WITTEN DEFORMATION OF PRODUCT STRUCTURES ON DERHAM COMPLEX

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Abstract. Wedge product on deRham complex of a Riemannian manifold M can be pulled back to $H^*(M)$ via explicit homotopy, constructed using Green's operator, to give higher product structures. We prove Fukaya's conjecture which suggests that Witten deformation of these higher product structures have semiclassical limits as operators defined by counting gradient flow trees with respect to Morse functions, which generalizes the remarkable Witten deformation of deRham differential from a statement concerning homology to one concerning rational homotopy type of M. Various applications of this conjecture to mirror symmetry are also suggested by Fukaya in [5].

1. Introduction

Let $f : M \to \mathbb{R}$ be a Morse function on an oriented compact Riemannian manifold M. Morse theory studies the homology of the manifold by the Morse complex CM^*_f, which is a finite dimensional vector space freely generated by critical points of f, equipped with the Morse differential δ defined by counting gradient flow lines of f. In an influential paper [13], Witten suggested a differential geometric approach toward Morse theory by deforming the exterior differential operator d with

$$d_{f,\hbar} := e^{-f/\hbar}(\hbar d)e^{f/\hbar} = \hbar d + df\wedge,$$

where $\hbar \in \mathbb{R}^+$. We can obtain the formal adjoint of $d_{f,\hbar}$ defined by

$$d^*_{f,\hbar} := e^{f/\hbar}(\hbar d^*)e^{-f/\hbar} = \hbar d^* + \iota_{\nabla f}$$

and the Laplace operator defined by

$$\Delta_{f,\hbar} := d_{f,\hbar}d^*_{f,\hbar} + d^*_{f,\hbar}d_{f,\hbar}. \tag{1.1}$$

Witten argued that if we consider eigenvalues of the operator $\Delta_{f,h}$ lying inside a small interval $[0, \hbar^{3/2})$, the sum of corresponding eigensubspaces $\Omega^*(M,\hbar)_{sm} \subset \Omega^*(M,\hbar)$ could be identified with the Morse complex CM^*_f

$$\phi = \phi(\hbar) : CM^*_f \to \Omega^*(M,\hbar)_{sm}. \tag{1.2}$$

The eigenform corresponding to a critical point concentrated near that critical point, when $\hbar$ small enough. Furthermore, the Witten differential has an asymptotic expansion $d_{f,\hbar} \sim (\delta + \mathcal{O}(\hbar))$ under the above identification. Readers may see [14] for a detailed introduction. The complete proof can be found in [7, 8, 9].

A natural question is whether the Witten's approach can be extended to study the wedge product structure on differential forms, which will be an enhancement from a statement concerning homology to one concerning rational homotopy type, as these informations are captured in the differential graded algebra $(\Omega^*(M), d, \wedge)$ (if $\pi_1(M) = 0$) from [11, 12].

It is first conjectured by Fukaya in [5] that similar asymptotic expansions hold for higher products, which are combinations of $d^*_{f,\hbar}$, wedge product, Green operator and projection to small eigensubspaces, in the Witten twisted deRham theory. The leading order terms in the asymptotic expansions are conjectured to be operators defined by counting gradient trees, which are A_∞ products $\{m_k^{Morse}\}_{k\in\mathbb{Z}_+}$ in the Morse category defined in [4], whose morphism space from f_i to f_j is the Morse complex CM^*_{ij} with respect to $f_{ij} = f_j - f_i$.

To be more precise, we are forced to consider more than one Morse function in order to satisfy the Leibniz rule. This leads to the notation of the differential graded (dg) category $DR_\hbar(M)$, with objects being smooth functions on M. The corresponding morphism complex relating f_i to f_j is given by the Witten twisted complex $\Omega^*_{ij}(M,\hbar) = (\Omega^*(M), d_{ij} := e^{-f_{ij}/\hbar}(\hbar d)e^{f_{ij}/\hbar})$. The finite dimensional subcomplex $\Omega^*_{ij}(M,\hbar)_{sm} \subset \Omega^*_{ij}(M,\hbar)$ is a homotopy retract under explicit homotopy involving Green operator. We can pull back the wedge product in the deRham category $DR_\hbar(M)$ via the homotopy, making use of homological perturbation lemma in [10], to give a deformed A_∞ category $DR_\hbar(M)_{sm}$ with A_∞ structure $\{m_k(\hbar)\}_{k\in\mathbb{Z}_+}$.

Fukaya's conjecture says that the A_∞ structure $\{m_k(\hbar)\}_{k\in\mathbb{Z}_+}$, expressed explicitly in terms of Witten twisted Green operator and wedge product, has leading order given by $\{m_k^{Morse}\}_{k\in\mathbb{Z}_+}$ defined by counting gradient flow trees, via the isomorphism ϕ.

Conjecture (Fukaya [5])**.** *For generic sequence of functions $\vec{f} = (f_0, \dots, f_k)$, with corresponding sequence of points $\vec{q} = (q_{01}, q_{12}, \dots, q_{(k-1)k})$ such that q_{ij} is a critical point of $f_{ij} = f_j - f_i$, we have*

$$m_k(\hbar)(\phi(\vec{q})) = e^{-\frac{A(\vec{q})}{\hbar}}(\phi(m_k^{Morse}(\vec{q})) + \mathcal{O}(\hbar^{1/2})), \tag{1.3}$$

where $A(\vec{q}) = f_{0k}(q_{0k}) - f_{01}(q_{01}) - \cdots - f_{(k-1)k}(q_{(k-1)k})$.

We prove Fukaya's conjecture in this paper.

Theorem (Main Theorem)**.** *Fukaya's conjecture is true.*

If we rescale $\phi(q_{ij})$ by multiplication of $e^{-\frac{f_{ij}(q_{ij})}{\hbar}}$, the above statement simply reads

$$\lim_{\hbar\to 0} m_k(\hbar) = m_k^{Morse}.$$

As A_∞ relations of $\{m_k(\hbar)\}_{k\in\mathbb{Z}_+}$ are obvious from their algebraic constructions while those of $\{m_k^{Morse}\}_{k\in\mathbb{Z}_+}$ require studies for boundaries of moduli spaces of gradient flow trees (see e.g. [1, 4]), we obtain an alternative proof for A_∞ relations of $\{m_k^{Morse}\}_{k\in\mathbb{Z}_+}$ as an corollary.

The original Witten-Morse theory is exactly the case $k = 1$, involving detailed estimate of operator d_{ij} along gradient flow lines. Starting from $k \geq 3$, our theorem involves the semi-classical analysis of the Witten twisted Green operator which is not included in the original theory.

Our Main Theorem for $k = 2$ involves three functions f_0, f_1, f_2, having q_{01}, q_{12}, q_{02} being critical points of f_{01}, f_{12}, f_{02} respectively, and can be proven using the analytical techniques in [7, 9]. We compute the leading order term in the matrix coefficients of $m_2(\hbar)$, which is essentially the integral

$$\int_M \langle m_2(\hbar)(\phi(q_{01}), \phi(q_{12})), \frac{\phi(q_{02})}{\|\phi(q_{02})\|^2}\rangle. \tag{1.4}$$

First, we make use of the global a priori estimate of the form $\phi(q_{ij}) \sim \mathcal{O}(e^{\frac{\rho(q_{ij},\cdot)}{\hbar}})$ (lemma 16), with ρ being the Agmon distance defined in definition 9, to cut off the integrand to neighborhoods of gradient trees appeared in m_2^{Morse}. After cutting off the integrand, we need to compute the leading order contribution from each gradient tree. The WKB approximation (lemma 25) of the eigenforms $\phi(q_{ij})$ is used to compute the leading order contribution of (1.4).

When $k \geq 3$, what we need is an WKB approximation of G_{ij} along a gradient flow line of f_{ij} in §5. More precisely, we need to study the local behaviour of the inhomogeneous Witten Laplacian equation of the form

$$\Delta_{ij}\zeta_E = d_{ij}^*(e^{-\frac{\psi_S}{\hbar}}\nu) \tag{1.5}$$

along a gradient flow line segment of f_{ij} from x_S to x_E, and obtain an approximation of ζ_E of the form

$$\zeta_E \sim e^{-\psi_E/\hbar}\hbar^{-1/2}(\omega_{E,0} + \omega_{E,1}\hbar^{1/2} + \dots).$$

The *key* step in our proof is to determine ψ_E from ψ_S and detailed construction is given in §5. A naive guess $\tilde{\psi}_E(x) := \inf_y(\psi_S(y) + \rho(y,x))$ captures the desired behaviors of ψ_E near x_E but is singular along a hypersurface U_S containing x_S, which cannot be used to solve (1.5) iteratively. We solve the minimal configuration in variational problem associated to $\inf_y(\psi_S(y) + \rho(y,x))$ and find that the point y is forced to lie on U_S, with a unique geodesic joining to x which realizes $\rho(y,x)$, for those x closed enough to x_E. This family of geodesics $\{\gamma_y\}_{y\in U_S}$ gives a foliation of a neighborhood of the flow line segment. Therefore we can use $\psi_E(\gamma_y(t)) = \psi_S(y) + t$ as an extension of $\tilde{\psi}_E$ across U_S. Provided the analytic results for G_{ij} (§4 and §5), the proof of the general case is similar to the $k = 3$ case, but with more involved combinatorics.

This paper consists of two parts. The first part gives the basic setup and definitions in §2 and the proof modulo technical analysis in §3. The second part is a study of Witten twisted Green operator in §4 and §5 which is used in previous sections.

2. Setting

In this section, we introduce the definitions and notations we need and state our main theorem. We begin with the definition of deRham category.

2.1. **deRham category.**

Given a compact oriented Riemannian manifold M, we can construct the deRham category $DR_\hbar(M)$ depending on a small real parameter $\hbar$. Objects of the category are smooth functions

$$f : M \to \mathbb{R}.$$

For any two objects f_i and f_j, we define the space of morphisms between them to be

$$\mathrm{Hom}^*_{DR_\hbar(M)}(f_i, f_j) = \Omega^*(M),$$

with differential $\hbar d + df_{ij}\wedge$, where $f_{ij} := f_j - f_i$. The composition of morphisms is defined to be the wedge product of differential forms on M. This composition is associative and hence the resulted category is a dg category. We denote the complex corresponding to $\mathrm{Hom}^*_{DR_\hbar(M)}(f_i, f_j)$ by $\Omega^*_{ij}(M, \hbar)$ and the differential $\hbar d + df_{ij}$ by d_{ij}. We then consider the Morse category which is closely related to the deRham category.

2.2. **Morse category.**

The Morse category $Morse(M)$ has the same class of objects as the deRham category $DR_\hbar(M)$, with the space of morphisms between two objects given by

$$Hom^*_{Morse(M)}(f_i, f_j) = CM^*(f_{ij}) = \sum_{q\in Crit(f_{ij})} \mathbb{C} \cdot e_q.$$

It is the Morse complex which is defined when f_{ij} is Morse. In this complex, e_q's are declared to be an orthonormal basis and graded by the Morse index of corresponding critical point q, which is the dimension of unstable submanifold V_q^-. The Morse category $Morse(M)$ is an A_∞-category equipped with higher products m_k^{Morse} for every $k \in \mathbb{Z}_+$, or simply denoted by m_k, which are given by counting gradient flow trees. To describe that, we first need some terminologies about directed trees.

2.2.1. *Directed trees.*

Definition 1. *A trivalent directed d-leafed tree T means an embedded tree in $\mathbb{R}^2$, together with the following data:*

(1) *a finite set of vertices $V(T)$;*
(2) *a set of internal edges $E(T)$;*
(3) *a set of d semi-infinite incoming edges $E_{in}(T)$;*
(4) *a semi-infinite outgoing edge e_{out}.*

Every vertex is required to be trivalent, having two incoming edges and one outgoing edge.

For simplicity, we will call it a d-tree. They are identified up to continuous map preserving the vertices and edges. Therefore, the topological class for d-trees will be finite.

Given a d-tree, by fixing the anticlockwise orientation of $\mathbb{R}^2$, we have cyclic ordering of all the semi-infinite edges. We can label the incoming edges by pairs of consecutive integers $(d-1)d, (d-1)(d-2), \ldots, 01$ and the outgoing edges by $0d$ such that the cyclic ordering $01, \ldots, (d-1)d, 0d$ agrees with the induced cyclic ordering of $\mathbb{R}^2$. Furthermore, we can extend this labeling to all the internal edges, by induction along the directed tree. If we have an vertex v with two incoming edges labelled ij and jk, then we assign labeling ik to the outgoing edge. For example, there are two different topological types for 3-tree, with corresponding labelings for their edges as shown in the following figure.

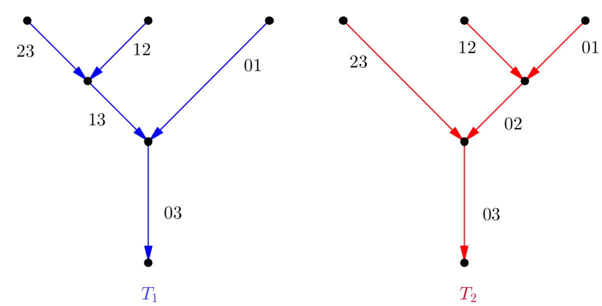


FIGURE 1. two different types of 3-trees

A pair (e, v), with e being an edge (either finite or semi-infinite) and v being an adjacent vertex, is called a flag. The unique vertex attached to the outgoing semi-infinite edge is called the root vertex. The following figure shows different flags on the tree T_1.

For the purpose of Morse homology, we need the following notation of metric trees.

Definition 2. *A metric d-tree $\tilde{T}$ is a d-tree together with a length function $l : E(T) \to (0, +\infty)$.*

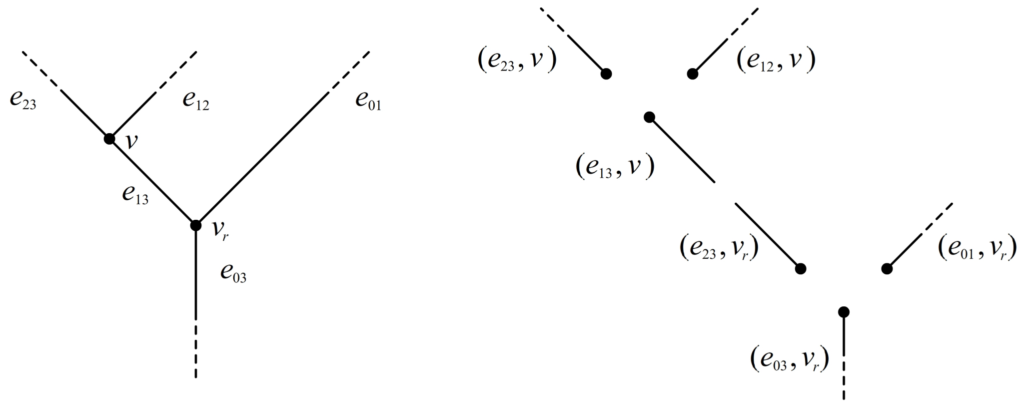


FIGURE 2. An example of a 3-tree and the corresponding flags.

Metric d-trees are identified up to homeomorphism preserving the length functions. The space of metric d-trees has finite number of components, with each component corresponding to a topological type T. The component corresponding to T, denoted by $\mathcal{S}(T)$, is a copy of $(0,+\infty)^{|E(T)|}$, where $|E(T)|$ is the number of internal edges and equals to $d-2$. The space $\mathcal{S}(T)$ can be partially compactified to a manifold with corners $(0,+\infty]^{|E(T)|}$, by allowing the length of internal edges going to be infinity. In particular, it has codimension-1 boundary

$$\partial\overline{\mathcal{S}(T)} = \coprod_{T=T'\sqcup T''} \mathcal{S}(T^{'}) \times \mathcal{S}(T^{''}),$$

where $\sqcup$ means joining the outgoing edge of $T^{'}$ with one of the incoming edges of $T^{''}$ to give an internal edge of infinite length.

2.2.2. *Morse A_∞ structure.* We are going to describe the product m_k of the Morse category. First of all, one may notice that the morphisms between two objects f_i and f_j is only defined when f_{ij} is Morse. Therefore, when we consider a sequence of functions $f_0,\dots,f_k$, we said the sequence is Morse if f_{ij} are Morse for all $i \neq j$. Given a Morse sequence $\vec{f} = (f_0,\dots,f_k)$, with a sequence of points $\vec{q} = (q_{01},\dots q_{(k-1)k}, q_{0k})$ such that q_{ij} is a critical point of f_{ij}, we have the following definition of gradient flow tree.

Definition 3. *A gradient flow tree Γ of $\vec{f}$ with endpoints at $\vec{q}$ is a continuous map $\mathbf{f} : \tilde{T} \to M$ such that it is a upward gradient flow lines of f_{ij} when restricted to the edge ij, the semi-infinite incoming edge $i(i+1)$ begins at the critical point $q_{i(i+1)}$ and the semi-infinite outgoing edge $0k$ ends at the critical point q_{0k}.*

We use $\mathcal{M}(\vec{f},\vec{q})$ to denote the moduli space of gradient trees (in the case $k=1$, the moduli of gradient flow line of a single Morse function has an extra $\mathbb{R}$ symmetry given by translation in the domain. We will use this notation for the reduced moduli, that is the one after taking quotient by $\mathbb{R}$). It has a decomposition according to topological types

$$\mathcal{M}(\vec{f},\vec{q}) = \coprod_{T} \mathcal{M}(\vec{f},\vec{q})(T).$$

This space can be endowed with smooth manifold structure if we put generic assumption on the Morse sequence. For an incoming critical point $q_{i(i+1)}$, with corresponding stable submanifold $V^{+}_{q_{i(i+1)}}$, we define a map

$$\mathbf{f}_{T,i(i+1)} : V^{+}_{q_{i(i+1)}} \times \mathcal{S}(T) \to M.$$

Fixing a point x in $V^{+}_{q_{i(i+1)}}$ together with a metric tree $\tilde{T}$, we need to determine a point in M. First, suppose v is the vertex connected to the edge labelled $i(i+1)$, there is a unique path following the directed graph joining v to the root vertex v_r. To determine the image of our function, we flow the point x by gradient flow with respect to Morse function according to labeling of edges in the path, with time determined by the length of the edge.

The maps are then put together to give a map

$$\mathbf{f}_T : V^{-}_{q_{0k}} \times V^{+}_{q_{(k-1)k}} \times \cdots \times V^{+}_{q_{01}} \times \mathcal{S}(T) \to \prod_{k+1} M, \tag{2.1}$$

where we use the embedding $\iota : V^{-}_{q_{0k}} \to M$ for the first component. There is a generic assumption on $\vec{f}$.

Definition 4. *A Morse sequence $\vec{f}$ is said to be generic if the image of $\mathbf{f}_T$ intersect transversally with the diagonal submanifold $\Delta \cong M \hookrightarrow M^{k+1}$, for any sequence of critical point $\vec{q}$ and any topological type T.*

When the sequence is generic, the moduli space $\mathcal{M}(\vec{f},\vec{q})$ is of dimension

$$\dim_{\mathbb{R}}(\mathcal{M}(\vec{f},\vec{q})) = \deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2,$$

where $\deg(q_{ij})$ is the Morse index of the critical point. Therefore, we can define m_k^{Morse}, or simply denoted by m_k, using the signed count $\#\mathcal{M}(\vec{f},\vec{q})$ of points in $\dim_{\mathbb{R}}(\mathcal{M}(\vec{f},\vec{q}))$ when it is of dimension 0. In order to have a signed count, we have to get an orientation of the space $\mathcal{M}(\vec{f},\vec{q})$. We will come to that later in definition 35.

We now give the definition of the higher products in the Morse category.

Definition 5. *Given a generic Morse sequence $\vec{f}$ with sequence of critical points $\vec{q}$, we define*

$$m_k : CM^{*}_{k(k-1)} \otimes \cdots \otimes CM^{*}_{01} \to CM^{*}_{0k}$$

given by

$$\langle m_k(q_{(k-1)k}, \dots, q_{01}), q_{0k}\rangle = \#\mathcal{M}(\vec{f}, \vec{q}), \tag{2.2}$$

when

$$\deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2 = 0.$$

Otherwise, the m_k is defined to be zero.

One may notice m_k^{Morse} can only be defined when $\vec{f}$ is a Morse sequence satisfying the generic assumption in definition 4. The Morse category is indeed a A_∞ pre-category instead of an honest category. We will not go into detail about the algebraic problem on getting an honest category from this structures. For details about this, readers may see [1, 4].

