SUBSONIC IRROTATIONAL FLOWS IN A FINITELY LONG NOZZLE WITH VARIABLE END PRESSURE*

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Abstract. In this paper, we characterize a set of physically acceptable boundary conditions that ensure the existence and uniqueness of a subsonic irrotational flow in a finitely long flat nozzle. Our results show that if the incoming flow is horizontal at the inlet and an appropriate pressure is prescribed at the exit, then there exist two positive constants m_0 and m_1 with $m_0 < m_1$, such that a global smooth irrotational subsonic flow exists uniquely in the nozzle, provided that the incoming mass flux $m \in [m_0, m_1)$. The horizontal velocity of the flow is always positive in the whole nozzle and the maximum speed of the flow will approach the sonic speed as the mass flux m tends to m_1. The flow is governed by the steady compressible irrotational Euler equations with nonlocal and nonlinear mixed boundary conditions. A new key issue is that the Bernoulli's constant of the irrotational flow is not given a priori, which can be determined uniquely by the end pressure and the incoming mass flux. To handle the nonlocal boundary condition raised from the mass flux, we introduce an auxiliary problem with Bernoulli's constant as a parameter, instead of the mass flux. Furthermore, the monotonicity between the mass flux and the Bernoulli's constant is established for given end pressure.

2000 Mathematics Subject Classification: 35J25; 76G25; 35Q35; 76N10.

Key words: Subsonic flows, Finitely long nozzle, Euler equations, Existence and uniqueness, Elliptic system of first order.

1. Introduction

In this paper, we consider the inflow-outflow problem for steady subsonic irrotational gases in a two-dimensional finitely long cylindrical nozzle, aiming at finding intrinsic (physically acceptable) boundary conditions in upstream and downstream, which arises from the classical aerodynamics. In the Section 143 in the book [4], Courant and Friedrichs had presented some facts and conjectures about boundary value problems for steady compressible flows and indicated difficulties in formulating suitable boundary conditions.

* Du is supported in part by NNSF of China (No. 11171236) and Fundamental Research Funds for the Central Universities. Xin is supported in part by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK4041/11P, CUHK 4042/08P, and a grant from Croucher foundation.

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For compressible subsonic irrotational flows in infinitely long nozzles, proposed by L. Bers [2], one may expect that there exists a critical value of the incoming mass flux such that a global irrotational subsonic flow exists uniquely in a nozzle, provided that the incoming mass flux is less than the critical value. Some progress have been made for giving a rigorous mathematical proof to the assertion. Xie and Xin [23] proved that such a conjecture is valid for two-dimensional infinitely long curved nozzles. In [7], the authors showed that the result holds for arbitrary dimensional infinitely long nozzles. Concerning compressible subsonic flows with non-vanishing vorticity in 2D infinitely long nozzles, Xie and Xin [25] explored the special structure of 2D Euler system and reduced the Euler system to a single second order quasi-linear equation, which yields the existence of subsonic flows when the variations of Bernoulli's function in the upstream are sufficiently small and mass flux is in a suitable regime with an upper critical value. Furthermore, one may refer [6, 24] for similar results on the subsonic flows in 3D asymmetric infinitely long nozzles.

In these works mentioned above, the subsonic flows problem can be formulated mathematically into an infinitely long nozzle problem, under the assumption that the length of the nozzle is usually much larger than their cross-sections in practical applications. Compared with the works about subsonic flows in an infinitely long nozzle, the main difficulty of the subsonic problem in a finitely long nozzle is how to prescribe the parameters of the flow in the upstream and the downstream physically. It is of great importance to know that under what circumstances a steady smooth subsonic flow in a finitely long nozzle is uniquely determined by the conditions at entrance, and when further conditions at the exit are appropriate.

There have been several types works on steady flow patterns in finitely long nozzles in the literature. One is the transonic shock problem in de Lavel nozzles in [4, 3, 16, 17, 18, 26, 27]. As proposed by Courant and Friedrichs in Section 147 in [4], a natural and physical boundary condition at the exit for transonic shock problems is to prescribe the various end pressure, called receiver pressure. In that case the boundary conditions behind the shock wave are determined naturally by the Rankine-Hugoniot conditions. So the solution in subsonic region is governed by the compressible Euler equations with the impermeable boundary condition on nozzle walls, the Rankine-Hugoniot conditions on shock surface and the end pressure at the exit. The main difficulty in transonic shock problems is that the shock surface is a free boundary, which should be determined by the transonic shock solution simultaneously. Another important wave pattern is smooth transonic flows through a finitely long nozzle. In view of the works [20, 21, 22] on irrotational smooth transonic flows in de Lavel nozzles, the angle and the mass flux of the incoming flows are imposed at the entrance and the sonic condition is prescribed naturally on the sonic line. Thus, the irrotational flows in subsonic regions are governed by the potential equation with given flow angles at the inlet, the Neumann boundary condition on the nozzle walls and the sonic condition on the possible sonic line. On the other hand, it should be mentioned again that as proposed by Bers in [2], the mass flux is an important physical quantity not only in infinitely long nozzles but also in finitely long nozzles, although it induces a nonlocal boundary condition which may cause some difficulties mathematically.

Motivated by these studies on boundary conditions for steady flow patterns in finite nozzles, in this paper, we will characterize a class of physical boundary conditions and establish the well-posedness of the subsonic flows through a finitely long nozzle under these boundary conditions.

Roughly speaking, we consider a finitely long flat nozzle given by $\Omega = [0,1]\times[0,1]$. Assume that the incoming flow is horizontal at the entrance and an appropriate pressure $p_e(x)$ is prescribed at the exit. We will look for a unique smooth subsonic flow through the nozzle with the given incoming mass flux m which lies in an appropriate interval $[m_0, m_1)$. Here m_1 is expected to be the critical mass flux in the sense that the maximal speed of the subsonic flow in the nozzle approaches to the sound speed as m tends to m_1.

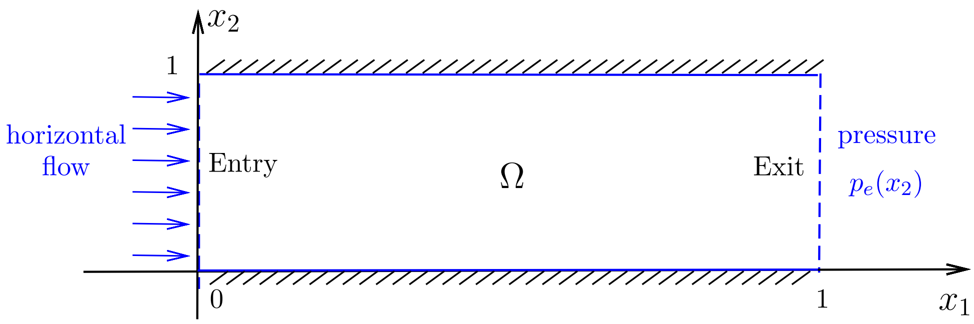

FIGURE 1. Subsonic problem in a finitely long nozzle

More precisely, we can formulate the boundary value problem as follows: Steady isentropic ideal gases are governed by the compressible Euler equations

$$\begin{cases} (\rho u_1)_{x_1} + (\rho u_2)_{x_2} = 0, \\ (\rho u_1^2)_{x_1} + (\rho u_1 u_2)_{x_2} + p_{x_1} = 0, \\ (\rho u_1 u_2)_{x_1} + (\rho u_2^2)_{x_2} + p_{x_2} = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the velocity field, p is the pressure and ρ is the density. For ideal polytropic gases, the equation of state is

$$p(\rho) = A\rho^\gamma, \quad \text{for} \quad \rho > 0, \tag{1.2}$$

with A being a positive constant depending on the specific entropy and $\gamma \in (1,3)$ being the adiabatic exponent. For simplicity, we take $A = \frac{1}{\gamma}$ throughout this paper.

The quantity $c(\rho) = \sqrt{p'(\rho)} = \sqrt{\rho^{\gamma-1}}$ is the sound speed, and the subsonic flow means that $|u| < c(\rho)$.

Moreover, the flow is assumed to be irrotational, that is $\partial_1 u_2 - \partial_2 u_1 \equiv 0$. Then one can obtain the strong form of the Bernoulli's Law(see Section 14 in [4])

$$\frac{1}{2}(u_1^2 + u_2^2) + h(\rho) = \text{constant}, \tag{1.3}$$

where $h(\rho)$ is the enthalpy with $h'(\rho) = \frac{p'(\rho)}{\rho}$. And it is easy to prove that if the Bernoulli's function is a uniform constant and the flow does not contain any stagnation points and vacuum, then the flow will be irrotational. Since we are looking for a subsonic irrotational flow without stagnation points and vacuum, throughout this paper, we will use the strong form of the Bernoulli' law instead of the zero vorticity condition.

At the inlet of the nozzle $\Omega = (0,1) \times (0,1)$, the incoming flow is assumed to be horizontal, i.e.

$$u_2(0,x_2) = 0 \quad \text{and} \quad u_1(0,x_2) > 0 \text{ for } \ x_2 \in [0,1]. \tag{1.4}$$

The positivity of the horizontal velocity guarantees that the flow goes into the nozzle at the entrance.

The nozzle walls are assumed to be solid so that

$$u_2 = 0 \qquad \text{for } \ x_2 = 0 \ \text{ or } \ x_2 = 1. \tag{1.5}$$

At the exit of the nozzle, the receiver pressure is imposed

$$p(\rho) = p_e(x_2) \quad \text{on} \quad x_1 = 1, \tag{1.6}$$

with compatibility conditions $p_e'(0) = p_e'(1) = 0$.

It follows from the continuity equation in (1.1) and the slip boundary condition (1.5) that

$$m(x_1) = \int_0^1 \rho(x_1,x_2)u_1(x_1,x_2)dx_2 = m \tag{1.7}$$

is a constant for any $x_1 \in [0,1]$, which is called the mass flux.

The subsonic flow problem in this finite flat nozzle is stated as follows.

Problem 1. (Subsonic problem in finitely long nozzles)
For a given appropriate receiver pressure $p_e(x_2)$ at the exit and mass flux $m > 0$, find a unique smooth subsonic solution (ρ, u_1, u_2), which satisfies the Euler equations (1.1)*, the equation of state* (1.2)*, the Bernoulli's law* (1.3)*, the boundary conditions* (1.4)-(1.6)*, the mass flux condition* (1.7)*, and the subsonic condition $u_1^2 + u_2^2 < c^2(\rho) = \rho^{\gamma-1}$.*

The main results in this paper can be stated as follows. Suppose that the incoming flow is horizontal at the entrance, for given end pressure function $p_e(x_2) \in C^{2,\alpha}([0,1])$, whose variation is not too large (see condition (2.7)), there exists a unique irrotational subsonic flow throughout the finitely long flat nozzle, provided that the incoming mass flux m lies in an appropriate interval $[m_0, m_1)$. The maximum speed of the flow will approach the sonic speed as the mass flux m tends to m_1. This interval $[m_0, m_1)$ depends on the receiver pressure, the equation of state.

Before getting into the details, we would like to give some comments on the key issues in this paper.

