AN UPDATE OF QUANTUM COHOMOLOGY OF HOMOGENEOUS VARIETIES

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Abstract. We describe recent progress on $QH^*(G/P)$ with special emphasis of our own work.

1. Introduction

How many intersection points are there for two given lines in a plane? How many lines pass through two given points in a plane? It is the main concern in enumerative geometry to find solutions to such questions of counting numbers of geometric objects that satisfy certain geometric conditions. There are two issues here. First, we should impose conditions so that the expected solution to a counting problem is a finite number. Second, we will work in the complex projective space, so that Schubert's *principle of conservation of number* holds. Then we ask for this invariant. For instance to either of the aforementioned questions, the solution is a constant number 1, if the condition is imposed precisely as "two distinct given lines (resp. points) in a complex projective plane $\mathbb{P}^2$".

Information on counting numbers of geometric objects may be packaged to form an algebra. For instance for the case of a complex projective line $\mathbb{P}^1$, basic geometric objects are either a point or the line $\mathbb{P}^1$ itself. There are only two pieces of non-trivial enumerative information among three basic geometric objects. Namely, (a) $\langle \mathrm{pt}, \mathrm{line}, \mathrm{line}\rangle_{0,3,0} = 1$, telling us that a point and two (same) lines intersect at a unique intersection point; (b) $\langle \mathrm{pt}, \mathrm{pt}, \mathrm{pt}\rangle_{0,3,1} = 1$, telling us that there is a unique line passing through three (distinct) given points. Incorporating all these enumerative information together, we obtain the algebra $\mathbb{C}[x, q]/\langle x^2 - q\rangle$. Here the identity 1 and the element x stand for the basic geometric objects, a line and a point, respectively; q stands for a line, the geometric object to be counted; the aforementioned counting numbers are then read off directly from the algebraic relations $1 * x = x$ and $x * x = q$, in terms of the coefficient $N^{x,0}_{1,x}$ of x and the coefficient $N^{1,q}_{x,x}$ of q, respectively. In modern language, we are saying that the quantum cohomology ring $QH^*(\mathbb{P}^1)$ of $\mathbb{P}^1$ is isomorphic to $\mathbb{C}[x, q]/\langle x^2 - q\rangle$ as an algebra.

The concept of quantum cohomology of a smooth projective complex variety arose from the subject of string theory in theoretic physics in 1990s, and the terminology was introduced by the physicists [99]. The coefficients for the quantum multiplication are genus zero Gromov-Witten invariants, which were rigorously defined later (by means of virtual fundamental classes in general) via symplectic geometry [95] and via algebraic geometry [54]. As a first surprising application of the *big* quantum cohomology, Kontsevich solved an old problem in enumerative geometry on counting the number of rational curves of degree d passing through $3d-1$ points in general position in $\mathbb{P}^2$, by giving a recursive formula in 1994. We note that a rational curve of degree 1 is a complex projective line. The space $\mathbb{P}^1$ can be written as the quotient of the special linear group $SL(2,\mathbb{C})$ by its subgroup of upper triangular matrices $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, and it is a special case of the so-called homogeneous variety G/P. In the present article, we will be concerned only with the (*small*) quantum cohomology ring $QH^*(G/P)$ of G/P, which is an algebra deforming the classical cohomology ring $H^*(G/P)$ by incorporating all genus zero, three-point *Gromov-Witten invariants* as the coefficients for the (small) quantum multiplication. Namely, it consists of all information on counting numbers of rational curves of various (fixed) degrees that pass through three given projective subvarieties. The study of the ring structure of $QH^*(G/P)$ is referred to as the quantum Schubert calculus, which is not only a branch of algebraic geometry, but also of great interest in mathematical physics, algebraic combinatorics, representation theory, and so on.

In the present article, we will give a brief review of the developments on quantum Schubert calculus since Fulton's beautiful lecture [35], with a focus on the authors' work [71, 72, 73, 74, 75]. We apologize to the many whose work less related with the four problems listed in section 2.4 has not been cited, and apologize to those whose work we did not notice.

2. A brief review of Schubert calculus and generalizations

2.1. **Classical Schubert calculus for homogeneous varieties G/P.** Like Grimm's fairy tales for the children, the next enumerative problem is known to everybody in the world of Schubert calculus:

How many lines in $\mathbb{P}^3$ intersect four given lines in general position?

The solution to it is 2, in modern language, obtained by a calculation in the cohomology ring of the complex Grassmannian $Gr(2,4)$ (see e.g. [52]).

A complex Grassmannian $Gr(k,n)$ consists of k-dimensional vector subspaces in $\mathbb{C}^n$, i.e., one-step flags $V \leqslant \mathbb{C}^n$. In particular, $\mathbb{P}^{n-1} = Gr(1,n)$. A direct generalization of it is the variety of partial flags

$$F\ell_{n_1,\cdots,n_r;n} := \{V_{n_1} \leqslant \cdots \leqslant V_{n_r} \leqslant \mathbb{C}^n \mid \dim V_{n_i} = n_i, \forall 1 \le i \le r\},$$

where $[n_1,\cdots,n_r]$ is a fixed subsequence of $[1,2,\cdots,n-1]$. Every partial flag variety $F\ell_{n_1,\cdots,n_r;n}$ is a quotient $SL(n,\mathbb{C})/P$ of $SL(n,\mathbb{C})$ by one parabolic subgroup P that consists of block-upper triangular matrices of the following type:

$$\begin{pmatrix} M_1 & * & * & * \\ \mathbf{0} & M_2 & * & * \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & M_{r+1} \end{pmatrix}_{n\times n},$$

where M_i is an $(n_i - n_{i-1}) \times (n_i - n_{i-1})$ invertible matrix for $1 \le i \le r$, and $n_0 := 0, n_{r+1} := n$. These are called homogeneous varieties (or flag varieties) of Lie type A_{n-1}.

In general, homogeneous varieties are $X = G/P$, where G is a simply-connected complex simple Lie group, and P a parabolic subgroup. These are classified by data from Dynkin diagrams, i.e., by a pair (Δ, Δ_P) of sets $\Delta_P \subset \Delta$. In particular, the parabolic subgroup corresponding to the empty subset of Δ is called a Borel subgroup, denoted as B instead. Let W denote the Weyl group of G, and W_P denote the Weyl subgroup corresponding to P. There is always an accompanied combinatorial subset $W^P \subset W$ bijective to W/W_P, which consists of minimal length representatives of the cosets in W/W_P with respect to a canonical length function $\ell : W \to \mathbb{Z}_{\geq 0}$. Every σ^u is the class of a Schubert variety Ω_u of complex codimension $\ell(u)$, and therefore is in $H^{2\ell(u)}(X, \mathbb{Z})$. The coefficients $N_{u,v}^w$ in the cup product in $H^*(X)$,

$$\sigma^u \cup \sigma^v = \sum_w N_{u,v}^w \sigma^w,$$

count the number of intersection points of three Schubert varieties $g\Omega_u, g'\Omega_v, g''\Omega_{w^\sharp}$ with generic elements $g, g', g'' \in G$, where $w^\sharp \in W^P$ parameterizes the dual basis of $H^*(G/P)$ to $\{\sigma^u\}$ with respect to the bilinear form $(\sigma^u, \sigma^v) := \int_{[G/P]} \sigma^u \cup \sigma^v$. In particular, the structure constants $N_{u,v}^w$ are nonnegative integers, which are known as Littlewood-Richardson coefficients in the special case of complex Grassmannians. Those unfamiliar with the general theory may just note the following two points, and then refer to [35] for a very nice introduction to $Gr(k, n)$.

