ORIENTABILITY FOR GAUGE THEORIES ON CALABI-YAU MANIFOLDS

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Abstract. We study orientability issues of moduli spaces from gauge theories on Calabi-Yau manifolds. Our results generalize and strengthen those for Donaldson-Thomas theory on Calabi-Yau manifolds of dimensions 3 and 4. We also prove a corresponding result in the relative situation which is relevant to the gluing problem in DT theory.

1. Introduction

Donaldson invariants count anti-self-dual connections on closed oriented 4-manifolds [14]. The definition requires an orientablity result proved by Donaldson in [15]. Indeed, Donaldson theory fits into the 3-dimensional TQFT structure in the sense of Atiyah [1]. In particular, relative Donaldson invariants for $(X, Y = \partial X)$ take values in the instanton Chern-Simons-Floer (co)homology $HF^*_{CS}(Y)$ [16]. The Euler characteristic of $HF^*_{CS}(Y)$ is the Casson invariant which counts flat connections on a closed 3-manifold Y.

As was proposed by Donaldson and Thomas [18], we are interested in the *complexification* of the above theory. Namely, we consider holomorphic vector bundles (or general coherent sheaves) over Calabi-Yau manifolds [43]. The complex analogs of (i) Donaldson invariants, (ii) Chern-Simons-Floer (co)homology $HF^*_{CS}(Y)$, and (iii) Casson invariants are (i) DT_4 invariants, (ii) DT_3 (co)homology $H^*_{DT_3}(Y)$, and (iii) DT_3 invariants.

As a complexification of Casson invariants, Thomas defined Donaldson-Thomas invariants for Calabi-Yau 3-folds [39]. DT_3 invariants for ideal sheaves of curves are related to many other interesting subjects including Gopakumar-Vafa conjecture on BPS numbers in string theory [20], [21], [25] and MNOP conjecture [29], [30], [31], [35] which relates DT_3 invariants and Gromov-Witten invariants. The generalization of DT_3 invariants to count strictly semi-stable sheaves is due to Joyce and Song [24] using Behrend's result [4]. Kontsevich and Soibelman proposed generalized as well as motivic DT theory for Calabi-Yau 3-categories [26], which was later studied by Behrend, Bryan and Szendröi [5] for Hilbert schemes of points. The wall-crossing formula [26], [24] is an important structure for Bridgeland's stability condition [9] and Pandharipande-Thomas invariants [36], [40].

As a complexification of Chern-Simons-Floer theory, Brav, Bussi, Dupont, Joyce and Szendroi [7] and Kiem, Li [25] recently defined a cohomology theory on Calabi-Yau 3-folds whose Euler characteristic is the DT_3 invariant. The point is that moduli spaces of simple sheaves on Calabi-Yau 3-folds are locally critical points of holomorphic functions [8], [24], and we could consider perverse sheaves of vanishing cycles of these functions. They glued these local perverse sheaves and defined its hypercohomology as DT_3 cohomology. In general, gluing these perverse sheaves requires a square root of the determinant line bundle of the moduli space. Nekrasov and Okounkov proved its existence in [34]. The square root is called an orientation data if it is furthermore compatible with wall-crossing (or Hall algebra structure) [26] whose existence was proved by Hua on simply-connected torsion-free CY_3's [22].

As a complexification of Donaldson theory, Borisov and Joyce [6] and the authors [11], [12] developed DT_4 invariants (or 'holomorphic Donaldson invariants') which count stable sheaves on Calabi-Yau 4-folds. The orientation issue here is whether $c_1(\mathcal{L}_{\mathcal{M}}) = 0$ for the determinant line bundle $\mathcal{L}_{\mathcal{M}}$ of the moduli space. It was solved by the authors in [12] for Calabi-Yau 4-fold X which satisfies $H^{odd}(X, \mathbb{Z}) = 0$ (for instance, complete intersections in product of projective spaces). Later, we generalized this result to torsion-free (i.e. homologies are torison-free) Calabi-Yau 4-folds [13].

