COX RINGS OF RATIONAL SURFACES AND
FLAG VARIETIES OF ADE-TYPES

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Abstract. The Cox rings of del Pezzo surfaces are closely related to the Lie groups \( E_n \). In this paper, we generalize the definition of Cox rings to \( G \)-surfaces defined by us earlier, where the Lie groups \( G = A_n, D_n \) or \( E_n \). We show that the Cox ring of a \( G \)-surface \( S \) is closely related to an irreducible representation \( V \) of \( G \), and is generated by degree one elements. The Proj of the Cox ring of \( S \) is a sub-variety of the orbit of the highest weight vector in \( V \), and both are closed sub-varieties of \( \mathbb{P}(V) \) defined by quadratic equations. The GIT quotient of the Spec of such a Cox ring by a natural torus action is considered.

1. Introduction

This is a continuation of our studies in which flat \( G \)-bundles over an elliptic curve are related to rational surfaces \( S \) of type \( G \), where \( G \) is a Lie group of simply laced type in \([11]\) and non-simply laced type in \([12]\). The affine \( E_n \) case is considered in \([13]\). These studies generalize a classical result of Looijenga (\([16]\), \([17]\)), Friedman-Morgan-Witten (\([6]\)), Donagi (\([5]\)) and so on, about the case of \( G = E_n \) and del Pezzo surfaces.

For instance, an \( E_n \)-surface \( S \) is simply a blowup of a del Pezzo surface \( X_n \) of degree \( 9 - n \) at a general point, where \( X_n \) is a blowup of \( \mathbb{P}^2 \) at \( n \) points in general position. The del pezzo surface \( X_n \) is well-known to be closely linked to \( E_n \) (\([3]\), \([18]\)). For example, the orthogonal complement of the canonical class \( K_{X_n} \) in \( H^2(X_n, \mathbb{Z}) \), equipped with the natural intersection product, is the root lattice of \( E_n \) (\([3]\), \([18]\)), where we extend the exceptional \( E_n \)-series to \( 0 \leq n \leq 8 \) by setting \( E_0 = 0, E_1 = \mathbb{C}, E_2 = A_1 \times \mathbb{C}, E_3 = A_2 \times A_1, E_4 = A_4, \) and \( E_5 = D_5 \). Recall that for the del Pezzo surface \( X_n \), a curve \( l \) is called a line if \( l^2 = l \cdot K_{X_n} = -1 \) (which is really of degree 1 under the anti-canonical morphism for \( n \leq 7 \)). In \([11]\), we use these root lattices and lines to construct an adjoint principal \( E_n \)-bundle \( E_n \) over \( X_n \) and its representation bundle (that is, an associated principal \( E_n \)-bundle) \( L_{E_n} \) over \( X_n \) (corresponding to the left-end node in the Dynkin diagram, see Figure 1).

In Section 2.1, we describe a \( D_n \)-surface (resp. an \( A_n \)-surface) \( S \) as a rational surface with a fixed ruling \( S \to \mathbb{P}^1 \) (resp. a fixed birational morphism \( S \to \mathbb{P}^2 \)). Note that the description of \( A_n \)-surfaces is slightly different from the description in \([11]\), where it is more indirect. Here we use a more direct description to obtain the same root lattice. The results about \( A_n \)-surfaces cited from \([11]\) are all about lattice structures and hence keep true. Similar to the \( E_n \)-surface case, there is an adjoint principal \( D_n \)-bundle \( D_n \) (resp. an adjoint principal \( A_n \)-bundle \( A_n \)) over a \( D_n \)-surface (resp. an \( A_n \)-surface) and an associated bundle \( L_{D_n} \) (resp. \( L_{A_n} \))
determined by the lines on this surface. For simplicity, we also use $\mathcal{L}_G$ to denote the bundle $\mathcal{L}_{E_7}, \mathcal{L}_{D_n}$, or $\mathcal{L}_{A_n}$, in the context.

Moreover, both the vector space $V = H^0(S, \mathcal{L}_G)$ and any fiber of the bundle $\mathcal{L}_G$ are representations of $G$. The vector space $V$, or a subspace of it (denoted still by $V$), is just the corresponding fundamental representation of $G$ determined by the left-end node $\alpha_L = \alpha_n$. Thus we have $G/P \subset \mathbb{P}(V)$, where $P$ is the maximal parabolic subgroup of $G$ associated with $\alpha_n$.

In the classical $G = E_n$ case, the representations and the flag varieties $G/P$ are related to the Cox rings of the del Pezzo surfaces $X_n$.

The notion of Cox rings is introduced by D. Cox in [2] and formulated by Hu-Keel in [7]. Let $X$ be an algebraic variety. Assume that the Picard group Pic$(X)$ is freely generated by the classes of divisors $D_0, D_1, \cdots, D_r$. Then the total homogeneous coordinate ring, or the Cox ring of $X$ with respect to this basis is given by

$$\text{Cox}(X) := \bigoplus_{(m_0, \ldots, m_r) \in \mathbb{Z}^{r+1}} H^0(X, \mathcal{O}_X(m_0D_0 + \cdots + m_rD_r)),$$

with multiplication induced by the multiplication of functions in the function field of $X$. Different choices of bases yield (non-canonically) isomorphic Cox rings.

The Cox ring of $X$ is naturally graded by Pic$(X)$. Moreover, in the two-dimensional case, it is also graded by $\text{deg}(D) := (-K_X)D$, where $-K_X$ is the anti-canonical class of $X$.

In [2], it is shown that for a toric variety $X$, Cox$(X)$ is a polynomial ring with generators $t_E$, where $E$ runs over the irreducible components of the boundary $X \setminus U$ and $U$ is the open torus orbit. For a smooth del Pezzo surface $X_n$ of degree at most 6, Cox$(X_n)$ is finitely generated by sections of degree one elements (which are sections of $-1$ curves for $n \leq 7$; and in $X_8$ case, sections of $-1$ curves and two linearly independent sections of $-K_{X_8}$), and these generators satisfy a collection of quadratic relations (see [1], [4], [10], [21] etc). Thus in particular, a smooth del Pezzo surface is a Mori Dream Space in the sense of Hu-Keel ([7]), and as a result, the GIT quotient of Spec$(\text{Cox}(X_n))$ by the action of the Néron-Severi torus $T_{NS}$ of $X_n$ is isomorphic to $X_n$ ([7]).

The Cox rings of del Pezzo surfaces are closely related to universal torsors and homogeneous varieties (see for example [3], [8], [14], [20] etc). For the Lie group $G = E_n$ with $4 \leq n \leq 8$, it is shown that

$$\text{Proj}(\text{Cox}(X_n)) \to G/P \hookrightarrow \mathbb{P}(V),$$

where $V$ is the fundamental representation associated with the left-end node $\alpha_L$ in the Dynkin diagrams (see Figure 1,2,3), and the Proj is considered with respect to the anti-canonical grading.

Motivated from above, we want to give a geometric description of above results in terms of the representation bundle $\mathcal{L}_G$ and also generalize these results to all $ADE$ cases. In this paper, we show how the Lie groups, the representations and the flag varieties are tied together with the rational surfaces.

For this, let $S$ be a $G$-surface (Definition 3) with $G$ a simple Lie group of simply laced type. Let $\mathcal{L}_G$ be the fundamental representation bundle over $S$ determined by lines. Let $\mathcal{W}$ be the fundamental representation bundles determined by rulings (see Section 2.2). Let $\text{Sym}^2 \mathcal{L}_G$ be the second symmetric power of $\mathcal{L}_G$. Let $P$ be the maximal parabolic subgroup of $G$ associated with $\mathcal{L}_G$. 
Our main results are the following:

**Theorem 1.** (Theorem 9) Let $S$, $G$, $L_G$, and $W$ be as above. There is a canonical fiberwise quadratic form $Q$ on $L_G$, $$Q : L_G \to \text{Sym}^2 L_G \to W,$$ such that $\ker(Q) \subset \mathbb{P}(L_G)$ is a fiber bundle over $S$ with fiber being the homogeneous variety $G/P$, where $\ker(Q)$ is the subscheme of $\mathbb{P}(L_G)$ defined by $x \in \mathbb{P}(L_G)$, such that $Q(x) = 0$.

