DONALDSON-THOMAS THEORY FOR CALABI-YAU FOUR-FOLDS

YALONG CAO AND NAICHUNG CONAN LEUNG

Abstract. Let X be a compact complex Calabi-Yau four-fold. Under certain assumptions, we define Donaldson-Thomas type deformation invariants (DT_4 invariants) by studying moduli spaces of solutions to the Donaldson-Thomas equations on X. We also study sheaves counting problem on local Calabi-Yau four-folds. We relate DT_4 invariants of K_Y to the Donaldson-Thomas invariants of the associated Fano three-fold Y. In some special cases, we prove a DT_4/GW correspondence for X. When the Calabi-Yau four-fold is toric, we use the virtual localization formula to define the equivariant DT_4 invariants. We also discuss the non-commutative version of DT_4 invariants for four dimensional Calabi-Yau algebras coming from path algebras of certain quivers with relations. Finally, we compute DT_4 invariants for certain Calabi-Yau four-folds when moduli spaces are smooth.

Contents

Notations and conventions. Throughout this paper, unless specified otherwise, $(X, \mathcal{O}(1))$ will be a polarized compact complex Calabi-Yau four-fold [82] equipped with a Ricci-flat Kähler metric g, a Kähler form ω and a holomorphic four-form Ω such that $\Omega \wedge \overline{\Omega} = dvol$ and $c_1(\mathcal{O}(1)) = [\omega]$, where $dvol$ is the volume form of g.

We denote (E, h) to be a complex vector bundle with a Hermitian metric over X and G to be the structure group of E with center $C(G)$. We will restrict to the case when $G = U(r)$, where r is the rank of E.

We denote $\mathcal{A}$ to be the space of all L^2_k (Sobolev norm) unitary connections on E and $\mathcal{G}$ to be the L^2_{k+1} unitary gauge transformation group, where k is a large enough positive integer. $\Omega^{0,i}(X, EndE)_k$ is denote to be the completion of $\Omega^{0,i}(X, EndE)$ by L^2_k norm.

We denote the space of irreducible L^2_k unitary connections by

$$\mathcal{A}^* = \{A \in \mathcal{A} \mid \Gamma_A = C(G)\},$$

where $\Gamma_A = \{u \in \mathcal{G} \mid u(A) = A\}$ is the isotropic group at A. $\mathcal{A}^*$ is a dense open subset of $\mathcal{A}$ [21]. Let $\mathcal{G}^0 = \mathcal{G}/C(\mathcal{G})$ be the reduced gauge group. We know the action $\mathcal{G}^0$ on $\mathcal{A}^*$ is free and define $\mathcal{B}^* \triangleq \mathcal{A}^*/\mathcal{G}^0$, which is a Banach manifold [19], [26].

We denote $\mathcal{M}_c(X, \mathcal{O}(1))$ or simply $\mathcal{M}_c$ to be the Gieseker moduli space of $\mathcal{O}(1)$-stable sheaves with given Chern character c. We always assume $\mathcal{M}_c$ is compact, i.e. $\mathcal{M}_c = \overline{\mathcal{M}}_c$ ($\overline{\mathcal{M}}_c$ is the Gieseker moduli space of semi-stable sheaves) which is satisfied under the coprime condition of degree and rank of coherent sheaves [36].

We take $\mathcal{M}_c^{bdl}$ to be the analytic open subspace of $\mathcal{M}_c$ consisting of slope-stable holomorphic bundles which is possibly empty. We will not distinguish $\mathcal{M}_c^{bdl}$ with the moduli space of holomorphic Hermitian-Yang-Mills connections by Donaldson-Uhlenbeck-Yau's theorem [79].

In this paper, when we say $\mathcal{M}_c$ is smooth, we always mean it in the strong sense, namely all Kuranishi maps are zero.

1. Introduction

In this paper, we study Donaldson-Thomas theory for Calabi-Yau four-folds. Originally, Floer studied Chern-Simons theory for closed oriented three-manifolds and defined the instanton Floer homology generalizing the Casson invariant for moduli spaces of flat connections. For closed oriented four-manifolds, Donaldson [19],[21] defined polynomial invariants by studying moduli spaces of anti-self-dual connections on $SU(2)$ bundles over four-manifolds. Obviously, flat connections are anti-self-dual connections. The converse is also true if $ch_2(E) = 0$.

Over complex-oriented manifolds, i.e. Calabi-Yau manifolds [82], flat bundles are replaced by holomorphic bundles. Thomas [76] then studied complex Chern-Simons gauge theory on Calabi-Yau three-folds and defined the so-called Donaldson-Thomas invariant for moduli spaces of stable holomorphic bundles (more generally, for Gieseker moduli spaces of stable sheaves).

As complex analogue of Donaldson theory, we study complex anti-self-dual equations written by Donaldson and Thomas on Calabi-Yau four-folds. Holomorphic Hermitian-Yang-Mills connections are complex anti-self-dual connections. The converse is also true if $ch_2(E) \in H^{2,2}(X)$. By Donaldson-Uhlenbeck-Yau's theorem [79], the moduli space of holomorphic Hermitian-Yang-Mills connections is the moduli space of slope-stable bundles which has natural Gieseker compactification. However, Gieseker moduli spaces on CY_4's generally do not have perfect obstruction theory [55], [6] and don't carry virtual fundamental classes. One of our aims in this paper is to modify the obstruction theory of Gieseker moduli spaces, construct virtual fundamental classes and define the corresponding invariants under certain assumptions.

We start with a compact Calabi-Yau four-fold (X, g, ω, Ω) and define

$$*_4 : \Omega^{0,2}(X) \to \Omega^{0,2}(X),$$

$$\alpha \wedge *_4\beta = (\alpha, \beta)_g \overline{\Omega}.$$

Coupled with bundle (E, h), it is extended to

$$*_4 : \Omega^{0,2}(X, EndE) \to \Omega^{0,2}(X, EndE)$$

with $*_4^2=1$ [22]. Then we use this $*_4$ operator define the (anti-) self-dual subspace of $\Omega^{0,2}(X,EndE)$. In fact, $*_4$ also splits the corresponding harmonic subspace into self-dual and anti-self-dual parts.

The DT_4 equations are defined to be

$$\begin{cases} F_+^{0,2}=0 \\ F\wedge\omega^3=0, \end{cases}$$

where the first equation is $F^{0,2}+*_4F^{0,2}=0$ and we assume $c_1(E)=0$ for simplicity in the moment map equation $F\wedge\omega^3=0$.

We denote $\mathcal{M}^{DT_4}(X,g,[\omega],c,h)$ or simply $\mathcal{M}_c^{DT_4}$ to be the space of gauge equivalence classes of solutions to the DT_4 equations on a complex vector bundle E with Chern character $ch(E)=c$.

To define Donaldson type invariants using $\mathcal{M}_c^{DT_4}$, we need

(1) compactness, (2) orientation, (3) transversality.

Compactness. To compactify $\mathcal{M}_c^{DT_4}$, we start with its local Kuranishi structure.

Theorem 1.1. *(Theorem 3.13) We assume $\mathcal{M}_c^{bdl}\neq\emptyset$, then*
(1) $\mathcal{M}_c^{bdl}\cong\mathcal{M}_c^{DT_4}$ as sets.
(2) We fix $d_A\in\mathcal{M}_c^{DT_4}$, then there exists a Kuranishi map $\tilde{\tilde{\kappa}}$ of $\mathcal{M}_c^{bdl}$ at $\overline{\partial}_A$ (the (0,1) part of d_A) such that $\tilde{\tilde{\kappa}}_+$ is a Kuranishi map of $\mathcal{M}_c^{DT_4}$ at d_A, where

$$\tilde{\tilde{\kappa}}_+=\pi_+(\tilde{\tilde{\kappa}}):H^{0,1}(X,EndE)\xrightarrow{\tilde{\tilde{\kappa}}}H^{0,2}(X,EndE)\xrightarrow{\pi_+}H_+^{0,2}(X,EndE)$$

and π_+ is projection to the self-dual forms.
(3) The closed imbedding between analytic spaces possibly with non-reduced structures $\mathcal{M}_c^{bdl}\hookrightarrow\mathcal{M}_c^{DT_4}$ is also a homeomorphism between topological spaces.

In general, we want to obtain a compactification of $\mathcal{M}_c^{DT_4}$, denoted to be $\overline{\mathcal{M}}_c^{DT_4}$, such that the homeomorphism $\mathcal{M}_c^{bdl}\cong\mathcal{M}_c^{DT_4}$ can be extended to $\mathcal{M}_c\cong\overline{\mathcal{M}}_c^{DT_4}$ while the local analytic structure of $\overline{\mathcal{M}}_c^{DT_4}$ is given by $\kappa_+^{-1}(0)$, where

$$\kappa_+=\pi_+(\kappa):Ext^1(\mathcal{F},\mathcal{F})\to Ext^2_+(\mathcal{F},\mathcal{F}),$$

κ is a Kuranishi map of $\mathcal{M}_c$ at $\mathcal{F}$ and $Ext^2_+(\mathcal{F},\mathcal{F})$ is a half-dimensional real subspace of $Ext^2(\mathcal{F},\mathcal{F})$ on which the Serre duality quadratic form is real and positive definite.

Although beginning with connections on bundles, we notice that $\mathcal{M}_c$ may not contain any locally free sheaf (like the moduli space of ideal sheaves of points) and the above gluing approach to define an analytic space $\overline{\mathcal{M}}_c^{DT_4}$ still makes sense. We then call $\overline{\mathcal{M}}_c^{DT_4}$ the generalized DT_4 moduli space (Definition 4.5). The name comes from the fact that it may not parameterize any locally free sheaf in general while the DT_4 moduli space consists of connections on bundles only.

It is then obvious that if $\mathcal{M}_c=\mathcal{M}_c^{bdl}\neq\emptyset$, $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4}=\mathcal{M}_c^{DT_4}$. We have the following less obvious gluing results.

Proposition 1.2. *(Proposition 4.7, 4.8)*
If (i) $\mathcal{M}_c$ is smooth or (ii) for any closed point $\mathcal{F}\in\mathcal{M}_c$, there exists a complex vector space $V_{\mathcal{F}}$ and a linear isometry

$$(Ext^2(\mathcal{F},\mathcal{F}),Q_{Serre})\cong(T^*V_{\mathcal{F}},Q_{std})$$

such that the image of a Kuranishi map κ of $\mathcal{M}_c$ at $\mathcal{F}$ satisfies

$$\mathrm{Image}(\kappa)\subseteq V_{\mathcal{F}},$$

then the generalized DT_4 moduli space exists and $\overline{\mathcal{M}}_c^{DT_4}\cong\mathcal{M}_c$ as real analytic spaces. Q_{Serre} is the Serre duality pairing and Q_{std} is the standard pairing between $V_{\mathcal{F}}$ and $V_{\mathcal{F}}^*$.

Remark 1.3. *Under any one of the above assumptions, we will no longer need to assume $\mathcal{M}_c$ contains any stable bundle.*

Orientation. The orientability issue for $\mathcal{M}_c^{DT_4}$ is concerning the determinant line bundle $\mathcal{L}$ of the index bundle of twisted Dirac operators. We recall that if $\mathcal{M}_c^{bdl}\neq\emptyset$, then $\mathcal{M}_c^{DT_4}\cong\mathcal{M}_c^{bdl}$ as topological spaces (Theorem 1.1). In this case,

$$\mathcal{L}|_E=\left(\wedge^{top}Ext^2_+(E,E)\right)^{-1}\otimes\wedge^{top}Ext^1(E,E),$$

where $Ext^2_+(E,E)$ is the self-dual subspace of $Ext^2(E,E)$.

We note that $w_1(\mathcal{L}) = w_1(Ind)$, where Ind is the index virtual bundle over $\mathcal{M}_c^{DT_4}$ for the operator

$$\Omega^1(X, g_E)_k \to \Omega_+^{0,2}(X, EndE)_{k-1} \oplus \Omega^0(X, g_E)_{k-1} \oplus \Omega^0(X, g_E)_{k-1},$$
$$a = a^{1,0} + a^{0,1} \mapsto (\pi_+ \overline{\partial}_A a^{0,1}, d_A^* a, d_A^{c\,*} a),$$

where $d_A \in \mathcal{M}_c^{DT_4}$. Explicitly, we have

$$Ind|_E = Ext^1(E,E) - Ext_+^2(E,E).$$

We define the complexified index bundle $Ind_{\mathbb{C}} \triangleq Ind \otimes_{\mathbb{R}} \mathbb{C}$. By Remark 2.3,

$$Ind_{\mathbb{C}}|_E \cong Ext^1(E,E) - Ext^2(E,E) + Ext^3(E,E),$$

which has the Serre duality quadratic form Q_{Serre}. Similar to a construction in K-theory [1], there exists a trivial bundle with the trivial standard quadratic form $(\mathbb{C}^N, q)$ such that $(Ind_{\mathbb{C}}, Q_{Serre}) \oplus (\mathbb{C}^N, q)$ becomes a quadratic bundle (vector bundle with a non-degenerate quadratic form). If $c_1(Ind_{\mathbb{C}}) = 0$, then the structure group of the quadratic bundle can be reduced to $SO(n, \mathbb{C})$ which makes the structure group of the corresponding real bundle inside $SO(n, \mathbb{R})$. The above reduction of the structure group to $SO(n, \mathbb{C})$ involves a choice on each component of $\mathcal{M}_c^{DT_4}$ [24], which corresponds exactly to a choice of an orientation of $\mathcal{L}$ on each component of $\mathcal{M}_c^{DT_4}$.

Definition 1.4. *If $c_1(Ind_{\mathbb{C}}) = 0$, the above $SO(n, \mathbb{C})$-reduction is called a choice of an orientation of $Ind_{\mathbb{C}}$. If the corresponding real bundle has a complex orientation, $Ind_{\mathbb{C}}$ is said to have a natural complex orientation, denoted by $o(\mathcal{O})$.*

Remark 1.5. *By the above discussion, a choice of an orientation of $Ind_{\mathbb{C}}$ is equivalent to a choice of an orientation of $\mathcal{L}$ on $\mathcal{M}_c^{DT_4}$.*

We note that, $Ind_{\mathbb{C}}$ has the advantage over Ind by being well defined also on $\mathcal{M}_c$ whereas $\mathcal{L}$ can be defined even when $c \notin \bigoplus_k H^{k,k}(X)$.

To pick a coherent choice of orientations for all components of the moduli space, as in Donaldson theory [21], we need to extend the index bundle and its determinant line bundle to some big connected space such that $\mathcal{M}_c^{DT_4}$ (or $\mathcal{M}_c$) embeds inside with induced index (or determinant line) bundle. We know $\mathcal{M}_c^{DT_4} \hookrightarrow \mathcal{B}^*$, where the determinant line bundle $\mathcal{L}$ extends naturally.

For $\mathcal{M}_c$ with $Hol(X) = SU(4)$, by Seidel-Thomas twists [39] [72], we can identify it with some component(s) of $\mathcal{M}_{si}$, a moduli space of simple holomorphic bundles with some fixed Chern classes, where the index bundle $Ind_{\mathbb{C}}$ also extends. By choosing a Hermitian metric, we can further imbed $\mathcal{M}_{si}$ into the space of gauge equivalence classes of irreducible unitary connections $\mathcal{B}^*$, where the determinant line bundle $\mathcal{L}$ of twisted Dirac operators mentioned before is defined. Note that, one choice of an orientation of $\mathcal{L}$ gives an orientation of $Ind_{\mathbb{C}}$ on $\mathcal{M}_{si}$.

By [21], $\mathcal{B}^*$ is connected, there are only two orientations for any orientable bundle. We assume from now on the determinant line bundle $\mathcal{L}$ on $\mathcal{B}^*$ is oriented.

Definition 1.6. *An orientation data of DT_4 theory, denoted by $o(\mathcal{L})$, is a choice of an orientation of $Ind_{\mathbb{C}}$ on $\mathcal{M}_c$ (Definition 1.4) induced from an orientation of $\mathcal{L}$ on $\mathcal{B}^*$ via Seidel-Thomas twists.*

Remark 1.7.
1. The orientation data may involve choices of Seidel-Thomas twists for $\mathcal{M}_c$.
2. Making a choice of orientation on $\mathcal{L}$ from the ambient space $\mathcal{B}^*$ is for the purpose of deformation invariance of the theory. If we have some natural orientation of $Ind_{\mathbb{C}}$ on $\mathcal{M}_c$, such as $Ind_{\mathbb{C}}$ has a natural complex orientation (Definition 1.4, we will see there are many such examples), then we will just use that natural orientation without referring to $\mathcal{B}^*$.

The existence of the orientation data is partially given by the following theorem.

Theorem 1.8. *(Theorem 10.14) For any compact simply connected Calabi-Yau four-folds X such that $H_3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$, and $U(r)$ bundle $E \to X$, the determinant line bundle $\mathcal{L}$ of the index bundle of twisted Dirac operators over the space $\mathcal{B}^*$ is trivial.*

Corollary 1.9. *(Corollary 10.17) Let X be a compact simply connected Calabi-Yau four-fold such that $Hol(X) = SU(4)$ and $H_3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$. Then the index bundle $Ind_{\mathbb{C}}$ over $\mathcal{M}_c$ satisfies $c_1(Ind_{\mathbb{C}}) = 0$. Furthermore, the orientation data of DT_4 theory (Definition 1.6) exists in this case.*

Transversality. Finally, we come to the transversality issue, i.e. making sense of the fundamental class of the moduli space despite it may contain many components of different dimensions. Firstly, we show that when $\mathcal{M}_c^{DT_4}$ is compact, its virtual fundamental class exists.

Theorem 1.10. *(Theorem 5.7)*
We assume $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$ and there exists an orientation data $o(\mathcal{L})$, then $\mathcal{M}_c^{DT_4}$ is compact and its virtual fundamental class exists as a cycle,

$$[\mathcal{M}_c^{DT_4}]^{vir} \in H_r(\mathcal{B}^*, \mathbb{Z}),$$

where $r = 2 - \chi(E,E)$ is the real virtual dimension and $\chi(E,E) = \sum_i (-1)^i h^i(X, EndE)$.

Furthermore, if the above assumptions are satisfied by a continuous family of Calabi-Yau four-folds X_t parameterized by $t \in [0,1]$, then the virtual cycle in $H_r(\mathcal{B}^*, \mathbb{Z})$ is independent of t.

We also define virtual fundamental classes of $\overline{\mathcal{M}}_c^{DT_4}$'s (Definition 5.12, 5.14) for the above two gluable cases where $\overline{\mathcal{M}}_c^{DT_4} \cong \mathcal{M}_c$ as real analytic spaces.

Axioms of DT_4 invariants. Similar to the case of Donaldson theory for four-manifolds [21], we can use μ-maps to cut down degrees of virtual fundamental classes and define the corresponding DT_4 invariants (Definition 5.10, 5.15).

At the moment, we can only define DT_4 invariants in several cases under different assumptions, i.e. under any one of the following assumptions and the assumption on the existence of the orientation data $o(\mathcal{L})$ (which is partially solved in Corollary 1.9 and cases (iii) and (iii') are always naturally oriented), we can define DT_4 invariants,
(i) if the Gieseker moduli space consists of slope-stable bundles only, i.e. $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$, or
(ii) if the Gieseker moduli space is smooth and consists of stable sheaves only, i.e. $\overline{\mathcal{M}}_c = \mathcal{M}_c$ is smooth, or
(iii) if the Gieseker moduli space of compactly supported sheaves $\overline{\mathcal{M}}_c(K_Y, \pi^*\mathcal{O}_Y(1))$ consists of slope-stable sheaves only, where $(Y, \mathcal{O}_Y(1))$ is a polarized Fano threefold, or more generally,
(iii') if $\overline{\mathcal{M}}_c = \mathcal{M}_c$ and there exists a perfect obstruction theory [6]

$$\phi: \quad \mathcal{V}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}_c},$$

such that

$$H^0(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}} \cong Ext^1(\mathcal{F},\mathcal{F}),$$
$$H^{-1}(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}} \oplus H^{-1}(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}}^* \cong Ext^2(\mathcal{F},\mathcal{F}),$$

and $H^{-1}(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}}$ is a maximal isotropic subspace of $Ext^2(\mathcal{F},\mathcal{F})$ with respect to the Serre duality pairing.

To make all these cases consistent, we propose several axioms that DT_4 invariants should satisfy. Axioms (6)-(8) are showed in the paper and axioms (1)-(5) are verified when we have definitions of virtual fundamental classes of (generalized) DT_4 moduli spaces.

Axiom 1.11. *Given a triple $(X, \mathcal{O}(1), c)$ and an auxiliary choice of an orientation data $o(\mathcal{L})$, where $(X, \mathcal{O}(1))$ is a polarized Calabi-Yau four-fold, $c \in H_c^{even}(X, \mathbb{Q})$ is a (compactly supported) cohomology class, the DT_4 invariant (i.e. Donaldson-Thomas four-fold invariant) of this quadruple, denoted by $DT_4(X, \mathcal{O}(1), c, o(\mathcal{L}))$ is a map*

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) : Sym^*\big(H_*(X,\mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{Z},$$

(Sym means graded symmetric with respect to the parity of the degree of $H_*(X)$) such that:
(1) Orientation reversed

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = -DT_4(X, \mathcal{O}(1), c, -o(\mathcal{L})),$$

where $-o(\mathcal{L})$ denotes the opposite orientation of $o(\mathcal{L})$.
(2) Deformation invariance

$$DT_4(X_0, \mathcal{O}(1)|_{X_0}, c, o(\mathcal{L}_0)) = DT_4(X_1, \mathcal{O}(1)|_{X_1}, c, o(\mathcal{L}_1)),$$

where $(X_t, \mathcal{O}(1))$ is a continuous family of complex structures and $o(\mathcal{L}_t)$ is an orientation data on the family determinant line bundle with $t \in [0,1]$.
(3) Vanishing for certain virtual dimensions

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = 0,$$

if any one of the following two conditions is satisfied,
(i) $\chi(\mathcal{F},\mathcal{F}) > 2$, or (ii) $\chi(\mathcal{F},\mathcal{F})$ is odd and $H^{odd}(X,\mathbb{Z}) = 0$. $\chi(\mathcal{F},\mathcal{F})$ is the holomorphic Euler

characteristic uniquely determined by c *and the topology of* X*.*
(4) Vanishing for certain choices of c

$$DT_4(X,\mathcal{O}(1),c,o(\mathcal{L}))=0,$$

if any one of the following two conditions is satisfied,
(i) $c|_{H^4(X,\mathbb{Q})}$ *has no component in* $H^{0,4}(X)$ *and* $c\notin\bigoplus_{i=0}^4 H^{i,i}(X)$*, or*
(ii) $c\in\bigoplus_{i=0}^4 H^{i,i}(X)$ *and* $\exists\ \varphi\in H^1(X,TX)$ *such that* $\varphi\lrcorner\left(c|_{H^{2,2}(X,\mathbb{Q})}\right)\neq 0$ *(Proposition 5.26).*
(5) Vanishing for compact hyper-Kähler manifolds

$$DT_4(X,\mathcal{O}(1),c,o(\mathcal{L}))=0,$$

when X *is compact hyper-Kähler (18).*
(6) DT_4/DT_3 ***correspondence***

$$DT_4(X,\pi^*\mathcal{O}_Y(1),c,o(\mathcal{O}))=DT_3(Y,\mathcal{O}_Y(1),c'),$$

if $c=(0,c|_{H^2_c(X)}\neq 0,c|_{H^4_c(X)},c|_{H^6_c(X)},c|_{H^8_c(X)})$ *and the Gieseker moduli space of compactly supported sheaves* $\overline{\mathcal{M}}_c(X,\pi^*\mathcal{O}_Y(1))$ *consists of slope-stable sheaves, where* $\pi: X=K_Y\to Y$ *is projection and* $(Y,\mathcal{O}_Y(1))$ *is a polarized compact Fano threefold.*

In this setup, sheaves in $\overline{\mathcal{M}}_c(X,\pi^*\mathcal{O}_Y(1))$ *are of type* $\iota_*(\mathcal{F})$ *where* $\iota: Y\to K_Y$ *is the zero section and* $c'=ch(\mathcal{F})\in H^{even}(Y)$ *is uniquely determined by* c*.* $o(\mathcal{O})$ *is the natural complex orientation of* $Ind_{\mathbb{C}}$ *over* $\mathcal{M}_c$*.*

$DT_3(Y,\mathcal{O}_Y(1),c')$ *is the* DT_3 *invariant of* $(Y,\mathcal{O}_Y(1))$ *with certain insertion fields (Theorem 6.5).*
(7) Normalization 1

$$DT_4(X,\mathcal{O}(1),c,o(\mathcal{L}))=DT_4^{\mu_1}(X,\mathcal{O}(1),c,o(\mathcal{L})),$$

if X *is compact and* $\overline{\mathcal{M}}_c=\mathcal{M}_c^{bdl}\neq\emptyset$*.*

$DT_4^{\mu_1}(X,\mathcal{O}(1),c,o(\mathcal{L}))$ *is defined using virtual fundamental class of* $\mathcal{M}_c^{DT_4}$ *and the corresponding* μ*-map (15).*
(8) Normalization 2

$$DT_4(X,\mathcal{O}(1),c,o(\mathcal{L}))=DT_4^{\mu_2}(X,\mathcal{O}(1),c,o(\mathcal{L})),$$

if $\overline{\mathcal{M}}_c=\mathcal{M}_c\neq\emptyset$ *is smooth or satisfies the condition in Definition 5.14.*

$DT_4^{\mu_2}(X,\mathcal{O}(1),c,o(\mathcal{L}))$ *is defined using virtual fundamental class of* $\overline{\mathcal{M}}_c^{DT_4}$ *and the corresponding* μ*-map (17).*

In normalization axioms, the construction depends on the existence of virtual fundamental classes and μ-map descendent fields as mentioned above. Throughout the paper, we will often only mention DT_4 virtual cycles (virtual fundamental classes of generalized DT_4 moduli spaces) instead of using the corresponding DT_4 invariants for convenience purposes.

Computational examples. Li-Qin [58] had provided examples when $\overline{\mathcal{M}}_c=\mathcal{M}_c^{bdl}\neq\emptyset$. By studying their examples, we have

Theorem 1.12. *(Theorem 7.2)*
Let X *be a generic smooth hyperplane section in* $\mathbb{P}^1\times\mathbb{P}^4$ *of bi-degree* $(2,5)$*. Let*

$$cl=[1+(-1,1)|_X]\cdot[1+(\epsilon_1+1,\epsilon_2-1)|_X],$$
$$k=(1+\epsilon_1)\begin{pmatrix}6-\epsilon_2\\4\end{pmatrix},\quad \epsilon_1,\epsilon_2=0,1.$$

We denote $\overline{\mathcal{M}}_c(L_r)$ *to be the moduli space of Gieseker* L_r*-semistable rank-2 torsion-free sheaves with Chern character* c *(which can be easily read from the total Chern class* cl*), where* $L_r=\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(1,r)|_X$*.*
(1) *If*

$$\frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2}<r<\frac{15(2-\epsilon_2)}{\epsilon_1(1+2\epsilon_2)},$$

then $\overline{\mathcal{M}}_c^{DT_4}$ *exists and* $\overline{\mathcal{M}}_c^{DT_4}\cong\overline{\mathcal{M}}_c(L_r)\cong\mathbb{P}^k$, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}=[\mathbb{P}^k]$.

(2) *If*

$$0<r<\frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2},$$

then $\overline{\mathcal{M}}_c^{DT_4}=\emptyset$ *and* $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}=0$.

We also study the DT_4/GW correspondence for compact Calabi-Yau four-folds in some specific cases.

Theorem 1.13. *(Theorem 7.8, 7.16)*
Let X be a compact Calabi-Yau four-fold. We assume $\mathcal{M}_c$ with given Chern character $c = (1,0,0,-PD(\beta),-1)$ is smooth and consists of ideal sheaves of smooth connected genus zero imbedded curves only, then $Ind_{\mathbb{C}}$ on $\mathcal{M}_c$ has a natural complex orientation $o(\mathcal{O})$. Assume the GW moduli space $\overline{\mathcal{M}}_{0,0}(X,\beta) \cong \mathcal{M}_c$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_{0,0}(X,\beta)$. Furthermore,

(1) *if $Hol(X) = SU(4)$, then $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = [\overline{\mathcal{M}}_{0,0}(X,\beta)]^{vir}$.*

(2) *if $Hol(X) = Sp(2)$, i.e. irreducible hyper-Kähler, then $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0$.*

Furthermore, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}_{hyper-red} = [\overline{\mathcal{M}}_{0,0}(X,\beta)]^{vir}_{red}$ (see theorem 7.16).

When $X = T^*S$, we only consider sheaves with scheme theoretical support inside S (i.e. it is of type $\iota_*\mathcal{F}$, where $\iota : S \hookrightarrow X$ is the zero section). To ensure they form component(s) of a moduli space of sheaves on X. We make the following assumption

$$Ext^0_S(\mathcal{F},\mathcal{F}\otimes\Omega^1_S) = 0, \quad Ext^2_S(\mathcal{F},\mathcal{F}) = 0, \tag{1}$$

which is satisfied when (i) $S = \mathbb{P}^2$, $\mathcal{F}$ is torsion-free slope stable or (ii) S is del-Pezzo, $\mathcal{F}$ is an ideal sheaf of points.