In this paper, we generalize and strengthen these results concerning orientability to Calabi-Yau manifolds of any dimension.

Theorem 1.1. *(Theorem 2.2, Theorem 3.2)*
Let X be a compact Calabi-Yau n-fold with $Hol(X) = SU(n)$ and $Tor(H_*(X, \mathbb{Z})) = 0$, $\mathcal{M}_X$ be a moduli space of simple sheaves with fixed Chern classes, and we denote its determinant line bundle as $\mathcal{L}_X$ with $\mathcal{L}_X|_{\mathcal{F}} = det(Ext^{odd}(\mathcal{F}, \mathcal{F})) \otimes det(Ext^{even}(\mathcal{F}, \mathcal{F}))^{-1}$. Then, we have

(1) if n is even, $c_1(\mathcal{L}_X)=0$,
(2) if n is odd, $\mathcal{L}_X$ has a square root.

The proof of this main theorem relies on varies tricks in gauge theory: firstly we use a machinery called Seidel-Thomas twist [24], [38] to transform the problem to a problem on some moduli spaces of simple holomorphic bundles; secondly, we use Donaldson's argument [15] to further reduce the problem to complex vector bundles with high ranks and vanishing first Chern classes; thirdly, we calculate the torsion part of the second cohomology group (as an abelian group) of the base of the index bundle and prove it vanishes; finally, we apply the Atiyah-Singer family index theorem to calculate Chern classes of the index bundle directly (modulo torsion). We remark that in the proof of (2) in the above theorem, we further need to use a brilliant idea due to Maulik, Nekrasov and Okounkov [34].

Along this line, we also prove an orientability result for the relative situation where we have Calabi-Yau manifolds as anti-canonical divisors of even dimension projective manifolds. This generalizes the orientability result in the relative DT_4 theory [13] (taken as a complexification of the Donaldson-Floer TQFT theory for 4-3 dimensional manifolds).

Theorem 1.2. *(Theorem 4.2)*
Let Y be a smooth anti-canonical divisor in a projective $2n$-fold X with $Tor(H_*(X,\mathbb{Z}))=0$. In particular, Y is a Calabi-Yau $(2n-1)$-fold. Let $\mathcal{M}_X$ be a moduli space of simple bundles on X with fixed Chern classes which has a well-defined restriction morphism

$$r:\mathcal{M}_X\to\mathcal{M}_Y,$$

to a moduli space of simple bundles on Y with fixed Chern classes.

Then there exists a square root $\mathcal{L}_Y^{\frac{1}{2}}$ of $\mathcal{L}_Y$ (i.e. $\mathcal{L}_Y^{\frac{1}{2}}\otimes\mathcal{L}_Y^{\frac{1}{2}}\cong\mathcal{L}_Y$) such that

$$c_1(\mathcal{L}_X)=r^*c_1(\mathcal{L}_Y^{\frac{1}{2}}),$$

where $\mathcal{L}_X$ (resp. $\mathcal{L}_Y$) is the determinant line bundle of $\mathcal{M}_X$ (resp. $\mathcal{M}_Y$).

We remark that these results might be viewed as the orientability for corresponding derived schemes with shifted symplectic structures in the sense of Pantev-Töen-Vaquié-Vezzosi [37], [10].

Acknowledgement: The first author expresses his deep gratitude to Professor Simon Donaldson for useful discussions on orientations in Yang-Mills theory. We thank Zheng Hua for varies helpful discussions and comments. The work of the second author was substantially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK401411 and CUHK14302714).