2.3. **From deRham to Morse.** To relate $DR_\hbar(M)$ and $Morse(M)$, we need to apply homological perturbation to $DR_\hbar(M)$. Fixing two functions f_i and f_j, we consider the Witten Laplacian

$$\Delta_{ij} = d_{ij}d_{ij}^* + d_{ij}^*d_{ij},$$

where $d_{ij}^* = \hbar d^* + \iota_{\nabla f_{ij}}$. We take the interval $I(\hbar) = [0, \hbar^{3/2})$ and denote the span of eigenspaces with eigenvalues contained in $I(\hbar)$ by $\Omega_{ij}^*(M,\hbar)_{sm}$.

By the result of [9], we have a map

$$\phi = \phi_{ij}(\eta, \hbar) : CM^*(f_{ij}) \to \Omega_{ij}^*(M, \hbar)_{sm}$$

depending on η, $\hbar \in \mathbb{R}_+$ such that it is an isomorphism when η, $\hbar$ are small enough. Here η, which can be arbitrarily small, is the radius of some cut off function will be introduced in section 3.

Furthermore, under the identification $\phi_{ij}(\eta,\hbar)$, we have the identification of differential d_{ij} and Morse differential m_1 from [9] as

$$\langle d_{ij}\phi_{ij}(p), \phi_{ij}(q)\rangle = e^{-\frac{f_{ij}(q)-f_{ij}(p)}{\hbar}} \langle m_1(p), q\rangle(1 + \mathcal{O}(\hbar)) \tag{2.3}$$

for $\hbar$ small enough, if p, q are critical points of f_{ij}. This is originally proposed by Witten to understand Morse theory using twisted deRham complex.

It is natural to ask whether the product structures of two categories are related via this identification, and the answer is definite. The first observation is that the Witten's approach indeed produces an A_∞ category, denoted by $DR_\hbar(M)_{sm}$, with A_∞ structure $\{m_k(\hbar)\}_{k\in\mathbb{Z}_+}$. It has the same class of objects as $DR_\hbar(M)$. However, the space of morphisms between two objects f_i, f_j is taken to be $\Omega_{ij}^*(M,\hbar)_{sm}$, with $m_1(\hbar)$ being the restriction of d_{ij} to the eigenspace $\Omega_{ij}^*(M,\hbar)_{sm}$.

The natural way to define $m_2(\hbar)$ for any three objects f_0, f_1 and f_2 is the operation given by

$$\begin{tikzcd}
\Omega^*_{12}(M,\hbar)_{sm} \otimes \Omega^*_{01}(M,\hbar)_{sm} \arrow[r, "(\iota_{12},\iota_{01})"] & \Omega^*_{12}(M,\hbar) \otimes \Omega^*_{01}(M,\hbar) \arrow[d, "\wedge"] \\
& \Omega^*_{02}(M,\hbar) \arrow[d, "P_{02}"] \\
& \Omega^*_{02}(M,\hbar)_{sm},
\end{tikzcd}$$

where ι_{12} and ι_{01} are inclusion maps and $P_{ij} : \Omega^*_{ij}(M,\hbar) \to \Omega^*_{ij}(M,\hbar)_{sm}$ is the orthogonal projection.

Notice that $m_2(\hbar)$ is not associative, and we need a $m_3(\hbar)$ to record the non-associativity. To do this, let us consider the Green's operator G^0_{ij} corresponding to Witten Laplacian Δ_{ij}. We let

$$G_{ij} = (I - P_{ij})G^0_{ij} \tag{2.4}$$

and

$$H_{ij} = d^*_{ij} G_{ij}. \tag{2.5}$$

Then H_{ij} is a linear operator from $\Omega^*_{ij}(M,\hbar)$ to $\Omega^{*-1}_{ij}(M,\hbar)$ and we have

$$d_{ij}H_{ij} + H_{ij}d_{ij} = I - P_{ij}.$$

Namely $\Omega^*_{ij}(M,\hbar)_{sm}$ is a homotopy retract of $\Omega^*_{ij}(M,\hbar)$ with homotopy operator H_{ij}. Suppose f_0, f_1, f_2 and f_3 are smooth functions on M and let $\varphi_{ij} \in \Omega^*_{ij}(M,\hbar)_{sm}$, the higher product

$$m_3(\hbar) : \Omega^*_{23}(M,\hbar)_{sm} \otimes \Omega^*_{12}(M,\hbar)_{sm} \otimes \Omega^*_{01}(M,\hbar)_{sm} \to \Omega^*_{03}(M,\hbar)_{sm}$$

is defined by

$$\begin{multline} m_3(\hbar)(\varphi_{23},\varphi_{12},\varphi_{01}) = \\ P_{03}(H_{13}(\varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01}) + P_{03}(\varphi_{23} \wedge H_{02}(\varphi_{12} \wedge \varphi_{01})). \end{multline} \tag{2.6}$$

In general, construction of $m_k(\hbar)$ can be described using k-tree. For $k \geq 2$, we decompose $m_k(\hbar) := \sum_T m^T_k(\hbar)$, where T runs over all topological types of k-trees.

$$m^T_k(\hbar) : \Omega^*_{(k-1)k}(M,\hbar)_{sm} \otimes \cdots \otimes \Omega^*_{01}(M,\hbar)_{sm} \to \Omega^*_{0k}(M,\hbar)_{sm}$$

is an operation defined along the directed tree T by

(1) applying inclusion map $\iota_{i(i+1)} : \Omega^*_{i(i+1)}(M,\hbar)_{sm} \to \Omega^*_{i(i+1)}(M,\hbar)$ at semi-infinite incoming edges;
(2) applying wedge product $\wedge$ to each interior vertex;
(3) applying homotopy operator H_{ij} to each internal edge labelled ij;
(4) applying projection P_{0k} to the outgoing semi-infinite edge.

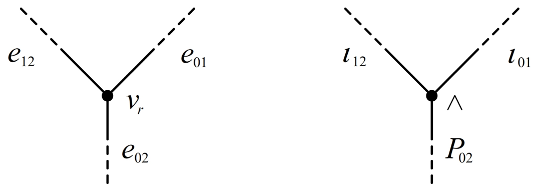


FIGURE 3. The unique 2-tree and the corresponding assignment of operators for defining $m_2(\hbar)$.

The following graph shows the operation associated to the unique 2-tree.

The higher products $\{m_k(\hbar)\}_{k\in\mathbb{Z}_+}$ satisfies the generalized associativity relation which is the so called A_∞ relation. One may treat the A_∞ product as a pullback of the wedge product under the homotopy retract $P_{ij} : \Omega^*_{ij}(M,\hbar) \to \Omega^*_{ij}(M,\hbar)_{sm}$. This proceed is called the homological perturbation lemma. For details about this construction, readers may see [10]. As a result, we obtain an A_∞ pre-category $DR_\hbar(M)_{sm}$.

Finally, we restate our Main Theorem with the notations from this section.

Theorem 6 (Main Theorem). *Given $f_0,\dots,f_k$ satisfying generic assumption 4, with $q_{ij} \in CM^*(f_{ij})$ be corresponding critical points, there exist η_0, $\hbar_0 > 0$ and $C_0 > 0$, such that $\phi_{ij}(\eta,\hbar) : CM^*(f_{ij}) \to \Omega^*_{ij}(M,\hbar)_{sm}$ are isomorphism for all $i \neq j$ when $\eta < \eta_0$ and $\hbar < \hbar_0$. If we write $\phi(q_{ij}) = \phi_{ij}(\eta,\hbar)(q_{ij})$, then we have*

$$\begin{aligned}
&\langle m_k(\hbar)(\phi(q_{(k-1)k}),\dots,\phi(q_{01})), \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2}\rangle \\
= \;& \hbar^{2-k} e^{-A/\hbar}(\langle m_k^{Morse}(q_{(k-1)k},\dots,q_{01}), q_{0k}\rangle + R(\hbar)),
\end{aligned}$$

with

$$|R(\hbar)| \le C_0 \hbar^{1/2}$$

and $A = f_{0k}(q_{0k}) - f_{01}(q_{01}) - \cdots - f_{(k-1)k}(q_{(k-1)k})$.

Remark 7. *The constants η_0, C_0 and $\hbar_0$ depend on the functions $f_0,\dots,f_k$. In general, we cannot choose fixed constants that the above statement holds true for all $m_k(\hbar)$ and all sequences of functions.*

Remark 8. *The constant A has a geometric meaning. If we consider the cotangent bundle T^*M of a manifold M which equips the canonical symplectic form ω_{can}, and take $L_i = \Gamma_{df_i}$ to be the Lagrangian sections. Then $q_{ij} \in L_i \pitchfork L_j$ and A would be the symplectic area of a degenerated holomorphic disk passing through the intersection points q_{ij} and having boundary lying on L_i. For details, one may consult* [10]

3. Proof of Main Theorem

In the proof, we fix a generic sequence $\vec{f}$ of $k+1$ functions, with corresponding sequence of critical points $\vec{q}$. First of all, we have

$$\deg(m_k(\hbar)(\phi(q_{(k-1)k}),\dots,\phi(q_{01})) = \sum_{i=0}^{k-1}\deg(q_{i(i+1)}) - k + 2,$$

so $\langle m_k(\hbar)(\phi(q_{(k-1)k}),\dots,\phi(q_{01}),\phi(q_{0k})\rangle$ is non-trivial only when the equality

$$\sum_{i=0}^{k-1}\deg(q_{i(i+1)}) - k + 2 = \deg(q_{0k}) \tag{3.1}$$

holds, which is exactly the condition for m_k^{Morse} in the Morse category to be non-trivial. We will therefore assume condition (3.1) and consider the integral

$$\begin{aligned}&\langle m_k(\hbar)(\phi(q_{(k-1)k}),\dots,\phi(q_{01})),\frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2}\rangle\\ = &\int_M m_k(\hbar)(\phi(q_{(k-1)k}),\dots,\phi(q_{01}))\wedge *\frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2}.\end{aligned}$$

Recall that each directed tree T gives an operation $m_k^T(\hbar)$ and $m_k(\hbar) = \sum_T m_k^T(\hbar)$ which is also the case in Morse category. Therefore, we just have to consider each $m_k^T(\hbar)$ separately.

The first step uses the a priori estimate and resolvent estimate to show that there is an expected exponential decay $e^{-A/\hbar}$ as described in the theorem. We can therefore drop out terms with faster exponential decay. It turns out that the integral localizes to gradient flow trees of corresponding Morse functions. The second step is to replace input eigenforms and the homotopy operators H_{ij} by their WKB approximations. The WKB approximations are governed by ODEs which make computations possible. The final step is to carry out the explicit computations for the leading order term.

3.1. **Results for a single Morse function.** We start with stating the results on Witten deformation for a single Morse function. These results come from [9], with a few modifications to fit our content.

Definition 9. *For a Morse function f_{ij}, the Agmon distance ρ_{ij}, or simply denoted by ρ, is the distance function with respect to the degenerated Riemannian metric $\langle\cdot,\cdot\rangle_{f_{ij}} = |df_{ij}|^2\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the background metric.*

Readers may see [6] for its basic properties. We denote the set of critical points by C_{ij}^*. For each $q \in C_{ij}^l$ we let

$$M_{q,\eta} = M \setminus \bigcup_{p\in C_{ij}^l\setminus\{q\}} B(p,\eta),$$

where $B(p,\eta)$ is the open ball centered at p with radius η with respect to the Agmon metric. $M_{q,\eta}$ is a manifold with boundary.

For each $q \in C^l_{ij}$, we use $\Omega^*_{ij}(M_{q,\eta},\hbar)_0$ to denote the space of differential forms with Dirichlet boundary condition, acting by Witten Lacplacian $\Delta_{ij,q,0}$. We have the following spectral gap lemma, saying that eigenvalues in the interval $I(\hbar)$ are well separated from the rest of the spectrum.

Lemma 10. *For any ϵ, $\eta > 0$ small enough, there is $\hbar_0 = \hbar_0(\epsilon,\eta) > 0$ and $C_\epsilon > 0$ such that when $\hbar < \hbar_0$, we have*

$$\mathrm{Spec}(\Delta_{ij,q,0}) \cap [\hbar^{3/2}, \hbar^{3/2} + C_\epsilon e^{-\epsilon/\hbar}) = \emptyset,$$

and also

$$\mathrm{Spec}(\Delta_{ij}) \cap [\hbar^{3/2}, \hbar^{3/2} + C_\epsilon e^{-\epsilon/\hbar}) = \emptyset.$$

The eigenforms with corresponding eigenvalue in $I(\hbar)$ are what we concentrated on, and we have the following decay estimate for them.

Lemma 11. *For any ϵ, $\eta > 0$ small enough, we have $\hbar_0 = \hbar_0(\epsilon,\eta) > 0$ such that when $\hbar < \hbar_0$, $\Delta_{ij,q,0}$ has one dimensional eigenspace in $I(\hbar)$. If we let $\varphi_q \in \Omega^*_{ij}(M_{q,\eta},\hbar)_0$ be the coresponding unit length eigenform, we have*

$$\varphi_q = \mathcal{O}_\epsilon(e^{-(\rho_{ij}(q,x)-\epsilon)/\hbar}), \tag{3.2}$$

where $\mathcal{O}_\epsilon$ stands for C^0 bound with a constant depending on ϵ. Same estimate holds for $d_{ij}\varphi_q$ and $d^*_{ij}\varphi_q$ as well.

We are now ready to give the definition of $\phi_{ij}(\eta,\hbar)$. For each critical point p, we take a cut off function θ_p such that $\theta_p \equiv 1$ in $\overline{B(p,\eta)}$ and compactly supported in $B(p,2\eta)$. Given a critical point $q \in C^l$, we let

$$\chi_q = 1 - \sum_{p \in C^l \setminus \{q\}} \theta_p.$$

Proposition 12. *For $\eta > 0$ small enough, there exists $h_0 = h_0(\eta) > 0$, such that when $h < h_0$, we have a linear isomorphism*

$$\hat{\phi}_{ij} = \hat{\phi}_{ij}(\eta,\hbar) : CM^*(f_{ij}) \to \Omega^*_{ij}(M,\hbar)_{sm}$$

defined by

$$\hat{\phi}_{ij}(\eta,\hbar)(q) = P_{ij}\chi_q\varphi_q, \tag{3.3}$$

where $P_{ij} : \Omega^*_{ij}(M,\hbar) \to \Omega^*_{ij}(M,\hbar)_{sm}$ is the projection to the small eigenspace.

Remark 13. *One may notice that φ_q is defined only up to $\pm$ sign. Recall that in the definition of Morse category, we fix an orientation for unstable submanifold V^-_q and stable submanifold V^+_q at q. The sign of φ_q is chosen such that it agrees with the orientation of V^-_q at q.*

Definition 14. *We renormalize $\hat{\phi}_{ij}(\eta,\hbar)$ to give a map $\phi_{ij}(\eta,\hbar)$ defined by*

$$\phi_{ij}(\eta,\hbar)(q)=\frac{|\lambda_-|^{\frac{1}{4}}}{|\lambda_+|^{\frac{1}{4}}}(\frac{\pi\hbar}{2})^{\frac{1}{2}(\frac{n}{2}-\deg(q))}\hat{\phi}_{ij}(\eta,\hbar)(q), \tag{3.4}$$

where λ_+ and λ_- are products of positive and negative eigenvalues of $\nabla^2 f$ at q respectively.

Remark 15. *The meaning of the normalization is to get the following asymptotic expansion*

$$\int_{V_q^-} e^{f_{ij}/\hbar}\phi_{ij}(\eta,\hbar)(q)=1+\mathcal{O}(\hbar), \tag{3.5}$$

which is the one appeared in [14].

From the estimate (3.2), we also have a similar estimate for $\hat{\phi}_{ij}(q)$.

Lemma 16. *For any ϵ, $\eta>0$ small enough, there exists $\hbar_0=\hbar_0(\epsilon,\eta)>0$ such that for $\hbar<\hbar_0$, we have*

$$\hat{\phi}_{ij}(q)=\mathcal{O}_{\epsilon,\eta}(e^{-(\rho_{ij}(q,x)-\epsilon-2\eta)/\hbar}), \tag{3.6}$$

and same estimate holds for $d_{ij}\hat{\phi}_{ij}(q)$ and $d^*_{ij}\hat{\phi}_{ij}(q)$.

The estimate for derivatives of $\hat{\phi}_{ij}(q)$ can be strengthened. Notice that $d_{ij}\hat{\phi}_{ij}(q)$ is an eigenform of degree $l+1$, similarily $d^*_{ij}\hat{\phi}_{ij}(q)$ is an eigenform of degree $l-1$. Making use of these, we have the following lemma.

Lemma 17. *For any ϵ, $\eta>0$ small enough, there exists $\hbar_0=\hbar_0(\epsilon,\eta)>0$ such that for $\hbar<\hbar_0(\epsilon,\eta)$, we have*

$$\begin{aligned} d_{ij}\hat{\phi}_{ij}(q) &= \mathcal{O}_{\epsilon,\eta}(e^{-(\alpha_q(x)-\epsilon-4\eta)/\hbar}),\\ d^*_{ij}\hat{\phi}_{ij}(q) &= \mathcal{O}_{\epsilon,\eta}(e^{-(\beta_q(x)-\epsilon-4\eta)/\hbar}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \alpha_q(x) &= \min_{p\in C^{l+1}\cup C^l\setminus\{q\}}(\rho_{ij}(q,p)+\rho_{ij}(p,x)),\\ \beta_q(x) &= \min_{p\in C^{l-1}\cup C^l\setminus\{q\}}(\rho_{ij}(q,p)+\rho_{ij}(p,x)). \end{aligned}$$

Furthermore, we can compare the inner products on CM^*_{ij} and $\Omega^*_{ij}(M,\hbar)_{sm}$. If we define a square matrix $\mathcal{D}$ with

$$\mathcal{D}_{pq}=\begin{cases} 0 & \text{if } p=q\\ e^{-\rho_{ij}(p,q)/\hbar} & \text{if } p\neq q \end{cases},$$

then we obtain the following estimates for inner product among eigenforms.

Lemma 18. *For any ϵ, $\eta>0$ small enough, there exists $\hbar_0=\hbar_0(\epsilon,\eta)>0$ such that for $\hbar<\hbar_0(\epsilon,\eta)$, we have*

$$\langle\hat{\phi}_{ij}(p),\hat{\phi}_{ij}(q)\rangle-\langle\chi_p\varphi_p,\chi_q\varphi_q\rangle=\mathcal{O}_{\epsilon,\eta}(e^{(\epsilon+4\eta)/\hbar}(\mathcal{D}^2+\mathcal{D}^3)_{pq}). \tag{3.8}$$

Furthermore, we have an estimate

$$\langle \chi_p\varphi_p, \chi_q\varphi_q\rangle = \begin{cases} 1+\mathcal{O}_{\epsilon,\eta}(e^{-(2S_0-\epsilon-4\eta)/\hbar}) & \text{if } p=q \\ \mathcal{O}_{\epsilon,\eta}(e^{-\rho_{ij}(p,q)/\hbar}) & \text{if } p\neq q \end{cases}, \tag{3.9}$$

where

$$S_0 = \min_{p'\neq q'\in C^l} \rho_{ij}(p',q').$$

Finally, we need the following resolvent estimate will be proved in section 4 for our operator G_{ij} defined in (2.4).

Lemma 19. *For any $j\in\mathbb{Z}_+$ and $\epsilon>0$, there is $k_j\in\mathbb{Z}_+$ and $\hbar_0=\hbar_0(\epsilon)>0$ such that for any two points $x_0,y_0\in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon}>0$ such that*

$$\|\nabla^j(G_{ij}u)\|_{C^0(V)} \leq C_{j,\epsilon}e^{-(\rho_{ij}(x_0,y_0)-\epsilon)/\hbar}\|u\|_{W^{k_j,2}(U)}, \tag{3.10}$$

for all $\hbar<\hbar_0$ and $u\in C^0_c(U)$, where $W^{k,p}$ refers to the Sobolev norm.

3.2. **A priori estimates.** So far we have been considering a fixed Morse function f_{ij}. From now on, we will consider a fixed generic sequence $\vec{f}$ with corresponding sequence of critical points $\vec{q}$ as in the beginning of section 3.

Notations 20. *We use q_{ij} to denote a critical point of f_{ij}. The eigenform $\phi_{ij}(\eta,\hbar)(q_{ij})$ associated to q_{ij} is abbreviated by ϕ_{ij}.*

We will use the result in the previous section to localize the integral

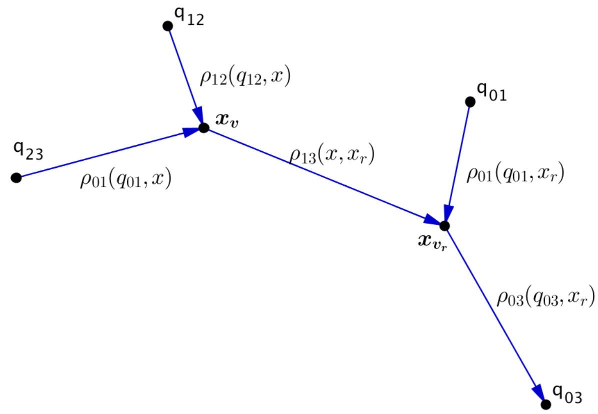
$$\int_M m_k(\hbar)(\phi(q_{(k-1)k}),\ldots,\phi(q_{01}))\wedge * \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2} \tag{3.11}$$

to gradient flow trees, when the degree condition (3.1) holds. We begin with the $m_3(\hbar)$ case which involves less combinatorics to illustrate the analytic argument.