We then denote $\mathcal{M}_c^{S_{cpn}} = \{\iota_*\mathcal{F} \mid \mathcal{F} \in \mathcal{M}_c(S)\}$ to be component(s) of the moduli space of sheaves on X which can be identified with $\mathcal{M}_c(S)$ (the moduli space of stable sheaves on S with Chern character $c \in H^{even}(S)$). Under the above assumptions, $\mathcal{M}_c^{S_{cpn}}$ is smooth with negative virtual dimension. After taking away the trivial part of the obstruction bundle and consider the reduced virtual cycle (Definition 6.12), we have

Theorem 1.14. *(Theorem 6.17) Let $X = T^*S$, where S is a compact algebraic surface with $q(S) = 0$. Under assumption (1), we have $[\mathcal{M}_c^{S_{cpn}}]^{vir} = 0$. Furthermore,*

(1) *$[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red} = 0$, when $c|_{H^0(S)} \geq 2$.*

(2) *$[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red} = 1$, when $c = (1, c|_{H^2(S)}, 0)$.*

(3) *$[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red} = e(Hilb^n(S))$, when $c = (1,0,-n)$, $n \geq 1$.*

Moreover, they fit into the following generating function

$$\sum_{n\geq 0}[\mathcal{M}^{S_{cpn}}_{(1,0,-n)}]^{vir}_{red}q^n = \prod_{k\geq 1}\frac{1}{(1-q^k)^{e(S)}}.$$

Lastly, for the case of ideal sheaves of one point, one have

Proposition 1.15. *(Proposition 7.18)*
If $Hol(X) = SU(4)$ and $c = (1,0,0,0,-1)$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong X$, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = \pm PD(c_3(X))$.

Equivariant DT_4 invariants. We also study the equivariant DT_4 theory for ideal sheaves of surfaces, curves and points on toric CY_4. We define the corresponding equivariant DT_4 invariants by the virtual localization formula [30]. Because torus fixed points of corresponding Gieseker moduli spaces are discrete, we do not need $\overline{\mathcal{M}}_c^{DT_4}$ to define invariants. We get the definition without any constraint on moduli spaces.

By studying the simplest example for $\mathbb{C}^4$, we get

Proposition 1.16. *(Proposition 8.8)*
Let $X = \mathbb{C}^4$, for some choice of toric orientation data (Definition 8.4), we have

$$Z_{DT_4}\big(X, 1\ |(1,1)\big)_0 = \frac{\sigma_1\sigma_2 - \sigma_3}{\sigma_1\sigma_3},$$

where σ_i is the i-th elementary symmetric polynomial of variables $\lambda_1,\lambda_2,\lambda_3$ and $Z_{DT_4}\big(X,1\ |(1,1)\big)_0$ is the equivariant DT_4 invariant (Definition 8.6) of $\mathbb{C}^4$ for ideal sheaves of one point without any insertion field.

Non-commutative DT_4 invariants. In the non-commutative world, where sheaves on manifolds are replaced by representations of algebras, we also have a definition of Donaldson-Thomas type theory for four dimensional Calabi-Yau algebras. We define the $NCDT_4$ invariants and compute some examples. The detail is left to Section 9.

Comparisons with Borisov-Joyce's work. A related work was done by Borisov and Joyce [8]. They used local 'Darboux charts' in the sense of Brav, Bussi and Joyce (BBJ [9]), the machinery of homotopical algebra and C^{∞}-algebraic geometry to obtain a compact derived C^{∞}-scheme with the same underlying topological structure as the Gieseker moduli space of stable sheaves. In our language, their results proved the existence of generalized DT_4 moduli spaces in general (C^{∞}-scheme version). Furthermore, they defined the virtual fundamental class of the derived C^{∞}-scheme.

In the appendix, we will first give another proof of BBJ's 'Darboux theorem' for Gieseker moduli spaces of stable sheaves using gauge theory and Seidel-Thomas twists [39], [72]. We then introduce a weaker condition on their local 'Darboux charts' to include local models induced from DT_4 equations (i.e. the map $\tilde{\kappa}$ in Theorem 1.1). It turns out that the weaker condition is already sufficient for the gluing requirement in Borisov and Joyce's work [8] which then indicates the equivalence of their approach to virtual fundamental classes and our DT_4 virtual cycles.

Content of the paper : In section 2, we study the $*_4$ operator which is needed in the definition of DT_4 equations. In section 3, we study local analytic structures of DT_4 moduli spaces. In section 4, we compactify DT_4 moduli spaces under certain assumptions. Under the gluing assumption 4.4, which is verified in some cases, we define the generalized DT_4 moduli space. In section 5, we construct DT_4 virtual cycles, study actions of global monodromy on them and also give some vanishing results. In section 6, we study DT_4 invariants for compactly supported sheaves on local Calabi-Yau four-folds. In section 7, we compute some DT_4 invariants when $\mathcal{M}_c$'s are smooth. In section 8, we define equivariant DT_4 invariants on toric CY_4. In section 9, we discuss a definition of non-commutative DT_4 invariants for CY_4 algebras. In the last section, we list some useful facts as appendix. In particular, we prove the existence of the orientation data of DT_4 theory in a large number of cases. We also give another proof of 'Darboux theorem' (in the sense of Brav, Bussi and Joyce [9] Corollary 5.20) in the case when $\mathbb{K}=\mathbb{C}$ and $\mathcal{M}$ is the Gieseker moduli space of stable sheaves using gauge theory and Seidel-Thomas twists [39], [72].

Acknowledgement: The first author would like to express his deep gratitude to Huai-Liang Chang, Zheng Hua, Wei-Ping Li, Dennis Sullivan and Jiu-Kang Yu for many useful discussions and encouragements. We are grateful to Dennis Borisov and Dominic Joyce for their interests in this work and informing us their related work and thoughts. Special thanks to Dennis Borisov for explaining his joint work with Joyce [8]. We also hope to express our deep gratitude to Simon Donaldson for many useful discussions and inviting the first named author to visit him at the Simons Center for Geometry and Physics on May 2014. The work of the second author was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK401411).

2. The $*_4$ operator

2.1. **The $*_4$ operator for bundles.** In this section, we introduce the $*_4$ operator on the space of bundle valued differential forms which is the key to the definition of DT_4 equations and the construction of DT_4 moduli spaces. We define

$$*_4 : \Omega^{0,k}(X) \to \Omega^{0,4-k}(X),$$
$$\alpha \wedge *_4\alpha = |\alpha|^2\overline{\Omega},$$

which satisfies $*_4^2=1$ when $k=2$. Note that $*_4$ is a complex anti-linear map.

Given any Hermitian metric on E, we can extend $*_4$ to $\Omega^{0,k}(X,EndE)$. Using $*_4^2=1$, we have the following orthogonal decomposition,

$$\Omega^{0,2}(X,EndE)=\Omega^{0,2}_{+}(X,EndE)\oplus\Omega^{0,2}_{-}(X,EndE),$$

given by $\alpha=\frac{1}{2}(\alpha+*_4\alpha)+\frac{1}{2}(\alpha-*_4\alpha)$, where $\Omega^{0,2}_{\pm}(X,EndE)=\{\alpha\in\Omega^{0,2}(X,EndE)\mid *_4\alpha=\pm\alpha\}$. The following lemma shows that $*_4$ descends to corresponding harmonic subspaces.

Lemma 2.1. *$\overline{\partial}_E^{*}=\overline{\partial}_E^{*_4}$ on $\Omega^{0,k}(X,EndE)$, where $\overline{\partial}_E$ is a holomorphic $(0,1)$ connection and $\overline{\partial}_E^{*_4}\triangleq -*_4\overline{\partial}_E *_4$.*

Proof. This follows directly from $*=\Omega\wedge *_4$ and Ω being a norm one parallel form. □

Corollary 2.2. *The $*_4$ operator splits the space of harmonic forms $H^{0,2}(X,EndE)$ into*

$$H^{0,2}(X,EndE)=H^{0,2}_{+}(X,EndE)\oplus H^{0,2}_{-}(X,EndE)$$

according to (± 1) *eigenvalues.*

Remark 2.3. *It is obvious that* $\sqrt{-1}H^{0,2}_{+} = H^{0,2}_{-}$. *Thus they have the same dimension.*

2.2. **The $*_4$ operator for general coherent sheaves.** The construction of the $*_4$ operator on $H^{0,2}(X, EndE) = Ext^2(E, E)$ has a generalization to $Ext^2(\mathcal{F}, \mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on X: Since X is smooth and projective, we can resolve $\mathcal{F}$ by a complex of holomorphic vector bundles $(E^\bullet, \delta) \to \mathcal{F} \to 0$. We consider the following double complex [36],

$$(\Omega^{0,q}(X, \mathcal{H}om^p(E^\bullet, E^\bullet)), \overline{\partial}, \delta).$$

On the total complex (C^*, D), where $C^n = \bigoplus_{l\in\mathbb{Z}} \bigoplus_{p+q=n} \Omega^{0,q}(X, \mathcal{H}om(E^l, E^{l+p}))$ and $D = \overline{\partial} + (-1)^q\delta$, there exists two natural filtrations which induce two spectral sequences converging to $Ext^*(\mathcal{F}, \mathcal{F})$.

We fix Hermitian metrics on $\{E^l\}$ and similarly define the $*_4$ operator

$$*_4 : \Omega^{0,q}(X, \mathcal{H}om(E^l, E^{l+p})) \to \Omega^{0,4-q}(X, \mathcal{H}om(E^{l+p}, E^l))$$

as above. We then $\mathbb{R}$-linearly extend $*_4$ to C^*.

Lemma 2.4. (C^*, D) *is an elliptic complex.*

Proof. D is a C^∞-differential operator. The twisted Laplacian $\Delta_D = D^*D + DD^*$ has an expansion $\Delta_D = \Delta_{\overline{\partial}}+$ 1st order terms of $\overline{\partial}$ + 0 order terms, which shows D is elliptic [80]. □

Corollary 2.5. *The above* $*_4$ *operator on* C^* *descends to the D-harmonic subspace of* C^*. *Furthermore, it splits* $H^2(C^*, D) \cong Ext^2(\mathcal{F}, \mathcal{F})$ *into*

$$Ext^2(\mathcal{F}, \mathcal{F}) = Ext^2_+(\mathcal{F}, \mathcal{F}) \oplus Ext^2_-(\mathcal{F}, \mathcal{F})$$

according to (± 1) *eigenvalues.*

Proof. The operator $D^{*_4} \triangleq - *_4 D *_4$ equals to D^* as in Lemma 2.1. By a similar argument as in Corollary 2.2, we are done. □

Remark 2.6. *The subspaces* $Ext^2_\pm(\mathcal{F}, \mathcal{F})$ *depend on the choice of metrics on* $E^\bullet$.

3. Local Kuranishi structures of DT_4 moduli spaces

To study gauge theory on a general Kähler manifold X, we consider the moduli space of polystable holomorphic structures on E. By the renowned theorem of Donaldson-Uhlenbeck-Yau [79], this space equals to the moduli space of holomorphic Hermitian-Yang-Mills connections on E, i.e. the space of gauge equivalence classes of solutions to

$$\begin{cases} F^{0,2} = 0 \\ F \wedge \omega^3 = 0, \end{cases}$$

where we have assumed $c_1(E) = 0$ in the moment map equation $F \wedge \omega^3 = 0$ for simplicity. However, the holomorphic HYM equations are overdetermined when $dim_{\mathbb{C}}(X) \geq 3$.

The DT_4 moduli space. When X is a CY_4, Donaldson and Thomas [22] used the calibrated form Ω to cut down the number of equations in the holomorphic HYM equations to obtain an elliptic system, i.e. Donaldson and Thomas's complex ASD equations (DT_4 equations for short),

$$\begin{cases} F^{0,2}_+ = 0 \\ F \wedge \omega^3 = 0, \end{cases} \tag{2}$$

where the first equation is $F^{0,2} + *_4 F^{0,2} = 0$.

Remark 3.1.
1. The group of unitary gauge transformations preserves the above equations.
2. From the viewpoint of deformation-obstruction theory, we will see that this defines a perfect-obstruction theory, which leads to the construction of a virtual fundamental class for its moduli.

Then the definition of DT_4 moduli spaces follows from the above DT_4 equations.

Definition 3.2. *The* DT_4 *moduli space* $\mathcal{M}^{DT_4}_c$ *is the space of gauge equivalence classes of solutions to equations (2).*

Note that $\mathcal{M}^{DT_4}_c \hookrightarrow \mathcal{B}^* = \mathcal{A}^*/\mathcal{G}^0$ embeds as the zero loci of a section $s = (\wedge F, F^{0,2}_+)$ of a Banach bundle $\mathcal{E}$ over $\mathcal{B}^*$, where $\mathcal{E} = \mathcal{A}^* \times_{\mathcal{G}^0} (\Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1})$. This gives the DT_4 moduli space a natural real analytic structure.

Remark 3.3. *The Banach manifold $\mathcal{A}^*/\mathcal{G}^0$ involves a choice of a large integer k in the L^2_k Sobolev norm completion. By the ellipticity of DT_4 equations and Proposition 4.2.16* [21]*, DT_4 moduli spaces are independent of the choice of k. Thus we omit k in the notation.*

By the definition of $\mathcal{M}_c^{DT_4}$, we have an obvious inclusion between two sets
$$\mathcal{M}_c^{bdl} \to \mathcal{M}_c^{DT_4}.$$
If $\mathcal{M}_c^{bdl} \neq \emptyset$, by Lemma 4.1, the inclusion is a bijection.

Local structures of DT_4 moduli spaces. Even though we have identified $\mathcal{M}_c^{DT_4}$ and $\mathcal{M}_c^{bdl}$ as sets, they could have different possibly non-reduced analytic structures. We will show there exists a closed imbedding $\mathcal{M}_c^{bdl} \hookrightarrow \mathcal{M}_c^{DT_4}$ between these two real analytic spaces.

We start with an unitary connection $d_A \in \mathcal{M}_c^{DT_4}$, denote its $(0,1)$ part by $\overline{\partial}_A$ (we call $\overline{\partial}_A$ a $(0,1)$ connection). We have $F_A^{0,2} = 0$ under the assumption $\mathcal{M}_c^{bdl} \neq \emptyset$.

By the Hodge decomposition theorem, we have
$$\Omega^{0,2}(X, EndE)_{k-1} = H^{0,2}(X, EndE) \oplus \overline{\partial}_A \Omega^{0,1}(X, EndE)_k \oplus \overline{\partial}_A^* \Omega^{0,3}(X, EndE)_k,$$
$$I = \mathbb{H}^{0,2} + P_{\overline{\partial}_A} + P_{\overline{\partial}_A^*},$$
where the RHS consists of projections to the corresponding components.

Meanwhile,
$$*_4 : \overline{\partial}_A \Omega^{0,1}(X, EndE)_k \cong \overline{\partial}_A^* \Omega^{0,3}(X, EndE)_k$$
and
$$*_4 : H^{0,2}(X, EndE) \cong H^{0,2}(X, EndE)$$
induce
$$H^{0,2}(X, EndE) = H_+^{0,2}(X, EndE) \oplus H_-^{0,2}(X, EndE)$$
and
$$F_+^{0,2}(d_A + a) = 0 \Leftrightarrow a'' \triangleq (a)^{0,1} \textit{satisfies the following}$$
$$\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'') = 0, \tag{3}$$
$$\pi_+ \circ \mathbb{H}^{0,2}(a'' \wedge a'') = 0. \tag{4}$$
Here $a \in \Omega^1(X, g_E)_k$ and $a'' \triangleq (a)^{0,1} \in \Omega^{0,1}(X, EndE)_k$ is the (0,1) part of a.

Using the gauge fixing $d_A^* a = 0$, a neighbourhood of $d_A \in \mathcal{M}_c^{DT_4}$ can be described as
$$\{a \in \Omega^1(X, g_E)_k \big| \ \|a\|_k < \epsilon, \ d_A^* a = 0, \ d_A + a \ \textit{satisfies} \ (2)\},$$
where ϵ is a small positive number. We introduce a new space $\mathcal{M}_A^+$ to help establish relations of local structures between $\mathcal{M}_c^{DT_4}$ and $\mathcal{M}_c^{bdl}$.
$$\mathcal{M}_A^+ \triangleq \{a'' \in \Omega^{0,1}(X, EndE)_k \ \big| \ \|a''\|_k < \epsilon'', \ a'' \ \textit{satisfies} \ (3), (4), (5)\}$$
where (5) is defined to be
$$\overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a') = 0, \tag{5}$$
ϵ'' is a small positive number and $a' \triangleq (a)^{1,0}$ is the $(1,0)$ part of a.

In fact, $\mathcal{M}_A^+$ is locally isomorphic to a neighbourhood of $d_A \in \mathcal{M}_c^{DT_4}$.

Lemma 3.4. *The map which takes unitary connections to their $(0,1)$ parts*
$$d_A + a \longmapsto \overline{\partial}_A + a''$$
induces a local isomorphism near the origin
$$\{a \in \Omega^1(X, g_E)_k \big| \ \|a\|_k < \epsilon, \ d_A^* a = 0, \ d_A + a \text{ satisfies } (2)\} \cong \mathcal{M}_A^+.$$

Proof. Locally, we consider an ambient space of the DT_4 moduli space
$$\{a \in \Omega^1(X, g_E)_k \big| \ \|a\|_k < \epsilon, \ d_A^* a = 0, \ \wedge (d_A a + a \wedge a) = 0, \ a'' \ \textit{satisfies} \ (3)\}.$$
By the isomorphism sending unitary connections to $(0,1)$ connections, we identify it with an open subset of Q_A by implicit function theorem ($\epsilon \ll 1$), where
$$Q_A = \{a'' \in \Omega^{0,1}(X, EndE)_k \ \big| \ \|a''\|_k < \epsilon'', \ a'' \ \textit{satisfies} \ (3), (5)\}.$$

Because $F_+^{0,2} = 0$ is preserved by the map $d_A + a \longmapsto \overline{\partial}_A + a''$, we get isomorphic analytic subspaces after adding it to both of the above two ambient spaces. $\square$

The ambient space Q_A of $\mathcal{M}_A^+$ can be locally identified with $H^{0,1}(X, EndE)$.

Lemma 3.5. *We denote*

$$Q_A = \{a'' \in \Omega^{0,1}(X, EndE)_k \mid \|a''\|_k < \epsilon'',\ a'' \text{ satisfies } (3), (5)\},$$

then the harmonic projection map

$$\mathbb{H}^{0,1} : Q_A \to H^{0,1}(X, EndE)$$

is a local analytic isomorphism if ϵ'' is small.

Proof. We define

$$q : \Omega^{0,1}(EndE)_k \to H^{0,1}(EndE) \oplus \overline{\partial}_A^*\Omega^{0,1}(EndE)_k \oplus \overline{\partial}_A^*\Omega^{0,2}(EndE)_{k-1},$$

$$q(a'') = \Big(\mathbb{H}(a''), \overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a'), \overline{\partial}_A^*\big(\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'')\big)\Big).$$

We show q is well-defined: Firstly, we have

$$\begin{aligned} \wedge(\varphi) &= \wedge(\mathbb{H}(\varphi)) + \wedge(\overline{\partial}_A\overline{\partial}_A^* G\varphi) + \wedge(\overline{\partial}_A^*\overline{\partial}_A G\varphi) \\ &= \wedge(\mathbb{H}(\varphi)) + \overline{\partial}_A \wedge (\overline{\partial}_A^* G\varphi) + \overline{\partial}_A^*(\wedge\overline{\partial}_A G\varphi) \\ &= \wedge(\mathbb{H}(\varphi)) + 0 + \overline{\partial}_A^*(\wedge\overline{\partial}_A G\varphi), \end{aligned}$$

where $\varphi = a' \wedge a'' + a'' \wedge a' \in \Omega^{1,1}(X, End_0E)$.

Meanwhile, we have

$$\overline{\partial}_A \wedge \mathbb{H}(\varphi) = \wedge\overline{\partial}_A\mathbb{H}(\varphi) \pm \overline{\partial}_A^*\mathbb{H}(\varphi) = 0, \quad \overline{\partial}_A^* \wedge \mathbb{H}(\varphi) = 0.$$

Thus $\wedge\mathbb{H}(\varphi) \in H^0(End_0E) = 0$ by the simpleness of $(E, \overline{\partial}_A)$. Hence q is well-defined.

We take the differentiation of q at 0,

$$dq_0(v) = (\mathbb{H}(v), \overline{\partial}_A^* v, \overline{\partial}_A^*\overline{\partial}_A v),$$

which is a diffeomorphism whose inverse is given by

$$dq_0^{-1}(u_0, u_1, u_2) = u_0 + G\overline{\partial}_A u_1 + Gu_2.$$

By the implicit function theorem, q is a local analytic isomorphism near the origin. $\square$

By Lemma 3.4, $\mathcal{M}_c^{DT_4}$ is locally identified with $\mathcal{M}_A^+$. To compare $\mathcal{M}_A^+$ and $\mathcal{M}_c^{bdl}$, we define,

(6) $$\mathcal{M}_A = \{a'' \in \Omega^{0,1}(X, EndE)_k \mid \|a''\|_k < \epsilon'',\ \mathbb{H}^{0,2}(a'' \wedge a'') = 0, a'' \textit{ satisfies } (3), (5)\}.$$

By Lemma 3.5, $\mathcal{M}_A \hookrightarrow Q_A$ embeds as a closed analytic subspace of a finite dimensional smooth manifold Q_A. We will show $\mathcal{M}_A$ is isomorphic to an analytic neighbourhood of $\overline{\partial}_A$ in $\mathcal{M}_c^{bdl}$ with analytic topology. To achieve this, we denote local analytic maps

$$P : \Omega^{0,1}(X, EndE)_k \to \Omega^{0,2}(X, EndE)_{k-1},$$
$$P(a'') \triangleq \overline{\partial}_A a'' + a'' \wedge a''$$

and

$$\lambda : Q_A \to Q_A \times \Omega^{0,2}(X, EndE)_{k-1},$$
$$\lambda(a'') \triangleq (a'', P(a'')).$$

Note that

(7) $$Q_A \cap P^{-1}(0) = \{a'' \mid \|a''\|_k < \epsilon'',\ F^{0,2}(\overline{\partial}_A + a'') = 0, a'' \textit{ satisfies } (5)\}$$

gives a neighbourhood of $\overline{\partial}_A$ in $\mathcal{M}_c^{bdl}$ and $Q_A \cap P^{-1}(0) \hookrightarrow \mathcal{M}_A$ embeds as a closed analytic subspace. We are thus left to show $Q_A \cap P^{-1}(0) = \mathcal{M}_A$.

The image of the above map λ satisfies

Lemma 3.6.

$$Im(\lambda) \subseteq F,$$

where

$$F = \Big\{\ (a'', \theta) \in Q_A \times \Omega_{k-1}^{0,2} \ \Big|\ \ \overline{\partial}_A^*\theta = \overline{\partial}_A^*(*_4\overline{\partial}_A^* G([a'', \theta])), \quad \overline{\partial}_A^*(\overline{\partial}_A\theta + [a'', \theta]) = 0\ \Big\}.$$

Proof. Let

$$\begin{aligned}\theta &= \overline{\partial}_A a'' + a'' \wedge a'' \\ &= \overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + \mathbb{H}^{0,2}(a'' \wedge a'') + P_{\overline{\partial}^*_A}(a'' \wedge a'').\end{aligned}$$

By definition

$$a'' \in Q_A \Rightarrow \overline{\partial}_A \alpha + P_{\overline{\partial}_A}(\alpha \wedge \alpha) + *_4 P_{\overline{\partial}^*_A}(\alpha \wedge \alpha) = 0.$$

Hence

$$\theta - \mathbb{H}^{0,2}(a'' \wedge a'') = P_{\overline{\partial}^*_A}(a'' \wedge a'') - *_4 P_{\overline{\partial}^*_A}(a'' \wedge a'') \tag{8}$$

$$= \overline{\partial}^*_A \overline{\partial}_A G(a'' \wedge a'') - *_4 \overline{\partial}^*_A \overline{\partial}_A G(a'' \wedge a''). \tag{9}$$

Taking $\overline{\partial}_A$ to both sides of $\theta = \overline{\partial}_A a'' + a'' \wedge a''$, we get

$$\overline{\partial}_A \theta = \overline{\partial}_A (a'' \wedge a'').$$

Combined with the Bianchi identity $\overline{\partial}_A \theta + [a'', \theta] = 0$, we have

$$\overline{\partial}_A (a'' \wedge a'') = -[a'', \theta]. \tag{10}$$

Using (8) and (10), we have

$$\theta - \mathbb{H}^{0,2}(a'' \wedge a'') = \overline{\partial}^*_A \overline{\partial}_A G(a'' \wedge a'') + *_4 \overline{\partial}^*_A G([a'', \theta]).$$

After taking $\overline{\partial}^*_A$, we get

$$\overline{\partial}^*_A \theta = \overline{\partial}^*_A \big(*_4 \overline{\partial}^*_A G([a'', \theta]) \big).$$

□

Lemma 3.7. *We denote*

$$F = \left\{ \ (a'', \theta) \in Q_A \times \Omega^{0,2}_{k-1} \ \middle| \ \ \overline{\partial}^*_A \theta = \overline{\partial}^*_A (*_4 \overline{\partial}^*_A G([a'', \theta])), \quad \overline{\partial}^*_A (\overline{\partial}_A \theta + [a'', \theta]) = 0 \ \right\},$$

then the harmonic projection map

$$(\mathbb{H}^{0,1} \times \mathbb{H}^{0,2}) : F \to H^{0,1}(X, EndE) \times H^{0,2}(X, EndE)$$

is a local analytic isomorphism.

Proof. We take a map f

$$f : Q_A \times \Omega^{0,2}(X, EndE)_{k-1} \to Q_A \times \Omega^{0,2}(X, EndE)_{k-3},$$

$$f(a'', \theta) = \Big(a'', \mathbb{H}^{0,2}(\theta) + \overline{\partial}_A \overline{\partial}^*_A \big(\theta - *_4 \overline{\partial}^*_A G([a'', \theta]) \big) + \overline{\partial}^*_A (\overline{\partial}_A \theta + [a'', \theta]) \Big).$$

It is easy to check that f is a local analytic isomorphism by using implicit function theorem and noticing

$$df_{0,0}(v_1, v_2) = (v_1, \mathbb{H}^{0,2} v_2 + \overline{\partial}_A \overline{\partial}^*_A v_2 + \overline{\partial}^*_A \overline{\partial}_A v_2)$$

whose inverse is given by

$$df^{-1}_{0,0}(u_1, u_2) = (u_1, \mathbb{H}^{0,2} u_2 + G u_2).$$

Hence $F = f^{-1}(Q_A \times H^{0,2}(X, EndE))$ and the projection map

$$(\mathbb{H}^{0,1} \times \mathbb{H}^{0,2}) : F \to H^{0,1}(X, EndE) \times H^{0,2}(X, EndE)$$

gives a local chart of F. □

Lemma 3.8. *If $\mathbb{H}^{0,2}(\theta) = 0$, $(a'', \theta) \in F$ and $\|a''\|_k \ll 1$, then $\theta = 0$.*

Proof. By the Hodge decomposition and $(a'', \theta) \in F$,

$$\begin{aligned}\theta &= \mathbb{H}^{0,2}(\theta) + \overline{\partial}^*_A \overline{\partial}_A G\theta + \overline{\partial}_A \overline{\partial}^*_A G\theta \\ &= -G\overline{\partial}^*_A([a'', \theta]) + G\overline{\partial}_A \overline{\partial}^*_A (*_4 \overline{\partial}^*_A G([a'', \theta])),\end{aligned}$$

then

$$\|\theta\|_{k-1} \leq C_1 \|a''\|_k \|\theta\|_{k-1} + C_2 \|a''\|_k \|\theta\|_{k-1} = C \|a''\|_k \|\theta\|_{k-1}.$$

C is a constant independent of a'', θ. Hence we can get $\theta = 0$ if $\|a''\|_k \ll 1$. □

Corollary 3.9. *The following three analytic spaces are set theoretically identical*

$$\mathcal{M}_A = \{a'' \in \Omega^{0,1}(X, EndE)_k \mid \|a''\|_k < \epsilon'',\ \mathbb{H}^{0,2}(a'' \wedge a'') = 0, a'' \text{ satisfies } (3), (5)\},$$

$$Q_A \cap P^{-1}(0) = \{a'' \mid \|a''\|_k < \epsilon'',\ F^{0,2}(\overline{\partial}_A + a'') = 0, a'' \text{ satisfies } (5)\},$$

$$Q_A \cap P^{-1}(0) = \left\{ a'' \;\middle|\; \begin{array}{c} \|a''\|_k < \epsilon'',\ a'' \text{ satisfies } (5) \\ \overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') = 0 \\ \mathbb{H}^{0,2}(a'' \wedge a'') = 0 \end{array} \right\}.$$

We use the same notation for the second and the third spaces because they are isomorphic as analytic spaces by the standard Kuranishi theory.

Proof. We only need to show $Q_A \cap P^{-1}(0)$ contains $\mathcal{M}_A$, i.e.

$$\forall\ \overline{\partial}_A + a'' \in \mathcal{M}_A \Rightarrow F^{0,2}(\overline{\partial}_A + a'') = 0.$$

Since $\mathcal{M}_A$ is a subset of Q_A, we can apply Lemma 3.6 to its image under the map λ. Combined with Lemma 3.8, we are done. $\square$

Furthermore, we can identify the above three spaces as analytic spaces possibly with non-reduced structures. Let us first recall a lemma due to Miyajima [63].

Lemma 3.10. *(Miyajima [63]). Let E, G be Banach spaces with direct sum decomposition $E = F_1 + F_2$. If a local analytic map*

$$h : E \to G$$

vanishes identically on F_2, then there exists a local analytic map

$$f : E \to L(F_2, G),$$

such that $h(t, s) =< f(t, s), s >$.

Proposition 3.11. *We have the following identification*

$$Q_A \cap P^{-1}(0) = \mathcal{M}_A$$

as analytic spaces possibly with non-reduced structures.

Proof. Obviously, $Q_A \cap P^{-1}(0) \hookrightarrow \mathcal{M}_A$, we are left to show that up to change of variables the analytic map P can be expressed analytically in terms of $\mathbb{H}^{0,2} \circ P$ and coordinates of Q_A.