2. Orientability for even dimensional Calabi-Yau

We move to even complex dimensional Calabi-Yau manifolds and extend the story of orientations for moduli spaces of sheaves on Calabi-Yau 4-folds. We fix a compact spin manifold X of real dimension $4m$, a complex vector bundle $(E,h)\to X$ with Hermitian metric, and define the twisted Dirac operator $\not{D}_{EndE}:\Gamma(\$_{\mathbb{C}}^{+}(X)\otimes EndE)\to\Gamma(\$_{\mathbb{C}}^{-}(X)\otimes EndE)$. We introduce the space $\mathcal{B}_X^*=\mathcal{A}^*/\mathcal{G}^0$ of gauge equivalence classes of irreducible unitary connections on $(E,h)\to X$, where $\mathcal{G}^0=\mathcal{G}/C(U(r))$ is the reduced gauge group and $C(U(r))$ is the center of $U(r)$ which acts trivially on $\mathcal{A}$ [19]. $\mathcal{B}_X^*$ is an open subset of the orbit space $\mathcal{B}_X=\mathcal{A}/\mathcal{G}^0$ whose complement is of infinite codimension [17]. There exists a universal family $\mathcal{E}\to X\times\mathcal{B}_X^*$ and an index bundle $Ind(\not{\mathbb{D}}_{End\mathcal{E}})\to\mathcal{B}_X^*$ with $Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_E=ker(\not{D}_{EndE})-coker(\not{D}_{EndE})$.

Theorem 2.1. *For any compact spin manifold X of real dimension $4m$ with $Tor(H_*(X,\mathbb{Z}))=0$, and a Hermitian vector bundle $E\to X$, the index bundle $Ind(\not{\mathbb{D}}_{End\mathcal{E}})$ satisfies*

$$c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}}))=0.$$

Proof. As in Donaldson theory [15], by considering $E^{'}=E\oplus(detE)^{-1}\oplus\mathbb{C}^p$, we are left to show $c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}}))=0\in H^2(\mathcal{B}_X^*,\mathbb{Z})$ for a $SU(N)$ complex vector bundle on X with $N\gg 0$. Analogs to Theorem 10.14 of [12], we apply the Federer spectral sequence [32],

$$E_2^{p,q}\cong H^p(X,\pi_{p+q}(BSU(N)))\Rightarrow\pi_q(Map_E(X,BSU(N))).$$

For $N\gg 0$, we get $\pi_1(Map_E(X,BSU(N)))\cong\bigoplus_{k\geq 1}H^{2k+1}(X,\mathbb{Z})$. It is torsion-free by the universal coefficient theorem and our assumptions. From Atiyah-Bott (Proposition 2.4 [2]), we have a homotopy equivalence

$$B\mathcal{G}\simeq Map_E(X,BSU(N)),$$

which shows $\pi_0(\mathcal{G})$ is torsion-free. From the exact sequence

$$\pi_0(C(SU(N))) \to \pi_0(\mathcal{G}) \to \pi_0(\mathcal{G}/C(SU(N))) \to 0,$$

we know $\pi_0(\mathcal{G}) \cong \pi_0(\mathcal{G}/C(SU(N)))$ as there is no homomorphism from the torsion group $\pi_0(C(SU(N)))$ to the torsion-free group $\pi_0(\mathcal{G})$. Then

$$H_1(\mathcal{B}^*_X, \mathbb{Z}) \cong \pi_1(\mathcal{B}^*_X) \cong \pi_1(\mathcal{B}_X) \cong \pi_0(\mathcal{G}/C(SU(N))) \cong \pi_1(Map_E(X, BSU(N)))$$

and $H^2(\mathcal{B}^*_X, \mathbb{Z})$ is also torsion-free. We are thus left to show for any embedded surface $C \subseteq \mathcal{B}^*_X$ $c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_C) = 0$. We denote $\mathcal{E}|_C$ to be the universal bundle on $X \times C$, and apply the Atiyah-Singer family index theorem [3],

$$c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_C) = [ch(End\mathcal{E}|_C) \cdot \hat{A}(X)]^{(4m+2)}/[X] = 0,$$

as $ch_{odd}(End\mathcal{E}|_C) = 0$. □

We fix a compact Calabi-Yau $2n$-fold X, and denote the determinant line bundle of a moduli space $\mathcal{M}_X$ of simple sheaves by $\mathcal{L}_X$ with $\mathcal{L}_X|_\mathcal{F} = det(Ext^{odd}(\mathcal{F}, \mathcal{F})) \otimes det(Ext^{even}(\mathcal{F}, \mathcal{F}))^{-1}$.