3.2.1. $m_3(\hbar)$ *case.* There are two 3-leafed directed trees, which are denoted by T_1 and T_2. We simply consider $m_3^{T_1}(\hbar)$ for T_1 which is the tree shown in figure 2.2.1 and relate this operation to counting gradient trees of type T_1. T_1 has two interior vertices, which are denoted by v and v_r as in the figure. According to the combinatorics of T_1, we define $\vec{\rho}_{T_1}: M^{|V(T_1)|}\to\mathbb{R}_+$ which is given by

$$\begin{aligned} & \vec{\rho}_{T_1}(x_v,x_{v_r}) \\ = \; & \rho_{13}(x_v,x_{v_r})+\rho_{01}(x_{v_r},q_{01})+\rho_{12}(x_v,q_{12})+\rho_{23}(x_v,q_{23})+\rho_{03}(x_{v_r},q_{03}). \end{aligned}$$

Roughly speaking, it is the length of the geodesic tree of type T_1 with interior vertices x_v,x_{v_r} and end points of semi-infinite edges e_{ij}'s laying on q_{ij}'s as shown in the following figure.



Notice that $\vec{\rho}_{T_1}(x_v, x_{v_r}) \geq A = f_{03}(q_{03}) - f_{01}(q_{01}) - f_{12}(q_{12}) - f_{23}(q_{23})$ and the equality holds if and only if (x_v, x_{v_r}) are interior vertices of a gradient flow tree of the type T_1.

The term

$$\langle m_3^{T_1}(\phi_{23}, \phi_{12}, \phi_{01}), \frac{\phi_{03}}{\|\phi_{03}\|^2}\rangle = \int_M H_{13}(\phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge *(\frac{\phi_{03}}{\|\phi_{03}\|^2}), \tag{3.12}$$

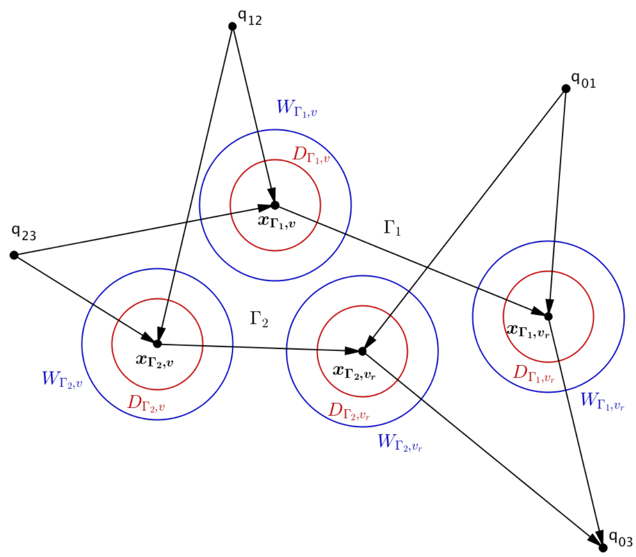
can be controlled by $\vec{\rho}_{T_1}$. More precisely, fixing two points $x_v, x_{v_r} \in M$ and ϵ small enough, lemma 19 holds for operator G_{13} and hence H_{13} with U and V being balls centering at x_v and x_{v_r} (with respect to ρ_{13}) of radius r_1. If we have two cut off functions χ and χ_r supported in $B(x_v, r_1)$ and $B(x_{v_r}, r_1)$ respectively, then we have

$$\begin{aligned} & \|\chi_r H_{13}(\chi\phi_{23} \wedge \phi_{12})\|_{L^\infty} \\ \leq \ & C_{\epsilon,\eta} e^{-(\rho_{23}(q_{23},x_v)+\rho_{12}(q_{12},x_v)+\rho_{13}(x_v,x_{v_r})-2r_1-3\epsilon-4\eta)/\hbar} \end{aligned}$$

for those small enough $\hbar$. Here the decay factors $\rho_{23}(q_{23}, x_v)$ and $\rho_{12}(q_{12}, x_v)$ come from the a priori estimate in lemma 16 for the input forms ϕ_{23} and ϕ_{12} respectively, while the decay factor $\rho_{13}(x_v, x_{v_r})$ comes from the resolvent estimate lemma 19. Combining with the decay estimates for ϕ_{01} and ϕ_{03}, we obtain

$$\begin{aligned} & \|\chi_r H_{13}(\chi\phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge *\phi_{03}\|_{L^\infty} \\ \leq \ & C_{\epsilon,\eta} e^{-(\vec{\rho}_{T_1}(x_v,x_{v_r})-4r_1-5\epsilon-6\eta)/\hbar}. \end{aligned}$$

We assume there are gradient trees $\Gamma_1, \dots \Gamma_l$ of the type T_1. For each tree Γ_i, we take open neighborhoods $D_{\Gamma_i,v}$ and $W_{\Gamma_i,v}$ of interiors vertices $x_{\Gamma,v}$ with $\overline{D_{\Gamma_i,v}} \subset W_{\Gamma_i,v}$, and similarly D_{Γ_i,v_r} and W_{Γ_i,v_r} for x_{Γ,v_r}. The following figure illustrates the situation.
We can assume there is a constant C such that $\vec{\rho}_{T_1} \geq A+C$ in $M^{|V(T_1)|} \setminus D_{\Gamma_i}$, where $D_{\Gamma_i} = D_{\Gamma_i,v} \times D_{\Gamma_i,v_r}$. If $\vec{B}(\vec{x}, r_1) = B(x_v, r_1) \times B(x_{v_r}, r_1)$ is away from



the D_{Γ_i}, we will have

$$
\begin{aligned}
& \|\chi_r H_{13}(\chi\phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge *\phi_{03}\|_{L^\infty} \\
\leq \; & C_{\epsilon,\eta} e^{-(A+\frac{C}{2})/\hbar}.
\end{aligned}
$$

Therefore, we can take cut off functions $\chi_{\Gamma_i,v}$, χ_{Γ_i,v_r} associating to each tree Γ_i, with support in $W_{\Gamma_i,v}$, W_{Γ_i,v_r} and equal to 1 on $\overline{D_{\Gamma_i,v}}$, $\overline{D_{\Gamma_i,v_r}}$ respectively, to get

$$
\begin{aligned}
& \langle m_3^{T_1}(\phi_{23},\phi_{12},\phi_{01}), \frac{\phi_{03}}{\|\phi_{03}\|^2} \rangle \\
= \; & \sum_i \int_M \{\chi_{\Gamma_i,v_r} H_{13}(\chi_{\Gamma_i,v}\phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge * \frac{\phi_{03}}{\|\phi_{03}\|^2}\} + \mathcal{O}(e^{-\frac{(A+\frac{C}{2})}{\hbar}}).
\end{aligned}
$$

This localizes the integral computing $m_3^{T_1}$ to gradient trees of type T_1. Notice that the neighborhood D_{Γ_i} and W_{Γ_i} can be chosen to be arbitrarily small in the previous argument. Next, we will consider $m_k(\hbar)$ with arbitrary k which has more complicated notations.

3.2.2. *$m_k(\hbar)$ case.* We fix a k-leafed tree T and consider the operation corresponding to it, denoted by $m_k^T(\hbar)$. We try to relate this operation to counting of gradient trees of type T. We have the function $\vec{\rho}_T : M^{|V(T)|} \to \mathbb{R}_+$ defined according to the combinatorics of T given by

$$\vec{\rho}_T(\vec{x}) = \sum_{e_{ij}\in E(T)} \rho_{ij}(x_S(e_{ij}), x_E(e_{ij})) + \sum_{i=0}^{k-1} \rho_{i(i+1)}(q_{i(i+1)}, x_E(e_{i(i+1)})) + \rho_{0k}(q_{0k}, x_S(e_{0k})). \tag{3.13}$$

Here the variables $\vec{x}$ are labelled by the vertices of T. ($x_S(e)$ and $x_E(e)$ refer to the variables corresponding to vertices which are starting point and endpoint of the edge e respectively.) Recall that $E(T)$ is the set of internal edges of T and each interior edge e has a unique label by two integers as e_{ij}, corresponding to the Morse function $f_{ij} = f_j - f_i$. The notation ρ_{ij} refers to the Agmon distance corresponding to the Morse function f_{ij}.

$\vec{\rho}_T(\vec{x})$ is the length function of a geodesic tree (may not be unique) with topological type T, with interior vertices $\vec{x}$ and semi-infinite edges ending on critical points q_{ij}. Similar to the case of $m_3(\hbar)$, we have the following lemma.

Lemma 21. *The function $\vec{\rho}_T$ is bounded below by $A = f_{01}(q_{01}) + \cdots + f_{(k-1)k}(q_{(k-1)k}) - f_{0k}(q_{0k})$, and it attains minimum at $\vec{x}$ if and only if $\vec{x}$ is the vector consisting of interior vertices of a gradient flow tree of $\vec{f}$ of type T ended at corresponding critical points $\vec{q}$.*

Proof. The proof relies on the fact (see [9]) that we have

$$|f_{ij}(x) - f_{ij}(y)| \leq \rho_{ij}(x, y),$$

if f_{ij} is a Morse function on M, and $\rho_{ij}(x, y)$ is the Agmon distance. Furthermore, the equality $f_{ij}(x) - f_{ij}(y) = \rho_{ij}(x, y)$ forces the geodesic from y to x to be a generalized integral curve of ∇f_{ij}. We apply this fact to each term in (3.13) and the result follows. $\square$

Every gradient flow tree $\Gamma \in \mathcal{M}(\vec{f}, \vec{q})(T)$ is associated with a unique minimum point $\vec{x}_\Gamma \in M^{|V(T)|}$. For each tree, we take a covering W_Γ of $\vec{x}_\Gamma$, given by a product $W_\Gamma = \prod_{v\in V(T)} W_{\Gamma,v}$, where each $W_{\Gamma,v}$ is an open subsets in M containing x_v such that all $W_{\Gamma,v}$'s are disjoint from each other. If we further take $D_\Gamma = \prod_{v\in V(T)} D_{\Gamma,v}$ such that $\overline{D_{\Gamma,v}} \subset W_{\Gamma,v}$, we have a constant $C > 0$ such that $\vec{\rho}_T \geq A + C$ on $M^{|V(T)|} \setminus D_\Gamma$. We are going to show that the integral (3.11) can be localized.

We take a finite covering of M with balls $\{B(x, r)\}_{B(x,r)\in\mathcal{J}}$ of radius r centering at x, with a partition of unity $\{\chi_B\}_{B\in\mathcal{J}}$ subordinating to it. We choose a covering $\{B_r(\vec{x})\}_{B\in\mathcal{I}}$ of $M^{|V(T)|}$ given by product $B_r(\vec{x}) = \prod_{v\in V(T)} B(x_v, r)$, where $B(x_v, r) \in \mathcal{J}$. We decompose $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $B \in \mathcal{I}_2$ are those having empty intersection with $\overline{D_\Gamma}$, and $B \in \mathcal{I}_1$ satisfying $\overline{B} \subset W_\Gamma$. These can be achieved by choosing r small enough.

We can take cut off functions subordinate to the covering $\{B\}_{\mathcal{I}}$, given by product of functions χ_B on M. We write $\vec{\chi}_B = \prod_{v\in V(T)} \chi_{B(x_v,r)}$ for the function supported in B. We will use $\vec{\chi}_B$ to cut off the following integral

$$m_k^T(\hbar)(\vec{q}) := \int_M m_k^T(\hbar)(\phi(q_{(k-1)k}),\ldots,\phi(q_{01})) \wedge * \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2}. \tag{3.14}$$

Recall that the $m_k^T(\hbar)$ is defined using wedge product and the homotopy operator H_{ij}, following the combinatorics of the tree T. We cut off the operation $m_k^T(\hbar)$ using the function $\chi_{B(x_v,r)}$ whenever taking wedge product at the vertex v. We will write $m_k^T(\hbar,\vec{\chi})$ for the integral after cutting off by $\vec{\chi}$. Therefore we have

$$m_k^T(\hbar)(\vec{q}) = \sum_{B\in\mathcal{I}_1} m_k^T(\hbar,\vec{\chi}_B)(\vec{q}) + \sum_{B\in\mathcal{I}_2} m_k^T(\hbar,\vec{\chi}_B)(\vec{q}), \tag{3.15}$$

where $m_k^T(\hbar,\vec{\chi}_{\vec{B}})(\vec{q})$ stand for the integral after cutting off by $\vec{\chi}_{\vec{B}}$. Applying the resolvent estimate in section 4 and the estimate (16), we obtain the following lemma.

Lemma 22. *For any $\epsilon > 0$, there exist positive $r(\epsilon)$, $\eta(\epsilon)$ and $\hbar(\epsilon)$ such that*

$$m_k^T(\hbar,\vec{\chi}_B) = \mathcal{O}_{r,\epsilon,\eta}(e^{-(\vec{\rho}_T(\vec{x})-\epsilon)/\hbar}) \tag{3.16}$$

for $\hbar < \hbar(\epsilon)$, if we take the covering of radius $r < r(\epsilon)$ and $\eta < \eta(\epsilon)$. Here $\vec{x}$ is the center of the ball B.

The proof is essentially the same as the case for $m_3(\hbar)$. Similarly, we can have

$$\sum_{B\in\mathcal{I}_2} m_k^T(\hbar,\vec{\chi}_B) = \mathcal{O}_{r,\epsilon,\eta}(e^{-(A+\frac{C}{2})/\hbar}),$$

for $\hbar$ small enough. It follows from the fact that $\vec{\rho}_T(\vec{x}) \geq A + C$ for those covering in $\mathcal{I}_2$. This result basically says that the integral $m_k^T(\hbar)$ can be localized to gradient flow tree using the cut off mentioned above. To summarize, we have the following proposition.

Proposition 23. *For each gradient flow tree Γ, there is a sequence of cutoff functions $\{\vec{\chi}_\Gamma\}$ which is supported in W_Γ and satisfy $\vec{\chi}_\Gamma \equiv 1$ on $\overline{D_\Gamma}$ such that*

$$m_k^T(\hbar)(\vec{q}) = \sum_{\Gamma\in\mathcal{M}(\vec{f},\vec{q})(T)} m_k^T(\hbar,\vec{\chi}_\Gamma)(\vec{q}) + \mathcal{O}(e^{-(A+\frac{C}{2})/\hbar}), \tag{3.17}$$

for $\hbar$ small enough.

Remark 24. *In the above argument, the neighborhood W_Γ can be chosen to be arbitrary small. We will obtain a smaller constant C if we shrink the neighborhood W_Γ.*

After localizing the integral, we move on to the section concerning WKB approximation which helps to compute of the leading order contribution of $m_k^T(\hbar,\vec{\chi}_\Gamma)$.

3.3. **WKB method.** In this section, we will state the results of WKB methods from section 5 and argue the WKB approximations can be used in the computation of leading order contribution of $m_k(\hbar)$. The WKB solutions are necessary for explicit computation as they are governed by ODEs instead of PDEs. We fix a gradient flow tree Γ to consider as the integrand in (3.11) is localized to gradient trees.

3.3.1. *WKB method for Witten Laplace operators.* We first state the result from [9].

Lemma 25. *There is a WKB approximation of the eigenform ϕ_{ij} of the form*

$$\phi_{ij} \sim e^{-\psi_{ij}/\hbar}\hbar^{-\frac{\deg(q_{ij})}{2}}(\omega_{ij,0} + \omega_{ij,1}\hbar^{1/2} + \dots) \tag{3.18}$$

in any small enough open set W containing $V^+_{q_{ij}} \cup V^-_{q_{ij}}$, where $\psi_{ij} = \rho_{ij}(q_{ij}, \cdot)$ is the Agmon distance function from q_{ij}.

Remark 26. *The precise meaning of this WKB approximation is given in section 5.6. Roughly speaking, it is in the sense of C^∞ approximation on every compact subset of W.*

Remark 27. $g^+_{ij} = \psi_{ij} - (f_{ij} - f_{ij}(q_{ij}))$ *is a nonnegative function which is Bott-Morse in a neighborhood of $V^+_{q_{ij}}$ with zero set $V^+_{q_{ij}}$.*

Remark 28. *There is also a similar approximation for $*\phi_{ij}$, in a neighborhood of the unstable submanifold $V^+_{q_{ij}} \cup V^-_{q_{ij}}$. In that case, $g^-_{ij} = \psi_{ij} + (f_{ij} - f_{ij}(q_{ij}))$ is a nonnegative function which is Bott-Morse in a neighborhood of $V^-_{q_{ij}}$ with zero set $V^-_{q_{ij}}$.*

3.3.2. *WKB method for homotopy operators.* Here we state a WKB method needed for the homotopy operators appearing in the higher products $m_k(\hbar)$. The proof can be found in section 5.

We begin by giving the setup of the lemma. Let $\gamma(t)$ be a flow line of $\nabla f_{ij}/|\nabla f_{ij}|_{\rho_{ij}}$ starts at $\gamma(0) = x_S$ and $\gamma(T) = x_E$ for a fixed $T > 0$. We consider an input form ζ_S defined in a neighborhood W_S of x_S. Suppose we are given a WKB approximation of ζ_S in W_S, which is an approximation of ζ_S according to order of $\hbar$ of the form

$$\zeta_S \sim e^{-\psi_S/\hbar}(\omega_{S,0} + \omega_{S,1}\hbar^{1/2} + \omega_{S,2}\hbar^1 + \dots) \tag{3.19}$$

(The precise meaning of this infinite series approximation can be found in section 5.6). We further assume that $g_S = \psi_S - f_{ij}$ is a nonnegative Bott-Morse function in W_S with zero set V_S. We consider the equation

$$\Delta_{ij}\zeta_E = (I - P_{ij})d^*_{ij}(\chi_S\zeta_S), \tag{3.20}$$

where χ_S is a cutoff function compactly supported in W_S, $P_{ij} : \Omega^*_{ij}(M, \hbar) \to \Omega^*_{ij}(M, \hbar)_{sm}$ is the projection. We want to have a WKB approximation of $\zeta_E = H_{ij}(\chi_S\zeta_S)$

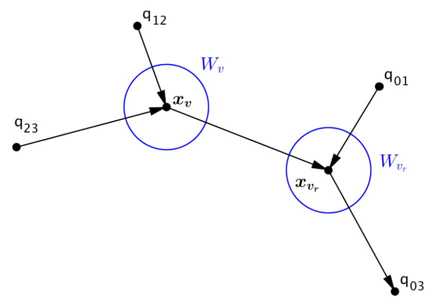
Lemma 29. *For* $\mathrm{supp}(\chi_S)$ *small enough, there is a WKB approximation of* ζ_E *in a small enough neighborhood* W_E *of* E*, of the form*

$$\zeta_E \sim e^{-\psi_E/\hbar}\hbar^{-1/2}(\omega_{E,0} + \omega_{E,1}\hbar^{1/2} + \dots). \tag{3.21}$$

Furthermore, the function $g_E := \psi_E - f_{ij}$ *is a nonnegative function which is Bott-Morse in* W_E *with zero set* $V_E = (\bigcup_{-\infty<t<+\infty} \sigma_t(V_S)) \cap W_E$ *which is closed in* W_E*, where* σ_t *is the time* t *flow of* $\nabla f_{ij}/|\nabla f_{ij}|_{\rho_{ij}}$*.*

Provided above lemmas, we are going to show that the WKB approximations can be used to compute the leading order contribution in $m_k^T(\hbar, \vec{\chi}_\Gamma)$. We proceed in the same way as the last subsection by first considering the $m_3(\hbar)$ case.

3.3.3. *$m_3(\hbar)$ case.* We fix a gradient tree Γ of type T_1 as in the subsection 3.2.1, with interior vertices $x_{\Gamma,v}$ and x_{Γ,v_r}. Since the gradient tree Γ is fixed, we trend to omit the dependence on Γ in our notations. We take neighborhoods W_v and W_{v_r} of x_v and x_{v_r} respectively, with cut off functions χ_v and χ_{v_r} supported in W_v and W_{v_r} respectively as shown in the following figure.



