Global Well-Posedness and Large Time Asymptotic Behavior of Classical Solutions to the Compressible Navier-Stokes Equations with Vacuum

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Abstract

This paper concerns the global well-posedness and large time asymptotic behavior of strong and classical solutions to the Cauchy problem of the Navier-Stokes equations for viscous compressible barotropic flows in two or three spatial dimensions with vacuum as far field density. For strong and classical solutions, some a priori decay with rates (in large time) for both the pressure and the spatial gradient of the velocity field are obtained provided that the initial total energy is suitably small. Moreover, by using these key decay rates and some analysis on the expansion rates of the essential support of the density, we establish the global existence and uniqueness of classical solutions (which may be of possibly large oscillations) in two spatial dimensions, provided the smooth initial data are of small total energy. In addition, the initial density can even have compact support. This, in particular, yields the global regularity and uniqueness of the re-normalized weak solutions of Lions-Feireisl to the two-dimensional compressible barotropic flows for all adiabatic number $\gamma > 1$ provided that the initial total energy is small.

1 Introduction

We consider the Navier-Stokes equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

for viscous compressible barotropic flows. Here, $t \geq 0$ is time, $x \in \Omega \subset \mathbb{R}^N (N = 2, 3)$ is the spatial coordinate, and $\rho = \rho(x,t)$, $u = (u^1, \cdots, u^N)(x,t)$, and

$$P(\rho) = R\rho^\gamma \ (R > 0, \gamma > 1) \tag{1.2}$$

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are the fluid density, velocity and pressure, respectively. Without loss of generality, it is assumed that $R = 1$. The constant viscosity coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad 2\mu + N\lambda \geq 0. \tag{1.3}$$

Let $\Omega = \mathbb{R}^N$ and we consider the Cauchy problem for (1.1) with (ρ, u) vanishing at infinity (in some weak sense) with given initial data ρ_0 and u_0, as

$$\rho(x,0) = \rho_0(x), \quad \rho u(x,0) = \rho_0 u_0(x), \quad x \in \Omega = \mathbb{R}^N. \tag{1.4}$$

There are huge literatures on the large time existence and behavior of solutions to (1.1). The one-dimensional problem has been studied extensively, see [9, 18, 29, 30] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions are known in [25, 31] in the absence of vacuum and recently, for strong solutions also, in [3–5,19,28] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura-Nishida [24] for initial data close to a non-vacuum equilibrium in some Sobolev space H^s. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from vacuum. Later, Hoff [10, 11, 13] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [22] (see also Feireisl [6, 7]), where the global existence of weak solutions when the exponent γ is suitably large are achieved. The main restriction on initial data is that the initial total energy is finite, so that the density vanishes at far fields, or even has compact support. However, little is known on the structure of such weak solutions, in particular, the regularity and the uniqueness of such weak solutions remain open. This is a subtle issue, as Xin [32] showed that in the case that the initial density has compact support, any smooth solution in $C^1([0,T] : H^s(\mathbb{R}^d))(s > [d/2]+2)$ to the Cauchy problem of the full compressible Navier-Stokes system without heat conduction blows up in finite time for any space dimension $d \geq 1$, and the same holds for the isentropic case (1.1), at least in one-dimension. The assumptions of [32] that the initial density has compact support and that the smooth solution has finite energy are removed recently by Xin-Yan [33] for a large class of initial data containing vacuum. However, this blow-up theory does not apply to the isentropic flows in general, at least in the case of $\mathbb{R}^3$. Indeed, very recently, for the case that the initial density is allowed to vanish and even has compact support, Huang-Li-Xin [16] established the quite surprising global existence and uniqueness of classical solutions with constant state as far field which could be either vacuum or non-vacuum to (1.1)-(1.4) in three-dimensional space with smooth initial data which are of small total energy but possibly large oscillations. Moreover, it was also showed in [16] that for any $p > 2$,

$$\lim_{t \to \infty} \left(\|P(\rho) - P(\tilde{\rho})\|_{L^p(\mathbb{R}^3)} + \|\nabla u\|_{L^2(\mathbb{R}^3)} \right) = 0, \tag{1.5}$$

where $\tilde{\rho}$ is the constant far field density. This not only generalizes the classical results of Matsumura-Nishida [24], but also yields the regularity and uniqueness of the weak solutions of Lions and Feireisl [6, 7, 22] with initial data of small total energy. Then a natural question arises whether the theory of Huang-Li-Xin [16] remains valid for the case of $\mathbb{R}^2$. This is interesting partially due to the following reasons: First, a positive answer would yield immediately the regularity and uniqueness of weak solutions of Lions-Feireisl with small initial total energy whose existence has been proved for all

$\gamma > 1$, see [6, 7]. Second, this question may be subtle due to the recent blow-up result in [23] where it is shown that non-trivial two-dimensional spherically symmetric solution in $C^1([0,T];H^s(\mathbb{R}^2)(s>2))$ with initial compactly supported density blows up in finite time. Technically, it is not easy to modify the three-dimensional analysis of [16] to the two-dimensional case with initial density containing vacuum since the analysis of [16] depends crucially on the a priori L^6-bound on the velocity. For two-dimensional problems, only in the case that the far field density is away from vacuum, the techniques of [16] can be modified directly since at this case, for any $p \in [2,\infty)$, the L^p-norm of a function u can be bounded by $\|\rho^{1/2}u\|_{L^2}$ and $\|\nabla u\|_{L^2}$, and the similar results can be obtained ([23]). However, when the far field density is vacuum, it seems difficult to bound the L^p-norm of u by $\|\rho^{1/2}u\|_{L^2}$ and $\|\nabla u\|_{L^2}$ for any $p \geq 1$, so the global existence and large time behavior of strong or classical solutions to the Cauchy problem are much subtle and remain open. Therefore, the main aim of this paper is to study the global existence and large time behavior of strong or classical solutions to (1.1)-(1.4) in some homogeneous Sobolev spaces in two-dimensional space with vacuum as far field density, and at the same time to investigate the decay rates of the pressure and the gradient of velocity in both two and three dimensional spaces provided the initial energy is suitably small, which turn out to be one of the keys for the two-dimensional global well-posedness theory.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For $R > 0$ and $\Omega = \mathbb{R}^N (N = 2,3)$, set

$$B_R \triangleq \{x \in \Omega |\ |x| < R\}, \quad \int f dx \triangleq \int_\Omega f dx.$$

Moreover, for $1 \leq r \leq \infty, k \geq 1$, and $\beta > 0$, the standard homogeneous and inhomogeneous Sobolev spaces are defined as follows:

$$\begin{cases} L^r = L^r(\Omega), \quad D^{k,r} = D^{k,r}(\Omega) = \{v \in L^1_{\mathrm{loc}}(\Omega) | \nabla^k v \in L^r(\Omega)\}, \\ D^1 = D^{1,2}, \quad W^{k,r} = W^{k,r}(\Omega), \quad H^k = W^{k,2}, \\ \dot{H}^\beta = \left\{ f : \Omega \to \mathbb{R} \left| \|f\|^2_{\dot{H}^\beta} = \int |\xi|^{2\beta} |\hat{f}(\xi)|^2 d\xi < \infty \right. \right\}, \end{cases}$$

where $\hat{f}$ is the Fourier transform of f. Next, we also give the definition of strong solutions as follows:

Definition 1.1 *If all derivatives involved in* (1.1) *for* (ρ, u) *are regular distributions, and equations* (1.1) *hold almost everywhere in* $\Omega \times (0,T)$, *then* (ρ, u) *is called a strong solution to* (1.1).

For $\Omega = \mathbb{R}^N (N = 2,3)$, the initial total energy is defined as:

$$C_0 = \int_\Omega \left(\frac{1}{2}\rho_0 |u_0|^2 + \frac{1}{\gamma - 1} P(\rho_0) \right) dx.$$

We consider first the two-dimensional case, that is, $\Omega = \mathbb{R}^2$. Without loss of generality, assume that the initial density ρ_0 satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1, \tag{1.6}$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2}\int \rho_0 dx = \frac{1}{2}. \tag{1.7}$$

We can now state our first main result, Theorem 1.1, concerning the global existence of strong solutions.

Theorem 1.1 *Let $\Omega = \mathbb{R}^2$. In addition to (1.6) and (1.7), suppose that the initial data (ρ_0, u_0) satisfy for any given numbers $M > 0$, $\bar\rho \geq 1$, $a > 1, q > 2$, and $\beta \in (0,1]$,*

$$0 \leq \rho_0 \leq \bar\rho, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad u_0 \in \dot{H}^\beta \cap D^1, \quad \rho_0^{1/2} u_0 \in L^2, \tag{1.8}$$

and

$$\|u_0\|_{\dot{H}^\beta} + \|\rho_0 \bar{x}^a\|_{L^1} \leq M, \tag{1.9}$$

where

$$\bar{x} \triangleq (e + |x|^2)^{1/2} \log^2(e + |x|^2). \tag{1.10}$$

Then there exists a positive constant ε depending on $\mu, \lambda, \gamma, a, \bar\rho, \beta, N_0$, and M such that if

$$C_0 \leq \varepsilon, \tag{1.11}$$

the problem (1.1)-(1.4) *has a unique global strong solution (ρ, u) satisfying for any $0 < T < \infty$,*

$$0 \leq \rho(x,t) \leq 2\bar\rho, \quad (x,t) \in \mathbb{R}^2 \times [0,T], \tag{1.12}$$

$$\begin{cases} \rho \in C([0,T]; L^1 \cap H^1 \cap W^{1,q}), \\ \bar{x}^a \rho \in L^\infty(0,T; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho} u,\ \nabla u,\ \bar{x}^{-1} u,\ \sqrt{t}\sqrt{\rho} u_t \in L^\infty(0,T; L^2), \\ \nabla u \in L^2(0,T; H^1) \cap L^{(q+1)/q}(0,T; W^{1,q}), \\ \sqrt{t} \nabla u \in L^2(0,T; W^{1,q}), \\ \sqrt{\rho} u_t,\ \sqrt{t} \nabla u_t,\ \sqrt{t} \bar{x}^{-1} u_t \in L^2(\mathbb{R}^2 \times (0,T)), \end{cases} \tag{1.13}$$

and

$$\inf_{0 \leq t \leq T} \int_{B_{N_1(1+t)\log^\alpha(e+t)}} \rho(x,t) dx \geq \frac{1}{4}, \tag{1.14}$$

for any $\alpha > 1$ and some positive constant N_1 depending only on α, N_0, and M. Moreover, (ρ, u) has the following decay rates, that is, for $t \geq 1$,

$$\begin{cases} \|\nabla u(\cdot,t)\|_{L^p} \leq C(p) t^{-1+1/p}, \ \text{for } p \in [2,\infty), \\ \|P(\cdot,t)\|_{L^r} \leq C(r) t^{-1+1/r}, \ \text{for } r \in (1,\infty), \\ \|\nabla\omega(\cdot,t)\|_{L^2} + \|\nabla F(\cdot,t)\|_{L^2} \leq C t^{-1}, \end{cases} \tag{1.15}$$

where

$$\omega \triangleq \partial_1 u^2 - \partial_2 u^1, \quad F \triangleq (2\mu + \lambda)\mathrm{div} u - P, \tag{1.16}$$

are respectively the vorticity and the effective viscous flux, and $C(\alpha)$ depends on α besides $\mu, \lambda, \gamma, a, \bar\rho, \beta$, N_0, and M.

Remark 1.1 *It should be noted here that the decay rate estimates* (1.15) *combined with the estimate on upper bound of the expansion rate of the essential support of the density* (1.14) *play a crucial role in deriving the global existence of strong and classical solutions to the two-dimensional problem* (1.1)-(1.4). *This is in contrast to the three-dimensional case ([16]) where the global existence of classical solutions to* (1.1)-(1.4) *was achieved without any bounds on the decay rates of the solutions partially due to the a priori L^6-bounds on the velocity field. As will be seen in the proof, the key observation is the decay with a rate for the mean-square norm of the pressure in* (1.15).