(1) When $G/P = F\ell_{n_1,\cdots,n_r;n}$, the Weyl group W of G is the group S_n of permutations of $\{1, \cdots, n\}$. The subset Δ_P is $\Delta \setminus \{\alpha_{n_1}, \cdots, \alpha_{n_r}\}$, provided that $\Delta = \{\alpha_1, \cdots, \alpha_{n-1}\}$ is a base of simple roots of Lie(G) whose Dynkin diagram is canonical, i.e., given by $\underset{\alpha_1}{\circ}\!\!-\!\!\underset{\alpha_2}{\circ}\cdots\underset{\alpha_{n-1}}{\circ\!\!-\!\!\circ}$. In particular, $G/B = F\ell_{1,2,\cdots,n-1;n}$.

(2) When $G/P = Gr(k, n)$, the combinatorial subset W^P is given by

$$W^P := \{w \in S_n \mid w(1) < w(2) < \cdots < w(k); w(k+1) < w(k+2) < \cdots < w(n)\}.$$

Take a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ with $n - k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, and a complete flag $F_\bullet$ of vector subspaces $\{0\} =: F_0 \leqslant F_1 \leqslant \cdots \leqslant F_{n-1} \leqslant F_n := \mathbb{C}^n$ with $\dim_{\mathbb{C}} F_i = i$ for $0 \leq i \leq n$. We have the Schubert variety

$$\Omega_\lambda(F_\bullet) := \{V \leqslant \mathbb{C}^n \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i, \forall 1 \leq i \leq k\} \subset Gr(k, n).$$

Then $N_{u,v}^w$ counts the number of intersection points of three Schubert varieties $\Omega_{\lambda(u)}(F_\bullet)$, $\Omega_{\lambda(v)}(F'_\bullet)$, $\Omega_{\lambda(w^\sharp)}(F''_\bullet)$ with respect to three general flags and the partitions $\lambda(u) = (u(k)-k, \cdots, u(2)-2, u(1)-1)$, $\lambda(v) = (v(k)-k, \cdots, v(2)-2, v(1)-1)$ and $\lambda(w^\sharp) = (n-k+1-w(1), n-k+2-w(2), \cdots, n-w(k))$.

A thorough study of $H^*(X)$ includes (but not limited to) a combinatorial description of the Littlewood-Richardson coefficients, a ring presentation of $H^*(X)$ with certain generators, and an expression of every Schubert class σ^u in terms of polynomial in the aforementioned generators. In the case of complex Grassmannians, there have been nice answers to all the above through various approaches. Much is also known about $H^*(G/P)$. However, a *manifestly positive* combinatorial rule for the structure constants $N_{u,v}^w$ is still lacking in general. To the authors' knowledge, such rules have been shown only for two-step flag varieties $F\ell_{n_1,n_2;n}$ [29] (and there is a preprint [30]), besides (co)minuscule Grassmannians. One may have noted that every Schubert variety admits a resolution by an associated Bott-Samelson manifold, which is a tower of $\mathbb{P}^1$-fibrations. Using the topology of the Bott-Samelson resolution, Duan obtained a nice algorithm for computing $N_{u,v}^w$ for general G/P in [33], though it involves sign cancellation.

2.2. **Equivariant Schubert calculus.** Every parabolic subgroup P of G contains a Borel subgroup B. Let K be a maximal compact Lie subgroup of G, such that $T = K \cap B$ is a maximal torus of K. We consider the T-equivariant cohomology $H_T^*(G/P) = H_T^*(G/P, \mathbb{Q})$, which is a module over the ring $S := H_T^*(\text{pt})$ which is a polynomial ring $\mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ freely generated by simple roots α_j's of G. $H_T^*(G/P)$ has an S-basis of Schubert classes σ^u's, indexed by the same combinatorial set W^P. The structure constants $p_{u,v}^w$'s in the equivariant product in $H_T^*(G/P)$,

$$\sigma^u \circ \sigma^v = \sum_w p_{u,v}^w \sigma^w,$$

are homogeneous polynomials of degree $\ell(u) + \ell(v) - \ell(w)$ in S (by which we mean the zero polynomial if the degree is negative). The evaluation of the polynomial $p_{u,v}^w(\alpha_1, \cdots, \alpha_n)$ at the origin coincides with the intersection number $N_{u,v}^w$. Geometric meanings of $p_{u,v}^w$'s are not completely clear. Nevertheless, $p_{u,v}^w$ enjoys the positivity property, lying inside $\mathbb{Q}_{\geq 0}[\alpha_1, \cdots, \alpha_n]$ (or enjoys a form of positivity conjectured by Peterson and proved by Graham [44], as an element in $\mathbb{Q}_{\geq 0}[-\alpha_1, \cdots, -\alpha_n]$, up to the choice of Schubert varieties that determine the equivariant Schubert classes). The study of $H_T^*(G/P)$ is referred to as the *equivariant Schubert calculus*.

2.3. **Affine Schubert calculus.** The *affine Kac-Moody group* $\mathcal{G}$ associated to G is an algebraic analog of the free (smooth) loop space $LK = Map\left(\mathbb{S}^1, K\right)$. The Dynkin diagram of $\mathcal{G}$ is precisely the affine Dynkin diagram of G, namely it has an extra simple root, called the *affine root* α_0, attached to the simple root corresponding to the adjoint representation of G. In particular $\Delta_{\text{aff}} = \Delta \cup \{\alpha_0\}$ is a base of $\mathcal{G}$.

A natural generalization of G/P is the infinite dimensional projective ind-variety $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ denotes a parabolic subgroup of $\mathcal{G}$. The two extreme cases $\mathcal{B}$, $\mathcal{P}_{\max}$ correspond to the subsets $\emptyset$, Δ of Δ_{aff} respectively[1]. The corresponding ind-varieties $\mathcal{G}/\mathcal{B}$ and $\mathcal{G}/\mathcal{P}_{\max}$ are called *affine flag manifolds* and *affine Grassmannians* respectively. Indeed $\mathcal{G}/\mathcal{B}$ is homotopy equivalent to LK/T and $\mathcal{G}/\mathcal{P}_{\max}$ is homotopy equivalent to the *base loop group* $\Omega K = \{f \in LK \mid f(\mathrm{id}_{\mathbb{S}^1}) = \mathrm{id}_K\}$. Both of them inherit the T-action. One may study the T-equivariant cohomology rings of LK/T and ΩK. Both of them admit an S-basis of affine Schubert classes.

