le C\big(\|\rho\|_\infty^{\frac{1+\beta\varepsilon}{4}}+1\big)\Big[\|\frac{F}{\sqrt{2\mu+\lambda(\rho)}}\|_2^{\frac12}\|\sqrt\rho\dot u\|_2^{\frac12}+\|\omega\|_2^{\frac12}\|\sqrt\rho\dot u\|_2^{\frac12}+1\Big]\\&\le C\big(\|\rho\|_\infty^{\frac{1+\beta\varepsilon+\beta}{4}}+1\big)(e+\|\nabla u\|_2)\Big(\frac{\varphi^2(t)}{e+Z^2(t)}\Big)^{\frac14},\end{aligned}$$

where in the fourth inequality one has used the fact (3.58). Then it holds that

$$\begin{aligned}&\|[u,R_iR_j](\rho u)\|_\infty\le C\big(\|\rho\|_\infty^{1+\frac\beta4+\frac{1+\beta\varepsilon+\beta}{p}}+1\big)(e+\|\nabla u\|_2^2)^{1-\frac1p}\Big(\frac{\varphi^2(t)}{e+Z^2(t)}\Big)^{\frac1p}\\&\le C\Big(\frac{\varphi^2(t)}{e+Z^2(t)}+1\Big)+C\big(\|\rho\|_\infty^{\left[1+\frac\beta4+\frac{1+\beta\varepsilon+\beta}{p}\right]\frac{p}{p-1}}+1\big)(e+\|\nabla u\|_2^2).\end{aligned}$$

Thus it holds that for any $\varepsilon>0$, one can choose sufficiently large $p>2$ such that

$$\int_0^T\|[u,R_iR_j](\rho u)\|_\infty(t)dt\le C\Phi_T^{1+\frac\beta4+\varepsilon}.\tag{3.66}$$

By Lemma 3.4, it holds that for suitably large but fixed $m > 1$,

$$\|\xi_1+\xi_2\|_{2m} \le Cm^{\frac{1}{2}}\Big[\|\rho-\bar\rho\|_{2m} + \|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_2^{\frac{2}{m\alpha}}\Big] \le C_m.$$

Then

$$\|\nabla(\xi_1+\xi_2)\|_2 \le C\|\rho u\|_2 \le C\|\rho\|_\infty^{\frac{1}{2}}\|\sqrt{\rho}u\|_2 \le C\|\rho\|_\infty^{\frac{1}{2}},$$

and then

$$\log^{\frac{1}{2}}(e+\|\nabla(\xi_1+\xi_2)\|_{2m}) \le C\,\log^{\frac{1}{2}}(e+\|\rho u\|_{2m}) \le C_m\,\log^{\frac{1}{2}}(e+\|\nabla u\|_2) \le C\Phi_T^{\frac{1+\beta\varepsilon}{2}}.$$

Therefore, it holds that

$$\|\xi_1+\xi_2\|_\infty \le C\big(\|\xi_1+\xi_2\|_{2m} + \|\nabla(\xi_1+\xi_2)\|_2\big)\log^{\frac{1}{2}}(e+\|\nabla(\xi_1+\xi_2)\|_{2m}) \le C\Phi_T^{1+\frac{\beta\varepsilon}{2}}. \quad (3.67)$$

Finally, substituting (3.66) and (3.67) into (3.65), it holds that

$$\Phi_T^\beta \le C\Phi_T^{1+\frac{\beta}{4}+\varepsilon} + C.$$

Therefore, if $\beta > \frac{4}{3}$ and choose ε suitably small, then

$$\sup_{t\in[0,T]}\|\rho\|_\infty(t) \le C_1, \quad (3.68)$$

for some positive constant C_1. Again by (3.65), (3.66), (3.67) and (3.68), it holds that

$$\sup_{t\in[0,T]}\|\ln\rho(t,\cdot)\|_\infty \le C,$$

which implies that there exists some positive constant c_1 such that

$$\rho(t,x) \ge c_1 > 0, \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^2.$$

Thus the proof of Lemma 3.8 is completed. □

4 Proof of main results

In this section, we give a sketch of proof of our main results.