We consider $\lambda : Q_A \to F$, $\lambda(\alpha) = (\alpha, P(\alpha))$. By Lemma 3.6, it is well defined. By Lemma 3.5 and 3.7, we have the following commutative diagram

$$\begin{tikzcd}
Q_A \arrow[r, "\lambda(\alpha)=(\alpha,P(\alpha))"] \arrow[d, "\wr\ \mathbb{H}^{0,1}"] \arrow[rr, bend left, "P(\alpha)=\overline{\partial}_A\alpha+\alpha\wedge\alpha"] & F \arrow[r, "\pi_2(\alpha,\theta)=\theta"] \arrow[d, "\wr\ \mathbb{H}^{0,1}\times\mathbb{H}^{0,2}"] & \Omega^{0,2}(X, EndE)_{k-1} . \\
H^{0,1}(X, EndE) \arrow[r, "\lambda^{'}"] \arrow[urr, bend right, "P^{'}(t)=\pi_2^{'}\circ\lambda^{'}(t)=\pi_2^{'}(t,\mathbb{H}^{0,2}\circ P(\alpha(t)))"'] & H^{0,1}(X, EndE) \times H^{0,2}(X, EndE) \arrow[ur, "\pi_2^{'}"'] &
\end{tikzcd}$$

With respect to local charts $(Q_A, \mathbb{H}^{0,1})$ and $(F, \mathbb{H}^{0,1} \times \mathbb{H}^{0,2})$, λ is expressed by $\lambda^{'} = \big(t, \mathbb{H}^{0,2} \circ P(\alpha(t))\big)$, π_2 is expressed by $\pi_2^{'}$ and P is expressed by $P^{'}$, where $t \in H^{0,1}(X, EndE)$ and $\alpha(t) = (\mathbb{H}^{0,1})^{-1}(t)$.

By Lemma 3.8, $\mathbb{H}^{0,2}(\theta) = 0$ and $(t, \theta) \in F$ imply $\theta = 0$. Hence $\pi_2^{'}(t, 0) = 0$. Applying Lemma 3.10, we get

$$\begin{aligned} P^{'}(t) &= \pi_2^{'}\big(t, \mathbb{H}^{0,2} \circ P(\alpha(t))\big) \\ &= < \eta\big(t, \mathbb{H}^{0,2} \circ P(\alpha(t))\big),\ \mathbb{H}^{0,2} \circ P(\alpha(t)) > \end{aligned}$$

for some local analytic map $\eta : H^{0,1} \times H^{0,2} \to L(H^{0,2}, \Omega^{0,2}_{k-1})$, where $L(H^{0,2}, \Omega^{0,2}_{k-1})$ is the space of analytic maps between Banach spaces $H^{0,2}$ and $\Omega^{0,2}_{k-1}$. $\square$

We have proved that $\mathcal{M}_A$ is isomorphic to an analytic neighbourhood of $\overline{\partial}_A$ in $\mathcal{M}_c^{bdl}$. We note that the above isomorphism is between real analytic spaces. We make the following definition generalizing the definition of Kuranishi maps.

Definition 3.12. *Given an integrable* $(0,1)$*-connection* $\overline{\partial}_A$ *with trivial isotropy subgroup, a real analytic map*

$$\kappa : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE)$$

is a Kuranishi map of the moduli space of holomorphic bundles if there exists an open analytic neighbourhood U_A *of* $\overline{\partial}_A$ *in the moduli space such that* $\kappa^{-1}(0) \cong U_A$ *locally as real analytic spaces possibly with non-reduced structures.*

Finally, we get the following local Kuranishi model of $\mathcal{M}_c^{DT_4}$.

Theorem 3.13. *We assume* $\mathcal{M}_c^{bdl} \neq \emptyset$ *and fix* $d_A \in \mathcal{M}_c^{DT_4}$*, then there exists a Kuranishi map* $\tilde{\tilde{\kappa}}$ *of* $\mathcal{M}_c^{bdl}$ *at* $\overline{\partial}_A$ *(the (0,1) part of* d_A*) such that* $\tilde{\tilde{\kappa}}_+$ *is a Kuranishi map of* $\mathcal{M}_c^{DT_4}$ *at* d_A*, where*

$$\tilde{\tilde{\kappa}}_+ = \pi_+(\tilde{\tilde{\kappa}}) : H^{0,1}(X, EndE) \cap B_\epsilon \xrightarrow{\kappa} H^{0,2}(X, EndE) \xrightarrow{\pi_+} H_+^{0,2}(X, EndE),$$

B_ϵ *is a small open ball containing the origin of the deformation space and* π_+ *is projection to self-dual two forms.*

Proof. By Lemma 3.4, $\mathcal{M}_c^{DT_4}$ is locally isomorphic to $\mathcal{M}_A^+$. From the definition of $\mathcal{M}_A$ (6), $\mathcal{M}_A = \tilde{\tilde{\kappa}}^{-1}(0)$, where $\tilde{\tilde{\kappa}} : H^{0,1}(X, EndE) \cap B_\epsilon \to H^{0,2}(X, EndE)$ is

$$\tilde{\tilde{\kappa}}(\alpha) = \mathbb{H}^{0,2}\big(q^{-1}(\alpha) \wedge q^{-1}(\alpha)\big),$$

where

$$q : \Omega^{0,1}(EndE)_k \to H^{0,1}(EndE) \oplus \overline{\partial}_A^*\Omega^{0,1}(EndE)_k \oplus \overline{\partial}_A^*\Omega^{0,2}(EndE)_{k-1},$$

$$q(a'') = \left(\mathbb{H}(a''), \overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a'), \overline{\partial}_A^*\big(\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'')\big)\right).$$

By Proposition 3.11, $\tilde{\tilde{\kappa}}$ is a Kuranishi map of $\mathcal{M}_c^{bdl}$. Composing with π_+, $\mathcal{M}_A^+ = \big(\pi_+ \tilde{\tilde{\kappa}}\big)^{-1}(0)$. □

Remark 3.14.
1. Under the assumption $\mathcal{M}_c^{bdl} \neq \emptyset$*, we have a bijective map* $\mathcal{M}_c^{bdl} \to \mathcal{M}_c^{DT_4}$*. The map can be enhanced to be a closed imbedding between analytic spaces by Theorem 3.13. Note that this map is then also a homeomorphism between topological spaces.*
2. For simplicity, we will always restrict to a small neighbourhood of the origin in $Ext^1(\mathcal{F}, \mathcal{F})$ *when we talk about a Kuranishi map* $\kappa : Ext^1(\mathcal{F}, \mathcal{F}) \to Ext^2(\mathcal{F}, \mathcal{F})$ *for any coherent sheaf* $\mathcal{F}$ *and abbreviate* B_ϵ *in the notation from now on.*

4. The compactification of DT_4 moduli spaces

We come to the issue of compactification of DT_4 moduli spaces. As Uhlenbeck, or generally Tian [77] have shown, we need to consider connections with singularities supported on codimension 4 subspaces to compactify moduli spaces of holomorphic HYM connections. This becomes very difficult when the real dimension of the underlying manifold is bigger than 4. Even if one could compactify it, as Tian showed in his paper, one still does not understand the local analytic structure of the compactified moduli space very well.

Instead of using this compactification, our attempted approach here is algebro-geometric, using Gieseker moduli spaces of semi-stable sheaves.

4.1. The stable bundles compactification of DT_4 moduli spaces.

In this subsection, assuming $\overline{\mathcal{M}}_c \neq \emptyset$ consists of slope-stable bundles only, we prove that $\mathcal{M}_c^{DT_4}$ is compact.

We take a connection on E with curvature F. By Chern-Weil theory, we have

$$-8\pi^2 \int ch_2(E) \wedge \Omega = \int Tr(F^{0,2} \wedge F^{0,2}) \wedge \Omega$$

$$= \int Tr(F_+^{0,2} \wedge F_+^{0,2}) \wedge \Omega + \int Tr(F_-^{0,2} \wedge F_-^{0,2}) \wedge \Omega$$

$$+ \int Tr(F_+^{0,2} \wedge F_-^{0,2}) \wedge \Omega + \int Tr(F_-^{0,2} \wedge F_+^{0,2}) \wedge \Omega$$

$$= \int |F_+^{0,2}|^2 \wedge \Omega \wedge \overline{\Omega} - \int |F_-^{0,2}|^2 \wedge \Omega \wedge \overline{\Omega} + \int \sqrt{-1}\chi \wedge \Omega \wedge \overline{\Omega}, \tag{11}$$

where χ is some real valued function.

Lemma 4.1. *(Lewis [53]) If $ch_2(E) \in H^{2,2}(X,\mathbb{C})$ or has no component of type $(0,4)$, then $F^{0,2}_+ = 0$ implies $F^{0,2} = 0$.*

Proof. Note that Ω is $(4,0)$ form and χ is a real valued function. □

Corollary 4.2. *If $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$, then $\mathcal{M}_c^{DT_4}$ is compact.*

Proof. By the assumptions and Lemma 4.1. □

From the viewpoint of local Kuranishi models (i.e. Theorem 3.13), Lemma 4.1 says

$$\pi_+\tilde{\tilde{\kappa}} = 0 \Rightarrow \tilde{\tilde{\kappa}} = 0,$$

which gives restrictions to $\tilde{\tilde{\kappa}}$.

Proposition 4.3. *Given a map*

$$\kappa : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE)$$

such that $\kappa_+ = 0 \Rightarrow \kappa = 0$ and $\kappa(0) = 0$, where

$$\kappa_+ = \pi_+ \circ \kappa : H^{0,1}(X, EndE) \to H^{0,2}_+(X, EndE),$$

then the image of κ can not be a neighbourhood of the origin.

Proof. By assumptions, $\kappa(U(0)) \cap H^{0,2}_-(X, EndE) = \{0\}$. □

4.2. **The attempted general compactification of DT_4 moduli spaces.** In this subsection, we propose an attempted approach to the general compactification of $\mathcal{M}_c^{DT_4}$. Under the gluing assumptions, we define the generalized DT_4 moduli space $\overline{\mathcal{M}}_c^{DT_4}$ by gluing local models. We then show in some cases we can get rid of the gluing assumption and prove the existence of $\overline{\mathcal{M}}_c^{DT_4}$.

We recall that if we assume $\mathcal{M}_c^{bdl} \neq \emptyset$, we have a homeomorphism

$$\mathcal{M}_c^{bdl} \to \mathcal{M}_c^{DT_4},$$

which is a closed imbedding between analytic spaces possibly with non-reduced structures. The idea of general compactification is to extend the above map to a homeomorphism

$$\mathcal{M}_c \to \overline{\mathcal{M}}_c^{DT_4},$$

where $\overline{\mathcal{M}}_c^{DT_4}$ comes from gluing local models, i.e. locally at a stable sheaf $\mathcal{F}$, $\overline{\mathcal{M}}_c^{DT_4}$ is $\kappa_+^{-1}(0)$, where

$$\kappa_+ = \pi_+ \circ \kappa : Ext^1(\mathcal{F},\mathcal{F}) \to Ext^2_+(\mathcal{F},\mathcal{F}),$$

κ is a Kuranishi map of $\mathcal{M}_c$ at $\mathcal{F}$ and π_+ is the projection map.

However, the Kuranishi map κ is unique only up to change of variables. Meanwhile, the $*_4$ is a real operator and if we use different re-parametrization, the resulting models may be different in general, i.e.

$$(\pi_+ \circ \kappa_1)^{-1}(0) \ncong (\pi_+ \circ \kappa_2)^{-1}(0)$$

for different κ_i, $i = 1, 2$.

For the purpose of gluing, we need to pick a coherent choice of local Kuranishi models for $\mathcal{M}_c$. In the case when $\mathcal{M}_c = \mathcal{M}_c^{bdl}$, the moment map equation in DT_4 equations (2) gives such a choice. In general, we need a similar moment map equation for $\mathcal{M}_c$. This is achieved by a quiver representation of $\mathcal{M}_c$ due to [5]. We then proposed a candidate local model at each $\mathcal{F} \in \mathcal{M}_c$ based on their work. As we do not know how to glue such local models at the moment, we will not discuss them in detail. The interested reader could refer to the appendix of the first named author's master thesis [16].

We will always make the following assumptions if $\mathcal{M}_c \neq \mathcal{M}_c^{bdl}$.

Assumption 4.4. *We assume there exists a real analytic space $\overline{\mathcal{M}}_c^{DT_4}$ and a homeomorphism*

$$\mathcal{M}_c \to \overline{\mathcal{M}}_c^{DT_4}$$

such that at each closed point of $\mathcal{M}_c$, say $\mathcal{F}$, $\overline{\mathcal{M}}_c^{DT_4}$ is locally isomorphic to $\kappa_+^{-1}(0)$, where

$$\kappa_+ = \pi_+ \circ \kappa : Ext^1(\mathcal{F},\mathcal{F}) \to Ext^2_+(\mathcal{F},\mathcal{F}),$$

κ is a Kuranishi map at $\mathcal{F}$ and $Ext^2_+(\mathcal{F},\mathcal{F})$ is a half dimensional real subspace of $Ext^2(\mathcal{F},\mathcal{F})$ on which the Serre duality quadratic form is real and positive definite.

Definition 4.5. *Under the Assumption 4.4, we obtain a real analytic space $\overline{\mathcal{M}}_c^{DT_4}$ which is compact if $\overline{\mathcal{M}}_c = \mathcal{M}_c$. We call it the generalized DT_4 moduli space.*

Remark 4.6. *In* [8] *Borisov and Joyce used local 'Darboux charts' in the sense of Brav, Bussi and Joyce* [9]*, the machinery of homotopical algebra and C^∞-algebraic geometry to construct a compact derived C^∞-scheme with the same underlying topological structure as the Gieseker moduli space of stable sheaves in general. In our language, this C^∞-scheme is the C^∞-scheme (instead of the real analytic space) version of the hoped generalized DT_4 moduli space. Thus generalized DT_4 moduli spaces always exist at least as C^∞-schemes by their gluing result.*

We show in several good cases, generalized DT_4 moduli spaces exist as real analytic spaces. The first case is when $\mathcal{M}_c$ is smooth.

Proposition 4.7. *If the Gieseker moduli space $\mathcal{M}_c$ is smooth, the generalized DT_4 moduli space exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \mathcal{M}_c$ as real analytic spaces.*

Proof. By the assumption, all Kuranishi maps are zero. The conclusion is obvious. □

There is another interesting case when we can get $\overline{\mathcal{M}}_c^{DT_4}$ as a real analytic space via gluing.

Proposition 4.8. *We assume for any closed point $\mathcal{F} \in \mathcal{M}_c$, there is a splitting of obstruction space*

$$Ext^2(\mathcal{F},\mathcal{F}) = V_{\mathcal{F}} \oplus V_{\mathcal{F}}^*$$

such that $V_{\mathcal{F}}$ is its maximal isotropic subspace with respect to the Serre duality pairing and the image of a Kuranishi map κ at $\mathcal{F}$ satisfies

$$\mathrm{Image}(\kappa) \subseteq V_{\mathcal{F}}.$$

Then the generalized DT_4 moduli space exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \mathcal{M}_c$ as real analytic spaces.

Proof. We pick a Hermitian metric h on $V_{\mathcal{F}}$ which induces a Hermitian metric on $V_{\mathcal{F}}^*$. We abuse the notation h for the direct sum Hermitian metric on $Ext^2(\mathcal{F},\mathcal{F}) = V_{\mathcal{F}} \oplus V_{\mathcal{F}}^*$. We define $*_4 : Ext^2(\mathcal{F},\mathcal{F}) \to Ext^2(\mathcal{F},\mathcal{F})$ such that $Q_{Serre}(\alpha, *_4\beta) = h(\alpha,\beta)$, where Q_{Serre} denotes the Serre duality pairing. Then for $\kappa(\alpha) \in V_{\mathcal{F}}$, we have $*_4\big(\kappa(\alpha)\big) \in V_{\mathcal{F}}^*$ which implies $\kappa_+ = 0 \Rightarrow \kappa = 0$ by the assumption $Image(\kappa) \subseteq V_{\mathcal{F}}$. □

Remark 4.9. *We will see the above conditions are satisfied for compactly supported sheaves on certain local CY_4 manifolds.*

5. Virtual cycle constructions

In algebraic geometry, we can use GIT to construct moduli spaces. If one wants to define invariants associated to them, we need to make sense of their fundamental classes. However, because of the lack of transversality, moduli spaces are in general very singular and not of expected dimensions. The way to obtain correct fundamental cycles (deformation invariant) originated from the idea of Fulton-MacPherson's localized top Chern class [28]. It is generalized to Fredholm Banach bundles over Banach manifolds by many people (such as Brussee [12], Cieliebak-Mundet i Riera-Salamon [17] in the equivariant case) and developed in moduli problems by Li-Tian [55], Behrend-Fantechi [6] in full generality. The equivalence of these works were proved in [57], [43].

5.1. **The virtual cycle construction of DT_4 moduli spaces.** We start with

$$\begin{array}{ccc} & \mathcal{E} = & \mathcal{A}^* \times_{\mathcal{G}^0} (\Omega^0(X,g_E)_{k-1} \oplus \Omega^{0,2}_+(X,EndE)_{k-1}) \\ & & \downarrow \\ \mathcal{M}_c^{DT_4} & \hookrightarrow & \mathcal{B}^* = \mathcal{A}^*/\mathcal{G}^0, \end{array} \tag{12}$$

where $\mathcal{M}_c^{DT_4} \hookrightarrow \mathcal{B}^*$ embeds as the zero loci of section $s = (\wedge F, F_+^{0,2})$ of $\mathcal{E}$. We assume $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$ to get compactness of $\mathcal{M}_c^{DT_4}$.

Remark 5.1. *The orientability of the Banach bundle is proved when $H^{odd}(X,\mathbb{Z}) = 0$ in Theorem 10.14. Then we can choose an orientation data $o(\mathcal{L})$ (Definition 1.6) in this case.*

We check the Fredholm property of the above Banach bundle.

Lemma 5.2. *The above Banach bundle $\mathcal{E} \to \mathcal{B}^*$ is a Fredholm bundle.*

Proof. We take an open cover $\{U_i\}$ of $s^{-1}(0)$ in $\mathcal{A}^*/\mathcal{G}^0$, where

$$U_i = \{d_{A_i} + a \mid \|a\|_k < \epsilon,\ d^*_{A_i} a = 0\}.$$

Note that

$$E\mid_{U_i} = U_i \times (\Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1}).$$

On the intersection of two charts, we have a commutative diagram

$$\begin{tikzcd} E\mid_{U_i} \arrow[r, "\Phi_{ij}"] \arrow[d, "\pi_i"'] & E\mid_{U_j} \arrow[d, "\pi_j"] \\ U_i \arrow[r, "\phi_{ij}"] & U_j, \end{tikzcd}$$

where ϕ_{ij} is the gauge transformation on U_{ij}, and Φ_{ij} is the adjoint action in the fiber direction.

The section s near d_A with $\overline{\partial}^2_A = 0$ is given by

$$\mathcal{G}^0 \curvearrowright \Omega^1(X, g_E)_k \to \Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1},$$

$$a = a^{0,1} + a^{1,0} \mapsto (\wedge F(d_A + a^{0,1} + a^{1,0}), F^{0,2}_+(\overline{\partial}_A + a^{0,1})),$$

where we identify unitary connections with $(0,1)$ connections $\Omega^1(X, g_E)_k \cong \Omega^{0,1}(X, EndE)_k$. After gauge fixing, we get $ker(d^*_A) \subseteq \Omega^1(X, g_E)_k$. By the Kähler identity, $[\wedge, d_A] = i(\overline{\partial}^*_A - \partial^*_A)$, we have

$$ker(ds)\mid_A \cong H^{0,1}(X, EndE),$$

$$\begin{aligned} coker(ds)\mid_A &= \frac{\Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1}}{(i\overline{\partial}^*_A a^{0,1} - i\partial^* a^{1,0}) \oplus \overline{\partial}^+_A a^{0,1}} \\ &= \frac{\Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1}}{(2i\overline{\partial}^*_A a^{0,1}) \oplus \overline{\partial}^+_A a^{0,1}} \\ &= H^0(X, g_E) \oplus H^{0,2}_+(X, EndE), \end{aligned}$$

where the second equality uses $d^*_A(a) = \overline{\partial}^*_A a^{0,1} + \partial^*_A a^{1,0} = 0$. □

Then by Proposition 14 of [12], the Euler class of the above Fredholm Banach bundle $e([s : \mathcal{B}^* \to \mathcal{E}])$ exists. We define it to be the virtual fundamental class of $\mathcal{M}^{DT_4}_c$.

Definition 5.3. *We assume $\overline{\mathcal{M}}_c = \mathcal{M}^{bdl}_c \neq \emptyset$ and there exists an orientation data $o(\mathcal{L})$, then the virtual fundamental class of $\mathcal{M}^{DT_4}_c$ (DT_4 virtual cycle for short) is the Euler class of the above oriented Fredholm Banach bundle with $[\mathcal{M}^{DT_4}_c]^{vir} \in H_r(\mathcal{B}^*, \mathbb{Z})$, where $r = 2 - \chi(E, E)$ is the virtual dimension.*

Remark 5.4. *One can similarly define the DT_4 virtual cycle as a homology class $[\widetilde{\mathcal{M}^{DT_4}_c}]^{vir} \in H_*(\widetilde{\mathcal{B}}^*)$. Here, $\widetilde{\mathcal{M}^{DT_4}_c}$ is the framed DT_4 moduli space, i.e. the zero loci of section $\tilde{s} = (\wedge F, F^{0,2}_+)$ of the following Banach bundle*

$$\begin{array}{rcl} \widetilde{\mathcal{E}} = & & (\mathcal{A}^* \times Hom(U(r), P_{x_0})) \times_{\mathcal{G}} (\Omega^0(X, g_E)_{k-1} \oplus \Omega^{0,2}_+(X, EndE)_{k-1}) \\ & & \downarrow \\ \widetilde{\mathcal{M}^{DT_4}_c} & \hookrightarrow & \widetilde{\mathcal{B}}^* = (\mathcal{A}^* \times Hom(U(r), P_{x_0}))/\mathcal{G}, \end{array}$$

where $x_0 \in X$ is a base-point, $P \to X$ is the principal $U(r)$-bundle associated with $E \to X$ and $\widetilde{\mathcal{B}}^*$ is the space of framed irreducible connections which admits a base-point $PU(r)$-fibration $\beta : \widetilde{\mathcal{B}}^* \to \mathcal{B}^*$ [21]. Furthermore, we have a $PU(r)$-bundle map $\widetilde{\mathcal{E}} \to \mathcal{E}$ covering β. Thus we can choose the representative sub-manifold of $[\widetilde{\mathcal{M}^{DT_4}_c}]^{vir}$ in $\widetilde{\mathcal{B}}^*$ to be $PU(r)$ equivariant and the free $PU(r)$ quotient will represent $[\mathcal{M}^{DT_4}_c]^{vir} \in H_r(\mathcal{B}^*, \mathbb{Z})$ [17].

The deformation invariance. We show $[\mathcal{M}^{DT_4}_c]^{vir}$ is independent of the choice of
(1) the holomorphic top form Ω,
(2) the Hermitian metric h on E,
(3) the parameter t of any deformation of complex structures X_t when $\overline{\mathcal{M}}_c = \mathcal{M}^{bdl}_c \neq \emptyset$ for all X_t.

Lemma 5.5. $[\mathcal{M}^{DT_4}_c]^{vir}$ *is independent of the choice of Ω and h.*

Proof. We choose two holomorphic top forms Ω, $e^{i\theta}\Omega$ which give $*$, $*_1=e^{i\theta}*$ respectively (we use $*$ to denote $*_4$ for simplicity here). As $\Omega^0(X,g_E)_{k-1}$ is independent of the choice of Ω, we omit it in the expression of the Banach bundle $\mathcal{E}$. There exists a bundle isomorphism

$$\begin{tikzcd}
\Omega^{0,2}_{+}(X,EndE)_{k-1} \arrow[rr, "f_1"] \arrow[dr, "\pi"'] & & \Omega^{0,2}_{+_1}(X,EndE)_{k-1} \arrow[dl, "\pi_1"] \\
& \mathcal{B}^* &
\end{tikzcd}$$

where f_1 is fiberwise multiplication by $\frac{1}{2}(cos\theta+1+sin\theta\sqrt{-1})$ if $\theta\neq\pi$ (if $\theta=\pi$, f_1 is defined to be multiplication by $\sqrt{-1}$). We denote $s_1'=f_1^{-1}\circ s_1$, where $s_1=F^{0,2}_{+_1}$ is the complex ASD equation with respect to $e^{i\theta}\Omega$. By the functorial property of the Euler class, we are reduced to prove

$$e([s_1':\mathcal{B}^*\to\Omega^{0,2}_{+}(X,EndE)_{k-1}])=e([s:\mathcal{B}^*\to\Omega^{0,2}_{+}(X,EndE)_{k-1}]),$$

where $s=F^{0,2}_{+}$ is the complex ASD equation with respect to Ω.

We consider a family of sections

$$s_t':\mathcal{B}^*\to\Omega^{0,2}_{+}(X,EndE)_{k-1},$$
$$s_t'=f_t^{-1}\circ s_t\triangleq\big(\frac{1}{2}(\sqrt{1-t^2sin^2\theta}+1+t\cdot sin\theta\sqrt{-1})\big)^{-1}\cdot s_t$$

and $s_t=F^{0,2}_{+_t}$ is the complex ASD equation with respect to $(\sqrt{1-t^2sin^2\theta}+t\cdot sin\theta\sqrt{-1})\cdot\Omega$.

It is easy to check we have the following commutative relation

$$f_t\circ *\circ F^{0,2}=*_t\circ f_t\circ F^{0,2},$$

where $*_t\triangleq(\sqrt{1-t^2sin^2\theta}+t\cdot sin\theta\sqrt{-1})\circ *$. Then, we get

$$s_t'=f_t^{-1}\circ s_t=f_t^{-1}\circ\pi_{+_t}F^{0,2}=\pi_+\circ(f_t^{-1}F^{0,2}),$$

which connects $s_0'=s$ and s_1'. We define

$$S:\mathcal{B}^*\times[0,1]\to\mathcal{A}^*\times_{\mathcal{G}^0}(\Omega^0(X,g_E)_{k-1}\oplus\Omega^{0,2}_{+}(X,EndE)_{k-1}),$$
$$S(A,t)=(\wedge F(A),s_t'),$$

which is an oriented Fredholm Banach bundle of index $r+1$ with $S|_{\mathcal{B}^*\times 0}=s$ and $S|_{\mathcal{B}^*\times 1}=s_1'$. Note that, by topological reasons, the above family miss the case when $*_1=-*$ which can be covered by moving $*$ in the S^1 family. Then by [12], we are done.

The space of all Hermitian metrics on E is connected, the above argument goes through. The only difference is we also need to identify $\mathcal{B}^*$ for different choices of h which is standard. $\square$

Similarly, we have

Lemma 5.6. $[\mathcal{M}_c^{DT_4}]^{vir}$ *is a deformation invariant of* X.

Proof. We fix a Hermitian metric, a continuous deformations of complex structures J_t of X and an orientation data $o(\mathcal{L})$ (it does not depend on t). We consider Fredholm Banach bundle

$$s:\mathcal{B}^*\times[0,1]\to\mathcal{A}^*\times_{\mathcal{G}^0}(\Omega^0(X,g_E)_{k-1}\oplus\Omega^{0,2}_{+}(X,EndE)_{k-1}),$$
$$s_t=(\wedge F,f_t^{-1}\circ F^{0,2}_{+_t}),$$

where $*_t$ is the $*_4$ operator with respect to the holomorphic structure J_t and

$$f_t:\Omega^{0,2}_{+}(X,EndE)_{k-1}\to\Omega^{0,2}_{+_t}(X,EndE)_{k-1}$$

is a Banach bundle isomorphism which commutes with the adjoint action of $\mathcal{G}$. f_t exists because the complex structure only affects the differential forms part of the underling manifold, not the topological bundle, while the unitary gauge transformations act on bundle E only.

We have

$$f_t(*F^{0,2})=*_tf_t(F^{0,2})$$

by extending f_t to $\Omega^{0,2}_{-}(X,EndE)_{k-1}$ using $f_t(\sqrt{-1}\alpha)\triangleq\sqrt{-1}f_t(\alpha)$, where $\alpha\in\Omega^{0,2}_{+}(X,EndE)_{k-1}$. Then by [12], we prove the deformation invariance. $\square$

To sum up, we have the following result.

Theorem 5.7. *We assume* $\overline{\mathcal{M}}_c=\mathcal{M}_c^{bdl}\neq\emptyset$ *and there exists an orientation data* $o(\mathcal{L})$. *Then* $\mathcal{M}_c^{DT_4}$ *is compact and its virtual fundamental class exists as a cycle* $[\mathcal{M}_c^{DT_4}]^{vir}\in H_r(\mathcal{B}^*,\mathbb{Z})$, *where* $r=2-\chi(E,E)$ *is the virtual dimension.*

Furthermore, if the above assumptions are satisfied by a continuous family of Calabi-Yau four-folds X_t *parameterized by* $t\in[0,1]$, *then the cycle in* $H_r(\mathcal{B}^*,\mathbb{Z})$ *is independent of* t.

Remark 5.8.
1. The Banach manifold $\mathcal{B}^*=\mathcal{A}^*/\mathcal{G}^0$ involves a choice of a large integer k in L^2_k norm completion. As stated before, the DT_4 moduli space is independent of the choice of k. Meanwhile, the homotopy-invariant properties of $\mathcal{B}^*$ are insensitive to k [21] *and it is easy to show the virtual fundamental class does not depend on the choice of k.*
2. By Donaldson-Thomas [22]*, the Calabi-Yau four-fold X is also a $Spin(7)$ manifold ($SU(4)\subset Spin(7)$) and*

$$\Omega^2(X)=\Omega^2_7(X)\oplus\Omega^2_{21}(X),$$
$$\Omega^2(X)\otimes_{\mathbb{R}}\mathbb{C}=\Omega^{1,1}_0(X)\oplus\Omega^{0,0}(X)<\omega>\oplus\Omega^{0,2}(X)\oplus\Omega^{2,0}(X).$$

Coupled with bundles, the deformation complex of $Spin(7)$ instantons [53]

$$\Omega^0(X,g_E)\to\Omega^1(X,g_E)\to\Omega^2_7(X,g_E)$$

is the same as

$$\Omega^0(X,g_E)\to\Omega^{0,1}(X,EndE)\to\Omega^{0,2}_+(X,EndE)\oplus\Omega^0(X,g_E).$$

Correspondingly, the $Spin(7)$ instanton equation

$$\pi_7(F)=0$$

is equivalent to DT_4 equations (2), where

$$\pi_7:\Omega^2(X,g_E)\to\Omega^2_7(X,g_E)$$

is the projection map. Thus $Spin(7)$ instanton counting is just the DT_4 invariant (defined later) when the base manifold is a Calabi-Yau four-fold.