As $x_v \in V^+_{q_{12}} \cap V^+_{q_{23}}$, we can assume that the WKB approximations

$$\phi_{12} \sim e^{-\psi_{12}/\hbar}\hbar^{-\frac{\deg(q_{12})}{2}}(\omega_{12,0} + \omega_{12,1}\hbar^{1/2} + \dots),$$

and

$$\phi_{23} \sim e^{-\psi_{23}/\hbar}\hbar^{-\frac{\deg(q_{23})}{2}}(\omega_{23,0} + \omega_{23,1}\hbar^{1/2} + \dots)$$

hold in W_v, by taking a smaller W_v if necessary.

We apply lemma 5.1 with Morse function f_{13}, input form $\zeta_S = \phi_{23} \wedge \phi_{12}$, starting vertex $x_S = x_v$, ending vertex $x_E = x_{v_r}$, with neighborhood $W_S = W_v$ and $W_E = W_{v_r}$ (This can be done by shrinking W_v and W_{v_r} if necessary). As a result, we obtain the WKB approximation

$$H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \sim e^{-\psi_{13}/\hbar}\hbar^{-\frac{\deg(q_{23})+\deg(q_{12})+1}{2}}(\omega_{13,0} + \omega_{13,1}\hbar^{1/2} + \dots),$$

by taking $\psi_E = \psi_{13}$ and $\omega_{E,i} = \omega_{13,i}$ in the lemma.

In order to compute

$$m_3^{T_1}(\hbar, \vec{\chi}_\Gamma) = \int_M \chi_{v_r} H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge * \frac{\phi_{03}}{\|\phi_{03}\|^2}$$

up to an error of order $e^{-\frac{A}{\hbar}}\mathcal{O}(\hbar^{1/2})$, we can simply compute the integral

$$\begin{aligned} (3.22) \quad & \int_M \{\chi_{v_r}(e^{-\psi_{13}/\hbar}\hbar^{-\frac{\deg(q_{23})+\deg(q_{12})+1}{2}}\omega_{13,0}) \wedge (e^{-\psi_{01}/\hbar}\hbar^{-\frac{\deg(q_{01})}{2}}\omega_{01,0}) \\ & \qquad \wedge (\hbar^{\frac{\deg(q_{03})}{2}} \frac{e^{-\psi_{03}/\hbar} * \omega_{03,0}}{\|e^{-\psi_{03}/\hbar}\omega_{03,0}\|^2})\} \\ & = \frac{\hbar^{-1}}{\|e^{-\psi_{03}/\hbar}\omega_{03,0}\|^2} \int_M \{\chi_{v_r}(e^{-(\psi_{13}+\psi_{01}+\psi_{03})/\hbar}\omega_{13,0} \wedge \omega_{01,0} \wedge *\omega_{03,0}). \end{aligned}$$

What we obtained is an integral involving $\omega_{ij,0}$'s, which are governed by ODEs. This is easier for explicit computations. Next, we move on to show this also happens in the $m_k(\hbar)$ case for any k.

3.3.4. $m_k(\hbar)$ *case.* We consider a gradient tree Γ of type T, with k semi-infinite incoming edges. Recall in section 2.2.1 that each edge in T is assigned with a label by two integer ij. We will use ij to represent an edge in T and denote the corresponding edge in the gradient tree Γ by e_{ij}. The vertex in the gradient tree corresponding to v in T will be denoted by x_v. We again omit the dependence on Γ in our notations as it is fixed. We are going to associate $\phi_{(ij,v)} \in \Omega^*_{ij}(M,\hbar)$, together with its WKB approximation

$$\phi_{(ij,v)} \sim e^{-\psi_{(ij,v)}/\hbar}\hbar^{r_{(ij,v)}}(\omega_{(ij,v),0} + \omega_{(ij,v),1} + \dots)$$

in some neighborhood W_v of x_v to each flag (ij,v) as shown in the following figure 4. We also fix cut off functions χ_v's supported in W_v and restrict our attention to integral $m_k^T(\hbar,\vec{\chi})(\vec{q})$, using the arguments in section 3.2.

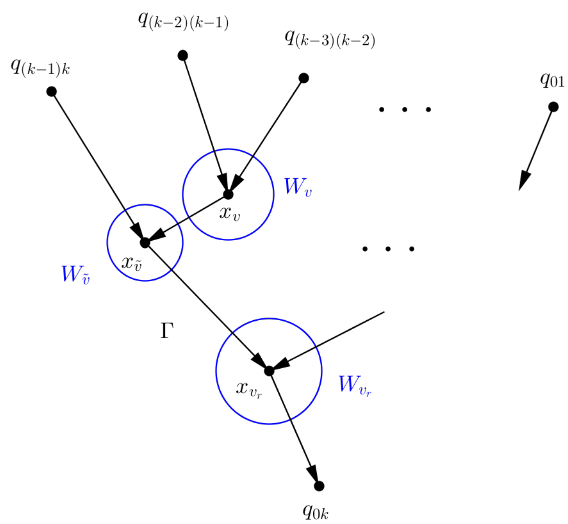


FIGURE 4.

We define the followings inductively.

(1) for a semi-infinite incoming edge $i(i+1)$ and its ending vertex v, we take $\phi_{(i(i+1),v)}$ to be the input eigenform $\phi_{i(i+1)}$, with its the WKB approximation in W_v as in lemma 25. We also let $g_{(i(i+1),v)} = \psi_{(i(i+1),v)} - (f_{i(i+1)} - f_{i(i+1)}(q))$. We choose W_v small enought such that the WKB approximation of input eigenform $\phi_{i(i+1)}$ holds in W_v. For example in the above figure 4, we require the WKB approximation of $\phi_{(k-1)k}$ in $W_{\tilde{v}}$ and WKB approximations of $\phi_{(k-2)(k-1)}, \phi_{(k-3)(k-2)}$ in W_v hold;

(2) for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting il at v as shown in figure 5, we

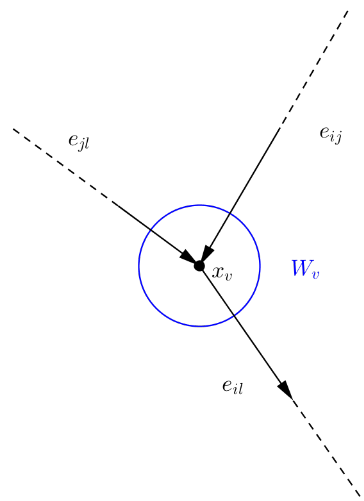


FIGURE 5.

take $\phi_{(il,v)} = \phi_{(jl,v)} \wedge \phi_{(ij,v)}$. The WKB expression of $\phi_{(il,v)}$ comes from the expression of $\phi_{(jl,v)} \wedge \phi_{(ij,v)}$, which means

$$\begin{aligned} \psi_{(il,v)} &= \psi_{(ij,v)} + \psi_{(jl,v)}, \\ \omega_{(e_{il},v),n} &= \sum_{m+m'=n} \omega_{(jl,v),m} \wedge \omega_{(ij,v),m'}, \\ r_{(il,v)} &= r_{(jl,v)} + r_{(ij,v)}. \end{aligned}$$

We also let $g_{(il,v)} = g_{(ij,v)} + g_{(jl,v)}$;

(3) for an internal edge ij with its starting vertex v_S and ending vertex v_E as shown in figure 6, we take the WKB approximation in lemma

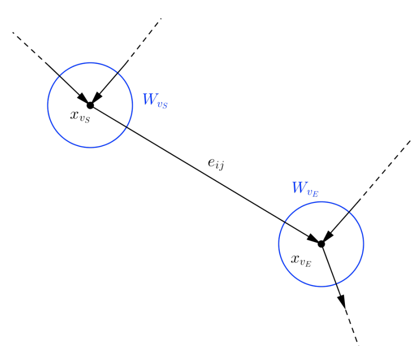


FIGURE 6.

29 of $\phi_{(ij,v_E)} = H_{ij}(\chi_{v_S}\phi_{(ij,v_S)})$ in W_{v_E} by taking $\mathrm{supp}(\chi_{v_S})$ and W_{v_E} small enough if necessary. We also let $g_{(ij,v_E)} = \psi_{(ij,v_E)} - f_{ij} +$

$\sum_{i<m\leq j} f_{(m-1)m}(q_{(m-1)m})$ and $r_{(ij,v_E)} = r_{(ij,v_S)} - \frac{1}{2}$;

(4) for the semi-infinite outgoing edge $0k$ with the root vertex v_r, we take $\phi_{(0k,v_r)}$ to be the eigenform ϕ_{0k}, with the WKB approximation of $*\phi_{0k}$ in lemma 25 holds in W_{v_r}. We also let $g_{(0k,v_r)} = \psi_{(0k,v_r)} + (f_{0k} - f_{0k}(q_{0k}))$.

Remark 30. *When we apply lemma 29 along an internal edge ij with starting vertex v_S and ending vertex v_E, the size of* $\mathrm{supp}(\chi_{v_S})$ *and* W_{v_E} *depends on the input form $\phi_{(ij,v_S)}$, or more precisely, properties of the function $\psi_{(ij,v_S)}$.*

From the definition of $m_k^T(\hbar, \vec{\chi}_\Gamma)$, we see that

$$\begin{aligned} &\langle m_k^T(\phi_{(k-1)k}, \dots, \phi_{01}), \frac{\phi_{0k}}{\|\phi_{0k}\|^2} \rangle \\ = \ & \int_M \phi_{(jk,v_r)} \wedge \phi_{(0j,v_r)} \wedge \frac{*\phi_{(0k,v_r)}}{\|\phi_{(0k,v_r)}\|^2}, \end{aligned}$$

if three edges $0j$, jk and $0k$ are meeting at the root vertex v_r. Applying lemma 25 to input eigenforms and lemma 29 to homotopy operators H_{ij} along internal edges e_{ij}'s, we prove that each WKB approximation

$$\phi_{(ij,v)} \sim e^{-\psi_{(ij,v)}/\hbar} \hbar^{r_{(ij,v)}} (\omega_{(ij,v),0} + \omega_{(ij,v),1} + \dots)$$

is an C^∞ approximation with error $e^{-\psi_{(ij,v)}/\hbar}\mathcal{O}(\hbar^\infty)$. Therefore, we can replace ϕ's by first term in its WKB approximation for computing the leading order contribution. We obtain

$$\begin{aligned} (3.23) \quad & \langle m_k^T(\phi_{(k-1)k}, \dots, \phi_{01}), \frac{\phi_{0k}}{\|\phi_{0k}\|^2} \rangle \\ & = \{\hbar^{r_{(jk,v_r)} + r_{(0j,v_r)} + r_{(0k,v_r)}} \int_M e^{-\frac{\psi_{(jk,v_r)} + \psi_{(0j,v_r)} + \psi_{(0k,v_r)}}{\hbar}} \\ & \qquad \chi_{v_r}(\omega_{(jk,v_r),0} \wedge \omega_{(0j,v_r),0} \wedge \frac{*\omega_{(0k,v_r),0}}{\|\phi_{0k}\|^2})\}(1 + \mathcal{O}(\hbar^{1/2})). \end{aligned}$$

3.4. **Explicit computations.** We are going to compute the leading order contribution of the above integral in the this subsection. This is possible since ψ's are explicit geometric functions and ω's are determined by ODEs. We again begin with the computation for $m_3(\hbar)$ case which involves less combinatorics.

3.4.1. $m_3(\hbar)$ *case:* We have to compute the leading order contribution from the integral (3.22). We first take a look on the exponential decay factor

$e^{-\frac{\psi_{13}+\psi_{01}+\psi_{03}}{\hbar}}$ in the integral. We recall that g_{13}, g_{01}^{+} and g_{03}^{-} are defined by

$$\begin{aligned}\psi_{13} &= g_{13} + f_{13} - f_{23}(q_{23}) - f_{12}(q_{12}),\\ \psi_{01} &= g_{01}^{+} + f_{01} - f_{01}(q_{01}),\\ \psi_{03} &= g_{03}^{-} - f_{03} + f_{03}(q_{03}).\end{aligned}$$

Therefore we have

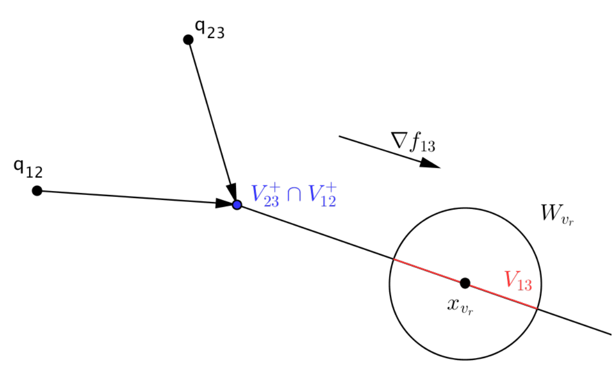
$$\psi_{13} + \psi_{01} + \psi_{03} = g_{13} + g_{01}^{+} + g_{03}^{-} + A,$$

where $A = f_{03}(q_{03}) - f_{01}(q_{01}) - f_{12}(q_{12}) - f_{23}(q_{23})$. Therefore, what we have to compute is the leading order term from the integral

$$\int_M \chi_{v_r}(e^{-\frac{g_{13}+g_{01}^{+}+g_{03}^{-}}{\hbar}}\omega_{13,0}\wedge\omega_{01,0}\wedge *\frac{\omega_{03,0}}{\|\phi_{03}\|^2}). \tag{3.24}$$

We claim that the exponential decay $e^{-\frac{g_{13}+g_{01}^{+}+g_{03}^{-}}{\hbar}}$ will limit to the delta distribution concentrating at the root vertex x_{v_r} of the gradient tree Γ. This precisely means that $g_{13} + g_{01}^{+} + g_{03}^{-} \geq 0$ is a Bott-Morse function in W_{v_r} with zero at x_{v_r}.

We recall in remark 27 that g_{01}^{+} and g_{03}^{-} are Bott-Morse with absolute minimums on V_{01}^{+} and V_{03}^{-} respectively. We also recall from lemma 29 that g_{13} is also Bott-Morse in W_{v_r} with absolute minimum denoted by V_{13} (colored red in the following figure), which is the submanifold $(\bigcup_{-\infty<t<+\infty}\sigma_t(V_{23}^{+}\cap V_{12}^{+}))\cap W_{v_r}$ flowed out from $V_{23}^{+}\cap V_{12}^{+}$ (colored blue in the following figure), under the flow of $\frac{\nabla f_{13}}{|\nabla f_{13}|_{\rho_{13}}}$ which is denoted by σ_t.



The definition of gradient tree indicates that $\{x_{v_r}\} = V_{13}\cap V_{01}^{+}\cap V_{03}^{-}$ transversally which means $e^{-\frac{g_{13}+g_{01}^{+}+g_{03}^{-}}{\hbar}}$ concentrating at x_{v_r}. To deal with this situation, we have the following lemma which will be proven in section 5.8.

Lemma 31. *Let M be a n-dimensional manifold and S be a k-dimensional submanifold in M, with a neighborhood B of S which can be identified as the normal bundle $\pi : NS \to S$. Suppose $\varphi : B \to \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set S and $\beta \in \Omega^*(B)$ has vertical compactly support along the fiber of π, we have*

$$\pi_*(e^{-\varphi(x)/\hbar}\beta) = (2\pi\hbar)^{(n-k)/2}(\iota_{\mathrm{vol}(\nabla^2\varphi)}\beta)|_V(1+\mathcal{O}(\hbar)),$$

where π_* is the integration along fiber. Here $\mathrm{vol}(\nabla^2\varphi)$ *stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2\varphi$ along fibers of π.*

We find from the above lemma that the leading order contribution in the above integral (3.24) depend only on values of $\omega_{13,0}$, $\omega_{01,0}$ and $*\omega_{03,0}$ at the point v_r. We will see later in lemma 33 that $\omega_{13,0}$, $\omega_{23,0}$ and $*\omega_{03,0}$ are forms in $\bigwedge^{top}(NV_{13})^*$, $\bigwedge^{top}(NV_{23}^+)^*$ and $\bigwedge^{top}(NV_{03}^-)^*$ respectively. Therefore we have

$$\begin{aligned}
&\int_M \chi_{v_r} e^{-\frac{g_{13}+g_{01}^+ + g_{03}^-}{\hbar}}\omega_{13,0}\wedge\omega_{01,0}\wedge *\omega_{03,0}\\
= \ & \pm\Big(\int_{NV_{13,x_{v_r}}} e^{-\frac{g_{13}}{\hbar}}\chi_{v_r}\omega_{13,0}\Big)\Big(\int_{NV^+_{01,x_{v_r}}} e^{-\frac{g_{01}^+}{\hbar}}\chi_{v_r}\omega_{01,0}\Big)\cdot\\
&\Big(\int_{NV^-_{03,x_{v_r}}} e^{-\frac{g_{03}^-}{\hbar}}\chi_{v_r} * \omega_{03,0}\Big)(1+\mathcal{O}(\hbar)),
\end{aligned}$$

where the sign depends on whether the orientations of $NV_{13}\oplus NV_{01}^+\oplus NV_{03}^-$ and TM at the point x_{v_r} match or not. We will compute the above three integrals one by one. We begin with equalities

$$\hbar^{-\frac{\deg(q_{01})}{2}}\int_{NV^+_{01,x_{v_r}}} e^{-\frac{g_{01}^+}{\hbar}}\chi_{v_r}\omega_{01,0} = 1+\mathcal{O}(\hbar)$$

and

$$\frac{\hbar^{-\frac{\deg(q_{03})}{2}}}{\|\phi_{03}\|^2}\int_{NV^-_{03,x_{v_r}}} e^{-\frac{g_{03}^-}{\hbar}}\chi_{v_r} * \omega_{03,0} = 1+\mathcal{O}(\hbar).$$

These come from [9] which is reformulated as the following lemma.

Lemma 32. *For an eigenform ϕ_{ij} with its WKB approximation*

$$\phi_{ij} \sim e^{-\psi_{ij}/\hbar}\hbar^{-\frac{\deg(q_{ij})}{2}}(\omega_{ij,0}+\omega_{ij,1}\hbar^{1/2}+\dots)$$

in any small enough open set W containing $V_{q_{ij}}^+\cup V_{q_{ij}}^-$ as in lemma 25, we have

$$\hbar^{-\frac{\deg(q_{ij})}{2}}\int_{NV^+_{q_{ij},x}} e^{-\frac{g_{ij}^+}{\hbar}}\chi\omega_{ij,0} = 1+\mathcal{O}(\hbar)$$

for any point $x \in V^+_{q_{ij}}$ and $\chi \equiv 1$ around x compactly supported in W. Similarily, we also have

$$\frac{\hbar^{-\frac{\deg(q_{ij})}{2}}}{\|\phi_{ij}\|^2}\int_{NV^-_{q_{ij},x}} e^{-\frac{g^-_{ij}}{\hbar}}\chi * \omega_{ij,0} = 1+\mathcal{O}(\hbar),$$

for any point $x \in V^-_{q_{ij}}$ and χ compactly supported in W and $\chi \equiv 1$ around x.

Finally, we have an equality

$$\hbar^{-\frac{\deg(q_{23})+\deg(q_{12})+1}{2}}\int_{NV_{13,x_{v_r}}} e^{-\frac{g_{13}}{\hbar}}\chi_{v_r}\omega_{13,0} = \frac{1}{\hbar}(1+\mathcal{O}(\hbar^{1/2})).$$

This depends on the fact that

$$\begin{aligned}
& \hbar^{-\frac{\deg(q_{23})+\deg(q_{12})}{2}}\int_{N(V^+_{23}\cap V^+_{12})_{x_v}} e^{-\frac{g^+_{23}+g^+_{12}}{\hbar}}\chi_v\omega_{23,0}\wedge\omega_{12,0} \\
= \;& (\hbar^{-\frac{\deg(q_{23})}{2}}\int_{N(V^+_{23})_{x_v}} e^{-\frac{g^+_{23}}{\hbar}}\chi_v\omega_{23,0})(\hbar^{-\frac{\deg(q_{12})}{2}}\int_{N(V^+_{12})_{x_v}} e^{-\frac{g^+_{12}}{\hbar}}\chi_v\omega_{12,0})(1+\mathcal{O}(\hbar)) \\
= \;& 1+\mathcal{O}(\hbar),
\end{aligned}$$

and the following lemma.

Lemma 33. *Using same notations in lemma 29 and suppose χ_S and χ_E are cut off function supported in W_S and W_E respectively, then we have*

(3.25)
$$\int_{N(V_E)_{v_E}} e^{-\frac{g_E}{\hbar}}\hbar^{-1/2}\chi_E\omega_{E,0} = \frac{1}{\hbar}(\int_{N(V_S)_{v_S}} e^{-\frac{g_S}{\hbar}}\chi_S\omega_{S,0})(1+\mathcal{O}(\hbar^{1/2})).$$

Furthermore, suppose $\omega_{S,0}(x_S) \in \bigwedge^{top} N(V_S)^*_{x_S}$, we have $\omega_{E,0}(x_E) \in \bigwedge^{top} N(V_E)^*_{x_E}$.

