On the Boundary Regularity for the 6D Stationary Navier-Stokes Equations

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Abstract

It is shown in this paper that suitable weak solutions to the 6D steady incompressible Navier-Stokes equations are Hölder continuous near boundary provided that either $r^{-3}\int_{B_r^+}|u(x)|^3dx$ or $r^{-2}\int_{B_r^+}|\nabla u(x)|^2dx$ is sufficiently small, which implies that the 2D Hausdorff measure of the set of singular points near the boundary is zero. This generalizes recent interior regularity results by Dong-Strain [4].

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1 Introduction

In this paper, we consider the following 6D steady incompressible Navier-Stokes equations on $\Omega \subset \mathbb{R}^6$:

$$\text{(SNS)} \quad \begin{cases} -\Delta u + u\cdot\nabla u = -\nabla\pi + f, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where u represents the fluid velocity field, π is a scalar pressure, and the boundary condition of u is given as a no-slip condition, namely

$$u = 0, \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

The main interests are in the boundary partial regularity for suitable weak solutions to the equations (1.1).

Recall the development of interior and boundary regularity criteria for the Navier-Stokes equations in brief. For the three dimensional time-dependent Navier-Stokes equations, partial regularity of weak solutions satisfying the local energy inequality was proved in a series of papers by Scheffer [29, 30, 32]. Later, the notion of suitable weak solutions was first introduced in a celebrated paper by Caffarelli-Kohn-Nirenberg [1], showing that the set $\mathcal{S}$ of possible interior singular points of a suitable weak solution is one-dimensional parabolic Hausdorff measure zero. Simplified proofs and improvements can be found in many works by Lin [21], Ladyzhenskaya-Seregin [22], Tian-Xin [36], Escauriaza-Seregin-Šverák [5], Seregin [25], Gustafson-Kang-Tsai [17], Vasseur [38], Kukavica [20], Wang-Zhang [37] and the references therein. Some similar boundary regularity results are also proved, see Seregin [24, 26, 27], Kang [18, 19], Gustafson-Kang-Tsai [16] and so on. Moreover, it's worth to mention that general curved boundary regularity was obtained by Seregin-Shilkin-Solonnikov in [28].

There are only fewer results available in the literature for the 4D and higher dimensional time-dependent Navier-Stokes equations. In [31], Scheffer showed that there exists a weak solution in $R^4 \times R^+$, whose singular set has vanishing 3D Hausdorff measure. Later, Dong-Du [2] proved that, for any local-in-time smooth solution to the 4D Navier-Stokes equations, the 2D Hausdorff measure of the set of singular points at the first potential blow-up time is equal to zero, and we refer to [3] for recent results with general suitable weak solutions. As it's commented in [4], four (similarly, six for stationary Navier-Stokes equations) is the highest dimension in which all the existing methods on partial regularity could be applied.

Now we turn to the steady Navier-Stokes equations. In a series of papers by Frehse and Ruzicka [6, 7, 8, 9], the existence on a class of special regular solutions of (1.1) was obtained for the five-dimensional and higher dimensional case. Gerhardt [12] obtained the regularity of weak solutions under the four-dimensional case. For $N \geq 5$, it is not known yet whether weak solutions are regular. In [34, 35], Struwe obtained partial regularity for $N = 5$ by regularity methods of elliptic systems (c.f. Morrey [23] and Giaqinta [13]). Later, the result of Struwe was extended to the boundary case by Kang [18]. In a recent paper, Dong-Strain [4] extended the interior regularity result of Struwe to the six-dimensional space. Their main idea is to apply an iteration method and a bootstrap argument to get a suitable decay estimate of $L^{3/2}$ norm of ∇u, then the Morrey lemma implies the required regularity. In this paper, we generalized the result in [4] and proved the boundary regularity of six-dimensional steady Navier-Stokes equations. At last, we refer to [10] by Farwig and Sohr for existence and regularity criteria for weak solutions to inhomogeneous Navier-Stokes equations.

Let us introduce the definition of suitable weak solutions near the boundary.

Definition 1.1 *Let $\Omega \subset \mathbb{R}^6$ be an open domain, and $\Gamma \subset \partial\Omega$ be an open set. (u, π) is said to be a suitable weak solution to the steady Navier-Stoks equations (1.1) in Ω near the boundary Γ, if the following conditions hold.*

(i) $u \in H^1(\Omega)$, $\pi \in L^{\frac{3}{2}}(\Omega)$, $\nabla\pi \in L^{\frac{6}{5}}(\Omega)$, $f \in L^6(\Omega)$;

(ii) (u, π) satisfies the equations(1.1) in the sense of distribution sense and the boundary condition $u|_\Gamma = 0$ holds;

(iii) u and π satisfy the local energy inequality

$$2\int_\Omega |\nabla u|^2 \phi dx \leq \int_\Omega \left[|u|^2 \triangle\phi + u \cdot \nabla\phi(|u|^2 + 2\pi)\right] + 2fu\phi dx \tag{1.3}$$

for any nonnegative C^∞ test function ϕ vanishing at the boundary $\partial\Omega\backslash\Gamma$.

The existence of such a suitable weak solution can be found in [8]. The major concern of this paper is the regularity and the main results can be stated as follows:

Theorem 1.2 *Let (u, π) be a suitable weak solution to (1.1) in B_1^+ near the boundary $\{x \in B_1, x_6 = 0\}$. Then 0 is a regular point of u, if there exists a small positive constant ε such* *that one of the following conditions holds,*

$$i) \limsup_{r\to 0_+} r^{-3}\int_{B_r^+}|u(x)|^3dx<\varepsilon,$$
$$ii) \limsup_{r\to 0_+} r^{-2}\int_{B_r^+}|\nabla u(x)|^2dx<\varepsilon.$$

Remark 1.3 *The boundary regularity criteria above for the 6D steady Navier-Stokes equations generalize recent interior regularity results by Dong-Strain [4] and boundary regularity results for the 5D case by Kang in [19]. Here we consider the flat boundary for simplicity, the results hold true for general C^2 boundary, which follows from the analysis here and the standard techniques as in [28].*

Theorem 1.4 *Let (u,π) be a suitable weak solution to (1.1) in B_1^+ near the boundary $\{x\in B_1, x_6=0\}$. Then the 2D Hausdorff measure of the set of singular points of (u,π) in B_1^+ is equal to zero.*

Remark 1.5 *This theorem follows directly from Theorem 1.2 by the standard arguments from the geometric measure theory, which is explained for example in [1].*

As it will be shown later, Theorem 1.2 will follow from the following partial regularity criteria.

Proposition 1.6 *Let (u,π) be a suitable weak solution to (1.1) in B_1^+ near the boundary $\{x\in B_1, x_6=0\}$. If there exists $\rho_0>0$ and a small positive constant ε_1 such that*

$$\rho_0^{-3}\|u\|^3_{L^3(B^+_{\rho_0})}+\rho_0^{-2}\|\nabla\pi\|_{L^{6/5}(B^+_{\rho_0})}+\rho_0^{3}\|f\|^3_{L^3(B^+_{\rho_0})}<\varepsilon_1$$

Then 0 is a regular point of u.

Remark 1.7 *There are several remarks in order concerning the proof of the main results. First, it should be pointed out that it seems difficult to adapt the boundary regularity theory for 5D steady Navier-Stokes system by Kang in [19] to our case, since the key blow-up arguments there (Lemma 4.6, [19]) are based on the compact imbedding $W^{1,2}(B_1)\hookrightarrow L^3(B_1)$ which fails in the 6D case. Our analysis is motivated by a bootstrap argument due to Dong-Strain [4] for the interior regularity theory. However, due to the boundary, new difficulties arise. In particular, there are slow decaying terms involving $E^{1/2}(\rho)$ in the pressure decomposition in the presence of boundaries (see Lemma 3.2), where*

$$E(\rho)=\rho^{-2}\int_{B_\rho^+}|\nabla u(x)|^2dx.$$

This means that $E^{1/2}(\rho)$ and $D_1(\rho)\equiv\rho^{-2}\parallel\nabla\pi\parallel_{L^{\frac{6}{5}}(B^+_\rho)}$ are the same order in the standard iterative scheme, which seems impossible to obtain an effective iterative estimate by the local energy inequality as in [4] (more details see Remark 4.2). To overcome the difficulty, we first derive a revised local energy inequality, see Proposition 2.4, such that there exists a free parameter in the coefficient of the scaled energy on the large ball in the local energy inequality, which plays an important role in the required estimate in Lemma 4.1 and yields an effective iteration scheme.