On the other hand the (equivariant) homology $H_*^T(\Omega K)$ (in the sense of Borel-Moore homology) also admits a natural ring structure, called the *Pontryagin product* which is induced by the group multiplication of K. The space $H_*^T(\Omega K)$ is further equipped with a Hopf algebra structure, with the coproduct structure induced from the (equivariant) cohomology ring structure of ΩK. The study of this structure is referred to as the (equivariant) *affine Schubert calculus*. A surprising fact, as will be discussed in the next section, is that $H_*^T(\Omega K)$ is essentially the same as the equivariant *quantum cohomology* of G/B.

2.4. **Quantum Schubert calculus.** The (small) *quantum cohomology* ring $QH^*(G/P)$ is part of the intersection theory on the moduli space of stable maps to G/P (see e.g. [36]). It contains the ordinary $H^*(G/P)$ as G/P is a connected component of the moduli space of stable maps to G/P, namely the component of constant maps. $QH^*(G/P) = (H^*(G/P) \otimes \mathbb{Q}[q_1, \cdots, q_r], *)$ also has a basis of Schubert classes σ^u over $\mathbb{Q}[q_1, \cdots, q_r]$, where $r = \dim H_2(G/P)$. We identify $H_2(G/P, \mathbb{Z})$ with $\mathbb{Z}^r$ with basis given by two (real) dimensional Schubert cycles in G/P.

[1] If the subset of Δ_{aff} contains α_0, then $\mathcal{G}/\mathcal{P}$ reduces to G/P.

Given $\mathbf{d} = (d_1, \cdots, d_r) \in \mathbb{Z}^r = H_2(G/P, \mathbb{Z})$, we denote $q^{\mathbf{d}} := q_1^{d_1} \cdots q_r^{d_r}$. The structure constants $N_{u,v}^{w,\mathbf{d}}$ in the quantum multiplication,

$$\sigma^u * \sigma^v = \sum_{w \in W^P, \mathbf{d} \in H_2(G/P,\mathbb{Z})} N_{u,v}^{w,\mathbf{d}} \sigma^w q^{\mathbf{d}},$$

are genus zero, three-point Gromov-Witten invariants. Geometrically, $N_{u,v}^{w,\mathbf{d}}$ counts the cardinality of the set

$$\{f : \mathbb{P}^1 \to G/P \mid f_*([\mathbb{P}^1]) = \mathbf{d}; f(0) \in g\Omega_u, f(1) \in g'\Omega_v, f(\infty) \in g''\Omega_{w^\sharp}; f \text{ is holomorphic}\}$$

with generic $g, g', g'' \in G$. In particular, $N_{u,v}^{w,\mathbf{d}} \in \mathbb{Z}_{\geq 0}$, and it is zero unless $d_i \geq 0$ for all i. A holomorphic map $f : \mathbb{P}^1 \to G/P$ of degree $\mathbf{0}$ is a constant map. Therefore $N_{u,v}^{w,\mathbf{0}} = N_{u,v}^w$ for the cup product.

The study of $QH^*(G/P)$ is referred to as the *quantum Schubert calculus*, which includes at least the following as pointed out in [35].

(1) A presentation of the ring, $QH^*(G/P) = \mathbb{Q}[x_1, \cdots, x_N, q_1, \cdots, q_r]/(\text{relations})$. See section 3.1.
(2) A *manifestly positive* combinatorial rule for the structure constants $N_{u,v}^{w,\mathbf{d}}$.
(3) A "quantum Giambelli" formula, which expresses each Schubert class σ^u as a polynomial in the generators x_i and q_j. See section 3.4.
Because of the lack of functoriality for *quantum* products in general, we also have the following problem.
(4) A comparison between $QH^*(G/B)$ and $QH^*(G/P)$. See section 3.2.

We can consider the T-equivariant version $QH_T^*(G/P)$ of the quantum cohomology in a similar fashion, which is a module over $S[q_1, \cdots, q_r]$ generated by Schubert classes σ^u's. The structure constants $\tilde{N}_{u,v}^{w,\mathbf{d}}$'s in the equivariant quantum product

$$\sigma^u \star \sigma^v = \sum_{w,\mathbf{d}} \tilde{N}_{u,v}^{w,\mathbf{d}} \sigma^w q^{\mathbf{d}},$$

are again homogeneous polynomials in S, enjoying the positivity property (provided that a choice of positive/negative simple roots is chosen properly) [82]. They contain information on both $QH^*(G/P)$ and $H_T^*(G/P)$: (i) $\tilde{N}_{u,v}^{w,\mathbf{0}}$ coincides with $p_{u,v}^w$ for the T-equivariant product $\sigma^u \circ \sigma^v$ and (ii) the evaluation of the polynomial $\tilde{N}_{u,v}^{w,\mathbf{d}} \in S$ at the origin coincides with $N_{u,v}^{w,\mathbf{d}}$ for the quantum product.

We remark that it is possible to define $QH_T^*(G/P)$ over $\mathbb{Z}$, instead of $\mathbb{C}$.

3. An overview on (equivariant) quantum Schubert calculus

3.1. **Ring presentations.** In order to find a ring presentation of $QH^*(G/P)$, it is natural to start with an explicit ring presentation of the ordinary cohomology ring $H^*(G/P)$, say

$$H^*(G/P) \cong \frac{\mathbb{Q}[x_1, \cdots, x_N]}{\langle f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})\rangle}.$$

Such a presentation is know in many cases. Then a lemma of Siebert and Tian [96] (or Proposition 11 of [36]) shows that the ring structure on $QH^*(G/P) = H^*(G/P) \otimes \mathbb{Q}[q_1, \cdots, q_r]$ can be obtained by deforming the relations f_i's, i.e.

$$QH^*(G/P) \cong \frac{\mathbb{Q}[x_1, \cdots, x_N, q_1, \cdots, q_r]}{\langle f_1(\mathbf{x}) - g_1(\mathbf{x}, \mathbf{q}), \cdots, f_m(\mathbf{x}) - g_m(\mathbf{x}, \mathbf{q})\rangle}.$$

Here each $g_i(\mathbf{x}, \mathbf{q})$ is computed by f_i after replacing the cup product by the quantum product. This is usually difficult to compute.

For instance, $H^*(\mathbb{P}^1) \stackrel{\cong}{\to} \mathbb{Q}[x]/\langle x^2\rangle$, which sends the hyperplane class H (Poincaré dual to [pt]) to x. Since $H * H = q$ in $QH^*(\mathbb{P}^1)$, we have $QH^*(\mathbb{P}^1) \cong \mathbb{Q}[x]/\langle x^2 - q\rangle$.

Most known ring presentations of $QH^*(G/P)$ are obtained in this way. Such G/P's include $F\ell_{n_1,\cdots,n_r;n}$ [39],[48],[3] and various Grassmannians [96],[58],[59],[20],[15],[23]. Some Grassmannians of exceptional Lie types are still unknown.