Putting the above together, we get the following

$$m_3^{T_1}(\hbar,\vec{\chi}_\Gamma) = \pm\hbar^{-1}e^{-\frac{A}{\hbar}}(1+\mathcal{O}(\hbar^{1/2})), \tag{3.26}$$

where the sign depends on matching the orientations of $NV_{13}\oplus NV^+_{01}\oplus NV^-_{03}$ and TM at the point x_{v_r}. The proof for $m_3(\hbar)$ is completed and we move on to the $m_k(\hbar)$ case for any k.

3.4.2. *$m_k(\hbar)$ case:* The argument of the general case is similar to the case $k=3$, with more combinatorics involved. As in section 3.3.4, we fix a gradient tree Γ of type T. Similar to the previous section, we may drop the dependence of Γ in our notations. We are going to show that

$$m_k^T(\hbar,\vec{\chi}_\Gamma) = \pm\hbar^{2-k}e^{-\frac{A}{\hbar}}(1+\mathcal{O}(\hbar^{1/2})), \tag{3.27}$$

where the sign agrees with that associated to the gradient tree Γ in Morse category. We begin with some notations associated to Γ.

Notations 34. *Given a gradient tree Γ, we inductively associate to each flag (ij,v) an oriented closed submanifold $V_{(ij,v)} \subset W_v$ by specifying orientation of its normal bundle. We require:*

(1) *for each semi-infinite incoming edge $i(i+1)$ with ending vertex v, we let $V_{(i(i+1),v)} := V^+_{q_{i(i+1)}} \cap W_v$, where $V^+_{q_{i(i+1)}}$ is the stable submanifold of $f_{i(i+1)}$ from the critical point $q_{i(i+1)}$ with the chosen orientation $\nu_{(i(i+1),v)}$ equals to that in the Morse category;*
(2) *for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting e_{il} at v as in the section 5. We let $V_{(il,v)} = V_{(ij,v)} \cap V_{(jl,v)}$ (the intersections is transversal from the generic assumption) and $\nu_{(il,v)} = \nu_{(jl,v)} \wedge \nu_{(ij,v)}$, if $\nu_{(ij,v)}$ and $\nu_{(jl,v)}$ are two corresponding orientation forms;*
(3) *for an internal edge ij with its starting vertex v_S and ending vertex v_E, we define $V_{(ij,v_E)}$ to be V_E obtained from applying lemma 29 to the homotopy operator H_{ij}. The orientation form $\nu_{(ij,v_E)}$ is chosen such that $[\nu_{(ij,v_E)}] = [df_{ij} \wedge \nu_{(ij,v_S)}]$, under the identification by flow of ∇f_{ij};*
(4) *for the semi-infinite incoming edge $0k$ with root vertex v_r, we let $V_{(0k,v)} := V^-_{q_{0k}} \cap W_{v_r}$, where $V^-_{q_{0k}}$ is the unstable submanifold of f_{0k} from critical point q_{0k} with the chosen orientation $\nu_{(0k,v_r)}$ equals to that in the Morse category;*

We further choose an isomorphism and projection map for every flag (ij,v)

$$\begin{tikzcd} W_v \arrow[r, "\cong"] \arrow[d, "\pi_{(e,v)}"'] & NV_{(ij,v)} \arrow[d, "\pi_{NV_{(ij,v)}}"'] \\ V_{(ij,v)} \arrow[r, equal] & V_{(ij,v)}, \end{tikzcd} \tag{3.28}$$

by further shrinking W_v suitably.

We can therefore assign a sign to the gradient tree Γ in the following way.

Definition 35. *For a generic sequence of Morse function $\vec{f}$ with corresponding critical points $q_{01}, \ldots, q_{(k-1)k}, q_{0k}$ satisfying the degree condition (3.1), with a gradient tree Γ, we define*

$$sign(\Gamma) = sign\left(\frac{\nu_{(jk,v_r)} \wedge \nu_{(0j,v_r)} \wedge \nu_{(0k,v_r)}}{\nu_M}\right), \tag{3.29}$$

where $0j, jk$ and $0k$ are edges joining the root vertex v_r as in section 5, $\nu_{(ij,v)}$ is the orientation of normal bundle defined in notation 34 and ν_M is the orientation of M.

We are going to argue that

$$\int_{N(V_{(ij,v)})_{x_v}} (e^{-\frac{g_{(ij,v)}}{\hbar}} \hbar^{r_{(ij,v)}} \chi_v \omega_{(ij,v),0}) = \hbar^{-r_v}(1 + \mathcal{O}(\hbar^{1/2})),$$

for any flag (ij, v) except the outgoing edge $0k$, where r_v is the number of internal edge before the vertex v. This can be seen inductively along the tree T. We see that:

(1) it is true for the semi-infinite incoming edge $i(i+1)$ by lemma 32;

(2) for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting il at v, we have

$$\begin{aligned}
& \hbar^{r_{(il,v)}} \int_{N(V_{(il,v)})_{x_v}} e^{-\frac{g_{(il,v)}}{\hbar}} \chi_v \omega_{(il,v),0} \\
\equiv\ & \hbar^{r_{(jl,v)}+r_{(ij,v)}} \int_{N(V_{(jl,v)} \cap V_{(ij,v)})_{x_v}} e^{-\frac{g_{(jl,v)}+g_{(ij,v)}}{\hbar}} \chi_v \omega_{(jl,v),0} \wedge \omega_{(ij,v),0} \\
\equiv\ & (\hbar^{r_{(jl,v)}} \int_{N(V_{(jl,v)})_{x_v}} e^{-\frac{g_{(jl,v)}}{\hbar}} \chi_v \omega_{(jl,v),0})(\hbar^{r_{(ij,v)}} \int_{N(V_{(ij,v)})_{x_v}} e^{-\frac{g_{(ij,v)}}{\hbar}} \chi_v \omega_{(ij,v),0}) \\
\equiv\ & 1,
\end{aligned}$$

modulo an error of order $\mathcal{O}(\hbar^{1/2})$;

(3) for an internal edge ij with its starting vertex v_S and ending vertex v_E, we make use of the lemma 33 to show that an extra $\hbar^{-1}$ is created.

Together with the fact that

$$\frac{1}{\|\phi_{0k}\|^2} \int_{N(V_{(0k,v_r)})_{x_{v_r}}} (e^{-\frac{g_{(0k,v_r)}}{\hbar}} \hbar^{r_{(0k,v_r)}} \chi_{v_r} * \omega_{(0k,v_r),0}) = 1 + \mathcal{O}(\hbar)$$

for the outgoing edge $0k$, we can now calculate the leading contribution from the integral (3.23). Recall that

$$\begin{aligned}
\psi_{(0j,v_r)} &= g_{(0j,v_r)} + f_{0j} - \sum_{0<m\leq j} f_{(m-1)m}(q_{(m-1)m}), \\
\psi_{(jk,v_r)} &= g_{(jk,v_r)} + f_{jk} - \sum_{j<m\leq k} f_{(m-1)m}(q_{(m-1)m}), \\
\psi_{(0k,v_r)} &= g_{(0k,v_r)} + f_{0k}(q_{0k}) - f_{0k}.
\end{aligned}$$

Therefore we have

$$\psi_{(0j,v_r)} + \psi_{(jk,v_r)} + \psi_{(0k,v_r)} = g_{(0j,v_r)} + g_{(jk,v_r)} + g_{(0k,v_r)} + A. \tag{3.30}$$

Finally we have

$$\begin{aligned}
&\int_M \Big(\frac{\hbar^{r_{(0j,v_r)}+r_{(jk,v_r)}+r_{(0k,v_r)}}}{e^{\frac{\psi_{(0j,v_r)}+\psi_{(jk,v_r)}+\psi_{(0k,v_r)}}{\hbar}}}\Big)\chi_{v_r}\cdot(\omega_{(jk,v_r),0}\wedge\omega_{(0j,v_r),0}\wedge\frac{*\omega_{(0k,v_r),0}}{\|\phi_{0k}\|^2})\\
= \;& e^{-\frac{A}{\hbar}}\{\int_M \Big(\frac{\hbar^{r_{(0j,v_r)}+r_{(jk,v_r)}+r_{(0k,v_r)}}}{e^{\frac{g_{(0j,v_r)}+g_{(jk,v_r)}+g_{(0k,v_r)}}{\hbar}}}\Big)\chi_{v_r}(\omega_{(jk,v_r),0}\wedge\omega_{(0j,v_r),0}\wedge\frac{*\omega_{(0k,v_r),0}}{\|\phi_{0k}\|^2})\}\\
= \;& \pm e^{-\frac{A}{\hbar}}\hbar^{2-k}(1+\mathcal{O}(\hbar^{1/2})),
\end{aligned}$$

which means

$$m_k^T(\hbar,\vec{\chi}_\Gamma)=\pm\hbar^{2-k}e^{-\frac{A}{\hbar}}(1+\mathcal{O}(\hbar^{1/2})). \tag{3.31}$$

The sign $\pm$ comes from matching the orientation $[\nu_{(jk,v_r)}\wedge\nu_{(0j,v_r)}\wedge\nu_{(0k,v_r)}]$ against that of M, which agrees with the sign in Morse category. This completes the proof of our Main Theorem.

4. Resolvent estimate

In this section, we obtain a rough estimate for the Green operator G^0_{ij} associated to Witten Laplacian Δ_{ij}, which is used in section 3.2. More precisely, we are looking at the twisted Green operator $G_{ij}=(I-P_{ij})G^0_{ij}$ after projecting to the orthogonal complement of the space of small eigenforms. Roughly speaking, it is an estimate of the form

$$G_{ij}(x,y)\sim\mathcal{O}(e^{-\frac{\rho_{ij}(x,y)}{\hbar}}) \tag{4.1}$$

for the kernel function on $M\times M$. We first recall some of our setting from section 2. We fix ij from now on and consider the case of a single Morse function $f=f_{ij}:M\to\mathbb{R}$. We define

$$\begin{aligned}
d_{f,\hbar} &:= e^{-f/\hbar}\hbar d e^{f/\hbar}=\hbar d+df\wedge,\quad\text{and}\\
d^*_{f,\hbar} &:= e^{f/\hbar}\hbar d^* e^{-f/\hbar}=\hbar d^*+\iota_{\nabla f},
\end{aligned}$$

where $\hbar$ is a small positive real number. The twisted Laplace operator can be defined as

$$\Delta_{f,\hbar}:=d_{f,\hbar}d^*_{f,\hbar}+d^*_{f,\hbar}d_{f,\hbar}.$$

A direct computation shows that

$$\begin{aligned}
\Delta_f &= \hbar^2\Delta+\hbar(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f})+|\nabla f|^2\\
&= \hbar^2\Delta+\hbar M_f+|\nabla f|^2,
\end{aligned} \tag{4.2}$$

where $\mathcal{L}_{\nabla f}$ is the Lie derivative and $M_f=\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}$ is a tensor on M.

The eigenvalues in the interval $I(\hbar)=[0,\hbar^{3/2})$ are called small eigenvalues. The direct sum of corresponding finite dimensional eigenspaces is denoted by $\Omega(M,\hbar)^*_{sm}$. We consider the twisted Green operator $G_{f,\hbar}=(I-P_{f,\hbar})G^0_{f,\hbar}$ which is the ordinary Green operator after projecting to the orthogonal complement of small eigenspace. The main result we have in this section is

Proposition 36. *For any $\epsilon > 0$, we have $\hbar_0 = \hbar_0(\epsilon)$ such that*

$$G_{f,\hbar} = \hat{\mathcal{O}}_\epsilon(e^{-\frac{\rho_f(x,y)-\epsilon}{\hbar}})$$

for all $\hbar < \hbar_0$, where ρ_f is the Agmon metric f defined in 9.

To really explain what the above notation stands for, we recall from [8] the following

Definition 37. *Let $A = A_\hbar : L^2(M) \to W^{1,2}(M)$ be a family of bounded operators with $\hbar \in (0,1]$. We say the $A \in \hat{\mathcal{O}}(e^{-\frac{f(x,y)}{\hbar}})$ for a continuous $f \in C^0(M \times M; \mathbb{R})$, if for any x_0, $y_0 \in M$, there exist neighborhoods V and U in M of x_0 and y_0 and a constant $C > 0$ such that*

$$\|Au\|_{W^{1,2}(V)} \leq Ce^{-\frac{f(x,y)}{\hbar}}\|u\|_{L^2(U)}$$

for all $\hbar$ small enough and $u \in L^2(M)$ such that $\operatorname{supp} u \subset U$.

Remark 38. *We will use subscript on $\hat{\mathcal{O}}$., if we want to emphasize what the constant C depends on.*

Remark 39. *If $A \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{f(x,y)-\epsilon}{\hbar}})$ and $B \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{g(x,y)-\epsilon}{\hbar}})$ for all ϵ small enough, then we have*

$$B \circ A \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{h(x,y)-\epsilon}{\hbar}})$$

for all ϵ small enough, where

$$h(x,y) = \min_{z\in M}(g(x,z) + f(z,y))$$

Remark 40. *For convenience, we write $\hat{\mathcal{O}}(e^{-f(x,y)/\hbar} + e^{-g(x,y)/\hbar})$ to stand for $\hat{\mathcal{O}}(e^{-(\min_{x,y}\{f(x,y),g(x,y)\})/\hbar})$.*

Remark 41. *We will use the same notation for a family of operators from $L^2(M)$ to $L^2(M)$ as well.*

We can generalize proposition 36 easily to L^∞ norm of all derivatives.

Proposition 42. *For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\hbar_0 = \hbar_0(\epsilon) > 0$ such that for any two point $x_0, y_0 \in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon} > 0$ such that*

$$\|\nabla^j(G_{ij}u)\|_{L^\infty(V)} \leq C_{j,\epsilon}e^{-(\rho(x_0,y_0)-\epsilon)/\hbar}\|u\|_{W^{k_j,2}(U)}, \tag{4.3}$$

for any $\hbar < \hbar_0$ and $u \in C_c^0(U)$. Here $W^{k,p}$ refers to the Sobolev norm.

The rest of the section is devoted to the proof of proposition 36 and 42. In this section, we will drop the dependence of $\hbar$ in our notations for simplicity, e.g. we will write Δ_f to denote $\Delta_{f,\hbar}$. We will use skills from [8] with suitable modifications. We will make use of the following equality as a tool for various kinds of estimates.

Lemma 43. *Let $\Omega \subset M$ be a domain with smooth boundary. If $v \in C^2(\bar{\Omega}, \wedge^* T_{\bar{\Omega}})$ with $v|_{\partial\Omega} = 0$ and $\psi \in C^2(\bar{\Omega}, \mathbb{R})$, we have*

$$(4.4)\ \mathrm{Re}\langle e^{2\psi/\hbar}\Delta_f v, v\rangle$$
$$= \hbar^2(\|d(e^{\psi/h}v)\|^2 + \|d^*(e^{\psi/h}v)\|^2) + \langle(|df|^2 - |d\psi|^2 + \hbar M_f)e^{\psi/\hbar}v, e^{\psi/\hbar}v\rangle.$$

Let C_f^* be the set of critical points of f, we let $B(p,\eta)$ be the open ball centered at p with radius η with respect to the Agmon metric. Then we define

$$\hat{M}_p := M \backslash \bigcup_{q \in C_f^* \backslash \{p\}} B(q,\eta).$$

Hence p is the only critical point of f in $\hat{M}_p$. We further fix $\delta > 0$ and define

$$\hat{M}_{p,0} := \hat{M}_p \backslash B(p,\delta),$$

and so there is no critical point of f in $\hat{M}_{p,0}$. We use $G_{f,p}$ and $G_{f,p,0}$ to stand for the twisted Green operators on $\hat{M}_p$ and $\hat{M}_{p,0}$ respectively, aftering projecting to the orthogonal complement of small eigenspaces. We first have the corresponding estimate for $G_{f,p,0}$ which is

Lemma 44. *For $\epsilon > 0$, there exists $\hbar_0 = \hbar_0(\epsilon,\eta,\delta) > 0$, such that*

$$G_{f,p,0} \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}),$$

for those $\hbar < \hbar_0$.

Proof. It follows from [8], since there is no small eigenvalue in $\hat{M}_{p,0}$. □

Next, we let θ_p and $\hat{\theta}_p$ be functions on M such that

$$(4.5)\qquad \theta_p \equiv 1 \text{ in a neighborhood of } \overline{B(p,\delta)} \text{ and } \operatorname{supp}\theta_p \subset B(p,2\delta),$$

$$(4.6)\qquad \hat{\theta}_p \equiv 1 \text{ in a neighborhood of } \overline{B(p,2\delta)} \text{ and } \operatorname{supp}\hat{\theta}_p \subset B(p,4\delta).$$

We will use these functions to cut off the Green operator and consider its behavior near one critical point. First we have

Lemma 45. *For any small $\epsilon > 0$, there exists $\hbar_0 = \hbar_0(\epsilon,\eta,\delta) > 0$, such that*

$$G_{f,p}\hat{\theta}_p \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,p)-4\delta-\epsilon}{\hbar}}),$$

for any $\hbar < \hbar_0$.

Proof. Fixing two points x_0, $y_0 \in \hat{M}_p$ with $V = B(x_0,\epsilon)$ and $U = B(y_0,\epsilon)$. We consider the relation $v = G_{f,p}\hat{\theta}_p u$ with $u \in C_0^\infty(\Omega^*(U))$. Putting $\psi(x) = (1-\epsilon)\rho(x,p)$ into the equality (4.4), we get

$$\begin{aligned} &\mathrm{Re}\langle (I - P_{f,p})\hat{\theta}_p u, e^{2\psi/\hbar}v\rangle \\ = \ &\hbar^2(\|d(e^{\psi/\hbar}v)\|^2 + \|d^*(e^{\psi/\hbar}v)\|^2) + \hbar\langle M_f e^{\psi/\hbar}v, e^{\psi/\hbar}v\rangle \\ &+\langle((2-\epsilon)\epsilon|df|^2)e^{\psi/\hbar}v, e^{\psi/\hbar}v\rangle. \end{aligned}$$

There exists $m = m(\epsilon, \eta, \delta) > 0$ such that $|df|^2 \geq m\epsilon$ in $\hat{M}_{p,0}$ and hence we have

$$\langle((2-\epsilon)\epsilon|df|^2)e^{\psi/\hbar}v, e^{\psi/\hbar}v\rangle \geq \frac{3}{2}m\epsilon^2\|e^{\psi/\hbar}v\|^2_{\hat{M}_{p,0}}.$$

Therefore, we get

$$\begin{aligned} & \mathrm{Re}\langle(I-P_{f,p})\hat{\theta}_p u, e^{2\psi/\hbar}v\rangle \\ \geq\ & \hbar^2(\|d(e^{\psi/\hbar}v)\|^2 + \|d^*(e^{\psi/\hbar}v)\|^2) + m\epsilon^2\|e^{\psi/\hbar}v\|^2_{\hat{M}_{p,0}}, \end{aligned}$$

since the term $\hbar\langle M_f e^{\psi/\hbar}v, e^{\psi/\hbar}v\rangle$ can be absorbed by taking $\hbar$ small enough. Therefore, we have

$$\begin{aligned} & \frac{1}{2c}\|e^{\psi/\hbar}(I-P_{f,p})\hat{\theta}_p u\|^2 + m\epsilon^2\|e^{\psi/\hbar}v\|^2_{B(p,\delta)} \\ \geq\ & h^2(\|d(e^{\psi/\hbar}v)\|^2 + \|d^*(e^{\psi/\hbar}v)\|^2) + \tilde{c}\|e^{\psi/\hbar}v\|^2, \end{aligned} \tag{4.7}$$

for some $\tilde{c} = \tilde{c}(\epsilon, \eta, \delta) > 0$, if $\hbar$ and c are small enough (both depending on ϵ, η, δ).