The rest of the paper is organized as follows. In section 2, we introduce some notations and some technical lemmas about Stokes estimates and local energy estimates. In section 3, we prove Theorem 1.2 under the assumption of Proposition 1.6. Section 4 is devoted to the proof of Proposition 1.6.

2 Notations and some technical lemmas

Throughout this article, C_0 denotes an absolute constant independent of u, ρ, r and may be different from line to line.

Let (u, π) be a solution to the steady Navier-Stokes equations (1.1). Set the following scaling:

$$u^\lambda(x) = \lambda u(\lambda x), \quad \pi^\lambda(x) = \lambda^2 \pi(\lambda x), \quad f^\lambda(x) = \lambda^3 f(\lambda x), \tag{2.1}$$

for any $\lambda > 0$, then the family (u^λ, π^λ) is also a solution of (1.1) with f replaced by $f^\lambda(x)$. Now define some quantities which are invariant under the scaling (2.1):

$$A(r) = r^{-4} \int_{B_r^+} |u(x)|^2 dx, \quad C(r) = r^{-3} \int_{B_r^+} |u(x)|^3 dx;$$

$$E(r) = r^{-2} \int_{B_r^+} |\nabla u(x)|^2 dx, \quad D_1(r) = r^{-2} \parallel \nabla \pi \parallel_{L^{\frac{6}{5}}(B_r^+)};$$

$$D(r) = r^{-3} \int_{B_r^+} |\pi - \pi_{B_r^+}|^{\frac{3}{2}} dx, \quad \pi_{B_r^+} = \frac{1}{|B_r^+|} \int_{B_r^+} \pi dx;$$

$$F(r) = r^3 \int_{B_r^+} |f(x)|^3 dx,$$

where $B_r^+(x_0)$ is the semi-ball of radius r centered at x_0, and we denote $B_r^+(0)$ by B_r^+. Moreover, a solution u is said to be regular at x_0 if $u \in L^\infty(B_r^+(x_0))$ for some $r > 0$.

The following two lemmas about L^p estimates on solutions to boundary value problems for the Stokes equations are well-known, for example, seeing Solonnikov [33], Giga-Sohr [15], Galdi [11], Seregin [24] and Kang [18].

Lemma 2.1 *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open domain , $n \geq 3$, $1 < p < \infty$, and $f \in L^p(\Omega)$. (v, q) satisfies the following conditions:*

$$\begin{cases} v \in W^{2,p}(\Omega), \quad q \in W^{1,p}(\Omega), \\ -\triangle v + \nabla q = f, \quad \nabla \cdot v = 0, \\ \displaystyle\int_\Omega q dx = 0, \quad v|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Then there holds

$$\|v\|_{W^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)} \leq C(p, \Omega) \|f\|_{L^p(\Omega)}.$$

Lemma 2.2 *Assume that $B_1^+ \subset \mathbb{R}^n$ is a bounded open domain , $n \geq 3$, $1 < p_0 \leq p < \infty$, and $f \in L^p(B_1^+)$. (v, q) satisfies the following conditions:*

$$\begin{cases} v \in W^{2,p_0}(B_1^+), \quad q \in W^{1,p_0}(B_1^+), \\ -\triangle v + \nabla q = f, \quad \nabla \cdot v = 0, \\ v|_{\{x_n=0\}} = 0. \end{cases} \tag{2.3}$$

Then there holds

$$\|v\|_{W^{2,p}(B_{1/2}^+)} + \|q\|_{W^{1,p}(B_{1/2}^+)} \leq C(p_0, p)\big(\|f\|_{L^p(B_1^+)} + \|q\|_{L^{p_0}(B_1^+)} + \|v\|_{W^{1,p_0}(B_1^+)}\big).$$

Recall the interior regular result by Dong-Strain (Proposition 3.7 and (3.33), [4]), which is necessary in the proof of Theorem 1.2. We write it in a slightly different form.

Proposition 2.3 *There exists $\varepsilon_0 > 0$ satisfying the following property. Suppose that (u, π) is a suitable weak solution of (1.1) in $B_1(x_0)$, and for some $\rho_0 \in (0, 1)$, it holds that*

$$\rho_0^{-3} \int_{B_{\rho_0}(x_0)} |u|^3 dx + \rho_0^{-3} \int_{B_{\rho_0}(x_0)} |\pi - \pi_{B_{\rho_0}(x_0)}|^{\frac{3}{2}} dx + \rho_0^3 \int_{B_{\rho_0}(x_0)} |f|^3 dx \leq \varepsilon_2.$$

Then, for $0 < \rho < \rho_0/8$, the following inequality will hold uniformly

$$\int_{B_\rho(x_0)} |\nabla u|^{\frac{6}{5}} dx \leq C_0 \rho^{\frac{24}{5} + \frac{2}{25}}, \tag{2.4}$$

where C_0 is a positive constant independent of ρ.

Next, modifying the analysis in [4] by choosing a new test function, we will derive the following revised local energy inequality, which improves the local energy inequality in [4] and plays a crucial role in our proof later.

Proposition 2.4 *Let $0 < 4r < \rho \leq 1$. It holds that*

$$\begin{aligned} &k^{-2} A(r) + E(r) \\ &\leq C_0 k^4 \big(\frac{r}{\rho}\big)^2 A(\rho) + C_0 k^{-1} \big(\frac{\rho}{r}\big)^3 \big[C(\rho) + C^{\frac{1}{3}}(\rho) D^{\frac{2}{3}}(\rho)\big] + C_0 \big(\frac{\rho}{r}\big)^2 C^{\frac{1}{3}}(\rho) F^{\frac{1}{3}}(\rho), \end{aligned} \tag{2.5}$$

where $1 \leq k \leq \frac{\rho}{r}$ and the constant C_0 is independent of k, r, ρ.

Proof. Let ζ be a cutoff function, which vanishes outside of B_ρ and equals 1 in $B_{\rho/2}$, satisfying

$$|\nabla \zeta| \leq C_0 \rho^{-1}, \quad |\nabla^2 \zeta| \leq C_0 \rho^{-2}.$$

Introduce a smooth function as

$$\Gamma(x) = \frac{1}{(k^2 r^2 + |x|^2)^2}, \quad 1 \leq k \leq \frac{\rho}{r},$$

which clearly satisfies

$$\triangle \Gamma = \frac{-24 k^2 r^2}{(k^2 r^2 + |x|^2)^4} < 0.$$

Taking the test function $\phi = \Gamma\zeta$ in the local energy inequality (1.3), we obtain that

$$
\begin{aligned}
&-\int_{B_r^+} |u|^2\zeta\triangle\Gamma dx + 2\int_{B_r^+} |\nabla u|^2\zeta\Gamma dx \\
&\le \int_{B_\rho^+} \big[|u|^2(\Gamma\triangle\zeta + 2\nabla\Gamma\cdot\nabla\zeta) + u\cdot\nabla\phi(|u|^2 + 2\pi - 2\pi_{B_\rho^+})\big]dx + 2\int_{B_\rho^+} fu\Gamma\zeta dx
\end{aligned}
$$