If the initial data (ρ_0, u_0) satisfy some additional regularity and compatibility conditions, the global strong solutions obtained by Theorem 1.1 become classical ones, that is,

Theorem 1.2 *Let $\Omega = \mathbb{R}^2$. In addition to the assumptions in Theorem 1.1, assume further that (ρ_0, u_0) satisfy*

$$\nabla^2\rho_0,\, \nabla^2 P(\rho_0) \in L^2 \cap L^q, \quad \bar{x}^{\delta_0}\nabla^2\rho_0,\, \bar{x}^{\delta_0}\nabla^2 P(\rho_0),\, \nabla^2 u_0 \in L^2, \tag{1.17}$$

for some constant $\delta_0 \in (0,1)$, and the following compatibility condition:

$$-\mu\triangle u_0 - (\mu+\lambda)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g, \tag{1.18}$$

with some $g \in L^2$. Then, in addition to (1.12)-(1.15)*, the strong solution (ρ, u) obtained by Theorem 1.1 satisfies for any $0 < T < \infty$,*

$$\begin{cases} \nabla^2\rho,\ \nabla^2 P(\rho) \in C([0,T]; L^2 \cap L^q), \\ \bar{x}^{\delta_0}\nabla^2\rho,\ \bar{x}^{\delta_0}\nabla^2 P(\rho),\, \nabla^2 u \in L^\infty(0,T; L^2), \\ \sqrt{\rho}u_t,\, \sqrt{t}\nabla u_t,\, \sqrt{t}\bar{x}^{-1}u_t,\, t\sqrt{\rho}u_{tt},\, t\nabla^2 u_t \in L^\infty(0,T;L^2), \\ t\nabla^3 u \in L^\infty(0,T; L^2 \cap L^q), \\ \nabla u_t,\, \bar{x}^{-1}u_t,\, t\nabla u_{tt},\, t\bar{x}^{-1}u_{tt} \in L^2(0,T;L^2), \\ t\nabla^2(\rho u) \in L^\infty(0,T; L^{(q+2)/2}). \end{cases} \tag{1.19}$$

Remark 1.2 *The solution obtained in Theorem 1.2 becomes a classical one for positive time ([19]). Although it has small energy, yet whose oscillations could be arbitrarily large. In particular, both interior and far field vacuum are allowed. There is no requirement on the size of the set of vacuum states. Therefore, the initial density may have compact support. Moreover, by the strong-weak uniqueness theorem of Lions [22], Theorem 1.1 and Theorem 1.2 can be regarded as uniqueness and regularity theory of Lions-Feireisl's weak solutions with small initial energy, whose existence has been proved for all $\gamma > 1$ in [6, 22].*

Remark 1.3 *It is worth noting that the conclusions in Theorem 1.2 and Theorem 1.1 are somewhat surprising since for the isentropic compressible Navier-Stokes equations* (1.1)*, any non-trivial two-dimensional spherically symmetric solution $(\rho, u) \in C^1([0,T], H^s)(s > 2)$ with initial compact supported density blows up in finite time ([23]). Indeed, as in [32], the key point of [23] to prove the blowup phenomena is based on the fact that the support of the density will not grow in time in the space $C^([0,T]; H^m)$. However, in the current case, though the density $\rho \in C([0,T]; H^2)$, yet the velocity u satisfies only $\nabla u \in C((0,T]; H^k)$. Note that the function $u \in \{\nabla u \in H^k\}$*

decays much slower for large values of the spatial variable x than $u \in H^{k+1}$. Therefore, it seems that it is the slow decay of the velocity field for large values of the spatial variable x that leads to the global existence of smooth solutions. Unfortunately, such an argument cannot be valid for the full compressible Navier-Stokes system since the blow-up results of Xin-Yan in [33] work for any classical solutions with compactly supported initial density.

For the three-dimensional case, that is, $\Omega = \mathbb{R}^3$, we have the following results concerning the decay properties of the global classical solutions whose existence is essentially due to [16].

Theorem 1.3 *Let $\Omega = \mathbb{R}^3$. For given numbers $M > 0$, $\bar\rho \ge 1$, $\beta \in (1/2, 1]$, and $q \in (3,6)$, suppose that the initial data (ρ_0, u_0) satisfy*

$$\rho_0,\ P(\rho_0) \in H^2 \cap W^{2,q}, \quad P(\rho_0),\ \rho_0|u_0|^2 \in L^1, \quad u_0 \in \dot H^{\beta}, \quad \nabla u_0 \in H^1, \tag{1.20}$$

$$0 \le \rho_0 \le \bar\rho, \quad \|u_0\|_{\dot H^{\beta}} \le M, \tag{1.21}$$

and the compatibility condition

$$-\mu \triangle u_0 - (\mu+\lambda)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g, \tag{1.22}$$

for some $g \in L^2$. Moreover, if $\gamma > 3/2$, assume that

$$\rho_0 \in L^1. \tag{1.23}$$

Then there exists a positive constant ε depending on $\mu, \lambda, \gamma, \bar\rho, \beta$, and M such that if

$$C_0 \le \varepsilon, \tag{1.24}$$

the Cauchy problem (1.1)-(1.4) has a unique global classical solution (ρ, u) in $\mathbb{R}^3 \times (0,\infty)$ satisfying for any $0 < \tau < T < \infty$,

$$0 \le \rho(x,t) \le 2\bar\rho, \quad x \in \mathbb{R}^3,\ t \ge 0, \tag{1.25}$$

$$\begin{cases} \rho \in C([0,T]; L^{3/2} \cap H^2 \cap W^{2,q}), \\ P \in C([0,T]; L^1 \cap H^2 \cap W^{2,q}), \quad u \in L^\infty(0,T; L^6), \\ \nabla u \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2) \cap L^\infty(\tau,T; H^2 \cap W^{2,q}), \\ \nabla u_t \in L^2(0,T; L^2) \cap L^\infty(\tau,T; H^1) \cap H^1(\tau,T; L^2). \end{cases} \tag{1.26}$$

Moreover, for $r \in (1,\infty)$, there exist positive constants $C(r)$ and C depending on $\mu, \lambda, \gamma, \bar\rho, \beta$, and M such that for $t \ge 1$,

$$\begin{cases} \|\nabla u(\cdot,t)\|_{L^p} \le C t^{-1+1/p}, \ \text{for } p \in [2,6], \\ \|P(\cdot,t)\|_{L^r} \le C(r) t^{-1+1/r}, \ \text{for } r \in (1,\infty), \\ \|\nabla(\nabla \times u)(\cdot,t)\|_{L^2} + \|\nabla((2\mu+\lambda)\mathrm{div} u - P)(\cdot,t)\|_{L^2} \le C t^{-1}, \end{cases} \tag{1.27}$$

where if $\gamma > 3/2$, $C(r)$ and C both depend on $\|\rho_0\|_{L^1(\mathbb{R}^3)}$ also.

Remark 1.4 *It should be pointed out that the large time asymptotic decay with rates of the global strong or classical solutions,* (1.15) *and* (1.27)*, are completely new for the multi-dimensional compressible Navier-Stokes equations* (1.1) *in the presence of vacuum. They show in particular that the L^2-norm of both the pressure and the gradient of the velocity decay in time with a rate $t^{-1/2}$, and the gradient of the vorticity and the effective viscous flux decay faster than themselves. However, whether the second derivatives of the velocity field decay or not remains open. This is an interesting problem and left for the future.*

We now make some comments on the analysis of this paper. Note that for initial data in the class satisfying (1.8), (1.9), (1.17), and (1.18) except $u_0 \in \dot{H}^\beta$, the local existence and uniqueness of classical solutions to the Cauchy problem, (1.1)-(1.4), have been established recently in [19]. Thus, to extend the classical solution globally in time, one needs some global a priori estimates on smooth solutions to (1.1)-(1.4) in suitable higher norms. It turns out that as in the three-dimensional case [16], the key issue here is to derive both the time-independent upper bound for the density and the time-depending higher norm estimates of the smooth solution (ρ, u), so some basic ideas used in [16] will be adapted here, yet new difficulties arises in the two-dimensional case. Indeed, the analysis in [16] relies heavily on the basic fact that, for the three-dimensional case, the L^6-norm of $v \in D^1(\mathbb{R}^3)$ can be bounded by the L^2-norm of the gradient of v which fails for $v \in D^1(\mathbb{R}^2)$. In fact, for two-dimensional case, some of the main new difficulties are due to the appearance of vacuum at far field and the lack of integrability of the velocity and its material derivatives in the whole two-dimensional space. To overcome these difficulties, first, using the L^1-integrability of the density, we observe that the L^2-norm in both space and time of the pressure is time-independent (see (3.18)). This is crucial to show that the H^1-norm of the effective viscous flux decays at the rate of $t^{-1/2}$ for large time (see (3.61)) which plays a key role in obtaining the decay property of the L^∞-norm of the effective viscous flux. Then, after some careful estimates of the expansion rates of the essential support of the density (see (3.39)), we succeed in obtaining that, for large time, the L^p-norm of the gradient of the effective viscous flux (see (1.16) for the definition) can be bounded by the product of $(1+t)^5$ and some function $g(t)$ whose temporal L^2-norm is independent of time (see (3.59)). Based on these key ingredients, we are able to obtain the desired estimates on $L^1(0, \min\{1,T\}; L^\infty(\mathbb{R}^2))$-norm and the time-independent ones on $L^4(\min\{1,T\}, T; L^\infty(\mathbb{R}^2))$-norm of the effective viscous flux (see (3.62)). Then, motivated by [20], we deduce from these estimates and Zlotnik's inequality (see Lemma 2.6) that the density admits a time-uniform upper bound which is the key for global estimates of classical solutions. The next main step is to bound the gradients of the density and the velocity. Similar to [14–16], such bounds can be obtained by solving a logarithm Gronwall inequality based on a Beale-Kato-Majda type inequality (see Lemma 2.7) and the a priori estimates we have just derived, and moreover, such a derivation yields simultaneously also the bound for $L^1(0,T; L^\infty(\mathbb{R}^2))$-norm of the gradient of the velocity, see Lemma 4.1 and its proof. Finally, with these a priori estimates on the gradients of the density and the velocity at hand, one can estimate the higher order derivatives by using the same arguments as in [14, 19] to obtain the desired results.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 and 4 are devoted to deriving the necessary a priori estimates on classical solutions which are needed to extend the local solution to all time. Then finally, the main results, Theorems 1.1-1.3, are proved in Section 5.

2 Preliminaries

In this section, for $\Omega = \mathbb{R}^2$, we will recall some known facts and elementary inequalities which will be used frequently later.

We begin with the local existence of strong and classical solutions whose proof can be found in [19].

Lemma 2.1 *Let $\Omega = \mathbb{R}^2$. Assume that (ρ_0, u_0) satisfies (1.8) except $u_0 \in \dot{H}^\beta$. Then there exist a small time $T > 0$ and a unique strong solution (ρ, u) to the problem (1.1)-(1.4) in $\mathbb{R}^2 \times (0, T)$ satisfying (1.13) and (1.14). Moreover, if (ρ_0, u_0) satisfies (1.17) and (1.18) besides (1.8), (ρ, u) satisfies (1.19) also.*

Next, the following well-known Gagliardo-Nirenberg inequality (see [26]) will be used later.

Lemma 2.2 (Gagliardo-Nirenberg) *For $p \in [2, \infty), q \in (1, \infty)$, and $r \in (2, \infty)$, there exists some generic constant $C > 0$ which may depend on p, q, and r such that for $f \in H^1(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2)$, we have*

$$\|f\|_{L^p(\mathbb{R}^2)}^p \le C\|f\|_{L^2(\mathbb{R}^2)}^2 \|\nabla f\|_{L^2(\mathbb{R}^2)}^{p-2}, \tag{2.1}$$

$$\|g\|_{C\left(\overline{\mathbb{R}^2}\right)} \le C\|g\|_{L^q(\mathbb{R}^2)}^{q(r-2)/(2r+q(r-2))} \|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+q(r-2))}. \tag{2.2}$$

The following weighted L^p bounds for elements of the Hilbert space $D^1(\mathbb{R}^2)$ can be found in [21, Theorem B.1].

Lemma 2.3 *For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant C such that we have for all $v \in D^{1,2}(\mathbb{R}^2)$,*

$$\left(\int_{\mathbb{R}^2} \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} dx \right)^{1/m} \le C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.3}$$

The combination of Lemma 2.3 with the Poincaré inequality yields

Lemma 2.4 *For $\bar{x}$ as in (1.10), suppose that $\rho \in L^\infty(\mathbb{R}^2)$ is a function such that*

$$0 \le \rho \le M_1, \quad M_2 \le \int_{B_{N_*}} \rho dx, \quad \rho \bar{x}^\alpha \in L^1(\mathbb{R}^2), \tag{2.4}$$

for $N_* \ge 1$ and positive constants M_1, M_2, and α. Then, for $r \in [2, \infty)$, there exists a positive constant C depending only on M_1, M_2, α, and r such that

$$\left(\int_{\mathbb{R}^2} \rho |v|^r dx \right)^{1/r} \le C N_*^3 (1 + \|\rho \bar{x}^\alpha\|_{L^1(\mathbb{R}^2)}) \left(\|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)} \right), \tag{2.5}$$

for each $v \in \left\{ v \in D^1(\mathbb{R}^2) \middle| \, \rho^{1/2} v \in L^2(\mathbb{R}^2) \right\}$.