Moreover, by taking global sections, we realize $G/P$ as a subvariety of $\mathbb{P}(H^0(S, L_G))$ cut out by quadratic equations, for $G \neq E_8$. For $G = E_8$, we should replace $H^0(S, L_G)$ by a subspace $V$ of dimension 248.

We have a uniform definition for an $ADE$-surface in [11] (see also Section 2). Using this definition, we can give a uniform definition of the Cox ring of a $G$-surface $S$ (Definition 11), where $G$ is the $ADE$ Lie group. For $G = E_n$, it turns out that the Cox ring of an $E_n$-surface $S$ is the same as the Cox ring of a del Pezzo surface $X_n$ of degree $9 - n$. Let $\text{Cox}(S, G)$ be the Cox ring of a $G$-surface $S$. Let $T_G \subseteq P$ be the maximal subtorus of $G$, and $T_{S,G}$ be the torus defined in Section 3.3.

**Theorem 2.** (Theorem 14, 15, 16 and Proposition 18, 19)

1. The Cox ring of an $ADE$-surface $S$ is generated by degree 1 elements, and the ideal of relations between the degree 1 generators is generated by quadrics.

2. We have $\mathbb{C}^\times \times T_G$-equivariant embeddings:
   $$\text{Spec}(\text{Cox}(S, G)) \hookrightarrow C(G/P) \hookrightarrow H^0(S, L_G).$$

Taking the Proj, we have $T_G$-equivariant embeddings:
   $$\text{Proj}(\text{Cox}(S, G)) \hookrightarrow G/P \hookrightarrow \mathbb{P}(H^0(S, L_G)).$$

Both of the first two spaces are embedded into the last space as sub-varieties defined by quadratic equations.

3. The GIT quotient of $\text{Spec}(\text{Cox}(S, G))$ by the action of the torus $T_{S,G}$ is respectively $X_n$ for $G = E_n$, $\mathbb{P}^1$ for $G = D_n$, and a point for $G = A_n$.

Thus, we have a uniform description for Cox rings of $ADE$-surfaces and their relations to configurations of curves, representation theory and flag varieties, as is the purpose of this paper.

Note that in the $E_6$ and $E_7$ cases, the proof of the embedding $\text{Proj}(\text{Cox}(S, G)) \hookrightarrow G/P$ was achieved by Derenthal in [4] with the help of a computer program. Trying to simplify this proof is also a very interesting question. For $G = E_8$, the embedding was proved by Serganova and Skorobogatov (20). These results about $E_n(4 \leq n \leq 8)$ answer a conjecture of Batyrev and Popov (11). Here we just cite their results without new proofs.

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2. ADE-surfaces and Associated Principal G-bundles

Let $G = A_n, D_n,$ or $E_n$ be a complex (semi-)simple Lie groups. In this section, we first briefly recall the definitions and constructions of $G$-surfaces and associated principal $G$-bundles from [11]. After that, we study the quadratic forms defined fiberwise over these associated principal $G$-bundles.

2.1. ADE-surfaces. The definition of ADE-surfaces is motivated from the classical del Pezzo surfaces ([11]). According to the results of [11] and [11], over a del Pezzo surface $X_n(0 \leq n \leq 8)$ of degree $9 - n$, there is a root lattice structure of the Lie group $E_n$, and the lines and the rulings in $X_n$ can be related to the fundamental representations associated with the endpoints of the Dynkin diagram, via a natural way. Inspired by these, we can consider general $G$-surfaces, where $G = A_n, D_n,$ or $E_n$.

When the simply laced Lie group $G$ is simple, that is, $G = E_n$ for $4 \leq n \leq 8$, $A_n$ for $n \geq 1$, or $D_n$ for $n \geq 3$, we gave a uniform definition of ADE-surfaces in [11], using the pair $(S, C)$. It turns out that when $G = E_n$, after blowing down an exceptional curve, we obtain the classical del Pezzo surfaces $X_n$.

Notations. Let $h$ be the (divisor, the same below) class of a line in $\mathbb{P}^2$. Fix a ruled surface structure of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ over $\mathbb{P}^1$, and let $f, s$ be the classes of a fiber and a section in the natural projection from $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ to $\mathbb{P}^1$. If $S$ is a blowup of one of these surfaces, then we use the same notations to denote the pullback class of $h, f, s$, and use $l_1$ to denote the exceptional class corresponding to the blowup at a point $x_i$. Let $K_S$ be the canonical class of $S$. Since for $S$ the Picard group and the divisor class group are isomorphic, we use $Pic(S)$ to denote the divisor class group of $S$. The Picard group $Pic(S)$ is generated by $h, l_1, \ldots, l_n$ or by $f, s, l_1, \ldots, l_n$, respectively.

Definition 3. Let $(S, C)$ be a pair consisting of a smooth rational surface $S$ and a smooth rational curve $C \subset S$ with $C^2 \neq 4$. The pair $(S, C)$ is called an ADE-surface, or a $G$-surface for the Lie group $G = A_n, D_n$, or $E_n$ if it satisfies the following two conditions:

(i) any rational curve on $S$ has a self-intersection number at least $\; -1;

(ii) the sub-lattice $\langle K_S, C \rangle ^\perp$ of $Pic(S)$ is an irreducible root lattice of rank equal to $r - 2$, where $r$ is the rank of $Pic(S)$.

The following proposition shows that such surfaces can be classified into three types, and the curve $C$ in fact sits in the negative part of the Mori cone.

Proposition 4. ([11], Proposition 2.6) Let $(S, C)$ be an ADE-surface. Let $n = \text{rank}(\text{Pic}(S)) - 2$. Then $C^2 \in \{ -1, 0, 1 \}$ and

(i) when $C^2 = -1$, $\langle K_S, C \rangle ^\perp$ is of $E_n$-type, where $4 \leq n \leq 8$;

(ii) when $C^2 = 0$, $\langle K_S, C \rangle ^\perp$ is of $D_n$-type, where $n \geq 3$;

(iii) when $C^2 = 1$, $\langle K_S, C \rangle ^\perp$ is of $A_n$-type.

In the following corollary, $n$ points on $\mathbb{F}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ are said to be in general position, if the surface obtained by blowing up these points contains no irreducible rational curves with self-intersection number less than or equal to $-2$.

Corollary 5. Let $(S, C)$ be an ADE-surface.

(i) In the $E_n$ case, blowing down the $(-1)$ curve $C$ of $S$, we obtain a del Pezzo surface $X_n$ of degree $9 - n$. 


(ii) In the $D_n$ case, $S$ is just a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at $n$ points in general position with $C$ as the natural ruling.

(iii) In the $A_n$ case, the linear system $|C|$ defines a birational map $\varphi_C : S \to \mathbb{P}^2$. Therefore $S$ is just the blowup of $\mathbb{P}^2$ at $n+1$ points in general position, and $C$ is a smooth curve which represents the class determined by lines in $\mathbb{P}^2$.

**Corollary 6.** Let $(S, C)$ be an ADE-surface, and $G$ be the corresponding simple Lie group. The lattice $\text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ is the corresponding weight lattice. Hence its dual $\text{Hom}(\text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S), \mathbb{C}^*)$ is a maximal torus of $G$.