Remark 1.1. In view of the recent works ([7, 23, 24]) on irrotational subsonic flows through infinitely long nozzles, the nozzles are assumed to be asymptotically flat at the far field and the asymptotic behavior of the flows at the entrance and the exit can be derived by the mass flux and the shape of the nozzles. In another words, the results on subsonic irrotational flows in infinitely long nozzles indicate that the flows are determined uniquely by the mass flux and the geometry of the infinitely long nozzles. However, in finitely long nozzles except for the mass flux condition, some additional physically acceptable conditions should be imposed not only at the inlet but also at the exit. These additional boundary conditions make it difficult to adapt the analysis for the infinitely long nozzle in [7, 23, 24] to the current case. Some new approaches are needed to obtain some key properties such as positivity of the horizontal velocity and uniqueness of the flow, which play key roles in establishing the well-posedness of Problem 1.

Remark 1.2. In the previous works on the subsonic irrotational flows in unbounded domain (including the subsonic flows past a given obstacle [1, 5, 10, 11, 12] and the subsonic flows through an infinitely long nozzle [7, 23, 24]), the Bernoulli's constant can be determined by the asymptotic behavior at far field, which is always normalized to be 1. In present situation, the Bernoulli's constant should not be determined a priori, in fact, it should be determined by the mass flux and the receiver pressure at the exit. However, the mass flux condition (1.7) is a nonlocal boundary condition, which is difficult to use. To overcome this difficulty, we first introduce an auxiliary problem regarding the Bernoulli's constant as a parameter as follows. If the upstream flow remains horizontal at the inlet and a proper end pressure $p_e(x_2)$ is imposed at the exit, the global smooth subsonic flow exists uniquely in the nozzle, provided that the Bernoulli's constant B lies in some appropriate interval $[B_0, B_1)$. Fortunately, a monotonic relationship between the mass flux and the Bernoulli's constant is established and Problem 1 can be solved.

Remark 1.3. In the works [7, 23, 24, 27] on irrotational flows, the potential function formulation or the stream function formulation was introduced to analyze the irrotational flows problem. However, the arguments can not be employed directly to deal with the existence of the Problem 1 in this paper, due to the possible absence of the obliqueness of the boundary condition at the exit. In other words, one has to find a mechanism to guarantee that the flow possesses the positive horizontal velocity in the whole nozzle. The key ingredient in this paper is based on the reformulation of the Euler equations in terms of density and angular velocity, which has been introduced in [16] to deal with the transonic shock problem. (See also [3] for transonic shock wave in a tube). Unlike the case in the study of the transonic shock problem in a divergent nozzle [16], the elliptic system of first order to the density and the angular velocity here always satisfies the compatibility conditions, so it is possible to apply the standard Leray-Schauder fixed point argument to obtain the existence of the subsonic solutions. However, the positivity of the horizontal velocity is crucial to this argument. For the transonic shock problem in [3, 16, 26, 27], the transonic shock solution is expected to be adjacent to some background solution, which possesses the positive horizontal velocity. It implies the absence of the vanishing horizontal velocity of the flows. However, for subsonic flows established in this paper, there is no background solution to start with. One of the main difficulties here is how to guarantee the positivity of the horizontal velocity in the whole nozzle. In this paper, we apply a standard Moser iteration to obtain the L^∞-estimate of the angular velocity, which gives the positivity of the horizontal velocity.

Remark 1.4. For steady compressible ideal flows with non-vanishing vorticity, the main difficulty is that the 2D steady Euler equations is a hyperbolic-elliptic coupled system for subsonic flows. In this paper, the analysis depends crucially on the standard theory of the elliptic equations, which can not be used to deal with the mixed-type equations directly. In [25], a stream function formulation is introduced to solve the mixed-type equations effectively in infinitely long nozzles. However, we have not been able to adapt this approach for finitely long nozzles, due to the mixed boundary conditions being imposed, especially at the exit.

The rest of this paper is organized as follows. In Section 2, we reduce the original problem to an auxiliary problem with a parameter B. Main results for the auxiliary problem and Problem 1 are stated at the end of Section 2. In Section 3, we give the rigorous proof to the existence and the uniqueness for the auxiliary problem. A reformulation of the 2D isentropic Euler equations involving the density and the angular velocity is given. The existence of the auxiliary problem is based on the Leray-Schauder fixed point arguments. One of the key points is to establish the a priori estimate of the flow angle, which guarantees the positivity of the horizontal velocity. On the other hand, the uniqueness of the subsonic irrotational flow is obtained by a maximum principle argument to the velocity potential function. In Section 4, for given end pressure at the exit, the monotonic property between the mass flux and the Bernoulli's constant is established. Hence, it follows from the monotonic property that we can obtain the proof of the main results to the Problem 1.

2. Reduction of the problem and main results

In this section, we will introduce an auxiliary problem to simplify the Problem 1, and state the main results in this paper.

2.1. **An auxiliary problem.**

One of the main difficulties to solve the Problem 1 is how to localize the nonlocal condition (1.7) on the mass flux. Our strategy is to introduce the following auxiliary problem involving the Bernoulli's constant as a parameter and try to find a monotonic property between the mass flux and the Bernoulli's constant.

Problem 2. (Auxiliary subsonic problem in finitely long nozzles.) *For given an appropriate receiver pressure $p_e(x_2)$ at the exit and the Bernoulli's constant $B > 0$, find a unique smooth subsonic solution (ρ, u_1, u_2), which satisfies the Euler equations* (1.1)*, the equation of state* (1.2)*, the Bernoulli's law*

$$\frac{u_1^2 + u_2^2}{2} + \frac{1}{\gamma - 1}\rho^{\gamma-1} = B, \tag{2.1}$$

the boundary conditions (1.4)-(1.6) *and the subsonic condition $u_1^2 + u_2^2 < c^2(\rho)$.*

2.2. **Restrictions on the end pressure.**

Before stating the main results in this paper, we would like to mention that the receiver pressure should satisfy some necessary conditions to guarantee the subsonicity of the flows at the exit.

For the given end pressure $p_e(x_2)$ at the exit, the end density and the speed at the exit are defined to be

$$\rho_e(x_2) = (\gamma p_e(x_2))^{1/\gamma} \quad \text{and} \quad q_e(x_2) = \sqrt{2\Big(B - \frac{1}{\gamma - 1}\rho_e^{\gamma-1}(x_2)\Big)},$$

respectively.

Denote the maximal and minimal pressure, density, speed at the exit, respectively by

$$\bar{p} = \max_{x_2\in[0,1]} p_e(x_2), \quad \underline{p} = \min_{x_2\in[0,1]} p_e(x_2), \tag{2.2}$$

$$\bar{\rho} = \max_{x_2\in[0,1]} \rho_e(x_2) = (\gamma\bar{p})^{1/\gamma}, \quad \underline{\rho} = \min_{x_2\in[0,1]} \rho_e(x_2) = (\gamma\underline{p})^{1/\gamma}, \tag{2.3}$$

and

$$\bar{q} = \max_{x_2\in[0,1]} q_e(x_2), \quad \underline{q} = \min_{x_2\in[0,1]} q_e(x_2). \tag{2.4}$$

Thus, it follows from the strong form of the Bernoulli's law in (1.3) that

$$B = \frac{1}{2}\bar{q}^2 + \frac{1}{\gamma-1}\underline{\rho}^{\gamma-1} = \frac{1}{2}\underline{q}^2 + \frac{1}{\gamma-1}\bar{\rho}^{\gamma-1}. \tag{2.5}$$

Since we are looking for a subsonic flow without any stagnation points in the whole nozzle, the following conditions are required,

$$\underline{q} > 0, \quad \bar{q}^2 < c^2(\underline{\rho}) = \underline{\rho}^{\gamma-1}.$$

Hence, combining this with (2.5) yields

$$\frac{1}{\gamma-1}\bar{\rho}^{\gamma-1} < B < \frac{1}{2}\underline{\rho}^{\gamma-1} + \frac{1}{\gamma-1}\underline{\rho}^{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}\underline{\rho}^{\gamma-1},$$

which implies that the end density should be restricted with the following condition

$$\bar{\rho}/\underline{\rho} < \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}. \tag{2.6}$$

Hence one can conclude that if the irrotational flow remains subsonic at the exit, the ratio of the maximum to the minimum of the end pressure remains below a certain critical value, i.e.

$$\bar{p}/\underline{p} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}, \quad \underline{p} > 0, \tag{2.7}$$

which means the oscillation of the receiver pressure is not too large.

Remark 2.1. The condition (2.7) above is necessary for the existence of the smooth subsonic irrotational flows. In other words, if the condition (2.7) is not valid then there does not exist a smooth subsonic irrotational flow, even for the curved nozzles. We also note that the condition (2.7) involves no restriction on the magnitude of the receiver pressure $p_e(x_2)$ other than its oscillation.

2.3. Main results.

Now, we state the main results in this paper as follows.

Theorem 2.1. *(Unique solvability of Problem 2)*
Given two constants $\bar{p}$ and $\underline{p}$ satisfying (2.7), define

$$B_0 = \frac{1}{\gamma-1}(\gamma\bar{p})^{\frac{\gamma-1}{\gamma}} + \delta \quad \textit{and} \quad B_1 = \frac{\gamma+1}{2(\gamma-1)}(\gamma\underline{p})^{\frac{\gamma-1}{\gamma}}, \tag{2.8}$$

where $0 < \delta < \dfrac{1}{(\gamma-1)}\left(\dfrac{\gamma+1}{2}(\gamma\underline{p})^{\frac{\gamma-1}{\gamma}} - (\gamma\bar{p})^{\frac{\gamma-1}{\gamma}}\right)$. *Fix a constant* $\Theta_0 \in \left(1, \dfrac{\pi}{2}\right)$, *suppose that the Bernoulli's constant* $B \in [B_0, B_1)$ *and the end pressure at the exit* $p_e(x_2) \in C^{2,\alpha}([0,1])$ *satisfies*

$$p_e'(0) = p_e'(1) = 0, \quad \bar{p} = \max_{x_2\in[0,1]} p_e(x_2), \quad \underline{p} = \min_{x_2\in[0,1]} p_e(x_2), \tag{2.9}$$

and

$$C(\underline{p}, \bar{p}, \Omega)\delta^{-5}\|p_e'\|_{L^\infty}^4 \|p_e'\|_{L^2} \leq \left(1 - \frac{1}{\Theta_0}\right)^4 \Theta_0, \tag{2.10}$$

where $C(\underline{p}, \bar{p}, \Omega)$ *is a positive constant depending on* $\underline{p}$, $\bar{p}$ *and* Ω, *then the auxiliary problem has a unique smooth subsonic irrotational solution* $(\rho, u_1, u_2) \in \left(C^{2,\alpha}(\bar{\Omega})\right)^3$, *which satisfies the following properties,*

(1). the horizontal velocity of the flow is positive in the whole nozzle, namely $u_1 > 0$ *in* $\bar{\Omega}$;

(2). the flow angle is bounded by Θ_0, *ie.* $\sup_{x\in\bar{\Omega}}\left|\arctan\dfrac{u_2}{u_1}\right| \leq \Theta_0$;

(3). the minimal speed of the flow in the nozzle is greater than $\sqrt{\delta} > 0$, *which achieves at the exit;*

(4). the maximal speed of the flow will approach the sonic speed as the Bernoulli's constant B *tends to* B_1.