Proof. First, for $f \in L^1(B_{N_*})$, denote the average of f over B_{N_*} by

$$f_{B_{N_*}} \triangleq \frac{1}{|B_{N_*}|} \int_{B_{N_*}} f(x)dx.$$

It then follows from (2.4) that

$$\begin{aligned}
\left|\rho_{B_{N_*}} v_{B_{N_*}}\right| =& \left| \frac{1}{|B_{N_*}|} \int_{B_{N_*}} \left(\rho_{B_{N_*}} - \rho\right)(v - v_{B_{N_*}})dx + \frac{1}{|B_{N_*}|} \int_{B_{N_*}} \rho v dx \right| \\
\leq & 2M_1 N_*^{-1} \|v - v_{B_{N_*}}\|_{L^2(B_{N_*})} + M_1^{1/2} N_*^{-1} \|\rho^{1/2} v\|_{L^2(B_{N_*})} \\
\leq & 8M_1 \|\nabla v\|_{L^2(B_{N_*})} + M_1^{1/2} N_*^{-1} \|\rho^{1/2} v\|_{L^2(B_{N_*})},
\end{aligned} \tag{2.6}$$

where in the last inequality one has used the following Poincaré inequality ([8, (7.45)])

$$\|v - v_{B_{N^*}}\|_{L^2(B_{N^*})} \leq 4N^* \|\nabla v\|_{L^2(B_{N^*})}. \tag{2.7}$$

Then, it follows from (2.6) and (2.4) that

$$\left|v_{B_{N_*}}\right| \leq C(M_1, M_2) N_*^2 \|\nabla v\|_{L^2(B_{N_*})} + C(M_1, M_2) N_* \|\rho^{1/2} v\|_{L^2(B_{N_*})},$$

which together with (2.7) leads to

$$\begin{aligned}
\int_{B_{N_*}} |v|^2 dx \leq & 2 \int_{B_{N_*}} |v - v_{B_{N_*}}|^2 dx + 2|B_{N_*}||v_{B_{N_*}}|^2 \\
\leq & C(M_1, M_2) N_*^6 \|\nabla v\|_{L^2(B_{N_*})}^2 + C(M_1, M_2) N_*^4 \|\rho^{1/2} v\|_{L^2(B_{N_*})}^2.
\end{aligned} \tag{2.8}$$

Finally, it follows from Holder's inequality, (2.3), (2.8), and (2.4) that for $r \in [2, \infty)$ and $\sigma = 4/(4+\alpha) \in (0,1)$,

$$\begin{aligned}
\int_{\mathbb{R}^2} \rho |v|^r dx \leq & \|(\rho \bar{x}^\alpha)^\sigma\|_{L^{1/\sigma}(\mathbb{R}^2)} \||v|^r \bar{x}^{-\alpha\sigma}\|_{L^{4/(\alpha\sigma)}(\mathbb{R}^2)} \|\rho\|_{L^\infty(\mathbb{R}^2)}^{1-\sigma} \\
\leq & C\left(1 + \|\rho \bar{x}^\alpha\|_{L^1(\mathbb{R}^2)}\right) \left(N_*^3 \left(\|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)}\right)\right)^r,
\end{aligned}$$

which gives (2.5). This completes the proof of Lemma 2.4.

Next, for $\nabla^\perp \triangleq (-\partial_2, \partial_1)$, denoting the material derivative of f by $\dot{f} \triangleq f_t + u \cdot \nabla f$, we state some elementary estimates which follow from (2.1) and the standard L^p-estimate for the following elliptic system derived from the momentum equations in (1.1):

$$\triangle F = \operatorname{div}(\rho \dot{u}), \quad \mu \triangle \omega = \nabla^\perp \cdot (\rho \dot{u}), \tag{2.9}$$

where F and ω are as in (1.16).

Lemma 2.5 *Let $\Omega = \mathbb{R}^2$ and (ρ, u) be a smooth solution of (1.1). Then for $p \geq 2$ there exists a positive constant C depending only on p, μ, and λ such that*

$$\|\nabla F\|_{L^p(\mathbb{R}^2)} + \|\nabla \omega\|_{L^p(\mathbb{R}^2)} \leq C \|\rho \dot{u}\|_{L^p(\mathbb{R}^2)}, \tag{2.10}$$

$$\|F\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)} \leq C \|\rho \dot{u}\|_{L^2(\mathbb{R}^2)}^{1-2/p} \left(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)}\right)^{2/p}, \tag{2.11}$$

$$\|\nabla u\|_{L^p(\mathbb{R}^2)} \leq C \|\rho \dot{u}\|_{L^2(\mathbb{R}^2)}^{1-2/p} \left(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)}\right)^{2/p} + C\|P\|_{L^p(\mathbb{R}^2)}. \tag{2.12}$$

Proof. On the one hand, the standard L^p-estimate for the elliptic system (2.9) yields (2.10) directly, which, together with (2.1) and (1.16), gives (2.11). On the other hand, since $-\Delta u = -\nabla \mathrm{div} u - \nabla^{\perp}\omega$, we have

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla \mathrm{div} u - \nabla(-\Delta)^{-1}\nabla^{\perp}\omega. \tag{2.13}$$

Thus applying the standard L^p-estimate to (2.13) shows

$$\begin{aligned}\|\nabla u\|_{L^p(\mathbb{R}^2)} &\le C(p)(\|\mathrm{div} u\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)})\\ &\le C(p)\|F\|_{L^p(\mathbb{R}^2)} + C(p)\|\omega\|_{L^p(\mathbb{R}^2)} + C(p)\|P\|_{L^p(\mathbb{R}^2)},\end{aligned}$$

which, along with (2.11), gives (2.12). The proof of Lemma 2.5 is completed.

Next, to get the uniform (in time) upper bound of the density ρ, we need the following Zlotnik inequality.

Lemma 2.6 ([34]) *Let the function y satisfy*

$$y'(t) = g(y) + b'(t) \text{ on } [0,T], \quad y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.14}$$

for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\left\{y^0, \overline{\zeta}\right\} + N_0 < \infty \text{ on } [0,T],$$

where $\overline{\zeta}$ is a constant such that

$$g(\zeta) \le -N_1 \quad \text{for} \quad \zeta \ge \overline{\zeta}. \tag{2.15}$$

Finally, the following Beale-Kato-Majda type inequality, which was proved in [1, 17] when $\mathrm{div} u \equiv 0$, will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^q} (q > 2)$.

Lemma 2.7 *For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^2) \cap D^{1,q}(\mathbb{R}^2)$,*

$$\begin{aligned}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \le\, & C\left(\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)}\right)\log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^2)})\\ &+ C\|\nabla u\|_{L^2(\mathbb{R}^2)} + C.\end{aligned}$$

3 A priori estimates(I): lower order estimates

In this section, for $\Omega = \mathbb{R}^2$, we will establish some necessary a priori bounds for smooth solutions to the Cauchy problem (1.1)-(1.4) to extend the local strong and classical solutions guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and (ρ, u) be the smooth solution to (1.1)-(1.4) on $\mathbb{R}^2 \times (0,T]$ with smooth initial data (ρ_0, u_0) satisfying (1.8) and (1.9).

Set $\sigma(t) \triangleq \min\{1, t\}$. Define

$$A_1(T) \triangleq \sup_{0\le t\le T}\left(\sigma\|\nabla u\|_{L^2}^2\right) + \int_0^T \sigma \int \rho|\dot{u}|^2 dxdt, \tag{3.1}$$

and

$$A_2(T) \triangleq \sup_{0\le t\le T} \sigma^2 \int \rho|\dot{u}|^2 dx + \int_0^T \int \sigma^2 |\nabla \dot{u}|^2 dxdt. \tag{3.2}$$

We have the following key a priori estimates on (ρ, u).

Proposition 3.1 *Under the conditions of Theorem 1.1, there exists some positive constant* ε *depending on* $\mu, \lambda, \gamma, a, \bar\rho, \beta,\ N_0,$ *and* M *such that if* (ρ, u) *is a smooth solution of (1.1)-(1.4) on* $\mathbb{R}^2 \times (0, T]$ *satisfying*

$$\sup_{\mathbb{R}^2\times[0,T]} \rho \le 2\bar\rho, \quad A_1(T) + A_2(T) \le 2C_0^{1/2}, \tag{3.3}$$

the following estimates hold

$$\sup_{\mathbb{R}^2\times[0,T]} \rho \le 7\bar\rho/4, \quad A_1(T) + A_2(T) + \int_0^T \sigma\|P\|_{L^2}^2 dt \le C_0^{1/2}, \tag{3.4}$$

provided $C_0 \le \varepsilon$.

The proof of Proposition 3.1 will be postponed to the end of this section.

In the following, we will use the convention that C denotes a generic positive constant depending on μ, λ, γ, a, $\bar\rho$, β, N_0, and M, and use $C(\alpha)$ to emphasize that C depends on α.

We begin with the following standard energy estimate for (ρ, u) and preliminary L^2 bounds for ∇u and $\rho\dot u$.

Lemma 3.2 *Let* (ρ, u) *be a smooth solution of (1.1)-(1.4) on* $\mathbb{R}^2 \times (0, T]$. *Then there is a positive constant* C *depending only on* μ, λ, *and* γ *such that*

$$\sup_{0\le t\le T} \int \left(\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}P\right) dx + \mu \int_0^T \int |\nabla u|^2 dxdt \le C_0, \tag{3.5}$$

$$A_1(T) \le CC_0 + C \sup_{0\le t\le T} \|P\|_{L^2}^2 + C \int_0^T \sigma \int \left(|\nabla u|^3 + P|\nabla u|^2\right) dxdt, \tag{3.6}$$

and

$$A_2(T) \le CA_1(T) + C \int_0^T \sigma^2 \left(\|\nabla u\|_{L^4}^4 + \|P\|_{L^4}^4\right) dt. \tag{3.7}$$

Proof. First, the standard energy inequality reads:

$$\sup_{0\le t\le T} \int \left(\frac{1}{2}\rho|u|^2 + \frac{P}{\gamma-1}\right) dx + \int_0^T \int \left(\mu|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}u)^2\right) dxdt \le C_0,$$

which together with (1.3) shows (3.5).

Next, multiplying $(1.1)_2$ by $\dot u$ and then integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\int \rho|\dot u|^2 dx = -\int \dot u \cdot \nabla P dx + \mu \int \triangle u \cdot \dot u dx + (\mu+\lambda) \int \nabla \mathrm{div}u \cdot \dot u dx. \tag{3.8}$$

Since P satisfies

$$P_t + u\cdot\nabla P + \gamma P \mathrm{div}u = 0, \tag{3.9}$$

integration by parts yields that

$$\begin{aligned} -\int \dot u\cdot\nabla P dx &= \int ((\mathrm{div}u)_t P - (u\cdot\nabla u)\cdot\nabla P) dx \\ &= \left(\int \mathrm{div}u P dx\right)_t + \int \left((\gamma-1)P(\mathrm{div}u)^2 + P\partial_i u_j \partial_j u_i\right) dx \\ &\le \left(\int \mathrm{div}u P dx\right)_t + C\int P|\nabla u|^2 dx. \end{aligned} \tag{3.10}$$

Integration by parts also implies that

$$
\begin{aligned}
\mu \int \triangle u \cdot \dot{u} dx &= -\frac{\mu}{2} \left(\|\nabla u\|_{L^2}^2 \right)_t - \mu \int \partial_i u_j \partial_i (u_k \partial_k u_j) dx \\
&\le -\frac{\mu}{2} \left(\|\nabla u\|_{L^2}^2 \right)_t + C \int |\nabla u|^3 dx,
\end{aligned} \tag{3.11}
$$

and that

$$
\begin{aligned}
(\mu+\lambda) \int \nabla \mathrm{div} u \cdot \dot{u} dx &= -\frac{\lambda+\mu}{2} \left(\|\mathrm{div} u\|_{L^2}^2 \right)_t - (\lambda+\mu) \int \mathrm{div} u \mathrm{div}(u \cdot \nabla u) dx \\
&\le -\frac{\lambda+\mu}{2} \left(\|\mathrm{div} u\|_{L^2}^2 \right)_t + C \int |\nabla u|^3 dx.
\end{aligned} \tag{3.12}
$$

Putting (3.10)-(3.12) into (3.8) leads to

$$
B'(t) + \int \rho |\dot{u}|^2 dx \le C \int P |\nabla u|^2 dx + C \|\nabla u\|_{L^3}^3, \tag{3.13}
$$

where

$$
B(t) \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda+\mu}{2} \|\mathrm{div} u\|_{L^2}^2 - \int \mathrm{div} u P dx \tag{3.14}
$$

satisfies

$$
\frac{\mu}{4} \|\nabla u\|_{L^2}^2 - C \|P\|_{L^2}^2 \le B(t) \le C \|\nabla u\|_{L^2}^2 + C \|P\|_{L^2}^2. \tag{3.15}
$$

Then, integrating (3.13) multiplied by σ over $(0,T)$ and using (3.15) and (3.5) yield (3.6) directly.