**Proof.** The intersection pairing $\langle C, K_S \rangle \perp \times \text{Pic}(S) \to \mathbb{Z}$ induces a perfect non-degenerate pairing

$$\langle C, K_S \rangle \perp \times \text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S) \to \mathbb{Z}. $$

Since $\langle C, K_S \rangle \perp$ is the (simply laced) root lattice of $G$, $\text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ is the weight lattice of $G$. And the last statement follows since $G$ is simply connected. $\square$

For convenience, we draw the Dynkin diagrams of the root lattices $\langle K_S, C \rangle \perp$ for the given ADE-surfaces $(S, C)$ as Figures 1-3.

![Figure 1. The root system $E_n; \alpha_1 = -h + l_1 + l_2 + l_3, \alpha_i = l_i - l_{i-1}, 2 \leq i \leq n$](image)

![Figure 2. The root system $D_n; \alpha_1 = -f + l_1, \alpha_i = l_i - l_{i-1}, 2 \leq i \leq n$](image)

![Figure 3. The root system $A_n; \alpha_i = l_i - l_{i-1}, 1 \leq i \leq n$](image)

In these Dynkin diagrams, we specify three special nodes: the top node $\alpha_T$, the right-end node $\alpha_R$ and the left-end node $\alpha_L$, if any. These special nodes determine three fundamental representation bundles.
Definition 7. Let \((S,C)\) be an ADE-surface.

1. A class \(l \in \text{Pic}(S)\) is called a line if \(l^2 = lK_S = -1\) and \(lC = 0\).
2. A class \(r \in \text{Pic}(S)\) is called a ruling if \(r^2 = 0, rK_S = -2\) and \(rC = 0\).
3. A section \(sD \in H^0(S, \mathcal{O}_S(D))\) is called of degree \(d\), if \(D(-K_S) = d\).

We denote the root system of the root lattice in Proposition 4 (respectively, the set of lines, the set of rulings) by \(R(S,C)\) (respectively, \(I(S,C), J(S,C)\)).

Note that there is a \(Z\)-basis for Pic\((S)\), such that all these sets and the curve \(C\) can be written down concretely (see [11] for details). The adjoint principal \(G\)-bundle (where \(G\) is of rank \(n\)) is

\[ G := \mathcal{O}_S^\oplus n \bigoplus_{\alpha \in R(S,C)} \mathcal{O}_S(\alpha). \]

The fundamental representation bundles determined by \(\alpha_L\), denoted by \(\mathcal{L}_G\), are the following (see [11] for details):

For \(G = E_n\) with \(3 \leq n \leq 7\),

\[ \mathcal{L}_{E_n} := \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l); \]

and for \(G = E_8\),

\[ \mathcal{L}_{E_8} := \mathcal{O}_S(-K_S)^\oplus 8 \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l) \cong \mathcal{E}_8 \otimes \mathcal{O}_S(-K_S). \]

For \(G = D_n\) and \(A_n\),

\[ \mathcal{L}_G := \bigoplus_{l \in I(S,C)} \mathcal{O}_S(l). \]

For \(G = E_n, 3 \leq n \leq 7\), the rulings in corresponding surfaces are used to construct the fundamental representation bundles \(\mathcal{R}_{E_n}\) determined by \(\alpha_R\) (see [11] for details):

For \(G = E_n\) with \(3 \leq n \leq 6\),

\[ \mathcal{R}_{E_n} := \bigoplus_{D \in J(S,C)} \mathcal{O}_S(D). \]

For \(G = E_7\),

\[ \mathcal{R}_{E_7} := \mathcal{O}_S(-K_S)^\oplus 7 \bigoplus_{D \in J(S,C)} \mathcal{O}_S(D) \cong \mathcal{E}_7 \otimes \mathcal{O}_S(-K_S). \]

We summarize some facts from [11] about these representation bundles in the following lemma.

Lemma 8. For any irreducible representation \(V_\lambda\) of \(G\) with the highest weight \(\lambda\), denote by \(\Pi(\lambda)\) or \(\Pi(V_\lambda)\) the set of all weights of \(V_\lambda\).

(i) For \(G = A_n-1, D_n,\) or \(E_n\), the exceptional class \(l_n\) represents the highest weight associated with \(\alpha_L\). Therefore \(\Pi(l_n) = I(S,C)\) for \(G \neq E_8\); and \(\Pi(l_8) = I(S,C) \cup \{-K_S\}\) for \(G = E_8\).

(ii) For \(G = E_n\), the class \(h - l_1\) represents the highest weight associated with \(\alpha_R\). Therefore \(\Pi(h - l_1) = J(S,C)\) for \(3 \leq n \leq 6\); \(\Pi(h - l_1) = J(S,C) \cup \{-K_S\}\) for \(n = 7\); and \(J(S,C) \subset \Pi(h - l_1)\) for \(n = 8\).

Proof. (i) According to Figure 1, 2 and 3, by the definition of the pairing between weights and roots in Page 759 of [11], we see that \(l_n(\alpha_L) = -l_n \cdot \alpha_L = 1\), while
\( l_n(\alpha_i) = -l_n \cdot \alpha_i = 0 \), if \( \alpha_i \neq \alpha_L \). Thus \( l_n \) represents the highest weight associated with \( \alpha_L \).

For \( G \neq E_8 \), \( l_n \) is minuscule (that is, \( W(G) \) acts on \( \Pi(l_n) \) transitively), and by Lemma 11, \( W(G) \) acts on \( I(S, C) \) transitively. Therefore \( \Pi(l_n) = I(S, C) \).

For \( G = E_8 \), \( -K_S \in \Pi(l_8) \) because \( -K_S = l_8 - (3h + l_1 + \cdots + l_7 + 2l_8) \) and \( -3h + l_1 + \cdots + l_7 + 2l_8 \) is a positive root of \( E_8 \). In fact, \( -K_S \) is the zero weight in \( \Pi(l_8) \) (that is, \( W(E_8) \) acts on \( -K_S \) trivially). Now \( l_8 \) is quasi-minuscule (that is, \( W(G) \) acts on non-zero weights of \( \Pi(l_8) \) transitively), and by Lemma 18 or Lemma 11, \( W(G) \) acts on \( I(S, C) \) transitively. Therefore \( \Pi(l_8) = I(S, C) \cup \{-K_S\} \).

(ii) The proof is similar. \( \square \)

2.2. Quadratic forms over associated bundles. Let \( V_\lambda \) be a fundamental representation of a semisimple Lie group \( G \) with the fundamental weight \( \lambda \). Let \( \text{Sym}^2 V_\lambda \) be the second symmetric product of \( V \). Since \( 2\lambda \) is the highest weight in the weight set of \( \text{Sym}^2 V_\lambda \), \( V_{2\lambda} \) is a summand of the representation \( \text{Sym}^2 V_\lambda \), where \( V_{2\lambda} \) is the fundamental representation associated with the highest weight \( 2\lambda \). Therefore there is another representation \( W \) such that \( \text{Sym}^2 V_\lambda = W \bigoplus V_{2\lambda} \).

With the help of the program LiE ([15]), we list the decomposition of \( \text{Sym}^2 V_\lambda \) for simply laced Lie group \( G \) with \( \lambda \) the fundamental weight associated with \( \alpha_L = \alpha_n \) (see Figure 1, 2 and 3).

In the \( G = E_n \) case, for \( 4 \leq n \leq 6 \), \( W \) is a non-trivial irreducible \( G \)-module of the least dimension, which is a minuscule representation of \( G \). If \( r = 7 \), then \( W \) is the adjoint representation, which is quasi-minuscule (that is, all the non-zero weights have multiplicity 1 and form one orbit of the Weyl group \( W(E_7) \) of \( E_7 \)). If \( r = 8 \), then \( W = W_1 \bigoplus \mathbb{C} \), where \( W_1 \) is the irreducible representation associated with the node \( \alpha_R \) (of dimension 3875), and \( \mathbb{C} \) is the trivial representation.

In the \( G = D_n \) case, \( W = \mathbb{C} \) is the trivial representation.