Since $\|G_{f,p}\|_{\mathcal{L}(L^2,L^2)} \leq C_\epsilon e^{\epsilon/\hbar}$ for arbitrary ϵ due to the fact that small eigenvalue are taken away, we have

$$\|v\|^2_{B(p,\delta)} \leq C_\epsilon e^{2\epsilon/\hbar}\|\hat{\theta}_p u\|^2 \leq C_\epsilon e^{2\epsilon/\hbar}\|u\|^2.$$

As $\psi(x) \leq (1-\epsilon)\delta$ in $B(p,\delta)$ and $\operatorname{supp} u \subset U$, we get

$$m\epsilon^2\|e^{\psi/\hbar}v\|^2_{B(p,\delta)} \leq Ce^{(2(1-\epsilon)\delta+2\epsilon)/\hbar}\|u\|^2_U \tag{4.8}$$

by replacing another constant $C = C(\epsilon, \eta, \delta)$. The next term to be controlled will be

$$\|e^{\psi/\hbar}(I-P_{f,p})\hat{\theta}_p u\|^2.$$

As $\operatorname{supp}\hat{\theta} \subset B(p, 4\delta)$ and $\operatorname{supp} u \subset U$, we have

$$\begin{aligned} \|e^{\psi/\hbar}(I-P_{f,p})\hat{\theta}_p u\|^2 &\leq \|e^{\psi/\hbar}\hat{\theta}_p u\|^2_{B(p,4\delta)\cap U} + \|e^{\psi/\hbar}\langle\hat{\theta}_p u, \varphi_p\rangle\varphi_p\|^2 \\ &\leq \|e^{\psi/\hbar}\hat{\theta}_p u\|^2_{B(p,4\delta)\cap U} + \|\hat{\theta}_p u\|^2\|e^{\psi/\hbar}\varphi_p\|^2, \end{aligned} \tag{4.9}$$

where φ_p is the unique small eigenform (as $\hat{M}_p$ has only one critical point p) corresponding to small eigenvalue of $\Delta_{f,p}$. We make use of the fact about the eigenform φ_p in lemma 3.2 which says

$$\varphi_p = \mathcal{O}_\epsilon(e^{-(\rho(p,x)-\epsilon)/\hbar})$$

for arbitrary ϵ and get

$$\|e^{\psi/\hbar}(I-P_{f,p})\hat{\theta}_p u\|^2 \leq \tilde{C}e^{\frac{8\delta(1-\epsilon)+2\epsilon}{\hbar}}\|u\|^2_U$$

for some constant $\tilde{C}$. Combining (4.7), (4.8), (4.9), we have

$$\|e^{\psi/\hbar}dv\| + \|e^{\psi/\hbar}d^*v\| + \|e^{\psi/\hbar}v\| \leq Ce^{(4\delta(1-\epsilon)+\epsilon)/\hbar}\|u\|_U.$$

Finally, we have an estimate for ψ in V and get

$$e^{(1-\epsilon)(\rho(x_0,p)-\epsilon)/\hbar}\|v\|_{W^{1,2}(V)} \leq Ce^{(4\delta(1-\epsilon)+\epsilon)/\hbar}\|u\|_{L^2(U)},$$

which is the desired result by choosing a suitable ϵ to start with. $\square$

We have the decomposition from [8]

$$G_{f,p} = (I-P_{f,p})(1-\theta_p)G_{f,p,0}(1-\hat{\theta}_p)+G_{f,p}\hat{\theta}_p+G_{f,p}\hat{\theta}_p[\Delta_f,\theta_p]G_{f,p,0}(1-\hat{\theta}_p), \tag{4.10}$$

which can be verfied by taking $\Delta_{f,p}$ to both sides of the equation. Combining lemmas 44, 45 and (4.10), we have

Lemma 46. *For any $\epsilon > 0$, there exists $\hbar_0 = \hbar_0(\epsilon,\eta) > 0$, such that*

$$G_{f,p} \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}),$$

for $\hbar < \hbar_0$.

Proof. We will estimate the right hand side of (4.10) term by term. From lemma 45 we have

$$(1-\theta_p)G_{f,p,0}(1-\hat{\theta}_p) \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}).$$

Making use of lemma 3.2, we see that

$$P_{f,p} \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}})$$

for small ϵ. Using remark 39 and triangle inequality, we get the desired estimate for the first term. For the second term, recall from lemma 45 that

$$G_{f,p}\hat{\theta}_p \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,p_j)-4\delta}{\hbar}}).$$

For the operator to be non-trivial, we need the support of input to intersect $\mathrm{supp}(\hat{\theta}_p)$. Therefore we can have

$$G_{f,p}\hat{\theta}_p \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,y)-8\delta-\epsilon}{\hbar}}).$$

Finally, we have

$$[\Delta_f,\theta]G_{f,p,0}(1-\hat{\theta}_p) \in \hat{\mathcal{O}}_{\epsilon,\eta,\delta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}})$$

as an operator from $L^2(\hat{M}_p)$ to itself. We use remark 39 to obtain an estimate for the last term. To finish the proof, we may choose suitable δ and ϵ to obtain the desired statement. $\square$

Now, we move to the estimate of G_f on the whole manifold M. We are going to use various cut off functions to relate it to $G_{f,p}$ on $\hat{M}_p$. We let ϑ_p be a function on M such that

$$\vartheta_p \equiv 1 \text{ in a neighborhood of } \overline{B(p,\eta)} \text{ and } \operatorname{supp}\vartheta_p \subset B(p,2\eta).$$

We define two sets of cut off functions, one of them is given by

$$\chi_p = 1 - \sum_{q\in C_f^*\setminus\{p\}} \vartheta_q.$$

Another one is $\tilde{\chi}_p \in C_c^\infty(\mathrm{int}(\hat{M}_p))$, which is independent of how large is the ball $B(p,\eta)$, and satisfies

$$\sum_p \tilde{\chi}_p = 1.$$

If η is small enough, we can assume

$$\mathrm{supp}\,\tilde{\chi}_p \cap \mathrm{supp}(1-\chi_p) = \emptyset.$$

We first take an approximation $G_{app} : L^2(\wedge^* T_M^*) \to W^{1,2}(\wedge^* T_M^*)$ defined by

$$G_{app} = (I - P_f) \sum_p \chi_p G_{f,p} \tilde{\chi}_p. \tag{4.11}$$

Then,

$$\begin{aligned} \Delta_f G_{app} &= (I-P_f)\left(\sum_p \Delta_f \chi_p G_{f,p}\tilde{\chi}_p\right) \\ &= (I-P_f)\left(I - \sum_p \chi_p P_{f,p}\tilde{\chi}_p - \sum_{p\neq q}[\Delta_f, \vartheta_q] G_{f,p}\tilde{\chi}_p\right) \\ &= (I-P_f)\left(I - (I-P_f)\sum_j \chi_p P_{f,p}\tilde{\chi}_p - \sum_{p\neq q}[\Delta_f, \vartheta_q] G_{f,p}\tilde{\chi}_p\right) \\ &= \Delta_f G_f (I - L - K), \end{aligned}$$

where

$$L = (I-P_f)\sum_p \chi_p P_{f,p}\tilde{\chi}_p \text{ and } K = \sum_{p\neq q}[\Delta_f, \vartheta_q] G_{f,p}\tilde{\chi}_p.$$

Notice that images of operators G_{app} and $G_f(I-L-K)$ are orthogonal to the kernel of Δ_f, therefore we have

$$G_{app} = G_f(I-L-K).$$

We can express G_f in term of G_{app} if we are able to invert $(I-L-K)$.

By checking the convergence of the series, we define

$$\hat{K} = \sum_{j=1}^{\infty} K^j, \tag{4.12}$$

$$\hat{L} = \sum_{j=1}^{\infty} (L(I+\hat{K}))^j), \tag{4.13}$$

and we can write

$$G_f = G_{app}(I+\hat{K})(I+\hat{L}). \tag{4.14}$$

We are going to estimate each term on the right hand side and obtain an estimate of G_f. First, we have

$$G_{app} \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}). \tag{4.15}$$

Next, we look at the operator

$$K = \sum_{p \neq q} [\Delta_f, \vartheta_q] G_{f,p} \tilde{\chi}_p : L^2(\wedge^* T_M^*) \to L^2(\wedge^* T_M^*).$$

Applying lemma 46 to $G_{f,p}$, we obtain

$$K \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}).$$

If we look at the second term in $\hat{K}$,

$$K^2 = \sum_{p_0 \neq p_1,\, p_1 \neq p_2} [\Delta_f, \vartheta_{p_0}] \cdot (G_{f,p_1}[\Delta_f, \vartheta_{p_1}]) \cdot G_{f,p_2} \tilde{\chi}_{p_2},$$

by applying lemma 46 to G_{f,p_1} and G_{f,p_2} with U and V having diameters less than ϵ, we obtain

$$K^2 \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\min_z(\rho(x,z)+\rho(z,y))-2\epsilon}{\hbar}}).$$

If we further take $\mathrm{supp}(\vartheta_{p_1})$ into account, we have

$$K^2 \in \hat{\mathcal{O}}_{\epsilon,\eta}(e^{-\frac{\rho(x,p_1)+\rho(p_1,y)-4\eta-4\epsilon}{\hbar}}).$$

In general, for $l \geq 2$, we have

$$K^l = \sum_{\substack{p_0 \neq p_1, \\ \dots, \\ p_{l-1} \neq p_l}} [\Delta_f, \vartheta_{p_0}] \cdot (G_{f,p_1}[\Delta_f, \vartheta_{p_1}]) \cdots (G_{f,p_{l-1}}[\Delta_f, \vartheta_{p_{l-1}}]) \cdot G_{f,p_l} \tilde{\chi}_{p_l},$$

and similarly we get

(4.16)

$$K^l \in \hat{\mathcal{O}}_{\epsilon,\eta}\Big(\sum_{\substack{p_1 \neq p_2, \\ \dots, \\ p_{l-2} \neq p_{l-1}}} \exp(\frac{-1}{\hbar}(\rho(x,p_1)+\rho(p_1,p_2)+\cdots+\rho(p_{l-1},y)-4(l-1)\eta-(3l-2)\epsilon))\Big).$$

To summarize, we let D be the matrix defined by

$$D_{p,q} = \begin{cases} 0 & \text{if } p \neq q, \\ e^{-\rho(p,q)/\hbar} & \text{if } p = q. \end{cases}$$

and the column vector $T(x)$ with componets given by

$$T_p(x) = (e^{-\rho(p,x)/\hbar}).$$

Then we can rewrite (4.16) as

$$K^l \in \hat{\mathcal{O}}_{\epsilon,\eta}\Big(e^{\frac{4(l-1)\eta+(3l-2)\epsilon}{\hbar}} T(x)^t \cdot D^{l-2} \cdot T(y)\Big). \tag{4.17}$$

Since there are only N critical points, for $M \geq N$, we have an entry-wise inequality

$$D^M \leq e^{-2T_0/\hbar} C_M (I + D + \cdots + D^{N-1}) \tag{4.18}$$

for some C_M, where $T_0 = \min_{p\neq q} \rho(p,q)$. We can absorb the error terms involving $e^{\frac{4(l-1)\eta}{\hbar}}$ and $e^{\frac{(3l-2)\epsilon}{\hbar}}$ into the decay term $e^{-2T_0/\hbar}$ to show convergence for $\hat{K}$. Combining (4.12), (4.17) and (4.18), we obtain

$$\hat{K} \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}) \tag{4.19}$$

for any ϵ small enough.

Now, it comes to the estimate of the term L. We learn from [8] that

$$(I - P_f)\chi_p\varphi_p = \mathcal{O}_{\epsilon,\eta}(\sum_{q\neq p} e^{-\frac{\rho(x,q)+\rho(q,p)-4\eta-2\epsilon}{\hbar}}), \tag{4.20}$$

where φ_p is the unique eigenform of Δ_p on $\hat{M}_p$. Therefore, we get

$$L \in \hat{\mathcal{O}}_{\epsilon,\eta}\Big(\sum_{q\neq p} \exp(\frac{-1}{\hbar}(\rho(x,q)+\rho(q,p)+\rho(p,y)-4\eta-3\epsilon))\Big). \tag{4.21}$$

Similar to the $\hat{K}$ case, we have the series in (4.13) converges and has

$$\hat{L} \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}), \tag{4.22}$$

by choosing suitable η and ϵ to start with.

Finally, combining the estimates in (4.15), (4.19), (4.22) and applying formula (4.14), we conclude that

$$G_f \in \hat{\mathcal{O}}_\epsilon(e^{-\frac{\rho(x,y)-\epsilon}{\hbar}}). \tag{4.23}$$

This completes the proof of proposition 36.

Proposition 42 can be proved by having higher derivatives estimate using the same argument as above, and applying Sobolev embedding.

5. WKB for Green operator

In the previous section, we have a rough estimate for the twisted Green operator, or the homotopy operator $H_f = d_f^* G_f(I - P_f)$, with an error of order $\mathcal{O}(e^{\frac{\epsilon}{\hbar}})$. In a neighborhood of gradient flow line segment of f, we are going to improve the results in section 4 to estimate with error $\mathcal{O}(\hbar^\infty)$. This is done by the WKB method for *inhomogeneous* Laplace equation (3.20).

We study the local behavior of the homotopy operator H_f along a normalized gradient flow line segment

$$\begin{aligned} \gamma : [0,T] &\longrightarrow M, \\ \frac{d\gamma}{dt} &= \frac{\nabla f}{|\nabla f|_f}, \\ \gamma(0) = x_S \quad &, \quad \gamma(T) = x_E, \end{aligned}$$

as shown in the following figure. We consider the relation

$$\zeta_E = H_f(\chi_S \zeta_S).$$

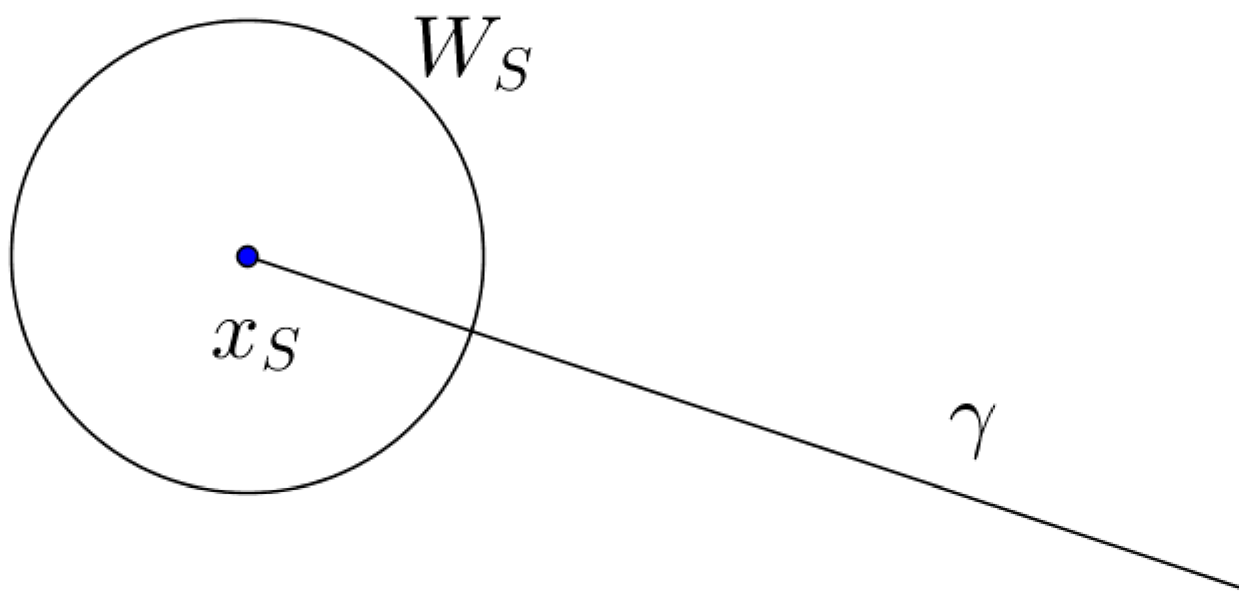
Suppose we have a WKB approximation of ζ_S in W_S of the form

$$\zeta_S \sim e^{-\psi_S/\hbar}(\omega_{S,0} + \omega_{S,1}\hbar^{1/2} + \omega_{S,2}\hbar^{1} + \dots), \tag{5.1}$$

we need to establish a similar expression

$$\zeta_E \sim e^{-\psi_E/\hbar}\hbar^{-1/2}(\omega_{E,0} + \omega_{E,1}\hbar^{1/2} + \dots), \tag{5.2}$$

of ζ_E in a some open neighborhood W_E of x_E.



The key step is to determine ψ_E, which is given in the following subsection. As a first trial, we consider the function

$$\tilde{\psi}_E(x) := \inf_{y\in W_S}\{\psi_S(y) + \rho_f(y,x)\},$$

since $e^{-\frac{\tilde{\psi}_E}{\hbar}}$ is the expected exponential decay suggested by the resolvent estimate in section 4.

$\tilde{\psi}_E$ is not the correct function since it is singular along a hypersurface U_S through x_S, and cannot be used for the iteration process as we keep on differentiating it.

In the coming section 5.1, we will solve the minimal configuration in variational problem associated to $\inf_{y\in W_S}(\psi_S(y)+\rho_f(y,x))$ and find that the point y is forced to lie on U_S, with a unique geodesic joining to x which realizes $\rho(y,x)$, for those x closed enough to x_E. These family of geodesics $\{\gamma_y\}_{y\in U_S}$ will give a foliation of a neighborhood of γ. Therefore we can use $\psi_E(\gamma_y(t)) = \psi_S(y)+t$ as an extension of $\tilde{\psi}_E$ across U_S. We then use ψ_E in the iteration similar to classical WKB approximation to obtain the above expansion 5.2.

5.1. **The phase function ψ_E.**

We apply variational method to study the function $\tilde{\psi}(x)$. Fixing $x\in M$, we take $\alpha(\epsilon,t) := \alpha_\epsilon(t) : (-\epsilon_0,\epsilon_0)\times[0,1]\to M$ such that $\alpha_\epsilon(1)\equiv x$ for all ϵ. To minimize the functional

$$L(\epsilon) = \psi_S(\alpha_\epsilon(0)) + \int_0^1 |\partial_t\alpha_\epsilon|_f dt,$$

we take derivatives and get

Lemma 47. *(First variation formula)*

$$\frac{dL}{d\epsilon} = \langle\nabla\psi_S(\alpha_\epsilon),\partial_\epsilon\alpha_\epsilon\rangle|_{t=0} + \int_0^1 \frac{1}{|\partial_t\alpha|_f}\langle\tilde{\nabla}_t\partial_\epsilon\alpha,\partial_t\alpha\rangle_f dt. \tag{5.3}$$

Here $\tilde{\nabla}_t$ is the Levi-Civita connection corresponding to the Agmon metric $\langle\cdot,\cdot\rangle_f$ in definition 9.

If we assume α_0 is a geodesic with $|\alpha_0'(t)|\equiv const.$, the Euler-Lagrange equation for $L(\epsilon)$ is

$$\left.\frac{dL}{d\epsilon}\right|_{\epsilon=0} = \left.\langle\nabla\psi_S(\alpha_0) - \frac{\alpha_0'}{|\alpha_0'|}|\nabla f(\alpha_0)|,\partial_\epsilon\alpha\rangle\right|_{\substack{t=0\\ \epsilon=0}} = 0.$$

Since $\partial_\epsilon\alpha(0,0)$ can be chosen arbitrarily, we have

$$\left.\Big(\nabla\psi_S(\alpha_0) - \frac{\alpha_0'}{|\alpha_0'|}|\nabla f(\alpha_0)|\Big)\right|_{t=0} = 0. \tag{5.4}$$

Taking norm we obtain the equation

$$|\nabla\psi_S| = |\nabla f|,$$

or equivalently, $|\tilde{\nabla}\psi_S|_f = 1$.

Definition 48.

$$U_S := \{|\tilde{\nabla}\psi_S|_f = 1\} \cap W_S.$$

If α_0 is a local extrema of L, it forces $\alpha_0(0) \in U_S$. To obtain nice properties of U_S, we are going to assume the following throughout the whole section.

Assumption 49. *We assume $g_S : W_S \to \mathbb{R}_{\geq 0}$, defined by $g_S = \psi_S - f$, be a Bott-Morse function with zero set V_S such that $v_S \in V_S$.*

Lemma 50. *U_S is a hypersurface containing V_S, if we shrink W_S suitably.*

Proof. Since we have $\nabla g_S \equiv 0$ on V_S and hence $|\nabla\psi_S| = |\nabla f|$ on V_S. This gives $V_S \subset U_S$. Moreover, U_S can be defined by the equation

$$\Phi(x) = 2\langle\nabla f(x), \nabla g_S(x)\rangle + |\nabla g_S(x)|^2 = 0.$$

If $v \in T_pM$ where $p \in V_S$, then we have

$$\begin{aligned}\nabla_v\Phi(p) &= 2\nabla^2 f(p)(v, \nabla g_S(p)) + 2\nabla^2 g_S(p)(v, \nabla f(p)) + 2\nabla^2 g_S(p)(v, \nabla g_S(p)) \\ &= 2\nabla^2 g_S(p)(v, \nabla f(p)),\end{aligned}$$

since $\nabla g_S(p) = 0$ on V_S. As g_S is a Bott-Morse function with critical set V_S, $\nabla^2 g_S(p)$ is nondegenerate when restricted to the orthogonal complement of T_pV_S in T_pM. Therefore, there exists v such that $\nabla_v\Phi(p) \neq 0$. □

We are going to parametrize a neighborhood of γ by $U_S \times (-\delta, T+\delta)$ such that $U_S \times \{0\} \to M$ is the embedding and $\nu_S \times [0, T]$ is γ. ψ_E is defined to be the coordinate function corresponding to the last variable.