It follows from some straightforward computations that

$$
\begin{aligned}
&\zeta\Gamma(x,t) \ge C_0(kr)^{-4}, \quad -\zeta\triangle\Gamma(x,t) \ge C_0(kr)^{-6} \quad \text{in } B_r^+, \\
&|\nabla\phi| \le |\nabla\Gamma|\zeta + \Gamma|\nabla\zeta| \le C_0(kr)^{-5} \quad \text{in } B_\rho^+, \\
&|\Gamma\triangle\zeta| + 2|\nabla\Gamma\cdot\nabla\zeta| \le C_0\rho^{-6} \quad \text{in } B_\rho^+,
\end{aligned}
$$

from which and the Hölder inequality, one derives that

$$
\begin{aligned}
&k^{-2}A(r) + E(r) \\
&\le C_0k^4\big(\frac{r}{\rho}\big)^2 A(\rho) + C_0k^{-1}\big(\frac{\rho}{r}\big)^3\rho^{-3}\int_{B_\rho^+}(|u|^3 + |u||\pi - \pi_{B_\rho^+}|)dx \\
&+C_0r^{-2}\rho^2\big(\int_{B_\rho^+}|f|^3dx\big)^{\frac{1}{3}}\big(\int_{B_\rho^+}|u|^3dx\big)^{\frac{1}{3}} \\
&\le C_0k^4\big(\frac{r}{\rho}\big)^2 A(\rho) + C_0k^{-1}\big(\frac{\rho}{r}\big)^3\big[C(\rho) + C^{\frac{1}{3}}(\rho)D^{\frac{2}{3}}(\rho)\big] + C_0\big(\frac{\rho}{r}\big)^2 C^{\frac{1}{3}}(\rho)F^{\frac{1}{3}}(\rho).
\end{aligned}
$$

The proof is completed. □

Remark 2.5 *If $k = 1$ in (2.5), then we obtain the local energy inequality as the interior case in [4]:*

$$
\begin{aligned}
&A(r) + E(r) \\
&\le C_0\big(\frac{r}{\rho}\big)^2 A(\rho) + C_0\big(\frac{\rho}{r}\big)^3\big[C(\rho) + C^{\frac{1}{3}}(\rho)D^{\frac{2}{3}}(\rho)\big] + C_0\big(\frac{\rho}{r}\big)^2 C^{\frac{1}{3}}(\rho)F^{\frac{1}{3}}(\rho). \qquad (2.6)
\end{aligned}
$$

However, it seems difficult for us to construct an effective iteration scheme based on (2.6) due to the fast decay of the second term in the right hand side of (2.6) (for more details see Remark 4.2). Yet, by choosing a suitable k in Proposition 2.4, we can overcome the difficulty, see the proof of Lemma 4.1.

3 Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2 assuming that Proposition 1.6 holds ture, and the next section is devoted to the proof of Proposition 1.6. In fact, due to the Sobolev inequality in Lemma 3.1, it suffices to assume that

$$
C(r) < \varepsilon, \quad \text{for any} \quad 0 < r < r_0 < 1. \qquad (3.1)
$$

First, we need some technical lemmas on the scaling quantities introduced before.

Lemma 3.1 *For any $0 < r < r_0$, there holds*

$$
C(r) \le C_0E(r)^{\frac{3}{2}}, \quad D(r) \le C_0D_1^{\frac{3}{2}}(r). \qquad (3.2)
$$

Proof. These follow from the Sobolev imbedding inequality in the six dimensional space. □

Lemma 3.2 *For any $0 < 4r < \rho < r_0$, there holds*

$$D_1(r) \le C_0\big(\frac{r}{\rho}\big)^{3-\frac{6}{p}}\big(E^{\frac12}(\rho) + D_1(\rho)\big) + C_0\big(\frac{\rho}{r}\big)^2\big(E^{\frac12}(\rho)C^{\frac13}(\rho) + F^{\frac13}(\rho)\big), \tag{3.3}$$

where $p \ge 12$, and C_0 depends on p.

Proof. Choose a domain $\widetilde{B}^+$ with a smooth boundary such that $B^+_{\rho/2} \subset \widetilde{B}^+ \subset B^+_\rho$. Let v and π_1 be the unique solution to the following initial boundary value problem for the Stokes system:

$$\begin{cases} -\Delta v + \nabla \pi_1 &= f - u\cdot\nabla u, \quad \mathrm{div} v = 0 \quad \text{in} \quad \widetilde{B}^+, \\ (\pi_1)_{\widetilde{B}^+} &= \frac{1}{|\widetilde{B}^+|}\int_{\widetilde{B}^+}\pi_1 dx = 0, \\ v &= 0 \quad \text{on} \quad \partial\widetilde{B}^+. \end{cases} \tag{3.4}$$

Then by the estimates for the steady Stokes system in Lemma 2.1, one can obtain

$$\begin{aligned} &\frac{1}{\rho^2}\| v \|_{L^{\frac65}(\widetilde{B}^+)} + \frac{1}{\rho}\| \nabla v \|_{L^{\frac65}(\widetilde{B}^+)} + \| \nabla^2 v \|_{L^{\frac65}(\widetilde{B}^+)} + \frac{1}{\rho}\| \pi_1 \|_{L^{\frac65}(\widetilde{B}^+)} + \| \nabla \pi_1 \|_{L^{\frac65}(\widetilde{B}^+)} \\ \le\ & C_0\big(\| u\cdot\nabla u \|_{L^{\frac65}(\widetilde{B}^+)} + \| f \|_{L^{\frac65}(\widetilde{B}^+)} \big) \\ \le\ & C_0\rho^2\big(E^{\frac12}(\rho)C^{\frac13}(\rho) + F^{\frac13}(\rho)\big) \end{aligned} \tag{3.5}$$

with C_0 depending on B^+_1.

On the other hand, let

$$w = u - v, \quad \pi_2 = \pi - \pi_{B^+_{\rho/2}} - \pi_1 \tag{3.6}$$

Then w, π_2 solve the following boundary value problem:

$$-\Delta w + \nabla\pi_2 = 0, \quad \mathrm{div} w = 0 \quad \text{in} \quad \widetilde{B}^+,$$

$$w = 0 \quad \text{on} \quad \partial\widetilde{B}^+ \cap \{x_6 = 0\}.$$

Then the local estimate near the boundary for the steady Stokes system in Lemma 2.2 for $p \ge 12$ yields that

$$\rho^{3-\frac6p}\| \nabla\pi_2 \|_{L^p(B^+_{\rho/4})} \le C_0\big(\frac{1}{\rho^4}\| w \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^3}\| \nabla w \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^3}\| \pi_2 \|_{L^{\frac65}(B^+_{\rho/2})}\big).$$

It follows from the Sobolev inequality, (3.6), and (3.5) that

$$\begin{aligned} &\rho^{3-\frac6p}\| \nabla\pi_2 \|_{L^p(B^+_{\rho/4})} \\ \le\ & C_0\big(\frac{1}{\rho^3}\| \nabla w \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^3}\| \pi_2 \|_{L^{\frac65}(B^+_{\rho/2})}\big) \\ \le\ & C_0\big(\frac{1}{\rho^3}\| \nabla u \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^3}\| \nabla v \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^2}\| \nabla \pi \|_{L^{\frac65}(B^+_{\rho/2})} + \frac{1}{\rho^3}\| \pi_1 \|_{L^{\frac65}(B^+_{\rho/2})}\big) \\ \le\ & C_0\big(E^{\frac12}(\rho) + E^{\frac12}(\rho)C^{\frac13}(\rho) + F^{\frac13}(\rho) + D_1(\rho)\big). \end{aligned} \tag{3.7}$$