Proof of Theorem 1.1:

Under the assumptions of the theorem, the local existence of the classical solution can be proved in a similar way as in [34, 42] and we omit it for simplicity. In view of the lower and upper bound of the density obtained in Section 3, the compressible Navier-Stokes equations (1.1) are a hyperbolic-parabolic coupled system. One can get the higher order a priori estimates. Using these a priori estimates, one can extend the local solution to the global one in a standard way(see [25, 26] for more details). The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2:

To use Theorem 1.1, we first construct the approximation of the initial data in (1.8) as follows. Since $\lim_{|x|\to+\infty}\rho_0(x) = \bar\rho > 0$, there exists a large number $M > 0$ such that if $|x| \ge M$,

$\rho_0(x) \geq \frac{\bar{\rho}}{2}$. Then for any $0 < \delta < \frac{\bar{\rho}}{2}$, we define

$$\rho_0^\delta(x) = \begin{cases} \rho_0(x) + \delta, \quad \text{if } |x| \leq M, \\ \rho_0(x) + \delta s(x), \quad \text{if } M \leq |x| \leq M+1, \\ \rho_0(x), \ \text{if } |x| \geq M+1, \end{cases} \tag{4.1}$$

where $s(x) = s(|x|)$ is a smooth and decreasing function satisfying $s(x) \equiv 1$ if $|x| \leq M$ and $s(x) = 0$ if $|x| \geq M + 1$. Similarly, one can construct the approximation of the initial pressure denoted by $P_0^\delta(x)$. Then it follows that $(\rho_0^\delta, P_0^\delta)(x)$ are regular functions satisfying $\rho_0^\delta(x) > \delta, P_0^\delta(x) > P(\delta)$ for any $x \in \mathbb{R}^2$ and $(\rho_0^\delta, P_0^\delta)(x) = (\rho_0, P_0)(x)$ if $|x| \geq M + 1$. Moreover, one has

$$(\rho_0^\delta - \bar{\rho}, P_0^\delta - P(\bar{\rho})) \to (\rho_0 - \bar{\rho}, P(\rho_0) - P(\bar{\rho})) \ \text{ in } W^{2,q}(\mathbb{R}^2) \times W^{2,q}(\mathbb{R}^2),$$

and

$$\Psi(\rho_0^\delta, \bar{\rho})(1 + |x|^\alpha) \to \Psi(\rho_0, \bar{\rho})(1 + |x|^\alpha) \ \text{ in } L^1(\mathbb{R}^2),$$

as $\delta \to 0$. To construct the approximation of the initial velocity, we define u_0^δ as

$$u_0^\delta = \begin{cases} \tilde{u}_0^\delta, & |x| \leq M+1, \\ u_0, & |x| \geq M+1, \end{cases} \tag{4.2}$$

where $\tilde{u}_0^\delta$ is the unique solution to the following elliptic problem

$$\begin{cases} \mathcal{L}_{\rho_0^\delta} \tilde{u}_0^\delta = \nabla P_0^\delta + \sqrt{\rho_0} g, & \text{in } \ \Omega_M := \{x| \ |x| < M+1\}, \\ \tilde{u}_0^\delta|_{|x|=M+1} = u_0. \end{cases} \tag{4.3}$$

From (4.3), one has

$$\mathcal{L}_{\rho_0} \tilde{u}_0^\delta = -\nabla\big[(\lambda(\rho_0^\delta) - \lambda(\rho_0))\text{div}\tilde{u}_0^\delta\big] + \nabla P_0^\delta + \sqrt{\rho_0} g, \ \text{ in } \Omega_M. \tag{4.4}$$

By the elliptic regularity, one has

$$\begin{aligned} &\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} \\ &\leq C\Big[\|\lambda(\rho_0^\delta) - \lambda(\rho_0)\|_\infty \|\nabla(\text{div}\tilde{u}_0^\delta)\|_2 + \|\nabla(\lambda(\rho_0^\delta) - \lambda(\rho_0))\|_\infty \|\text{div} u_0^\delta\|_2 + \|\nabla P_0^\delta\|_2 + \|\sqrt{\rho_0} g\|_2 + 1\Big] \\ &\leq C\Big[\delta\|\nabla^2 \tilde{u}_0^\delta\|_2 + \|\nabla P_0^\delta\|_2 + \|\sqrt{\rho_0}\|_{L^\infty(\mathbb{R}^2)}\|g\|_2 + 1\Big] \\ &\leq C\Big[\delta\|\nabla^2 \tilde{u}_0^\delta\|_2 + 1\Big]. \end{aligned} \tag{4.5}$$

where the generic positive constant C is independent of $\delta > 0$. Therefore, it follows from (4.5) that

$$\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} \leq C \tag{4.6}$$

where the positive constant C is independent of $0 < \delta \ll 1$.