Moreover, for given receiver pressure $p_e(x_2)$ with (2.7), one can show that the incoming mass flux m is a continuous and monotone increasing function of B, provided that the horizontal velocity is always positive in the nozzle, which yields the following theorem for the original problem 1.

Theorem 2.2. *(Unique solvability of Problem 1)*
Given two constants $\underline{p}$ *and* $\bar{p}$ *satisfying (2.7), one can choose* δ *as in Theorem 2.1. Fix a constant* $\Theta_0 \in \left(1, \dfrac{\pi}{2}\right)$, *suppose that the end pressure at the exit* $p_e(x_2) \in C^{2,\alpha}([0,1])$ *satisfies the conditions* (2.9) *and* (2.10), *then there exist two constants* $m_1 > m_0 > 0$ *depending on* $p_e(x_2)$, δ, γ *and* Ω, *such that if the incoming mass flux* $m \in [m_0, m_1)$, *then there exists a unique smooth subsonic irrotational flow* $(\rho, u_1, u_2) \in \left(C^{2,\alpha}(\bar{\Omega})\right)^3$ *to the subsonic problem 1, which satisfies the following properties,*

(1). the horizontal velocity of the flow is positive in the whole nozzle, namely $u_1 > 0$ *in* $\bar{\Omega}$;

(2). the flow angle is bounded by Θ_0, *ie.* $\sup_{x\in\bar{\Omega}}\left|\arctan\dfrac{u_2}{u_1}\right| \leq \Theta_0$;

(3). the minimal speed of the flow in the nozzle is greater than $\sqrt{\delta} > 0$, *which achieves at the exit;*

(4). the maximal speed of the flow will approach the sonic speed as the mass flux m *tends to* m_1.

Remark 2.2. In fact, since the incoming flow is horizontal at the entrance, it follows from the standard maximum principle and Hopf's lemma that both of the maximum and minimum of the density can be only achieved at the exit of the nozzle. Moreover, by Bernoulli's law the speed of the fluid achieves its maximum and minimum only at the exit. This property guarantees immediately that the flow established here is subsonic throughout the whole nozzle, provided that the end pressure satisfies (2.7) at the exit. It suffices to restrict the Bernoulli's constant $B = \frac{1}{2}\bar{q}^2 + \frac{1}{\gamma-1}\underline{\rho}^{\gamma-1}$ is less than B_1, automatically, the flow remains subsonic globally in the nozzle, which avoids the operation of subsonic truncation. Meanwhile, the square of the minimal speed of the flows is larger than the positive parameter δ in the Theorems, which excludes the stagnation points in the whole nozzle, provided that the Bernoulli's constant or mass flux lies in some proper intervals. Furthermore, for given the end pressure at the exit, the maximum and minimum of flow speed in the whole nozzle are monotone increasing with respect to the incoming mass flux.

Remark 2.3. To guarantee the positivity of the horizontal velocity of the fluid in the whole nozzle, a sufficient condition (2.10) is imposed to the receiver pressure, which implies the flow angle in the nozzle remains bounded by some fixed constant Θ_0.

Remark 2.4. The results are still valid for the general two-dimensional finitely long nozzles with variable cross sections, provided that the nozzle walls are perpendicular to the entrance or exit. For simplicity, we only consider the cylinder case in this paper.

Remark 2.5. We remark that it is reasonable to give an upper and lower bound on the mass flux in order to obtain a global smooth subsonic flow. Actually, if the mass flux is too large, then a transonic bubble may emerge in the nozzle [14], and a global subsonicity is not available anymore. If the mass flux is too small, one can not expect that the horizontal velocity remains to be positive in the whole nozzle, the flow may turn around and stagnation point may appear in the flow region. However, a rigorous mathematical proof for such interesting phenomena seems to be a big challenge.

3. Unique solvability of the auxiliary problem

In this section, we will establish the existence and the uniqueness of the subsonic flow to the auxiliary problem.

3.1. **A reformulation of the auxiliary problem.** It is a usual practice to describe irrotational flows in terms of either velocity potential functions or stream functions. However, they are not convenient to deal with the auxiliary problem in this paper due to the mixed boundary conditions.

In fact, if one introduces a velocity potential function φ such that $(u_1, u_2) = (\partial_1\varphi, \partial_2\varphi)$, then the auxiliary problem can be rewritten as follows,

$$\begin{cases} \operatorname{div}\Big(\rho\left(|D\varphi|^2; B\right) D\varphi\Big) = 0, & \text{in} \quad \Omega, \\ \varphi = 0, \quad \partial_1\varphi > 0, & \text{on} \quad x_1 = 0, \\ \partial_2\varphi = 0, & \text{on} \quad x_2 = 0 \cup x_2 = 1, \\ (\partial_1\varphi)^2 + (\partial_2\varphi)^2 = q_e^2(x_2; B), & \text{on} \quad x_1 = 1, \\ \sup\limits_{x\in\bar{\Omega}} |D\varphi(x)|^2 < c^2(\rho), \end{cases} \tag{3.1}$$

where the density ρ can be regarded as a function of speed by strong form of Bernoulli's Laws with parameter $B > 0$, and $q_e^2(x_2; B) = 2\bigg(B - \dfrac{1}{\gamma-1}\rho_e^{\gamma-1}(x_2)\bigg)$. Then the main difficulty is the loss of obliqueness of the boundary condition at the exit. To avoid it, one way to proceed may be to introduce a truncated boundary condition

$$T(\partial_1\varphi)\partial_1\varphi + (\partial_2\varphi)^2 = q_e^2(x_2; B), \tag{3.2}$$

where $T(x)$ is a smooth truncated function. Then solve the problem (3.1) with truncated boundary condition (3.2) instead of the original boundary condition at the exit. This can be done easily. However, to recover the original boundary condition on $x_1 = 1$, one has to require that $\|p_e'\|_{C^{1,\alpha}([0,1])}$ to be sufficiently small, which implies the variation of the end pressure is sufficiently small. As a consequence, the flow established has to be adjacent to some constant state. Similar difficulties arise for the stream function formulation.

In this subsection, we will use a different reformulation to avoid the difficulty caused by the loss of obliqueness and give a sufficient condition (2.10) instead of the smallness of $\|p_e'\|_{C^{1,\alpha}([0,1])}$ to guarantee the positivity of the horizontal velocity. Motivated by the discussion in [16] for transonic shock problem, the Euler system (1.1) can be reformulated into a first order elliptic system in terms of the angular velocity $w = \dfrac{u_2}{u_1}$ and the density, provided that the horizontal velocity $u_1 > 0$ in the nozzle and the flow is subsonic in the whole nozzle.

Multiplying the first equation in (1.1) by $\dfrac{-u_{i-1}}{\rho u_1^2}$, dividing the i-th equation in (1.1) by ρu_1^2 and adding them together for $i = 2, 3$ yield

$$\begin{cases} \partial_1 w - \dfrac{w}{\rho}\partial_1\rho + \dfrac{1}{\rho}\left(\dfrac{c^2(\rho)}{u_1^2} - w^2\right)\partial_2\rho = 0, \\ \dfrac{1}{\rho}\left(\dfrac{c^2(\rho)}{u_1^2} - 1\right)\partial_1\rho - \partial_2 w - \dfrac{w}{\rho}\partial_2\rho = 0. \end{cases} \tag{3.3}$$

It follows from the Bernoulli's law (2.1) that

$$u_1^2 = \frac{2(B - h(\rho))}{1 + w^2}. \tag{3.4}$$

Substituting this into (3.3) yields

$$
\begin{cases}
\partial_1 w - \dfrac{w}{\rho}\partial_1\rho + \dfrac{1}{\rho}\left(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - w^2\right)\partial_2\rho = 0,\\
\dfrac{1}{\rho}\left(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - 1\right)\partial_1\rho - \partial_2 w - \dfrac{w}{\rho}\partial_2\rho = 0,\\
w(0,x_2)=0, \quad \text{and} \quad u_1(0,x_2)>0, & \text{for } \ x_2\in[0,1],\\
w(x_1,0)=w(x_1,1)=0, & \text{for } \ x_1\in[0,1],\\
\rho(1,x_2)=\rho_e(x_2), & \text{for } \ x_2\in[0,1].
\end{cases}
\tag{3.5}
$$

We will apply the following Leray-Schauder theorem to prove the existence theorem for the problem (3.5).

Theorem 3.1. *(Theorem 11.3 in* [13]*) Let T be a compact mapping of a Banach space $\mathbb{B}$ into itself, and suppose there exists a constant M such that*

$$\|x\|_{\mathbb{B}} \le M.$$

for all $x\in\mathbb{B}$ and $\sigma\in[0,1]$ satisfying $x=\sigma Tx$. Then T has a fixed point.

Remark 3.1. The above theorem holds also for a compact mapping from a closed convex set to itself by checking the proof in [13] carefully.

3.2. **Existence of the subsonic flows.**

Fix a constant $B\in[B_0,B_1)$ and define a mapping T of the set

$$
\begin{aligned}
\Sigma = \Bigg\{(\rho,w)\in\Big(C^{1,\alpha}(\bar\Omega)\Big)^2 : \underline{\rho}\le\rho\le\bar\rho,\ w(0,x_2)=w(x_1,0)=w(x_1,1)=0,\\
\partial_2\rho(x_1,0)=\partial_2\rho(x_1,1)=0\Bigg\},
\end{aligned}
\tag{3.6}
$$

by letting $(\rho,w)=T(\tilde\rho,\tilde w)$ be the unique solution of the linear problem

$$
\begin{cases}
\partial_1 w - \dfrac{\tilde w}{\tilde\rho}\partial_1\rho + \dfrac{1}{\tilde\rho}\left(\dfrac{c^2(\tilde\rho)(1+\tilde w^2)}{2(B-h(\tilde\rho))} - \tilde w^2\right)\partial_2\rho = 0,\\
\dfrac{1}{\tilde\rho}\left(\dfrac{c^2(\tilde\rho)(1+\tilde w^2)}{2(B-h(\tilde\rho))} - 1\right)\partial_1\rho - \partial_2 w - \dfrac{\tilde w}{\tilde\rho}\partial_2\rho = 0,\\
w=0, & \text{on } \ x_1=0,\\
w=0, & \text{on } \ x_2=0 \ \text{ or } \ x_2=1,\\
\rho=\rho_e(x_2), & \text{on } \ x_1=1,
\end{cases}
\tag{3.7}
$$

for $(\tilde\rho,\tilde w)\in\Sigma$.

Next, we first show that T maps the convex set Σ into itself.