Finally, to prove (3.7), we will use the basic estimates of $\dot{u}$ due to Hoff [10]. Operating $\partial/\partial t + \mathrm{div}(u \cdot)$ to $(1.1)_2^j$, one gets by some simple calculations that

$$
\begin{aligned}
&\rho (\dot{u}^j)_t + \rho u \cdot \nabla \dot{u}^j - \mu \triangle \dot{u}^j - (\mu+\lambda) \partial_j (\mathrm{div} \dot{u}) \\
&= \mu \partial_i (-\partial_i u \cdot \nabla u^j + \mathrm{div} u \partial_i u^j) - \mu \mathrm{div}(\partial_i u \partial_i u^j) \\
&\quad - (\mu+\lambda) \partial_j \left(\partial_i u \cdot \nabla u^i - (\mathrm{div} u)^2 \right) - (\mu+\lambda) \mathrm{div}(\partial_j u \mathrm{div} u) \\
&\quad + (\gamma-1) \partial_j (P \mathrm{div} u) + \mathrm{div}(P \partial_j u).
\end{aligned} \tag{3.16}
$$

Multiplying (3.16) by $\dot{u}$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$
\left(\int \rho |\dot{u}|^2 dx \right)_t + \mu \int |\nabla \dot{u}|^2 dx \le C \|\nabla u\|_{L^4}^4 + C \|P\|_{L^4}^4, \tag{3.17}
$$

which multiplied by σ^2 gives (3.7) and completes the proof of Lemma 3.2.

Remark 3.1 *It is easy to check that the estimates* (3.13) *and* (3.17) *also hold for* $\Omega = \mathbb{R}^3$.

Next, we give a key observation that pressure decays in time.

Lemma 3.3 *Let (ρ, u) be a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2 \times (0,T]$ satisfying (3.3). Then there exists a positive constant $C(\bar{\rho})$ depending only on μ, λ, γ, and $\bar{\rho}$ such that*

$$
A_1(T) + A_2(T) + \int_0^T \sigma \|P\|_{L^2}^2 dt \le C(\bar{\rho}) C_0. \tag{3.18}
$$

Proof. First, it follows from (2.12), (3.5), and (3.3) that

$$
\begin{aligned}
&\int_0^T \sigma^2 \left(\|\nabla u\|_{L^4}^4 + \|P\|_{L^4}^4\right) dt \\
&\le C \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \left(\sigma \|\nabla u\|_{L^2}^2 + \sigma \|P\|_{L^2}^2\right) dt + C \int_0^T \sigma^2 \|P\|_{L^4}^4 dt \\
&\le C(\bar\rho)\left(A_1(T) + C_0\right) \int_0^T \sigma \|\rho^{1/2}\dot{u}\|_{L^2}^2 dt + C(\bar\rho) \int_0^T \sigma^2 \|P\|_{L^2}^2 dt.
\end{aligned}
\tag{3.19}
$$

To estimate the last term on the right-hand side of (3.19), noticing that $(1.1)_2$ gives

$$
P = (-\Delta)^{-1}\mathrm{div}(\rho \dot{u}) + (2\mu + \lambda)\mathrm{div} u,
\tag{3.20}
$$

we obtain from Hölder's and Sobolev's inequalities that

$$
\begin{aligned}
\int P^2 dx \le & C\|(-\Delta)^{-1}\mathrm{div}(\rho\dot{u})\|_{L^{4\gamma}} \|P\|_{L^{4\gamma/(4\gamma-1)}} + C\|\nabla u\|_{L^2}\|P\|_{L^2} \\
\le & C\|\rho\dot{u}\|_{L^{4\gamma/(2\gamma+1)}} \|\rho\|_{L^1}^{1/2} \|\rho\|_{L^{2\gamma}}^{\gamma-1/2} + C\|\nabla u\|_{L^2}\|P\|_{L^2} \\
\le & C\|\rho^{1/2}\|_{L^{4\gamma}} \|\rho^{1/2}\dot{u}\|_{L^2} \|\rho\|_{L^1}^{1/2} \|\rho\|_{L^{2\gamma}}^{\gamma-1/2} + C\|\nabla u\|_{L^2}\|P\|_{L^2} \\
\le & C\|P\|_{L^2}\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2}\|P\|_{L^2},
\end{aligned}
$$

where in the last inequality, one has used

$$
\int \rho dx = \int \rho_0 dx = 1,
\tag{3.21}
$$

due to the mass conservation equation $(1.1)_1$. Thus, we arrive at

$$
\|P\|_{L^2} \le C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2},
\tag{3.22}
$$

which, along with (3.6), (3.7), (3.19), (3.5), and (3.3) gives

$$
A_1(T) + A_2(T) \le C(\bar\rho)C_0 + C(\bar\rho) \int_0^T \sigma \|\nabla u\|_{L^3}^3 dt.
\tag{3.23}
$$

Then, on the one hand, one deduces from (2.12), (3.5), and (3.3) that

$$
\begin{aligned}
\int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3 dt \le & C \int_0^{\sigma(T)} \sigma \|\rho^{1/2}\dot{u}\|_{L^2} \left(\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2\right) dt + C(\bar\rho)C_0 \\
\le & C A_2^{1/2}(\sigma(T)) \int_0^{\sigma(T)} \left(\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2\right) dt + C(\bar\rho)C_0 \\
\le & C(\bar\rho)C_0.
\end{aligned}
\tag{3.24}
$$

On the other hand, Hölder's inequality, (3.19), (3.3), and (3.22) imply

$$
\begin{aligned}
\int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3 dt \le & \delta \int_{\sigma(T)}^T \|\nabla u\|_{L^4}^4 dt + C(\delta) \int_{\sigma(T)}^T \|\nabla u\|_{L^2}^2 dt \\
\le & \delta C(\bar\rho) A_1(T) + C(\delta)C(\bar\rho)C_0.
\end{aligned}
\tag{3.25}
$$

Finally, putting (3.24) and (3.25) into (3.23) and choosing δ suitably small lead to

$$
A_1(T) + A_2(T) \le C(\bar\rho)C_0,
$$

which together with (3.22) and (3.5) gives (3.18) and completes the proof of Lemma 3.3.

Next, we derive the rates of decay for ∇u and P, which are essential to obtain the uniform (in time) upper bound of the density for large time.

Lemma 3.4 *For $p \in [2,\infty)$, there exists a positive constant $C(p,\bar\rho)$ depending only on p, μ, λ, γ, and $\bar\rho$ such that, if (ρ,u) is a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2\times(0,T]$ satisfying (3.3), then*

$$\sup_{\sigma(T)\le t\le T}\left(t^{p-1}(\|\nabla u\|_{L^p}^p+\|P\|_{L^p}^p)+t^2\|\rho^{1/2}\dot u\|_{L^2}^2\right)\le C(p,\bar\rho)C_0. \tag{3.26}$$

Proof. First, for $p\ge 2$, multiplying (3.9) by pP^{p-1} and integrating the resulting equality over $\mathbb{R}^2$, one gets after using $\mathrm{div}u=\frac{1}{2\mu+\lambda}(F+P)$ that

$$\begin{aligned}\left(\|P\|_{L^p}^p\right)_t+\frac{p\gamma-1}{2\mu+\lambda}\|P\|_{L^{p+1}}^{p+1}&=-\frac{p\gamma-1}{2\mu+\lambda}\int P^pFdx\\&\le\frac{p\gamma-1}{2(2\mu+\lambda)}\|P\|_{L^{p+1}}^{p+1}+C(p)\|F\|_{L^{p+1}}^{p+1},\end{aligned} \tag{3.27}$$

which together with (2.11) gives

$$\begin{aligned}\frac{2(2\mu+\lambda)}{p\gamma-1}\left(\|P\|_{L^p}^p\right)_t+\|P\|_{L^{p+1}}^{p+1}&\le C(p)\|F\|_{L^{p+1}}^{p+1}\\&\le C(p)\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right)\|\rho\dot u\|_{L^2}^{p-1}.\end{aligned} \tag{3.28}$$

In particular, choosing $p=2$ in (3.28) shows

$$\left(\|P\|_{L^2}^2\right)_t+\frac{2\gamma-1}{2(2\mu+\lambda)}\|P\|_{L^3}^3\le\delta\|\rho^{1/2}\dot u\|_{L^2}^2+C(\delta)\left(\|\nabla u\|_{L^2}^4+\|P\|_{L^2}^4\right). \tag{3.29}$$

Next, it follows from (3.13) and (2.12) that

$$\begin{aligned}B'(t)+\int\rho|\dot u|^2dx&\le C\|P\|_{L^3}^3+C\|\nabla u\|_{L^3}^3\\&\le C_1\|P\|_{L^3}^3+C\|\rho\dot u\|_{L^2}\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right)\\&\le C_1\|P\|_{L^3}^3+\delta\|\rho^{1/2}\dot u\|_{L^2}^2+C(\bar\rho,\delta)\left(\|\nabla u\|_{L^2}^4+\|P\|_{L^2}^4\right).\end{aligned} \tag{3.30}$$

Choosing $C_2\ge 2+2(2\mu+\lambda)(C_1+1)/(2\gamma-1)$ suitably large such that

$$\frac{\mu}{4}\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\le B(t)+C_2\|P\|_{L^2}^2\le C\|\nabla u\|_{L^2}^2+C\|P\|_{L^2}^2, \tag{3.31}$$

adding (3.29) multiplied by C_2 to (3.30), and choosing δ suitably small lead to

$$2\left(B(t)+C_2\|P\|_{L^2}^2\right)'+\int\left(\rho|\dot u|^2+P^3\right)dx\le C\|P\|_{L^2}^4+C\|\nabla u\|_{L^2}^4, \tag{3.32}$$

which multiplied by t, together with Gronwall's inequality, (3.31), (3.18), (3.5), and (3.3) yields

$$\sup_{\sigma(T)\le t\le T}t\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right)+\int_{\sigma(T)}^T t\int\left(\rho|\dot u|^2+P^3\right)dxdt\le C(\bar\rho)C_0. \tag{3.33}$$

Next, multiplying (3.17) by t^2 together with (2.12) gives

$$\begin{aligned} & \left(t^2\int\rho|\dot u|^2dx\right)_t+\mu t^2\int|\nabla\dot u|^2dx \\ & \le 2t\int\rho|\dot u|^2dx+C(\bar\rho)t^2\|\rho\dot u\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right)+\tilde C(\bar\rho)t^2\|P\|_{L^4}^4. \end{aligned} \tag{3.34}$$

Choosing $p=3$ in (3.28) and adding (3.28) multiplied by $(\tilde C+1)t^2$ to (3.34) lead to

$$\begin{aligned} & \left(t^2\int\rho|\dot u|^2dx+\frac{2(2\mu+\lambda)(\tilde C+1)}{3\gamma-1}t^2\|P\|_{L^3}^3\right)_t+\mu t^2\|\nabla\dot u\|_{L^2}^2+t^2\|P\|_{L^4}^4 \\ & \le Ct\int\left(\rho|\dot u|^2+P^3\right)dx+C(\bar\rho)t^2\|\rho^{1/2}\dot u\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right), \end{aligned}$$

which combined with Gronwall's inequality, (3.33), and (3.3) yields

$$\sup_{\sigma(T)\le t\le T}t^2\int\left(\rho|\dot u|^2+P^3\right)dx+\int_{\sigma(T)}^T t^2\left(\|\nabla\dot u\|_{L^2}^2+\|P\|_{L^4}^4\right)dt\le C(\bar\rho)C_0. \tag{3.35}$$

Finally, we claim that for $m=1,2,\cdots$,

$$\sup_{\sigma(T)\le t\le T}t^m\|P\|_{L^{m+1}}^{m+1}+\int_{\sigma(T)}^T t^m\|P\|_{L^{m+2}}^{m+2}dt\le C(m,\bar\rho)C_0, \tag{3.36}$$

which together with (2.12), (3.33), and (3.35) gives (3.26). We shall prove (3.36) by induction. In fact, (3.33) shows that (3.36) holds for $m=1$. Assume that (3.36) holds for $m=n$, that is,

$$\sup_{\sigma(T)\le t\le T}t^n\|P\|_{L^{n+1}}^{n+1}+\int_{\sigma(T)}^T t^n\|P\|_{L^{n+2}}^{n+2}dt\le C(n,\bar\rho)C_0. \tag{3.37}$$

Multiplying (3.28) where $p=n+2$ by t^{n+1}, one obtains after using (3.35)

$$\begin{aligned} & \frac{2(2\mu+\lambda)}{(n+2)\gamma-1}\left(t^{n+1}\|P\|_{L^{n+2}}^{n+2}\right)_t+t^{n+1}\|P\|_{L^{n+3}}^{n+3} \\ & \le C(n,\bar\rho)t^n\|P\|_{L^{n+2}}^{n+2}+C(n,\bar\rho)C_0\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2\right). \end{aligned} \tag{3.38}$$

Integrating (3.38) over $[\sigma(T),T]$ together with (3.37) and (3.18) shows that (3.36) holds for $m=n+1$. By induction, we obtain (3.36) and finish the proof of Lemma 3.4.