From the compatibility conditions (1.9), (4.3) and (4.4), it holds that

$$\begin{cases} \mathcal{L}_{\rho_0}(\tilde{u}_0^\delta - u_0) = -\nabla\big[(\lambda(\rho_0^\delta) - \lambda(\rho_0))\text{div} u_0^\delta\big] + \nabla(P_0^\delta - P_0) := \Theta^\delta, \quad \text{in } \ \Omega_M, \\ (\tilde{u}_0^\delta - u_0)|_{|x|=M+1} = 0. \end{cases} \tag{4.7}$$

It follows from (4.1), (4.6) and (4.7) that

$$\tilde{u}_0^\delta - u_0 \in H_0^1(\Omega_M) \cap H^2(\Omega_M), \tag{4.8}$$

and

$$\begin{aligned}
&\|\tilde{u}_0^\delta - u_0\|_{H^2(\Omega_M)} \le C\|\Theta^\delta\|_2 \\
&\le C\Big[\|\lambda(\rho_0^\delta) - \lambda(\rho_0)\|_{L^\infty(\Omega_M)}\|\nabla^2\tilde{u}_0^\delta\|_2 + \|\nabla(\lambda(\rho_0^\delta) - \lambda(\rho_0))\|_{L^\infty(\Omega_M)}\|\mathrm{div}\tilde{u}_0^\delta\|_2 + \|\nabla(P_0^\delta - P_0)\|_2\Big] \\
&\le C\Big[\|\lambda(\rho_0^\delta) - \lambda(\rho_0)\|_{L^\infty(\Omega_M)} + \|\nabla(\lambda(\rho_0^\delta) - \lambda(\rho_0))\|_{L^\infty(\Omega_M)} + \|\nabla(P_0^\delta - P_0)\|_2\Big] \\
&\le C\delta \ \to 0,
\end{aligned} \tag{4.9}$$

as $\delta \to 0$. Then it follows from (4.2), (4.8) and (4.9) that $u_0^\delta \in H^2(\mathbb{R}^2)$ and

$$u_0^\delta \to u_0, \quad \text{in } H^2(\mathbb{R}^2),$$

and

$$\sqrt{\rho_0^\delta} u_0^\delta |x|^{\frac{\alpha}{2}} \to \sqrt{\rho_0} u_0 |x|^{\frac{\alpha}{2}}, \quad \text{in } L^2(\mathbb{R}^2),$$

as $\delta \to 0$. By Theorem 1.1, there exists a unique classical solution (ρ^δ, u^δ) to the compressible Navier-Stokes equations (1.1) with the initial data $(\rho_0^\delta, P_0^\delta, u_0^\delta)$ such that $c_\delta \le \rho^\delta \le C$ for some positive constants c_δ depending on δ and $C > 0$. It should be noted that the estimates obtained in Section 3 are independent of the lower bound of the initial density $\rho_0(x)$ except the lower bound of the density $\rho(t, x)$ in Lemma 3.8. Then we can pass the limit $\delta \to 0$ to get the classical solution satisfying (1.7). It is referred to [26] for more details and the proof of Theorem 1.2 is finished.

References

[1] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.*, **238**(1-2), 211-223 (2003).

[2] D. Bresch, B. Desjardins, Chi-Kun Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations*, **28**(3-4), 843-868 (2003).

[3] Q. L. Chen, C. X. Miao, Z. F. Zhang, Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity, Comm. Pure Appl. Math., **63(9)**, 1173-1224, 2010.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, *Compositio Mathematica*, **53**, 259-275 (1984).

[5] F. Catrina, Z. Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions, *Comm. Pure Appl. Math.*, **LIV**, 229-258 (2001).

[6] Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manuscript Math.*, **120**, 91-129 (2006).

[7] M. DelPino, J. Dolbeault, Bestconstants for GagliardoCNirenberg inequalities and applications to nonlinear diffusions, *J. Math. Pures Appl.*, **81**, 847-875 (2002).

[8] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141**, 579-614 (2000).

[9] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, *Comm. PDEs*, **22**, 977-1008 (1997).

[10] S. J. Ding, H. Y. Wen, C. J. Zhu, Global classical large solutions to 1D compressible NavierCStokes equations with density-dependent viscosity and vacuum, *J. Diff. Equa.*, **251**, 1696-1725, (2011).

[11] E. Feireisl, Dynamics of viscous compressible fluids, Oxford University Press, Oxford, 2004.

[12] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1998.

[13] Z. H. Guo, Q. S. Jiu, Z. P. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.*, **39**(5), 1402-1427 (2008).

[14] D. Hoff, Discontinuous solution of the Navier-Stokes equations for multi-dimensional heat-conducting fluids, *Arch. Rat. Mech. Anal.*, **193**, 303-354 (1997).

[15] D. Hoff, Compressible flow in a half-space with Navier boundary conditions, *J. Math. Fluid Mech.*, **7**(3), 315-338 (2005).

[16] D. Hoff, J. Smoller, Non-formation of vacuum states for compressible Navier-Stokes equations, *Comm. Math. Phys.*, **216**(2), 255-276 (2001).

[17] X. D. Huang, J. Li, Existence and blowup behavior of global strong solutions to the two-dimensional baratropic compressible Navier-Stokes system with vacuum and large initial data, preprint, arxiv:1205.5342.