Theorem 2.2. *Let X be a compact Calabi-Yau $2n$-fold with $Tor(H_*(X, \mathbb{Z})) = 0$ and $Hol(X) = SU(2n)$. For any moduli space $\mathcal{M}_X$ of simple sheaves with fixed Chern classes, we have $c_1(\mathcal{L}_X) = 0$.*

Proof. By the Seidel-Thomas twist [38], [24], we are reduced to a problem for moduli spaces of simple holomorphic bundles. On Calabi-Yau manifolds, the index bundle $Ind(\not{\mathbb{D}}_{End\mathcal{E}})$ in Theorem 2.1 satisfies $Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_A = H^{0,odd}_{D_A}(X, EndE) - H^{0,even}_{D_A}(X, EndE)$, we are done. □

Remark 2.3.
1. The point here is: on Calabi-Yau manifolds, $\hat{A} = Td$ and $\not{D} = \bar{\partial}$.
2. Index bundles could be understood as tangent bundles of moduli spaces in the derived sense. Theorem 2.2 would then be understood as moduli spaces of simple sheaves on even complex dimensional Calabi-Yau manifolds are 'derived' Calabi-Yau spaces. When $n = 2$, it recovers the result on CY_4's. If $n = 1$, it reflects moduli spaces of simple sheaves on K3 surfaces are Calabi-Yau manifolds (in fact hyper-Kähler by Mukai [33]*).*
3. Smooth complete intersections in torsion-free toric varieties satisfy the above condition.

3. Orientability for odd dimensional Calabi-Yau

We move to complex odd dimensional Calabi-Yau manifolds and extend the story of spin structures on moduli spaces of sheaves on Calabi-Yau 3-folds. We first prove a general result for spin manifolds of dimension $2m$, which says moduli spaces of bundles on even dimensional spin manifolds are 'spin'.

Theorem 3.1. *For any compact spin manifold X of real dimension $2m$ with $Tor(H_*(X, \mathbb{Z})) = 0$, and a complex bundle $E \to X$, the determinant line bundle $det(Ind(\not{\mathbb{D}}_{End\mathcal{E}}))$ of the index bundle of twisted Dirac operators over $\mathcal{B}^*_X$ has a square root.*

Proof. As before, we only need to prove $det(Ind(\not{\mathbb{D}}_{End\mathcal{E}}))$ has a square root for a $SU(N)$ complex vector bundle on X with $N \gg 0$, By the Federer spectral sequence,

$$E^{p,q}_2 \cong H^p(X, \pi_{p+q}(BSU(N))) \Rightarrow \pi_q(Map_E(X, BSU(N))),$$

we get $\pi_1(Map_E(X, BSU(N))) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z})$ for $N \gg 0$, which is torsion-free. Then $H_1(\mathcal{B}^*_X, \mathbb{Z}) \cong \pi_1(\mathcal{B}^*_X, \mathbb{Z}) \cong \pi_1(Map_E(X, BSU(N)))$ and $H^2(\mathcal{B}^*_X, \mathbb{Z})$ is also torsion-free. We are thus left to show for any embedded surface $C \subseteq \mathcal{B}^*_X$ $c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_C) \in H^2(C, \mathbb{Q})$ is even.

We denote $\mathcal{E}|_C \to C \times X$ to be the universal bundle, and abuse the notation $Ind(\not{\mathbb{D}}_{End\mathcal{E}}) \to \mathcal{B}^*_X \times \mathcal{B}^*_X$ for the index bundle of twisted Dirac operators $\not{D}_{D^*_{A_1}, D_{A_2}}$'s. We consider $C \times C \subseteq \mathcal{B}^*_X \times \mathcal{B}^*_X$ and apply the Atiyah-Singer family index theorem [3],

$$c_1(Ind(\not{\mathbb{D}}_{End\mathcal{E}})|_{C\times C}) = [ch(p_1^*(\mathcal{E}|_C)^* \otimes p_2^*(\mathcal{E}|_C)) \cdot \hat{A}(X)]^{(2m+2)}/[X], \tag{1}$$

where $p_i : C \times C \times X \to C \times X$ are two natural projections with $i = 1, 2$.