Lemma 3.2. *For any $(\tilde{\rho}, \tilde{w}) \in \Sigma$, there exists a unique solution $(\rho, w) \in C^{2,\alpha}(\bar{\Omega})$ to (3.7).*

Proof. It is easy to derive that ρ satisfies the following second order elliptic equation,

$$\begin{cases} \partial_1\Big(\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-1\right)\partial_1\rho-\frac{\tilde{w}}{\tilde{\rho}}\partial_2\rho\Big)+\partial_2\Big(-\frac{\tilde{w}}{\tilde{\rho}}\partial_1\rho+\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-\tilde{w}^2\right)\partial_2\rho\Big)=0, \\ -\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-1\right)\partial_1\rho+\frac{\tilde{w}}{\tilde{\rho}}\partial_2\rho=0, \quad \text{on} \quad x_1=0, \\ -\frac{\tilde{w}}{\tilde{\rho}}\partial_1\rho+\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-\tilde{w}^2\right)\partial_2\rho=0, \quad \text{on} \quad x_2=0 \ \text{ or } \ x_2=1, \\ \rho=\rho_e(x_2), \quad \text{on} \quad x_1=1. \end{cases} \tag{3.8}$$

Consider the weak formulation for (3.8): To find a solution $\rho=\rho_e+W$, such that

$$\int_\Omega\Big\{\Big(\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-1\right)\partial_1\rho-\frac{\tilde{w}}{\tilde{\rho}}\partial_2\rho\Big)\partial_1\eta+\Big(-\frac{\tilde{w}}{\tilde{\rho}}\partial_1\rho+\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))}-\tilde{w}^2\right)\partial_2\rho\Big)\partial_2\eta\Big\}dx=0, \tag{3.9}$$

for $\eta \in W=\{\eta\in H^1(\Omega), \eta|_{x_1=1}=0\}$.

It follows from Lax-Milgram theorem that there exists a unique weak solution ρ to (3.9). By the standard elliptic estimates, we have $\rho\in C^{2,\alpha}(\bar{\Omega}_1)$ for any smooth domain $\bar{\Omega}_1\subset\bar{\Omega}\setminus S$ and $S=\{(0,0),(0,1),(1,0),(1,1)\}$. The boundary conditions are satisfied in the classical sense. We apply the reflection method to deal with the singularity near the corners. Rewrite the elliptic equation in (3.8) into the non-divergent form

$$\sum_{i,j=1}^{2}a_{ij}\partial_{ij}\rho+\sum_{i=1}^{2}b_i\partial_i\rho=0.$$

Since $(\tilde{\rho},\tilde{w})\in\Sigma$, it is easy to check that $a_{12}(x_1,0)=a_{12}(x_1,1)=0$ and $b_2(x_1,0)=b_2(x_1,1)=0$.

For $-1<x_2<0$, we can extend ρ as $\breve{\rho}(x_1,x_2)=\rho(x_1,-x_2)$. And then extend $\breve{\rho}$ periodically to $-\infty<x_2<+\infty$ with period 2, i.e.

$$\breve{\rho}(x_1,x_2)=\breve{\rho}(x_1,x_2+2n), \quad \text{if}-1\le x_2+2n\le 1.$$

Similarly, the coefficients can be extended as

$$\breve{a}_{ii}(x_1,x_2)=a_{ii}(x_1,-x_2), \quad \breve{a}_{12}(x_1,x_2)=-a_{12}(x_1,-x_2),$$

for $i=1,2$, and

$$\breve{b}_1(x_1,x_2)=b_1(x_1,-x_2), \quad \breve{b}_2(x_1,x_2)=-b_2(x_1,-x_2),$$

for $-1<x_2<0$ and then extend them periodically to $-\infty<x_2<+\infty$ with period 2.

It follows from $\rho_e'(0)=\rho_e'(1)=0$ that the end density function $\rho_e(x_2)$ can be extended to $\breve{\rho}_e(x_2)$ in a similar way. Hence, it is easy to check that the extended function $\breve{\rho}$ solves the following problem in $\Omega_e = \{(x_1, x_2) : 0 \le x_1 \le 1, x_2 \in \mathbb{R}\}$,

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{2} \breve{a}_{ij}\partial_{ij}\breve{\rho} + \sum_{i=1}^{2} \breve{b}_i\partial_i\breve{\rho} = 0, & \text{in } \ \Omega_e, \\
\partial_1\breve{\rho} = 0, & \text{on } \ x_1 = 0, \\
\breve{\rho} = \breve{\rho}_e(x_2), & \text{on } \ x_1 = 1.
\end{cases}
\tag{3.10}
$$

Then it follows from the standard elliptic estimates that $\breve{\rho} \in C^{2,\alpha}(\bar{\Omega}_e)$, which implies ρ solves the problem (3.8) uniquely in $C^{2,\alpha}(\bar{\Omega})$. Moreover, by maximum principle and Hopf's lemma, ρ only attains its maximum and minimum at the exit of the nozzle, hence one has $\underline{\rho} \le \rho \le \bar{\rho}$.

Once ρ is obtained, one can consider the following problem for w

$$
\begin{cases}
\partial_1 w = \dfrac{\tilde{w}}{\tilde{\rho}}\partial_1\rho - \dfrac{1}{\tilde{\rho}}\left(\dfrac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))} - \tilde{w}^2\right)\partial_2\rho, \\
\partial_2 w = \dfrac{1}{\tilde{\rho}}\left(\dfrac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))} - 1\right)\partial_1\rho - \dfrac{\tilde{w}}{\tilde{\rho}}\partial_2\rho, \\
w = 0, & \text{on } \ x_2 = 0 \ \text{ or } \ x_2 = 1, \\
w = 0, & \text{on } \ x_1 = 0.
\end{cases}
\tag{3.11}
$$

For the solvability of the problem (3.11), it suffices to check the compatibility conditions near the corners at the entrance.

In fact, at the entrance $x_1 = 0$, it holds that

$$
\frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))} - 1\right)\partial_1\rho - \frac{\tilde{w}}{\tilde{\rho}}\partial_2\rho = 0,
$$

which implies $\partial_2 w = 0$ on $x_1 = 0$. On the other hand, on the nozzle walls $x_2 = 0, 1$, one has

$$
\frac{\tilde{w}}{\tilde{\rho}}\partial_1\rho - \frac{1}{\tilde{\rho}}\left(\frac{c^2(\tilde{\rho})(1+\tilde{w}^2)}{2(B-h(\tilde{\rho}))} - \tilde{w}^2\right)\partial_2\rho = 0,
$$

which gives $\partial_1 w = 0$ for $x_2 = 0, 1$. These imply that the compatibility conditions for the system (3.11) are always satisfied. So, the system (3.11) has a unique solution $w \in C^{2,\alpha}(\bar{\Omega})$. Hence, the proof of Lemma 3.2 is completed. $\square$

Thanks to the Lemma 3.2 and the properties of (ρ, w), we can conclude that T maps the set Σ into itself. Furthermore, T is a continuous and compact operator due to the standard elliptic estimates.

Now we will apply Theorem 3.1 to obtain the existence result to the problem (3.5).

The equation $(\rho, w) = \sigma T(\rho, w)$ in Σ with $\sigma \in [0,1]$ is equivalent to the following problem with elliptic system of first order,

$$\begin{cases} \partial_1 w - \dfrac{w}{\rho}\partial_1\rho + \dfrac{1}{\rho}\Big(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - w^2\Big)\partial_2\rho = 0, \\ \dfrac{1}{\rho}\Big(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - 1\Big)\partial_1\rho - \partial_2 w - \dfrac{w}{\rho}\partial_2\rho = 0, \\ w = 0, & \text{on } \ x_1 = 0, \\ w = 0, & \text{on } \ x_2 = 0 \ \text{ or } \ x_2 = 1, \\ \rho = \sigma\rho_e(x_2), & \text{on } \ x_1 = 1. \end{cases} \tag{3.12}$$

We may conclude from the Schauder fixed point theorem that T has a fixed point $(\rho, w) = T(\rho, w)$ in Σ, provided that the solution (ρ, w) to the problem (3.12) is uniformly bounded in $C^{1,\alpha}$, which is independent of σ. First, we derive a L^∞-estimate of the density by the maximum principle arguments.

Lemma 3.3. *(L^∞-estimate of ρ) For any (ρ, w) satisfying (3.12) for some $\sigma \in [0,1]$, then the following estimate holds*

$$\|\rho\|_{L^\infty(\bar{\Omega})} \leq \|\rho_e(x_2)\|_{L^\infty([0,1])}.$$

Proof. It follows from the equations in (3.12) that ρ satisfies the following second order elliptic system

$$\begin{cases} \partial_1\Big(\dfrac{1}{\rho}\left(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - 1\right)\partial_1\rho - \dfrac{w}{\rho}\partial_2\rho\Big) + \partial_2\Big(- \dfrac{w}{\rho}\partial_1\rho + \dfrac{1}{\rho}\left(\dfrac{c^2(\rho)(1+w^2)}{2(B-h(\rho))} - w^2\right)\partial_2\rho\Big) = 0, \\ \partial_1\rho = 0, \qquad \text{on } \ x_1 = 0, \\ \partial_2\rho = 0, \qquad \text{on } \ x_2 = 0 \ \text{ or } \ x_2 = 1, \\ \rho = \sigma\rho_e(x_2), \qquad \text{on } \ x_1 = 1. \end{cases} \tag{3.13}$$

So the maximum principle and Hopf's lemma show that the maximum density in the nozzle is achieved only at the exit, namely,

$$\|\rho\|_{L^\infty(\Omega)} \leq \sigma\|\rho_e(x_2)\|_{L^\infty([0,1])} \leq \|\rho_e(x_2)\|_{L^\infty([0,1])}.$$

□

It remains to obtain the L^∞-bound for w, which implies the positivity of the horizontal velocity of the flows in the nozzle.