Next, the following Lemma 3.5 combined with Lemma 2.4 will be useful to estimate the L^p-norm of $\rho\dot u$ and obtain the uniform (in time) upper bound of the density for large time.

Lemma 3.5 *Let (ρ,u) be a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2\times(0,T]$ satisfying the assumptions in Theorem 1.1 and (3.3). Then for any $\alpha>0$, there exists a positive constant N_1 depending only on α, N_0, and M such that for all $t\in(0,T]$,*

$$\int_{B_{N_1(1+t)\log^\alpha(1+t)}}\rho(x,t)dx\ge\frac14. \tag{3.39}$$

Proof. First, multiplying $(1.1)_1$ by $(1+|x|^2)^{1/2}$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain after integration by parts and using both (3.5) and (3.21) that

$$\begin{aligned}\frac{d}{dt}\int \rho(1+|x|^2)^{1/2}dx &\le C\int \rho|u|dx\\ &\le C\left(\int \rho dx\right)^{1/2}\left(\int \rho|u|^2dx\right)^{1/2}\\ &\le C.\end{aligned}$$

This gives

$$\sup_{0\le s\le t}\int \rho(1+|x|^2)^{1/2}dx \le C(M)(1+t). \tag{3.40}$$

Next, for $\varphi(y)\in C_0^\infty(\mathbb{R}^2)$ such that

$$0\le\varphi(y)\le 1,\quad \varphi(y)=\begin{cases}1 & \text{if } |y|\le 1,\\ 0 & \text{if } |y|\ge 2,\end{cases}\quad |\nabla\varphi|\le 2,$$

multiplying $(1.1)_1$ by $\varphi(y)$ with $y=\tilde\delta x(1+t)^{-1}\log^{-\alpha}(e+t)$ for small $\tilde\delta>0$ which will be determined later, we obtain

$$\begin{aligned}\frac{d}{dt}\int\rho\varphi(y)dx &= \int\rho\nabla_y\varphi\cdot y_t dx+\frac{\tilde\delta}{(1+t)\log^\alpha(e+t)}\int\rho u\cdot\nabla_y\varphi dx\\ &\ge -\frac{C\tilde\delta}{(1+t)^2\log^\alpha(e+t)}\int\rho|x|dx-\frac{C\tilde\delta}{(1+t)\log^\alpha(e+t)}\\ &\ge -\frac{C(M)\tilde\delta}{(1+t)\log^\alpha(e+t)},\end{aligned}$$

where in the last inequality we have used (3.40). Since $\alpha>1$, this yields

$$\int\rho\varphi\,(y)\,dx\ge\int\rho_0(x)\varphi(x\tilde\delta)dx-C(\alpha,M)\tilde\delta\ge\frac{1}{4}, \tag{3.41}$$

where we choose $\tilde\delta=(N_0+4C(\alpha,M))^{-1}$.

Finally, it follows from (3.41) that for $N_1\triangleq 2\tilde\delta^{-1}=2(N_0+4C(\alpha,M))$,

$$\int_{B_{N_1(1+t)\log^\alpha(e+t)}}\rho dx\ge\int\rho\varphi\left(\tilde\delta x(1+t)^{-1}\log^{-\alpha}(e+t)\right)dx\ge\frac{1}{4},$$

which finishes the proof of Lemma 3.5.

Next, to obtain the upper bound of the density for small time, we still need the following lemma.

Lemma 3.6 *Let (ρ,u) be a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2\times(0,T]$ satisfying (3.3) and the assumptions in Theorem 1.1. Then there exists a positive constant K depending only on $\mu,\lambda,\gamma,a,\bar\rho,\beta$, N_0, and M such that*

$$\sup_{0\le t\le\sigma(T)}t^{1-\beta}\|\nabla u\|_{L^2}^2+\int_0^{\sigma(T)}t^{1-\beta}\int\rho|\dot u|^2dxdt\le K(\bar\rho,M). \tag{3.42}$$

Proof. First, set

$$\nu \triangleq \min\left\{\frac{\mu^{1/2}}{2(1+2\mu+\lambda)^{1/2}}, \frac{\beta}{1-\beta}\right\} \in (0,1/2].$$

If $\beta \in (0,1)$, Sobolev's inequality implies

$$\begin{aligned}\int \rho_0|u_0|^{2+\nu}dx &\le \int \rho_0|u_0|^2dx + \int \rho_0|u_0|^{2/(1-\beta)}dx \\ &\le C(\bar\rho) + C(\bar\rho)\|u_0\|_{\dot H^\beta}^{2/(1-\beta)} \le C(\bar\rho, M).\end{aligned} \tag{3.43}$$

For the case that $\beta = 1$, one obtains from (2.5) that

$$\int \rho_0|u_0|^{2+\nu}dx \le C(\bar\rho)\left(\int \rho_0|u_0|^2dx + \int |\nabla u_0|^2dx\right)^{(2+\nu)/2} \le C(\bar\rho, M). \tag{3.44}$$

Then, multiplying $(1.1)_2$ by $(2+\nu)|u|^\nu u$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned}&\frac{d}{dt}\int \rho|u|^{2+\nu}dx + (2+\nu)\int |u|^\nu\left(\mu|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}u)^2\right)dx \\ &\le (2+\nu)\nu\int(\mu+\lambda)|\mathrm{div}u||u|^\nu|\nabla u|dx + C\int \rho^\gamma|u|^\nu|\nabla u|dx \\ &\le \frac{2+\nu}{2}\int(\mu+\lambda)(\mathrm{div}u)^2|u|^\nu dx + \frac{(2+\nu)\mu}{4}\int |u|^\nu|\nabla u|^2dx \\ &\quad + C\int \rho|u|^{2+\nu}dx + C\int \rho^{(2+\nu)\gamma-\nu/2}dx,\end{aligned}$$

which together with Gronwall's inequality, (3.43), and (3.44) thus gives

$$\sup_{0\le t\le\sigma(T)}\int \rho|u|^{2+\nu}dx \le C(\bar\rho, M). \tag{3.45}$$

Next, as in [12], for the linear differential operator L defined by

$$\begin{aligned}(Lw)^j &\triangleq \rho w^j_t + \rho u\cdot\nabla w^j - (\mu\Delta w^j + (\mu+\lambda)\partial_j\mathrm{div}w) \\ &= \rho\dot w^j - (\mu\Delta w^j + (\mu+\lambda)\partial_j\mathrm{div}w), \quad j = 1,2,\end{aligned}$$

let w_1 and w_2 be the solution to:

$$Lw_1 = 0, \quad w_1(x,0) = w_{10}(x), \tag{3.46}$$

and

$$Lw_2 = -\nabla P(\rho), \quad w_2(x,0) = 0, \tag{3.47}$$

respectively. A straightforward energy estimate of (3.46) shows that:

$$\sup_{0\le t\le\sigma(T)}\int \rho|w_1|^2dx + \int_0^{\sigma(T)}\int |\nabla w_1|^2dxdt \le C(\bar\rho)\int |w_{10}|^2dx. \tag{3.48}$$

Then, multiplying (3.46) by w_{1t} and integrating the resulting equality over $\mathbb{R}^2$ yield that for $t \in (0, \sigma(T)]$,

$$\begin{aligned} &\frac{1}{2}\left(\mu\|\nabla w_1\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div} w_1\|_{L^2}^2\right)_t + \int \rho|\dot{w}_1|^2 dx \\ &= \int \rho \dot{w}_1 (u\cdot\nabla w_1) dx \\ &\le C(\bar\rho)\|\rho^{1/2}\dot{w}_1\|_{L^2}\|\rho^{1/(2+\nu)}u\|_{L^{2+\nu}}\|\nabla^2 w_1\|_{L^2}^{2/(2+\nu)}\|\nabla w_1\|_{L^2}^{\nu/(2+\nu)} \\ &\le \frac{1}{2}\int \rho|\dot{w}_1|^2 dx + C(\bar\rho, M)\|\nabla w_1\|_{L^2}^2, \end{aligned} \tag{3.49}$$

where in the last inequality we have used (3.45) and the following simple fact:

$$\|\nabla^2 w_1\|_{L^2} \le C\|\rho\dot{w}_1\|_{L^2},$$

due to the standard L^2-estimate of the elliptic system (3.46). Gronwall's inequality together with (3.49) and (3.48) gives

$$\sup_{0\le t\le\sigma(T)} \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)}\int \rho|\dot{w}_1|^2 dxdt \le C(\bar\rho, M)\|\nabla w_{10}\|_{L^2}^2, \tag{3.50}$$

and

$$\sup_{0\le t\le\sigma(T)} t\|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t\int \rho|\dot{w}_1|^2 dxdt \le C(\bar\rho, M)\|w_{10}\|_{L^2}^2. \tag{3.51}$$

Since the solution operator $w_{10} \mapsto w_1(\cdot, t)$ is linear, by the standard Stein-Weiss interpolation argument ([2]), one can deduce from (3.50) and (3.51) that for any $\theta \in [\beta, 1]$,

$$\sup_{0\le t\le\sigma(T)} t^{1-\theta}\|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta}\int \rho|\dot{w}_1|^2 dxdt \le C(\bar\rho, M)\|w_{10}\|_{\dot{H}^\theta}^2, \tag{3.52}$$

with a uniform constant C independent of θ.

Finally, we estimate w_2. It follows from a similar way as for the proof of (2.10) and (2.12) that

$$\|\nabla((2\mu+\lambda)\mathrm{div} w_2 - P)\|_{L^2} + \|\nabla(\nabla^\perp\cdot w_2)\|_{L^2} \le C\|\rho\dot{w}_2\|_{L^2}, \tag{3.53}$$

and that for $p \ge 2$,

$$\begin{aligned} \|\nabla w_2\|_{L^p} &\le C(\|(2\mu+\lambda)\mathrm{div} w_2 - P\|_{L^p} + C\|P\|_{L^p} + \|\nabla^\perp\cdot w_2\|_{L^p} \\ &\le \delta\|\rho\dot{w}_2\|_{L^2} + C(\bar\rho, p, \delta)\|\nabla w_2\|_{L^2} + C(\bar\rho, p, \delta)C_0^{1/p}. \end{aligned} \tag{3.54}$$

Multiplying (3.47) by w_{2t} and integrating the resulting equation over $\mathbb{R}^2$ yield that for $t \in (0, \sigma(T)]$,

$$\begin{aligned} &\frac{1}{2}\left(\mu\|\nabla w_2\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div} w_2\|_{L^2}^2 - 2\int P\mathrm{div} w_2 dx\right)_t + \int \rho|\dot{w}_2|^2 dx \\ &= \int \rho\dot{w}_2(u\cdot\nabla w_2)dx - \int P_t \mathrm{div} w_2 dx \\ &\le C(\bar\rho)\|\rho^{1/2}\dot{w}_2\|_{L^2}\|\rho^{1/(2+\nu)}u\|_{L^{2+\nu}}\|\nabla w_2\|_{L^{2(2+\nu)/\nu}} - \int P_t\mathrm{div} w_2 dx \\ &\le C(\bar\rho, M)\delta\|\rho^{1/2}\dot{w}_2\|_{L^2}^2 + C(\delta, \bar\rho, M)\left(\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1\right), \end{aligned} \tag{3.55}$$

where in the last inequality we have used (3.54), (3.45), and the following simple fact:

$$\begin{aligned}-\int P_t \mathrm{div} w_2 dx =& -\frac{1}{2\mu+\lambda}\int Pu\cdot\nabla((2\mu+\lambda)\mathrm{div} w_2 - P)dx \\ &+ \frac{1}{2(2\mu+\lambda)}\int P^2 \mathrm{div} u dx \\ \le& C\|Pu\|_{L^2}\|\rho\dot{w}_2\|_{L^2} + C\|P^2\|_{L^2}\|\nabla u\|_{L^2} \\ \le& \delta\|\rho^{1/2}\dot{w}_2\|_{L^2}^2 + C(\delta,\bar{\rho})\left(\|\nabla u\|_{L^2}^2+1\right),\end{aligned}$$

due to (3.9) and (3.53). Gronwall's inequality together with (3.55) gives

$$\sup_{0\le t\le\sigma(T)}\|\nabla w_2\|_{L^2}^2 + \int_0^{\sigma(T)}\int \rho|\dot{w}_2|^2 dxdt \le C(\bar{\rho},M). \tag{3.56}$$

Taking $w_{10}=u_0$ so that $w_1+w_2=u$, we then derive (3.42) from (3.52) and (3.56) directly. Thus, we finish the proof of Lemma 3.6.