Motivated by the idea of Maulik, Nekrasov and Okounkov [34], we consider an involution

$$\sigma : \mathcal{B}^*_X \times \mathcal{B}^*_X \to \mathcal{B}^*_X \times \mathcal{B}^*_X,$$

$$\sigma([A_1], [A_2]) = ([A_2], [A_1]).$$

We denote the determinant line bundle to be $\mathcal{L} = det(Ind(\not{D}_{End\mathcal{E}})|_{C\times C}) \to C \times C$. From the index formula (1), if m is even, $c_1(\mathcal{L}) = -c_1(\sigma^*\mathcal{L})$; if m is odd, $c_1(\mathcal{L}) = c_1(\sigma^*\mathcal{L})$. In both cases, we obtain

$$c_1(\mathcal{L}) \equiv c_1(\sigma^*\mathcal{L}) \ (mod\ 2) \in H^2(C \times C, \mathbb{Z}_2). \tag{2}$$

Now we are reduced to prove $c_1(\mathcal{L}|_\Delta) \equiv 0 \ (mod\ 2)$, where $\Delta \hookrightarrow C \times C$ is the diagonal.

By the Künneth formula,

$$H^2(C \times C, \mathbb{Z}_2) = H^0(C, \mathbb{Z}_2) \otimes H^2(C, \mathbb{Z}_2) \oplus H^2(C, \mathbb{Z}_2) \otimes H^0(C, \mathbb{Z}_2) \oplus H^1(C, \mathbb{Z}_2) \otimes H^1(C, \mathbb{Z}_2).$$

Assume $\{a_i\}$ is a basis of $H^0(C, \mathbb{Z}_2)$, $\{b_i\}$ is a basis of $H^2(C, \mathbb{Z}_2)$, $\{c_i\}$ is a basis of $H^1(C, \mathbb{Z}_2)$,

$$c_1(\mathcal{L}) \equiv \sum_{i,j} n_{ij} a_i \otimes b_j + \sum_{i,j} m_{ij} b_i \otimes a_j + \sum_{i,j} k_{ij} c_i \otimes c_j \ (mod\ 2).$$

Under the action of involution map,

$$\sigma^*\big(c_1(\mathcal{L})\big) \equiv \sum_{i,j} m_{ij} a_j \otimes b_i + \sum_{i,j} n_{ij} b_j \otimes a_i + \sum_{i,j} k_{ij} c_j \otimes c_i \ (mod\ 2).$$

By (2), we obtain $m_{ji} \equiv n_{ij} \ (mod\ 2)$, $k_{ji} \equiv k_{ij} \ (mod\ 2)$. When we restrict to the diagonal,

$$c_1(\mathcal{L}|_\Delta) \equiv \sum_{i,j} n_{ij}(a_i \cup b_j + b_j \cup a_i) \equiv 0 \ (mod\ 2).$$

□

We fix a compact Calabi-Yau $(2n+1)$-fold X, and denote the determinant line bundle of a moduli space $\mathcal{M}_X$ of simple sheaves by $\mathcal{L}_X$ with $\mathcal{L}_X|_{\mathcal{F}} = det(Ext^{odd}(\mathcal{F}, \mathcal{F})) \otimes det(Ext^{even}(\mathcal{F}, \mathcal{F}))^{-1}$.