There is another way to get a ring presentation of $QH^*(G/P)$, by finding "quantum differential equations". Givental's J-function is a $H^*(G/P)$-valued function, involving gravitational correlators (a class of invariants more general than Gromov-Witten invariants). It was introduced for any smooth projective variety X, and played an important role in mirror symmetry. Quantum differential equations are certain differential operators annihilating J. Every quantum differential equation gives rise to a relation in $QH^*(X)$. (See e.g. Example 10.3.1.1 of [32] for the case of $\mathbb{P}^1$). With this method, Kim studied the quantum D-module of G/B and obtained the following ring presentation of $QH^*(G/B)$ for general G.

Theorem 3.1 ([49])**.** *The small quantum cohomology ring $QH^*(G/B, \mathbb{C})$ is canonically isomorphic to*

$$\mathbb{C}[p_1, \cdots, p_l, q_1, \cdots, q_l]/I,$$

where l denotes the rank of G, and I is the ideal generated by the nonconstant complete integrals of motions of the Toda lattice for the Langlands-dual Lie group $(G^\vee, B^\vee, (T^{\mathbb{C}})^\vee)$ of $(G, B, T^{\mathbb{C}})$.

There have been the descriptions of J-function for general G/P [10],[11]. Nevertheless, even for complex Grassmannians, there are no closed formulas on the quantum differential equations, to the authors' knowledge.

In his unpublished lecture notes [89], Dale Peterson announced a uniform presentation of $QH^*(G/P)$ (and its T-equivariant extension) for all P. The so-called *Peterson variety* Y is subvariety in $G^\vee/B^\vee$, which is equipped with a $\mathbb{C}^*$-action. The $\mathbb{C}^*$-fixed points y_P's are isolated and parameterized by the finite set of (conjugacy classes of) parabolic subgroups P. (See [93],[45] for precise descriptions for type A case.) By considering $y \in Y$ with $z \cdot y$ approaching y_P with various P's as $z \in \mathbb{C}^*$ goes to 0 or infinity, we obtain two stratifications of Y by affine varieties Y_P^+'s or Y_P^-'s respectively.

Peterson claimed that the spectra of the quantum cohomology ring $QH^*(G/P)$ is Y_P^+, or equivalently $QH^*(G/P) \cong \mathbb{C}[Y_P^+]$ as algebras. This was proved by Rietsch [93] for all $SL(n, \mathbb{C})/P$, and by Cheong [26] when G/P is a Lagrangian Grassmannian or an orthogonal Grassmannian.

Peterson also interpreted all $\mathbb{C}[Y_P^-]$ as the homology of based loop groups. In particular, Y_G^- is birational to Y_B^+ and $\mathbb{C}[Y_G^-] \cong H_*(\Omega K)$, where K is a maximal compact subgroup of G. Its consequence has become the following theorem now, as was firstly proved by Lam and Shimozono. There is also an alternative proof by the authors.

Theorem 3.2 ([89],[65],[72])**.** *The equivariant quantum cohomology ring $QH_T^*(G/B)$ is isomorphic to the equivariant homology $H_*^T(\Omega K)$ as algebras, after localization.*

The above isomorphism is explicit, sending a (localized) S-basis $\sigma^w q^{\mathbf{d}}$ of $QH_T^*(G/B)$ to a (localized) S-basis of Schubert homology classes of $H_*^T(\Omega K)$. The proof in [65] also showed that $QH_T^*(G/P)$ is isomorphic to a quotient of $H_*^T(\Omega K)$, after localization.

The algebraic structures of $QH_T^*(G/B)$, or more generally $QH_T^*(G/P)$, is completely determined by the equivariant quantum Chevalley formula (i.e. quantum multiplication by elements in $H^2(G/P)$ in section 3.3) together with a few natural properties. This criterion was obtained by Mihalcea [83]. Using this criterion, the proof of the above theorem can be reduced to explicit computations of certain products in $H_*^T(\Omega K)$. These products were first obtained by Lam and Shimozono, mainly by using Peterson's j-isomorphism [61] together with properties of Kostant and Kumar's nilHecke algebras [57]. Later we realized that they can also be obtained by carefully analyzing a combinatorial formula on the structure constants (to be described in section 3.3). Despite the two aforementioned (combinatorial) proofs, a satisfactory understanding of the above theorem is still lacking.

When $G = SL(n, \mathbb{C})$, the connections among Givental-Kim's presentation [39, 49], Peterson's presentation above, and Kostant's solution to Toda lattice [56], are now becoming better understood [66],[67]. In this case, the quantum cohomology $QH^*(SL(n, \mathbb{C})/B)$ is also closely related with Dunkl elements, which leads to some relevant applications [50]. Some other characterizations for $QH^*(G/B)$ can be found for example in [77],[78].

Finally, we remark that there are some studies on the quantum differential equations for the cotangent bundle of a complete flag variety G/B [6] or of a partial flag variety $F\ell_{n_1,\cdots,n_r;n}$ [80], [42], [97]. This might lead to nice applications to the quantum cohomology of the corresponding flag variety by taking an appropriate limit. For instance, an application to the Chevalley-type formula was given in [6] (as will be discussed in section 3.3.2). The cotangent bundle of a partial flag variety is a Nakajima quiver variety. The quantum cohomology of it can be related with the integral systems and quantum groups [85],[6],[80]. The relation between such an approach and the aforementioned Peterson's approach for complex Grassmannians is studied in [41].

3.2. **Comparing $QH^*(G/P)$ with $QH^*(G/B)$.** The inclusions $B \subset P \subset G$ induce a fiber bundle

$$\pi : G/B \to G/P$$

with fiber P/B, which is again a complete flag variety.

Example 3.3. *For $G = SL(3, \mathbb{C})$ with $B \subsetneq P \subsetneq G$, we have $G/B = F\ell_{1,2;3}$ and $G/P = Gr(2, 3) \cong \mathbb{P}^2$; π coincides with the natural forgetful map, sending a flag $V_1 \leqslant V_2 \leqslant \mathbb{C}^3$ in G/B to the partial flag $V_2 \leqslant \mathbb{C}^3$ in G/P; the fiber of π is $P/B = \mathbb{P}^1$.*

For ordinary cohomologies, the Leray-Serre spectral sequence relates the cohomology ring of G/B with those of G/P and P/B. For instance the induced homomorphism $\pi^* : H^*(G/P) \to H^*(G/B)$ is injective, sending the Schubert class σ_P^u for G/P (where u varies over the combinatorial subset $W^P \subset W = W^B$) to the Schubert class σ_B^u for G/B.

For quantum cohomologies, there is no such functoriality in general. For instance, the pullback map π^* on H^* does not even have a *quantum* analog. Nevertheless, such functoriality does exist in our specific case. The analog of π^* is a comparison formula stated by Peterson [89] and proved by Woodward [98].