The μ_1-map. Because the virtual dimension of $\mathcal{M}_c^{DT_4}$ is not zero in general, we need the μ-map to cut down the dimension and define invariants.

We recall [21], if $G=SU(2)$, there exists a universal $SO(3)$ bundle

$$\mathcal{P}^{ad}\to\mathcal{B}^*\times X.$$

Then we define the μ_1-map using the slant product pairing,

$$\mu_1:H_*(X)\otimes\mathbb{Z}[x_1,x_2,...]\to H^*(\mathcal{B}^*),$$

$$\mu_1(\gamma,P)=P(0,-\frac{1}{4}p_1(\mathcal{P}^{ad}),0,...)/\gamma. \tag{13}$$

The μ_1-map for $U(r)$ bundles can be defined using higher Pontryagin classes of $PU(r)$ bundles with more complicated expression.

Remark 5.9. *There exists a universal framed $U(r)$ bundle $\widetilde{\mathcal{P}}\to\widetilde{\mathcal{B}}^*\times X$, where $\widetilde{\mathcal{B}}^*$ is the space of framed irreducible connections and we can define a $\widetilde{\mu}_1$-map*

$$\tilde{\mu}_1:H_*(X)\otimes\mathbb{Z}[x_1,x_2,...]\to H^*(\widetilde{\mathcal{B}}^*),$$

$$\tilde{\mu}_1(\gamma,P)=P(c_1(\widetilde{\mathcal{P}}),c_2(\widetilde{\mathcal{P}}),...)/\gamma. \tag{14}$$

For $SU(2)$ bundles, $\mu_1(\gamma,P)$ pulls back to be $\tilde{\mu}_1(\gamma,P)$ via $\beta:\widetilde{\mathcal{B}}^*\to\mathcal{B}^*$ [21]*. However, Pontryagin classes of $\mathcal{P}^{ad}$ can't recover Chern classes of $\widetilde{\mathcal{P}}$ for higher rank bundles in general.*

We use the pairing between DT_4 virtual cycles and μ_1-maps to define DT_4 invariants.

Definition 5.10. *Under the assumption in Theorem 5.7, the DT_4 invariant of $(X,\mathcal{O}(1))$ with respect to Chern character c and an orientation data $o(\mathcal{L})$ is a map*

$$DT_4^{\mu_1}(X,\mathcal{O}(1),c,o(\mathcal{L})):Sym^*\big(H_*(X,\mathbb{Z})\otimes\mathbb{Z}[x_1,x_2,...]\big)\to\mathbb{Z} \tag{15}$$

such that

$$DT_4^{\mu_1}(X,\mathcal{O}(1),c,o(\mathcal{L}))((\gamma_1,P_1),(\gamma_2,P_2),...)$$
$$=<\mu_1(\gamma_1,P_1)\cup\mu_1(\gamma_2,P_2)\cup...,[\mathcal{M}_c^{DT_4}]^{vir}>,$$

where $<,>$ denotes the natural pairing between homology and cohomology classes.

Remark 5.11.
1. The above DT_4 invariants can be viewed as partition functions of certain eight dimension quantum field theory [4] *because of the standard super-symmetry localization* [81].
2. If two Calabi-Yau four-folds under Mukai flops [65] *are deformation equivalent to each other* [34]*, DT_4 invariants will remain the same under these flops.*

5.2. **The virtual cycle construction of generalized DT_4 moduli spaces.** In this subsection, we construct virtual cycles of generalized DT_4 moduli spaces $\overline{\mathcal{M}}_c^{DT_4}$'s, when they are defined without the gluing assumption 4.4.

The first case is when $\mathcal{M}_c$ is smooth: the obstruction sheaf Ob such that $Ob|_{\mathcal{F}} = Ext^2(\mathcal{F},\mathcal{F})$ is a bundle with quadratic form Q_{Serre}, where Q_{Serre} is the Serre duality pairing. By Lemma 5 [24], there exists a real sub-bundle Ob_+ with positive definite quadratic form such that $Ob \cong Ob_+ \otimes_{\mathbb{R}} \mathbb{C}$ as vector bundles with quadratic form and $w_1(Ob_+) = 0 \Leftrightarrow c_1(Ob) = 0$. We call Ob_+ the self-dual obstruction bundle and choose an orientation data $o(\mathcal{L})$ for $\mathcal{M}_c$ which gives an orientation on Ob_+.

Definition 5.12. *We assume $\overline{\mathcal{M}}_c = \mathcal{M}_c$ is smooth, by Proposition 4.7, $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \mathcal{M}_c$. We assume there exists an orientation data $o(\mathcal{L})$. Then the virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$ (DT_4 virtual cycle for short) is the Poincaré dual of the Euler class of the self-dual obstruction bundle over $\mathcal{M}_c$, i.e.*

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} \triangleq PD(e(Ob_+)) \in H_r(\mathcal{M}_c, \mathbb{Z}),$$

where $r = 2 - \chi(\mathcal{F},\mathcal{F})$ is the real virtual dimension of $\overline{\mathcal{M}}_c^{DT_4}$.

When $\mathcal{M}_c$ is smooth, the following lemma will be useful for later computations.

Lemma 5.13. [24] *Let $E \to U$ be a complex vector bundle with a non-degenerate quadratic form. V is a maximal isotropic subbundle of E.*
(1) If $rk(E) = 2n$, then the structure group of E reduces to $SO(2n,\mathbb{C})$ and the half Euler class of E (i.e. the Euler class of the corresponding real quadratic bundle) is $\pm c_n(V)$ where the sign depends on the choice of the maximal isotropic family of V.
(2) If $rk(E) = 2n+1$ and the the structure group of E reduces to $SO(2n+1,\mathbb{C})$, then the class is zero.

The next case where we have $\overline{\mathcal{M}}_c^{DT_4}$ without the gluing assumption is the following.

Definition 5.14. *We assume $\overline{\mathcal{M}}_c = \mathcal{M}_c$ and there exists a perfect obstruction theory* [6]

$$\phi: \quad \mathcal{V}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}_c},$$

such that

$$H^0(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}} \cong Ext^1(\mathcal{F},\mathcal{F}),$$
$$H^{-1}(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}} \oplus H^{-1}(\mathcal{V}^\bullet)|^*_{\{\mathcal{F}\}} \cong Ext^2(\mathcal{F},\mathcal{F}),$$

and $H^{-1}(\mathcal{V}^\bullet)|_{\{\mathcal{F}\}}$ is a maximal isotropic subspace of $Ext^2(\mathcal{F},\mathcal{F})$ with respect to the Serre duality pairing, then by Proposition 4.8, $\overline{\mathcal{M}}_c^{DT_4}$ exists, $\overline{\mathcal{M}}_c^{DT_4} \cong \mathcal{M}_c$ and the index bundle $Ind_{\mathbb{C}}$ has a natural complex orientation $o(\mathcal{O})$ (Definition 1.4).

The virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$ (DT_4 virtual cycle for short) with respect to the natural complex orientation $o(\mathcal{O})$ is the virtual fundamental class of the above perfect obstruction theory, i.e.

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} \triangleq [\mathcal{M}_c, \mathcal{V}^\bullet]^{vir} \in A_{\frac{r}{2}}(\mathcal{M}_c),$$

where $r = 2 - \chi(\mathcal{F},\mathcal{F})$ is the real virtual dimension of $\overline{\mathcal{M}}_c^{DT_4}$.

The μ_2-map: We define a μ_2-map for the above two cases. We denote the universal sheaf of $\mathcal{M}_c$ by

$$\mathfrak{F} \to \mathcal{M}_c \times X.$$

The μ_2-map is similarly defined to be

$$\mu_2 : H_*(X) \otimes \mathbb{Z}[x_1, x_2, ...] \to H^*(\mathcal{M}_c),$$

$$\mu_2(\gamma, P) = P(c_1(\mathfrak{F}), c_2(\mathfrak{F}), ...)/\gamma. \tag{16}$$

Definition 5.15. *In Definitions 5.12, 5.14, the DT_4 invariant of $(X, \mathcal{O}(1))$ with respect to Chern character c and an orientation data $o(\mathcal{L})$ is a map*

$$DT_4^{\mu_2}(X, \mathcal{O}(1), c, o(\mathcal{L})) : Sym^*\big(H_*(X,\mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{Z} \tag{17}$$

such that

$$DT_4^{\mu_2}(X, \mathcal{O}(1), c, o(\mathcal{L}))((\gamma_1, P_1), (\gamma_2, P_2), ...)$$
$$= < \mu_2(\gamma_1, P_1) \cup \mu_2(\gamma_2, P_2) \cup ..., [\overline{\mathcal{M}}_c^{DT_4}]^{vir} >,$$

where $<,>$ denotes the natural pairing between homology and cohomology classes.

Actually, the above definition of DT_4 invariants is consistent with the definition before (15).

Proposition 5.16. *We assume $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$ is smooth, $c = (2, 0, *, 0, 0)$ i.e. $E \to X$ is a $SU(2)$ bundle, then*

$$DT_4^{\mu_1}(X, \mathcal{O}(1), c, o(\mathcal{L})) = DT_4^{\mu_2}(X, \mathcal{O}(1), c, o(\mathcal{L})).$$

Proof. By the assumption, we have $\overline{\mathcal{M}}_c \cong \mathcal{M}_c^{DT_4} \hookrightarrow \mathcal{B}^*$. It is not hard to check $[\mathcal{M}_c^{DT_4}]^{vir} = i_*(PD(e(Ob_+)))$, where Ob_+ is the self-dual obstruction bundle over $\overline{\mathcal{M}}_c$ and $i : \mathcal{M}_c^{DT_4} \hookrightarrow \mathcal{B}^*$ is the inclusion. Meanwhile, $i^*(p_1(\mathcal{P}^{ad})) = -4c_2(\mathcal{P})$, where $\mathcal{P} \to \overline{\mathcal{M}}_c \times X$ is the universal bundle and we abuse the notation $i : \overline{\mathcal{M}}_c \times X \hookrightarrow \mathcal{B}^* \times X$ for the product of the inclusion map and the identity map. □

Remark 5.17.
1. If $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl} \neq \emptyset$, $c = (2, 0, *, 0, 0)$ and conditions in Definition 5.14 are satisfied, we also have

$$DT_4^{\mu_1}(X, \mathcal{O}(1), c, o(\mathcal{L})) = DT_4^{\mu_2}(X, \mathcal{O}(1), c, o(\mathcal{L})).$$

This is proved by showing the equivalence of our DT_4 virtual cycles with Borisov-Joyce's virtual cycles (see the appendix) as their virtual cycles are shown to be independent of choices of local charts and splittings [8].
2. The condition $c = (2, 0, *, 0, 0)$, i.e. $G = SU(2)$ is to ensure Pontryagin classes of $\mathcal{P}^{ad}$ can recover Chern classes of $\mathcal{P}$. For higher rank bundles,

$$DT_4^{\mu_1}(X, \mathcal{O}(1), c, o(\mathcal{L}))((\gamma_1, P_1), (\gamma_2, P_2), ...) = DT_4^{\mu_2}(X, \mathcal{O}(1), c, o(\mathcal{L}))((\gamma_1, P_1), (\gamma_2, P_2), ...)$$

holds only for careful choices of insertions $((\gamma_1, P_1), (\gamma_2, P_2), ...)$.

5.3. **Monodromy group actions and DT_4 virtual cycles.** In this subsection, we prove a result which restricts the way DT_4 virtual cycles would sit inside $\widetilde{\mathcal{B}}^*$ (the space of framed irreducible connections). It is based on monodromy group actions determined by loops of complex structures on X and the deformation invariance of DT_4 virtual cycles. The idea is suggested to the authors by Simon Donaldson.

We first consider the case when $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl}$, i.e. the Gieseker moduli space consists of slope stable bundles only. Then we know $\mathcal{M}_c^{DT_4}$ is compact and its virtual fundamental class exists. For convenience purposes, we will take the DT_4 virtual cycle as a homology class in $H_*(\widetilde{\mathcal{B}}^*)$ as the rational cohomology of $\widetilde{\mathcal{B}}^*$ can be easily calculated.

Lemma 5.18. [21] *We assume $H^{odd}(X, \mathbb{Z}) = 0$, then $H^*(\widetilde{\mathcal{B}}^*, \mathbb{Q})$ is the polynomial algebra freely generated by $\tilde{\mu}_1(\gamma, x_l)$ (14), where $1 \leq l \leq rk(E)$ and γ runs through a basis of $H_{2i}(X)$ for $0 < i < l$.*

We restrict to the case when X is a smooth CY_4 complete intersections of k hypersurfaces of degree $d = (d_1, ..., d_k)$ inside $\mathbb{CP}^{4+k}$, where $k \geq 1$. We denote U to be the space of all such complete intersections and $\pi : V \to U$ to be the corresponding family. We assume $u \in U$ corresponds to the chosen X. It is well-known that $\pi_1(U, u)$ acts on $H^4_{prim}(X, \mathbb{Z})$ preserving the quadratic form q.

Definition 5.19. [23] *The global monodromy group $\Gamma = \Gamma_{4,d}$ is the image of the monodromy representation $\pi_1(U, u) \to O(H^4_{prim}(X, \mathbb{Z}), q)$, where $H^4_{prim}(X, \mathbb{Z})$ is the primitive cohomology.*

The following result characterizes the behavior of the monodromy group Γ inside $O(H^4_{prim}(X, \mathbb{Z}), q)$.

Lemma 5.20. *(Corollary 4.3.1* [23]*) The global monodromy group Γ is of finite index in $O(H^4_{prim}(X, \mathbb{Z}), q)$.*

We notice that a ring of polynomials invariant under Γ-action is the same as the ring invariant under the action of its Zariski closure in $SO(H^4_{prim}(X, \mathbb{C}), q)$. In fact, we have large Zariski closure in our case.

Lemma 5.21. *(Theorem 2.10, Chapter VI* [27]*)*
Let L be a lattice, i.e. a finite rank free $\mathbb{Z}$-module together with a non-degenerate quadratic form q. We denote $V \triangleq L \otimes \mathbb{C}$ and suppose q is of signature (r,s) with $r \geq 3$, $s \geq 2$. Let D be a subgroup of finite index of $SO(L, q)$. Then D is Zariski dense in $SO(V, q)$.

The following lemma gives the invariant subring explicitly.

Lemma 5.22. *(Lemma 2.1, Chapter VI* [27]*) Let V be a finite dimensional complex vector space with $dim_{\mathbb{C}} V \geq 2$, q is a non-degenerate quadratic form on V, then $Sym^*(V^*)^{SO(V,q)} = \mathbb{C}[q]$.*

Finally, we can describe the Γ-invariant part of $H^*(\widetilde{\mathcal{B}}^*, \mathbb{Q})$.

Theorem 5.23. *Let X be a smooth complete intersection Calabi-Yau four-fold inside $\mathbb{P}^N$, $E \to X$ be a complex vector bundle with $SU(r)$ structure group, where $r \geq 2$. Then we have*

$$H^*(\widetilde{\mathcal{B}}^*, \mathbb{C})^\Gamma = \mathbb{C}[\tilde{\mu}_1(PD[\omega^3], x_j)_{2\leq j\leq r}, \tilde{\mu}_1(PD[\omega^2], x_j)_{3\leq j\leq r}, \tilde{\mu}_1(PD[q], x_j)_{3\leq j\leq r},$$
$$\tilde{\mu}_1(PD[\omega], x_j)_{4\leq j\leq r}, \tilde{\mu}_1([X], x_j)_{5\leq j\leq r}],$$

where PD denotes the Poincaré dual operator.

Proof. By the Lefschetz hyperplane theorem, we know $H^{odd}(X, \mathbb{Z}) = 0$, $H_2(X) \cong \mathbb{Z}$. By [46], the signature of $(H^4_{prim}(X, \mathbb{Z}), q)$ is $(4h^{3,1} + 49, 2h^{3,1})$. We know $h^{3,1} \geq 1$ for complete intersection CY_4's as it is the dimension of the deformation space. Then we apply Lemma 5.18, 5.20, 5.21 and 5.22 to X. □

Corollary 5.24. *The DT_4 virtual cycle $\widetilde{[\mathcal{M}_c^{DT_4}]^{vir}} \in H_*(\widetilde{\mathcal{B}}^*)$ (Remark 5.4) can be expressed as a homogenous polynomial in terms of the above generators of $H^*(\widetilde{\mathcal{B}}^*, \mathbb{C})^\Gamma$.*

Proof. By the deformation invariance of DT_4 virtual cycles. □

Remark 5.25. *For general cases when $\overline{\mathcal{M}}_c \neq \mathcal{M}_c^{bdl}$, based on Definition 5.15, DT_4 invariants are defined by pairing DT_4 virtual cycles $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}$'s $\in H_r(\overline{\mathcal{M}}_c)$ with the corresponding μ_2-maps. These a priori have nothing to do with $H^*(\widetilde{\mathcal{B}}^*)$ and the global monodromy group action is not explicitly related to DT_4 invariants.*

However, we can use Seidel-Thomas twists to embed the Gieseker moduli space into $\mathcal{B}^*_{E'}$, the space of irreducible connections on some other topological bundle E'. We then use μ_1-maps associated with $\mathcal{B}^*_{E'}$ to define invariants and the invariants would share a similar property in Corollary 5.24. We notice that the way Gieseker moduli spaces are embedded into $\mathcal{B}^*_{E'}$ is not canonical and invariants defined in this way are a priori not the same as DT_4 invariants defined before in Definition 5.15.

5.4. **Some vanishing results of DT_4 virtual cycles.**

5.4.1. *Vanishing results for certain choices of c.* We know that DT_4 invariants are deformation invariants of Calabi-Yau four-fold X. Unlike the case of Calabi-Yau three-folds with $SU(3)$ holonomy [39], classes in $H^{2,2}(X)$ may not remain $(2,2)$-type when we deform the complex structure. When it is deformed to the case when non-algebraic stuff appears, i.e. $c \notin \bigoplus_i H^{i,i}(X)$, one has to use other nice analytic compactification to define invariants. However, DT_4 invariants turn out to be rather trivial in this case.

Proposition 5.26. *Let X be a compact Calabi-Yau four-fold. We fix cohomology classes $c = \bigoplus_{i=0}^4 c|_{H^{2i}(X,\mathbb{Q})}$, then*
(1) If $c|_{H^4(X,\mathbb{Q})}$ has no component in $H^{0,4}(X)$ and $c \notin \bigoplus_{i=0}^4 H^{i,i}(X)$, then $\mathcal{M}_c^{DT_4} = \emptyset$.
(2) If $c \in \bigoplus_{i=0}^4 H^{i,i}(X)$ and $\exists\ \varphi \in H^1(X, TX)$ such that $\varphi \lrcorner \left(c|_{H^{2,2}(X,\mathbb{Q})}\right) \neq 0$, then $\mathcal{M}_c = \emptyset$ for some complex structure X_t, where t is small and X_t is the family of Calabi-Yau four-fold such that $X_0 = X$ determined by φ.

Proof. (1) If there is no complex vector bundle E such that $ch(E) = c$, then it is trivial to get $\mathcal{M}_c^{DT_4} = \emptyset$. Thus we assume there exists a complex vector bundle E such that $ch(E) = c$, $ch_2(E)$ does not have $(0,4)$ component and $\bigoplus_{i=0}^4 ch_i(E) \notin \bigoplus_{i=0}^4 H^{i,i}(X)$. If $\mathcal{M}_c^{DT_4} \neq \emptyset$, we take an element d_A inside. Then by Lemma 4.1, the bundle is holomorphic which contradicts with the assumption.

(2) If $\mathcal{M}_c = \emptyset$ for X, we are done. Thus we assume $\mathcal{M}_c \neq \emptyset$ and take a stable sheaf $\mathcal{F}$ with $ch(\mathcal{F}) = c$. By the Tian-Todorov theorem [35], φ gives a one parameter family of complex structures X_t such that $X_0 = X$. Since $\varphi \lrcorner\, ch_2(\mathcal{F}) \neq 0$, we get $\varphi \circ At(\mathcal{F}) \neq 0$ as

$$tr\big(\varphi \circ At(\mathcal{F}) \circ At(\mathcal{F})\big) = 2\varphi \lrcorner\, ch_2(\mathcal{F}),$$

where $At(\mathcal{F})$ is the Atiyah class of $\mathcal{F}$ [15]. We notice that $\varphi \circ At(\mathcal{F}) \in Ext^2(\mathcal{F}, \mathcal{F})$ is the obstruction class for the deformation of $\mathcal{F}$ along φ. Hence we see that any sheaf in $\mathcal{M}_c$ can't be deformed to the nearby complex structures. Thus, $\mathcal{M}_c = \emptyset$ for X_t, where $t \neq 0$ and small. □

Remark 5.27. *Under the same assumption as in Proposition 5.26, any deformation invariant associated with $\mathcal{M}_c$ must vanish, for instance, DT_4 invariants defined in this paper and Borisov-Joyce's virtual fundamental classes [8] will vanish.*

5.4.2. *Vanishing results for hyper-Kähler four-folds.* We assume X to be a compact hyper-Kähler four-fold with a holomorphic symplectic two form $\sigma \in H^0(X, \Omega_X^2)$.

There is an obvious surjective cosection map of the obstruction sheaf of $\mathcal{M}_c$

$$\nu : Ob \twoheadrightarrow \mathcal{O}_{\mathcal{M}_c},$$

given by the trace map

$$\nu : Ext^2(\mathcal{F}, \mathcal{F}) \rightarrow H^2(X, \mathcal{O}_X) \cong \mathbb{C}.$$

Then we have the surjective cosection map for DT_4 obstruction space

$$\nu_+ : Ext_+^2(\mathcal{F}, \mathcal{F}) \twoheadrightarrow H_+^2(X, \mathcal{O}_X) \cong \mathbb{R}, \tag{18}$$

which leads to the vanishing of virtual fundamental classes of (generalized) DT_4 moduli spaces.

Remark 5.28. *If we fix the determinant of the torsion-free sheaf, there is a less obvious cosection map* [15]

$$\nu_{hyper} : Ext_0^2(\mathcal{F}, \mathcal{F}) \twoheadrightarrow H^{4,4}(X),$$

defined to be the composition of

$$Ext_0^2(\mathcal{F}, \mathcal{F}) \xrightarrow{\cdot \frac{(At(\mathcal{F}))^2}{2}} Ext^4(\mathcal{F}, \mathcal{F} \otimes \Omega_X^2) \xrightarrow{tr} H^4(X, \Omega_X^2) \xrightarrow{\wedge \sigma} H^{4,4}(X).$$

Similar to [61]*, one can show ν_{hyper} is surjective if $ch_3(\mathcal{F}) \neq 0$. However, it does not factor through $Ext_+^2(\mathcal{F}, \mathcal{F})$ and in general does not give a surjective cosection map of the trace-free DT_4 obstruction space.*

In fact, we will show later in the DT_4/GW correspondence that there exists a surjective cosection map for the trace-free DT_4 obstruction space which turns out to be the same as the cosection map of GW theory for hyper-Kähler four-folds (31).

6. DT_4 invariants for compactly supported sheaves on local CY_4

In this section, we study DT_4 invariants on local (non-compact) Calabi-Yau four-folds. We first consider compactly supported sheaves on Calabi-Yau four-folds of type K_Y, where Y is a compact Fano threefold. We define their DT_4 invariants and show that there is a DT_4/DT_3 correspondence. Secondly, we define DT_4 invariants for moduli spaces of sheaves of type $\iota_*(\mathcal{F})$, where $\iota : S \rightarrow T^*S$ is the zero section and $\mathcal{F}$ is stable on a compact algebraic surface S. We then relate DT_4 invariants on T^*S to some known invariants of S.

6.1. **The case of $X = K_Y$.** We first describe the stability of compactly supported sheaves on X. We denote $\iota : Y \rightarrow K_Y$ to be the zero section map and $\pi : K_Y \rightarrow Y$ to be the projection map. We pick an ample line bundle $\mathcal{O}_Y(1)$ on Y and define the Hilbert polynomial of a compactly supported coherent sheaf $\mathcal{F}$ on $X = K_Y$ to be $\chi(\mathcal{F} \otimes \pi^*\mathcal{O}_Y(k))$ for $k \gg 0$. Then we can talk about Gieseker $\pi^*\mathcal{O}_Y(1)$-stability on compactly supported sheaves over X [36].

Lemma 6.1. *Let Y be a compact Fano threefold, then any $\pi^*\mathcal{O}_Y(1)$-stable sheaf with three-dimensional compact support on local Calabi-Yau four-fold $X = K_Y$ is of type $\iota_*(\mathcal{F})$, where $\mathcal{F}$ is $\mathcal{O}_Y(1)$-stable on Y.*

Proof. The proof here is similar to the proof of Lemma 7.1 of [33]. We only need to show that any compactly supported stable sheaf is scheme theoretically supported on Y. We denote Z to be the scheme theoretical support of a compactly supported stable sheaf $\mathcal{E}$. By the trace map [36], we have

$$H^0(Z, \mathcal{O}_Z) \hookrightarrow Ext_Z^0(\mathcal{E}, \mathcal{E}).$$

It suffices to show $dim H^0(Z, \mathcal{O}_Z) > 1$ to get contradiction as stable sheaf $\mathcal{E}$ is always simple.

By our assumption, dimension of the support of $\mathcal{E}$ is three. Then Z is an order $n \geq 1$ thickening of Y in the normal direction inside X, i.e.

$$Z = \mathbf{Spec}\big(\bigoplus_{i=0}^{n} K_Y^{-i}\big).$$

There is a spectral sequence such that $E_\infty^{0,0} = H^0(Z, \mathcal{O}_Z)$ and $E_2^{0,0} = H^0(Y, \oplus_{i=0}^n K_Y^{-i})$ which implies

$$H^0(Z, \mathcal{O}_Z) \cong \oplus_{i=0}^n H^0(Y, K_Y^{-i}).$$

As $H^0(Y, K_Y^{-1}) \neq 0$ for any Fano threefold, $dim H^0(Z, \mathcal{O}_Z) \geq 2$, which leads to a contradiction. Thus $\mathcal{E}$ is of type $\iota_*\mathcal{F}$, where $\mathcal{F}$ is a sheaf on Y. Now we show $\mathcal{F}$ is stable with respect to $\mathcal{O}_Y(1)$.

By the projection formula, for any k, we have

$$\iota_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(k)) = \iota_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \iota^*\pi^*\mathcal{O}_Y(k)) = \iota_*\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{O}_Y(k).$$

Thus

$$H^*(Y, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(k)) = H^*(X, \iota_* \mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{O}_Y(k)).$$

Then we know the stability condition for $\iota_* \mathcal{F}$ on $X = K_Y$ is equivalent to the stability condition for $\mathcal{F}$ on Y. □

We compare obstruction theory of $\iota_*(\mathcal{F})$ on K_Y with the obstruction theory of $\mathcal{F}$ on Y.

Definition 6.2. [33] *Let $L = \oplus_{i=0}^{d} L^i$ be a finite dimensional L_∞ algebra over $\mathbb{C}$, with its $\mathbb{C}$ products μ_k. Let $\overline{L}$ be the graded vector space $L \oplus L[-d-1]$, i.e $\overline{L}^i = L^i \oplus (L^{d+1-i})^*$. We define the cyclic pairing and L_∞ products $\overline{\mu}_k : \wedge^k \overline{L} \to \overline{L}[2-k]$ according to rules,*
(1) define the bilinear form κ on $\overline{L}$ by the natural pairing between L and L^*,
(2) if the inputs of $\overline{\mu}_k$ all belong to L, then $\overline{\mu}_k = \mu_k$,
(3) if more than one input belong to L^*, then define $\overline{\mu}_k = 0$,
(4) if there is exactly one input $a_i^* \in L^*$, then define $\overline{\mu}_k$ by

$$\kappa(\overline{\mu}_k(a_1, ..., a_i, ..., a_k), b) = (-1)^\epsilon \kappa(\mu_k(a_{i+1}, ..., a_k, b, a_1, ..., a_{i-1}), a_i^*)$$

for arbitary $b \in L$ and ϵ depends on a_i, b only.
We then call the L_∞ algebra $(\overline{L}, \overline{\mu}_k, \kappa)$ the $d+1$ dimensional cyclic completion of L.

Lemma 6.3. [71] *Let Y be a smooth proper scheme of $\dim_{\mathbb{C}} = d-1$, $\iota : Y \to K_Y$ be the zero section map. Then for any $S \in D^b(Y)$ the A_∞ algebra $Ext^*_{K_Y}(\iota_* S, \iota_* S)$ is the d-dim cyclic completion of $Ext^*_Y(S, S)$.*

Using the above lemma, we have,

Lemma 6.4. *Let $\mathcal{F}$ be a torsion-free slope-stable sheaf on a compact Fano threefold Y. We denote $\iota : Y \to K_Y = X$ to be the zero section map. Then we have canonical isomorphisms*

$$Ext^1_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \cong Ext^1_Y(\mathcal{F}, \mathcal{F}),$$

$$Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \cong Ext^2_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^2_Y(\mathcal{F}, \mathcal{F})^*. \tag{19}$$

And a local Kuranishi map of a moduli space of sheaves of type $\iota_* \mathcal{F}$ on X

$$\kappa : Ext^1_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \to Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F})$$

can be identified with a local Kuranishi map of a moduli space of sheaves of type $\mathcal{F}$ on Y

$$Ext^1_Y(\mathcal{F}, \mathcal{F}) \to Ext^2_Y(\mathcal{F}, \mathcal{F}).$$

Furthermore, under the above identification, $Ext^2_Y(\mathcal{F}, \mathcal{F})$ is a maximal isotropic subspace of $Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F})$ with respect to the Serre duality pairing.