Combining the two estimates (3.5) and (3.7) shows that

$$
\begin{aligned}
D_1(r) &= \frac{1}{r^2}\Big(\int_{B_r^+}|\nabla p|^{\frac{6}{5}}dx\Big)^{\frac{5}{6}} \\
&\le C_0\frac{1}{r^2}\big(\parallel \nabla\pi_1 \parallel_{L^{\frac{6}{5}}(B_r^+)} + r^{5-\frac{6}{p}} \parallel \nabla\pi_2 \parallel_{L^p(B_r^+)}\big) \\
&\le C_0\big(\frac{\rho}{r}\big)^2\big(E^{\frac12}(\rho)C^{\frac13}(\rho)+F^{\frac13}(\rho)\big)+C_0\big(\frac{r}{\rho}\big)^{3-\frac{6}{p}}\big(E^{\frac12}(\rho)+E^{\frac12}(\rho)C^{\frac13}(\rho)+F^{\frac13}(\rho)+D_1(\rho)\big) \\
&\le C_0\big(\frac{r}{\rho}\big)^{3-\frac{6}{p}}\big(E^{\frac12}(\rho)+D_1(\rho)\big)+C_0\big(\frac{\rho}{r}\big)^2\big(E^{\frac12}(\rho)C^{\frac13}(\rho)+F^{\frac13}(\rho)\big).
\end{aligned}
$$

Thus the proof is completed. □

Proof of Theorem 1.2: Take $0<16r<4\rho<\kappa<r_0$ and set

$$
G(r)=A(r)+E(r)+\varepsilon^{\frac16}D_1(r)+\varepsilon^{\frac{1}{12}}F^{\frac13}(r).
$$

It follows from (3.2) and (3.3) for $p=12$ that

$$
\begin{aligned}
D_1(\rho) &\le C_0\big(\frac{\rho}{\kappa}\big)^{\frac52}\big(E^{\frac12}(\kappa)+D_1(\kappa)\big)+C_0\big(\frac{\kappa}{\rho}\big)^2\big(E^{\frac12}(\kappa)C^{\frac13}(\kappa)+F^{\frac13}(\kappa)\big) \\
&\le C_0\big(\frac{\kappa}{\rho}\big)^2\big(E(\kappa)+F^{\frac13}(\kappa)\big)+C_0\big(\frac{\rho}{\kappa}\big)^{\frac52}\big(D_1(\kappa)+1\big).
\end{aligned}
\tag{3.8}
$$

Then using the local energy inequality in Proposition 2.4 for $k=1$, (3.1) and (3.2), one can obtain that

$$
\begin{aligned}
G(r) &\le C_0\big(\frac{r}{\rho}\big)^2A(\rho)+C_0\big(\frac{\rho}{r}\big)^3\big[C(\rho)+C^{\frac13}(\rho)D^{\frac23}(\rho)\big]+C_0\big(\frac{\rho}{r}\big)^2C^{\frac13}(\rho)F^{\frac13}(\rho) \\
&\quad +C_0\varepsilon^{\frac16}D_1(r)+C_0\varepsilon^{\frac{1}{12}}F^{\frac13}(r) \\
&\le C_0\big(\frac{r}{\rho}\big)^2A(\rho)+C_0\big(\frac{\rho}{r}\big)^3\varepsilon^{\frac13}D_1(\rho)+C_0\big(\frac{\rho}{r}\big)^2\varepsilon^{\frac13}F^{\frac13}(\rho)+C_0\varepsilon^{\frac{1}{12}}F^{\frac13}(r) \\
&\quad +C_0\varepsilon^{\frac16}D_1(r)+C_0\big(\frac{\rho}{r}\big)^3\varepsilon.
\end{aligned}
$$

Applying the inequality (3.8) twice yields

$$
\begin{aligned}
G(r) &\le C_0\big(\frac{\kappa}{\rho}\big)^4\big(\frac{r}{\rho}\big)^2G(\kappa)+C_0\varepsilon^{\frac13}\big(\frac{\rho}{r}\big)^3\big(\frac{\kappa}{\rho}\big)^2\big(E(\kappa)+F^{\frac13}(\kappa)\big) \\
&\quad +C_0\varepsilon^{\frac13}\big(\frac{\rho}{r}\big)^3\big(\frac{\rho}{\kappa}\big)^{\frac52}\big(D_1(\kappa)+1\big)+C_0\varepsilon^{\frac13}\big(\frac{\rho}{r}\big)^2\big(\frac{\rho}{\kappa}\big)F^{\frac13}(\kappa)+C_0\varepsilon^{\frac{1}{12}}\big(\frac{r}{\kappa}\big)F^{\frac13}(\kappa) \\
&\quad +C_0\varepsilon^{\frac16}\big(\frac{\kappa}{r}\big)^2\big(E(\kappa)+F^{\frac13}(\kappa)\big)+C_0\varepsilon^{\frac16}\big(\frac{r}{\kappa}\big)^{\frac52}\big(D_1(\kappa)+1\big)+C_0\big(\frac{\rho}{r}\big)^3\varepsilon \\
&\le C_0\big[\big(\frac{\kappa}{\rho}\big)^4\big(\frac{r}{\rho}\big)^2+\big(\frac{r}{\kappa}\big)^{\frac52}+\big(\frac{r}{\kappa}\big)\big]G(\kappa)+C_0\varepsilon^{\frac13}\big(\frac{\kappa}{\rho}\big)^2\big(\frac{\rho}{r}\big)^3G(\kappa) \\
&\quad +C_0\varepsilon^{\frac14}\big[\big(\frac{\rho}{r}\big)^3\big(\frac{\kappa}{\rho}\big)^2+\big(\frac{\rho}{r}\big)^2\big(\frac{\rho}{\kappa}\big)\big]G(\kappa)+C_0\varepsilon^{\frac16}\big[\big(\frac{\rho}{r}\big)^3\big(\frac{\rho}{\kappa}\big)^{\frac52}+\big(\frac{\kappa}{r}\big)^2\big]G(\kappa) \\
&\quad +C_0\varepsilon^{\frac{1}{12}}\big(\frac{\kappa}{r}\big)^2G(\kappa)+C_0\varepsilon^{\frac13}\big(\frac{\rho}{r}\big)^3\big(\frac{\rho}{\kappa}\big)^{\frac52}+C_0\varepsilon^{\frac16}\big(\frac{r}{\kappa}\big)^{\frac52}+C_0\big(\frac{\rho}{r}\big)^3\varepsilon.
\end{aligned}
$$

Set $r=\theta^3\rho$, $\rho=\theta\kappa$ with $0<\theta<\frac18$. The above inequality yields that

$$
G(r)\le C_0\big[\theta^2+\varepsilon^{\frac13}\theta^{-11}+\varepsilon^{\frac14}\theta^{-11}+\varepsilon^{\frac16}\theta^{-8}+\varepsilon^{\frac{1}{12}}\theta^{-8}\big]G(\kappa)+C_0\varepsilon^{\frac13}\theta^{-\frac{13}{2}}+C_0\varepsilon^{\frac16}\theta^{10}+C_0\varepsilon\theta^{-9}.
$$

Thus choosing θ small at first, and then ε small, one can get

$$C_0\big[\theta^2+\varepsilon^{\frac{1}{3}}\theta^{-11}+\varepsilon^{\frac{1}{4}}\theta^{-11}+\varepsilon^{\frac{1}{6}}\theta^{-8}+\varepsilon^{\frac{1}{12}}\theta^{-8}\big]\le\frac{1}{2},$$
$$C_0\varepsilon^{\frac{1}{3}}\theta^{-\frac{13}{2}}+C_0\varepsilon^{\frac{1}{6}}\theta^{10}+C_0\varepsilon\theta^{-9}\le\varepsilon_1^3,$$

where ε_1 is the constant as in Proposition 1.6. Thus we obtain the following iterative inequality

$$G(\theta^4\kappa)\le\frac{1}{2}G(\kappa)+\varepsilon_1^3. \tag{3.9}$$

On the other hand, it is easy to see that

$$G(r_0)\le C_0,$$

where C_0 depends on r_0, $\|u\|_{H^1(B_1^+)}$, $\|\nabla p\|_{L^{\frac{6}{5}}(B_1^+)}$ and $\|f\|_{L^6(B_1^+)}$. Then the standard iteration arguments for (3.9) ensure that there exists $r_1>0$ such that

$$G(r)\le 3\varepsilon_1^3\quad\text{for all}\quad r\in(0,r_1). \tag{3.10}$$

Next, due to Lemma 3.2, (3.10) and $f\in L^6(B_1^+)$, there exists $r_2>0$ such that

$$D_1(r)\le C_0(\tfrac{r}{\rho})^2(\varepsilon_1^{3/2}+D_1(\rho))+C_0(\tfrac{\rho}{r})^2\varepsilon_1^3,\quad 0<r<\rho<r_2<r_1,$$
$$F(\rho)\le\varepsilon_1^3,\quad 0<\rho<r_2<r_1.$$