Theorem 3.2. *Let X be a compact Calabi-Yau $(2n+1)$-fold with $Tor(H_*(X, \mathbb{Z})) = 0$ and $Hol(X) = SU(2n+1)$. For any moduli space $\mathcal{M}_X$ of simple sheaves with fixed Chern classes, $\mathcal{L}_X$ has a square root.*

Proof. By the Seidel-Thomas twist [38], [24], we are reduced to a problem for moduli spaces of simple bundles. On Calabi-Yau manifolds, the index bundle in Theorem 3.1 satisfies $Ind(\not{D}_{End\mathcal{E}})|_{(A_1,A_2)} = H^{0,odd}_{D^*_{A_1}\otimes D_{A_2}}(X, E^* \otimes E) - H^{0,even}_{D^*_{A_1}\otimes D_{A_2}}(X, E^* \otimes E)$, we are done. □

Remark 3.3. *This result was proved directly (without going to gauge theory) by Nekrasov and Okounkov, see Lemma 6.1 of* [34]. *The advantage here is we could hope to choose compatible square roots for different components of $\mathcal{M}_X$ simultaneously, as the space $\mathcal{B}_X$ is (homotopically) independent of the choice of the complex structure and polarization of X.*

From the proof of Theorem 2.2, it's obvious that Theorem 3.2 also holds for even dimensional Calabi-Yau manifolds. Then one would be interested to know whether Theorem 2.2 holds for odd Calabi-Yau. In fact, it is not true in general.

Example 3.4. *Let X be a compact simply connected Calabi-Yau 3-fold such that $Tor(H_*(X, \mathbb{Z})) = 0$ (for instance, a generic quintic 3-fold). We consider the Hilbert scheme of two points on X (which is smooth), i.e. $Hilb^{(2)}(X) = Bl_\Delta(X \times X)/\mathbb{Z}_2$, where $\Delta \hookrightarrow X \times X$ is the diagonal. Its determinant line bundle satisfies $c_1(\mathcal{L}_X) = 2c_1(Hilb^{(2)}(X)) \neq 0$.*

4. Orientability for the relative case

We take a smooth (Calabi-Yau) $(2n-1)$-fold Y in a complex projective $2n$-fold X as its anti-canonical divisor, and denote $\mathcal{M}_X$ to be a moduli space of simple bundles on X with fixed Chern classes. We assume it has a natural restriction morphism

$$r : \mathcal{M}_X \to \mathcal{M}_Y,$$

to a moduli space of simple bundles on Y with fixed Chern classes. Motivated by the long exact sequence in relative DT_4 theory [13], we have

Lemma 4.1. *We take a simple bundle $E \in \mathcal{M}_X$, and assume Y is connected, then we have the following long exact sequence,*

$$0 \to H^{0,1}(X, EndE \otimes K_X) \to H^{0,1}(X, EndE) \to H^{0,1}(Y, EndE|_Y) \to \cdots$$
$$\cdots \to H^{0,n-1}(Y, EndE|_Y) \to H^{0,n}(X, EndE \otimes K_X) \to H^{0,n}(X, EndE) \to H^{0,n}(Y, EndE|_Y) \to \cdots$$
$$\cdots \to H^{0,2n-1}(X, EndE \otimes K_X) \to H^{0,2n-1}(X, EndE) \to 0.$$

Proof. We tensor $0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ with $EndE$ and take its cohomology. □

We denote the determinant line bundle of $\mathcal{M}_X$ by $\mathcal{L}_X$ with $\mathcal{L}_X|_E = det(H^{0,odd}(X, EndE)) \otimes det(H^{0,even}(X, EndE)))^{-1}$ (similarly for $\mathcal{L}_Y \to \mathcal{M}_Y$), and note that the transpose of the above LES with respect to Serre duality pairings on X and Y remains the same. This implies

$$2c_1(\mathcal{L}_X) = r^* c_1(\mathcal{L}_Y).$$

Meanwhile, if there exist a square root $\mathcal{L}_Y^{\frac{1}{2}}$, with $c_1(\mathcal{L}_Y) = 2c_1(\mathcal{L}_Y^{\frac{1}{2}})$, then $c_1(\mathcal{L}_X) = r^* c_1(\mathcal{L}_Y^{\frac{1}{2}})$ provided $H^2(\mathcal{M}_X)$ is free of even torison. In general, we have