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher order estimates and thus to extend the classical solution globally. We will use an approach motivated by our previous study on the two-dimensional Stokes approximation equations ([20]), see also [16].

Lemma 3.7 *There exists a positive constant $\varepsilon_0=\varepsilon_0(\bar{\rho},M)$ depending on $\mu,\lambda,\gamma,a,\bar{\rho},\beta$, N_0, and M such that, if (ρ,u) is a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2\times(0,T]$ satisfying (3.3) and the assumptions in Theorem 1.1, then*

$$\sup_{0\le t\le T}\|\rho(t)\|_{L^\infty}\le\frac{7\bar{\rho}}{4}, \tag{3.57}$$

provided $C_0\le\varepsilon_0$.

Proof. First, we rewrite the equation of the mass conservation $(1.1)_1$ as

$$D_t\rho = g(\rho)+b'(t), \tag{3.58}$$

where

$$D_t\rho\triangleq\rho_t+u\cdot\nabla\rho,\quad g(\rho)\triangleq-\frac{\rho^{\gamma+1}}{2\mu+\lambda},\quad b(t)\triangleq-\frac{1}{2\mu+\lambda}\int_0^t\rho F dt.$$

Next, it follows from (2.10), (3.40), (3.39), and (2.5) that for $t>0$ and $p\in[2,\infty)$,

$$\begin{aligned}\|\nabla F(\cdot,t)\|_{L^p}&\le C(p)\|\rho\dot{u}(\cdot,t)\|_{L^p} \\ &\le C(p,\bar{\rho},M)(1+t)^5\left(\|\rho^{1/2}\dot{u}(\cdot,t)\|_{L^2}+\|\nabla\dot{u}(\cdot,t)\|_{L^2}\right),\end{aligned} \tag{3.59}$$

which, together with the Gagliardo-Nirenberg inequality (2.2) for $q=2$, yields that for

$r \triangleq 4+4/\beta$ and $\delta_0 \triangleq (2r+(1-\beta)(r-2))/(3r-4) \in (0,1)$,

$$
\begin{aligned}
&|b(\sigma(T))| \\
&\le C(\bar\rho)\int_0^{\sigma(T)} \sigma^{(\beta-1)(r-2)/(4(r-1))}\left(\sigma^{1-\beta}\|F\|_{L^2}^2\right)^{(r-2)/(4(r-1))}\|\nabla F\|_{L^r}^{r/(2(r-1))}dt \\
&\le C(\bar\rho,M)\int_0^{\sigma(T)} \sigma^{-(2r+(1-\beta)(r-2))/(4(r-1))}\left(\sigma^2\|\nabla F\|_{L^r}^2\right)^{r/(4(r-1))}dt \\
&\le C(\bar\rho,M)\left(\int_0^{\sigma(T)}\sigma^{-\delta_0}dt\right)^{(3r-4)/(4(r-1))}\left(\int_0^{\sigma(T)}\sigma^2\|\nabla F\|_{L^r}^2dt\right)^{r/(4(r-1))} \\
&\le C(\bar\rho,M)\left(\int_0^{\sigma(T)}\left(\sigma^2\|\rho^{1/2}\dot u\|_{L^2}^2+\sigma^2\|\nabla\dot u\|_{L^2}^2\right)dt\right)^{r/(4(r-1))} \\
&\le C(\bar\rho,M)C_0^{r/(4(r-1))},
\end{aligned}
$$

where in the second, fourth, and last inequalities one has used respectively (3.42), (3.59), and (3.18). This combined with (3.58) yields that

$$
\sup_{t\in[0,\sigma(T)]}\|\rho\|_{L^\infty} \le \bar\rho + C(\bar\rho,M)C_0^{1/4} \le \frac{3\bar\rho}{2}, \tag{3.60}
$$

provided

$$
C_0 \le \varepsilon_1 \triangleq \min\{1,(\bar\rho/(2C(\bar\rho,M)))^4\}.
$$

Next, it follows from (2.10) and (3.26) that for $t\in[\sigma(T),T]$,

$$
\begin{aligned}
\|F(\cdot,t)\|_{H^1} &\le C\left(\|\nabla u(\cdot,t)\|_{L^2}+\|P(\cdot,t)\|_{L^2}+\|\rho\dot u(\cdot,t)\|_{L^2}\right) \\
&\le C(\bar\rho)C_0^{1/2}t^{-1/2},
\end{aligned} \tag{3.61}
$$

which together with (2.2) and (3.59) shows

$$
\begin{aligned}
&\int_{\sigma(T)}^T \|F(\cdot,t)\|_{L^\infty}^4dt \\
&\le C\int_{\sigma(T)}^T \|F(\cdot,t)\|_{L^{72}}^{35/9}\|\nabla F(\cdot,t)\|_{L^{72}}^{1/9}dt \\
&\le C(\bar\rho,M)C_0^{35/18}\int_{\sigma(T)}^T t^{-25/18}(\|\rho^{1/2}\dot u\|_{L^2}+\|\nabla\dot u\|_{L^2})^{1/9}dt \\
&\le C(\bar\rho,M)C_0^{35/18},
\end{aligned} \tag{3.62}
$$

where in the last inequality, one has used (3.3). This shows that for all $\sigma(T)\le t_1 \le t_2 \le T$,

$$
\begin{aligned}
|b(t_2)-b(t_1)| &\le C(\bar\rho)\int_{t_1}^{t_2}\|F(\cdot,t)\|_{L^\infty}dt \\
&\le \frac{1}{2\mu+\lambda}(t_2-t_1)+C(\bar\rho,M)\int_{\sigma(T)}^T\|F(\cdot,t)\|_{L^\infty}^4dt \\
&\le \frac{1}{2\mu+\lambda}(t_2-t_1)+C(\bar\rho,M)C_0^{35/18},
\end{aligned}
$$

which implies that one can choose N_1 and N_0 in (2.14) as:

$$N_1 = \frac{1}{2\mu+\lambda}, \quad N_0 = C(\bar{\rho}, M)C_0^{35/18}.$$

Hence, we set $\bar{\zeta} = 1$ in (2.15) since for all $\zeta \geq 1$,

$$g(\zeta) = -\frac{\zeta^{\gamma+1}}{2\mu+\lambda} \leq -N_1 = -\frac{1}{2\mu+\lambda}.$$

Lemma 2.6 and (3.60) thus lead to

$$\sup_{t\in[\sigma(T),T]} \|\rho\|_{L^\infty} \leq \frac{3\bar{\rho}}{2} + N_0 \leq \frac{7\bar{\rho}}{4}, \tag{3.63}$$

provided

$$C_0 \leq \varepsilon_0 \triangleq \min\{\varepsilon_1, \varepsilon_2\}, \quad \text{for } \varepsilon_2 \triangleq \left(\frac{\bar{\rho}}{4C(\bar{\rho}, M)}\right)^{18/35}.$$

The combination of (3.60) with (3.63) completes the proof of Lemma 3.7.

With Lemmas 3.3 and 3.7 at hand, we are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. It follows from (3.18) that

$$A_1(T) + A_2(T) + \int_0^T \sigma\|P\|_{L^2}^2 dt \leq C_0^{1/2}, \tag{3.64}$$

provided

$$C_0 \leq \varepsilon_3 \triangleq (C(\bar{\rho}))^{-2}.$$

Letting $\varepsilon \triangleq \min\{\varepsilon_0, \varepsilon_3\}$, we obtain (3.4) directly from (3.57) and (3.64) and finish the proof of Proposition 3.1.

4 A priori estimates (II): higher order estimates

Form now on, for smooth initial data (ρ_0, u_0) satisfying (1.8) and (1.9), assume that (ρ, u) is a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2 \times (0, T]$ satisfying (3.3). Then, we derive some necessary uniform estimates on the spatial gradient of the smooth solution (ρ, u).

Lemma 4.1 *There is a positive constant C depending only on $T, \mu, \lambda, \gamma, a, \bar{\rho}, \beta, N_0, M$, and $\|\rho_0\|_{H^1\cap W^{1,q}}$ such that*

$$\begin{aligned} &\sup_{0\leq t\leq T} \left(\|\rho\|_{H^1\cap W^{1,q}} + \|\nabla u\|_{L^2} + t\|\nabla^2 u\|_{L^2}^2\right) \\ &+ \int_0^T \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{(q+1)/q} + t\|\nabla^2 u\|_{L^q}^2\right) dt \leq C. \end{aligned} \tag{4.1}$$

Proof. First, it follows from (3.32), (3.31), Gronwall's inequality, and (3.5) that

$$\sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 dxdt \leq C, \tag{4.2}$$

which together with (2.12) shows

$$\int_0^T \|\nabla u\|_{L^4}^4 dt \le C. \tag{4.3}$$

Multiplying (3.17) by t and integrating the resulting inequality over $(0,T)$ combined with (4.2) and (4.3) lead to

$$\sup_{0\le t\le T} t\int \rho|\dot u|^2dx + \int_0^T t\|\nabla \dot u\|_{L^2}^2 dt \le C. \tag{4.4}$$

Next, we prove (4.1) by using Lemma 2.7 as in [15]. For $p\in[2,q]$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t + \mathrm{div}(|\nabla\rho|^p u) + (p-1)|\nabla\rho|^p \mathrm{div} u\\ &+ p|\nabla\rho|^{p-2}(\nabla\rho)^t\nabla u(\nabla\rho) + p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla \mathrm{div} u = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^p} &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}\\ &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\rho\dot u\|_{L^p},\end{aligned} \tag{4.5}$$

due to

$$\|\nabla^2 u\|_{L^p} \le C\left(\|\rho\dot u\|_{L^p} + \|\nabla P\|_{L^p}\right), \tag{4.6}$$

which follows from the standard L^p-estimate for the following elliptic system:

$$\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div} u = \rho\dot u + \nabla P, \quad u\to 0 \ \text{ as } |x|\to\infty.$$

Next, it follows from the Gargliardo-Nirenberg inequality, (4.2), and (2.10) that

$$\begin{aligned}\|\mathrm{div} u\|_{L^\infty} + \|\omega\|_{L^\infty} &\le C\|F\|_{L^\infty} + C\|P\|_{L^\infty} + C\|\omega\|_{L^\infty}\\ &\le C(q) + C(q)\|\nabla F\|_{L^q}^{q/(2(q-1))} + C(q)\|\nabla\omega\|_{L^q}^{q/(2(q-1))}\\ &\le C(q) + C(q)\|\rho\dot u\|_{L^q}^{q/(2(q-1))},\end{aligned} \tag{4.7}$$

which, together with Lemma 2.7, yields that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\left(\|\mathrm{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}\right)\log(e+\|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C\\ &\le C\left(1+\|\rho\dot u\|_{L^q}^{q/(2(q-1))}\right)\log(e+\|\rho\dot u\|_{L^q} + \|\nabla\rho\|_{L^q}) + C\\ &\le C\left(1+\|\rho\dot u\|_{L^q}\right)\log(e+\|\nabla\rho\|_{L^q}).\end{aligned} \tag{4.8}$$

Next, it follows from the Hölder inequality and (3.59) that

$$\begin{aligned}\|\rho\dot u\|_{L^q} &\le \|\rho\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\|\rho\dot u\|_{L^{q^2}}^{q(q-2)/(q^2-2)}\\ &\le C\|\rho\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\left(\|\rho^{1/2}\dot u\|_{L^2} + \|\nabla\dot u\|_{L^2}\right)^{q(q-2)/(q^2-2)}\\ &\le C\|\rho^{1/2}\dot u\|_{L^2} + C\|\rho^{1/2}\dot u\|_{L^2}^{2(q-1)/(q^2-2)}\|\nabla\dot u\|_{L^2}^{q(q-2)/(q^2-2)},\end{aligned}$$

which combined with (4.2) and (4.4) implies that

$$\begin{aligned}&\int_0^T\left(\|\rho\dot u\|_{L^q}^{1+1/q} + t\|\rho\dot u\|_{L^q}^2\right)dt\\ &\le C\int_0^T\left(\|\rho^{1/2}\dot u\|_{L^2}^2 + t\|\nabla\dot u\|_{L^2}^2 + t^{-(q^3-q^2-2q-1)/(q^3-q^2-2q)}\right)dt\\ &\le C.\end{aligned} \tag{4.9}$$

Then, substituting (4.8) into (4.5) where $p = q$, we deduce from Gronwall's inequality and (4.9) that

$$\sup_{0 \le t \le T} \|\nabla \rho\|_{L^q} \le C,$$

which, along with (4.6) and (4.9), shows

$$\int_0^T \left(\|\nabla^2 u\|_{L^q}^{(q+1)/q} + t\|\nabla^2 u\|_{L^q}^2 \right) dt \le C. \tag{4.10}$$

Finally, taking $p = 2$ in (4.5), one gets by using (4.10), (4.2), and Gronwall's inequality that

$$\sup_{0 \le t \le T} \|\nabla \rho\|_{L^2} \le C,$$

which, together with (4.6), (4.4), and (4.10), yields (4.1). The proof of Lemma 4.1 is completed.