Motivated from equation (5.4), we define a transversal vector field on U_S which is the initial tangent vector for minimizer of L.

Definition 51. *We define $\nu \in \Gamma(U_S, T_M)$*

$$\nu := \frac{\nabla\psi_S}{|\nabla\psi_S|_f} = \tilde{\nabla}\psi_S. \tag{5.5}$$

Notice that $\nu = \frac{\nabla f}{|\nabla f|_f} = \tilde{\nabla} f$ on V_S.

It follows from the Euler-Lagrange equation (5.4) that any local extrema α of L will have $\alpha(0) \in U_S$ and $\alpha'(0) = \nu(\alpha(0))$. For convenience, we assume that γ is extended to gradient flow line defined on (a, b) containing $[0, T]$.

Definition 52. *We define a map*

$$\sigma : W_0 \subset U_S \times (a, b) \to M, \tag{5.6}$$

given by

$$\sigma(u, t) = \exp_u(t\nu),$$

where W_0 is a suitable neighborhood of γ where the exponential map is well defined.

Lemma 53. *Restricting to a small open neighborhood of $\{x_S\} \times (a,b)$, σ is a diffeomorphism onto its image containing γ.*

This is achieved by showing there is no "conjugate point" along $\gamma(t)$ for certain type of geodesic family, and using the fact that γ being a global minimizer of functional L. Lemma 53 enable us to construct ψ_E needed for WKB approximation.

Definition 54. *We define ψ_E on $\sigma(U_S \times (a,b))$ by*

$$\psi_E(\sigma(u,t)) = \psi_S(u) + t, \tag{5.7}$$

for $(u,t) \in U_S \times (a,b)$.

5.2. **Proof of lemma 53.** We begin with the second variation formula of L. We assume $\alpha : (-\epsilon_0, \epsilon_0) \times [0,l] \to M$ is a family such that $\alpha_0(t)$ is arc-length parametrized geodesic satisfying the condition

$$\left(\tilde{\nabla}\psi_S(\alpha) - \frac{\partial_t \alpha}{|\partial_t \alpha|_f}\right)\Big|_{\substack{t=0\\ \epsilon=0}} = 0.$$

From the first variation formula

$$\frac{dL}{d\epsilon} = \langle \tilde{\nabla}\psi_S(\alpha_\epsilon(0)), \partial_\epsilon \alpha_\epsilon(0)\rangle_f + \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \frac{\partial_t \alpha}{|\partial_t \alpha|_f}\rangle_f \, dt,$$

we obtain

Lemma 55. *(Second variation formula)*

$$\begin{aligned} \frac{d^2 L}{d\epsilon^2}\Big|_{\epsilon=0} &= \langle \tilde{\nabla}_\epsilon \tilde{\nabla}\psi_S, \partial_\epsilon \alpha\rangle_f|_{t=0} + \langle \tilde{\nabla}\psi_S, \tilde{\nabla}_\epsilon \partial_\epsilon \alpha\rangle_f|_{t=0} + \langle \tilde{\nabla}_\epsilon \partial_\epsilon \alpha, \partial_t \alpha\rangle_f\Big|_0^l \\ &+ \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \tilde{\nabla}_t \partial_\epsilon \alpha\rangle_f + \langle \tilde{R}(\partial_\epsilon \alpha, \partial_t \alpha)\partial_\epsilon \alpha, \partial_t \alpha\rangle_f - \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_t \alpha\rangle_f^2 \, dt, \end{aligned} \tag{5.8}$$

where the right hand side is evaluated at $\epsilon = 0$. Here $\tilde{R}$ is the curvature tensor with respect to $\langle \cdot, \cdot \rangle_f$.

If we further impose the condition that $\alpha(\epsilon, 0) \in U_S$ and $\partial_\epsilon \alpha(\epsilon, l) \equiv 0$ for all ϵ, we have

$$\begin{aligned} \frac{d^2 L}{d\epsilon^2}\Big|_{\epsilon=0} &= \langle \tilde{\nabla}_\epsilon \tilde{\nabla}\psi_S, \partial_\epsilon \alpha\rangle_f|_{t=0} \\ &+ \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \tilde{\nabla}_t \partial_\epsilon \alpha\rangle_f + \langle \tilde{R}(\partial_\epsilon \alpha, \partial_t \alpha)\partial_\epsilon \alpha, \partial_t \alpha\rangle_f - \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_t \alpha\rangle_f^2 \, ds. \end{aligned} \tag{5.9}$$

Therefore we consider the bilinear form I associated to the above quadratic form.

Definition 56.

$$I(X,Y) = \tilde{\nabla}^2\psi_S(X,Y)(0) + \int_0^l \langle \tilde{R}(X,\partial_t\alpha)Y,\partial_t\alpha\rangle_f dt + \int_0^l \langle \tilde{\nabla}_t X - \langle \tilde{\nabla}_t X,\partial_t\alpha\rangle_f \partial_t\alpha, \tilde{\nabla}_t Y - \langle \tilde{\nabla}_t Y,\partial_t\alpha\rangle_f \partial_t\alpha\rangle_f dt, \tag{5.10}$$

for vector fields X, Y on α_0, satisfying $X(0), Y(0) \in TU_S$, $X(l) = 0 = Y(l)$.

For any such vector field X, we can find a family of curve α_ϵ satisfying the assumptions $\alpha(\epsilon,0) \in U_S$ and $\partial_\epsilon\alpha(\epsilon,l) \equiv 0$, with $\partial_\epsilon\alpha = X$. The same holds for piecewise smooth vector field with the same initial condition.

Proof of lemma 53. First, we notice that $d\sigma_{(x_S,t_0)}(0,\frac{\partial}{\partial t}) = \gamma'(t)$ for a fixed $t_0 \in [0,b)$. We have to compute $d\sigma_{(x_S,t_0)}(v,0)$ for arbitrary $(v,0) \in T_{(x_S,t_0)}(W_0)$. We claim that $\partial_\epsilon\alpha(0,t_0)$ can never be parallel to $\partial_t\alpha(0,t_0)$ for $v \neq 0$.

Taking a curve $\beta(\epsilon)$ in U_S with $\beta(0) = x_S$ and $\beta'(0) = v$, we can construct a family of arc-length parametrized geodesic α_ϵ by taking exponential map

$$\alpha(\epsilon,t) = \exp_{\beta(\epsilon)}(t\nu).$$

We have $\partial_\epsilon\alpha(0,t) = d\sigma_{(x_S,t)}(v,0)$ with $\partial_\epsilon\alpha$ being a Jacobi field on α_0. Suppose the contrary that $\partial_\epsilon\alpha(0,t_0) = c\partial_t\alpha(0,t_0)$ for some constant c, then we must have $\tilde{\nabla}_t\partial_\epsilon\alpha(0,t_0) \neq 0$, otherwise we must have $\partial_\epsilon\alpha \equiv c\partial_t\alpha$ which contradicts $v \neq 0$.

We argue that we can construct a path from U_S to the point $\sigma(v_S,t_0+\delta)$ which gives a smaller value of L comparing to the geodesic γ from v_S to the point $\sigma(v_S,t_0+\delta)$. This leads to contradiction because γ is an absolute minimum of L. We will denote $l = t_0+\delta$ to fit our previous discussion.

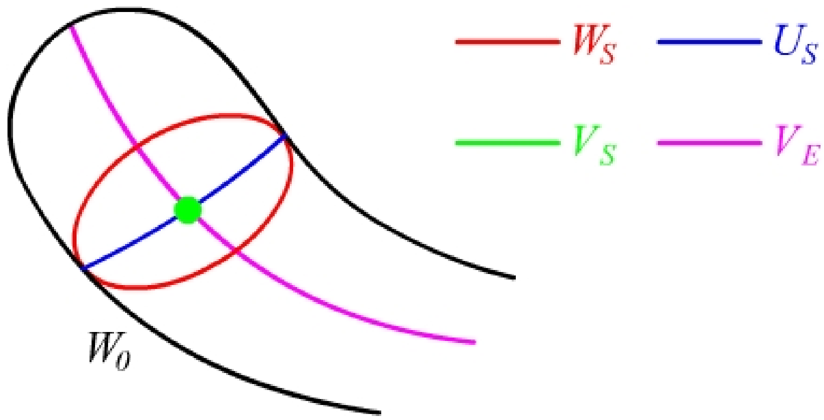
We construct the path by defining a variational vector field Y_η on γ, depending on a small $\eta > 0$ to be fixed. We take a vector field $Z(t)$ such that $Z(0) = 0$, $Z(l) = 0$, $\langle Z,\partial_t\rangle_f \equiv 0$ on $[t_0,l]$ and $Z(t_0) = -\tilde{\nabla}_t\partial_\epsilon(0,t_0)$. We define a piecewise smooth vector field

$$Y_\eta(t) := \begin{cases} \partial_\epsilon\alpha + \eta Z & \text{if } t \in [0,t_0], \\ \chi\langle\partial_\epsilon\alpha,\partial_t\alpha\rangle_f\partial_t\alpha + \eta Z & \text{if } t \in [t_0,l], \end{cases}$$

where χ is a cut off function in $[t_0,l]$ with $\chi(t_0) = 1$ and $\chi = 0$ in a neighborhood of l. Notice that $\tilde{\nabla}_t\langle\partial_\epsilon\alpha,\partial_t\alpha\rangle_f = 0$ from the fact that $|\partial_t\alpha|_f \equiv 1$. A direct computation shows

$$I(Y_\eta,Y_\eta) = -2\eta|\tilde{\nabla}_t\partial_\epsilon\alpha(0,t_0)|_f^2 + 2\eta^2 I(Z,Z).$$

We have $I(Y_\eta,Y_\eta) < 0$ for η small enough.



By taking the family of curves β_ϵ corresponding to Y_η, we obtain

$$\left.\frac{d^2 L_\beta}{d\epsilon^2}\right|_{\epsilon=0} < 0,$$

where $L_\beta(\epsilon) = L(\beta(\epsilon))$. For small enough ϵ, $\beta_\epsilon(t)$ will be a curve from U_S to $\sigma(0,l)$ which gives a smaller value of L comparing to $\beta_0 = \gamma$. This is impossible because we have

$$L_\beta(\epsilon) \geq f(\sigma(0,l))$$

and the lower bound is attained at γ.

For $t_0 \in (a, 0]$, the argument is similar by considering the variational problem associated to the functional

$$\hat{L}(\epsilon) = \psi_S(\alpha_\epsilon(0)) - \int_0^1 |\partial_t \alpha_\epsilon|_f dt.$$

As a conclusion, we can show that σ gives a local diffeomorphism onto its image by shrinking W_0 if necessary. Therefore it is injective in a contractible neighborhood of the gradient flow line γ. $\square$

Under the identification σ, we use the coordinate $u_1, \ldots, u_{n-1}$ for U_S and use $(u_1, \ldots, u_{n-1}, t)$, or simply (u, t), as coordinate for image of W_0 under σ. By shrinking W_0 if necessary, we assume that W_0 is a coordinate chart through the map σ. This justifies the definition 54 of ψ_E as a smooth function on $\sigma(W_0) \subset M$.

5.3. **Properties of ψ_E.** We are going to study the first and second derivatives of ψ_E which is necessary for having a WKB approximation for the equation (3.20). We define

$$V_E := \sigma((V_S \times (a,b)) \cap W_0) \subset \sigma(W_0)$$

as shown in the following picture.

Lemma 57. *In W_0, we have*

$$\tilde{\nabla}\psi_E = d\sigma_* \frac{\partial}{\partial t}.$$

In particular, we have $\nabla\psi_E = \nabla f$ on V_E and $|\nabla\psi_E| = |\nabla f|$.

Proof. We construct a family of geodesics α_ϵ using exponential map as before. Let $\beta(\epsilon)$ be a curve in U_S such that $\beta(0) = u$. Also, we let

$$\alpha(\epsilon, t) = \exp_{\beta(\epsilon)}(t\nu) = \sigma(\beta(\epsilon), t).$$

Applying the first variation together with the fact that α_ϵ satisfies Euler-Lagrange equation for L, we have

$$\begin{aligned} \langle \tilde{\nabla}\psi_E(\alpha(0,t)), \partial_\epsilon\alpha(0,t)\rangle_f &= \left.\frac{dL}{d\epsilon}\right|_{\epsilon=0} \\ &= \langle \partial_t\alpha(0,t), \partial_\epsilon\alpha(0,t)\rangle_f. \end{aligned}$$

As $\partial_\epsilon\alpha(0,t)$ can be chosen arbitrarily, we get

$$\tilde{\nabla}\psi_E(u,t) = \partial_t\alpha(0,t) = d\sigma_{(u,t)}\frac{\partial}{\partial t}.$$

Furthermore, we have $|\tilde{\nabla}\psi_E(u,t)|_f^2 = |d\sigma_{(u,t)}\frac{\partial}{\partial t}|_f^2 = 1$ which gives $|\tilde{\nabla}\psi_E(u,t)| = |\nabla f|$. As we know $\nabla\psi_S = \nabla f$ on V_S and flow lines of ∇f are geodesic after reparametrizations, we get $\nabla\psi_E = \nabla f$ on V_E. □

We now consider second derivatives of $g_E = \psi_E - f$.

Lemma 58. *By choosing a small enough W_0, we have*

1. *$g_E \geq 0$ and*
2. *g_E is a Bott-Morse function with critical set $V_E = \{g_E = 0\}$.*

Proof. The previous lemma implies that $\nabla g_E = 0$ on V_E. We are going to show $\nabla^2 g_E$ is positive definite in the normal bundle of V_E. Fixing any $t \in (a,b)$, we consider the submanifold $U_t = \sigma(U_S \times \{t\} \cap W_0)$. There is an isomorphism between the normal bundle of $V_t = \sigma(V_S \times \{t\} \cap W_0)$ in U_t and normal bundle of V_E in W_0. Therefore we restrict g_E to U_t and consider its Hessian.

We abuse the notations and write $u : W_0 \to U_S$ as the projection map. We take $h = g_E - g_S \circ u$. We have $h \geq 0$ on U_t and $\nabla h = 0 = h$ on V_t. Therefore we have h is positive semi-definite on the normal bundle of V_t in U_t. Moreover, we have $\nabla^2(g_S \circ u) = (\nabla^2 g_S) \circ u$ on V_S being positive definite in the normal bundle. This proves the lemma. □

Next, we consider the second order derivatives for $\Psi = \psi_E - \psi_S = g_E - g_S$ defined on W_S.

Lemma 59. *By choosing small enough neighborhood W_S of v_S if necessary, we have*

1. *$\Psi \leq 0$ on W_S and*
2. *Ψ is a Bott-Morse function with critical set $U_S = \{\Psi = 0\} \subset W_S$.*

Proof. We first have $\nabla\Psi = 0$ on U_S because $\nabla\psi_E = \nabla\psi_S$ on U_S. If we consider $\nabla^2\Psi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ on V_S, then we have $\nabla^2 g_E(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$ and $\nabla^2 g_S(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) > 0$. Therefore, there exists an neighborhood U of V_S in H so that

$$\nabla^2\Psi(x)(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) < 0$$

for all $x \in U$. $\square$

Remark 60. *We can extend the function Ψ from W_S to W_0 to be a non-negative function with critical set U_S which is also an absolute maximum. This is for our convenience in later arguments.*

5.4. **The WKB iteration.** After knowing these properties of ψ_E, we will describe the iteration procedure to define $\omega_{E,i}$ inductively.

First, by lemma 57, we have $|df|^2 = |d\psi_E|^2$ and hence the expansion

$$\begin{aligned} e^{\psi_E/\hbar}\Delta_f e^{-\psi_E/\hbar} &= \hbar^2\Delta + \hbar M_f + \hbar(\mathcal{L}_{\nabla\psi_E} - \mathcal{L}^*_{\nabla\psi_E}) \\ &= \hbar^2\Delta + \hbar(2\mathcal{L}_{\nabla\psi_E} - M_{g_E}), \end{aligned}$$

where $M_{g_E} = \mathcal{L}_{\nabla g_E} + \mathcal{L}^*_{\nabla g_E}$. Following [9], we let

$$\mathcal{T} = 2\mathcal{L}_{\nabla\psi_E} - M_{g_E},$$

and consider the following equation

$$(\hbar^2\Delta + \mathcal{T}\hbar)(\mu_0(\hbar) + \mu_1(\hbar) + \cdots) = e^{\Psi/\hbar}\nu,$$

order by order in $\hbar$ where $\mu_i(\hbar)$ is a function (depending on $\hbar$). We often write μ_i to simplify our notations. The first equation to be solved is

$$\hbar\mathcal{T}\mu_0(\hbar) = e^{\Psi/\hbar}\nu. \tag{5.11}$$

In order to solve the above equation involving $\mathcal{L}_{\nabla\psi_E}$, we need a map τ describing the flow of $\nabla\psi_E$. It is given by renormalising σ such that $d\tau_*(\frac{\partial}{\partial t}) = \nabla\psi_E$ and is of the form

$$\tau : W \subset U_S \times (-\infty, +\infty) \to M, \tag{5.12}$$

with the same image as σ. We use $(u_1, \ldots, u_{n-1}, t)$ as coordinate of $U_S \times \mathbb{R}$ as before and use it for coordinates of image as well from now on. We can assume that W also satisfies the property $\{u\} \times [t_0, t_1] \subset W$ for any $(u, t_0), (u, t_1) \in W$ under the identification by τ.

For the iteration process, we restrict our attention to

$$\Omega^*_0(W) = \{\beta \in \Omega^*(W) |\ \overline{supp(\beta)} \cap (U_S \times (-\infty, t_0])\ compact\ for\ all\ t_0\}.$$

This allows us to define the iteration operator.

Definition 61. *We let $I : \Omega^*_0(W) \to \Omega^*_0(W)$ given by*

$$I(\phi) := \int_{-\infty}^{0} e^{\int_s^0 \frac{1}{2}\tau^*_\epsilon(M_{g_E})\, d\epsilon}\, \tau^*_s(\phi) ds, \tag{5.13}$$

where $\tau_s(u, t) = \tau(u, t + s)$ is the flow of $\nabla\psi_E$ for time s.

To solve (5.11), we put

$$\mu_0 = \frac{1}{2\hbar} I(e^{\Psi/\hbar}\nu). \tag{5.14}$$

Then it can be checked that μ_0 is the solution to (5.11). The second equation to be solved is

$$\hbar \mathcal{T}\mu_1 = -\hbar^2 \Delta\mu_0. \tag{5.15}$$

Again, we put

$$\mu_1 = \frac{1}{2\hbar} I(-h^2\Delta\mu_0) = -\frac{\hbar}{2} I(\Delta\mu_0).$$

In general, we have the transport equation for $l \geq 0$

$$\mathcal{T}\mu_{l+1} = \hbar\Delta\mu_l. \tag{5.16}$$

This gives

$$\mu_{l+1} = -\frac{\hbar}{2} I(\Delta\mu_l). \tag{5.17}$$

as solutions in W.

5.5. **Estimate of the WKB iteration.** In this section, we are going to obtain norm estimates for μ_l's. We consider terms appearing in the iteration which are essentially of the form

$$I^j\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha\Psi)\beta\Big) \tag{5.18}$$

with $j \geq 0$ and $\beta \in \Omega_0^*(W)$, where I^j is the composition of I for j times. Here each $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index such that

$$\partial_\alpha\Psi = \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-1}}}{\partial u_{n-1}^{\alpha_{n-1}}} \frac{\partial^{\alpha_n}}{\partial t^{\alpha_n}} \Psi.$$

With

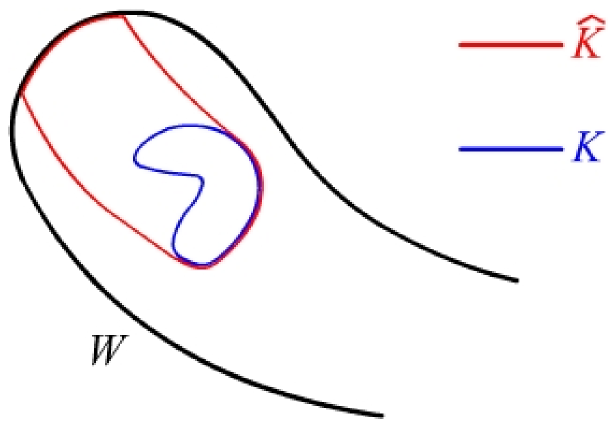
$$m(\alpha) := \max\{0, 2 - \alpha_n\},$$

we have

$$\nabla^j(\prod_\alpha \partial_\alpha\Psi)|_{U_S} \equiv 0, \tag{5.19}$$

for $j \leq \sum_\alpha m(\alpha)$ from lemma 59.