In the \( G = A_n \) case, \( W = \{0\} \), that is, \( \text{Sym}^2 V_\lambda = V_{2\lambda} \).

Let \( P \) be the maximal parabolic subgroup of \( G \) corresponding to the fundamental representation \( V_\lambda \). Then we have a homogeneous variety \( G/P \). It is well-known that \( G/P \hookrightarrow \mathbb{P}(V_\lambda) \) is a subvariety defined by quadratic relations ([14]). A way to write explicitly the quadratic relations is the following. Let \( C(G/P) \) be the affine cone over \( G/P \). Let \( pr \) be the natural projection \( \text{Sym}^2 V_\lambda \rightarrow W \), and \( \text{Ver} : V_\lambda \rightarrow \text{Sym}^2 V_\lambda \) be the Veronese map \( x \mapsto x^2 \), then it is well known that \( C(G/P) \) is the fibre \( (pr \circ \text{Ver})^{-1}(0) \) (as a scheme, see [1] Proposition 4.2 and references therein).

Thus the homogeneous variety \( G/P \) is defined by the quadratic form:
\[
Q : V_\lambda \rightarrow \text{Sym}^2 V_\lambda \rightarrow W.
\]

In fact, we can show that the quadratic form could be globally defined over fundamental representation bundles:
\[
Q : L_G \rightarrow \text{Sym}^2 L_G \rightarrow W,
\]
such that \( G/P \) is fiberwise defined by \( Q \).

Let \( L_G \) be the fundamental representation bundle defined as in the end of Section 2.1 by lines on an \( ADE \)-surface \( S \). That is, \( L_G \) corresponds to the left-end node \( \alpha_L \) (or equivalently, associated with the fundamental weight \( l_n \) corresponding to \( \alpha_L \) for \( G = A_{n-1}, D_n \) or \( E_n \), by Lemma 8). For a quadratic form over a vector bundle \( L_G \), we denote \( Q^{-1}(0) \) the subscheme of \( \mathbb{P}(L_G) \) defined by \( x \in \mathbb{P}(L_G) \), such that \( Q(x) = 0 \).
By Lemma \[8\] 
\[ \mathcal{L}_G = \bigoplus_{\mu \in \Pi(l_n) \subseteq \text{Pic}(S)} \mathcal{O}_S(\mu) \oplus k_\mu, \]
where the multiplicity \( k_\mu = 8 \) if \( \mu = -K_S \) and \( G = E_8 \); otherwise, \( k_\mu = 1 \). Let \( \Pi(Sym^2 \mathcal{L}_G) \) be the set of weights of \( Sym^2 \mathcal{L}_G \) which is saturated (see Section 13.4 of \[12\]). Then
\[ \Pi(Sym^2 \mathcal{L}_G) = \{ \lambda_1 + \lambda_2 | \lambda_1, \lambda_2 \in \Pi(l_n) \} \subseteq \text{Pic}(S), \]
and
\[ Sym^2 \mathcal{L}_G = \bigoplus_{\mu \in \Pi(Sym^2 \mathcal{L}_G) \subseteq \text{Pic}(S)} \mathcal{O}_S(\mu) \oplus m_\mu, \]
where \( m_\mu \) is the multiplicity uniquely determined by \( \mu \) and \( Sym^2 \mathcal{L}_G \). Since \( 2l_n \) occurs with multiplicity one, by the saturatedness, \( \Pi(2l_n) \subseteq \Pi(Sym^2 \mathcal{L}_G) \). Therefore \( Sym^2 \mathcal{L}_G \) contains a summand \( \mathcal{V}_{2l_n} \) which is an irreducible representation bundle associated with the highest weight \( 2l_n \). We write \( \mathcal{V}_{2l_n} \) as 
\[ \mathcal{V}_{2l_n} = \bigoplus_{\mu \in \Pi(Sym^2 \mathcal{L}_G) \subseteq \text{Pic}(S)} \mathcal{O}_S(\mu) \oplus n_\mu, \]
where \( n_\mu = 0 \) if \( \mu \notin \Pi(2l_n) \) and \( 1 \leq n_\mu \leq m_\mu \) if \( \mu \in \Pi(2l_n) \).

The other summand \( \mathcal{W} \) of \( Sym^2 \mathcal{L}_G \) is automatically a representation bundle:
\[ \mathcal{W} = \bigoplus_{\mu \in \Pi(Sym^2 \mathcal{L}_G) \subseteq \text{Pic}(S)} \mathcal{O}_S(\mu) \oplus (m_\mu - n_\mu). \]

We are mainly interested in the representation bundle \( \mathcal{W} \), which we discuss case by case according to \( G = E_n, D_n \) or \( A_{n-1} \).

(i) For \( G = E_n \), \( h - l_1 \in \Pi(Sym^2 \mathcal{L}_G) \). By \[15\], \( \mathcal{W} \) contains a weight space with the weight \( h - l_1 \). Thus the set \( J(S,C) \) of rullings on \( S \) are contained in the set \( \Pi(\mathcal{W}) \) of weights of \( \mathcal{W} \). Therefore, as a vector bundle, \( \mathcal{W} \) contains \( \bigoplus_{\mu \in J(S,C)} \mathcal{O}_S(\mu) \) as summands. By counting the rank of \( \mathcal{W} \) (\[15\]) and the number of the elements of \( J(S,C) \), we find that for \( 4 \leq n \leq 7 \),
\[ \mathcal{W} = \bigoplus_{\mu \in J(S,C)} \mathcal{O}_S(\mu) = \mathcal{R}_{E_n} \]
is the irreducible representation bundle associated with \( a_R \) (Lemma \[8\]).

Similarly, for \( G = E_8 \), by \[15\], \( \mathcal{W} \) is a direct sum of \( \mathcal{R}_{E_8} \) and a line bundle which is a trivial representation. Note that among the weights of \( Sym^2 \mathcal{L}_G \), only \(-2K_S \) appears as a zero weight (Lemma \[8\]). Thus the line bundle considered here is nothing but \( \mathcal{O}_S(-2K_S) \). Therefore
\[ \mathcal{W} = \mathcal{R}_{E_8} \bigoplus \mathcal{O}_S(-2K_S). \]

(ii) For \( G = D_n \), \( f \in \Pi(Sym^2 \mathcal{L}_G) \). By \[15\], \( \mathcal{W} \) is a line bundle which is a trivial representation bundle. Note that the only zero weight of \( Sym^2 \mathcal{L}_G \) is \( f \). Therefore \( \mathcal{W} \cong \mathcal{O}_S(f) \).

(iii) For \( G = A_{n-1} \), by a dimension counting, \( Sym^2 \mathcal{L}_G \cong \mathcal{V}_{2l_n} \). Therefore \( \mathcal{W} = 0 \).

Thus we achieved the first statement of the following theorem.

**Theorem 9.** The notations are as above.
(1) We have a decomposition of representation bundles:

$$\text{Sym}^2 L_G = W \bigoplus V_{2l_n}.$$  

Here $W = R_{E_n}$ for $G = E_n$ with $4 \leq n \leq 7$; $W = R_{E_8} \bigoplus \mathcal{O}_S(-2K_S)$ for $G = E_8$; $W = \mathcal{O}_S(f)$ for $G = D_n$; and $W = 0$ for $G = A_{n-1}$.

(2) The projection to the first summand defines a quadratic form on $L_G$

$$Q : L_G \to \text{Sym}^2 L_G \to W,$$

such that the homogeneous variety $G/P$ is the fiber of the subscheme (considered as a scheme defined over $S$) $\mathbb{P}(Q^{-1}(0)) \subseteq \mathbb{P}(L_G)$.