It follows from (3.5) that w satisfies the second order elliptic equation,

$$\begin{cases} \partial_1\rho = I\dfrac{1}{(1+w^2)}\Big[-\dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_1 w + \Big(c^2(\rho) - \dfrac{2(B-h(\rho))w^2}{(1+w^2)}\Big)\partial_2 w\Big], \\ \partial_2\rho = I\dfrac{1}{(1+w^2)}\Big[\dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_2 w - \Big(c^2(\rho) - \dfrac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1 w\Big], \end{cases} \tag{3.14}$$

where $I = \dfrac{2\rho(B-h(\rho))}{c^2(\rho)(c^2(\rho)-2(B-h(\rho)))}$. Combining the boundary conditions for w in (3.12), we obtain the following elliptic problem

$$\begin{cases} \partial_1\Big\{\dfrac{I}{(1+w^2)}\Big[\Big(c^2(\rho) - \dfrac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1 w - \dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_2 w\Big]\Big\} + \\ \partial_2\Big\{\dfrac{I}{(1+w^2)}\Big[-\dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_1 w + \Big(c^2(\rho) - \dfrac{2(B-h(\rho))w^2}{(1+w^2)}\Big)\partial_2 w\Big]\Big\} = 0, \\ w = 0, \quad \text{on } \ x_1 = 0, \\ w = 0, \quad \text{on } \ x_2 = 0 \ \text{ or } \ x_2 = 1, \\ \dfrac{I}{(1+w^2)}\Big[\dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_2 w - \Big(c^2(\rho) - \dfrac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1 w\Big] = \sigma\rho_e'(x_2), \quad \text{on } \ x_1 = 1. \end{cases} \tag{3.15}$$

Set

$$\begin{aligned} A = (A_1, A_2)^t &= \frac{I}{(1+w^2)}\begin{pmatrix} \Big(c^2(\rho) - \dfrac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1 w - \dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_2 w \\ -\dfrac{2(B-h(\rho))w}{(1+w^2)}\partial_1 w + \Big(c^2(\rho) - \dfrac{2(B-h(\rho))w^2}{(1+w^2)}\Big)\partial_2 w \end{pmatrix} \\ &= \frac{I}{(1+w^2)}\mathbb{M}\begin{pmatrix} \partial_1 w \\ \partial_2 w \end{pmatrix}, \end{aligned}$$

where the matrix $\mathbb{M}$ is

$$\begin{pmatrix} c^2(\rho) - \dfrac{2(B-h(\rho))}{(1+w^2)} & -\dfrac{2(B-h(\rho))w}{(1+w^2)} \\ -\dfrac{2(B-h(\rho))w}{(1+w^2)} & c^2(\rho) - \dfrac{2(B-h(\rho))w^2}{(1+w^2)} \end{pmatrix}.$$

A direct computation yields that the eigenvalues of the matrix $\mathbb{M}$ are

$$\lambda_1 = c^2(\rho) - 2(B-h(\rho)) \quad and \quad \lambda_2 = c^2(\rho),$$

hence

$$(\partial_1 w, \partial_2 w)\cdot A \geq \frac{2\rho(B-h(\rho))}{c^2(\rho)(1+w^2)}\big((\partial_1 w)^2 + (\partial_2 w)^2\big) \geq C(\underline{p}, \bar{p})\delta\frac{1}{1+w^2}|Dw|^2,$$

provided $B > B_0$. This implies that the equation in (3.15) for the angular velocity w is not uniformly elliptic apriorily. To deal with this difficulty, we introduce a new variable the flow angle $\Theta = \arctan w$, and fortunately $\Theta = \arctan w$ satisfies the following uniformly elliptic equation,

$$\begin{cases} \partial_1\Big\{I\Big[\Big(c^2(\rho)-\frac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1\Theta-\frac{2(B-h(\rho))w}{1+w^2}\partial_2\Theta\Big]\Big\}+ \\ \partial_2\Big\{I\Big[-\frac{2(B-h(\rho))w}{(1+w^2)}\partial_1\Theta+\Big(c^2(\rho)-\frac{2(B-h(\rho))w^2}{(1+w^2)}\Big)\partial_2\Theta\Big]\Big\}=0, \\ \Theta=0, \quad \text{on} \;\; x_1=0, \\ \Theta=0, \quad \text{on} \;\; x_2=0 \;\; \text{or} \;\; x_2=1, \\ I\Big[\frac{2(B-h(\rho))w}{(1+w^2)}\partial_2\Theta-\Big(c^2(\rho)-\frac{2(B-h(\rho))}{(1+w^2)}\Big)\partial_1\Theta\Big]=\sigma\rho_e'(x_2), \text{on} \;\; x_1=1. \end{cases} \tag{3.16}$$

The L^∞ estimate of w can be obtained by the standard Moser's iteration arguments.

Lemma 3.4. *(L^∞ estimate of w)*
For any (ρ, w) satisfying (3.12) for some $\sigma \in [0,1]$ and some fixed constant $\Theta_0 \in \left(1, \frac{\pi}{2}\right)$, if $C(\underline{p}, \bar{p}, \Omega)\delta^{-5}\|p_e'\|_{L^\infty}^4\|p_e'\|_{L^2} \leq \left(1-\frac{1}{\Theta_0}\right)^4\Theta_0$, then we have

$$\|w\|_{L^\infty(\bar{\Omega})} \leq \tan\Theta_0.$$

Proof. The proof is based on the Moser's iteration argument in [15]. Consider the weak formulation of the problem (3.16),

$$\int_\Omega I\mathbb{M}(\partial_1\Theta,\partial_2\Theta)^t\cdot D\eta dx+\int_0^1-\sigma\rho_e'(x_2)\eta(1,x_2)dx_2=0, \tag{3.17}$$

for any test function $\eta \in H^1(\Omega)\cap C(\bar{\Omega})$ with $\eta(0,x_2)=\eta(x_1,0)=\eta(x_1,1)=0$.

Set $\mu = 4q-3$ for $q>1$, and choose

$$\eta=\left(1-\frac{1}{|\Theta|}\right)_+^\mu|\Theta|^{q-2}\Theta,$$

where $\left(1-\frac{1}{|\Theta|}\right)_+=\max\left\{1-\frac{1}{|\Theta|},0\right\}$, then

$$D\eta=q_1\left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1}|\Theta|^{q-2}D\Theta, \tag{3.18}$$

where $q_1=(q-1)\left(1-\frac{1}{|\Theta|}\right)_++\mu\frac{1}{|\Theta|}$. Inserting η into (3.17) gives

$$\int_\Omega I\mathbb{M}(\partial_1\Theta,\partial_2\Theta)^t\cdot D\eta dx \geq C(\underline{p},\bar{p})\delta\int_{\Omega_1}q_1\left(1-\frac{1}{|\Theta|}\right)^{\mu-1}|\Theta|^{q-2}|D\Theta|^2dx, \tag{3.19}$$

and

$$\begin{aligned} \left|\int_0^1\sigma\rho_e'(x_2)\eta(1,x_2)dx_2\right| &\leq \int_0^1|\sigma\rho_e'(x_2)|\left(1-\frac{1}{|\Theta|}\right)_+^\mu|\Theta|^{q-1}dx_2 \\ &\leq \|\rho_e'(x_2)\|_{L^\infty}\int_0^1\left(1-\frac{1}{|\Theta|}\right)_+^\mu|\Theta|^q dx_2, \end{aligned} \tag{3.20}$$

where $\Omega_1 = \{x \in \Omega : |\Theta(x)| > 1\}$.

Using the trace theorem (see also Lemma 2.1 in [15]), one can obtain

$$\begin{aligned}
&\int_0^1 \left(1-\frac{1}{|\Theta|}\right)_+^{\mu} |\Theta|^q dx_2 \\
&\leq K(\Omega)\int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu} |\Theta|^q dx + \int_\Omega \left|D\left[\left(1-\frac{1}{|\Theta|}\right)_+^{\mu} |\Theta|^q\right]\right| dx \\
&\leq K(\Omega)\int_\Omega \left|D\left[\left(1-\frac{1}{|\Theta|}\right)_+^{\mu} |\Theta|^q\right]\right| dx \\
&\leq K(\Omega) q_2 \int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^{q-1}|D\Theta| dx,
\end{aligned} \tag{3.21}$$

where $q_2 = q\left(1-\frac{1}{|\Theta|}\right)_+ + \mu\frac{1}{|\Theta|}$.

Combining (3.17)-(3.21) and using Young's inequality yield

$$\begin{aligned}
&C(\underline{p},\bar{p})\delta\int_{\Omega_1} q_1 \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^{q-2}|D\Theta|^2 dx \\
&\leq \|\rho_e'(x_2)\|_{L^\infty}\int_0^1 \left(1-\frac{1}{|\Theta|}\right)_+^{\mu} |\Theta|^q dx_2 \\
&\leq C(\underline{p},\bar{p})\delta q_2\varepsilon\int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^{q-2}|D\Theta|^2 dx \\
&\quad + \frac{Cq_2}{C(\underline{p},\bar{p})\delta\varepsilon}\|\rho_e'(x_2)\|_{L^\infty}^2\int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^q dx.
\end{aligned} \tag{3.22}$$

That is

$$\begin{aligned}
&C(\underline{p},\bar{p})\delta\int_{\Omega_1} |D\Theta|^2|\Theta|^{q-2}\left(1-\frac{1}{|\Theta|}\right)^{\mu-1}(q_1-\varepsilon q_2) dx \\
&\leq \|\rho_e'\|_{L^\infty}^2\int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^q \frac{Cq_2}{C(\underline{p},\bar{p})\delta\varepsilon} dx.
\end{aligned} \tag{3.23}$$

Choosing $\varepsilon = \frac{1}{3}$ in (3.23) gives

$$\int_{\Omega_1} |D\Theta|^2|\Theta|^{q-2}\left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} dx \leq C(\underline{p},\bar{p},\Omega)\delta^{-2}\|\rho_e'\|_{L^\infty}^2\int_\Omega \left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1} |\Theta|^q dx. \tag{3.24}$$

The Poincaré inequality yields

$$\left(\int_\Omega h^4 dx\right)^{\frac{1}{2}} \leq C_0\left(\int_\Omega h^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |Dh|^2 dx\right)^{\frac{1}{2}} \leq C_0\int_\Omega |Dh|^2 dx. \tag{3.25}$$

Set $h^2 = \left(1-\frac{1}{|\Theta|}\right)_+^{\mu+1} |\Theta|^q$, and

$$Dh = \left(1-\frac{1}{|\Theta|}\right)_+^{\frac{\mu-1}{2}} |\Theta|^{\frac{q}{2}-2}\Theta\left[\frac{\mu+1}{2}\frac{1}{|\Theta|} + \frac{q}{2}\left(1-\frac{1}{|\Theta|}\right)_+\right] D\Theta. \tag{3.26}$$

By the inequalities (3.24)-(3.25), one can obtain

$$\begin{aligned}&\left(\int_\Omega\left(\left(1-\frac{1}{|\Theta|}\right)_+^{\mu+1}|\Theta|^q\right)^2dx\right)^{\frac12}\\ &\le C_0\left[(\max\{\mu+1,q\})^2\int_\Omega|D\Theta|^2|\Theta|^{q-2}\left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1}dx\right]\\ &\le(\max\{\mu+1,q\})^2\|\rho_e'\|_{L^\infty}^2C(\underline{p},\bar p,\Omega)\delta^{-2}\int_\Omega\left(1-\frac{1}{|\Theta|}\right)_+^{\mu-1}|\Theta|^qdx.\end{aligned}\tag{3.27}$$

Now set $\bar\Theta=\left(1-\frac{1}{|\Theta|}\right)_+^4|\Theta|, dx'=I_1\left(1-\frac{1}{|\Theta|}\right)_+^{-4}dx$, where $I_1(x)=1$ if $|\Theta|>1$, and $I_1(x)=0$ otherwise. Then the last inequality (3.27) can be rewritten as

$$\left(\int_\Omega|\bar\Theta|^{2q}dx'\right)^{\frac12}\le c_1q^2\int_\Omega|\bar\Theta|^qdx',$$

where $c_1=\|\rho_e'\|_{L^\infty}^2C(\underline{p},\bar p,\Omega)\delta^{-2}$.