Theorem 3.4 (Peterson-Woodward comparison formula)**.** *Every structure constant $N_{u,v}^{w,\lambda_P}$ for the quantum product $\sigma_P^u * \sigma_P^v$ in $QH^*(G/P)$ coincides with a structure constant $N_{u,v}^{w\tilde{w},\lambda_B}$ for $\sigma_B^u * \sigma_B^v$ in $QH^*(G/B)$, where $(\tilde{w}, \lambda_B) \in W_P \times H_2(G/B, \mathbb{Z})$ is uniquely and explicitly determined by $\lambda_P \in H_2(G/P, \mathbb{Z})$.*

We remark that the equivariant quantum extension of the above comparison formula is implicitly contained in Corollary 10.22 of [65] (see [46] for more details).

For ordinary cohomologies, $H^*(G/B)$ admits a $\mathbb{Z}^2$-filtration $\mathcal{F}$, which induces an isomorphism $Gr^{\mathcal{F}}(H^*(G/B)) \cong H^*(P/B) \otimes H^*(G/P)$ of $\mathbb{Z}^2$-graded algebras, by the spectral sequence. In order to generalize this to quantum cohomologies, we need to find a nice grading on $QH^*(G/B)$. This is quite tricky indeed.

Example 3.3. (continued) *The Weyl group W of $SL(3, \mathbb{C})$ is the permutation group S_3, generated by transpositions $s_1 = (12)$ and $s_2 = (23)$. The cohomology degree of a Schubert class σ^u is $2\ell(u)$. Explicitly $\ell(u) = 0, 1, 1, 2, 2, 3$* when $u = 1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1$ respectively. The $\mathbb{Z}^2$-grading $gr(\sigma^u)$ of $\sigma^u \in H^*(F\ell_{1,2;3})$, as given by the spectral sequence for $\pi : F\ell_{1,2;3} \to Gr(2, 3)$, are $(0, 0), (1, 0), (0, 1), (0, 2), (1, 1), (1, 2)$ respectively. This grading gives a number of nice consequences on $H^*(F\ell_{1,2;3})$.

The quantum cohomology $QH^*(F\ell_{1,2;3}) = H^*(F\ell_{1,2;3}) \otimes \mathbb{Q}[q_1, q_2]$ has a $\mathbb{Q}$-basis $\sigma^w q_1^a q_2^b$. The above $\mathbb{Z}^2$-grading map extends to $QH^*(F\ell_{1,2;3})$, by defining

$$gr(\sigma^u q_1^a q_2^b) := gr(\sigma^u) + (2a - b, 3b).$$

It is tricky to find this grading, e.g. $gr(q_2) = (-1, 3)$. This is the *correct* grading as all consequences from spectral sequence have natural quantum generalizations for $QH^*(F\ell_{1,2;3})$. (See Example 1.1 of [71] for precise descriptions.)

In general, we need to take a *maximal* chain of parabolic subgroups P_j, i.e.,

$$B := P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{r-1} \subsetneq P_r = P \subsetneq G,$$

where r is the rank of the Levi subgroup of P. This corresponds to a chain of subsets $\emptyset := \Delta_0 \subsetneq \Delta_1 \subsetneq \cdots \subsetneq \Delta_{r-1} \subsetneq \Delta_r = \Delta_P$ with $|\Delta_i| = i$.

We can always find P_j's such that P_j/P_{j-1}'s are all projective spaces $\mathbb{P}^{N_j}$, with at most one exception occurring at the last step P_r/P_{r-1}. For instance, for $P \subset Sp(8, \mathbb{C})$ with $Sp(8, \mathbb{C})/P \simeq \mathbb{P}^7$, we have $r = 3$ with $P_1/B = \mathbb{P}^1$, $P_2/P_1 = \mathbb{P}^2$ and $P/P_2 = LGr(3, 6)$ is a Lagrangian Grassmannian. Precise choices are made in Table 2 of [71] when P/B is of type A (and its associated Dynkin diagram is connected), and in section 3.5 of [71] (or Table 1 of [75]) for a general P/B.

The key point of the whole story, is to find a nice $\mathbb{Z}^{r+1}$-grading on $QH^*(G/B)$ with respect to the chosen chain. The Peterson-Woodward comparison formula plays a key role in defining such a grading[2]. With this grading, the authors obtained certain functorial properties among quantum cohomologies of homogeneous varieties in the following sense[3].

Theorem 3.5 ([71], [75])**.** *Let $\pi : G/B \to G/P$ denote the natural projection, and r denote the rank of the Levi subgroup of P.*

[2] The original definition of this grading was made recursively in [71]. It was greatly simplified in [75], by proving a conjecture due to a referee of [71].

[3] We remark that part of the statements were only proved for P/B of type A in [71], and are proved for all general cases in [75] recently.

(1) *There exists a $\mathbb{Z}^{r+1}$-filtration $\mathcal{F}$ on $QH^*(G/B)$, respecting the quantum product structure.*
(2) *There exist an ideal $\mathcal{I}$ of $QH^*(G/B)$ and a canonical algebra isomorphism*

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\simeq} QH^*(P/B).$$

(3) *There exists a subalgebra $\mathcal{A}$ of $QH^*(G/B)$ together with an ideal $\mathcal{J}$ of $\mathcal{A}$, such that $QH^*(G/P)$ is canonically isomorphic to $\mathcal{A}/\mathcal{J}$ as algebras.*
(4) *There exists a canonical injective morphism of graded algebras*

$$\Psi_{r+1}: \quad QH^*(G/P) \hookrightarrow Gr^{\mathcal{F}}_{(r+1)} \subset Gr^{\mathcal{F}}(QH^*(G/B))$$

together with an isomorphism of graded algebras after localization

$$Gr^{\mathcal{F}}(QH^*(G/B)) \cong \big(\bigotimes_{j=1}^{r} QH^*(P_j/P_{j-1})\big) \bigotimes Gr^{\mathcal{F}}_{(r+1)},$$

where P_j's are parabolic subgroups constructed in a canonical way, forming a chain $B := P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{r-1} \subsetneq P_r = P \subsetneq G$.

Furthermore, Ψ_{r+1} is an isomorphism if and only if P_j/P_{j-1} is a projective space for any $1 \leq j \leq r$.

All the relevant ideals, subalgebras and morphisms above could be described concretely.

When only quantum multiplication with an element in $H^2(G/B)$ is involved, this theorem is reduced to the quantum Chevalley formula. To prove the theorem, we use induction with respect to the length $\ell(u)$ of Schubert classes σ^u. The Peterson-Woodward comparison formula is used frequently and the positivity of the structure constants turns out to be needed as well. The most dedicated arguments occur in the proof of (1) and in the part to show Ψ_{r+1} is an morphism for the general case. The notion of virtual coroot was introduced to reduce many cases in general Lie types to type A, while there are still a number of cases that require individual discussions.

The functorial properties of $QH^*(G/P)$, as given in theorems 3.4 and 3.5, have many applications in finding combinatorial rules on certain $N_{u,v}^{w,\mathbf{d}}$'s, especially on the so-called *quantum to classical principle*. We could also relate certain Gromov-Witten invariants between G/P and G'/P' for $G \neq G'$. We will discuss these next.