Proof. By Lemma 6.3, we have

$$Ext^1_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \cong Ext^1_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^3_Y(\mathcal{F}, \mathcal{F})^* = Ext^1_Y(\mathcal{F}, \mathcal{F}),$$

$$Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \cong Ext^2_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^2_Y(\mathcal{F}, \mathcal{F})^*. \tag{20}$$

At $\iota_* \mathcal{F}$, any Kuranishi map

$$\kappa : Ext^1_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \to Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}),$$

can be described by the L_∞ products in Definition 6.2, we thus can identify κ with some Kuranishi map of a moduli space of stable sheaves on Y at $\mathcal{F}$.

By (20), we know the quadratic pairing

$$Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \otimes Ext^2_X(\iota_* \mathcal{F}, \iota_* \mathcal{F}) \to Ext^4_X(\iota_* \mathcal{F}, \iota_* \mathcal{F})$$

is just the quadratic pairing

$$(Ext^2_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^2_Y(\mathcal{F}, \mathcal{F})^*) \otimes (Ext^2_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^2_Y(\mathcal{F}, \mathcal{F})^*) \to Ext^3_Y(\mathcal{F}, \mathcal{F} \otimes K_Y).$$

When we restrict to $Ext^2_Y(\mathcal{F}, \mathcal{F})$, the pairing will produce an element in $Ext^4_Y(\mathcal{F}, \mathcal{F}) = 0$. □

By the above lemma and Lemma 6.1, for a polarized compact Fano threefold $(Y, \mathcal{O}_Y(1))$, the moduli space of $\pi^* \mathcal{O}_Y(1)$ slope-stable compactly supported sheaves on K_Y with compactly supported Chern character [39], $c = (0, c|_{H^2_c(X)} \neq 0, c|_{H^4_c(X)}, c|_{H^6_c(X)}, c|_{H^8_c(X)})$ can be identified with a moduli space of torsion-free $\mathcal{O}_Y(1)$ stable sheaves on Y with certain Chern character $c' \in H^{even}(Y)$ which is uniquely determined by c. Meanwhile the condition in Definition 5.14 is satisfied [76].

By Definition 5.14, the generalized DT_4 moduli space exists and we can identify its virtual fundamental class with the virtual fundamental class of a moduli space of stable sheaves on Y. Furthermore, if we use the same μ-map (16) to define invariants, we can also identify them.

Theorem 6.5. *(DT_4/DT_3)*
Let $\pi : X = K_Y \to Y$ be the projection map and $(Y, \mathcal{O}_Y(1))$ be a polarized compact Fano threefold. If $c = (0, c|_{H^2_c(X)} \neq 0, c|_{H^4_c(X)}, c|_{H^6_c(X)}, c|_{H^8_c(X)})$ and the Gieseker moduli space of compactly supported sheaves $\overline{\mathcal{M}}_c(X, \pi^*\mathcal{O}_Y(1))$ consists of slope-stable sheaves, then sheaves in $\mathcal{M}_c(X, \pi^*\mathcal{O}_Y(1))$ are of type $\iota_*(\mathcal{F})$, where $\iota : Y \to K_Y$ is the zero section and $c' = ch(\mathcal{F}) \in H^{even}(Y)$ is uniquely determined by c. Furthermore, the generalized DT_4 moduli space exists

$$\overline{\mathcal{M}}_c^{DT_4}(X, \pi^*\mathcal{O}_Y(1)) = \mathcal{M}_c(X, \pi^*\mathcal{O}_Y(1)) \cong \mathcal{M}_{c'}(Y, \mathcal{O}_Y(1))$$

and its virtual fundamental class (Definition 5.14) satisfies

$$[\overline{\mathcal{M}}_c^{DT_4}(X, \pi^*\mathcal{O}_Y(1))]^{vir} = [\mathcal{M}_{c'}(Y, \mathcal{O}_Y(1))]^{vir},$$

where $[\mathcal{M}_{c'}(Y, \mathcal{O}_Y(1))]^{vir}$ is the DT_3 virtual cycle defined by Thomas [76].

Since $H_*(X) \cong H_*(Y)$ and $H^*(\overline{\mathcal{M}}_c^{DT_4}(X, \pi^*\mathcal{O}_Y(1))) \cong H^*(\mathcal{M}_{c'}(Y, \mathcal{O}_Y(1)))$, we can use the same μ_2-map (16) to define invariants, then

$$DT_4^{\mu_2}(X, \pi^*\mathcal{O}_Y(1), c, o(\mathcal{O})) = DT_3(Y, \mathcal{O}_Y(1), c'),$$

where $o(\mathcal{O})$ is the natural complex orientation and $DT_3(Y, \mathcal{O}_Y(1), c')$ is defined by pairing $[\mathcal{M}_{c'}(Y, \mathcal{O}_Y(1))]^{vir}$ with the μ_2-map (17).

Remark 6.6. *If we have a compact complex smooth four-fold X (no need to be Calabi-Yau) containing a Fano threefold Y such that $\mathcal{N}_{Y/X} = K_Y$ and $\mathcal{N}^*_{Y/X}$ is ample, e.g. $X = P(K_Y \oplus \mathcal{O})$. Then we can define DT_4 invariants for stable sheaves supported in Y because X contains K_Y as its open subset by the renowned theorem of Grauert.*

6.2. **The case of $X = T^*S$.** We take $X = T^*S$ which is a hyper-Kähler four-fold when $S = \mathbb{P}^2$. We only consider counting torsion sheaves scheme theoretically supported on S in this subsection. We take $\mathcal{F}$ to be a torsion-free Gieseker stable sheaf on a compact algebraic surface $(S, \mathcal{O}_S(1))$ and denote $\pi : T^*S \to S$ to be the projection map, $\iota : S \to T^*S$ to be the inclusion map.

We relate the obstruction theory of sheaf $\iota_*(\mathcal{F})$ on X to the obstruction theory of $\mathcal{F}$ on S. By the projection formula [32],

$$\iota_*(\mathcal{F}) = \iota_*(\iota^*\pi^*\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S) = \pi^*\mathcal{F} \otimes_{\mathcal{O}_X} \iota_*\mathcal{O}_S,$$

where $\mathcal{F}$ is a complex of locally free sheaves on S. Then we have a local to global spectral sequence $E_2^{p,q} = Ext^q_S(\mathcal{F}, \wedge^p\Omega^1_S \otimes \mathcal{F}) \Rightarrow Ext^*_X(\iota_*\mathcal{F}, \iota_*\mathcal{F})$. We give a criterion for it to degenerate at E_2 terms.

Lemma 6.7. *Let $\mathcal{F}$ be a torsion free sheaf on S.*
(1) If $Ext^2_S(\mathcal{F}, \mathcal{F}) = 0$, the above spectral sequence degenerates at E_2.
(2) If the degree of K_S is negative with respect to the chosen polarization $\mathcal{O}_S(1)$ and $\mathcal{F}$ is slope-stable, then we have $Ext^2_S(\mathcal{F}, \mathcal{F}) = 0$.

Proof. (1) We denote $E_2^{p,q} = Ext^q_S(\mathcal{F}, \wedge^p\Omega^1_S \otimes \mathcal{F})$. We have

$$E_2^{p-2,q+1} \to E_2^{p,q} \to E_2^{p+2,q-1},$$

whose cohomology is $E_3^{p,q}$. Then $0 \to E_2^{1,q} \to 0$ yields $E_3^{1,q} \cong E_2^{1,q}$. Meanwhile, we have

$$E_2^{0,q+1} \to E_2^{2,q} \to 0, \quad 0 \to E_2^{0,q} \to E_2^{2,q-1}.$$

Under the assumption that $Ext^2_S(\mathcal{F}, \mathcal{F}) = 0$, we get

$$E_2^{2,0} = E_2^{0,2} = 0.$$

Thus the above spectral sequence degenerates at E_2.

(2) By Serre duality, we have

$$Ext^2_S(\mathcal{F}, \mathcal{F}) \cong Hom_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F} \otimes K_S).$$

By assumption,

$$\mu(\mathcal{F}) = \frac{deg(\mathcal{F})}{rk(\mathcal{F})} > \frac{deg(\mathcal{F} \otimes K_S)}{rk(\mathcal{F} \otimes K_S)} = \mu(\mathcal{F} \otimes K_S).$$

If the above homomorphism is not zero, we choose such a nonzero morphism

$$f : \mathcal{F} \to \mathcal{F} \otimes K_S,$$

then

$$0 \neq \mathcal{F}/ker(f) \hookrightarrow \mathcal{F} \otimes K_S.$$

By the stability of $\mathcal{F}$, we have

$$\mu(\mathcal{F}/ker(f)) \geq \mu(\mathcal{F}).$$

Thus

$$\mu(\mathcal{F} \otimes K_S) < \mu(\mathcal{F}) \leq \mu(\mathcal{F}/ker(f)),$$

which contradicts with the semi-stability of $\mathcal{F} \otimes K_S$. $\square$

We have to make sure that sheaves scheme theoretically supported on S can not move outside. This can be done by finding conditions such that

$$Ext^0_S(\mathcal{F}, \mathcal{F} \otimes \Omega^1_S) = 0.$$

If $S = \mathbb{P}^2$ and $\mathcal{F}$ is torsion-free slope stable, the condition is satisfied as Ω^1_S is stable. When $\mathcal{F} = I$ is an ideal sheaf of points, we have

Lemma 6.8. *Let $\mathcal{F} = I$ be an ideal sheaf of points on S. If $h^{0,1}(S) = 0$, then $Ext^0_S(\mathcal{F}, \mathcal{F} \otimes \Omega^1_S) = 0$.*

Proof. $0 \to I \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$ induces

$$\begin{aligned} 0 \to Ext^0_S(I, I \otimes \Omega^1_S) \to Ext^0_S(I, \Omega^1_S) &\cong Ext^2(\Omega^1_S, I \otimes K_S) \\ &= H^2(S, I \otimes K_S \otimes T_S), \end{aligned}$$

$$0 = H^1(S, \mathcal{O}_Z \otimes K_S \otimes T_S) \to H^2(S, I \otimes K_S \otimes T_S) \to H^2(S, K_S \otimes T_S) \cong H^0(S, \Omega^1_S).$$

Then $h^{1,0} = 0 \Rightarrow Ext^0_S(I, I \otimes \Omega^1_S) = 0$. $\square$

Proposition 6.9. *Under the following assumptions*

$$Ext^0_S(\mathcal{F}, \mathcal{F} \otimes \Omega^1_S) = 0, Ext^2_S(\mathcal{F}, \mathcal{F}) = 0, \tag{21}$$

which is satisfied when (i) S is del-Pezzo, $\mathcal{F}$ is an ideal sheaf of points on S or (ii) when $S = \mathbb{P}^2$, $\mathcal{F}$ is slope-stable torsion-free on S, we have canonical isomorphisms

$$Ext^1_X(\iota_*\mathcal{F}, \iota_*\mathcal{F}) \cong Ext^1_S(\mathcal{F}, \mathcal{F}),$$

$$Ext^2_X(\iota_*\mathcal{F}, \iota_*\mathcal{F}) \cong Ext^1_S(\mathcal{F}, \mathcal{F} \otimes \Omega^1_S).$$

Proof. By Lemma 6.7, Lemma 6.8 and the degenerate spectral sequence. $\square$

Under assumption (21), we denote

$$\mathcal{M}^{S_{cpn}}_c \triangleq \{\iota_*\mathcal{F} \mid \mathcal{F} \in \mathcal{M}_c(S, \mathcal{O}_S(1))\} \cong \mathcal{M}_c(S, \mathcal{O}_S(1))$$

to be the component(s) of a moduli space of sheaves on X which can be identified with $\mathcal{M}_c(S, \mathcal{O}_S(1))$ (the moduli space of $\mathcal{O}_S(1)$ stable sheaves on S with Chern character $c \in H^{even}(S)$). We notice that $\mathcal{M}_c(S, \mathcal{O}_S(1))$ is smooth by assumption (21) and use the philosophy of defining DT_4 virtual cycles (Definition 5.12) to define the virtual cycle of $\mathcal{M}^{S_{cpn}}_c$, i.e. we define the DT_4 virtual cycle of $\mathcal{M}^{S_{cpn}}_c$ to be the Poincaré dual of the Euler class of the self-dual obstruction bundle.

Proposition 6.10. *Under assumption (21), $\mathcal{M}^{S_{cpn}}_c \cong \mathcal{M}_c(S, \mathcal{O}_S(1))$ and the virtual dimension $v.d_{\mathbb{R}}(\mathcal{M}^{S_{cpn}}_c) \triangleq 2ext^1(\iota_*\mathcal{F}, \iota_*\mathcal{F}) - ext^2(\iota_*\mathcal{F}, \iota_*\mathcal{F}) < 0$, where $\mathcal{F} \in \mathcal{M}_c(S, \mathcal{O}_S(1))$.*

Proof. By the Hirzebruch-Riemann-Roch theorem and the assumption (21), we have

$$dim_{\mathbb{C}} Ext^1_S(\mathcal{F}, \mathcal{F} \otimes \Omega^1_S) = 2dim_{\mathbb{C}} Ext^1_S(\mathcal{F}, \mathcal{F}) + r^2 e(S) - 2,$$

where $r \geq 1$ is the rank of $\mathcal{F}$, $e(S)$ is the Euler characteristic of the surface S. Then,

$$v.d_{\mathbb{R}}(\mathcal{M}^{S_{cpn}}_c) \triangleq 2ext^1(\iota_*\mathcal{F}, \iota_*\mathcal{F}) - ext^2(\iota_*\mathcal{F}, \iota_*\mathcal{F}) = 2 - r^2 e(S) < 0.$$

$\square$

Remark 6.11. *The negative virtual dimension makes the DT_4 virtual cycle of $\mathcal{M}^{S_{cpn}}_c$ vanish.*

The reduced counting. The above vanishing result is because the obstruction bundle has a trivial subbundle. We consider the trace-free part of the obstruction bundle and define the reduced DT_4 virtual cycle.

Definition 6.12. *Let $X=T^*S$ where S is a compact algebraic surface with $h^{0,1}(S)=0$. We assume sheaves in $\mathcal{M}_c(S,\mathcal{O}_S(1))$ satisfy (21) and the self-dual trace-free obstruction bundle of $\mathcal{M}_c^{S_{cpn}}$ is orientable.*

The reduced DT_4 virtual cycle of $\mathcal{M}_c^{S_{cpn}}$ is the Poincaré dual of the Euler class of the self-dual trace-free obstruction bundle

$$[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red}\triangleq PD\big(e(Ob_{0,+})\big)\in H_{r_{red}}(\mathcal{M}_c\big(S,\mathcal{O}_S(1)\big),\mathbb{Z}),$$

where $Ob_{0,+}$ is the self-dual trace-free obstruction bundle, $r_{red}=h^{1,1}(S)+2-r^2(2+h^{1,1}(S))$ and $\mathcal{M}_c(S,\mathcal{O}_S(1))$ is the Gieseker moduli space of $\mathcal{O}_S(1)$ stable sheaves on S.

Remark 6.13.
1. The above Euler class involves a choice of an orientation on each connected component of $\mathcal{M}_c^{S_{cpn}}$. We will see for most interesting cases, a natural orientation exists.
2. Note that $h^{0,2}(S)=0$ as we have $Ext^2_S(\mathcal{F},\mathcal{F})=0$ by assumption (21).

We first show reduced virtual cycles for high rank sheaves in $\mathcal{M}_c(S,\mathcal{O}_S(1))$ vanish.

Proposition 6.14. $[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red}=0$*, if* $c|_{H^0(S)}\geq 2$.

Proof. The reduced virtual dimension $r_{red}=h^{1,1}+2-r^2(2+h^{1,1})<0$ if $r\geq 2$. □

Under assumption (21) and $r=1$, we have $r_{red}=0$. The corresponding reduced DT_4 virtual cycle is zero dimensional. For ideal sheaves of curves on S (line bundles on S)

$$Ext^1_S(\mathcal{F},\mathcal{F})=H^1(S,\mathcal{O})=0,$$

which shows that both the tangent space and the reduced obstruction space are zero. Then the moduli space is just one point and the reduced DT_4 invariant is 1 in this case.

Proposition 6.15. $[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red}=1$*, where* $c=(1,c|_{H^2(S)},0)$.

For ideal sheaves of points on S, we have

Lemma 6.16. *Let S be a compact algebraic surface with $h^{0,i}(S)=0$, $i=1,2$. We take I to be an ideal sheaf of points on S, then we have a canonical isomorphism*

$$Ext^1_S(I,I\otimes\Omega^1_S)_0\cong Ext^1_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S).$$

Furthermore, under this identification, $Ext^1_S(\mathcal{O}_Z,\mathcal{O}_Z)$ is a maximal isotropic subspace with respect to the Serre duality pairing.

Proof. We denote an ideal sheaf of n-points on S by I. Taking cohomology of the short exact sequence of sheaves

$$0\to I\otimes\Omega^1_S\to\Omega^1_S\to\mathcal{O}_Z\otimes\Omega^1_S\to 0,$$

where $\mathcal{O}_Z$ is the structure sheaf of n-points, we have

$$0\to H^0(S,\mathcal{O}_Z\otimes\Omega^1_S)\to H^1(S,I\otimes\Omega^1_S)\to H^1(S,\Omega^1_S)\to 0, \tag{22}$$

and

$$H^2(S,I\otimes\Omega^1_S)\cong H^2(S,\Omega^1_S)\cong H^0(S,\Omega^1_S)=0. \tag{23}$$

Applying $Hom_{\mathcal{O}_S}(\mathcal{O}_Z,\cdot)$ to

$$0\to I\otimes\Omega^1_S\to\Omega^1_S\to\mathcal{O}_Z\otimes\Omega^1_S\to 0,$$

we get

$$\begin{aligned}Ext^0_S(\mathcal{O}_Z,I\otimes\Omega^1_S)=0,\,&Ext^0_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S)\cong Ext^1_S(\mathcal{O}_Z,I\otimes\Omega^1_S),\\ &Ext^1_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S)\cong Ext^2_S(\mathcal{O}_Z,I\otimes\Omega^1_S).\end{aligned} \tag{24}$$

Applying $Hom_{\mathcal{O}_S}(\cdot,I\otimes\Omega^1_S)$ to

$$0\to I\to\mathcal{O}_S\to\mathcal{O}_Z\to 0,$$

we have

$$\to Ext^i_S(\mathcal{O}_Z,I\otimes\Omega^1_S)\to Ext^i_S(\mathcal{O}_S,I\otimes\Omega^1_S)\to Ext^i_S(I,I\otimes\Omega^1_S)\to.$$

By the condition $h^{0,1}(S)=0$, we have $Ext^0_S(I,I\otimes\Omega^1_S)=0$ by Lemma 6.8. Using (23), (24), we can get

$$\begin{aligned}0\to Ext^0_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S)\to H^1(S,I\otimes\Omega^1_S)\to\\ \to Ext^1_S(I,I\otimes\Omega^1_S)\to Ext^1_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S)\to 0.\end{aligned}$$

By (22), we get

$$0\to H^1(S,\Omega^1_S)\to Ext^1_S(I,I\otimes\Omega^1_S)\to Ext^1_S(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_S)\to 0.$$

where the first injective map is the inclusion of the trace factor.

Considering the Serre duality pairing,

$$Ext^1_S(I, I \otimes \Omega^1_S)_0 \otimes Ext^1_S(I, I \otimes \Omega^1_S)_0 \to Ext^2_S(I, I \otimes \Omega^2_S) \to H^{2,2}(S)$$

can be identified with

$$Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega^1_S) \otimes Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega^1_S) \to Ext^2_S(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega^2_S) \to H^{2,2}(S),$$

where the last map is taking trace. Furthermore it can be identified with

$$(Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z) \oplus Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega^2_S)) \otimes (Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z) \oplus Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega^2_S)) \to \mathbb{C},$$

as

$$Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z) \otimes Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z) \to Ext^2_S(\mathcal{O}_Z, \mathcal{O}_Z) \to H^2(S, \mathcal{O}_S) = 0,$$

$Ext^1_S(\mathcal{O}_Z, \mathcal{O}_Z)$ is a maximal isotropic subspace with respect to the Serre duality pairing. □

Thus, after taking away the trivial factor $H^1(S, \Omega^1_S)$, the maximal isotropic sub-bundle of the reduced obstruction bundle exists and can be identified with the tangent bundle of Hilbert scheme of points on S. Note that this gives a natural orientation on the self-dual trace-free obstruction bundle. By Lemma 5.13, the reduced DT_4 virtual cycle can be identified with the Euler characteristic of Hilbert scheme of n-points on S.

Theorem 6.17. *We take $X = T^*S$ and $c = (1, 0, -n)$, where S is a compact algebraic surface with $q(S) = 0$ and $n \geq 1$. We assume sheaves in $\mathcal{M}_c(S, \mathcal{O}_S(1))$ satisfy (21) which is true when S is del-Pezzo. Then the self-dual trace-free obstruction bundle (Definition 6.12) has a natural complex orientation and*

$$[\mathcal{M}_c^{S_{cpn}}]^{vir}_{red} = e(Hilb^n(S)).$$

Furthermore, they fit into the following generating function

$$\sum_{n \geq 0} [\mathcal{M}^{S_{cpn}}_{(1,0,-n)}]^{vir}_{red} q^n = \prod_{k \geq 1} \frac{1}{(1-q^k)^{e(S)}}.$$

Proof. By the above discussion and [18]. □

7. Computational examples

We compute DT_4 invariants for some examples when $\mathcal{M}_c$'s are smooth in this section. By Definition 5.12, the virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$ is the Poincaré dual of the Euler class of the self-dual obstruction bundle which has its origin in the theory of characteristic classes [24].

We take a complex vector bundle with a non-degenerate quadratic form, (V, q) on a projective manifold X [74]. We assume the structure group of the bundle to be $SO(n, \mathbb{C})$, $n = rk(V)$. By the homotopy equivalence $SO(n, \mathbb{C}) \sim SO(n, \mathbb{R})$, we have $H^*(BSO(n, \mathbb{C})) \cong H^*(BSO(n, \mathbb{R}))$. Meanwhile,

$$H^*(BSO(2r, \mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[p_1, ...p_{r-1}, e],$$

$$H^*(BSO(2r+1, \mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[p_1, ...p_r],$$

where $p_i = (-1)^i c_{2i}(\omega^+ \otimes \mathbb{C})$, ω^+ is the universal $SO(n)$ bundle and $e = e(\omega^+)$ is called the half Euler class of (V, q).

When $\mathcal{M}_c$ is smooth and $(V, q) = (Ob, Q_{Serre})$, where Ob is the obstruction bundle with $Ob|_{\mathcal{F}} = Ext^2(\mathcal{F}, \mathcal{F})$, Q_{Serre} is the Serre duality pairing, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} = \mathcal{M}_c$. Furthermore, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = PD(e)$ with an appropriate choice of orientations.

7.1. **Li-Qin's examples.** We have examples when $\overline{\mathcal{M}}_c = \mathcal{M}_c^{bdl}$. We consider moduli spaces of rank 2 bundles coming from non-trivial extensions of two line bundles with certain Chern classes. The construction is due to W. P. Li and Z.Qin [58].

Let X be a generic smooth hyperplane section in $\mathbb{P}^1 \times \mathbb{P}^4$ of bi-degree $(2, 5)$. Take

$$cl = [1 + (-1, 1)|_X] \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)|_X],$$

$$k = (1 + \epsilon_1) \begin{pmatrix} 6 - \epsilon_2 \\ 4 \end{pmatrix}, \quad \epsilon_1, \epsilon_2 = 0, 1, \quad L_r = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1, r)|_X.$$

We define $\overline{\mathcal{M}}_c(L_r)$ to be the moduli space of Gieseker L_r-semistable rank-2 torsion-free sheaves with Chern character c (which can be easily read from the total Chern class cl).

Lemma 7.1. *(Li-Qin, Theorem 5.7* [58]*)*
The moduli space of rank two bundles on X with the given Chern class stated above satisfies the following properties,
(i) The moduli space is isomorphic to $\mathbb{P}^k$ and consists of all the rank-2 bundles in the nonsplitting extensions

$$0 \to \mathcal{O}_X(-1,1) \to E \to \mathcal{O}_X(\epsilon_1+1,\epsilon_2-1) \to 0,$$

when

$$\frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2} < r < \frac{15(2-\epsilon_2)}{\epsilon_1(1+2\epsilon_2)}.$$

(ii) $\overline{\mathcal{M}}_c(L_r)$ is empty when

$$0 < r < \frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2}.$$

By the Hirzebruch-Riemann-Roch theorem,

$$\epsilon_1=0,\ \epsilon_2=1 \Rightarrow \chi(E,E)=-6,\quad k=4,$$

$$\epsilon_1=1,\ \epsilon_2=1 \Rightarrow \chi(E,E)=-16,\ k=9,$$

$$\epsilon_1=0,\ \epsilon_2=0 \Rightarrow \chi(E,E)=-26,\ k=14,$$

$$\epsilon_1=1,\ \epsilon_2=0 \Rightarrow \chi(E,E)=-56,\ k=29.$$

By Lemma 5.2 [58] and simple computations, $k = dimExt^1(E,E)$, $Ext^2(E,E)=0$ in all above four cases. Thus $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} = \overline{\mathcal{M}}_c(L_r)$ is compact smooth whose virtual fundamental class is the usual fundamental class of $\overline{\mathcal{M}}_c(L_r)$.

Theorem 7.2. *Let X be a generic smooth hyperplane section in $\mathbb{P}^1\times\mathbb{P}^4$ of $(2,5)$ type. Let*

$$cl = [1+(-1,1)|_X]\cdot[1+(\epsilon_1+1,\epsilon_2-1)|_X],$$

$$k=(1+\epsilon_1)\left(\begin{array}{c} 6-\epsilon_2 \\ 4 \end{array}\right),\quad \epsilon_1,\epsilon_2=0,1,\quad L_r=\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(1,r)|_X.$$

Denote $\overline{\mathcal{M}}_c(L_r)$ to be the moduli space of Gieseker L_r-semistable rank-2 torsion-free sheaves with Chern character c (which can be easily read from the total Chern class cl).
(1) *If*

$$\frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2} < r < \frac{15(2-\epsilon_2)}{\epsilon_1(1+2\epsilon_2)},$$

then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4}\cong\overline{\mathcal{M}}_c(L_r)\cong\mathbb{P}^k$, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}=[\mathbb{P}^k]$.

(2) *If*

$$0 < r < \frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2},$$

then $\overline{\mathcal{M}}_c^{DT_4}=\emptyset$ and $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}=0$.

Remark 7.3.
1. From the above example, we can see the $Spin(7)$ instanton moduli space is not spin in general.
2. Wall-crossing phenomenon exists in DT_4 theory.

Using μ-maps to define corresponding DT_4 invariants, we need the universal bundle of the moduli space which comes from the universal extension [49]

$$0 \to \pi_1^*L_2\otimes\pi_2^*\mathcal{O}(1) \to \mathcal{E} \to \pi_1^*L_1 \to 0,$$

$$L_1=\mathcal{O}_X(\epsilon_1+1,\epsilon_2-1), L_2=\mathcal{O}_X(-1,1),$$

where $\pi_1 : X\times\overline{\mathcal{M}}_c(L_r)\to X$, $\pi_2 : X\times\overline{\mathcal{M}}_c(L_r)\to\overline{\mathcal{M}}_c(L_r)$ are projection maps. The Chern class of the universal bundle $\mathcal{E}$ is given by $\big(1+\pi_1^*c_1(L_1)\big)\big(1+\pi_1^*c_1(L_2)+\pi_2^*c_1(\mathcal{O}(1))\big)$. We can then compute all DT_4 invariants in these examples.

7.2. DT_4/GW **correspondence in some special cases.** It is well known that we have equivalence between Donaldson-Thomas ideal sheaves invariants and Gromov-Witten invariants on Calabi-Yau three-folds [10] [60] [67] [78]. One may expect, there will be a similar gauge-string duality for Calabi-Yau four-folds. We will study such a correspondence in some special cases in this section. We fix $c = 1 - PD(\beta) - nPD(1)$, where $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. We first compute the real virtual dimension of $\overline{\mathcal{M}}_c^{DT_4}$ defined to be $2ext^1(I_C, I_C) - ext^2(I_C, I_C)$, $I_C \in \mathcal{M}_c$.

Lemma 7.4. *The real virtual dimension of* $\overline{\mathcal{M}}_c^{DT_4}$ *with* $c = 1 - PD(\beta) - nPD(1)$ *on a compact Calabi-Yau four-fold* X *satisfies*

$$v.d_{\mathbb{R}}(\overline{\mathcal{M}}_c^{DT_4}) = 2n, \qquad \text{if} \quad Hol(X) = SU(4),$$
$$v.d_{\mathbb{R}}(\overline{\mathcal{M}}_c^{DT_4}) = 2n - 1, \quad \text{if} \quad Hol(X) = Sp(2).$$

Proof. By the Hirzebruch-Riemann-Roch theorem,

$$\chi(I_C, I_C) = 2\int_X ch_4(I_C) + 2h^0(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X).$$

For $Hol(X) = Sp(2)$, we have

$$ext^1(I_C, I_C) - \frac{1}{2}ext^2(I_C, I_C) = n - \frac{1}{2}.$$

Similar calculations apply to the case when $Hol(X) = SU(4)$. ☐

Remark 7.5. *If* $Hol(X) = SU(4)$, $v.d_{\mathbb{R}}(\overline{\mathcal{M}}_c^{DT_4}) = 2n = 2(1 - g)$ *for ideal sheaves of smooth connected genus* g *curves.* DT_4 *invariants for such ideal sheaves with* $g \geq 2$ *vanish which coincides with the situation of Gromov-Witten invariants.*

We come to study DT_4/GW correspondence when $\mathcal{M}_c$ is smooth.