Then a standard iteration argumemnt yields that there exists $r_3>0$ such that

$$D_1(r)+F(r)\le C_0\varepsilon_1^{3/2}\quad\text{for}\quad 0<r<r_3<r_2.$$

Combining (3.10) and the above inequality, we get

$$C(r)+D_1(r)+F(r)\le C_0\varepsilon_1^{3/2}\quad\text{for}\quad 0<r<r_3.$$

We can assume that $C_0\varepsilon_1^{1/2}\le 1$ without loss of generality. Hence the proof of Theorem 1.2 is completed due to Proposition 1.6. □

4 Proof of Proposition 1.6

This section is devoted to the proof of Proposition 1.6. To overcome the lower order term $E^{1/2}(\rho)$ from the pressure decomposition, we'll make full use of the revised local energy inequality in Proposition 2.4 and the decay estimates (3.3) in order to derive an effective iteration (for more details, see the following lemma and Remark 4.2). Finally, we improve the decay of ∇u by a similar bootstrap argument as in [4], and carry out the boundary estimates as in [24].

Lemma 4.1 *Let $\rho > 0$ be a positive constant. Then there exists a θ_0, which is suitably small and independent of ρ, such that*

$$\begin{aligned} & \theta_0^{\frac{1}{2}} A(\theta_0\rho) + E(\theta_0\rho) + \theta_0^{-6+\frac{1}{10}} D_1^2(\theta_0\rho) \\ \leq \ & \frac{1}{4}\big[\theta_0^{\frac{1}{2}} A(\rho) + E(\rho) + \theta_0^{-6+\frac{1}{10}} D_1^2(\rho)\big] + C_0\big(E^{\frac{3}{2}}(\rho) + E^2(\rho) + F^{\frac{2}{3}}(\rho)\big), \end{aligned} \tag{4.1}$$

where C_0 is a positive constant independent of ρ.

Proof. For any $\theta \in (0, \frac{1}{4}]$, Proposition 2.4 with $r = \theta\rho$ and (3.2) yield that

$$\begin{aligned} & k^{-2} A(\theta\rho) + E(\theta\rho) \\ & \leq C_0 k^4 \theta^2 A(\rho) + C_0 k^{-1}\theta^{-3}\big[C(\rho) + C^{\frac{1}{3}}(\rho) D^{\frac{2}{3}}(\rho)\big] + C_0\theta^{-2} C^{\frac{1}{3}}(\rho) F^{\frac{1}{3}}(\rho), \\ & \leq C_0 k^4 \theta^2 E(\rho) + C_0 k^{-1}\theta^{-3}\big[E^{\frac{3}{2}}(\rho) + E^{\frac{1}{2}}(\rho) D_1(\rho)\big] + C_0\theta^{-2} E^{\frac{1}{2}}(\rho) F^{\frac{1}{3}}(\rho), \end{aligned} \tag{4.2}$$

By choosing $p = 600$ in Lemma 3.2, one hets

$$D_1(\theta\rho) \leq C_0 \theta^{3-\frac{1}{100}}\big(E^{\frac{1}{2}}(\rho) + D_1(\rho)\big) + C_0\theta^{-2}\big(E(\rho) + F^{\frac{1}{3}}(\rho)\big).$$

Hence

$$D_1^2(\theta\rho) \leq C_0 \theta^{6-\frac{1}{50}}\big(E(\rho) + D_1^2(\rho)\big) + C_0\theta^{-4}\big(E^2(\rho) + F^{\frac{2}{3}}(\rho)\big). \tag{4.3}$$

Set

$$G(r) = k^{-2} A(r) + E(r) + \gamma^{-1} D_1^2(r), \quad \text{and } r = \theta\rho,$$

with $\gamma > 0$ to be decided. It then follows from (4.2) and (4.3) that

$$\begin{aligned} G(\theta\rho) \ \leq \ & C_0 k^4\theta^2 E(\rho) + C_0 k^{-1}\theta^{-3}\big[E^{\frac{3}{2}}(\rho) + E^{\frac{1}{2}}(\rho) D_1(\rho)\big] + C_0\theta^{-2} E^{\frac{1}{2}}(\rho) F^{\frac{1}{3}}(\rho) \\ & + C_0\gamma^{-1}\theta^{6-\frac{1}{50}}\big(E(\rho) + D_1^2(\rho)\big) + C_0\gamma^{-1}\theta^{-4}\big(E^2(\rho) + F^{\frac{2}{3}}(\rho)\big) \\ \leq \ & C_0 k^4\theta^2 E(\rho) + C_0 k^{-1}\theta^{-3}\big(E^{\frac{3}{2}}(\rho) + \gamma^{\frac{1}{2}} E(\rho) + \gamma^{-\frac{1}{2}} D_1^2(\rho)\big) + C_0\gamma E(\rho) \\ & + C_0\gamma^{-1}\theta^{6-\frac{1}{50}}\big(E(\rho) + D_1^2(\rho)\big) + C_0\gamma^{-1}\theta^{-4}\big(E^2(\rho) + F^{\frac{2}{3}}(\rho)\big) \\ \leq \ & C_0[k^4\theta^2 + k^{-1}\theta^{-3}\gamma^{\frac{1}{2}} + \gamma + \gamma^{-1}\theta^{6-\frac{1}{50}} + \theta^{6-\frac{1}{50}}] G(\rho) \\ & + C_0 k^{-1}\theta^{-3} E^{\frac{3}{2}}(\rho) + C_0\gamma^{-1}\theta^{-4}\big(E^2(\rho) + F^{\frac{2}{3}}(\rho)\big) \end{aligned}$$

Choosing $k = \theta^{-\frac{1}{4}}$, $\gamma = \theta^{6-\frac{1}{10}}$, we get

$$C_0[k^4\theta^2 + k^{-1}\theta^{-3}\gamma^{\frac{1}{2}} + \gamma + \gamma^{-1}\theta^{6-\frac{1}{50}} + \theta^{6-\frac{1}{50}}] \leq C_0\theta^{\frac{4}{50}}.$$

Finally, θ is chosen so small such that the required inequality (4.1) holds. The lemma is proved. $\square$

Remark 4.2 *It should be noted that Proposition 2.4 is crucial in the proof of Lemma 4.1. Recall the local energy inequality (2.6) as in [4] and the pressure decomposition in Lemma 3.2:*

$$D_1(r) \le C_0\big(\frac{r}{\rho}\big)^{3-\frac{6}{p}}\big(E^{\frac12}(\rho)+D_1(\rho)\big)+C_0\big(\frac{\rho}{r}\big)^2\big(E^{\frac12}(\rho)C^{\frac13}(\rho)+F^{\frac13}(\rho)\big), \tag{4.4}$$

for $0<4r<\rho<1$ and $p\ge 12$. The two inequalities indicate that $A(r)$, $E(r)$, and $D_1^2(r)$ are the same order, hence it is natural to try to derive an iterative inequality for the following scaled quantity:

$$G(r)=A(r)+E(r)+\gamma^{-1}D_1^2(r), \qquad r=\theta\rho,$$

with $\gamma>0$ to be decided. It then follows from (2.6) and (4.4) that (as in Lemma 4.1)

$$\begin{aligned} G(\theta\rho) \quad &\le \quad C_0[\theta^2+\theta^{-3}\gamma^{\frac12}+\gamma+\gamma^{-1}\theta^{6-\frac{12}{p}}+\theta^{6-\frac{12}{p}}]G(\rho) \\ &\quad +C_0\theta^{-3}E^{\frac32}(\rho)+C_0\gamma^{-1}\theta^{-4}\big(E^2(\rho)+F^{\frac23}(\rho)\big) \end{aligned} \tag{4.5}$$

In order to obtain Proposition 1.6, one needs the smallness of

$$C_0[\theta^2+\theta^{-3}\gamma^{\frac12}+\gamma+\gamma^{-1}\theta^{6-\frac{12}{p}}+\theta^{6-\frac{12}{p}}],$$

however, which seems to be impossible.