(3) By taking global sections, for $G \neq E_8$, we realize $G/P$ as a subvariety of $\mathbb{P}(H^0(S, L_G))$, cut out by quadratic equations. For $G = E_8$, we replace $H^0(S, L_G)$ by a subspace $V$ of dimension 248, where $V = \mathbb{C}(s_K) \bigoplus \bigoplus_{\mu \in I(S,G)} H^0(S, \mathcal{O}_S(\mu))$

with $s_K$ a fixed non-zero global section of $\mathcal{O}_S(-K_S)$.

Proof. It remains to verify (2) and (3), which are essentially consequences of (1).

(2). Note that fiberwise, the map

$$Q : L_G \to \text{Sym}^2 L_G = W \bigoplus V_{2l_n} \to W,$$

is exactly the map (Lemma 8)

$$Q : V_{l_n} \to \text{Sym}^2 V_{l_n} \cong W \bigoplus V_{2l_n} \to W,$$

where $V_{l_n}, W$ and $V_{2l_n}$ are as in the beginning of Section 2.2.

By [14], $Q^{-1}(0)$ is the cone over $G/P$ in $V_{l_n}$, that is $\mathbb{P}(Q^{-1}(0)) = G/P \subseteq \mathbb{P}(V_{l_n})$.

(3). First by [11], every element $\mu \in I(S,G)$ is represented by a unique irreducible curve in an $ADE$-surface $S$ and hence $\dim H^0(S, \mathcal{O}_S(\mu)) = 1$. For $G \neq E_8$, recall that $L_G = \bigoplus_{\mu \in I(S,G)} \mathcal{O}_S(\mu)$. Therefore we can choose a unique global section for each summand of $L_G$ up to a constant.

By [14], $C(G/P) \subseteq V_{l_n}$ is defined by finitely many quadratic polynomials. Let $f(x_{\mu|\mu \in I(S,G)})$’s be such polynomials. Let $s_\mu$ be the global section of $\mathcal{O}_S(\mu), \mu \in I(S,G)$. Then

$$H^0(S, L_G) = \{ \sum_{\mu \in I(S,G)} x_\mu s_\mu | x_\mu \in \mathbb{C} \},$$

and the same polynomials $f(x_{\mu|\mu \in I(S,G)})$’s define $G/P$.

For $G = E_8$, since $H^0(S, \mathcal{O}_S(-K_S))$ is of dimension two, we should fix any one non-zero global section $s_K$ of $\mathcal{O}_S(-K_S)$. Similarly by [14], $C(G/P) \subseteq V_{l_8}$ is defined by finitely many quadratic polynomials. Let $f(x_{\mu|\mu \in I(S,G)})$’s be such polynomials. Thus, we take a subspace of $H^0(S, L_G)$ of dimension 248 as follows:

$$V = \mathbb{C}(s_K) \bigoplus \bigoplus_{\mu \in I(S,G)} H^0(S, \mathcal{O}_S(\mu)).$$

As a vector space $V = \{ x_1 s_{K,1} + \cdots + x_8 s_{K,8} + \sum x_\mu s_\mu | x_i, x_\mu \in \mathbb{C} \}$ where $s_{K,i} = s_K$ is the basis of the $i$-th $\mathbb{C}(s_K)$, and the same polynomials $f$’s define $G/P$.

\[ \square \]

Remark 10. The bundle $W$ appearing in Theorem 9 can be called the representation bundle determined by rulings, since in the $G = D_n$ and $E_n$ cases, it is constructed by using the rulings.
3. Cox rings of $ADE$-surfaces and flag varieties

3.1. Cox rings of $ADE$-surfaces. The notion of Cox rings is defined by Cox (2) for toric varieties and he shows that for a toric variety, its Cox ring is precisely its factorial projective varieties $X$. Then for the $E$ curve, let

Remark 12. Let $S$ be chosen as in Section 2.1. Then we define the Cox ring of $S$ as

$$Cox(S) := \bigoplus_{D \in Pic(S), DC=0} H^0(S, \mathcal{O}_S(D)),$$

with a well-defined multiplication (see Section 7).

Notice that $Cox(S)$ is naturally graded by the degree defined in Definition 7.

Remark 12. As usual, let $X_n$ be a del Pezzo surface of degree $9-n$ with $4 \leq n \leq 8$, let $S \to X_n$ be a blowup at a general point, and $C$ be the corresponding exceptional curve. Then for the $E_n$-surface $(S, C)$, we have

$$Cox(S, E_n) \cong \bigoplus_{D \in Pic(X_n)} H^0(X_n, \mathcal{O}_{X_n}(D)) = Cox(X_n).$$

Thus the definition of Cox rings of $E_n$-surfaces is the same as the classical definition of Cox rings for del Pezzo surfaces $X_n$. The reason for the displayed isomorphism is that the contraction morphism $\pi : S \to X_n$ induces an isomorphism $\pi^* : Pic(X_n) \to C^\perp \subseteq Pic(S)$ such that the pull-back of rational functions $H^0(X_n, \mathcal{O}_{X_n}(D)) \to H^0(S, \mathcal{O}_S(\pi^*D))$ is an isomorphism for any divisor $D$ of $X_n$.

Corollary 13. 1) For the $D_n$-surface $(S, C)$, $C \equiv f$ is a smooth fiber. Then we have

$$Cox(S, D_n) = \bigoplus_{D \in Pic(S), Df=0} H^0(S, \mathcal{O}_S(D)).$$

2) For the $A_n$-surface $(S, C)$, $C \equiv h$ (linear equivalence) is a twisted cubic. Then we have

$$Cox(S, A_n) = \bigoplus_{D \in Pic(S), Dh=0} H^0(S, \mathcal{O}_S(D)).$$

Theorem 14. Let $G = A_n$, $D_n (n \geq 3)$ or $E_n (4 \leq n \leq 8)$. The Cox ring $Cox(S, G)$ is finitely generated, and generated by degree 1 elements. For $G \neq E_8$, the generators of $Cox(S, G)$ are global sections of invertible sheaves defined by lines on $S$. For $G = E_8$, we should add to the above set of generators two linearly independent global sections of the anti-canonical sheaf on $X_8$. 

Proof. Let $f,s,h,l_i$’s be as in Section 2.1 (Notations).
1) For the $G = E_n$ case, see [1], [4], [10] and [19].
2) For the $G = D_n$ case, let $D \in \text{Pic}(S)$ and $DF = 0$. Assume that $D$ is effective. Then we can write $D \equiv \sum a_i D_i$ (here `$\equiv$’ means the linear equivalence) with $D_i$ irreducible curves and $a_i \geq 0$. Choose a smooth fiber $F$. Then $D_i F \geq 0$. Thus $DF = 0$ implies $a_i = 0$ or $D_i F = 0$ for all $i$. By the Hodge index theorem, $D_i F = 0$ implies that $D_i \equiv F$ or $D_i = l_j$ or $D_i = f - l_k$, for some $j,k$. Thus

$$D \equiv a_0 F + \sum a_i l_i + \sum_{j} b_j (f - l_j)$$

with $a_0, a_i, b_j \geq 0$. Moreover, we can assume that $\{i \mid a_i \neq 0\} \cap \{j \mid b_j \neq 0\} = \emptyset$.

Let $x_i$ (resp. $y_i$) be a nonzero global section of $O_S(l_i)$ (resp. $O_S(f - l_i)$).