7.2.1. *The case of* $Hol(X) = SU(4)$. We start with one dimensional closed subschemes of X.

Lemma 7.6. *Let* X *be a compact Calabi-Yau four-fold with* $Hol(X) = SU(4)$. *Let* $C \hookrightarrow X$ *be a closed subscheme with* $dim_{\mathbb{C}} C \leq 1$ *and* $H^1(X, \mathcal{O}_C) = 0$, *where* $\mathcal{O}_C$ *is the structure sheaf of* C. *Then we have canonical isomorphisms*

$$Ext^i(I_C, I_C) \cong Ext^i(\mathcal{O}_C, \mathcal{O}_C), \quad i = 1, 2,$$

where I_C *is the ideal sheaf of* C *in* X.

Proof. Taking cohomology of the following short exact sequence

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

we get

$$\to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_C) \to H^{i+1}(X, I_C) \to .$$

We have $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$ by $Hol(X) = SU(4)$. $H^i(X, \mathcal{O}_C) = 0$ for $i = 2, 3, 4$, because $dim_{\mathbb{C}} \mathcal{O}_C \leq 1$. Thus

$$H^1(X, \mathcal{O}_C) \cong H^2(X, I_C), H^3(X, I_C) = 0, H^4(X, I_C) \cong H^4(X, \mathcal{O}_X). \tag{25}$$

Applying $Hom(\mathcal{O}_C, \cdot)$ to $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we have

$$\to Ext^i(\mathcal{O}_C, \mathcal{O}_X) \to Ext^i(\mathcal{O}_C, \mathcal{O}_C) \to Ext^{i+1}(\mathcal{O}_C, I_C) \to .$$

By Serre duality, $Ext^i(\mathcal{O}_C, \mathcal{O}_X) \cong Ext^{4-i}(\mathcal{O}_X, \mathcal{O}_C) = 0$ if $i = 0, 1, 2$. Hence

$$Ext^0(\mathcal{O}_C, I_C) = 0, Ext^k(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^{k+1}(\mathcal{O}_C, I_C), k = 0, 1. \tag{26}$$

Meanwhile $Ext^3(\mathcal{O}_C, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_C) = 0$ by our assumption, thus

$$Ext^2(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^3(\mathcal{O}_C, I_C). \tag{27}$$

Using

$$H^0(X, \mathcal{O}_C) \cong Ext^4(\mathcal{O}_C, \mathcal{O}_X) \twoheadrightarrow Ext^4(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^0(\mathcal{O}_C, \mathcal{O}_C)$$

and the above long exact sequence, we have

$$Ext^1(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^3(\mathcal{O}_C, \mathcal{O}_C) \cong Ext^4(\mathcal{O}_C, I_C). \tag{28}$$

Applying the functor $Hom(\cdot, I_C)$ to $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we get

$$\to Ext^i(\mathcal{O}_X, I_C) \to Ext^i(I_C, I_C) \to Ext^{i+1}(\mathcal{O}_C, I_C) \to .$$

By (25), we know $H^2(X, I_C) \cong H^1(X, \mathcal{O}_C) = 0$ and $H^3(X, I_C) = 0$. Hence

$$Ext^2(I_C, I_C) \cong Ext^3(\mathcal{O}_C, I_C). \tag{29}$$

The remaining sequence of the long exact sequence is

$$Ext^3(I_C, I_C) \cong Ext^4(\mathcal{O}_C, I_C), \tag{30}$$

because $H^4(X, \mathcal{O}_X) \cong Ext^4(\mathcal{O}_X, I_C) \twoheadrightarrow Ext^4(I_C, I_C)$ and they have the same dimensions.

By (27),(28),(29),(30), we are done. □

If we further assume C to be a connected smooth imbedded curve inside X, we have

Lemma 7.7. *If C is a connected genus zero smooth imbedded curve inside X, then we have canonical isomorphisms*

$$Ext^1(I_C, I_C) \cong H^0(C, \mathcal{N}_{C/X}),$$
$$Ext^2(I_C, I_C) \cong H^1(C, \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{N}_{C/X})^*.$$

Furthermore, under this identification, $H^1(C, \mathcal{N}_{C/X})$ is a maximal isotropic subspace of $Ext^2(I_C, I_C)$ with respect to the Serre duality pairing.

Proof. We have the local to global spectral sequence which degenerates at E_2 terms

$$\begin{aligned} Ext^*_X(\mathcal{O}_C, \mathcal{O}_C) &\cong H^*(X, \mathcal{E}xt^*_X(\mathcal{O}_C, \mathcal{O}_C)) \\ &\cong H^*(X, \iota_*(\wedge^* \mathcal{N}_{C/X})) \\ &\cong H^*(C, \wedge^* \mathcal{N}_{C/X}), \end{aligned}$$

where $\iota : C \hookrightarrow X$ is the imbedding map. Then we have

$$Ext^1(\mathcal{O}_C, \mathcal{O}_C) \cong H^0(C, \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{O}_C),$$
$$Ext^2(\mathcal{O}_C, \mathcal{O}_C) \cong H^0(C, \wedge^2 \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{N}_{C/X}).$$

By $H^1(X, \mathcal{O}_C) = 0$ and the perfect pairing $\wedge^2 \mathcal{N}_{C/X} \otimes \mathcal{N}_{C/X} \to \wedge^3 \mathcal{N}_{C/X} \cong \Omega_C$, we get the hoped canonical isomorphisms. As for the pairing, we have

$$Ext^2(I_C, I_C) \otimes Ext^2(I_C, I_C) \to Ext^4(I_C, I_C) \to H^4(X, \mathcal{O}_X),$$

where the last trace map is a canonical isomorphism. The above pairing can be identified with

$$(H^0(C, \wedge^2 \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{N}_{C/X})) \otimes (H^0(C, \wedge^2 \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{N}_{C/X})),$$
$$\to H^1(C, \wedge^3 \mathcal{N}_{C/X}) \cong \mathbb{C}.$$

Meanwhile

$$H^1(C, \mathcal{N}_{C/X}) \otimes H^1(C, \mathcal{N}_{C/X}) \to 0,$$

which shows $H^1(C, \mathcal{N}_{C/X})$ is a maximal isotropic subspace under the above identification. □

We have canonically identified the deformation and obstruction spaces of DT_4 theory and GW theory in the above case. If we have some further assumptions on moduli spaces, we have the following DT_4/GW correspondence by Definition 5.12 and Lemma 5.13.

Theorem 7.8. *Let X be a compact Calabi-Yau four-fold with $Hol(X) = SU(4)$. If $\mathcal{M}_c$ with given Chern character $c = (1, 0, 0, -PD(\beta), -1)$ is smooth and consists of ideal sheaves of smooth connected genus zero imbedded curves only, then $Ind_{\mathbb{C}}$ on $\mathcal{M}_c$ has a natural complex orientation $o(\mathcal{O})$. Assume the GW moduli space $\overline{\mathcal{M}}_{0,0}(X, \beta) \cong \mathcal{M}_c$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists, $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_{0,0}(X, \beta)$ and $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = [\overline{\mathcal{M}}_{0,0}(X, \beta)]^{vir}$.*

Remark 7.9. *If there exists an embedding $i : \mathbb{CP}^3 \hookrightarrow X$ and $\beta = i_*[H^2]$, where $[H^2] \in H_2(\mathbb{CP}^3)$ is the generator, ideal sheaves of curves in class β are always of the form I_C, where $C \hookrightarrow \mathbb{CP}^3 \hookrightarrow X$, by the negativity of the normal bundle $\mathcal{N}_{\mathbb{P}^3/X}$. Meanwhile $\mathcal{M}_c$ with $c = (1, 0, 0, -PD(\beta), -1)$, $\beta = i_*[H^2] \in H_2(X)$ is smooth. Hence conditions in Theorem 7.8 are satisfied in this case.*

We explicitly compute DT_4 invariants in this case.

Proposition 7.10. *Let $c = (1, 0, 0, -PD(\beta), -1) \in H^{even}(X, \mathbb{Z})$. If there exists an embedding $i : \mathbb{P}^3 \hookrightarrow X$ and $\beta = i_*[H^2]$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_{0,0}(K_{\mathbb{P}^3}, [H^2])$. Furthermore, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = [\overline{\mathcal{M}}_{0,0}(K_{\mathbb{P}^3}, [H^2])]^{vir}$.*

Proof. By the positivity of $TC \cong T\mathbb{P}^1$, we have $H^1(C, \mathcal{N}_{C/X}) \cong H^1(C, \iota^* TX)$, where $\iota : C \hookrightarrow X$. Meanwhile,

$$0 \to T\mathbb{P}^3 \to TX\mid_{\mathbb{P}^3} \to \mathcal{N}_{\mathbb{P}^3/X} \to 0$$

induces $H^1(C, \iota^* TX) \cong H^1(C, \iota^* K_{\mathbb{P}^3})$ which is the obstruction space of Gromov-Witten theory of $K_{\mathbb{P}^3}$ as $H^1(C, \iota^* TK_{\mathbb{P}^3}) \cong H^1(C, \iota^* K_{\mathbb{P}^3})$. □

7.2.2. *The case of $Hol(X) = Sp(2)$.* When $Hol(X) = Sp(2)$, we similarly have

Lemma 7.11. *Let X be a compact irreducible hyper-Kähler four-fold. Under the assumption that $C \hookrightarrow X$ is a closed subscheme with $dim_{\mathbb{C}} C \leq 1$ and $H^1(X, \mathcal{O}_C) = 0$, we have*

$$Ext^1(I_C, I_C) \cong Ext^1(\mathcal{O}_C, \mathcal{O}_C),$$

$$0 \to H^2(X, \mathcal{O}_X) \to Ext^2(I_C, I_C) \to Ext^2(\mathcal{O}_C, \mathcal{O}_C) \to 0.$$

Lemma 7.12. *If C is a connected genus zero smooth imbedded curve inside X, then we have*

$$Ext^1(I_C, I_C) \cong H^0(C, \mathcal{N}_{C/X}),$$

$$0 \to H^2(X, \mathcal{O}_X) \to Ext^2(I_C, I_C) \to H^1(C, \mathcal{N}_{C/X}) \oplus H^1(C, \mathcal{N}_{C/X})^* \to 0.$$

The GW obstruction space $H^1(C, \mathcal{N}_{C/X}) \cong H^1(C, \iota^* TX)$, if $C \cong \mathbb{P}^1$. Meanwhile,

$$0 \to \mathcal{N}^*_{C/X} \to \iota^* \Omega_X \to \Omega_C \to 0$$

induces

$$0 \cong H^0(C, \Omega_C) \to H^1(C, \mathcal{N}^*_{C/X}) \to H^1(C, \iota^* \Omega_X) \cong H^1(C, \iota^* TX) \to H^1(C, \Omega_C) \to 0,$$

where the second isomorphism is by the existence of holomorphic symplectic 2-form $\sigma \in H^0(X, \Omega^2_X)$. The above sequence establishes the surjective cosection of the obstruction sheaf of GW theory for hyper-Kähler manifolds [41], [42].

As vector spaces,

$$Ext^2(I_C, I_C) \cong H^1(C, \mathcal{N}^*_{C/X}) \oplus H^1(C, \Omega_C) \oplus H^1(C, \mathcal{N}^*_{C/X})^* \oplus H^1(C, \Omega_C)^* \oplus H^2(X, \mathcal{O}_X). \tag{31}$$

Taking away trivial factors and restrict to the maximal isotropic subspace, we define the hyper-reduced DT_4 obstruction space.

Definition 7.13. *The hyper-reduced DT_4 obstruction space is*

$$Ext^2_{\text{hyper-red}}(I_C, I_C) \triangleq H^1(C, \mathcal{N}^*_{C/X})$$

Remark 7.14. *The hyper-reduced DT_4 obstruction space coincides with the reduced GW obstruction space* [42].

Definition 7.15. *The hyper-reduced virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$, denoted by $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}_{hyper-red}$ is the Poincaré dual of the Euler class of the hyper-reduced obstruction bundle.*

Theorem 7.16. *Let X be a compact irreducible hyper-Kähler four-fold. If $\mathcal{M}_c$ with given Chern character $c = (1, 0, 0, -PD(\beta), -1)$ is smooth and consists of ideal sheaves of smooth connected genus zero imbedded curves only, then $Ind_{\mathbb{C}}$ on $\mathcal{M}_c$ has a natural complex orientation $o(\mathcal{O})$. Assume the GW moduli space $\overline{\mathcal{M}}_{0,0}(X, \beta) \cong \mathcal{M}_c$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists, $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_{0,0}(X, \beta)$ and $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0$.*

Furthermore, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}_{hyper-red} = [\overline{\mathcal{M}}_{0,0}(X, \beta)]^{vir}_{red}$, where $[\overline{\mathcal{M}}_{0,0}(X, \beta)]^{vir}_{red}$ is the reduced virtual fundamental class of the GW moduli space [42].

A simple application is the following result due to Mukai [65].

Corollary 7.17. $\mathbb{P}^3$ *can't be embedded into any compact irreducible hyper-Kähler four-fold.*

Proof. We take away the trivial factor $H^2(X, \mathcal{O}_X)$ and consider the maximal isotropic subspace

$$\begin{aligned} Ext^2_{red}(I_C, I_C) &\triangleq H^1(C, \mathcal{N}_{C/X}) \\ &= H^1(C, \iota^* TX) \\ &= H^1(C, \iota^* K_{\mathbb{P}^3}). \end{aligned}$$

The Euler class of the GW obstruction bundle with fiber $H^1(C, \iota^* K_{\mathbb{P}^3})$ is not trivial by localization calculation [47].

By the hyper-Kähler condition, $H^1(C, \iota^* TX)$ has a surjective map to $H^1(C, \Omega_C)$ which leads to the vanishing of the virtual cycle. $\square$

7.3. **Moduli spaces of ideal sheaves of one point.** In this subsection, we consider the Hilbert scheme of one point on X, i.e. $\mathcal{M}_c = X$. Here we prefer considering the moduli space of structure sheaf of one point.

Proposition 7.18. *If $Hol(X) = SU(4)$ and $c = (1,0,0,0,-1)$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong X$, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = \pm PD\big(c_3(X)\big)$.*

Proof. By the standard Koszul resolution, we have

$$\cdots \to \mathcal{O} \otimes \wedge^2 T_p^* \to \mathcal{O} \otimes T_p^* \to \mathcal{O} \to \mathcal{O}_p \to 0.$$

Then

$$\begin{aligned} Ext^i(\mathcal{O}_p, \mathcal{O}_p) &\cong Ext^i(\mathcal{O} \otimes \wedge^\bullet T_p^*, \mathcal{O}_p) \\ &\cong H^i(\mathcal{O} \otimes \wedge^\bullet T_p \otimes \mathcal{O}_p) \\ &\cong \wedge^i T_p \otimes \mathcal{O}_p. \end{aligned}$$

The last equality is because differentials in Koszul complex are 0 at point p. The above isomorphism is canonical and we can identify the obstruction bundle as $\wedge^2 T\mathcal{M}_c \cong \wedge^2 TX$.

As $SU(4) = Spin(6)$, we take V to be a bundle of fundamental representations of $Spin(6)$ on X such that

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong \wedge^2 T^* X.$$

We take a complex bundle U such that V is its underlying real bundle, then the spinor bundle $S^+(V) = \wedge^{even} U \otimes K^{\frac{1}{2}}$, where $K = \wedge^3 U^*$ and $c_3(S^+(V)) = -c_3(U)$.

If we identify $T^*X \cong S^+(V)$, we get $e(V) = c_3(U) = c_3(X)$. Since $Ob_+ \triangleq \wedge^2_+ TX \cong V^*$, we have $e(Ob_+) = e(V^*) = -c_3(X)$. $\square$

Remark 7.19. *If $Hol(X) = Sp(2)$, $v.d_{\mathbb{R}}(\overline{\mathcal{M}}_c^{DT_4}) = 1$ by Lemma 7.4. Fixing determinants of ideal sheaves, $\overline{\mathcal{M}}_c^{DT_4}$ has real virtual dimension 2 with still trivial reduced virtual cycle.*

8. Equivariant DT_4 invariants on toric CY_4 via localization

In this section, we restrict to the moduli space of ideal sheaves of curves $I_n(X,\beta)$ (the whole section also works for moduli spaces of ideal sheaves of surfaces in X), where X is a toric Calabi-Yau four-fold. By definition, X admits a $(\mathbb{C}^*)^4$-action which can be naturally lifted to the moduli space. If we restrict to the three dimensional sub-torus $T \subseteq (\mathbb{C}^*)^4$ which preserves the holomorphic top form of X, the action will also preserve the Serre duality pairing.

By the philosophy of virtual localization [30], we will define the corresponding equivariant DT_4 invariants. Roughly speaking, we should have

$$\int_{[\overline{\mathcal{M}}_{n,\beta}^{DT_4}(X)]^{vir}} \prod_{i=1}^{r} \gamma_i \approx \sum_{[\mathcal{I}] \in I_n(X,\beta)^T} \int_{[S(\mathcal{I})]^{vir}} \prod_{i=1}^{r} \gamma_i|_{\mathcal{I}} \cdot \frac{\sqrt{e_T(Ext^2(\mathcal{I},\mathcal{I}))}}{e_T(Ext^1(\mathcal{I},\mathcal{I}))},$$

where $\overline{\mathcal{M}}_{n,\beta}^{DT_4}(X)$ denotes the generally undefined generalized DT_4 moduli space whose reduced structure is the same as the reduced structure of $I_n(X,\beta)$ and γ_i are certain insertion fields we only define on the right hand side.

By a similar argument as Lemma 6, 8 in [59], one can show that for $\mathcal{I} \in I_n(X,\beta)^T$ which is a T-fixed point, T-representations

$$Ext^1(\mathcal{I},\mathcal{I}), \quad Ext^2(\mathcal{I},\mathcal{I})$$

contain no trivial sub-representations. Hence when we are reduced to local contributions, we can get ride of the non-reduced structure and consider everything on $I_n(X,\beta)$ instead of on the undefined generalized DT_4 moduli space.

For $\mathcal{I} \in I_n(X,\beta)^T$, we form the following complex vector bundle over BT whose fiber is $V_{\mathcal{I}} \triangleq Ext^2(\mathcal{I},\mathcal{I})$,

$$\begin{array}{c} ET \times_T V_{\mathcal{I}} \\ \downarrow \\ ET \times_T \{\mathcal{I}\} = BT. \end{array}$$

The Serre duality pairing naturally induces a non-degenerate pairing Q_{Serre} on $ET \times_T V_{\mathcal{I}}$ as T preserves the holomorphic top form. Thus, $(ET \times_T V_{\mathcal{I}}, Q_{Serre})$ becomes a quadratic bundle (vector bundle with a non-degenerate quadratic form).

By the theory of characteristic classes of quadratic bundles [24], there exists a half Euler class $e(ET \times_T V_{\mathcal{I}}, Q_{Serre})$ if $c_1(ET \times_T V_{\mathcal{I}}) = 0$. In fact, we have

Lemma 8.1. *For any $\mathcal{I} \in I_n(X,\beta)^T$, $c_1(ET \times_T V_{\mathcal{I}}) = 0$.*

Proof. By the Serre duality, $ET \times_T V_{\mathcal{I}} \cong (ET \times_T V_{\mathcal{I}})^*$ which implies

$$2c_1(ET \times_T V_{\mathcal{I}}) = 0.$$

Meanwhile, we know that $H^2(BT,\mathbb{Z})$ is torsion free. □

Definition 8.2. *For $\mathcal{I} \in I_n(X,\beta)^T$,*

$$e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I})) \triangleq e(ET \times_T V_{\mathcal{I}}, Q_{Serre}) \in H^*_T(pt)$$

is the half Euler class of the above quadratic bundle [24]*, where Q_{Serre} denotes the induced Serre duality pairing.*

Remark 8.3.
1. If $dim_{\mathbb{C}} V_{\mathcal{I}}$ is odd, the half Euler class is zero. If $dim_{\mathbb{C}} V_{\mathcal{I}}$ is even, the half Euler class is unique up to a sign which is determined by an $SO(N,\mathbb{C})$ reduction of the structure group of the quadratic bundle

$$(ET \times_T V_{\mathcal{I}}, Q_{Serre}) \to ET \times_T \{\mathcal{I}\},$$

where $N = dim_{\mathbb{C}} V_{\mathcal{I}}$.
2. If $dim_{\mathbb{C}} V_{\mathcal{I}}$ is even, by Proposition 2 of [24]*,*

$$\big(e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I}))\big)^2 = (-1)^{\frac{dim_{\mathbb{C}} V_{\mathcal{I}}}{2}} e_T(Ext^2(\mathcal{I},\mathcal{I})) \in H^*(BT).$$

Meanwhile, $H^*(BT)$ is a polynomial ring, thus $e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I}))$ is uniquely determined by $e_T(Ext^2(\mathcal{I},\mathcal{I}))$ up to a sign corresponding to the choice of an orientation. Hence we are essentially reduced to calculate $e_T(Ext^i(\mathcal{I},\mathcal{I}))$ for $i = 1,2$.

As we do not have Seidel-Thomas twist for toric CY_4, we do not know how to give a compatible orientation for different components of $ET \times_T I_n(X,\beta)^T$. We just arbitrarily choose an orientation for each component at the moment.

Definition 8.4. *The toric orientation data is a choice of $SO(N,\mathbb{C})$ reduction of the structure group of the quadratic bundle*

$$(ET \times_T V_{\mathcal{I}}, Q_{Serre}) \to ET \times_T \{\mathcal{I}\},$$

for each $\mathcal{I} \in I_n(X,\beta)^T$.

Definition 8.5. *Given $[\mathcal{I}] \in I_n(X,\beta)^T$, $P \in \mathbb{Z}[x_1,x_2,...]$ and $\gamma \in H^*_T(X,\mathbb{Z})$,*

$$\pi_*([\mathcal{I}],P,\gamma) \triangleq \pi_*\Big(P\big(c_i^T(\mathfrak{I}|_{[\mathcal{I}]\times X})\big) \cup \gamma\Big) \in H^*_T(pt),$$

where $\pi_* : H^*_T(X) \to H^*_T(pt)$ is the equivariant push-forward and $\mathfrak{I} \to I_n(X,\beta) \times X$ is the universal ideal sheaf.

Definition 8.6. *Given a toric Calabi-Yau four-fold X, $\beta \in H_2(X,\mathbb{Z})$, $n, r \in \mathbb{Z}_+$, polynomials $P_i \in \mathbb{Z}[x_1,x_2,...]$, insertion fields $\gamma_i \in H^*_T(X,\mathbb{Z})$ for $i = 1,2,...,r$ and a toric orientation data, the equivariant DT_4 invariant for ideal sheaves of curves associated with the above data is*

$$Z_{DT_4}\big(X,n\big|(P_1,\gamma_1),...,(P_r,\gamma_r)\big)_\beta \triangleq \sum_{[\mathcal{I}]\in I_n(X,\beta)^T} \left(\prod_{i=1}^r \big(\pi_*([\mathcal{I}],P_i,\gamma_i)\big)\right) \cdot \frac{e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I}))}{e_T(Ext^1(\mathcal{I},\mathcal{I}))},$$

where the sign in $e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I}))$ is compatible with the chosen toric orientation data.

Remark 8.7.
1. *The above definition can also be applied to define equivariant DT_4 invariants for ideal sheaves of surfaces (two dimensional closed subschemes) in toric CY_4 as T-fixed locus of moduli spaces are also isolated.*
2. *The insertions in the above definition is just one plausible choice. If we want to match it with the corresponding GW invariants, we may need some adjustment as toric 3-folds cases* [60].
3. *By Remark 8.3,*

$$\left(\frac{e_T(Ext^2_{iso}(\mathcal{I},\mathcal{I}))}{e_T(Ext^1(\mathcal{I},\mathcal{I}))}\right)^2 = \pm\frac{e_T(Ext^2(\mathcal{I},\mathcal{I}))}{e_T(Ext^3(\mathcal{I},\mathcal{I}))e_T(Ext^1(\mathcal{I},\mathcal{I}))}.$$

The RHS has a generalization to any dimensional smooth toric varieties.
4. *If the base manifold is an algebraic surface, moduli spaces of stable sheaves are smooth with vanishing obstruction spaces at least for $K3$ and del-Pezzo surfaces. Euler characteristics of moduli spaces are virtual invariants. For Calabi-Yau threefolds, if $\mathcal{M}_c$ is smooth, the DT_3 invariant is the Euler characteristic of the moduli space up to a sign.*

In the above two cases, Euler characteristics of moduli spaces in some sense represent sheaves virtual countings which should be partition functions of certain topological quantum field theories.

But for CY_4, Euler characteristics can not reflect the corresponding DT_4 invariants in general. We should consider the Euler class of the self-dual obstruction bundle and the quadratic form takes its role. Maybe this is one of the reasons we did not get a closed formula for the generating function of Euler characteristics of Hilbert schemes of points on a complex four-fold [18]. *We are wondering whether there are closed formulas for generating functions of DT_4 invariants for ideal sheaves of points.*

We calculate the simplest example for $X = \mathbb{C}^4$: take the sub-torus $T = \{(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4 \mid t_1t_2t_3t_4 = 1\}$ which acts on $\mathbb{C}^4$ by

$$(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3, x_4) = (t_1x_1, t_2x_2, t_3x_3, t_4x_4).$$

Let ρ_i be the character of $(\mathbb{C}^*)^4$ defined by $\rho_i(t_1, t_2, t_3, t_4) = t_i$ and λ_i be its weight. Then

$$H^*_T(pt, \mathbb{C}) \cong \mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]/\sum_{i=1}^{4} \lambda_i.$$

We consider the moduli space of ideal sheaves of one point on $\mathbb{C}^4$, $I_1(\mathbb{C}^4, 0) \cong \mathbb{C}^4$ on which T acts with only one fixed point I_0. It is easy to check

$$e_T(Ext^1(I_0, I_0)) = \lambda_1\lambda_2\lambda_3\lambda_4,$$
$$e_T(Ext^2(I_0, I_0)) = \prod_{1\leq i<j\leq 4} (\lambda_i + \lambda_j).$$

By Remark 8.3, we are left to find a square root of $-e_T(Ext^2(I_0, I_0))$. It is obvious that

$$-e_T(Ext^2(I_0, I_0)) = \big((\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\big)^2,$$

which gives

Proposition 8.8. *Let $X = \mathbb{C}^4$, for some choice of toric orientation data (Definition 8.4), we have*

$$Z_{DT_4}\big(X, 1\ |(1, 1)\big)_0 = \frac{\sigma_1\sigma_2 - \sigma_3}{\sigma_1\sigma_3},$$

where σ_i is the i-th elementary symmetric polynomial of variables $\lambda_1, \lambda_2, \lambda_3$.

Remark 8.9. *We remark that $c_3^T(\mathbb{C}^4) = \sigma_3 - \sigma_1\sigma_2$ and $c_4^T(\mathbb{C}^4) = \sigma_1\sigma_3$. Thus*

$$Z_{DT_4}\big(X, 1\ |(1, 1)\big)_0 = \pm\frac{c_3^T(\mathbb{C}^4)}{c_4^T(\mathbb{C}^4)},$$

which relates to the fact that the DT_4 virtual cycle for the moduli space of ideal sheaves of one point on X is $\pm PD(c_3(X))$ (Proposition 7.18).

9. Noncommutative DT_4 invariants

In the non-commutative world, sheaves on manifolds are replaced by representations of algebras. When algebras are CY_4 coming from path algebras of quivers with relations, we have a corresponding theory counting their representations. Our constructions are motivated by previous works of Szendröi [75] and Mozgovoy-Reineke [64] on the non-commutative Donaldson-Thomas theory for CY_3 algebras.

9.1. **Basic facts on quivers with relations.**

9.1.1. *Moduli spaces of representations of quivers with relations.* Let $Q = (Q_0, Q_1, h, t)$ be a quiver, where Q_0 is a finite set of vertices indexed by $0, 1, ..., n$, Q_1 is a finite set of oriented edges, and $h, t : Q_1 \to Q_0$ are maps giving the head and tail of each edge. The path algebra $\mathbb{C}Q$ is an associative algebra with basis consisting of all paths of finite length. A quiver with relations (Q, I) is a quiver Q with a two-sided ideal I in $\mathbb{C}Q$. We denote $A = \mathbb{C}Q/I$.

Definition 9.1. [39],[44] *Let $Q = (Q_0, Q_1, h, t)$ be a quiver. A representation of Q is a pair $(V = \bigoplus_{i=0}^{n} V_i, \phi = \bigoplus_{a\in Q_1} \phi_a)$ which consists of a finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$ and a $\mathbb{C}$-linear map $\phi_a : V_{t(a)} \to V_{h(a)}$ for each edge $a \in Q_1$. If (Q, I) is a quiver with relations, a representation of (Q, I) is a representation of Q, denoted by (V, ϕ) such that $\phi\ |_I= 0$. The dimension vector of (V, ϕ) is denoted by $\underline{dim}(V) = \boldsymbol{d} = (d_0, d_1, ..., d_n)$, where $d_i = dim_{\mathbb{C}}V_i$. We say (Q, I) has bounded relations if I is finitely generated.*

Remark 9.2. *Representations of (Q, I) are in one-one correspondence with finite-dimensional $A = \mathbb{C}Q/I$-modules.*

We fix a dimension vector $\mathbf{d}$ and consider the vector space

$$R_{\mathbf{d}}(Q) \triangleq \bigoplus_{a\in Q_1} Hom_{\mathbb{C}}(V_{t(a)}, V_{h(a)}),$$

on which the algebraic reductive group $G_{\mathbf{d}} = \prod_{i\in Q_0} GL(V_i)$ acts via

$$(g_i)\circ(\phi_a) = (g_{h(a)}\phi_a g_{t(a)}^{-1}).$$

To define a proper quotient, we introduce a stability condition due to [25], [44].