To overcome the difficulty above, we make use of the free parameter in the revised local energy inequality (2.5) such that the term $\big(\frac{r}{\rho}\big)^2A(\rho)$ in (2.6) plays an important role, and consequently we obtain Lemma 4.1.

For $\alpha_0=\log_{\theta_0}^{\frac12}$ and any small $\delta>0$, we will prove that there exists an integer $m=m(\delta)$ and an increasing sequence of real number $\{\alpha_k\}_{k=1}^m\in(\alpha_0,2)$ with $\alpha_m>2-\delta$. For these fixed δ and $m(\delta)$, we obtain the following proposition.

Proposition 4.3 *For $\rho_0\in(0,1)$, there exits a small constant $\varepsilon_0>0$ satisfying the following property that if*

$$C(\rho_0)+D_1(\rho_0)+F(\rho_0)\le\varepsilon_0,$$

then, for any given $\delta>0$, any $0<\rho<\rho_0/8$ and $|x_0|\le\rho_0/8$ with $x_0=(x',x_6)$ and $x_6=0$, there exists an increasing sequence of real numbers $\{\alpha_k\}_{k=0}^m\in[\alpha_0,2)$ with $\alpha_m>2-\delta$ such that the following inequality holds uniformly

$$A(\rho,x_0)+E(\rho,x_0)+C^{\frac23}(\rho,x_0)+D_1^2(\rho,x_0)\le C_0\rho^{\alpha_k}, \quad k=0,1,\cdots,m \tag{4.6}$$

where C_0 is a positive constant depending on δ and k, but independent of ρ.

Proof. We will prove this by indiction. Without loss of generality, assume that $x'=0$. By Proposition 2.4 and (3.2), it holds that

$$\begin{aligned} & A(\rho_0/8)+E(\rho_0/8) \\ \le\ & C_0A(\rho_0)+C_0\big[C(\rho_0)+C^{\frac13}(\rho_0)D_1(\rho_0)\big]+C_0C^{\frac13}(\rho_0)F^{\frac13}(\rho_0) \\ \le\ & C_0C^{\frac23}(\rho_0)+C_0\big[C(\rho_0)+C^{\frac13}(\rho_0)D_1(\rho_0)\big]+C_0F^{\frac12}(\rho_0) \\ \le\ & C_0(\varepsilon_0+\varepsilon_0^{\frac12}+\varepsilon_0^{\frac23}+\varepsilon_0^{\frac43}). \end{aligned}$$

Thus we can choose $\varepsilon_3 > 0$ first, then $\varepsilon_0 = \varepsilon_0(\varepsilon_3) > 0$ small enough such that

$$C_0\big(\varepsilon_3^{\frac{1}{2}} + \varepsilon_3\big) < 1/4, \quad C_0(\varepsilon_0 + \varepsilon_0{}^{\frac{1}{2}} + \varepsilon_0^{\frac{2}{3}} + \varepsilon_0^{\frac{4}{3}}) < \varepsilon_3/4, \tag{4.7}$$

and

$$\varphi(\rho_0) = \theta_0^{\frac{1}{2}} A(\rho_0/8) + E(\rho_0/8) + \theta_0^{-6+\frac{1}{10}} D_1^2(\rho_0/8) \le \varepsilon_3. \tag{4.8}$$

It follows from (4.7)-(4.8) and (4.1) that

$$\varphi(\theta_0^k \rho_0) \le \varepsilon_3. \tag{4.9}$$

Furthermore, by (4.9) and (4.7), one has

$$C_0(E(\theta_0^{k-1}\rho_0/8)^{1/2} + E(\theta_0^{k-1}\rho_0/8)) < \frac{1}{4}.$$

Then (4.9) and (4.1) imply that

$$\varphi(\theta_0^k\rho_0) \le \frac{1}{2}\varphi(\theta_0^{k-1}\rho_0) + C_1(\theta_0^{k-1}\rho_0)^4,$$

where C_1 is a constant independent of k and we have used that fact that $f \in L^6$. One can iterate the above inequality to reach

$$\begin{aligned}
\varphi(\theta_0^k\rho_0) &\le (\frac{1}{2})^k\varphi(\rho_0) + C_1\rho_0^4\sum_{j=0}^{k-1}(\frac{1}{2})^j(\theta_0^{k-1-j})^4 \\
&\le (\frac{1}{2})^k\big[\varphi(\rho_0) + \frac{2C_1}{1-2\theta_0^4}\rho_0^4\big] \le (\frac{1}{2})^k\big[\varphi(\rho_0) + \frac{2C_1}{1-\theta_0}\rho_0^4\big].
\end{aligned}$$

For $\rho \in (0, \rho_0/8)$, we can find k such that $\theta_0^k\frac{\rho_0}{8} < \rho < \theta_0^{k-1}\frac{\rho_0}{8}$. Note that $\alpha_0 = \log_{\theta_0}^{\frac{1}{2}}$, then we have

$$\begin{aligned}
\theta_0^{\frac{1}{2}}A(\rho) + E(\rho) + \theta_0^{-6+\frac{1}{10}}D_1^2(\rho) &\le C(\theta_0)\varphi(\theta_0^{k-1}\rho_0) \\
&\le C(\theta_0)(\frac{1}{2})^k\big[\varphi(\rho_0) + \frac{2C_1}{1-\theta_0}\rho_0^4\big] \\
&\le C(\theta_0)(\theta_0^k)^{\alpha_0}\big[\varphi(\rho_0) + \frac{2C_1}{1-\theta_0}\rho_0^4\big] \\
&\le C(\theta_0)\rho^{\alpha_0}\big[\varphi(\rho_0) + \frac{2C_1}{1-\theta_0}\rho_0^4\big] \le C_0\rho^{\alpha_0},
\end{aligned}$$

where $C_0 = C_0(\theta_0, \varphi(\rho_0), C_1, \rho_0)$. By (3.2) in Lemma 3.1, we can get similar estimates for $C(\rho)$, thus the case $k = 0$ is proved.

Assume that (4.6) is true for $m = k$, i.e.

$$A(\rho) + E(\rho) + C^{\frac{2}{3}}(\rho) + D_1^2(\rho) \le C_0\rho^{\alpha_k}. \tag{4.10}$$

When $m = k + 1$, we will estimate $A(\rho) + E(\rho)$.