Thus, by induction, we can show that

$$\dim H^0(S, \mathcal{O}_S(D)) = \dim H^0(S, \mathcal{O}_S(a_0 F)) = a_0 + 1.$$ 

The proof goes as follows. We have a short exact sequence

$$0 \rightarrow \mathcal{O}_S(a_0 F + (a_i - 1) l_i) \rightarrow \mathcal{O}_S(a_0 F + a_i l_i) \rightarrow \mathcal{O}_l(a_0 F + a_i l_i) \rightarrow 0.$$ 

Note that $O_l(a_0 F + a_i l_i) \cong O_{F.l}(-a_i)$, since $l_i \cong \mathbb{P}^1$ and $l_i(a_0 F + a_i l_i) = -a_i$. Thus we have a long exact sequence

$$0 \rightarrow \begin{array}{c} H^0(S, \mathcal{O}_S(a_0 F + (a_i - 1) l_i)) \rightarrow \end{array} H^0(S, \mathcal{O}_S(a_0 F + a_i l_i)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) \rightarrow$$

$$\begin{array}{c} H^1(S, \mathcal{O}_S(a_0 F + (a_i - 1) l_i)) \rightarrow \cdots. \end{array}$$

When $a_i \geq 1$, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) = 0$, and therefore

$$H^0(S, \mathcal{O}_S(a_0 F + (a_i - 1) l_i)) \cong H^0(S, \mathcal{O}_S(a_0 F + a_i l_i)).$$

Hence by induction we have

$$H^0(S, \mathcal{O}_S(a_0 F)) \cong H^0(S, \mathcal{O}_S(a_0 F + a_i l_i)).$$

By repeating this process, we have

$$H^0(S, \mathcal{O}_S(D)) \cong H^0(S, \mathcal{O}_S(a_0 F)).$$

It remains to prove $\dim H^0(S, \mathcal{O}_S(a_0 F))) = a_0 + 1$. Also this comes from the following short exact sequence:

$$0 \rightarrow \mathcal{O}_S((a_0 - 1) F) \rightarrow \mathcal{O}_S(a_0 F) \rightarrow \mathcal{O}_F(a_0 F) \rightarrow 0.$$ 

Here $\mathcal{O}_F(a_0 F) \cong \mathcal{O}_{\mathbb{P}^1}$, since $F \cong \mathbb{P}^1$ and $(a_0 F) F = 0$. Taking the long exact sequence, we have

$$0 \rightarrow \begin{array}{c} H^0(S, \mathcal{O}_S((a_0 - 1) F)) \rightarrow \end{array} H^0(S, \mathcal{O}_S(a_0 F)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$$

$$\rightarrow \begin{array}{c} H^1(S, \mathcal{O}_S((a_0 - 1) F)) \rightarrow \end{array} H^1(S, \mathcal{O}_S(a_0 F)) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \rightarrow \cdots.$$ 

Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, we shall have $H^1(S, \mathcal{O}_S(a_0 F)) = 0$ if $H^1(S, \mathcal{O}_S((a_0 - 1) F)) = 0$. For $a_0 = 1$, we have $H^1(S, \mathcal{O}_S((a_0 - 1) F)) = H^1(S, \mathcal{O}_S) = 0$, since $S$ is a rational surface. Thus by induction, we have for all $a_0 \geq 0$, $H^1(S, \mathcal{O}_S(a_0 F)) = 0$.

Then from the last long exact sequence we have

$$\dim H^0(S, \mathcal{O}_S(a_0 F)) = \dim H^0(S, \mathcal{O}_S((a_0 - 1) F)) + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$$

$$= \dim H^0(S, \mathcal{O}_S((a_0 - 1) F)) + 1.$$ 

Therefore by induction, we have

$$\dim H^0(S, \mathcal{O}_S(a_0 F)) = a_0 + 1.$$
Let \( H^0(S, \mathcal{O}_S(F)) = \mathbb{C}(v_1, v_2) \), where \( v_1, v_2 \) are two linearly independent global sections of \( \mathcal{O}_S(F) \). Then the linearly independent generators of \( H^0(S, \mathcal{O}_S(D)) \) can be taken as \( u_k(\Pi_i x_i^{a_i}(\Pi_j y_j^{b_j})) \), where \( u_k = u_k^{a_0-k}, k = 0, \ldots, a_0 \).

Let \( n \geq 2 \). Thus we have at least two different singular fibers: \( l_1 + (f - l_1) \) and \( l_2 + (f - l_2) \). Then \( x_1y_1 \) and \( x_2y_2 \) are linearly independent elements in \( H^0(S, \mathcal{O}_S(F)) \). Thus we can take \( v_1 = x_1y_1 \), \( v_2 = x_2y_2 \).

Therefore, the Cox ring is generated by global sections of the invertible sheaves defined by lines (when \( n \geq 2 \)).

In fact, if \((x) = F'\) is a smooth fiber, then we must have
\[
x = a(x_1y_1) + b(x_2y_2),
\]
with \(a \neq 0\) and \(b \neq 0\).

3) For the \( G = A_n \) case, let \( D \in \text{Pic}(S) \), such that \( D bah = 0 \). Then obviously, \( D \equiv a_1l_1 + \cdots + a_{n+1}l_{n+1} \). \( D \geq 0 \) if and only if \( a_i \geq 0 \). Let \( x_i \neq 0 \) be a global section of \( \mathcal{O}_S(l_i), 1 \leq i \leq n+1 \). Note that
\[
\dim H^0(S, \mathcal{O}_S(a_1l_1 + \cdots + a_{n+1}l_{n+1})) = 1,
\]
and \( x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} \) generates the one-dimensional vector space \( H^0(S, \mathcal{O}_S(a_1l_1 + \cdots + a_{n+1}l_{n+1})) \). By Definition 7
\[
\deg(x_1^{a_1} \cdots x_{n+1}^{a_{n+1}}) := D(-K_S) = a_0 + \cdots + a_{n+1}.
\]
Thus, the Cox ring is in fact a polynomial ring with \( n+1 \) variables:
\[
\text{Cox}(S, A_n) = k[x_1, \cdots, x_{n+1}].
\]

By this theorem, the Cox ring \( \text{Cox}(S, G) \) of a \( G \)-surface \( S \) is a quotient of the polynomial ring \( P(S, G) = k[x_1, \cdots, x_{N_G}] \) by an ideal \( I(S, G) \):
\[
\text{Cox}(S, G) = k[x_1, \cdots, x_{N_G}] / I(S, G),
\]
where \( N_G \) is the number of lines (Definition 7) in the \( G \)-surface \( S \) for \( G \neq E_8 \); for \( \text{G} = E_8 \), \( N_G \) is the number of lines plus 8.

**Theorem 15.** For any ADE-surface \( S \), the ideal \( I(S, G) \) is generated by quadrics.

**Proof.**
1) For \( G = A_n \), the ideal \( I(S, G) = 0 \).
2) For \( G = E_n \), see [4] for \( 4 \leq n \leq 7 \) and [20], [21] for \( n = 8 \).
3) For \( G = D_n \), let \( x_i, y_i \) be as in the proof of Theorem 14. We want to show that
\[
\text{Cox}(S, D_n) = k[x_1, y_1, \cdots, x_n, y_n] / I(S, D_n),
\]
where
\[
I(S, D_n) = \langle a_{31}x_1y_1 + a_{32}x_2y_2 + a_{33}x_3y_3, \cdots, a_{n1}x_1y_1 + a_{n2}x_2y_2 + a_{n3}x_3y_3 \rangle,
\]
and all \( a_{ij} \neq 0 \).

By the proof of Theorem 14 we see that all the generating relations come from the ruling \( f \). The vector space \( H^0(S, \mathcal{O}_S(f)) \) is a two-dimensional space. Moreover, any two singular fibers are different. Therefore, when \( n \geq 3 \), any two elements of \( \{x_1y_1, \cdots, x_ny_n\} \) are linearly independent, and any three elements are linearly dependent. Thus, the ideal \( I(S, D_n) \) is of desired form. \( \square \)

### 3.2. Cox rings and flag varieties.

Let \( G \) be a complex simple Lie groups, and \( \Lambda \) be a fundamental weight. Let \( P \) be the corresponding maximal parabolic subgroup
For Proof.