Setting $q=2^k, k=2,3,4,\cdots$, then we obtain

$$\|\bar\Theta\|_{L^\infty(\Omega)}\le c_2\left(\int_\Omega|\bar\Theta|^{2^2}dx'\right)^{\frac{1}{2^2}},$$

where $c_2=c_1^\beta 2^{\beta_1}$,

$$\beta=\sum_{k=2}^\infty 2^{-k}=\frac12,\qquad \beta_1=2\sum_{k=2}^\infty k2^{-k}.$$

Now we estimate the L^p norm of the flow angle. Choosing $\eta=\Theta$, then we obtain

$$\int_\Omega\frac{2\rho(B-h(\rho))}{c^2(\rho)(c^2(\rho)-2(B-h(\rho)))}\mathbb{M}(\partial_1\Theta,\partial_2\Theta)^t\cdot D\Theta dx=\int_0^1\sigma\rho_e'\Theta(1,x_2)dx_2.$$

Hence

$$C(\underline{p},\bar p)\delta\int_\Omega|D\Theta|^2dx\le\|\rho_e'\|_{L^2([0,1])}\|\Theta(1,x_2)\|_{L^2([0,1])}\le C(\Omega)\|\rho_e'\|_{L^2([0,1])}\|D\Theta\|_{L^2(\Omega)}.$$

So, it gives the uniform H_0^1-estimate to Θ,

$$\|D\Theta\|_{L^2(\Omega)}\le C(\underline{p},\bar p,\Omega)\delta^{-1}\|\rho_e'\|_{L^2},$$

which implies a uniform L^q-estimate

$$\|\Theta\|_{L^q(\Omega)}\le C(\underline{p},\bar p,\Omega,q)\delta^{-1}\|\rho_e'\|_{L^2},$$

for $q\in[1,+\infty)$.

Since $\lim\limits_{q\to+\infty}\left(\int_\Omega|\bar\Theta|^qdx'\right)^{\frac1q}=\sup\limits_\Omega|\bar\Theta|$, we obtain

$$\sup_\Omega|\bar\Theta|\le c_2^{\frac{1}{2^2}}\left(\int_\Omega|\bar\Theta|^{2^2}dx'\right)^{\frac{1}{2^2}}\le c_2^{\frac{1}{2^2}}(\sup_\Omega|\bar\Theta|)^{1-\frac{1}{2^2}}(\int_\Omega|\bar\Theta|dx')^{\frac{1}{2^2}}.$$

Hence

$$\sup_\Omega|\bar\Theta|=\sup_\Omega\left(1-\frac{1}{|\Theta|}\right)_+^4|\Theta|\le c_2^4\int_\Omega|\bar\Theta|dx'\le c_2^4\int_\Omega|\Theta|dx.\tag{3.28}$$

If $|\Theta| \geq \Theta_0$, we have $1 - \dfrac{1}{|\Theta|} \geq 1 - \dfrac{1}{\Theta_0} > 0$, hence $|\Theta| \leq \left(1 - \dfrac{1}{\Theta_0}\right)^{-4} c_2^4 \displaystyle\int_\Omega |\Theta| dx$.

Finally, we obtain

$$\sup_\Omega |\Theta| \leq \max\left\{\Theta_0, \left(1 - \frac{1}{\Theta_0}\right)^{-4} c_2^4 \int_\Omega |\Theta| dx\right\}. \tag{3.29}$$

If

$$c_2^4 \int_\Omega |\Theta| dx \leq C(\underline{p}, \bar{p}, \Omega)\delta^{-5} \|\rho_e'\|_{L^\infty}^4 \|\rho_e'\|_{L^2} \leq C(\underline{p}, \bar{p}, \Omega)\delta^{-5} \|p_e'\|_{L^\infty}^4 \|p_e'\|_{L^2} \leq \left(1 - \frac{1}{\Theta_0}\right)^4 \Theta_0,$$

then one has $\|\Theta\|_{L^\infty(\bar{\Omega})} \leq \Theta_0$ and $\|w\|_{L^\infty(\bar{\Omega})} \leq \tan \Theta_0$.

□

Remark 3.2. *Since the flow does not contain any stagnation points, the boundedness of the angular velocity* $w = \dfrac{u_2}{u_1}$ *will guarantee that the horizontal velocity is always positive.*

In the following, we always assume that $C(\underline{p}, \bar{p}, \Omega)\delta^{-5} \|p_e'\|_{L^\infty}^4 \|p_e'\|_{L^2} \leq \left(1 - \dfrac{1}{\Theta_0}\right)^4 \Theta_0$, so we have $\|w\|_{L^\infty} \leq \tan \Theta_0$.

Next, we begin to estimate the gradient of ρ and w.

Lemma 3.5. *(Gradient estimate of ρ and w) For any (ρ, w) satisfying the problem (3.12) for some $\sigma \in [0,1]$, then the following estimate holds*

$$\|(D\rho, Dw)\|_{L^\infty(\bar{\Omega})} \leq C(\underline{p}, \bar{p}, \delta, \Theta_0)\left(\|p_e'\|_{L^\infty} + \nu e^{2\nu M}\right),$$

where $\nu = \nu(\|p_e\|_{C^2}, \delta, \Theta_0)$, $C(\underline{p}, \bar{p}, \delta, \Theta_0)$ are positive constants depending continuously on $\|p_e\|_{C^2}$, $\underline{p}$, $\bar{p}$, δ, Θ_0 and $M = C(\bar{p}, \underline{p}, \Theta_0)$.

Proof. Set $\check{\rho} = \rho - \sigma\rho_e$. Then $\check{\rho}$ solves the following problem,

$$\begin{cases} \check{Q}\check{\rho} = \displaystyle\sum_{i,j=1}^2 \check{a}^{ij}\partial_{ij}\check{\rho} + \sum_{k,l=1}^2 \check{b}^{kl}\partial_k\check{\rho}\partial_l\check{\rho} + \check{g} = 0, & \\ \partial_1\check{\rho} = 0, & \text{on } x_1 = 0, \\ \partial_2\check{\rho} = 0, & \text{on } x_2 = 0, 1, \\ \check{\rho} = 0, & \text{on } x_1 = 1, \end{cases} \tag{3.30}$$

where

$$\check{a}^{11}(x) = \frac{1}{\rho}\left(\frac{c^2(1+w^2)}{2(B - h(\rho))} - 1\right), \quad \check{a}^{12} = \check{a}^{21} = -\frac{w}{\rho}, \quad \check{a}^{22} = \frac{1}{\rho}\left(\frac{c^2(1+w^2)}{2(B - h(\rho))} - w^2\right),$$

and $\check{g}(x) = \sigma\check{a}^{22}\rho_e'' + \sigma^2\check{b}^{22}(\rho_e')^2$, $\check{b}^{kl}$ $(k, l = 1, 2)$ are functions of ρ and w. We omit the explicit forms of these functions here.

Next we will use the barrier function to obtain the boundary gradient estimate. Let

$$\check{b}(x, \mathbf{p}) = \check{b}^{kl}p_kp_l + \check{g}, \quad \text{and} \quad \Delta(\mathbf{p}) = \check{a}_{ij}p_ip_j,$$

for $\mathbf{P} = (p_1, p_2)$. Then there exists a positive constant $\nu = \nu(\|p_e\|_{C^2[0,1]}, \delta, \Theta_0)$ such that

$$|\check{b}(x, \mathbf{p})| \leq \nu \Delta(\mathbf{p}) \quad \text{for any } x \in \bar{\Omega} \ \text{ and } |\mathbf{p}| \geq \nu.$$

Define $v^+(x_1, x_2) = \psi(d) = \dfrac{1}{\nu} \ln(1 + \kappa d)$ in $\bar{\Omega}$, where $d = 1 - x_1$ is the distant function and $\kappa = \nu^2 e^{\nu M}$ for $M = C(\bar{p}, \underline{p}, \Theta_0) \geq \|\check{\rho}\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}$.

Then if $|Dv^+| \geq \nu$, a direct computation yields

$$\begin{aligned}\check{Q}v^+ &= \check{a}^{ij}\partial_{ij}v^+ + \check{b}(x, Dv^+) = \psi' \check{a}^{ij}\partial_{ij}d + \frac{\psi''}{(\psi')^2}\Delta(Dv^+) + \check{b}(x, Dv^+) \\ &= \frac{\psi''}{(\psi')^2}\Delta(Dv^+) + \check{b}(x, Dv^+) \leq \left(\frac{\psi''}{(\psi')^2} + \nu\right)\Delta(Dv^+).\end{aligned}$$

Note that $\psi'' + \nu(\psi')^2 = 0$, we have

$$\check{Q}v^+ \leq 0, \qquad \text{if} \quad |Dv^+| \geq \nu.$$

Restrict the domain in a small neighborhood $N = \{x \in \Omega \mid d(x) < a\}$ for some $a > 0$ to be determined later. On the interior boundary $\{x \in \Omega \mid d(x) = a\}$, one can choose a such that

$$v^+ = \psi(a) = \frac{1}{\nu}\ln(1 + \kappa a) = M \geq \|\check{\rho}\|_{L^\infty(\Omega)},$$

on the other boundary one has $v^+ = \psi \equiv 0$. And in the neighborhood N, we have

$$|Dv^+| = \psi'(d) = \frac{\kappa}{\nu(1 + \kappa d)} \geq \frac{\kappa}{\nu(1 + \kappa a)} = \frac{\kappa}{\nu e^{\nu M}} = \nu.$$

Hence, for any point $(1, x_{2,0})$ at the exit, the function $v^+ = \psi(d)$ is an upper barrier at $(1, x_{2,0})$ for the operator $\check{Q}$ and the function $\check{\rho}$.

Similarly, we can construct a lower barrier as $v^- = -\psi(d)$. Hence we obtain

$$|\partial_{x_1}\check{\rho}(1, x_{2,0})| \leq \psi'(0) = \nu e^{\nu M}.$$

While $\partial_{x_2}\check{\rho}(1, x_{2,0}) = 0$, one has

$$|D\rho(1, x_{2,0})| \leq (\nu e^{\nu M} + \|\rho_e'\|_{L^\infty}).$$

for any $x_{2,0} \in (0, 1)$. Then we obtain the gradient estimate of the density at the exit.

In view of the equations in (3.12), the gradient of w can be expressed as linear combinations of the gradient of ρ, one also obtains the gradient estimate at the exit

$$|Dw(1, x_{2,0})| \leq C(\underline{p}, \bar{p}, \delta, \Theta_0)(\nu e^{\nu M} + \|p_e'\|_{L^\infty}).$$

Since w satisfies the Dirichlet boundary condition at the inlet, one can construct a barrier function to obtain the boundary gradient estimate of w as the one of ρ as above. So, we obtain the gradient estimate on the boundary

$$\|(D\rho, Dw)\|_{L^\infty(\partial\Omega)} \leq C(\underline{p}, \bar{p}, \delta, \Theta_0)\left(\|p_e'\|_{L^\infty} + \nu e^{2\nu M}\right). \tag{3.31}$$

For the gradient estimate of ρ in the whole nozzle Ω, applying the Theorem 15.8 in [13] and Theorem 15.9 in [13] to (3.13), where the structure conditions listed in these two theorems hold for $\tau = 0$, yields

$$\|(D\rho, Dw)\|_{L^\infty(\Omega)} \leq C(\underline{p}, \bar{p}, \delta, \Theta_0)\left(\nu e^{\nu M} + \|p_e'\|_{L^\infty}\right). \tag{3.32}$$

□

Lemma 3.6. *(Hölder estimate of the gradient of ρ and w.) For any (ρ, w) satisfying (3.12) for some $\sigma \in [0,1]$, we have*

$$\|(D\rho, Dw)\|_{C^\tau(\bar\Omega)} \le C(\underline{p}, \bar p, \delta, \Theta_0, \|p_e\|_{C^2([0,1])}),$$

for some $\tau \in (0,1)$, where $C(\underline{p}, \bar p, \delta, \Theta_0, \|p_e\|_{C^2([0,1])})$ is a positive constant depending continuously on $\underline{p}$, $\bar p$, δ, Θ_0, $\|p_e\|_{C^2([0,1])}$.