Let $\alpha+\beta+\gamma=1$, with $\alpha,\beta,\gamma\in(0,1)$ to be decided. By Proposition 2.4, (3.2) and the assumption for $m=k$, one can obtain that

$$\begin{aligned}A(\rho)+E(\rho) &\leq C_0\rho^{2\alpha}A(\rho^{\beta+\gamma})+C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{3}{2}}\\ &\quad +C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}D_1(\rho^{\beta+\gamma})+C_0\rho^{-2\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}F^{\frac{1}{3}}(\rho^{\beta+\gamma})\\ &\leq C_0\rho^{2\alpha}A(\rho^{\beta+\gamma})+C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{3}{2}}+C_0\rho^{-\frac{3}{2}\alpha}F^{\frac{1}{2}}(\rho^{\beta+\gamma})\\ &\quad +C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}D_1(\rho^{\beta+\gamma}).\end{aligned}$$

It follows from (4.10) and $f\in L^6$ that

$$\begin{aligned}&A(\rho)+E(\rho)\\ \leq\ & C_0\rho^{2\alpha+(\beta+\gamma)\alpha_k}+C_0\rho^{-3\alpha+\frac{3(\beta+\gamma)\alpha_k}{2}}+C_0\rho^{-\frac{3}{2}\alpha+3(\beta+\gamma)}+C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}D_1(\rho^{\beta+\gamma})\\ \leq\ & C_0\rho^{\alpha_k+(2-\alpha_k)\alpha}+C_0\rho^{\alpha_k(1+\frac{\beta}{2}+\frac{\gamma}{2}-\alpha)-3\alpha}+C_0\rho^{-\frac{3}{2}\alpha+3(\beta+\gamma)}+C_0\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}D_1(\rho^{\beta+\gamma}).\end{aligned}\tag{4.11}$$

For the last term, by Lemma 3.2 with $p=12$ and (4.10), one can derive that

$$\begin{aligned}\rho^{-3\alpha}E(\rho^{\beta+\gamma})^{\frac{1}{2}}D_1(\rho^{\beta+\gamma}) &\leq C_0\rho^{-3\alpha}\rho^{\frac{(\beta+\gamma)\alpha_k}{2}}\big(\rho^{\frac{5\beta}{2}}\rho^{\frac{\gamma\alpha_k}{2}}+\rho^{-2\beta}\rho^{\gamma\alpha_k}+\rho^{-2\beta}\rho^{2\gamma}\big)\\ &\leq C_0\rho^{-3\alpha}\rho^{\frac{(\beta+\gamma)\alpha_k}{2}}\big(\rho^{\frac{5\beta}{2}}\rho^{\frac{\gamma\alpha_k}{2}}+\rho^{-2\beta}\rho^{\gamma\alpha_k}\big)\\ &\leq C_0\rho^{\alpha_k(1-\alpha-\frac{\beta}{2})+\frac{5}{2}\beta-3\alpha}+C_0\rho^{\alpha_k(1+\frac{\gamma}{2}-\alpha-\frac{\beta}{2})-3\alpha-2\beta}.\end{aligned}\tag{4.12}$$

Since $\alpha_k<2$, one can choose

$$\alpha=\frac{\alpha_k}{100+5\alpha_k},\quad \beta=\frac{4\alpha_k}{100+5\alpha_k},\quad \gamma=\frac{100}{100+5\alpha_k}.$$

Now we define α_{k+1} as :

$$\begin{aligned}\alpha_{k+1} &= \min\{\alpha_k+(2-\alpha_k)\alpha,\alpha_k(1+\frac{\beta}{2}+\frac{\gamma}{2}-\alpha)-3\alpha,-\frac{3}{2}\alpha+3(\beta+\gamma),\\ &\quad \alpha_k(1-\alpha-\frac{\beta}{2})+\frac{5}{2}\beta-3\alpha,\alpha_k(1+\frac{\gamma}{2}-\alpha-\frac{\beta}{2})-3\alpha-2\beta\}\\ &= \min\{\frac{102+4\alpha_k}{100+5\alpha_k}\alpha_k,\frac{147+6\alpha_k}{100+5\alpha_k}\alpha_k,\frac{300+\frac{21}{2}\alpha_k}{100+5\alpha_k},\frac{107+2\alpha_k}{100+5\alpha_k}\alpha_k,\frac{139+2\alpha_k}{100+5\alpha_k}\alpha_k\}\\ &= \frac{102+4\alpha_k}{100+5\alpha_k}\alpha_k\in(\alpha_k,2).\end{aligned}$$

Thus it follows from (4.11) and (4.12) that

$$A(\rho)+E(\rho)+C^{\frac{2}{3}}(\rho)\leq C_0\rho^{\alpha_{k+1}},\tag{4.13}$$

where $C(\rho)$ is estimated directly by using Lemma 3.1.

Next, we estimate $D_1(\rho)$. Since $\alpha_{k+1}<2$, by Lemma 3.2 with $p=12$, the inequality

$$\begin{aligned}D_1(\gamma\rho) &\leq C_0\gamma^{\frac{5}{2}}\big(E^{\frac{1}{2}}(\rho)+D_1(\rho)\big)+C_0\gamma^{-2}\big(E^{\frac{1}{2}}(\rho)C^{\frac{1}{3}}(\rho)+F^{\frac{1}{3}}(\rho)\big)\\ &\leq C_0\big(\gamma^{\frac{5}{2}}D_1(\rho)+\gamma^{\frac{5}{2}}\rho^{\frac{\alpha_{k+1}}{2}}+\gamma^{-2}\rho^{\alpha_{k+1}}+\gamma^{-2}\rho^2\big)\\ &\leq C_0\big(\gamma^{\frac{5}{2}}D_1(\rho)+\gamma^{-2}\rho^{\frac{\alpha_{k+1}}{2}}\big)\end{aligned}$$

holds for any $\gamma, \rho \in (0,1)$.

For fixed $\rho_1 \in (0, \frac{\rho_0}{8})$, let $\theta = \frac{5}{2} - \frac{\alpha_{k+1}}{2}$. Then we can iterate the above inequality to reach

$$\begin{aligned} D_1(\gamma^m \rho_1) &\leq C_0\big[\gamma^{\frac{5}{2}} D_1(\gamma^{m-1}\rho_1) + \gamma^{-2}(\gamma^{m-1}\rho_1)^{\frac{\alpha_{k+1}}{2}}\big] \\ &\leq C_0\big[\gamma^{\frac{5}{2}m} D_1(\rho_1) + \sum_{j=0}^{m-1} \gamma^{-2}\gamma^{\theta j}(\gamma^{m-1}\rho_1)^{\frac{\alpha_{k+1}}{2}}\big] \\ &\leq C_0\big[(\gamma^m\rho_1)^{\frac{\alpha_{k+1}}{2}}\gamma^{(\frac{5}{2}-\frac{\alpha_{k+1}}{2})m}\rho_1^{\frac{\alpha_k-\alpha_{k+1}}{2}} + \gamma^{-2}\gamma^{-\frac{\alpha_{k+1}}{2}}(\gamma^m\rho_1)^{\frac{\alpha_{k+1}}{2}}\sum_{j=0}^{m-1}\gamma^{\theta j}\big] \\ &\leq C_0\big[(\gamma^m\rho_1)^{\frac{\alpha_{k+1}}{2}} + \gamma^{-3}\frac{1}{1-\gamma^\theta}(\gamma^m\rho_1)^{\frac{\alpha_{k+1}}{2}}\big] \\ &\leq C_0(\gamma^m\rho_1)^{\frac{\alpha_{k+1}}{2}}, \end{aligned}$$

where we have used $\alpha_{k+1} < 2$, and C_0 is independent of m. This yields the desired estimate for $D(\rho)$. Combining this with (4.13) shows (4.10) for the case $k+1$. Hence (4.6) is proved.

Finally, it follows from the choice of α_k, that α_k is increasing as $k \to \infty$, and $\alpha_k \leq 2$. Thus there exists a limit $\bar{\alpha}$ of α_k satisfying $0 < \bar{\alpha} \leq 2$. By the definition of α_{k+1}, we claim that $\bar{\alpha} = 2$. The proof is completed. □

Proof of Proposition 1.6: It follows from Proposition 4.3 that, for any small $\delta > 0$ and $0 < \rho < \rho_0/8$, there hold

$$\int_{B_\rho^+} |u(x)|^2 \leq C_0 \rho^{6-\delta}, \tag{4.14}$$

$$\int_{B_\rho^+} |\nabla u(x)|^2 \leq C_0 \rho^{4-\delta}, \tag{4.15}$$

$$\int_{B_\rho^+} |u(x)|^3 dx \leq C_0 \rho^{6-\frac{3}{2}\delta}. \tag{4.16}$$

By (4.14), there exists $\rho_1 \in (\rho/2, \rho)$ such that

$$\int_{S_{\rho_1}^+} |u(x)|^2 dx \leq C_0 \rho^{5-\delta}, \tag{4.17}$$

where $S_{\rho_1}^+ = \{x; |x| = \rho_1, x_6 \geq 0\}$.