Proj

Let Theorem 16.

\[ X \text{ showed in } [4] \text{ that for the del Pezzo surfaces } X_n \text{ with } n = 6, 7, \]

\[ \text{Spec}(\text{Cox}(X_n)) \hookrightarrow C(E_n/P). \]

Here \( P \) is the maximal parabolic subgroup determined by the left-end node \( \alpha_L \) in the Dynkin diagram (Figure 1), and \( C(E_n/P) \) is the affine cone over the homogeneous space \( E_n/P \).

Given an \( ADE \)-surface \( S \), we let \( L_G \) be the representation bundle determined by lines on \( S \). The vector space of global sections \( H^0(S, L_G) \) is the fundamental representation of \( G \) associated with the node \( \alpha_L \) (for \( G = E_8 \) we should replace it by a subspace \( V \)).

The following result relates the Cox ring of a \( G \)-surface with the homogeneous variety \( G/P \). Thus we obtain a uniform description of the relation between the Cox rings \( \text{Cox}(S, G) \), the homogeneous space \( G/P \), and fundamental representation bundles defined by lines in \( S \), for any Lie group \( G = A_n, D_n, E_n \).

**Theorem 16.** Let \( G = A_n, D_n \) or \( E_n \). Let \( S, L_G \) and \( P \) be as above. We have an embedding: \( \text{Proj}(\text{Cox}(S, G)) \hookrightarrow G/P \).

**Proof.** For \( G = E_n \) and \( 4 \leq n \leq 7 \), the result is known, see [1] and [4]. For \( G = E_8 \), see [20].

For \( G = A_n \), in fact we have an isomorphism: \( \text{Proj}(\text{Cox}(S, G)) \cong A_n/P \cong \mathbb{P}^n \).

For \( G = D_n \), by the proof of Theorem [15]

\[ \text{Cox}(S, D_n) = k[x_1, y_1, \ldots, x_n, y_n]/\mathcal{I}(S, D_n), \]

where

\[ \mathcal{I}(S, D_n) = (a_{31}x_1y_1 + a_{32}x_2y_2 + a_{33}x_3y_3, \ldots, a_{n1}x_1y_1 + a_{n2}x_2y_2 + a_{n3}x_ny_n, \]

and all \( a_{ij} \neq 0 \). By Lemma [17] the affine coordinate ring of \( C(D_n/P) \) is

\[ k[x_1, y_1, \ldots, x_n, y_n]/(x_1y_1 + \cdots + x_ny_n). \]

There exist nonzero \( b_i, 3 \leq i \leq n \), such that the coefficient of the term \( x_iy_i \) in the sum \( \sum_{3 \leq i \leq n} b_i(a_{i1}x_1y_1 + a_{i2}x_2y_2 + a_{i3}x_iy_i) \) is nonzero, by dimension counting.

Therefore, we have a surjective homomorphism from the affine coordinate ring of \( C(G/P) \) to that of \( \text{Spec}(\text{Cox}(S, G)) \), which defines a closed embedding

\[ \text{Spec}(\text{Cox}(S, G)) \hookrightarrow C(G/P). \]

The following result is well-known. But since we can not find an appropriate reference, we include its proof here.

**Lemma 17.** Let \( V_n = \bigoplus_{\mu \in I(S, G)} V_\mu \) be the irreducible representation of \( D_n = SO(2n, \mathbb{C}) \) associated with the highest weight \( l_n \). Let \( P \) be the maximal hyperbolic subgroup of \( D_n \) associated with the highest weight \( l_n \).

Then there exists a basis \( \{ u_i, v_i \}_{i = 1, \ldots, n} \) for \( V_n \) such that \( D_n/P \subseteq P(V_n) \) is defined by the quadratic equation \( Q(x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^{n} x_iy_i = 0 \).

**Proof.** By [11], \( \Pi(l_n) = I(S, G) = \{ l_i, f - l_i \}_{i = 1, \ldots, n} \). Hence we can take a basis for \( V_n \) as \( \{ u_i, v_i \}_{i = 1, \ldots, n} \), where for \( 1 \leq i \leq n \), \( u_i \) (resp. \( v_i \)) is the basis for the one-dimensional weight space with the weight \( l_i \) (resp. \( f - l_i \)).
As in the beginning of Section 2.2, let $Q$ be the composite map 

$$Q : V_n \to Sym^2 V_n = V_{2n} \bigoplus W \to W,$$

where $W \cong \mathbb{C}$ is the trivial representation. By [14], $D_n/P \subseteq \mathbb{P}(V_n)$ is defined by the equation $Q = 0$. We only need to write down $Q$ explicitly.

The following map

$$Q' : V_n = \mathbb{C}^{2n} \langle u_i, v_i \mid 1 \leq i \leq n \rangle \to \mathbb{C} \quad \sum (x_i u_i + y_i v_i) \to \sum x_i y_i$$

defines a non-degenerate symmetric quadratic form which is $D_n$-invariant. (In fact, $D_n = SO(2n, \mathbb{C})$ is the Lie group preserving this non-degenerate symmetric quadratic form with determinant one.) Therefore by the Complete Reducibility Theorem for semi-simple Lie groups, $\mathbb{C}$ is a summand of the $D_n$-module $Sym^2 V_n$. Hence $W \cong \mathbb{C}$ and $Q = Q'$.

### 3.3. Cox rings and the GIT quotients.

Let $(S, C)$ be a $G$-surface. The subset $C^\perp$ of $\text{Pic}(S)$ is a free abelian group of rank equal to $\text{rank}(\text{Pic}(S)) - 1$. We have the following short exact sequence:

$$0 \to \mathbb{Z}C \to \text{Pic}(S) \to \text{Pic}(S)/\mathbb{Z}C \to 0.$$ 

Taking the dual, we have 

$$1 \to \text{Hom}(\text{Pic}(S)/\mathbb{Z}C, \mathbb{C}^*) \to T_{NS} \to \mathbb{C}^* \to 1.$$ 

We denote the torus $\text{Hom}(\text{Pic}(S)/\mathbb{Z}C, \mathbb{C}^*)$ by $T_{S,G}$. Note that, for $G = E_n$, $T_{S,G}$ is exactly the Néron-Severi torus $T_{NS}$ of the del Pezzo surface $X_n$ obtained by blowing down $C$ from $S$.

The torus $T_{S,G}$ is an extension of $\mathbb{C}^*$ by a maximal torus $T_G$ of $G$. One can see that the lattice $\langle C, K_S \rangle^\perp$ is a sublattice of $C^\perp$ of rank equal to $\text{rank}(C^\perp) - 1$. In fact we have a short exact sequence:

$$0 \to \mathbb{Z}K_S \to \text{Pic}(S)/\mathbb{Z}C \to \text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S) \to 0.$$ 

Since the character group $\chi(T_G)$ of $T_G$ is isomorphic to the weight lattice $\text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ (if we take $G$ to be the simply connected one), we have $T_G \cong \text{Hom}(\text{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S), \mathbb{C}^*)$, by Corollary [6]. Therefore the following sequence is exact:

$$1 \to T_G \to T_{S,G} \to \mathbb{C}^* \to 1.$$ 

The torus $T_{S,G}$ acts on $\text{Cox}(S, G)$ (and therefore acts on $\text{Spec}(\text{Cox}(S, G))$) naturally.

**Proposition 18.** The embeddings 

$$\text{Proj}(\text{Cox}(S, G)) \hookrightarrow G/P \hookrightarrow \mathbb{P}(V_n)$$

arising in Theorem [16] are $T_G$-equivariant.

**Proof.** This is known for $G = E_n$ by [11], [4] and [19].

For $G = D_n$, since our coordinate system $\{x_i, y_i \mid i = 1, \ldots, n\}$ is chosen by the weight vectors (see Lemma [17]), $T_G$ acts on these spaces as scalars on each coordinate. According to Lemma [17] and the proof of Theorem [16], these embeddings are $T_G$-equivariant.