Lemma 4.2 *There is a positive constant C depending only on $T, \mu, \lambda, \gamma, a, \bar{\rho}, \beta, N_0, M$, and $\|\nabla(\bar{x}^a \rho_0)\|_{L^2 \cap L^q}$ such that*

$$\sup_{0 \le t \le T} \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} \le C. \tag{4.11}$$

Proof. First, it follows from (2.3), (3.40), (3.39), and (2.8) that for any $\eta \in (0, 1]$ and any $s > 2$,

$$\|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \le C(\eta, s). \tag{4.12}$$

Multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\begin{aligned}
\frac{d}{dt} \int \rho \bar{x}^a dx &\le C \int \rho |u| \bar{x}^{a-1} \log^2(e + |x|^2) dx \\
&\le C \|\rho \bar{x}^{a-1+8/(8+a)}\|_{L^{(8+a)/(7+a)}} \|u\bar{x}^{-4/(8+a)}\|_{L^{8+a}} \\
&\le C \int \rho \bar{x}^a dx + C.
\end{aligned}$$

This gives

$$\sup_{0 \le t \le T} \int \rho \bar{x}^a dx \le C. \tag{4.13}$$

Then, one derives from $(1.1)_1$ that $v \triangleq \rho \bar{x}^a$ satisfies

$$v_t + u \cdot \nabla v - a v u \cdot \nabla \log \bar{x} + v \operatorname{div} u = 0,$$

which, together with some estimates as for (4.5), gives that for any $p \in [2, q]$

$$\begin{aligned}
(\|\nabla v\|_{L^p})_t &\le C(1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \bar{x}\|_{L^\infty}) \|\nabla v\|_{L^p} \\
&\quad + C\|v\|_{L^\infty} \left(\||\nabla u||\nabla \log \bar{x}|\|_{L^p} + \||u||\nabla^2 \log \bar{x}|\|_{L^p} + \|\nabla^2 u\|_{L^p} \right) \\
&\le C(1 + \|\nabla u\|_{W^{1,q}}) \|\nabla v\|_{L^p} \\
&\quad + C\|v\|_{L^\infty} \left(\|\nabla u\|_{L^p} + \|u\bar{x}^{-2/5}\|_{L^{4p}} \|\bar{x}^{-3/2}\|_{L^{4p/3}} + \|\nabla^2 u\|_{L^p} \right) \\
&\le C(1 + \|\nabla^2 u\|_{L^p} + \|\nabla u\|_{W^{1,q}})(1 + \|\nabla v\|_{L^p} + \|\nabla v\|_{L^q}),
\end{aligned} \tag{4.14}$$

where in the second and the last inequalities, one has used (4.12) and (4.13). Choosing $p = q$ in (4.14), we obtain after using Gronwall's inequality and (4.1) that

$$\sup_{0\le t\le T} \|\nabla(\rho\bar{x}^a)\|_{L^q} \le C. \tag{4.15}$$

Finally, setting $p = 2$ in (4.14), we deduce from (4.1) and (4.15) that

$$\sup_{0\le t\le T} \|\nabla(\rho\bar{x}^a)\|_{L^2} \le C,$$

which combined with (4.13) and (4.15) thus gives (4.11) and finishes the proof of Lemma 4.2.

Lemma 4.3 *There is a positive constant C depending only on $T, \mu, \lambda, \gamma, a, \bar{\rho}, \beta, N_0, M$, and $\|\nabla(\bar{x}^a\rho_0)\|_{L^2\cap L^q}$ such that*

$$\sup_{0\le t\le T} t\left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^2\right) + \int_0^T t\|\nabla u_t\|_{L^2}^2 dt \le C. \tag{4.16}$$

Proof. Differentiating $(1.1)_2$ with respect to t gives

$$\begin{aligned} &\rho u_{tt} + \rho u\cdot\nabla u_t - \mu\Delta u_t - (\mu+\lambda)\nabla\mathrm{div}u_t \\ &= -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \nabla P_t. \end{aligned} \tag{4.17}$$

Multiplying (4.17) by u_t and integrating the resulting equation over $\mathbb{R}^2$, we obtain after using $(1.1)_1$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \int \left(\mu|\nabla u_t|^2 + (\mu+\lambda)(\mathrm{div}u_t)^2\right)dx \\ &= -2\int \rho u\cdot\nabla u_t\cdot u_t dx - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx \\ &\quad - \int \rho u_t\cdot\nabla u\cdot u_t dx + \int P_t \mathrm{div}u_t dx \\ &\le C\int \rho|u||u_t|\left(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|\right)dx + C\int \rho|u|^2|\nabla u||\nabla u_t|dx \\ &\quad + C\int \rho|u_t|^2|\nabla u|dx + C(\delta)\|P_t\|_{L^2}^2 + \delta\|\nabla u_t\|_{L^2}^2. \end{aligned} \tag{4.18}$$

Each term on the right-hand side of (4.18) can be estimated as follows:

First, the combination of (4.12) with (4.11) gives that for any $\eta \in (0,1]$ and any $s > 2$,

$$\|\rho^\eta u\|_{L^{s/\eta}} + \|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \le C(\eta, s). \tag{4.19}$$

Moreover, it follows from (2.5), (3.40), and (3.39) that

$$\|\rho^{1/2}u_t\|_{L^6} \le C\|\rho^{1/2}u_t\|_{L^2} + C\|\nabla u_t\|_{L^2}, \tag{4.20}$$

which together with (4.19), (4.2), and Holder's inequality yields that for $\delta \in (0,1)$,

$$\begin{aligned} &\int \rho|u||u_t|\left(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|\right)dx \\ &\le C\|\rho^{1/2}u\|_{L^6}\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|\rho^{1/2}u_t\|_{L^6}^{1/2}\left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}^2\right) \\ &\quad + C\|\rho^{1/4}u\|_{L^{12}}^2\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|\rho^{1/2}u_t\|_{L^6}^{1/2}\|\nabla^2 u\|_{L^2} \\ &\le C\|\rho^{1/2}u_t\|_{L^2}^{1/2}\left(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right)^{1/2}\left(\|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} + 1\right) \\ &\le \delta\|\nabla u_t\|_{L^2}^2 + C(\delta)\left(\|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2}u_t\|_{L^2}^2 + 1\right). \end{aligned} \tag{4.21}$$

Next, Holder's inequality, (4.19), and (4.20) lead to

$$
\begin{aligned}
&\int \rho|u|^2|\nabla u||\nabla u_t|dx + \int \rho|u_t|^2|\nabla u|dx \\
&\le C\|\rho^{1/2}u\|_{L^8}^2\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}\|\rho^{1/2}u_t\|_{L^6}^{3/2}\|\rho^{1/2}u_t\|_{L^2}^{1/2} \\
&\le \delta\|\nabla u_t\|_{L^2}^2 + C(\delta)\left(\|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2}u_t\|_{L^2}^2 + 1\right).
\end{aligned}
\tag{4.22}
$$

Next, it follows from (4.19), (4.2), and (4.11) that

$$
\|P_t\|_{L^2} \le C\|\bar{x}^{-a}u\|_{L^{2q/(q-2)}}\|\rho\|_{L^\infty}^{\gamma-1}\|\bar{x}^a\nabla\rho\|_{L^q} + C\|\nabla u\|_{L^2} \le C. \tag{4.23}
$$

Finally, putting (4.21)-(4.23) into (4.18) and choosing δ suitably small, we obtain after using (4.6) and (4.1) that

$$
\frac{d}{dt}\int \rho|u_t|^2dx + \mu\int|\nabla u_t|^2dx \le C\int\rho|u_t|^2dx + C\int\rho|\dot{u}|^2dx + 1. \tag{4.24}
$$

It follows from (4.6) and (4.19) that

$$
\begin{aligned}
&\|\nabla u\|_{H^1} + \|\rho^{1/2}u\cdot\nabla u\|_{L^2} \\
&\le C + C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\rho^{1/2}u\|_{L^6}\|\nabla u\|_{L^2}^{2/3}\|\nabla^2 u\|_{L^2}^{1/3} \\
&\le C + C\|\rho^{1/2}\dot{u}\|_{L^2} + \frac{1}{2}\|\nabla^2 u\|_{L^2},
\end{aligned}
$$

which together with (4.2) shows

$$
\|\nabla u\|_{H^1} + \|\rho^{1/2}u_t\|_{L^2} \le C\|\rho^{1/2}\dot{u}\|_{L^2} + C. \tag{4.25}
$$

This combined with (4.24), (4.2), and Gronwall's inequality gives (4.16) and finishes the proof of Lemma 4.3.

¿From now on, assume that (ρ, u) is a smooth solution of (1.1)-(1.4) on $\mathbb{R}^2\times(0,T]$ satisfying (3.3) for smooth initial data (ρ_0, u_0) satisfying (1.8), (1.9), (1.17), and (1.18). Moreover, in addition to $T, \mu, \lambda, \gamma, a, \bar{\rho}, \beta, N_0, M$, and $\|\nabla(\bar{x}^a\rho_0)\|_{L^2\cap L^q}$, the generic positive constant C may depend on $\|\nabla^2 u_0\|_{L^2}$, $\|\bar{x}^{\delta_0}\nabla^2\rho_0\|_{L^2}$, $\|\bar{x}^{\delta_0}\nabla^2 P(\rho_0)\|_{L^2}$, and $\|g\|_{L^2}$, with g as in (1.18).

Lemma 4.4 *It holds that*

$$
\sup_{0\le t\le T}\left(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u\|_{H^1}\right) + \int_0^T\|\nabla u_t\|_{L^2}^2dt \le C. \tag{4.26}
$$

Proof. Taking into account on the compatibility condition (1.18), we can define

$$
\sqrt{\rho}\dot{u}(x, t=0) = g. \tag{4.27}
$$

Integrating (3.17) over $(0,T)$ together with (4.27) and (4.3) yields directly that

$$
\sup_{0\le t\le T}\|\rho^{1/2}\dot{u}\|_{L^2} + \int_0^T\|\nabla\dot{u}\|_{L^2}^2dt \le C,
$$

which, along with (4.25) and (4.24), gives (4.26) and finishes the proof of Lemma 4.4.

The following higher order estimates of the solutions which are needed to guarantee the extension of local classical solution to be a global one are similar to those in [19], so we omit their proofs here.

Lemma 4.5 *The following estimates hold:*

$$\sup_{0\le t\le T}\left(\|\bar{x}^{\delta_0}\nabla^2\rho\|_{L^2}+\|\bar{x}^{\delta_0}\nabla^2 P\|_{L^2}\right)\le C, \tag{4.28}$$

$$\sup_{0\le t\le T} t\|\nabla u_t\|_{L^2}^2+\int_0^T t\left(\|\rho^{1/2}u_{tt}\|_{L^2}^2+\|\nabla^2 u_t\|_{L^2}^2\right)dt\le C, \tag{4.29}$$

$$\sup_{0\le t\le T}\left(\|\nabla^2\rho\|_{L^q}+\|\nabla^2 P\|_{L^q}\right)\le C, \tag{4.30}$$

$$\begin{aligned}&\sup_{0\le t\le T} t\left(\|\rho^{1/2}u_{tt}\|_{L^2}+\|\nabla^3 u\|_{L^2\cap L^q}+\|\nabla u_t\|_{H^1}+\|\nabla^2(\rho u)\|_{L^{(q+2)/2}}\right)\\ &+\int_0^T t^2\left(\|\nabla u_{tt}\|_{L^2}^2+\|u_{tt}\bar{x}^{-1}\|_{L^2}^2\right)dt\le C.\end{aligned} \tag{4.31}$$

5 Proofs of Theorems 1.1-1.3

With all the a priori estimates in Sections 3 and 4 at hand, we are ready to prove the main results of this paper in this section.