Theorem 4.2. *Let Y be a smooth anti-canonical divisor in a projective $2n$-fold X with $Tor(H_*(X,\mathbb{Z})) = 0$. In particular, Y is a Calabi-Yau $(2n-1)$-fold. Let $\mathcal{M}_X$ be a moduli space of simple bundles on X with fixed Chern classes which has a well-defined restriction morphism*

$$r : \mathcal{M}_X \to \mathcal{M}_Y,$$

to a moduli space of simple bundles on Y with fixed Chern classes.

Then there exists a square root $\mathcal{L}_Y^{\frac{1}{2}}$ of $\mathcal{L}_Y$ (i.e. $\mathcal{L}_Y^{\frac{1}{2}} \otimes \mathcal{L}_Y^{\frac{1}{2}} \cong \mathcal{L}_Y$) such that

$$c_1(\mathcal{L}_X) = r^* c_1(\mathcal{L}_Y^{\frac{1}{2}}),$$

where $\mathcal{L}_X$ (resp. $\mathcal{L}_Y$) is the determinant line bundle of $\mathcal{M}_X$ (resp. $\mathcal{M}_Y$).

Proof. By the Lefschetz hyperplane theorem, $Tor(H_*(Y,\mathbb{Z})) = 0$. By Theorem 3.2, there exists a square root $\mathcal{L}_Y^{\frac{1}{2}}$ coming from the pull-back of a square root $det(Ind(\not{D}_{End\mathcal{E}_Y}))^{\frac{1}{2}} \to \mathcal{B}_Y^*$. As $\mathcal{B}_Y \backslash \mathcal{B}_Y^* \subseteq \mathcal{B}_Y$ has large codimension, $det(Ind(\not{D}_{End\mathcal{E}_Y}))^{\frac{1}{2}}$ uniquely extends to a complex line bundle $det(Ind(\not{D}_{End\mathcal{E}_Y}))^{\frac{1}{2}} \to \mathcal{B}_Y$. As before, we are left to show

$$c_1(Ind(\not{D}_{End\mathcal{E}_X})) - r^* c_1(det(Ind(\not{D}_{End\mathcal{E}_Y}))^{\frac{1}{2}}) = 0 \in H^2(\mathcal{B}_X^*, \mathbb{Z}), \tag{3}$$

for a $SU(N)$ complex vector bundle on X with $N \gg 0$, where $r : \mathcal{B}_X^* \to \mathcal{B}_Y$ is the restriction map to the orbit space of connections on Y, and $\mathcal{E}_X$ (resp. $\mathcal{E}_Y$) is the universal family over $\mathcal{B}_X^*$ (resp. $\mathcal{B}_Y^*$). The index bundle $Ind(\not{D}_{End\mathcal{E}_X})$, (defined by a lifting $c_1(X)$ of $w_2(X)$) satisfies $Ind(\not{D}_{End\mathcal{E}_X})|_A = H^{0,odd}_{D_A}(X, EndE) - H^{0,even}_{D_A}(X, EndE)$. The Federer spectral sequence

$$E_2^{p,q} \cong H^p(X, \pi_{p+q}(BSU(N))) \Rightarrow \pi_q(Map_E(X, BSU(N))).$$

gives $\pi_1(Map_E(X, BSU(N))) \cong \bigoplus_{k\geq 1} H^{2k+1}(X,\mathbb{Z})$ for $N \gg 0$, which is torsion-free. Then $H_1(\mathcal{B}_X^*, \mathbb{Z}) \cong \pi_1(\mathcal{B}_X^*, \mathbb{Z}) \cong \pi_1(Map_E(X, BSU(N)))$ and $H^2(\mathcal{B}_X^*, \mathbb{Z})$ is also torsion-free. Thus to prove (3), we only need

$$2c_1(Ind(\not{D}_{End\mathcal{E}_X})) - r^* c_1(det(Ind(\not{D}_{End\mathcal{E}_Y}))) = 0 \in H^2(\mathcal{B}_X^*, \mathbb{Q}).$$