For $G = A_{n-1}$, it is trivial, since $\text{Proj}(\text{Cox}(S, G)) \cong G/P \cong \mathbb{P}(V_n)$ and $T_G$ acts on these spaces as scalars on each coordinate.
Proposition 19. Let $X_n$ be the del Pezzo surface obtained from an $E_n$-surface $S$ by blowing down $C$. The GIT quotient of $\text{Spec}(\text{Cox}(S,G))$ by the action of $T_{S,G}$ is respectively $X_n$ for $G = E_n$, $\mathbb{P}^1$ for $G = D_n$, and a point for $G = A_n$.

Proof. For the case $G = E_n$, we can apply the result of Hu-Keel (Proposition 2.9 in [7]), since $\text{Cox}(S,E_n) \cong \text{Cox}(X_n)$ is finitely generated by [1], and since $T_{NS}(X_n) \cong T_{S,G}$.

It remains to prove the cases $G = D_n$ and $G = A_n$.

For $G = D_n$, we apply a linearization argument as in Hu-Keel ([7]). In this case $\text{Pic}(S) = \mathbb{Z}(f,s,l_1,\ldots,l_n)$ and $C \subseteq f$. Let $R = \text{Cox}(S,D_n)$. Note that $R$ is naturally graded by the lattice $\text{Pic}(S)/\mathbb{Z}f$. For example, $a_0f + \sum_{i=1}^n a_il_i \in f^\perp$ with $a_i \in \mathbb{Z}$ is graded by $a_0+s+\sum_{i=1}^n a_il_i \in \text{Pic}(S)/\mathbb{Z}f$. Then $R = \bigoplus_{v \in \text{Pic}(S)/\mathbb{Z}f} R_v$. According to Theorem [14], $R$ is finitely generated. Note that $T_{S,G}$ acts naturally on $R$. So $R = \bigoplus_{v \in \text{Pic}(S)/\mathbb{Z}f} R_v$ is the eigenspace decomposition for this action. Thus

$$H^n(\text{Spec}(\text{Cox}(S,G)), L_v)_{T_{S,G}} \cong R_v,$$

where $L_v$ is the line bundle determined by the linearization $v \in \text{Pic}(S)$. And the ring of invariants is

$$R(\text{Spec}(\text{Cox}(S,G)), L_v)_{T_{S,G}} = R(S,\mathcal{O}_S(v')),$$

where $v' \in f^\perp$ is graded by $v$, and $R(S,\mathcal{O}_S(v'))$ (similar for $R(\text{Spec}(\text{Cox}(S,G)), L_v)$) denotes the graded ring $\bigoplus_{n \geq 0} H^n(S,\mathcal{O}_S(nv'))$. (This notation is taken from [7]). Thus $\mathbb{P}^1 \cong \text{Proj}(R(S,\mathcal{O}_S(f)))$ is the GIT quotient for the linearization $v = s \in \chi(T_{S,G})/\mathbb{Q}$.

The proof for $G = A_n$ is similar.

APPENDIX: TWO NON-SIMPLE BUT SEMISIMPLE CASES

Note that $G = E_3 = A_2 \times A_1$ and $G = D_2 = A_1 \times A_1$ are not simple, but semisimple. For completeness, in these two cases, we define $G$-surfaces $(S,C)$ and the Cox rings $\text{Cox}(S,G)$ similarly as in Corollary [3] and Definition [11] and we compute briefly the coordinate rings of the $G/P$ and the Cox rings explicitly. It turns out that there is no embedding of $\text{Spec}(\text{Cox}(S,G))$ into $C(G/P)$.

(1) The case $G = E_3$. Let $(S,C)$ be an $E_3$-surface, that is, $S$ is a blowup of a del Pezzo surface $X_3$ of degree 6 at a general point, and $C$ is the exceptional curve. Note that $X_3$ is a blowup of $\mathbb{P}^2$ at 3 points in general position. The representation bundle $\mathcal{L}_{E_3}$ is the tensor product of the standard representation bundles of $A_2$ and $A_1$.

Precisely, $\mathcal{L}_{E_3} = \mathcal{V}_2 \otimes \mathcal{V}_1$ where $\mathcal{V}_i (i = 1, 2)$ is the standard representation of $A_i$.

By checking the highest weights, it is easy to see that

i) $\mathcal{L}_{E_3}$ is determined by the set of $-1$ curves

$$\{l_1, l_2, l_3, h - l_1 - l_2, h - l_1 - l_3, h - l_2 - l_3\}.$$
ii) $V_2$ is determined by $\{h-l_1, h-l_2, h-l_3\}$ and $V_3$ is determined by $\{-h-l_1, -h-l_2, -h-l_3\}$.

iii) $V_1$ is determined by $\{h, 2h-l_1-l_2-l_3\}$.

Note that $O_S(h) \oplus O_S(2h-l_1-l_2-l_3)$ is a standard representation bundle of the adjoint principal bundle $\mathcal{A}_1 := O_S \oplus O_S(\alpha_1) \oplus O_S(-\alpha_1)$ (recall that $\alpha_1 = -h + l_1 + l_2 + l_3$).

The $Cox(S, E_3)$ is defined as in Definition 11. Then $Cox(S, E_3) \cong Cox(X_3)$, and it is well-known that $Cox(X_3) \cong \mathbb{C}[y_1, \ldots, y_6]$, since $X_3$ is toric (2). Therefore

$$Proj(Cox(S, E_3)) \cong \mathbb{P}(H^0(S, \mathcal{L}_{E_3})) = \mathbb{P}^5.$$  

And $E_3/P = \mathbb{P}^2 \times \mathbb{P}^1$. Denote $V = H^0(S, \mathcal{L}_{E_3})$.

Then we have

$$Proj(Cox(S, E_3)) = \mathbb{P}(V) \ (\cong \mathbb{P}^5), \text{ and } E_3/P = \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(V),$$

where the embedding $G/P \hookrightarrow \mathbb{P}(H^0(S, \mathcal{L}_{E_3})) = \mathbb{P}^5$ corresponds to the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$.

(2) The case $G = D_2$. Let $(S, C)$ be a $D_2$-surface, that is, $S$ is a blowup of the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^1$ at two points in general position, and $C$ is a smooth fiber. The rank 4 representation bundle $\mathcal{L}_{D_2}$ is the tensor product of the standard representation bundles of $A_1$ (recall that $\mathcal{L}_2 = A_1 \times A_1$).

Note that $\mathcal{L}_{D_2} = O_S(l_1) \oplus O_S(l_2) \oplus O_S(f-l_1) \oplus O_S(f-l_2)$, and $\mathcal{L}_{V_1} := O_S(l_1) \oplus O_S(l_2)$ and $\mathcal{L}_{V_2} := O_S(s) \oplus O_S(s+f-l_1-l_2)$. Then we find that $\mathcal{L}_{D_2} = \mathcal{L}_{V_1} \otimes \mathcal{L}_{V_2}$. By comparing the highest weights, we see that $\mathcal{L}_{V_1}, \mathcal{L}_{V_2}$ are the corresponding standard representations.

Thus we have $G/P \cong \mathbb{P}^1 \times \mathbb{P}^1$. And the embedding

$$G/P \hookrightarrow \mathbb{P}(H^0(S, \mathcal{L}_{D_2})) \cong \mathbb{P}^3$$

corresponds to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. On the other hand, the Cox ring $Cox(S, D_2)$, defined as in Definition 11, is a sub-ring of $Cox(X_3)$ generated by degree 1 elements in $Cox(X_3)$, since $S$ is also a del Pezzo surface $X_3$. That $Cox(S, D_2)$ is a sub-ring of $Cox(X_3)$ follows directly from their definitions. Therefore, we have $Cox(S, D_2) = \mathbb{C}[x_1, \ldots, x_4]$, and hence

$$Proj(Cox(S, D_2)) \cong \mathbb{P}(H^0(S, \mathcal{L}_{D_2})) \cong \mathbb{P}^3.$$  

References


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