[18] X. D. Huang, J. Li, Global well-posedness of classical solutions to the Cauchy problem of two-dimesional baratropic compressible Navier-Stokes system with vacuum and large initial data, preprint, arxiv:1207.3746.

[19] X. D. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Comm. Pure Appl. Math.*, **65** (4), 549-585, (2012).

[20] N. Itaya, On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluids, *Kodia Math. Sem. Rep.*, **23**, 60-120 (1971).

[21] S. Jiang, Z. P. Xin and P. Zhang, Global weak solutions to 1D compressible isentropy Navier-Stokes with density-dependent viscosity, *Methods and Applications of Analysis*, **12** (3), 239-252 (2005).

[22] S. Jiang, P. Zhang, Global spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.*, **215**, 559-581 (2001).

[23] Q. S. Jiu, Y. Wang, Z. P. Xin, Stability of rarefaction waves to the 1D compressible Navier-Stokes equations with density-dependent viscosity, *Comm. Part. Diff. Equ.*, **36**, 602-634 (2011).

[24] Q. S. Jiu, Y. Wang, Z. P. Xin, Vacuum behaviors around the rarefaction waves to the 1D compressible Navier-Stokes equations with density-dependent viscosity, *http://arxiv.org/abs/1109.0871.*

[25] Q. S. Jiu, Y. Wang, Z. P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, *http://arxiv.org/abs/1202.1382.*

[26] Q. S. Jiu, Y. Wang, Z. P. Xin, Global well-posedness of the Cauchy problem of 2D compressible Navier-Stokes equations in weighted spaces, *http://arxiv.org/abs/1207.5874.*

[27] Q. S. Jiu, Z. P. Xin, The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients, *Kinet. Relat. Models*, **1** (2), 313-330 (2008).

[28] J. I. Kanel, A model system of equations for the one-dimensional motion of a gas (in Russian), *Diff. Uravn.*, **4**, 721-734 (1968).

[29] A. V. Kazhikhov, V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* **41**, 273-282 (1977); translated from *Prikl. Mat. Meh.* **41** , 282-291 (1977).

[30] H. L. Li, J. Li, Z. P. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations, *Comm. Math. Phys.*, **281**(2), 401-444 (2008).

[31] P. L. Lions, Mathematical Topics in Fluid Dynamics, Vol. 2, Compressible Models, Oxford Science Publication, Oxford, 1998.

[32] T. P. Liu, J. Smoller, On the vacuum state for the isentropic gas dynamics equations, *Adv. in Appl. Math.*, **1**(4), 345-359 (1980).

[33] T. P. Liu, Z. P. Xin, T. Yang, Vacuum states of compressible flow, *Discrete Contin. Dyn. Syst.*, **4**, 1-32 (1998).

[34] Z. Luo, Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum, to appear at *Comm. Math. Sci.*, 2011.

[35] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, **20**, 67-104 (1980).

[36] A. Mellet and A. Vasseur, On the barotropic compressible Navier-Stokes equation, *Comm. Partial Differential Equations*, **32**(3), 431-452 (2007).

[37] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France*, **90**, 487-497 (1962).

[38] M. Perepelitsa, On the global existence of weak solutions for the Navier-Stokes equations of compressible fluid flows. *SIAM J. Math. Anal.*, **38** (4), 1126C1153 (2006).

[39] O. Rozanova, Blow up of smooth solutions to the compressible Navier-Stokes equations with the data highly decreasing at infinity, *J. Differ. Eqs.*, **245**, 1762-1774 (2008).

[40] A. Tani, On the first initial-boundary value problem of compressible viscous fluid motion, *Publ. Res. Inst. Math. Sci. Kytt Univ.* , **13**, 193-253 (1971).

[41] V. A. Vaigant, A. V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid. (Russian) *Sibirsk. Mat. Zh.*, 36 (1995), no. 6, 1283–1316, ii; translation in *Siberian Math. J.*, **36**, no. 6, 1108-1141 (1995).

[42] V. A. Solonnikov, On solvability of an initial-boundary value problem for the equations motion of a viscous compressible fluid, LOMI, 56 (1976), 128-142.

[43] Z. P. Xin, Blow-up of smooth solution to the compressible Navier-Stokes equations with compact density, *Comm. Pure Appl. Math.*, **51**, 229-240 (1998).

[44] Z. P. Xin, W. Yan, On blowup of classical solutions to the compressible Navier-Stokes equations, preprint, arxiv:1204.3169.

[45] T. Yang, Z. A. Yao, C. J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Differential Equations*, **26** (5-6), 965-981 (2001).

[46] T. Yang, C. J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, *Comm. Math. Phys.*, **230** (2), 329-363 (2002).

[47] T. Zhang, D. Y. Fang, Compressible flows with a density-dependent viscosity coefficient. *SIAM J. Math. Anal.*, **41**, no. 6, 2453-2488 (2009/10).