Proof. We consider the interior estimate first. Since ρ satisfies (3.13), applying Theorem 13.1 in [13], we obtain

$$[D\rho]_{\tau;\Omega'} \le C(\underline{p}, \bar p, \delta, \Theta_0, K) d^{-\tau},$$

for any $\Omega' \subset\subset \Omega$, where $\tau = \tau(\underline{p}, \bar p, \delta, \Theta_0)$, $d = dist(\Omega', \partial\Omega)$ and $K = \|\rho\|_{C^1(\bar\Omega)} + \|w\|_{C^1(\bar\Omega)}$.

While for the boundary estimate, Theorem 13.2 in [13] gives

$$[D\rho]_{\tau;B_\epsilon(x_0)\bigcap\Omega} \le C(\underline{p}, \bar p, \delta, \Theta_0, K, \Phi),$$

where x_0 is a boundary point at the exit, $\tau = \tau(\underline{p}, \bar p, \delta, \Theta_0, \Omega)$, $\Phi = \|\rho_e\|_{C^2([0,1])}$.

On the other boundaries except the exit, note that w satisfies a uniformly elliptic equation with Dirichlet boundary conditions, we can also apply Theorem 13.2 in [13] to w and obtain a similar estimate as above. Then we use the relation between ρ and w to get a global estimate on the whole domain for ρ and w. □

Lemma 3.7. *(Hölder estimate of the second order derivative of ρ and w) For any (ρ, w) satisfying (3.12) for some $\sigma \in [0,1]$, it holds that*

$$\|(D^2\rho, D^2 w)\|_{C^\tau(\bar\Omega)} \le C\left(\underline{p}, \bar p, \delta, \Theta_0, \|p_e\|_{C^{2,\alpha}([0,1])}\right),$$

for some $\tau \in (0,1)$, where $C\left(\underline{p}, \bar p, \delta, \Theta_0, \|p_e\|_{C^{2,\alpha}([0,1])}\right)$ is a positive constant depending continuously on $\underline{p}$, $\bar p$, δ, Θ_0, $\|p_e\|_{C^{2,\alpha}([0,1])}$.

Proof. Note that $\check\rho = \rho - \sigma\rho_e$ solves the following problem

$$\begin{cases} \check Q\rho = \sum_{i,j=1}^{2} \check a^{ij}\partial_{ij}\check\rho + \sum_{k=1}^{2} \check b^k \partial_k \check\rho + \check g = 0, \\ \partial_1\check\rho = 0, \quad \text{on} \quad x_1 = 0, \\ \partial_2\check\rho = 0, \quad \text{on} \quad x_2 = 0, 1 \\ \check\rho = 0, \quad \text{on} \quad x_1 = 1, \end{cases} \tag{3.33}$$

where $\check b^k = \sum_{l=1}^{2} \check b^{kl}\partial_l\check\rho$. Thanks to the Lemma 3.7, it is easy to see that $\check a^{ij}$, $\check b^k$ and $\check g$ have a uniform Hölder estimate with exponent τ. It follows from the Corollary 6.3 in [13] for the $C^{2,\tau}$-estimates in the interior domain that

$$\begin{aligned} \|\check\rho\|_{C^{2,\tau}(\Omega')} &\le C\left(\underline{p}, \bar p, \delta, \Theta_0, \|\rho_e\|_{C^2}\right)\left(\|\check\rho\|_{L^\infty(\bar\Omega)} + \|\bar g\|_{C^\tau(\bar\Omega)}\right) d^{-(2+\tau)} \\ &\le C\left(\underline{p}, \bar p, \delta, \Theta_0, \|p_e\|_{C^{2,\alpha}([0,1])}\right) d^{-(2+\tau)}, \end{aligned}$$

for any $\Omega' \subset\subset \Omega$, where $d = dist(\Omega', \partial\Omega)$.

The boundary estimates can be derived by Lemma 6.5 in [13]. Hence we obtain the global estimates as follows

$$\|(\rho, w)\|_{C^{2,\tau}(\bar{\Omega})} \le C\left(\underline{p}, \bar{p}, \delta, \Theta_0, \|p_e\|_{C^{2,\alpha}([0,1])}\right).$$

□

The $C^{2,\tau}$ uniform estimate of ρ and w implies the uniform $C^{1,\alpha}$ estimate of ρ and w. hence we have the solvability of the problem (3.5) in Σ. Since the fluid does not contain stagnation points and vacuum, the angular velocity is also bounded, then u_1 can not change sign in the whole nozzle. Set $u_1 = \sqrt{\dfrac{2(B-h(\rho))}{1+w^2}}$, $u_2 = u_1 w$, then there exists a subsonic solution (ρ, u_1, u_2) to auxiliary problem as long as $B \in [B_0, B_1)$.

Thus, we have completed the proof of the existence results in Theorem 2.1.

3.3. **Uniqueness of subsonic flow.**

In this subsection, we will investigate the uniqueness of smooth subsonic solutions with positive horizontal velocity to the auxiliary problem. More precisely, we will show that there exists at most one subsonic irrotational flow, with same Bernoulli's constant and the end pressure being imposed at the exit.

Since the coefficients of the elliptic system (3.5) depend on the density and the angular velocity, the deviation of two possible densities does not satisfy the maximum principle due to the attendance of the lower order terms. So the reformulation of the subsonic problem in Subsection 3.1 is not suitable to deal with the uniqueness of the subsonic irrotational flow.

It turns out that the potential function formulation is more convenient to obtain the uniqueness, since the coefficients of the potential equation depend only on the gradient of the potential function, hence the difference of two velocity potential functions with same Bernoulli's constant satisfies the maximum principle. It is noted that the assumption that the incoming flow is horizontal at the entrance also plays a crucial role in the proof.

Proposition 3.8. *(Uniqueness of the auxiliary problem)*
Under the assumptions in Theorem 2.1, then the smooth subsonic solution with positive horizontal velocity and the same Bernoulli's constant to auxiliary problem in the whole nozzle is unique.

Proof. Suppose that $(\rho_k, u_{1,k}, u_{2,k})$ for $k = 1, 2$ are two subsonic solutions to Problem 2. It suffices to prove that

$$(\rho_1, u_{1,1}, u_{2,1}) = (\rho_2, u_{1,2}, u_{2,2}).$$

Since the flow is irrotational, we can introduce the velocity potential function φ_k such that $u_{i,k} = \partial_i \varphi_k$ for $i = 1, 2$. By the strong form of Bernoulli's law, we have

$$\rho_k = \rho_k(|D\varphi_k|) = (\gamma-1)^{\frac{1}{\gamma-1}}\left(B - \frac{1}{2}|D\varphi_k|\right)^{\frac{1}{\gamma-1}}.$$

It follows from the continuity equation that we obtain the following potential equation

$$\operatorname{div}\left(\rho\left(|D\varphi_k|^2\right) D\varphi_k\right) = 0.$$

Without loss of generality, we assume that $\varphi_k(0,0)=0$, and the auxiliary problem can be reformulated as follows

$$\begin{cases} \operatorname{div}\Big(\rho(|D\varphi_k|^2)D\varphi_k\Big)=0, & \text{in} \quad \Omega, \\ \varphi_k=0, \quad \partial_1\varphi_k>0 & \text{on} \quad x_1=0, \\ \partial_2\varphi_k=0, & \text{on} \quad x_2=0\cup x_2=1, \\ (\partial_1\varphi_k)^2+(\partial_2\varphi_k)^2=q_e^2(x_2), \quad \partial_1\varphi_k>0 & \text{on} \quad x_1=1, \\ \sup_{x\in\bar\Omega}|D\varphi_k(x)|^2<c^2(\rho_k), \end{cases} \tag{3.34}$$

where $q_e^2(x_2)=2\left(B-\dfrac{1}{\gamma-1}\rho_e^{\gamma-1}(x_2)\right)$.

Let $\psi=\varphi_1-\varphi_2$ be the deviation of the two solutions, which satisfies the following elliptic equation

$$\operatorname{div}\Big(\rho(|D\varphi_1|^2)D\psi+\big(\rho(|D\varphi_1|^2)-\rho(|D\varphi_2|^2)\big)\,D\varphi_2\Big)=0,$$

which contains no lower order terms of ψ.

On the boundaries,

$$\psi=0 \qquad \text{on} \quad x_1=0,$$
$$\partial_2\psi=0, \quad \text{on} \quad x_2=0,1,$$

and

$$\partial_1(\varphi_1+\varphi_2)\partial_1\psi+\partial_2(\varphi_1+\varphi_2)\partial_2\psi=0, \quad \text{on} \quad x_1=1. \tag{3.35}$$

It follows from the maximum principle that ψ can not achieve its minimum in the interior of Ω unless it is a constant.

Assume that the minimum of ψ is achieved at the point $(1,x_0)$, then Hopf's lemma implies that $\partial_1\psi(1,x_0)<0$ and $\partial_2\psi(1,x_0)=0$.

Thanks to the positivity of $\partial_1\varphi_k$ on $x_1=1$, then substituting these into (3.35) leads a contradiction.

Thus, the minimum of ψ must be achieved at the inlet of the nozzle. Similar argument leads to the same conclusion that the maximum of ψ must be achieved on $x_1=0$. Hence we have $\psi\equiv 0$. That is $\varphi_1(x)=\varphi_2(x)$ and $(\rho_1,u_{1,1},u_{2,1})=(\rho_2,u_{1,2},u_{2,2})$ in $\bar\Omega$. □

Collecting all results obtained in Subsection 3.2 and Subsection 3.3, we complete the proof of Theorem 2.1.

4. The relationship between the mass flux m and the Bernoulli's constant B

To solve the original subsonic problem in finitely long flat nozzles, we will establish the monotonic property between the mass flux m and the Bernoulli's constant B. Actually, for given receiver pressure $p_e(x_2)$ at the exit, the mass flux is a continuous, monotonic increasing function with respect to B, provided that the flow is subsonic and the horizontal velocity is positive in the nozzle.