Let v be the unique H^1 solution to the Laplace equation

$$\begin{cases} \triangle v = 0, & \text{in } B_{\rho_1}^+, \\ v = 0, & \text{on } \partial B_{\rho_1}^+ \cap \{x; x_6 = 0\}, \\ v = u, & \text{on } S_{\rho_1}^+. \end{cases}$$

Now we extend $v(x)$ from $B_{\rho_1}^+$ to B_{ρ_1}. For convenience, we still write $x = (x', x_6)$ and define $\widetilde{v}(x)$ to be the odd extension of $v(x)$ from $B_{\rho_1}^+$ to B_{ρ_1} as:

$$\widetilde{v}(x', x_6) = \begin{cases} v(x', x_6) & x_6 \geq 0, \\ -v(x', -x_6) & x_6 < 0. \end{cases}$$

Moreover, we set

$$\widetilde{u}(x', x_6) = \begin{cases} u(x', x_6) & x_6 \geq 0, \\ -u(x', -x_6) & x_6 < 0. \end{cases}$$

Obviously, $\widetilde{v}(x)$ satisfies

$$\left\{ \begin{array}{lll} \triangle \widetilde{v}(x) = 0, & \text{in} & B_{\rho_1}, \\ \widetilde{v} = \widetilde{u}, & \text{on} & S_{\rho_1} = \{x; |x| = \rho_1\}. \end{array} \right.$$

Then by the standard estimates for harmonic functions and (4.17), we get

$$\begin{aligned} \sup_{B^+_{\rho_1/2}} |\nabla v| &\leq \sup_{B_{\rho_1/2}} |\nabla \widetilde{v}| \leq C_0 \rho_1^{-6} \int_{S_{\rho_1}} |\widetilde{u}(x)| dx \\ &\leq C_0 \rho_1^{-6} \int_{S^+_{\rho_1}} |u(x)| dx \leq C_0 \rho^{-1-\delta/2}. \end{aligned} \tag{4.18}$$

On the other hand, we let $w = u - v \in H^1(B^+_{\rho_1})$, then w satisfies the stationary Stokes equation

$$\left\{ \begin{array}{lll} \triangle w - \nabla \pi = u \cdot \nabla u - f, & \text{in} & B^+_{\rho_1}, \\ w = 0, & \text{on} & \partial B^+_{\rho_1}. \end{array} \right.$$

Then by the classical L^p estimates for Stokes equations in Lemma 2.1, we have

$$\|\nabla w\|_{L^{6/5}(B^+_{\rho_1})} \leq C_0 \rho_1 \|u\|_{L^3(B^+_{\rho_1})} \|\nabla u\|_{L^2(B^+_{\rho_1})} + C_0 \rho_1 \|f\|_{L^{6/5}(B^+_{\rho_1})}.$$

This, together with the assumption that $f \in L^6(B^+_1)$ and (4.15)-(4.16), yields that

$$\|\nabla w\|_{L^{6/5}(B^+_{\rho_1})} \leq C_0 \rho_1^{5-\delta} + C_0 \rho_1^5 \leq C_0 \rho_1^{5-\delta}. \tag{4.19}$$

Since $|\nabla u| \leq |\nabla w| + |\nabla v|$, combining (4.18) and (4.19), for any $r \in (0, \rho/4)$, we obtain that

$$\int_{B^+_r} |\nabla u|^{\frac{6}{5}} dx \leq C_0 \rho^{6-\frac{6}{5}\delta} + C_0 r^6 \rho^{-\frac{6}{5}-\frac{3}{5}\delta}.$$

Taking $r = \frac{1}{4}\rho^{\frac{6}{5}-\frac{1}{10}\delta}$, we derive that

$$\int_{B^+_r} |\nabla u|^{\frac{6}{5}} dx \leq C_0 r^{\alpha}, \tag{4.20}$$

where

$$\alpha = \frac{6 - \frac{6}{5}\delta}{\frac{6}{5} - \frac{1}{10}\delta} > \frac{24}{5} + \frac{1}{10},$$

for a sufficiently small $\delta > 0$.

It should be noted that (4.20) and Proposition 2.3 imply that u is Hölder continuity in a neighborhood of x_0, where $x_0 \in B^+_{\rho_0/8}$. In fact, the following arguments are similar to that in [24] and we sketch its proof for completeness.

Let $\Gamma = \{x; |x| \leq \rho_0, x_6 = 0\}$. For any $x_0 \in B^+_{\rho_0/8}$, we define $r' = dist\{x_0, \Gamma\} = dist\{x_0, x^*\}$ with $x^* \in \Gamma$.

Case I: $r' \geq \rho_0/32$. Note that $B_{r'}(x_0) \subset B^+_{\rho_0/2}(x^*)$, then one has

$$r'^{-3}\int_{B_{r'}(x_0)}|u|^3dx + r'^{-3}\int_{B_{r'}(x_0)}|\pi - \pi_{B_{r'}(x_0)}|^{3/2}dx + r'^3\int_{B_{r'}(x_0)}|f|^3dx$$
$$\leq C_0\big(C(\rho_0) + D_1^{3/2}(\rho_0) + F(\rho_0)\big) \leq C_0(\varepsilon + \varepsilon^{3/2}) \leq \varepsilon_2.$$

Hence by (2.4) in Proposition 2.3, we obtain, for any $0 < \rho < r'$

$$\int_{B_\rho(x_0)}|\nabla u|^{\frac{6}{5}}dx \leq C_0\rho^{\frac{24}{5}+\frac{2}{25}}. \tag{4.21}$$

Case II: $r' \leq \rho_0/32$. First, for $r'/2 < \rho < \rho_0/32$, since $B_\rho(x_0) \subset B^+_{4\rho}(x^*)$, the boundary estimate (4.20) yields that

$$\int_{B_\rho(x_0)\cap B_1^+}|\nabla u|^{\frac{6}{5}}dx \leq C_0\int_{B^+_{4\rho}(x^*)}|\nabla u|^{\frac{6}{5}}dx \leq C_0\rho^{\frac{24}{5}+\frac{1}{10}}. \tag{4.22}$$

On the other hand, if $\rho \leq r'/2$, then $B_{r'}(x_0) \subset B^+_{4r'}(x^*)$. By (4.6) in Proposition 4.3, one has

$$r'^{-3}\int_{B_{r'}(x_0)}|u|^3dx + r'^{-3}\int_{B_{r'}(x_0)}|\pi - \pi_{B_{r'}(x_0)}|^{3/2}dx + r'^3\int_{B_{r'}(x_0)}|f|^3dx \leq C_0r' < \varepsilon_2,$$

if ρ_0 is sufficiently small. Again, one can use (2.4) in Proposition 2.3 to derive that

$$\int_{B_\rho(x_0)}|\nabla u|^{\frac{6}{5}}dx \leq C_0\rho^{\frac{24}{5}+\frac{2}{25}}, \quad \forall\ 0 < \rho \leq r'/2. \tag{4.23}$$

Combining the inequalities (4.20)-(4.23) shows that for any $x_0 \in B^+_{\rho_0/8}$ and $0 < \rho < \rho_0/8$,

$$\int_{B_\rho(x_0)\cap B_1^+}|\nabla u|^{\frac{6}{5}}dx \leq C_0\rho^{\frac{24}{5}+\frac{2}{25}}. \tag{4.24}$$

Then the Morrey lemma yields that u is Hölder continuity in a neighborhood of 0. Hence we complete the proof of Proposition 1.6. □

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