Proof of Theorem 1.1. By Lemma 2.1, there exists a $T_*>0$ such that the Cauchy problem (1.1)-(1.4) has a unique strong solution (ρ,u) on $\mathbb{R}^2\times(0,T_*]$. We will use the a priori estimates, Proposition 3.1 and Lemmas 4.1-4.3, to extend the local strong solution (ρ,u) to all time.

First, it follows from (3.1), (3.2), and (1.8) that

$$A_1(0)+A_2(0)=0,\quad \rho_0\le\bar{\rho}.$$

Therefore, there exists a $T_1\in(0,T_*]$ such that (3.3) holds for $T=T_1$.

Next, set

$$T^*=\sup\{T\,|\,(3.3)\text{ holds}\}. \tag{5.1}$$

Then $T^*\ge T_1>0$. Hence, for any $0<\tau<T\le T^*$ with T finite, one deduces from (4.16) that for any $q\ge 2$,

$$\nabla u\in C([\tau,T];L^2\cap L^q), \tag{5.2}$$

where one has used the standard embedding

$$L^\infty(\tau,T;H^1)\cap H^1(\tau,T;H^{-1})\hookrightarrow C\left([\tau,T];L^q\right),\qquad \text{for any } q\in[2,\infty).$$

Moreover, it follows from (4.1), (4.11), and [21, Lemma 2.3] that

$$\rho\in C([0,T];L^1\cap H^1\cap W^{1,q}). \tag{5.3}$$

Finally, we claim that

$$T^*=\infty. \tag{5.4}$$

Otherwise, $T^*<\infty$. Then by Proposition 3.1, (3.4) holds for $T=T^*$. It follows from (3.5), (4.11), (5.2) and (5.3) that $(\rho(x,T^*),u(x,T^*))$ satisfies (1.8) except $u(\cdot,T^*)\in\dot{H}^\beta$. Thus, Lemma 2.1 implies that there exists some $T^{**}>T^*$, such that (3.3) holds for $T=T^{**}$, which contradicts (5.1). Hence, (5.4) holds. Lemmas 2.1 and 4.1-4.3 thus show that (ρ, u) is in fact the unique strong solution defined on $\mathbb{R}^2 \times (0, T]$ for any $0 < T < T^* = \infty$. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, one can prove Theorem 1.2 by using Lemma 2.1, Proposition 3.1, and Lemmas 4.1-4.5.

To prove Theorem 1.3, we need the following elementary estimates similar to those of Lemma 2.5 whose proof can be found in [16, Lemma 2.3].

Lemma 5.1 *Let $\Omega = \mathbb{R}^3$ and (ρ, u) be a smooth solution of (1.1). Then there exists a generic positive constant C depending only on μ and λ such that for any $p \in [2, 6]$*

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{5.5}$$

$$\|F\|_{L^p} + \|\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)} \left(\|\nabla u\|_{L^2} + \|P\|_{L^2}\right)^{(6-p)/(2p)}, \tag{5.6}$$

$$\|\nabla u\|_{L^p} \le C\left(\|F\|_{L^p} + \|\omega\|_{L^p}\right) + C\|P\|_{L^p}, \tag{5.7}$$

where $F = (2\mu + \lambda)\mathrm{div}u - P$ and $\omega = \nabla \times u$ are the effective viscous flux and the vorticity respectively.

Proof of Theorem 1.3. It suffices to prove (1.27). In fact, it follows from [16, Proposition 3.1 and (3.6)] that there exists some ε depending only on $\mu, \lambda, \gamma, \bar{\rho}, \beta$, and M such that

$$\begin{aligned} &\sup_{1\le t<\infty}\left(\|\nabla u\|_{L^2} + \|\rho\|_{L^\gamma\cap L^\infty} + \|\rho^{1/2}\dot{u}\|_{L^2}\right) \\ &+ \int_1^\infty \left(\|\nabla u\|_{L^2}^2 + \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2\right) dt \le C, \end{aligned} \tag{5.8}$$

provided $C_0 \le \varepsilon$.

If $\gamma \le 3/2$, it then holds that

$$\sup_{1\le t<\infty} \|\rho\|_{L^{3/2}} \le C \sup_{1\le t<\infty} \|\rho\|_{L^\gamma}^{2\gamma/3} \le C. \tag{5.9}$$

If $\gamma > 3/2$, since $\rho_0 \in L^1$, $(1.1)_1$ yields that for $t \ge 0$,

$$\int \rho(x,t)dx = \int \rho_0(x)dx,$$

which combined with (5.8) implies

$$\sup_{1\le t<\infty} \|\rho\|_{L^{3/2}} \le C \sup_{1\le t<\infty} \|\rho\|_{L^1}^{2/3} \le C. \tag{5.10}$$

Similar to (3.20), one deduces from $(1.1)_2$ that

$$P = (-\Delta)^{-1}\mathrm{div}(\rho\dot{u}) + (2\mu+\lambda)\mathrm{div}u,$$

which together with the Sobolev inequality gives

$$\begin{aligned} \|P\|_{L^2} \le& C\|(-\Delta)^{-1}\mathrm{div}(\rho\dot{u})\|_{L^2} + C\|\nabla u\|_{L^2} \\ \le& C\|\rho\dot{u}\|_{L^{6/5}} + C\|\nabla u\|_{L^2} \\ \le& C\|\rho\|_{L^{3/2}}^{1/2}\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2} \\ \le& C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2}, \end{aligned}$$

where in the last inequality one has used (5.9) and (5.10). This combined with (5.8) leads to

$$\int_1^\infty \|P\|_{L^2}^2 dt \le C. \tag{5.11}$$

Next, similar to (3.27), for $p \ge 2$, we have

$$\left(\|P\|_{L^p}^p\right)_t + \frac{p\gamma - 1}{2\mu+\lambda}\|P\|_{L^{p+1}}^{p+1} = -\frac{p\gamma-1}{2\mu+\lambda}\int P^p F dx, \tag{5.12}$$

which together with Holder's inequality yields

$$\left(\|P\|_{L^p}^p\right)_t + \frac{p\gamma - 1}{2(2\mu+\lambda)}\|P\|_{L^{p+1}}^{p+1} \le C(p)\|F\|_{L^{p+1}}^{p+1}. \tag{5.13}$$

Next, for $B(t)$ defined as in (3.14), it follows from (3.13) and (5.7) that

$$\begin{aligned} B'(t) + \int \rho|\dot u|^2 dx &\le C\|P\|_{L^3}^3 + C\|\nabla u\|_{L^3}^3 \\ &\le C_1\|P\|_{L^3}^3 + C\|F\|_{L^3}^3 + C\|\omega\|_{L^3}^3. \end{aligned} \tag{5.14}$$

Choosing $C_2 \ge 2 + 2(2\mu+\lambda)(C_1+1)/(2\gamma-1)$ suitably large such that

$$\frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 \le B(t) + C_2\|P\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 + C\|P\|_{L^2}^2, \tag{5.15}$$

setting $p = 2$ in (5.13), and adding (5.13) multiplied by C_2 to (5.14) yield that for $t \ge 1$,

$$\begin{aligned} &2\left(B(t) + C_2\|P\|_{L^2}^2\right)' + 2\int \left(\rho|\dot u|^2 + P^3\right) dx \\ &\le C\|F\|_{L^3}^3 + C\|\omega\|_{L^3}^3 \\ &\le \|\rho^{1/2}\dot u\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^4 + \|P\|_{L^2}^4\right), \end{aligned} \tag{5.16}$$

where in the second inequality we have used (5.6) and (5.8). Multiplying (5.16) by t, along with Gronwall's inequality, (5.15), (5.8), and (5.11), gives

$$\sup_{1\le t<\infty} t\left(\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2\right) + \int_1^\infty t\int \left(\rho|\dot u|^2 + P^3\right) dxdt \le C. \tag{5.17}$$

Then, multiplying (3.17) by t^2 together with (5.7) gives

$$\begin{aligned} &\left(t^2\int \rho|\dot u|^2 dx\right)_t + \mu t^2\int |\nabla \dot u|^2 dx \\ &\le 2t\int \rho|\dot u|^2 dx + Ct^2\|F\|_{L^4}^4 + Ct^2\|\omega\|_{L^4}^4 + \tilde C t^2\|P\|_{L^4}^4. \end{aligned} \tag{5.18}$$

Setting $p = 3$ in (5.13) and adding (5.13) multiplied by $2(2\mu+\lambda)(\tilde C+1)t^2/(3\gamma-1)$ to (5.18) lead to

$$\begin{aligned} &\left(t^2\int \rho|\dot u|^2 dx + \frac{2(2\mu+\lambda)(\tilde C+1)}{3\gamma-1}t^2\|P\|_{L^3}^3\right)_t + \mu t^2\|\nabla \dot u\|_{L^2}^2 + t^2\|P\|_{L^4}^4 \\ &\le Ct\int \left(\rho|\dot u|^2 + P^3\right) dx + Ct^2\|F\|_{L^4}^4 + Ct^2\|\omega\|_{L^4}^4 \\ &\le Ct\int \left(\rho|\dot u|^2 + P^3\right) dx + Ct^2\|\rho^{1/2}\dot u\|_{L^2}^3\left(\|\nabla u\|_{L^2} + \|P\|_{L^2}\right) \\ &\le Ct\int \left(\rho|\dot u|^2 + P^3\right) dx + Ct^2\|\rho^{1/2}\dot u\|_{L^2}^2\left(\|\rho^{1/2}\dot u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2\right), \end{aligned}$$

where in the second inequality we have used (5.6). This combined with Gronwall's inequality, (5.17), (5.8), and (5.11) yields

$$\sup_{1\le t<\infty} t^2\int\left(\rho|\dot u|^2+P^3\right)dx+\int_1^\infty t^2\left(\|\nabla\dot u\|_{L^2}^2+\|P\|_{L^4}^4\right)dt\le C. \tag{5.19}$$

This combined with (2.12) gives (1.27) provided we show that for $m=1,2,\cdots,$

$$\sup_{1\le t<\infty} t^m\|P\|_{L^{m+1}}^{m+1}+\int_0^\infty t^m\|P\|_{L^{m+2}}^{m+2}dt\le C(m), \tag{5.20}$$

which will be proved by induction. Since (5.17) shows that (5.20) holds for $m=1$, we assume that (5.20) holds for $m=n$, that is,

$$\sup_{1\le t<\infty} t^n\|P\|_{L^{n+1}}^{n+1}+\int_1^\infty t^n\|P\|_{L^{n+2}}^{n+2}dt\le C(n). \tag{5.21}$$

Setting $p=n+2$ in (5.12) and multiplying (5.12) by t^{n+1} give

$$\begin{aligned}&\frac{2(2\mu+\lambda)}{(n+2)\gamma-1}\left(t^{n+1}\|P\|_{L^{n+2}}^{n+2}\right)_t+t^{n+1}\|P\|_{L^{n+3}}^{n+3}\\&\le C(n)t^n\|P\|_{L^{n+2}}^{n+2}+C(n)t^{n+1}\|P\|_{L^{n+2}}^{n+2}\|F\|_{L^\infty}.\end{aligned} \tag{5.22}$$

It follows from the Gagliardo-Nirenberg inequality, (5.5), and (5.19) that

$$\begin{aligned}\int_1^\infty\|F\|_{L^\infty}dt&\le C\int_1^\infty\|F\|_{L^6}^{1/2}\|\nabla F\|_{L^6}^{1/2}dt\\&\le C\int_1^\infty\|\rho\dot u\|_{L^2}^{1/2}\|\rho\dot u\|_{L^6}^{1/2}dt\\&\le C\int_1^\infty t^{-1/2}\|\nabla\dot u\|_{L^2}^{1/2}dt\\&\le C,\end{aligned}$$

which, along with (5.22), (5.21), and Gronwall's inequality, thus shows that (5.20) holds for $m=n+1$. By induction, we obtain (5.20) and finish the proof of (1.27). The proof of Theorem 1.3 is completed.

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