Definition 9.3. *We fix a linear form $\Theta \in (\mathbb{Z}^{n+1})^*$ (called a stability condition). The slope of a $\mathbb{C}Q$-module M with non-zero dimension vector $\underline{dim}M$ is*

$$\mu(M) \triangleq \mu(\underline{dim}M) \triangleq \frac{\Theta(\underline{dim}M)}{\sum_{i=0}^{n} dim_{\mathbb{C}} M_i}.$$

M is μ-stable (resp. μ-semi-stable), if

$$\mu(U) < \mu(M)\ (resp.\ \mu(U) \leq \mu(M))$$

for any non-zero proper sub-representation U of M.

We denote $R^s_{\mathbf{d}}(Q) \subseteq R_{\mathbf{d}}(Q)$ to be the subspace of μ-stable representations of Q. By [25], [44], there exists a smooth complex algebraic variety

$$\mathcal{M}^s_{\mathbf{d}}(Q) \triangleq R^s_{\mathbf{d}}(Q)/G_{\mathbf{d}}$$

parametrizing isomorphism classes of μ-stable representations of Q $\big(\text{not } (Q,I)\big)$ with dimension vector $\mathbf{d}$. We denote $\mathcal{M}^{ss}_{\mathbf{d}}(Q)$ to be the coarse moduli space of μ-semi-stable representations of Q. It contains $\mathcal{M}^s_{\mathbf{d}}(Q)$ as an open subvariety.

As in the case of moduli spaces of sheaves [36], we want to find out a condition so that $\mathcal{M}^s_{\mathbf{d}}(Q) = \mathcal{M}^{ss}_{\mathbf{d}}(Q)$.

Definition 9.4. [25] *A dimension vector $\boldsymbol{d} \in \mathbb{N}^{n+1}$ is coprime for $\Theta \in (\mathbb{Z}^{n+1})^*$ if $\mu(\boldsymbol{e}) \neq \mu(\boldsymbol{d})$ for all $0 < \boldsymbol{e} < \boldsymbol{d}$, where $\boldsymbol{e} < \boldsymbol{d}$ means that $e_i \leq d_i$ for all $0 \leq i \leq n$ and $e_i < d_i$ for some i.*

Lemma 9.5. [25]
(1) If $\boldsymbol{d}$ is coprime for Θ, then $\mathcal{M}^{ss}_{\boldsymbol{d}}(Q) = \mathcal{M}^s_{\boldsymbol{d}}(Q)$ is a connected smooth algebraic variety.
(2) If the quiver Q does not have oriented cycles, then $\mathcal{M}^{ss}_{\mathbf{d}}(Q)$ is a projective variety.

Proof. When $\Theta = 0$, $\mathcal{M}^{ss}_{\mathbf{d}}(Q)$ will be denoted by $\mathcal{M}^{ssimple}_{\mathbf{d}}(Q)$ which parametrizes semisimple representations of Q. It is affine and contains a special element,

$$\vec{0} \triangleq \bigoplus_{i=0}^{n} S_i \otimes V_i \in \mathcal{M}^{ssimple}_{\mathbf{d}}(Q).$$

We consider a quotient map $\pi : R_{\mathbf{d}}(Q) \to \mathcal{M}^{ssimple}_{\mathbf{d}}(Q)$, sending a representation to the isomorphism class of its semi-simplification (i.e. the direct sum of its Jordan-Hölder components). Then the fiber $\pi^{-1}(\vec{0})$ consists of nilpotent representations, i.e. representations with vanishing traces along all non-trivial oriented cycles.

By [25], π is descended to $p : \mathcal{M}^{ss}_{\mathbf{d}}(Q) \to \mathcal{M}^{ssimple}_{\mathbf{d}}(Q)$ and p is a projective morphism. If Q does not have oriented cycles, $\mathcal{M}^{ssimple}_{\mathbf{d}}(Q)$ is a single point and $\mathcal{M}^{ss}_{\mathbf{d}}(Q)$ is projective. $\square$

Our aim is to study moduli spaces of representations of (Q, I).

Definition 9.6.

$$\mathcal{M}^s_{\boldsymbol{d}}(Q,I) \triangleq R^s_{\boldsymbol{d}}(Q,I)/G_{\boldsymbol{d}},$$

where $R^s_{\boldsymbol{d}}(Q,I) \subseteq R^s_{\boldsymbol{d}}(Q)$ consists of μ-stable representations of (Q,I).

The obstruction theory of $\mathcal{M}^s_{\mathbf{d}}(Q,I)$ is governed by L_∞-algebra:

$$L^*_{\mathbf{d}} \triangleq Ext^*_A\Big(\bigoplus_{i=0}^{n} S_i \otimes V_i, \bigoplus_{i=0}^{n} S_i \otimes V_i\Big), \tag{32}$$

where S_i is the simple $A = \mathbb{C}Q/I$ module corresponding to node $i \in Q_0$, V_i is a complex vector space of dimension d_i [33], [40], [71].

Definition 9.7. *The obstruction bundle of $\mathcal{M}^s_{\boldsymbol{d}}(Q,I)$ is*

$$Ob \triangleq (L^1_{\boldsymbol{d}})^s \times_{G_{\boldsymbol{d}}} L^2_{\boldsymbol{d}}. \tag{33}$$

Remark 9.8. *Ob is over $\mathcal{M}^s_{\boldsymbol{d}}(Q)$ with fiber $L^2_{\boldsymbol{d}}$ as $L^1_{\boldsymbol{d}}$ is canonically identified with $R_{\boldsymbol{d}}(Q)$.*

The Maurer-Cartan equation of $L_{\mathbf{d}}$ converges provided (Q,I) has bounded relations as $L^2_{\mathbf{d}}$ computes relations in the quiver. In this case, we have

Lemma 9.9. [33], [71] *We assume $\boldsymbol{d}$ is coprime for Θ and (Q,I) has bounded relations, then $\mathcal{M}^s_{\boldsymbol{d}}(Q,I) \cong MC(L_{\boldsymbol{d}})//_{\Theta}G_{\boldsymbol{d}}$.*

Another way to phrase Lemma 9.9 is that there exists a section of the obstruction bundle Ob over $\mathcal{M}^s_{\mathbf{d}}(Q)$ whose zero loci is $\mathcal{M}^s_{\mathbf{d}}(Q,I)$, i.e.

$$\begin{array}{ccc} & & Ob \\ & & \downarrow \\ \mathcal{M}^s_{\mathbf{d}}(Q,I) & \hookrightarrow & \mathcal{M}^s_{\mathbf{d}}(Q), \end{array} \tag{34}$$

which will be used to construct virtual cycles of $\mathcal{M}^s_{\mathbf{d}}(Q,I)$ when (Q,I) satisfies the CY_4 condition.

9.1.2. *Framed quiver moduli spaces.* Moduli spaces of semi-stable representations are in general not schemes unless under the coprime assumption on the stability (Lemma 9.5). This will cause difficulties for applying the virtual theory developed by Li-Tian [55] and Behrend-Fantechi [6].

To overcome this, we introduce smooth models of quiver moduli spaces in the sense of Engel and Reineke [25].

Definition 9.10. [25] *Given a datum $(Q,\boldsymbol{d},\Theta)$ as before and an extra non-zero dimension vector $\boldsymbol{e}=(e_0,e_1,...,e_n)\in\mathbb{Z}^{n+1}$, we associate to them a new datum $(\hat{Q},\hat{\boldsymbol{d}},\hat{\Theta})$ as follows:*

- *the vertices of $\hat{Q}$ are those of Q, together with one additional vertex ∞,*
- *the edges of $\hat{Q}$ are those of Q, together with e_i edges from ∞ to i, for each vertex i of Q,*
- *we define $\hat{d}_i=d_i$ for all $i=0,1,...,n$ and $\hat{d}_\infty=1$,*
- *we define $\hat{\Theta}_i=\Theta_i$ for all $i=0,1,...,n$ and $\hat{\Theta}_\infty=\frac{\Theta(\boldsymbol{d})}{\sum_{i=0}^n d_i}+\epsilon$ for some sufficiently small positive $\epsilon\in\mathbb{Q}$.*

We consider the vector space,

$$R_{\mathbf{d},\mathbf{e}}(Q) \triangleq \bigoplus_{a\in Q_1} Hom_{\mathbb{C}}(V_{t(a)},V_{h(a)}) \times \bigoplus_{i=0}^{n} Hom_{\mathbb{C}}(W_i,V_i),$$

where $dim_{\mathbb{C}}V_i=d_i$ and $dim_{\mathbb{C}}W_i=e_i$. $G_{\mathbf{d}}=\prod_{i\in Q_0}GL(V_i)$ acts on $R_{\mathbf{d},\mathbf{e}}(Q)$ and the GIT quotient exists

$$\mathcal{M}^{ss}_{\mathbf{d}}(\hat{Q}) = R_{\mathbf{d},\mathbf{e}}(Q)//_{\Theta}G_{\mathbf{d}}.$$

Remark 9.11.
1. By Lemma 3.2 [25], *$\mathcal{M}^{ss}_{\boldsymbol{d}}(\hat{Q})=\mathcal{M}^{s}_{\boldsymbol{d}}(\hat{Q})$. We denote $\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q)\triangleq\mathcal{M}^{s}_{\boldsymbol{d}}(\hat{Q})$ and call it the smooth model for $\mathcal{M}^{ss}_{\boldsymbol{d}}(Q)$.*
2. By Proposition 3.8 [25], *$\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q)$ is a projective bundle over $\mathcal{M}^{s}_{\boldsymbol{d}}(Q)$ if $\boldsymbol{d}$ is coprime for Θ.*

Definition 9.12. *Given a quiver with relations (Q,I), dimension vectors $\boldsymbol{d},\boldsymbol{e}\neq\vec{0}\in\mathbb{Z}^{n+1}_{\geq 0}$ and a stability condition $\Theta\in(\mathbb{Z}^{n+1})^*$, the moduli space of framed representations of (Q,I) is*

$$\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I) \triangleq R_{\boldsymbol{d},\boldsymbol{e}}(Q,I)//_{\Theta}G_{\boldsymbol{d}},$$

where $R_{\boldsymbol{d},\boldsymbol{e}}(Q,I)\triangleq Z(I)\times\bigoplus_{i=0}^n Hom_{\mathbb{C}}(W_i,V_i)$ and $Z(I)\subseteq\bigoplus_{a\in Q_1}Hom_{\mathbb{C}}(V_{t(a)},V_{h(a)})$ consists of representations of (Q,I).

Similar to (33), we can define the framed obstruction bundle over $\mathcal{M}^{\Theta}_{\mathbf{d},\mathbf{e}}(Q)$ with fiber $L^2_{\mathbf{d}}$.

Definition 9.13. *The framed obstruction bundle is*

$$Ob_{fr} \triangleq R^s_{\boldsymbol{d},\boldsymbol{e}}(Q)\times_{G_{\boldsymbol{d}}} L^2_{\boldsymbol{d}}.$$

Remark 9.14. *As in (34), Ob_{fr} has a section whose zero loci is $\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I)$, i.e.*

$$\begin{array}{ccc} & & Ob_{fr} \\ & & \downarrow \\ \mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I) & \hookrightarrow & \mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q). \end{array} \tag{35}$$

9.2. **The definition of $NCDT_4$ invariants.** We first introduce the concept of CY_4 algebras.

Definition 9.15. [29], [75] *An associative $\mathbb{C}$-algebra A is 4-Calabi-Yau (CY_4 for short), if for all $M, N \in A$-Mod (the category of finitely generated A-modules) with at least one of them finite dimensional, there exist perfect bi-functorial pairings*

$$Ext^k_A(M,N) \times Ext^{4-k}_A(N,M) \to \mathbb{C}$$

between finite-dimensional $\mathbb{C}$-vector spaces.

We take a quiver with bounded relations (Q,I) such that $A = \mathbb{C}Q/I$ is CY_4, then Ob admits a non-degenerate quadratic form q. By the characteristic class theory of quadratic bundles [24], [74], the half Euler class $e(Ob,q) \in H^{rk(Ob)/2}(\mathcal{M}^s_{\mathbf{d}}(Q),\mathbb{Z})$ exists if $c_1(Ob) = 0$.

Lemma 9.16. *The obstruction bundle Ob over $\mathcal{M}^s_{\boldsymbol{d}}(Q)$ satisfies $c_1(Ob) = 0$ if $H^1(\mathcal{M}^s_{\boldsymbol{d}}(Q),\mathbb{Z})$ does not have even order torsion.*

Proof. By the CY_4 condition, $Ob \cong Ob^*$, then $2c_1(Ob) = 0$. □

Remark 9.17. *The condition on torsions holds when Q does not have oriented cycles* [45].

Note that when $A = \mathbb{C}Q/I$ is CY_4, (Q,I) has oriented cycles. Thus $\mathcal{M}^{ss}_{\mathbf{d}}(Q,I)$ is non-compact for a general dimension vector $\mathbf{d}$. To take care the non-compactness, we take the Borel-Moore homology $H^{BM}_r(\mathcal{M}^s_{\mathbf{d}}(Q)) \cong H^{top-r}(\mathcal{M}^s_{\mathbf{d}}(Q))$ [28]. By the philosophy of defining virtual cycles in DT_4 theory before, we define

Definition 9.18. *We take a quiver with bounded relations (Q,I) such that $A = \mathbb{C}Q/I$ is a CY_4 algebra. We fix a dimension vector $\boldsymbol{d}$ and a stability condition Θ such that $\boldsymbol{d}$ is coprime for Θ. Assume $c_1(Ob) = 0$ and choose a $SO(N,\mathbb{C})$ reduction of the structure group of Ob, where $N = rk(Ob)$. Then the virtual fundamental class of $\mathcal{M}^s_{\boldsymbol{d}}(Q,I)$ is*

$$[\mathcal{M}^s_{\boldsymbol{d}}(Q,I)]^{vir} \triangleq PD(e(Ob,q)) \in H^{BM}_{v.d}(\mathcal{M}^s_{\boldsymbol{d}}(Q)), \tag{36}$$

where $v.d = 2ext^1_A(W,W) - 2ext^0_A(W,W) - ext^2_A(W,W) + 2$ and $W = \bigoplus_{i=0}^n S_i \otimes V_i$.
We further call $[\mathcal{M}^s_{\boldsymbol{d}}(Q,I)]^{vir}$ the non-commutative DT_4 ($NCDT_4$ for short) virtual cycle associated to $(Q,I,\boldsymbol{d},\Theta)$.

Remark 9.19. *Because of the non-compactness of $\mathcal{M}^s_{\boldsymbol{d}}(Q)$ for general $\boldsymbol{d}$, $NCDT_4$ invariants may be zero in general. However, if we choose $\boldsymbol{d}$ to be of special form (killing oriented cycles in quivers), $NCDT_4$ invariants are shown to be nontrivial (Proposition 9.26).*

For the case of framed quiver moduli spaces, we similarly have

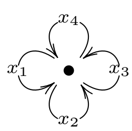
Definition 9.20. *We take a quiver with bounded relations (Q,I) such that $A = \mathbb{C}Q/I$ is a CY_4 algebra. We fix dimension vectors $\boldsymbol{d}, \boldsymbol{e} \neq \vec{0}$, and a stability condition Θ. Assume $c_1(Ob_{fr}) = 0$ and choose a $SO(N,\mathbb{C})$ reduction of the structure group of Ob_{fr}, where $N = rk(Ob_{fr})$. Then the virtual fundamental class of $\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I)$ is*

$$[\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I)]^{vir} \triangleq PD\big(e(Ob_{fr},q)\big) \in H^{BM}_{v.d+2\boldsymbol{d}\cdot\boldsymbol{e}-2}(\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q)), \tag{37}$$

where $v.d = 2ext^1_A(W,W) - 2ext^0_A(W,W) - ext^2_A(W,W) + 2$ for $W = \bigoplus_{i=0}^n S_i \otimes V_i$ and $\boldsymbol{d}\cdot\boldsymbol{e}$ is the inner product between vectors $\boldsymbol{d}$ and $\boldsymbol{e}$.
We further call $[\mathcal{M}^{\Theta}_{\boldsymbol{d},\boldsymbol{e}}(Q,I)]^{vir}$ the non-commutative DT_4 ($NCDT_4$ for short) virtual cycle associated to $(Q,I,\boldsymbol{d},\boldsymbol{e},\Theta)$.

9.3. **Computational examples of $NCDT_4$ invariants.**

9.3.1. *Quivers associated with $D^b(\mathbb{C}^4)$.* We consider a quiver Q consisting of a single vertex and 4 loops x_1, x_2, x_3, x_4 with trivial stability condition $\Theta = 0$,



We take an ideal $I = \langle x_i x_j - x_j x_i \rangle$ of $\mathbb{C}Q = \mathbb{C}\langle x_1,x_2,x_3,x_4\rangle$. Then the path algebra of (Q,I), $A \triangleq \mathbb{C}Q/I \cong \mathbb{C}[x_1,x_2,x_3,x_4]$ is a CY_4 algebra. By [25], [69], the smooth model $\mathcal{M}^0_{d,1}(Q)$ is the non-commutative Hilbert scheme $H^{(4)}_{d,1}$ containing the Hilbert scheme $\mathcal{M}^0_{d,1}(Q,I) \cong Hilb^d(\mathbb{C}^4)$.

Remark 9.21.
1. By Corollary 4.3 [69], $H^{odd}(\mathcal{M}^0_{d,1}(Q),\mathbb{Z}) = 0$. *By Lemma 9.16, the* $NCDT_4$ *virtual cycle exists (Definition 9.20).*
2. $[\mathcal{M}^0_{d,1}(Q,I)]^{vir}$ *could be viewed as* DT_4 *invariants of ideal sheaves of* d*-points on* $\mathbb{C}^4$ *(Section 1.6 of* [75]*).*

We first consider the simplest case of $[\mathcal{M}^0_{d,1}(Q,I)]^{vir}$. $R_{1,1}(Q) = \mathbb{C}^4 \times \mathbb{C}$, $G_1 = \mathbb{C}^*$ and

$$\mathcal{M}^0_{1,1}(Q) \triangleq R_{1,1}(Q)//G_1 \cong \mathbb{C}^4, \quad Ob_{fr} \cong \mathbb{C}^4 \times \wedge^2\mathbb{C}^4.$$

By $H^{\geq 1}(\mathbb{C}^4) = 0$, we get $[\mathcal{M}^0_{1,1}(Q,I)]^{vir} = 0$. In general, we have

Proposition 9.22. *We take a quiver* Q *consisting of a single vertex and* 4 *loops* x_1, x_2, x_3, x_4 *with relations* $I = \langle x_i x_j - x_j x_i \rangle$. *Then the path algebra*

$$A \triangleq \mathbb{C}Q/I \cong \mathbb{C}[x_1, x_2, x_3, x_4]$$

is a CY_4 *algebra. We fix the trivial stability condition* $\Theta = 0$ *and dimension vectors* $d \in \mathbb{Z}_{\geq 1}, e = 1$*, then the* $NCDT_4$ *virtual cycle of* $\mathcal{M}^0_{d,1}(Q,I)$ *exists and*

$$[\mathcal{M}^0_{d,1}(Q,I)]^{vir} = 0.$$

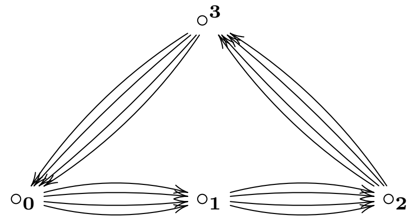
Proof. By [69], $\mathcal{M}^0_{d,1}(Q) = H^{(4)}_{d,1}$ is smooth irreducible of complex dimension $3d^2 + d$. By Definition 9.20, $[\mathcal{M}^0_{d,1}(Q,I)]^{vir} \in H^{BM}_{2d}(\mathcal{M}^0_{d,1}(Q))$.

The obstruction space $Ext^2(W,W) = (End(\mathbb{C}^d))^{\oplus 6}$ contains a trivial sub-representation $\mathbb{C}^{\oplus 6}$ of $GL_d(\mathbb{C})$, where $W = S_0 \otimes \mathbb{C}^d$ and S_0 is the simple module corresponding to the vertex. The sub-representation is preserved by the quadratic form q on $Ext^2(W,W)$ which gives $e(Ob,q) = 0$ (In fact, we furthermore know $H^{BM}_{2d}(\mathcal{M}^0_{d,1}(Q)) = 0$ by Corollary 4.3 [69]). $\square$

Remark 9.23. *The vanishing result is related to* $\mathbb{C}^4$ *is hyper-Kähler. The trivial sub-representation of* $Ext^2(W,W)$ *is the non-commutative analog of the cosection map of* DT_4 *theory on hyper-Kähler four-folds (18). Note the equivariant version of* $[\mathcal{M}^0_{d,1}(Q,I)]^{vir}$ *defined in Definition 8.6 is not trivial (Proposition 8.8).*

9.3.2. *Quivers associated with* $D^b(K_{\mathbb{P}^3})$. We take a quiver Q

(38)



with relations $I =< y_j x_i - y_i x_j, z_j y_i - z_i y_j, w_j z_i - w_i z_j, x_j w_i - x_i w_j >$, where x_i, y_i, z_i, w_i denote paths between vertices $0 \to 1$, $1 \to 2$, $2 \to 3$, $3 \to 0$ in the above figure respectively. The path algebra is CY_4 and associates with $K_{\mathbb{P}^3}$ by the existence of the full exceptional collection on $\mathbb{P}^3$ [11], [29].

Fixing the dimension vector $\mathbf{d} = (d_0, d_1, d_2, d_3)$ and a stability condition Θ, we get

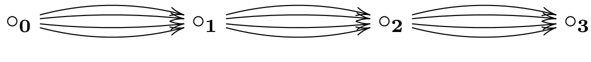
$$\begin{array}{rcl} Ext^2_A(W,W) \to & Ob & \\ & \downarrow & \\ \mathcal{M}^s_{\mathbf{d}}(Q,I) \hookrightarrow & \mathcal{M}^s_{\mathbf{d}}(Q) & = (Ext^1_A(W,W))^s/G_{\mathbf{d}}, \end{array}$$

where $W = \oplus_{i=0}^3 S_i \otimes V_i$, S_i is the simple A-module corresponding to vertex i, V_i is a complex vector space of dimension d_i. Explicitly, we have

$$Ext^1_A(W,W) \cong (V_0^* \otimes V_1)^{\oplus 4} \oplus (V_1^* \otimes V_2)^{\oplus 4} \oplus (V_2^* \otimes V_3)^{\oplus 4} \oplus (V_3^* \otimes V_0)^{\oplus 4},$$
$$Ext^2_A(W,W) \cong [(V_0^* \otimes V_2) \oplus (V_2^* \otimes V_0)]^{\oplus 6} \oplus [(V_1^* \otimes V_3) \oplus (V_3^* \otimes V_1)]^{\oplus 6}.$$

The obstruction space has a natural quadratic form q with $(V_0^* \otimes V_2)^{\oplus 6} \oplus (V_1^* \otimes V_3)^{\oplus 6}$ as a maximal isotropic subspace. Thus, $c_1(Ob) = 0$ [24] and the $NCDT_4$ virtual cycle of $(Q, I, \mathbf{d}, \Theta)$ exists by Definition 9.18.

For convenience of latter discussions, we introduce the $NCDT_3$ virtual cycle of the quiver (Q', I'), where (Q', I') is the quiver obtained from (Q, I) by eliminating edges connecting the vertex 3 and the vertex 0, i.e.

 .

Definition 9.24. *Given the above quiver with relations (Q', I'), dimension vector $\boldsymbol{d}$ and a stability condition Θ such that $\boldsymbol{d}$ and Θ are coprime, the $NCDT_3$ virtual cycle associated with $(Q', I', \boldsymbol{d}, \Theta)$ is*

$$[\mathcal{M}^s_{\boldsymbol{d}}(Q', I')]^{vir} \triangleq PD\big(e(Ob)\big) \in H_{v.d}(\mathcal{M}^s_{\boldsymbol{d}}(Q'), \mathbb{Z}).$$

where $Ob = Ext^2_{A'}(W,W) \times_{G_{\boldsymbol{d}}} (Ext^1_{A'}(W,W))^s$ and $A' = \mathbb{C}Q'/I'$ is the path algebra of (Q', I'). $v.d \triangleq 2ext^1_{A'}(W,W) - 2ext^0_{A'}(W,W) - ext^2_{A'}(W,W) + 2$.

Remark 9.25. *In the above definition, $\mathcal{M}^s_{\boldsymbol{d}}(Q')$ is compact since the quiver does not have oriented cycles.*

We compute $NCDT_4$ virtual cycles for some dimension vectors and stability conditions.

Proposition 9.26. $(NCDT_4/NCDT_3)$
We take the quiver with relations (Q, I) in figure 38, a stability condition Θ which is coprime to $\boldsymbol{d} = (d_0, d_1, d_2, d_3)$, where $d_0, d_1, d_2 \geq 1$, $d_3 = 0$, then $\mathcal{M}^s_{\boldsymbol{d}}(Q) \cong \mathcal{M}^s_{\boldsymbol{d}}(Q')$ as smooth complex projective varieties. Using the natural complex orientation, we further have

$$[\mathcal{M}^s_{(d_0,d_1,d_2,0)}(Q, I)]^{vir} = [\mathcal{M}^s_{(d_0,d_1,d_2,0)}(Q', I')]^{vir},$$

where (Q', I') is the quiver obtained from Q by eliminating edges connecting the vertex 3 and the vertex 0. $[\mathcal{M}^s_{(d_0,d_1,d_2,0)}(Q', I')]^{vir}$ is the $NCDT_3$ virtual cycle of (Q', I') (Definition 9.24).

Proof. We obviously have $\mathcal{M}^s_{\mathbf{d}}(Q) \cong \mathcal{M}^s_{\mathbf{d}}(Q')$, $Ext^2_A(W,W) = [(V_0^* \otimes V_2) \oplus (V_2^* \otimes V_0)]^{\oplus 6}$. Meanwhile the obstruction space of (Q', I'), $Ext^2_{A'}(W,W) = (V_0^* \otimes V_2)^{\oplus 6}$ is an isotropic subspace of $Ext^2_A(W,W)$. By Lemma 5.13, we are done. $\square$

Remark 9.27. *The quiver (Q', I') is associated with $\mathbb{P}^3$ by the existence of full exceptional collections* [11]. *Proposition 9.26 is the non-commutative version of DT_4/DT_3 correspondence proved in Theorem 6.5.*

10. Appendix

10.1. **Local Kuranishi models of $\mathcal{M}^{bdl}_c$.** We review several local Kuranishi models of $\mathcal{M}^{bdl}_c$.

A Kuranishi model of $\mathcal{M}^{bdl}_c$ with gauge fixing condition $\overline{\partial}^*_A a'' = 0$. By [27], we define

$$\kappa : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE),$$

$$\kappa(\alpha) = \mathbb{H}^{0,2}\big(g^{-1}(\alpha) \wedge g^{-1}(\alpha)\big), \tag{39}$$

where

$$g : \Omega^{0,1}(X, EndE)_k \to \Omega^{0,1}(X, EndE)_k,$$
$$g(a'') \triangleq a'' + \overline{\partial}_A^{-1} P_{\overline{\partial}_A}(a'' \wedge a'').$$

Note that $\overline{\partial}_A : Im(\overline{\partial}^*_A) \to \Omega^{0,2}(X, EndE)_{k-1}$ is an isomorphism onto its image and $\overline{\partial}_A^{-1}$ is defined as its inverse. By the standard Kuranishi theory, we have a local isomorphism

$$\kappa^{-1}(0) \cong \Big\{ \; a'' \;\Big|\; \|a''\|_k < \epsilon'', F^{0,2}(\overline{\partial}_A + a'') = 0, \; \overline{\partial}^*_A a'' = 0 \; \Big\}.$$

A Kuranishi model of $\mathcal{M}^{bdl}_c$ with gauge fixing condition $F \wedge \omega^3 = 0$. If we use gauge fixing $F \wedge \omega^3 = 0$, $d^*_A a = 0$, we have another Kuranishi model of $\mathcal{M}^{bdl}_c$ at $\overline{\partial}_A$: We define

$$\widetilde{\kappa} : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE),$$

$$\widetilde{\kappa}(\alpha) = \mathbb{H}^{0,2}\big(\widetilde{g}^{-1}(\alpha) \wedge \widetilde{g}^{-1}(\alpha)\big), \tag{40}$$

where

$$\widetilde{g} : \Omega^{0,1}(EndE)_k \to H^{0,1}(EndE) \oplus \overline{\partial}^*_A \Omega^{0,1}(EndE)_k \oplus \overline{\partial}^*_A \Omega^{0,2}(EndE)_{k-1},$$
$$\widetilde{g}(a'') \triangleq \bigg(\mathbb{H}(a''), \overline{\partial}^*_A a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a'), \overline{\partial}^*_A\big(\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'')\big)\bigg).$$

By a suitable complex gauge transformation, we have $\widetilde{\kappa}^{-1}(0) \cong \kappa^{-1}(0)$.

Another Kuranishi model of $\mathcal{M}^{bdl}_c$ with gauge fixing condition $F \wedge \omega^3 = 0$. Induced from the DT_4 equations, we have the following local Kuranishi model of $\mathcal{M}^{bdl}_c$ at $\overline{\partial}_A$: We define

$$\tilde{\tilde{\kappa}} : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE),$$

$$\tilde{\tilde{\kappa}}(\alpha) = \mathbb{H}^{0,2}\big(q^{-1}(\alpha) \wedge q^{-1}(\alpha)\big), \tag{41}$$

where

$$q : \Omega^{0,1}(EndE)_k \to H^{0,1}(EndE) \oplus \overline{\partial}_A^* \Omega^{0,1}(EndE)_k \oplus \overline{\partial}_A^* \Omega^{0,2}(EndE)_{k-1},$$

$$q(a'') \triangleq \Big(\mathbb{H}(a''), \overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a'), \overline{\partial}_A^* \big(\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'')\big)\Big).$$

By Proposition 3.11 and Definition 3.12, $\tilde{\tilde{\kappa}}$ is a Kuranishi map of $\mathcal{M}_c^{bdl}$ at $\overline{\partial}_A$.