3.3. **Combinatorial rules.** The problem of finding a manifestly positive combinatorial rule for the Gromov-Witten invariants, or equivalently the structure constants, $N_{u,v}^{w,\mathbf{d}}$ of $QH^*(G/P)$ is open. Even the special case when $\mathbf{d} = 0$, namely the counterpart for classical cohomology $H^*(G/P)$, is still not solved except in very limited cases, including complex Grassmannians.

3.3.1. *A combinatorial formula on $N_{u,v}^{w,\mathbf{d}}$ with signs cancellations.* If we do not require coefficients of the combinatorial rule to be all positive, namely we allow signs cancellations, then it suffices to find one for $QH^*(G/B)$ because of the Peterson-Woodward comparison formula. Using the relationship between $QH^*(G/B)$ and $H_*(\Omega K)$, we obtained such a formula as we explain next.

Let $\Delta = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ denote a base of simple roots of G and ρ (resp. $\rho^\vee$) denote the sum of fundamental (co)weights. Then $H_2(G/B, \mathbb{Z})$ is canonically identified with the coroot lattice $Q^\vee := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$. For the affine flag manifold LK/T,

$H_T^*(LK/T)$ has an S-basis of Schubert classes $\mathfrak{S}^x$, with $x = wt_\lambda$, parametrized by the affine Weyl group $W_{\rm af} = W \ltimes Q^\vee$. Here t_λ denotes the translation by λ.

To describe the combinatorial rule, we define the Kostant (homogeneous) polynomial $d_{y,[x]}$ and a rational function $c_{x,[y]}$ as below: (i) $d_{x,y}$ is the coefficient $p^y_{x,y}$ of $\mathfrak{S}^y$ in $\mathfrak{S}^x \cdot \mathfrak{S}^y = \sum p^z_{x,y}\mathfrak{S}^z$; (ii) $c_{x,[y]} = \sum_{w\in W} c_{x,yw}$ with $\left(c_{x',y'}\right)$ being the transpose inverse of the (infinitely dimensional invertible) matrix $\left(d_{x,y}\right)$. They both have combinatorial descriptions [57, 60].

Theorem 3.6 ([72])**.** *The structure constants $N^{w,\lambda}_{u,v}$ in the quantum product of $QH^*(G/B)$,*

$$\sigma^u * \sigma^v = \sum_{w\in W, \lambda\in Q^\vee} N^{w,\lambda}_{u,v}\sigma^w q^\lambda,$$

are given by the constant function

$$N^{w,\lambda}_{u,v} = \sum_{\lambda_1,\lambda_2\in Q^\vee} c_{ut_A,[t_{\lambda_1}]}c_{vt_A,[t_{\lambda_2}]}d_{wt_{2A+\lambda},[t_{\lambda_1+\lambda_2}]}$$

where $A := -12n(n+1)\rho^\vee$, provided that $\langle 2\rho,\lambda\rangle = \ell(u)+\ell(v)-\ell(w)$ and $\lambda \succcurlyeq 0$, and zero otherwise.

The proof of this theorem uses the fact that $H_*^T(\Omega K)$ has a basis ψ_{t_λ} parametrized by translation representatives of $W_{\rm af}/W$ for which the Pontryagin product is simple: $\psi_{t_{\lambda_1}}\cdot\psi_{t_{\lambda_2}} = \psi_{t_{\lambda_1+\lambda_2}}$; the change of bases to Schubert classes is known explicitly by localization and the relationship between $QH^*(G/B)$ and $H_*(\Omega K)$.

3.3.2. *Combinatorial formulae of Pieri-Chevalley type.* There are manifestly positive combinatorial formulas for quantum multiplication by *special* Schubert classes σ^v. When $\ell(v) = 1$, i.e. $v = s_i$ is a simple refection in W, it is called the Chevalley formula for $H^*(G/P)$. The quantum version of Chevalley formula for $QH^*(G/P)$ was conjectured by Peterson [89], and was first proved by Fulton and Woodward [37]. It is also called the quantum Monk formula in the special case of $G/P = F\ell_{1,\cdots,n-1;n}$ [34], and it has an alternative description when G/P is a (co)minuscule Grassmannian [20].

The T-equivariant generalization was also conjectured by Peterson [89]. It was first proved by Mihalcea [81, 83] in a combinatorial way, and there is also a geometric approach [18] by the technique of curve neighborhoods. A special case of it is stated as follows, for which there is another proof by using the Springer resolution [6].

Theorem 3.7 (Equivariant quantum Chevalley formula for G/B)**.** *In $QH_T^*(G/B)$,*

$$\sigma^u \star \sigma^{s_i} = (\chi_i - u(\chi_i))\sigma^u + \sum\langle\chi_i,\gamma^\vee\rangle\sigma^{us_\gamma} + \sum\langle\chi_i,\gamma^\vee\rangle q^{\gamma^\vee}\sigma^{us_\gamma},$$

where χ_i denotes a fundamental weight, the first sum is over positive roots γ with $\ell(us_\gamma) = \ell(u)+1$, and the second sum is over positive roots γ with $\ell(us_\gamma) = \ell(u)+1-\langle 2\rho,\gamma^\vee\rangle$.

The corresponding formula for G/P is slightly more complicated.

When G/P is a complex Grassmannian $Gr(k,n)$, there is an exact sequence of tautological bundles over it: $0\to\mathcal{S}\to\underline{\mathbb{C}}^n\to\mathcal{Q}\to 0$. The fiber of $\mathcal{S}$ over $[V]\in Gr(k,n)$ is given by the vector subspace V in $\mathbb{C}^n$ itself. The ordinary cohomology $H^*(Gr(k,n))$, as a ring, is generated by Chern classes $c_p(\mathcal{S})$'s (or $c_p(\mathcal{Q})$'s). A (manifestly positive) combinatorial formula on the multiplication by either sets of

Chern classes is referred to as a *Pieri rule*. The quantum version of a Pieri rule was first given by Bertram [5].

The analog of $Gr\,(k,n)$ for other Lie types is $G/P_{\max}$ with $P_{\max}$ a maximal parabolic subgroup of G. When G is a classical group, $G/P_{\max}$ parameterizes linear subspaces which are isotropic with respect to a non-degenerate bilinear form which is skew-symmetric (for type C) or symmetric (for type B and D). Therefore such a $G/P_{\max}$ is usually called an *isotropic Grassmannian*. The corresponding quantum Pieri rules with respect to Chern classes of tautological quotient bundles have been obtained by Buch, Kresch and Tamvakis [58, 59], [15].

For instance when $G/P = IG(k, 2n)$ is a (non-maximal) isotropic Grassmannian of type C, Schubert classes can also labeled by $(n-k)$-strict partitions, with $c_p(\mathcal{Q})$ corresponding to a class σ^p of the special $(n-k)$-strict partition p. In terms of $(n-k)$-strict partitions, one has a quantum Pieri rule with respect to $c_p(\mathcal{Q})$'s in the following form.