We are furthermore left to show

$$2c_1(Ind(\not{D}_{End\mathcal{E}_X})|_C) - c_1((r^* det(Ind(\not{D}_{End\mathcal{E}_Y})))|_C) = 0 \in H^2(C, \mathbb{Q})$$

for any embedded surface $C \subseteq \mathcal{B}_X^*$.

We denote the universal bundle over C to be $\mathcal{E} \to X \times C$, $\pi_X : X \times C \to C$, $\pi_Y : Y \times C \to C$ to be projection maps, and $i = (i_Y, Id) : Y \times C \to X \times C$. The commutative diagram

$$\begin{array}{ccc} Y \times C & \xrightarrow{\quad i \quad} & X \times C \\ & {\scriptstyle \pi_Y} \searrow \quad \swarrow {\scriptstyle \pi_X} & \\ & C & \end{array}$$

implies that $\pi_{X_!} \circ i_! = \pi_{Y_!}$ for Gysin homomorphisms on cohomologies. We apply the Atiyah-Singer family index theorem [3],

$$\begin{aligned} c_1(Ind(\not{D}_{End\mathcal{E}_X})|_C) &= \pi_{X_!}([ch(End\mathcal{E}) \cdot Td(X)]^{(2n+1)}) \\ &= (\sum_{i=1}^{n} ch_{2i}(End\mathcal{E}) \cdot Td_{2n-2i+1}(X))/[X], \end{aligned}$$

$$\begin{aligned} c_1((r^*det(Ind(\not{D}_{End\mathcal{E}_Y})))|_C) &= \pi_{Y!}([ch(End(i^*\mathcal{E}))\cdot Td(Y)]^{(2n)}) \\ &= \pi_{X!}\circ i_!([i^*ch(End\mathcal{E})\cdot \frac{Td(X)|_Y}{Td(\mathcal{N}_{Y/X})}]^{(2n)}) \\ &= \pi_{X!}([i_!\circ i^*(ch(End\mathcal{E})\cdot \frac{Td(X)}{Td(K_X^{-1})})]^{(2n+1)}) \\ &= \pi_{X!}([ch(End\mathcal{E})\cdot \frac{Td(X)}{Td(K_X^{-1})}\cdot c_1(K_X^{-1})]^{(2n+1)}) \\ &= [ch(End\mathcal{E})\cdot Td(X)\cdot (1-e^{-c_1(X)})]^{(2n+1)}/[X]. \end{aligned}$$

We introduce $\widetilde{Td}(X)=Td(X)\cdot(1-e^{-c_1(X)})$. To prove $c_1((r^*det(Ind(\not{D}_{End\mathcal{E}_Y})))|_C)=2c_1(Ind(\not{D}_{End\mathcal{E}_X})|_C)$, we are left to show

$$\widetilde{Td}_{2i-1}(X)=2\ Td_{2i-1}(X), \text{ for } 1\leq i\leq n, \tag{4}$$

i.e. $\widetilde{Td}(X)-2\ Td(X)$ consists of even index classes. Note that the $\hat{A}$-class satisfies

$$Td(X)=e^{\frac{c_1(X)}{2}}\cdot \hat{A}(X),$$

and

$$\begin{aligned} \widetilde{Td}(X)-2Td(X) &= \hat{A}(X)(e^{\frac{c_1(X)}{2}}-e^{-\frac{c_1(X)}{2}})-2\hat{A}(X)\cdot e^{\frac{c_1(X)}{2}} \\ &= -\hat{A}(X)(e^{\frac{c_1(X)}{2}}+e^{-\frac{c_1(X)}{2}}), \end{aligned}$$

which is of even index as both factors in the RHS are so. □

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