10.2. **Seidel-Thomas twists.** In this subsection, we recall Seidel-Thomas twists [39], [72].

Definition 10.1. *Let $(X, \mathcal{O}_X(1))$ be a projective Calabi-Yau m-fold with $Hol(X) = SU(m)$. For each $n \in \mathbb{Z}$, the Seidel-Thomas twist $T_{\mathcal{O}_X(-n)}$ by $\mathcal{O}_X(-n)$ is the Fourier-Mukai transform from D(X) to D(X) with kernel*

$$K = cone(\mathcal{O}_X(n) \boxtimes \mathcal{O}_X(-n)) \to \mathcal{O}_\Delta.$$

In general, $T_n \triangleq T_{\mathcal{O}_X(-n)}[-1]$ maps sheaves to complexes of sheaves. But for $n \gg 0$, we have

Lemma 10.2. [39] *Let U be a finite type $\mathbb{C}$-scheme and $\mathcal{F}_U$ is a coherent sheaf on $U \times X$ flat over U i.e. it is a U-family of coherent sheaves on X. Then for $n \gg 0$, $T_n(\mathcal{F}_U)$ is also a U-family of coherent sheaves on X.*

Sufficiently many compositions of Seidel-Thomas twists map sheaves to vector bundles.

Definition 10.3. *For a nonzero coherent sheaf $\mathcal{F}$, the homological dimension $hd(\mathcal{F})$ is the smallest $n \geq 0$ for which there exists an exact sequence in $coh(X)$*

$$0 \to E_n \to E_{n-1} \cdots \to E_0 \to \mathcal{F} \to 0$$

with $\{E_i\}_{i=0,...,n}$ are vector bundles.

Lemma 10.4. [39] *Let $\mathcal{F}_U$, $n \gg 0$ be the same as in Lemma 10.2, then for any $u \in U$, we have $hd(T_n(\mathcal{F}_u)) = max(hd(\mathcal{F}_u) - 1, 0)$.*

Corollary 10.5. [39] *Let U be a finite type $\mathbb{C}$-scheme and $\mathcal{F}_U$ is a U-family of coherent sheaves on X. Then there exists $n_1, ...n_m \gg 0$ such that for $T_{n_m} \circ \cdots \circ T_{n_1}(\mathcal{F}_U)$ is a U-family of vector bundles on X.*

By successive Seidel-Thomas twists, we obtain an isomorphism between $\mathcal{M}_c$ and some component(s) of a moduli space of simple holomorphic bundles. We also have

$$Ext^i(\mathcal{F}, \mathcal{F}) \cong Ext^i(E_b, E_b), \ i = 1, 2,$$

$$\kappa^{-1}(0) \cong \kappa'^{-1}(0), \tag{42}$$

where $\mathcal{F} \in \mathcal{M}_c$, κ is a Kuranishi map at $\mathcal{F}$ and E_b is the corresponding holomorphic bundle under Seidel-Thomas twists with κ' to be a Kuranishi map at E_b.

One possible way to prove the vanishing result $\kappa_+ = 0 \Rightarrow \kappa = 0$ for $\mathcal{M}_c$ in Assumption 4.4 is to use Seidel-Thomas twists transform stable sheaves to holomorphic bundles and then prove it using gauge theory. We will make it work in the next subsection.

10.3. **Comparisons of Borisov-Joyce's virtual fundamental classes with DT_4 virtual cycles.** We first recall local Kuranishi models used by Borisov and Joyce in their constructions of virtual fundamental classes for moduli spaces of stable sheaves on Calabi-Yau four-folds.

Theorem 10.6. *(Brav, Bussi and Joyce* [9] *Corollary 5.20)*
Suppose X is a Calabi-Yau four-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves, or simple complexes of coherent sheaves, on X. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme U with $dimU = dimExt^1(F, F)$, a vector bundle $E \to U$ with $rankE = dimExt^2(F, F)$, a non-degenerate quadratic form Q on E, a section $s \in H^0(E)$ with $Q(s, s) = 0$, and an isomorphism from $s^{-1}(0) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.

Their original proof is based on the theory of cyclic homology and derived algebraic geometry.

As mentioned before, Borisov and Joyce [8] used the above local 'Darboux charts', the machinery of homotopical algebra and C^∞-algebraic geometry to obtain a compact derived C^∞-scheme with the same underlying topological structure as the Gieseker moduli space of stable sheaves. In our language, this C^∞-scheme is called a generalized DT_4 moduli space (C^∞-scheme version, see Definition 4.5). Furthermore, they defined its virtual fundamental class.

In the next theorem, we give a gauge theoretical proof of the above local 'Darboux theorem' in the case when $\mathbb{K} = \mathbb{C}$ and $\mathcal{M} = \mathcal{M}_c$, the Gieseker moduli space of stable sheaves. We then introduce a weaker condition on their local 'Darboux charts' to include local models induced from DT_4 equations (the map $\tilde{\tilde{\kappa}}$ in Theorem 3.13). It turns out that the weaker condition is already sufficient for the gluing requirement in Borisov and Joyce's work [8].

Theorem 10.7. *Let $\mathcal{M}_c$ be the Gieseker moduli space of stable sheaves with fixed Chern character c on a compact Calabi-Yau four-fold X with $Hol(X) = SU(4)$.*
Then for any closed point $\mathcal{F} \in \mathcal{M}_c$, there exists an analytic neighborhood $U_{\mathcal{F}} \subseteq \mathcal{M}_c$, a holomorphic map near the origin

$$\kappa : Ext^1(\mathcal{F},\mathcal{F}) \to Ext^2(\mathcal{F},\mathcal{F})$$

such that $Q(\kappa,\kappa) = 0$ and $\kappa^{-1}(0) \cong U_{\mathcal{F}}$ as complex analytic spaces possibly with non-reduced structures, where Q is the Serre duality pairing on $Ext^2(\mathcal{F},\mathcal{F})$.

Proof. We use Seidel-Thomas twists [39],[72] transfer the problem to a problem on moduli spaces of holomorphic bundles. We take a connection on E with curvature F. By Chern-Weil theory,

$$-8\pi^2 \int ch_2(E) \wedge \Omega = \int Tr(F^{0,2} \wedge F^{0,2}) \wedge \Omega. \tag{43}$$

We take the $(0,1)$ part of the connection to be $\overline{\partial}_A + a''$, where $\overline{\partial}_A^2 = 0$. Then

$$F^{0,2}(\overline{\partial}_A + a'') = \overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + P_{\overline{\partial}_A^*}(a'' \wedge a'') + \mathbb{H}^{0,2}(a'' \wedge a'').$$

To describe Kuranishi theory, we take a'' satisfies $\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') = 0$. After gauge fixing $\overline{\partial}_A^* a'' = 0$, we have

$$F^{0,2}(\overline{\partial}_A + a'') = P_{\overline{\partial}_A^*}(g^{-1}(\alpha) \wedge g^{-1}(\alpha)) + \mathbb{H}^{0,2}(g^{-1}(\alpha) \wedge g^{-1}(\alpha)),$$

where $\alpha \triangleq g(a'')$, $g(a'') \triangleq a'' + \overline{\partial}_A^{-1} P_{\overline{\partial}_A}(a'' \wedge a'')$. Note that $\alpha \in H^{0,1}(X, EndE)$ and $\kappa(\alpha) \triangleq \mathbb{H}^{0,2}(g^{-1}(\alpha) \wedge g^{-1}(\alpha))$ is a Kuranishi map of moduli spaces of holomorphic bundles (39). We denote $\Delta(\alpha) \triangleq P_{\overline{\partial}_A^*}(g^{-1}(\alpha) \wedge g^{-1}(\alpha))$ and apply it into (43),

$$-8\pi^2 \int ch_2(E) \wedge \Omega = \int Tr\big(\kappa(\alpha) \wedge \kappa(\alpha) + 2\kappa(\alpha) \wedge \Delta(\alpha) + \Delta(\alpha) \wedge \Delta(\alpha)\big) \wedge \Omega.$$

As $\kappa(\alpha) \in H^{0,2}(X, EndE)$, $\Delta(\alpha) \in \overline{\partial}_A^* \Omega^{0,3}(X, EndE)_k$, we get

$$\int Tr(\kappa(\alpha) \wedge \Delta(\alpha)) \wedge \Omega = \int \big(\kappa(\alpha), *_4\Delta(\alpha)\big)_h \wedge \Omega \wedge \overline{\Omega} = 0$$

by the Hodge decomposition theorem. Similarly, we have $\int Tr\big(\Delta(\alpha) \wedge \Delta(\alpha)\big) \wedge \Omega = 0$. Thus,

$$-8\pi^2 \int ch_2(E) \wedge \Omega = \int Tr\big(\kappa(\alpha) \wedge \kappa(\alpha)\big) \wedge \Omega.$$

$ch_2(E) \in H^{2,2}(X)$ for holomorphic bundle E which gives $ch_2(E) \wedge \Omega = 0$. □

Remark 10.8. *One can check the Kuranishi map $\tilde{\kappa}$ (40) also satisfies $Q(\tilde{\kappa},\tilde{\kappa}) = 0$.*

The above theorem has an application to the unobstructedness of Gieseker moduli spaces.

Corollary 10.9. *If for any closed point $\mathcal{F} \in \mathcal{M}_c$, $dim_{\mathbb{C}} Ext^2(\mathcal{F},\mathcal{F}) \leq 1$, then $\mathcal{M}_c$ is smooth, i.e. all Kuranishi maps are zero.*

Then it is natural to ask whether the Kuranishi map $\tilde{\tilde{\kappa}}$ (41) induced from the DT_4 equations satisfies $Q(\tilde{\tilde{\kappa}},\tilde{\tilde{\kappa}}) = 0$.

Proposition 10.10. *The Kuranishi map $\tilde{\tilde{\kappa}}$ in (41) satisfies $Q(\tilde{\tilde{\kappa}},\tilde{\tilde{\kappa}}) \geq 0$.*

Proof. We denote the $(0,1)$ part of a unitary connection to be $\overline{\partial}_A + a''$, where $\overline{\partial}_A^2 = 0$. Then

$$F^{0,2}(\overline{\partial}_A + a'') = \overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + P_{\overline{\partial}_A^*}(a'' \wedge a'') + \mathbb{H}^{0,2}(a'' \wedge a'').$$

To describe $\tilde{\tilde{\kappa}}$, we take a'' satisfies $\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'') = 0$. After gauge fixing,

$$\overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a') = 0.$$

We get

$$F^{0,2}(\overline{\partial}_A + a'') = P_{\overline{\partial}_A^*}(q^{-1}(\alpha) \wedge q^{-1}(\alpha)) - *_4 P_{\overline{\partial}_A^*}(q^{-1}(\alpha) \wedge q^{-1}(\alpha)) + \mathbb{H}^{0,2}(q^{-1}(\alpha) \wedge q^{-1}(\alpha)),$$

where

$$q : \Omega^{0,1}(EndE)_k \to H^{0,1}(EndE) \oplus \overline{\partial}_A^* \Omega^{0,1}(EndE)_k \oplus \overline{\partial}_A^* \Omega^{0,2}(EndE)_{k-1},$$

$$q(a'') = \Big(\mathbb{H}(a''), \overline{\partial}_A^* a'' - \frac{i}{2} \wedge (a' \wedge a'' + a'' \wedge a'), \overline{\partial}_A^*\big(\overline{\partial}_A a'' + P_{\overline{\partial}_A}(a'' \wedge a'') + *_4 P_{\overline{\partial}_A^*}(a'' \wedge a'')\big)\Big)$$

and $q(a'') = \alpha \in H^{0,1}(EndE)$. Similar as before, we get

$$-8\pi^2 \int ch_2(E) \wedge \Omega = \int Tr\big(\tilde{\tilde{\kappa}}(\alpha) \wedge \tilde{\tilde{\kappa}}(\alpha)\big) \wedge \Omega - 2 \int \left\| P_{\overline{\partial}_A^*}\big(q^{-1}(\alpha) \wedge q^{-1}(\alpha)\big) \right\|_h^2 \wedge \Omega \wedge \overline{\Omega},$$

where $\tilde{\tilde{\kappa}} = \mathbb{H}^{0,2}(q^{-1}(\alpha) \wedge q^{-1}(\alpha))$. Thus $Q(\tilde{\tilde{\kappa}}, \tilde{\tilde{\kappa}}) \geq 0$. □

Although $Q(\tilde{\tilde{\kappa}}, \tilde{\tilde{\kappa}}) \geq 0$ instead of identically zero, it is already enough for the proof of the following promised vanishing result (see also Proposition 12 of [8]).

Lemma 10.11. *We take a Kuranishi map κ at $\mathcal{F} \in \mathcal{M}_c$,*

$$\kappa : Ext^1(\mathcal{F}, \mathcal{F}) \to Ext^2(\mathcal{F}, \mathcal{F})$$

such that $Q(\kappa, \kappa) \geq 0$, a half dimension real subspace $Ext^2_+(\mathcal{F}, \mathcal{F}) \subseteq Ext^2(\mathcal{F}, \mathcal{F})$ such that $Q|_{Ext^2_+(\mathcal{F},\mathcal{F})}$ is real and positive definite, where Q is the Serre duality pairing.

Then $\kappa_+^{-1}(0) \cong \kappa^{-1}(0)$ as topological spaces, where $\kappa_+ \triangleq \pi_+ \circ \kappa$ and π_+ is projection to $Ext^2_+(\mathcal{F}, \mathcal{F})$.

Proof. By the above assumption,

$$Q(\kappa, \kappa) = Q(\kappa_+, \kappa_+) + Q(\kappa_-, \kappa_-) + 2Q(\kappa_+, \kappa_-) \geq 0.$$

If $\kappa_+(a) = 0$, then $Q(\kappa_-(a), \kappa_-(a)) \geq 0$. However, Q is negative definite on $iExt^2_+(\mathcal{F}, \mathcal{F})$. Thus $Q(\kappa_-, \kappa_-) \leq 0$ which implies that $\kappa_-(a) = 0$. □

Remark 10.12. *One can check the condition $Q(\kappa, \kappa) = 0$ in BBJ's 'Darboux theorem' used by Borisov and Joyce* [8] *for gluing can be replaced by the weaker condition $Q(\kappa, \kappa) \geq 0$. Thus the local Kuranishi map $\tilde{\tilde{\kappa}}$ (41) for moduli spaces of stable bundles induced from DT_4 equations (differential geometrical nature) fits into the gluing data of* [8]*. This then indicates the equivalence of DT_4 virtual cycles defined using purely gauge theory (Theorem 5.7) and Borisov-Joyce's virtual cycles defined using derived C^∞-geometry, as their virtual cycles are independent of the choices of local charts and splittings* [8].

10.4. **A proof of the orientability of determinant line bundles for moduli spaces of $Spin(7)$ instantons.** In this subsection, we prove the orientability of determinant line bundles on spaces of gauge equivalence classes of connections on $Spin(7)$ manifolds under certain assumptions. We also prove the existence of the orientation data of DT_4 theory in those cases. We remark that we gave some partial results on the orientability of determinant line bundles for (generalized) DT_4 moduli spaces before in [16]. They are summarized as follows.

Proposition 10.13. *(*[16] *Corollary 9.5, 9.7)*
(1) If $\mathcal{M}_c^{bdl} \neq \emptyset$, $H^*(\mathcal{M}_c^{bdl}, \mathbb{Z}_2)$ is finitely generated and $H_1(\mathcal{M}_c^{bdl}, \mathbb{Z})$ does not have torsion of type $\mathbb{Z}_{4k}$, then the determinant line bundle of $\mathcal{M}_c^{DT_4}$ is orientable.
(2) If the Gieseker moduli space of stable sheaves $\mathcal{M}_c$ is smooth and $H_1(\mathcal{M}_c, \mathbb{Z})$ does not have torsion of type $\mathbb{Z}_{4k}$, where $k \geq 1$. Then the index bundle of the generalized DT_4 moduli space is oriented.

The above assumptions on moduli spaces are quite annoying. We state a result with assumptions only on base manifolds.

Theorem 10.14. *For any compact oriented real eight dimensional manifold X with $Hol(X) \subseteq Spin(7)$ and $H^{odd}(X, \mathbb{Z}) = 0$, and $U(r)$ bundle $E \to X$, the determinant line bundle $\mathcal{L}$ of the index bundle of the twisted Dirac operators over the space $\mathcal{B}^*$ is trivial.*

Proof. The proof is essentially due to Donaldson, Corollary 3.22 of [20]. By the standard argument as in [20], we are reduced to the case of $SU(N)$ bundles with $N \gg 0$. We only need to show $\pi_0(\mathcal{G}) = 0$ as $\pi_1(\mathcal{B}^*) \cong \pi_0(\mathcal{G})$. By Atiyah-Bott's Proposition 2.4 [3], we have homotopy equivalence

$$B\mathcal{G} \simeq Map_P(X, BSU),$$

where Map_P denotes the component of the principal bundle P in the mapping space. Thus we are left to show $\pi_1(Map_P(X, BSU)) = 0$. By Theorem 11.7 of [62], we have the Federer spectral sequence such that

$$E_2^{p,q} \cong H^q(X, \pi_{p+q}(BSU)) \Rightarrow \pi_p(Map_P(X, BSU)).$$

By our assumptions and direct calculations, we get $\pi_1(Map_P(X, BSU)) = 0$. □

Corollary 10.15. *If X is a compact simply connected Calabi-Yau four-fold with $h^{1,2}(X) = 0$ and $H_2(X, \mathbb{Z})$, $H_3(X, \mathbb{Z})$ are torsion-free, then $H^{odd}(X, \mathbb{Z}) = 0$.*

Proof. By the universal coefficient theorem and the classification of simply connected Calabi-Yau manifolds [50]. □

Remark 10.16. *There are many examples which satisfy assumptions in Theorem 10.14. For instance, smooth complete intersection Calabi-Yau four-folds in product of projective spaces.*

For Calabi-Yau four-folds with $SU(4)$ holonomy, we can also prove $Ind_{\mathbb{C}}$ over $\mathcal{M}_c$ satisfies $c_1(Ind_{\mathbb{C}}) = 0$.

Corollary 10.17. *Let X be a compact simply connected Calabi-Yau four-fold such that $Hol(X) = SU(4)$ and $H_3(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$. Then the index bundle $Ind_{\mathbb{C}}$ over the Gieseker moduli space $\mathcal{M}_c$ satisfies $c_1(Ind_{\mathbb{C}}) = 0$. Furthermore, the orientation data of DT_4 theory (Definition 1.6) exists in this case.*

Proof. By the Seidel-Thomas twist, $c_1(Ind_{\mathbb{C}}) = 0$ is reduced to Theorem 10.14. □

10.5. **Some remarks on Cayley submanifolds and $Spin(7)$ instantons.** By Remark 5.8, $Spin(7)$ instanton countings on Calabi-Yau four-folds are DT_4 invariants. Combining Borisov-Joyce's construction [8] and our results, we can define DT_4 invariants for complex vector bundles when the corresponding Chern character satisfies $c \in \oplus_k H^{k,k}(X)$ and also for general stable coherent sheaves (we need Corollary 10.17 to guarantee invariants to be defined in $\mathbb{Z}$ instead of $\mathbb{Z}_2$). If $c \notin \oplus_k H^{k,k}(X)$, we do not have a definition of DT_4 invariants at this moment.

From perspectives of SYZ mirror transformations [73], [51] and the blow-up analysis of moduli spaces of $Spin(7)$ instantons [77], [53], it would be interesting to consider the counting problem of Cayley submanifolds with ASD connections.

We fix (L, A), a special Lagrangian submanifold with an ASD connection on certain vector bundle over L. If $b_1(L) = 0$, L is rigid, i.e. it can't be deformed as a Lagrangian submanifold inside X. Then deformations of (L, A) are deformations of A and the component of the moduli space $\mathfrak{M}^L_{SLag+ASD} \triangleq \{(L, A)\}$ can be identified with $\mathfrak{M}_{ASD}(L)$, the moduli space of ASD connections on L. Thus, naively speaking, DT_4*-type inv* $(\mathfrak{M}^L_{SLag+ASD}) =$ *Donaldson inv* (L) if $b_1(L) = 0$.

References

[1] M. Atiyah, *K-theory*, Benjamin, New York (1967).
[2] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology 23 (1984) 1-28.
[3] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A 308 (1982) 523-615.
[4] L. Baulieu, H. Kanno and I. M. Singer, *Special quantum field theories in eight and other dimensions*, Commun. Math. Phys. 194, 149-175 (1998).
[5] K. Behrend, I. Ciocan-Fontanine, J. Hwang and M. Rose, *The derived moduli space of stable sheaves*, arXiv:1004.1884v1, 2010.
[6] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), 45-88.
[7] K. Behrend and B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra Number Theory, Vol. 2, 313-345, 2008.
[8] D. Borisov and D. Joyce *Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds*, preprint in the homepage of D. Borisov, 2014.
[9] C. Brav, V. Bussi and D. Joyce *A 'Darboux theorem' for derived schemes with shifted symplectic structure*, arXiv:1305.6302, 2013.
[10] T. Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. 24, 969-998, 2011.
[11] T. Bridgeland, *T-structures on some local Calabi-Yau varieties*, Journal of Algebra 289 (2005) 453-483.
[12] R. Brussee, *The canonical class and the C^∞ properties of Kähler surfaces*, New York J. Math. 2 (1996) 103-146.
[13] J. Bryan and N. C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. 13, 371-410, 2000.
[14] J. Bryan and N. C. Leung, *Generating functions for the number of curves on abelian surfaces*, Duke Math. J. 99 (1999), no. 2, 311-328.
[15] R. O. Buchweitz and H. Flenner, *A semiregularity map for modules and applications to deformations*, Compositio Math. 137, 135-210, 2003.
[16] Y. Cao, *Donaldson-Thomas theory for Calabi-Yau four-folds*, MPhil thesis, arXiv:1309.4230, 2013.
[17] K. Cieliebak, I. Mundet i Riera and D. Salamon, *Equivariant moduli problems, branched manifolds, and the Euler class*, Topology 42 (2003), no. 3, 641-700.
[18] J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. Vol. 5, 479-511, 1996.
[19] S. K. Donaldson *An application of gauge theory to 4-dimensional topology*, J. Diff. Geom. 18 (1983), 279-315.
[20] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Diff. Geom. 26 (1987), no. 3, 397-428.
[21] S. K. Donaldson and P. B. Kronheimer, *The geometry of four manifolds*, Oxford University Press,(1990).
[22] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions* in The Geometric Universe (Oxford, 1996), Oxford Univ. Press,Oxford, 1998, 31-47.

[23] W. Ebeling, *The Monodromy Groups of Isolated Singularities of Complete Intersections*, Lecture Notes in Mathematics 1293, Springer-Verlag, 1987.

[24] D. Edidin and W. Graham *Charateristic classes and quadric bundles*, Duke Math. J. 78 (1995), no. 2, 277-299.

[25] J. Engel, M. Reineke, *Smooth models of quiver moduli*, Mathematische Zeitschrift August 2009, Volume 262, Issue 4, pp 817-848.

[26] D. Freed and K. K. Uhlenbeck *Instantons and four-manifolds*, Springer-Verlag, New York, 1984.

[27] R. Friedman and J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergeb. der Math. und ihrer Grenzgebiete 27, Springer-Verlag, 1994.

[28] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[29] V. Ginzburg, *Calabi-Yau algebra*, arXiv:math/0612139, 2006.

[30] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. 135 (1999), 487-518.

[31] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Interscience, New York, 1978.

[32] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. 52, Springer.

[33] Z. Hua, *Chern-Simons functions on toric Calabi-Yau threefolds and Donaldson-Thomas theory*, arXiv:1103.1921v5, 2012.

[34] D. Huybrechts, *Birational symplectic manifolds and their deformations*, J. Diff. Geom. 45, 488-513 (1997).

[35] D. Huybrechts, *Complex Geometry: An Introduction*, Springer-Verlag, Berlin (2005).

[36] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Math. E31, Vieweg, Braunschweig/Wiesbaden, 1997.

[37] D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira Spencer classes*, Math. Ann. 346 (2010), 545-569.

[38] L. Illusie, *Complexe cotangent et d'eformations I*, Springer Lecture Notes in Math. 239, Springer-Verlag, Berlin, 1971.

[39] D. Joyce and Y. N. Song, *A theory of generalized Donaldson-Thomas invariants*, Memoirs of the AMS, arXiv:0810.5645, 2010.

[40] B. Keller, *A-infinity algebras, modules and functor categories*, arXiv:math/0510508v3, 2006.

[41] Y. H. Kiem and J. Li, *Localizing virtual cycles by cosections*, J. Amer. Math. Soc. 26, 2013.

[42] Y. H. Kiem and J. Li, *Gromov-Witten invariants of varieties with holomorphic 2-forms*, arXiv: 0707.2986.

[43] B. Kim, A. Kresch and T. Pantev, *Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee*, J. Pure Appl. Algebra 179 (2003), no. 1-2, 127-136.

[44] A. D. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math Oxford 45, 515-530 (1994).

[45] A. D. King and C. H. Walter, *On Chow Rings of Fine Moduli Spaces of Modules*, J. reine. angew. Math. Volume 1995, Issue 461, Pages 179-188.

[46] A. Klemm, B. Lian, S. S. Roan and S. T. Yau, *Calabi-Yau fourfolds for M-and F-Theory compactifications*, arXiv:hep-th/9701023.

[47] A. Klemm and R. Pandharipande, *Enumerative geometry of Calabi-Yau 4-folds*, Commun. Math. Phys. 281, 621-653 (2008).

[48] M. Kool, *Moduli spaces of sheaves on toric varieties*, PhD thesis, Lincoln College Oxford University, 2010.

[49] H. Lange, *Universal families of extensions*, Journal of Algebra 83, 101-112 (1983).

[50] N. C. Leung, *Geometric structures on Riemannian manifolds*, Surveys in Differential Geometry, Vol 16, no.1, 161-264 (2011).

[51] N.C. Leung, S. T. Yau, and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. 4 (2000), no. 6, 1319-1341.

[52] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math. Vol. 176, 63-130, 2009.

[53] C. Lewis, *Spin(7) instantons*, PhD thesis, 1998.

[54] J. Li, *Zero dimensional Donaldson-Thomas invariants of threefolds*, Geom. Topol. Vol. 10, 2117-2171, 2006.

[55] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, Jour. Amer. Math. Soc. 11 119-174.

[56] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47-83, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 199.

[57] J. Li and G. Tian, *Comparison of the algebraic and the symplectic Gromov-Witten invariants*, Asian J. Math. 3(3) (1999) 689-728.

[58] W. P. Li and Z. Qin, *Stable rank-2 bundles on Calabi-Yau manifolds*, Internat. J. Math. 14 (2003), 1097-1120.

[59] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, Compositio Math. 142 (2006) 1263-1285

[60] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. 186, 435-479, 2011.

[61] D. Maulik, R. Pandharipande and R. P. Thomas, *Curves on K3 surfaces and modular forms*, J. Topology (2010) 3 (4): 937-996.

[62] J. McCleary, *A user's guide to spectral sequences*, Mathematics Lecture Series, Vol. 12, Publish or Perish, Wilmington, DE, 1985.

[63] K. Miyajima, *Kuranishi family of vector bundles and algebraic description of Einstein-Hermitian connections*, Publ. RIMS, Kyoto Univ. 25 (1989).

[64] S. Mozgovoy, M. Reineke, *On the noncommutative Donaldson-Thomas invariants arising from brane tilings*, Advances in Mathematics, Volume 223, Issue 5, 20 March 2010, 1521-1544.

[65] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. math. 77, 101-116 (1984).

[66] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhauser (1980).

[67] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs correspondence for the quintic 3-fold*, arXiv:1206.5490. 2012.

[68] T. Pantev, B. Töen, M. Vaquié and G. Vezzosi, *Quantization and Derived Moduli Spaces I: Shifted Symplectic Structures*, Publications mathématiques de l'IHÉS, June 2013, Volume 117, Issue 1, pp 271-328.

[69] M. Reineke, *Cohomology of non-commutative Hilbert schemes*, Algebr. Represent. Theory 8 (2005), no. 4, 541-561.

[70] M. Reineke, *Framed quiver moduli, cohomology, and quantum groups*, Journal of Algebra. Volume 320, Issue 1, 1 July 2008, Pages 94-115.

[71] E. Segal, *The A-infinity deformation theory of a point and the derived categories of local Calabi-Yaus*, arXiv:math/0702539.

[72] P. Seidel and R.P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), 37-108.

[73] A. Strominger, S. T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243-259.

[74] R. M. Switzer, *Algebraic topology homology and homotopy*, Springer-Verlag, 1975.

[75] B. Szendröi, *Non-commutative Donaldson-Thomas invariants and the conifold*, Geom Topol. 12 (2008), no. 2, 1171-1202.

[76] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, J. Diff. Geometry. 54(2000), 367-438.

[77] G. Tian, *Gauge theory and calibrated geometry I*, Ann. Math. 151, 193-268, (2000).

[78] Y. Toda, *Curve counting theories via stable objects I: DT/PT correspondence*, J. Amer. Math. Soc. 23, 1119-1157, 2010.

[79] K. Uhlenbeck and S. T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. 39 (1986) 257-293.

[80] R. O. Wells, *Differential analysis on complex manifolds*, Graduate Texts in Math. 65, Springer.

[81] E. Witten, *Topological quantum field theory*, Commun. Math. Phys. 117, 353 (1988).

[82] S. T. Yau, *On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978) 339-411.

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: `ylcao@math.cuhk.edu.hk`

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: `leung@math.cuhk.edu.hk`