Theorem 3.8 ([15])**.**

$$\sigma^p * \sigma^\lambda = \sum_\mu 2^{N(\lambda,\mu)}\sigma^\mu + \sum_\nu 2^{N(\lambda,\nu^\sharp)-1}\sigma^\nu q,$$

where $\nu^\sharp$ is an $(n-k)$-strict partition for $IG(k+1, 2n+2)$, associated to the $(n-k)$-partition ν for $IG(k, 2n)$.

The classical part of the above formula is new even for the classical cohomology $H^*(IG(k, 2n))$.

The quantum Pieri rules with respect to $c_p(\mathcal{S})$'s have been studied by the authors in [74]. For instance, when $G/P = IG(k, 2n)$, there is another parameterization of the Schubert classes by *shapes*, which are pairs of partitions. In terms of shapes, every Chern class $c_p(\mathcal{S}^*)$ corresponds to a class σ^p of special shape p, and one has a quantum Pieri rule with respect to these Chern classes in the following form.

Theorem 3.9 ([74])**.**

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum_{\mathbf{b}} 2^{e(\mathbf{a},\mathbf{b})}\sigma^{\mathbf{b}} + \sum_{\mathbf{c}} 2^{e(\tilde{\mathbf{a}},\tilde{\mathbf{c}})}\sigma^{\mathbf{c}} q,$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{c}}$ are shapes for $IG(k-1, 2n)$, associated to the shapes $\mathbf{a}$ and $\mathbf{c}$ for $IG(k, 2n)$ respectively.

The classical part of the above formula is the classical Pieri rule of Pragacz and Ratajski [92]. We remark that when G/P is a non-maximal isotropic Grassmanian of type B or D, the above formula does involve sign cancellations even for some degree *one* Gromov-Witten invariants, thus it is not quite satisfactory.

For homogeneous varieties of type A, namely partial flag varieties, there are natural forgetting maps to complex Grassmannians $\pi_{n_i} : F\ell_{n_1,\cdots,n_r;n} \to Gr(n_i, n)$. The ring $QH^*(F\ell_{n_1,\cdots,n_r;n})$ is generated by the pull-back of Chern classes of the tautological subbundle (or quotient bundle) over $Gr(n_i, n)$ for all i. The quantum Pieri rule with respect to these classes was obtained by Ciocan-Fontanine [27]. The equivariant quantum version of it has been obtained in [46] recently.

All these quantum Pieri rules are obtained by determining relevant Gromov-Witten invariants $N_{u,v}^{w,\mathbf{d}}$ of $QH^*(G/P)$ explicitly.

3.3.3. *Calculations of $N_{u,v}^{w,\mathbf{d}}$.* Each $N_{u,v}^{w,\mathbf{d}}$ is an intersection number of cycles in the moduli space of stable maps, which is the stable maps compactification of the space of morphisms from $\mathbb{P}^1$ to G/P. Bertram [5] used a different compactification, namely the quot schemes compactification, in order to calculate the relevant Gromov-Witten invariants. This method was further used in [27], [58, 59].

For $Gr(k,n)$, because of $H_2 = \mathbb{Z}$, $N_{u,v}^{w,d}$ counts the number of rational curves C of degree d passing through three Schubert subvarieties of $Gr(k,n)$. Buch [12] introduced the *span* (resp. *kernel*) of C as the smallest (resp. largest) subspace of $\mathbb{C}^n$ containing (resp. contained in) all the k-dimensional subspaces parametrized by points of C. Buch showed that the span (resp. kernel) of C determines a point in a related Schubert subvariety of $Gr(k+d,n)$ (resp. $Gr(k-d,n)$). Furthermore, he related Gromov-Witten invariants involved in a quantum Pieri rule for $Gr(k,n)$ to classical intersection numbers (of Pieri type) of $Gr(k+1,n)$, thus giving an elementary proof [12].

This idea was later used by Buch, Kresch and Tamvakis to show that all Gromov-Witten invariants $N_{u,v}^{w,d}$ for complex Grassmannians, Lagrangian Grassmannians and (maximal) orthogonal Grassmannians are classical. Such a phenomenon is now referred to as the *quantum to classical* principle. It was further shown to hold for the remaining two (co)minuscule Grassmannians of exceptional Lie type, i.e., Cayley plane E_6/P_1 and the Freudenthal variety E_7/P_7. Combining both statements, we have

Theorem 3.10 ([14], [20]). *All genus zero, three-point Gromov-Witten invariants for a (co)minuscule Grassmannian G/P are equal to classical intersection numbers on some auxiliary homogeneous varieties of G .*

Such a statement has been extended to the equivariant quantum K-theory setting in [17], [24]. The above theorem leads to a manifestly positive combinatorial formula for all $N_{u,v}^{w,d}$ for $Gr(k,n)$, because of the known positive formula on the classical intersection numbers on two-step partial flag varieties [29]. The kernel-span technique was also used in [15] to derive the quantum Pieri rules with respect to $c_p(\mathcal{Q})$'s for non-maximal isotropic Grassmannians.

There is another (combinatorial) approach to show the "quantum to classical" principle by the authors [73]: As a consequence of the special case of Theorem 3.5 with $\pi : G/B \to G/P$ being a $\mathbb{P}^1$-bundle, the authors obtained vanishing and identities among various Gromov-Witten invariants. For any simple root α, we introduce a map $\mathrm{sgn}_\alpha : W \to \{0,1\}$ defined by $\mathrm{sgn}_\alpha(w) := 1$ if $\ell(w) - \ell(ws_\alpha) > 0$, and 0 otherwise.

Theorem 3.11 ([73]). *For any $u, v, w \in W$ and for any $\lambda \in Q^\vee \simeq H_2(G/B,\mathbb{Z})$, we have the following for $QH^*(G/B)$*

(1) $N_{u,v}^{w,\lambda} = 0$ *unless* $\mathrm{sgn}_\alpha(w) + \langle\alpha,\lambda\rangle \le \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v)$ *for all* $\alpha \in \Delta$.
(2) *Suppose* $\mathrm{sgn}_\alpha(w) + \langle\alpha,\lambda\rangle = \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v) = 2$ *for some* $\alpha \in \Delta$, *then*

$$N_{u,v}^{w,\lambda} = N_{us_\alpha,vs_\alpha}^{w,\lambda-\alpha^\vee} = \begin{cases} N_{u,vs_\alpha}^{ws_\alpha,\lambda-\alpha^\vee}, & \text{if } \mathrm{sgn}_\alpha(w) = 0 \\ N_{u,vs_\alpha}^{ws_\alpha,\lambda}, & \text{if } \mathrm{sgn}_\alpha(w) = 1 \end{cases}.$$

Combining the above theorem with the Peterson-Woodward comparison formula, the authors obtained a lot of nice applications, including the quantum Pieri rules with respect to $c_p(\mathcal{S})$'s for isotropic Grassmannians [74]. There are T-equivariant generalization [46] of these results, giving nice applications on equivariant quantum Schubert calculus, including equivariant quantum Pieri rules for all $SL(n,\mathbb{C})/P$'s.