Proposition 4.1. (*Continuous dependence between m and B*) *Suppose that the end pressure $p_e(x_2) \in C^{2,\alpha}([0,1])$ satisfies (2.7). Let $(\rho, u_1, u_2) \in C^{2,\alpha}(\bar{\Omega})$ be a smooth subsonic solution to the auxiliary problem with the Bernoulli's constant $B \in [B_0, B_1)$. Then the mass flux m defined in (1.7) is a continuous function of B.*

Proof. Consider the normalized Bernoulli's law

$$\frac{1}{2}\left(\hat{u}_1^2 + \hat{u}_2^2\right) + \frac{1}{\gamma - 1}\hat{\rho}^{\gamma-1} = 1, \tag{4.1}$$

where

$$\hat{u}_1 = \frac{u_1}{\sqrt{B}}, \quad \hat{u}_2 = \frac{u_2}{\sqrt{B}} \quad \text{and} \quad \hat{\rho} = \frac{\rho}{B^{\frac{1}{\gamma-1}}}.$$

Also we have

$$\hat{w} = \frac{\hat{u}_2}{\hat{u}_1} = w \quad \text{and} \quad \hat{u}_1^2 = \frac{2(1 - h(\hat{\rho}))}{(1 + \hat{w}^2)}.$$

As in Section 3, one can get that the normalized density and angular velocity $(\hat{\rho}, \hat{w})$ satisfy the following first order elliptic system,

$$\begin{cases} \partial_1 \hat{w} - \dfrac{\hat{w}}{\hat{\rho}}\partial_1\hat{\rho} + \dfrac{1}{\hat{\rho}}\left(\dfrac{c^2(\hat{\rho})(1+\hat{w}^2)}{2(1-h(\hat{\rho}))} - \hat{w}^2\right)\partial_2\hat{\rho} = 0, \\ \dfrac{1}{\hat{\rho}}\left(\dfrac{c^2(\hat{\rho})(1+\hat{w}^2)}{2(1-h(\hat{\rho}))} - 1\right)\partial_1\hat{\rho} - \partial_2\hat{w} - \dfrac{\hat{w}}{\hat{\rho}}\partial_2\hat{\rho} = 0, \\ \hat{w} = 0 & \text{on } x_1 = 0, \\ \hat{w} = 0 & \text{on } x_2 = 0 \text{ or } x_2 = 1, \\ \hat{\rho} = B^{-\frac{1}{\gamma-1}}\rho_e(x_2) & \text{on } x_1 = 1. \end{cases} \tag{4.2}$$

For any fixed $B \in [B_0, B_1)$, we can take a sequence $B_k \in [B_0, B_1)$ for $k = 1, 2, \ldots$ such that $B_k \to B$. Thanks to the uniform $C^{1,\alpha}$-estimate for the solutions $(\hat{\rho}_k(x; B_k), \hat{w}_k(x; B_k))$ in Section 3, one can obtain

$$\|\hat{\rho}_k, \hat{w}_k\|_{C^{1,\alpha}(\bar{\Omega})} \leq C,$$

where C depends on $B, \rho_e(x_2), \Omega$, but independent of k. Furthermore, by the algebraic relationships (4.1), we have the following uniform estimates

$$\|\hat{\rho}_k, \hat{u}_{1,k}, \hat{u}_{2,k}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C.$$

Hence, it follows from Arzela-Ascoli lemma and a diagonal procedure that there exists a subsequence $\left\{\hat{\rho}_{k_j}, \hat{u}_{1,k_j}, \hat{u}_{2,k_j}\right\}_{j=1}^{\infty}$ such that

$$(\hat{\rho}_{k_j}, \hat{u}_{1,k_j}, \hat{u}_{2,k_j}) \to (\hat{\rho}_0, \hat{u}_{1,0}, \hat{u}_{2,0}) \quad \text{in } C^{1,\mu}(\bar{\Omega}) \quad \text{as } j \to \infty, \quad \text{for some } \mu \in (0, \alpha).$$

Clearly, $(\hat{\rho}_0, \hat{u}_{1,0}, \hat{u}_{2,0})$ solves the problem (4.2). However, recalling that $(\hat{\rho}, \hat{u}_1, \hat{u}_2)$ is also a solution of problem (4.2), by the uniqueness, we have

$$(\hat{\rho}_0, \hat{u}_{1,0}, \hat{u}_{2,0}) = (\hat{\rho}, \hat{u}_1, \hat{u}_2).$$

Thus,

$$m(B_{k_j}) = B_{k_j}^{\frac{1}{2}} \int_0^1 \hat{u}_{1,k_j}(1, x_2)\rho_e(x_2)dx_2 \to B^{\frac{1}{2}} \int_0^1 \hat{u}_1(1, x_2)\rho_e(x_2)dx_2 = m(B),$$

which implies the continuity of the mass flux $m(B)$ with respect to B. □

Moreover, we can establish the following monotonic property between the Bernoulli's constant and the mass flux.

Proposition 4.2. (*Monotonicity of* $m(B)$) *For given end pressure* $p_e(x_2)$*, the mass flux* $m(B)$ *defined in (1.7) is monotonic increasing with respect to* B*, provided that the subsonic flow possesses positive horizontal velocity at the exit.*

Proof. For any given constants B_2 and B_3 such that $B_0 \leq B_2 < B_3 < B_1$, let

$$(\rho_k(x), u_{1,k}(x), u_{2,k}(x)) = (\rho(x; B_k), u_1(x; B_k), u_2(x; B_k)) \quad \text{for} \quad k = 2, 3,$$

solve the auxiliary problem with the Bernoulli's constant B_k, respectively.

Consider the normalized flows $\hat{u}_{1,k} = \dfrac{u_{1,k}}{\sqrt{B_k}}$, $\hat{u}_{2,k} = \dfrac{u_{2,k}}{\sqrt{B_k}}$ and $\hat{\rho}_k = \dfrac{\rho_k}{B_k^{\frac{1}{\gamma-1}}}$, for $k = 2, 3$.

In this case, the mass flux can be expressed as

$$m(B_k) = B_k^{\frac{\gamma+1}{2(\gamma-1)}} \int_0^1 \hat{\rho}_k(x_1, x_2)\hat{u}_{1,k}(x_1, x_2)dx_2,$$

for any $x_1 \in [0, 1]$.

Consider the mass flux at the entrance, we claim that the momentum at the entrance is strictly increasing respect to the Bernoulli's constant, that is

$$\hat{\rho}_2(0, x_2)\hat{u}_{1,2}(0, x_2) < \hat{\rho}_3(0, x_2)\hat{u}_{1,3}(0, x_2)$$

for $x_2 \in [0, 1]$. In view of the fact that $\hat{\rho}(q^2)q$ is monotone increasing in q for subsonic flows and the incoming flow is horizontal at the entrance, it suffices to show that

$$\hat{u}_{1,2}(0, x_2) < \hat{u}_{1,3}(0, x_2) \quad \text{for} \quad x_2 \in [0, 1].$$

In fact, consider the velocity potential function formulation again, and define a velocity potential $\hat{\varphi}_k$, such that $(\partial_1\hat{\varphi}_k, \partial_2\hat{\varphi}_k) = (u_{1,k}, u_{2,k})$ with $\hat{\varphi}_k(0, 0) = 0$ for $k = 2, 3$. By the normalized Bernoulli's law (4.1), one obtains

$$\frac{1}{2}|D\hat{\varphi}_k|^2 + \frac{1}{\gamma - 1}\hat{\rho}_k^{\gamma-1} = 1.$$

Hence, the density can be regarded as a function of the speed of the subsonic flows

$$\hat{\rho}_k = \hat{\rho}(|D\hat{\varphi}_k|^2) = (\gamma - 1)^{\frac{1}{\gamma-1}} \left(1 - \frac{|D\hat{\varphi}_k|^2}{2}\right)^{\frac{1}{\gamma-1}}.$$

So $\hat\varphi_k$ solves the following quasilinear elliptic equation with mixed boundary conditions

$$\begin{cases} \operatorname{div}\Big(\hat\rho(|D\hat\varphi_k|^2)D\hat\varphi_k\Big) = 0, & \text{in} \quad \Omega, \\ \hat\varphi_k = 0, & \text{on} \quad x_1 = 0, \\ \partial_2\hat\varphi_k = 0, & \text{on} \quad x_2 = 0,1, \\ (\partial_1\hat\varphi_k)^2 + (\partial_2\hat\varphi_k)^2 = 2 - \dfrac{2}{\gamma-1}\dfrac{\rho_e^{\gamma-1}(x_2)}{B_k}, & \text{on} \quad x_1 = 1, \end{cases} \tag{4.3}$$

for $k = 2,3$.

Set $\hat\psi = \hat\varphi_3 - \hat\varphi_2$. Then $\hat\psi$ satisfies the following elliptic equation

$$\operatorname{div}\Big(\hat\rho(|D\hat\varphi^3|^2)D\hat\psi + \big(\hat\rho(|D\hat\varphi^3|^2) - \hat\rho(|D\hat\varphi^2|^2)\big)\, D\hat\varphi^2\Big) = 0,$$

which does not contain the zero order terms of $\hat\psi$.

Furthermore, the deviation $\hat\psi$ satisfies the following boundary conditions

$$\hat\psi = 0 \qquad \text{on} \quad x_1 = 0,$$
$$\partial_2\hat\psi = 0, \quad \text{on} \quad x_2 = 0,1,$$

and

$$\partial_1(\hat\varphi_2+\hat\varphi_3)\partial_1\hat\psi + \partial_2(\hat\varphi_2+\hat\varphi_3)\partial_2\hat\psi = -\frac{2}{\gamma-1}\rho_e^{\gamma-1}\left(\frac{1}{B_3}-\frac{1}{B_2}\right) > 0, \quad \text{on} \quad x_1 = 1. \tag{4.4}$$

Clearly, maximum principle for $\hat\psi$ yields that $\hat\psi$ can not achieve its minimum in the interior of Ω unless it is a constant.

Assume that the minimum of $\hat\psi$ is achieved at the point $(1, x_0)$, then Hopf's lemma implies that $\partial_1\hat\psi(1,x_0) < 0$ and $\partial_2\hat\psi(1,x_0) = 0$.

Thanks to the positivity of $\partial_1\hat\varphi_k$ on $x_1 = 1$, then substituting these into (4.4) leads a contradiction.

Thus, the minimum of $\hat\psi$ must be achieved at the entrance $x_1 = 0$. Applying Hopf's lemma again, one has

$$\partial_1\hat\psi(x) > 0 \quad \text{on} \quad x_1 = 0,$$

which completes the proof of the claim. As a direct consequence, the proof of the monotonicity is completed. □

Consequently, one can conclude that the mass flux of the flow is strictly increasing and dependent continuously with respect to the Bernoulli's constant for the given end pressure. Hence, we can define $m_0 = \lim\limits_{B\to B_0} m(B)$, $m_1 = \lim\limits_{B\to B_1} m(B)$, then for any $m \in [m_0, m_1)$, there exists a unique $B(m) \in [B_0, B_1)$. The solution (ρ, u_1, u_2) to the auxiliary problem associated with $B(m)$ is the unique solution to the original Problem 1. Thus the proof of Theorem 2.2 is completed.

Acknowledgements. This work was initiated when the first author was a research associate at The Institute of Mathematical Science, The Chinese University of Hong Kong in the spring

2010. The second author would like to thank Dr. Chunjing Xie and Dr. Wei Yan for the wonderful discussions.

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