There are some other ways to compute (part of) $N_{u,v}^{w,\mathbf{d}}$ in a few cases (see [1], [9], [91], [55], [53], [40] etc.). For instance for $Gr(k,n)$, the structure constants $N_{u,v}^{w,d}$ can also be computed from the classical intersection numbers on $Gr(k,2n)$ [8]. The equivariant quantum situation of this method is studied in [7]. For two-step flag variety $F\ell_{n_1,n_2;n}$, the quantum to classical principle holds for certain $N_{u,v}^{w,\mathbf{d}}$'s as well [31].

3.4. **Quantum Giambelli formulae.** These are formulae to express each Schubert class σ^u as a polynomial in the generators x_i and q_j in a presentation $QH^*(G/P) = \mathbb{Q}[\mathbf{x},\mathbf{q}]/(\text{relations})$. There are limited results as even such a ring presentation is still open in many cases, as discussed in section 3.1.

Schubert classes of $H^*(Gr(k,n))$ are labeled by partitions. Partitions corresponding $c_p(\mathcal{S}^*)$'s (or $c_p(\mathcal{Q})$'s) are called special. The Giambelli formula expresses Schubert classes in terms of determinants with special Schubert classes as entries. The first quantum version of Giambelli formula, due to Bertram [5], was obtained by evaluation in the classical cohomology ring of quot schemes. Similar ideas were applied to the case of Lagrangian Grassmannians and orthogonal Grassmannians [58, 59]. There is an alternative way to obtain a quantum Giambelli formula, by using a quantum Pieri rule and the classical Giambelli formula. This was used to reprove a quantum Giambelli formula for complex Grassmannians by Buch [12], and to obtain one for non-maximal isotropic Grassmannian by Buch, Kresch and Tamvakis [16]. There are also known formulas for some Grassmannians of exceptional types by using software [25].

For complete flag variety $SL(n,\mathbb{C})/B$, Formin, Gelfand, and Postnikov constructed quantum Schubert polynomials [34] that represent the quantum Schubert classes. For partial flag varieties $SL(n,\mathbb{C})/P$, the quantum Giambelli formula are obtained by Ciocan-Fontanine [27] by using a geometric tool of moving lemma for quot scheme. In general, there could be a third way to get a quantum Giambelli formula, by studying the equivariant quantum version first and then taking the non-equivariant limit. The equivariant quantum cohomology ring $QH_T^*(G/P)$ contains more information than the quantum cohomology, but behaves more simply than $QH^*(G/P)$ in the sense that it is essentially determined by the equivariant quantum Chevalley formula due to a criterion by Mihalcea [83]. Suppose that we already have an expectation on a ring presentation of $QH_T^*(G/P)$, together with an expected formula on the equivariant quantum Schubert classes. Then we can prove our expectation by checking that these candidate satisfy the equivariant quantum Chevalley formula. In this way, Mihalcea obtained the equivariant quantum Giambelli formula for complex Grassmannians [84], and Ikeda, Mihalcea and Naruse have achieved the case of maximal isotropic Grassmannians recently [47]. For $SL(n,\mathbb{C})/B$, there are quantum double Schubert polynomials studied by Kirillov and Maeno [51], and by Ciocan-Fontanine and Fulton [28]. Lam and Shimozono define the analogues for $SL(n,\mathbb{C})/P$ and show them to represent the equivariant quantum Schubert classes by using the third approach [69]. Such a result has also been independently obtained by Anderson and Chen by deriving an equivariant moving lemma for quot schemes [2].

3.5. **A few remarks.** So far we have mainly focused on an overview of the developments on the four problems listed in section 2.4. There are many other interesting problems in quantum Schubert calculus, for instance, the study of the symmetry among (part of) $QH^*(G/P)$ or its T-equivariant extension [4], [90], [19],[22], [21], [31]. While there are fewer results on $QH^*_T(G/P)$, especially on combinatorial rules of the structure constants and on equivariant quantum Giambelli formulas, our discussions are not made systematic.

On the other hand, because of Theorem 3.2, (equivariant) quantum Schubert calculus is essentially part of (equivariant) affine Schubert calculus. For instance, (equivariant) quantum Pieri rules [63], [68] can be obtained this way. We apologize for not mentioning all such applications.

A natural generalization of (equivariant) quantum Schubert calculus is (equivariant) *quantum K-theory* of homogeneous varieties [38], [70]. However, very little is known, including [17], [24], [13], [76], etc. Peterson's approach to the homology of affine Grassmannians could be generalized to obtain a K-theoretic analogue [64]. It is interesting to know:

Is there a K-theoretic analogue of Theorem 3.2, or more generally, of the statements on strata data $Y_P^{\pm}$ of the Peterson variety?[4]

There are other generalizations of the quantum Schubert calculus, say for some *inhomogeneous* varieties, for instance odd symplectic Grassmannians [86].

The notion of quantum cohomology arose in string theory in mathematical physics. It is natural to ask

What is mirror symmetry of quantum Schubert calculus?

Mirror symmetry predicts that the quantum cohomology ring $QH^*(G/P)$ is isomorphic to the Jacobian ring $\mathrm{Jac}(W)$ of a mirror Landau-Ginzberg model $(X^\vee, W)$. Recall that a *Landau-Ginzberg model* is pair $(X^\vee, W)$, consisting a non-compact Kähler manifold $X^\vee$ together with a holomorphic function $W : X^\vee \to \mathbb{C}$, which is called a *superpotential*. For instance when $G/P = \mathbb{P}^1$, we have $X^\vee = \mathbb{C}^*$ and $W : \mathbb{C}^* \to \mathbb{C}$ is defined by $W(z) = z + \frac{q}{z}$, where the quantum parameter q is treated as a fixed nonzero complex number. Then we have $QH^*(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}[x]/\langle x^2 - q\rangle \cong \mathbb{C}[z, z^{-1}]/\langle 1 - \frac{q}{z^2}\rangle = \mathrm{Jac}(W)$.

As we have already seen, such a mirror statement is about a ring presentation of $QH^*(G/P)$. Using the presentation of $QH^*(G/P)$ announced by Peterson, Rietsch constructed a mirror Landau-Ginzberg model of G/P [94]. We refer our readers to [62] for some recent developments in the relations between the quantum Schubert calculus and mirror symmetry (as well as the Whittaker functions) from an algebro-combinatorial perspective.

Mirror symmetry also predicts an isomorphism between $QH^*(G/P)$ and $\mathrm{Jac}(W)$ as Frobenius manifolds, matching flat coordinates of both sides. It would be very interesting if one could get combinatorial rules on $N_{u,v}^{w,\mathbf{d}}$ by using the corresponding basis of $\mathrm{Jac}(W)$. In certain cases, Schubert classes do play special roles in mirror symmetry [43], [87, 88],[79].

[4]Recently, Lam, Mihalcea, Shimozono and the second author have made such a conjectural K-theoretic analogue.

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