Given any compact subset K of W, we let $\hat{K} = (\bigcup_{(u,t)\in K}\{u\} \times (-\infty, t]) \cap W$ to be the union of backward flow line from K as shown in the following figure. We use the notations $(\{u\} \times \mathbb{R}) \cap K = K_u$ and $(\{u\} \times \mathbb{R}) \cap \hat{K} = \hat{K}_u$ to denote the intersection of K and $\hat{K}$ with the flow lines.



Lemma 62. *For any $\beta \in \Omega_0^*(W)$ and $j \geq 0$,*

$$\int_{K_u} |I^j\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)\beta\Big)|^2_{(u,t)} dt \leq Ch^{1/2+\sum_\alpha m(\alpha)} \|\beta\|^2_{L^\infty(\hat{K}_u)},$$

where C depends on j, $\operatorname{supp}(\beta) \cap \hat{K}$ and derivatives of Ψ up to order $2\sum_\alpha m(\alpha)$.

Proof. First of all, from the fact that $\beta \in \Omega_0^*(W)$ and K being compact, we can reduce the lower limit in the integral from $-\infty$ to $-c$. If we denote the norm of $\wedge^* T^*_{(u,t)}M$ by $|\cdot|_{(u,t)}$, we have for any $\phi \in \Omega_0^*(W)$

$$\begin{aligned}
|I(\phi)|^2_{(u,t)} &\leq \Big(\int_{-c}^0 |e^{\int_s^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})d\epsilon}\ \tau_s^*(\phi)|_{(u,t)}\ ds\Big)^2 \\
&\leq c\int_{-c}^0 |e^{\int_s^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})d\epsilon}\ \tau_s^*(\phi)|^2_{(u,t)}\ ds \\
&\leq c^2 \|e^{\int_s^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})d\epsilon}\|^2_{L^\infty(\hat{K}_u)} \int_{-c}^0 |\tau_s^*(\phi)|^2_{(u,t)}\ ds \\
&\leq C\|\phi\|^2_{L^\infty(\hat{K}_u)}.
\end{aligned}$$

Therefore, we conclude that $\|I(\phi)\|^2_{L^\infty(\hat{K}_u)} \leq C\|\phi\|^2_{L^\infty(\hat{K}_u)}$.

For $j > 0$, we have

$$\begin{aligned}
&\|I^j\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)\beta\Big)\|^2_{L^\infty(K_u)} \\
\leq\ & C\|I^{j-1}\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)\beta\Big)\|^2_{L^\infty(\hat{K}_u)} \\
\leq\ & C^{j-1}\Big(\int_{\hat{K}_u} e^{2\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)^2\Big)\Big(\|\beta\|^2_{L^\infty(\hat{K}_u)}\Big).
\end{aligned}$$

By lemma 59, $\Psi \leq 0$ is a Morse function with zero set U_S, so using (5.19) and the following lemma 63, we have

$$\int_{\hat{K}_u} e^{2\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)^2 ds \leq C' h^{1/2+\sum_\alpha m(\alpha)},$$

for some constant C' depending on derivatives of Ψ up to order $2\sum_\alpha m(\alpha)$. Therefore we indeed have the L^∞ norm estimate

$$\|I^j\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha \Psi)\beta\Big)\|^2_{L^\infty(K_u)} \leq \tilde{C}\hbar^{1/2+\sum_\alpha m(\alpha)}\|\beta\|^2_{L^\infty(\hat{K}_u)},$$

for $j > 0$. The case of $j = 0$ is obtained in a similar way. This is the only case that we have to consider the integration of $\|I^j\Big(e^{\Psi/\hbar}(\prod_\alpha \partial_\alpha\Psi)\beta\Big)\|^2$ over K_u to obtain the estimate. $\square$

The following semi-classical approximation lemma 63 used in the above proof, will also be used extensively in the rest of this section. Readers can refer to [3] for details.

Lemma 63. *Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 with coordinates $x_1, \ldots, x_n$. Let $\varphi : U \to \mathbb{R}_{\geq 0}$ be a Morse function with unique minimum $\varphi(0) = 0$ in U. Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be a Morse coordinates near 0 such that*

$$\varphi(x) = \frac{1}{2}(\tilde{x}_1^2 + \cdots + \tilde{x}_n^2).$$

For every compact subset $K \subset U$, there exists a constant $C = C_{K,N}$ such that for every $u \in C^\infty(U)$ with $\mathrm{supp}(u) \subset K$, we have

$$\begin{aligned} &|(\int_K e^{-\varphi(x)/\hbar}u) - (2\pi\hbar)^{n/2}\Big(\sum_{k=0}^{N-1}\frac{\hbar^k}{2^k k!}\tilde{\Delta}^k(\frac{u}{\Im})(0)\Big)| \\ \leq\ & C\hbar^{n/2+N}\sum_{|\alpha|\leq 2N+n+1}\sup|\partial^\alpha u|, \end{aligned} \tag{5.20}$$

where

$$\tilde{\Delta} = \sum \frac{\partial^2}{\partial \tilde{x}_j^2}, \qquad \Im = \pm\det(\frac{d\tilde{x}}{dx}),$$

and $\Im(0) = (\det \nabla^2\varphi(0))^{1/2}$.

In particular, if u vanishes at 0 up to order L, then we can take $N = \lceil L/2\rceil$ and get

$$|\int_K e^{-\varphi(x)/\hbar}u| \leq C\hbar^{n/2+\lceil L/2\rceil}.$$

Remark 64. *The same argument holds true for terms of the form*

$$I\Bigg(F_l I\Big(F_{l-1}I(\cdots F_1 I(e^{\Psi/\hbar}(\prod_k \partial_{\alpha_k}\Psi)\beta)\cdots)\Big)\Bigg),$$

where F_i's are tensors involving the function g_E and the metric. The proof can be obtained simply by induction.

We now investigate the norm of μ_l concerning its order in $\hbar$.

Proposition 65. *For $l \geq 0$, we have*

$$\|\mu_l(\hbar)\|^2_{L^\infty(K_u)} \leq C\hbar^{2l-2\lfloor \frac{l}{2}\rfloor -1}\|\nu\|^2_{C^{2l}(\hat{K}_u)},$$

for some constant C depending on derivatives of g_E and the metric tensor up to order $2l+2$.

Proof. W.L.O.G., we assume that the metric is standard under the coordinate $u_1, \dots, u_{n-1}, t$ since the metric tensor as well as its derivatives are bounded.

We let $Q(u,t,s) = e^{\int_s^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})\, d\epsilon}$ and consider the effect of first order derivatives acting on $I(e^{\Phi/\hbar}\nu) = \int_{-\infty}^0 Q(\tau_s^*(e^{\Phi/\hbar}\nu))ds$, we have

$$\begin{aligned}
2\nabla_t I(e^{\Phi/\hbar}\nu) &= M_{g_E} I(e^{\Phi/\hbar}\nu) + e^{\Phi/\hbar}\nu \\
\nabla_{u_i} I(e^{\Phi/\hbar}\nu) &= \int_{-\infty}^0 (\nabla_{u_i} Q(\tau_s^*(e^{\Phi/\hbar}\nu)))ds + \hbar^{-1} I(e^{\Phi/\hbar}(\nabla_{u_i}\Phi)\nu) + I(e^{\Phi/\hbar}\nabla_{u_i}\nu).
\end{aligned}$$

For second order derivatives acting on $I(e^{\Phi/\hbar}\nu)$, we have

$$\begin{aligned}
\nabla_t\nabla_t I(e^{\Phi/\hbar}\nu) &= \frac{1}{2}(\nabla_t(M_{g_E})I(e^{\Phi/\hbar}\nu) + \frac{1}{2}(M_{g_E})^2 I(e^{\Phi/\hbar}\nu) + \hbar^{-1}e^{\Phi/\hbar}(\nabla_t\Phi)\nu + e^{\Phi/\hbar}\nabla_t\nu) \\
\nabla_{u_i}\nabla_t I(e^{\Phi/\hbar}\nu) &= \frac{1}{2}\{(\nabla_{u_i}M_{g_E})I(e^{\Phi/\hbar}\nu) + \hbar^{-1}M_{g_E}I(e^{\Phi/\hbar}(\nabla_{u_i}\Phi)\nu) + M_{g_E}I(e^{\Phi/\hbar}(\nabla_{u_i}\nu)) \\
&+ M_{g_E}(\int_{-\infty}^0 \nabla_{u_i}Q(\tau_s^*(e^{\Phi/\hbar}\nu))ds) + \hbar^{-1}e^{\Phi/\hbar}(\nabla_{u_i}\Phi)\nu + e^{\Phi/\hbar}\nabla_{u_i}\nu\} \\
\nabla_{u_i}\nabla_{u_j} I(e^{\Phi/\hbar}\nu) &= \int_{-\infty}^0 (\nabla_{u_i}\nabla_{u_j}Q)(\tau_s^*(e^{\Phi/\hbar}\nu))ds + \hbar^{-1}\int_{-\infty}^0 \nabla_{u_j}Q(\tau_s^*(e^{\Phi/\hbar}(\nabla_{u_i}\Phi)\nu))ds \\
&+ \int_{-\infty}^0 \nabla_{u_j}Q(\tau_s^*(e^{\Phi/\hbar}\nabla_{u_i}\nu))ds + \hbar^{-1}\int_{-\infty}^0 \nabla_{u_i}Q(\tau_s^*(e^{\Phi/\hbar}(\nabla_{u_j}\Phi)\nu))ds \\
&+ \int_{-\infty}^0 \nabla_{u_i}Q(\tau_s^*(e^{\Phi/\hbar}(\nabla_{u_j}\Phi)\nu))ds + \hbar^{-1}I(e^{\Phi/\hbar}(\nabla_{u_i}\nabla_{u_j}\Phi)\nu) \\
&+ \hbar^{-1}I(e^{\Phi/\hbar}(\nabla_{u_i}\Phi)(\nabla_{u_j}\nu)) + \hbar^{-1}I(e^{\Phi/\hbar}(\nabla_{u_j}\Phi)(\nabla_{u_i}\nu)) \\
&+ \hbar^{-2}I(e^{\Phi/\hbar}(\nabla_{u_i}\Phi)(\nabla_{u_j}\Phi)\nu) + I(e^{\Phi/\hbar}\nabla_{u_i}\nabla_{u_j}\nu)
\end{aligned}$$

Using similar arguments as in the previous lemma, we obtain estimates

$$\begin{aligned}
\|I(\nabla_t\nabla_t I(e^{\Phi/\hbar}\nu))\|^2_{L^\infty(K_u)} &= \mathcal{O}(\hbar)\|\nu\|^2_{C^1(\hat{K}_u)}, \\
\|I(\nabla_{u_i}\nabla_t I(e^{\Phi/\hbar}\nu))\|^2_{L^\infty(K_u)} &= \mathcal{O}(\hbar)\|\nu\|^2_{C^1(\hat{K}_u)}, \\
\|I(\nabla_{u_i}\nabla_{u_j} I(e^{\Phi/\hbar}\nu))\|^2_{L^\infty(K_u)} &= \mathcal{O}(\hbar)\|\nu\|^2_{C^2(\hat{K}_u)}.
\end{aligned}$$

We conclude that $\|\mu_1\|^2_{L^\infty(K_u)} = \|\frac{1}{4}I(\Delta I(e^{\Phi/\hbar}\nu))\|^2_{L^\infty(K_u)} = \mathcal{O}(\hbar)\|\nu\|^2_{C^2(\hat{K}_u)}$.

In general, despite μ_l having a complicated expression with terms of the form as in remark 64, what we need to know is the vanishing order of the integrand along U as the integrand carries an exponential term $e^{\Phi/\hbar}$. Notice that applying differentiation to the term $e^{\Phi/\hbar}$ will give a term of the form $\hbar^{-1}\nabla\Phi$. $\nabla_t\Phi$ and $\nabla_{u_i}\Phi$ vanish up to first order and second order, both contribute $\hbar^2$ to the square norm.

In the iteration process, if we write $\Delta = \nabla_t\nabla_t + \sum_i \nabla_{u_i}\nabla_t + \sum_{i,j}\nabla_{u_i}\nabla_{u_j}$, we find that the terms involving $\nabla_t\nabla_t$ will give the lowest order in $\hbar$. Therefore, we look at terms obtained from applying $I\nabla_t\nabla_t$ iteratively. For $\mu_2(\hbar)$, that is the term $\frac{\hbar}{8}(I\nabla_t\nabla_t)(I\nabla_t\nabla_t)I(e^{\Phi/\hbar}\nu)$. Since ∇_t essentially cancels with I, the lowest order term is $\frac{\hbar}{8}I(\hbar^{-1}(\nabla_t\nabla_t\Phi)\nu e^{\Phi/\hbar}) = \mathcal{O}(\hbar^0)$.

For μ_k, we consider the term $(I\nabla_t\nabla_t)^k I(e^{\Phi/\hbar}\nu)$. When k is even, the lowest order term is $\frac{\hbar^{\frac{k}{2}-1}}{2^{k+1}}(\nabla_t\nabla_t\Phi)^{k/2}\nu e^{\Phi/\hbar}$, which has order $\mathcal{O}(\hbar^{\frac{k}{2}-1})$. When k is odd, the lowest order term becomes $\frac{\hbar^{\frac{k-1}{2}}}{2^{k+1}}(\nabla_t\Phi)(\nabla_t\nabla_t\Phi)^{(k-1)/2}\nu e^{\Phi/\hbar}$, which has order $\mathcal{O}(\hbar^{\frac{k+1}{2}})$. Combining these two cases, we obtain the statement in the proposition. $\square$

Remark 66. *Using the same argument, we have similar estimates*

$$\int_{K_u} |\nabla^j \mu_l(\hbar)|^2 \leq C_j \hbar^{2l-2\lfloor\frac{l+j}{2}\rfloor-1}\|\nu\|^2_{C^{2l+j}(\hat{K}_u)},$$

for all j. The integration along t would be necessary for $j>0$ as ∇_t may cancel with the operator I.

Making use of Sobolev embedding in dimension 1 along the t direction, we obtain the estimate

$$\|\nabla^j \mu_l(\hbar)\|^2_{L^\infty(K_u)} \leq C_j \hbar^{2l-2\lfloor\frac{l+j+1}{2}\rfloor-1}\|\nu\|^2_{C^{2l+j+1}(\hat{K}_u)}.$$

5.6. **A priori estimate.** We make use of the WKB iteration to construct the WKB expansion and prove that it does give a desired approximate to the solution in the rest of section 5. Before that, we obtain an a priori estimate for the solution in this subsection.

We consider the equation

$$\Delta_f \zeta_E = (I - P_f) d_f^*(\chi_S \zeta_S) \tag{5.21}$$

in W, where $\zeta_S \in \Omega^*(W_S)$ is the input form depending on $\hbar$ and $\chi_S \in C_c^\infty(W_S)$ is some cut off function to be chosen later. We assume ζ_S has a WKB approximation on W_S of the form

$$\zeta_S \sim e^{-\psi_S/\hbar}(\omega_{S,0} + \omega_{S,1}\hbar^{1/2} + \omega_{S,2}\hbar^1 + \dots), \tag{5.22}$$

where $\omega_{S,i} \in \Omega^*(W_S)$ and $\psi_S = f + g_S$. It is an approximation in the sense that

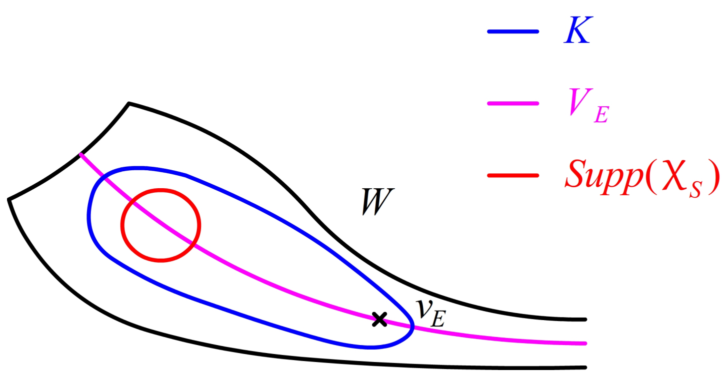
$$\|e^{\psi_S/\hbar}\zeta_S - (\sum_{i=0}^{N} \omega_{S,i}\hbar^{i/2})\|^2_{L^\infty(W_S)} \le C_N \hbar^{N+1}, \tag{5.23}$$

for N large enough, where C_N is a constant depending on N. We also require similar norm estimates for its derivatives

$$\|e^{\psi_S/\hbar}\nabla^j(\zeta_S - e^{-\psi_S/\hbar}(\sum_{i=0}^{N} \omega_{S,i}\hbar^{i/2}))\|^2_{L^\infty(W_S)} \le C_{j,N}\hbar^{N+1-2j}, \tag{5.24}$$

with $C_{j,N}$ depending on j, N.

We want to get a similar expansion for ζ_E, using the iteration defined in the section 5.4. We fix a cut off function χ supported in W with $\chi \equiv 1$ on the flow line γ from x_S to x_E. We consider any small enough compact neighborhood $K \subset W$ of the flow line γ with $\chi \equiv 1$ on K. χ_S is chosen so that $\mathrm{supp}(\chi_S) \subset K$. The following figure illustrates the situation.



For small enough K, we have an a priori estimate of ζ_E in K, the technique is similar to the method used for eigenform in [7].

Lemma 67. *For small enough* $\mathrm{supp}(\chi_S)$ *and* K*, and any* $j \in \mathbb{Z}_+$*, there exists* $\hbar_{j,0} > 0$ *such that for any* $\hbar < \hbar_{j,0}$*, we have*

$$\|e^{\psi_E/\hbar}\nabla^j\zeta_E\|^2_{L^\infty(K)} \le C_j\hbar^{-N_j}, \tag{5.25}$$

where N_j *is an positive integer depending on* j*.*

In order to prove the above lemma, we need to know certain special properties about χ and our chosen compact set K. Letting $\tilde\psi := \inf_{y\in\mathrm{supp}(\chi_S)}\{\psi_S + \rho_f(y,x)\}$, we have the following lemma.

Lemma 68. *There exists* $\epsilon > 0$ *such that for all* K *small enough, we have*

$$\tilde\psi(x) + \rho(y,x) \ge \psi_E(y) + \epsilon, \tag{5.26}$$

for all $y \in K$ and $x \in \mathrm{supp}(\nabla\chi)$.

Proof. Using the fact that $\psi_E = f$ on V_E, we have $|\psi_E - f| \leq \epsilon$ on K. We can simply prove

$$\tilde{\psi}(x) + \rho(y,x) \geq f(y) + \epsilon,$$

by choosing small enough K and ϵ. From properties of Agmon distance ρ, we have

$$\tilde{\psi}(x) \geq \min_{z\in\mathrm{supp}(\chi_S)} (f(z) + f(x) - f(z)) = f(x),$$

with equality holds only if $z \in V_S$ and there is a generalized gradient line joining z to x. Therefore, we have

$$\tilde{\psi}(x) + \rho(y,x) \geq f(x) + f(y) - f(x) = f(y),$$

with equality holds only if there is a generalized gradient line joining a point $z \in V_S$ to $x \in \mathrm{supp}(\chi)$ and then to $y \in K$. This is impossible by for our choices of χ and K. Hence we always have strict inequality and therefore we can find small ϵ by compactness argument. $\square$

We consider a closed neighborhood $\tilde{W}$ of $\mathrm{supp}(\chi)$ in W with smooth boundary. We let $\tilde{G}$ to be the twisted Green's operator on $\tilde{W}$ using Dirchlet boundary condition. We first argue that ζ_E can be replaced by $\tilde{\zeta}_E = d_f^*\tilde{G}\chi_S\zeta_S$.

Lemma 69. *There is a $\delta > 0$ such that*

$$\|e^{\frac{\psi_E}{\hbar}}\nabla^j(\chi\zeta_E - \tilde{\zeta}_E)\|_{L^\infty(K)} \leq C_j e^{-\frac{\delta}{\hbar}},$$

whenever $\mathrm{supp}(\chi_S)$ and K are chosen small enough and $j \in \mathbb{Z}_+$.

Proof. We let $r_\hbar = \chi\zeta_E - \tilde{\zeta}_E$. First, $r_\hbar$ satisfies the equation

$$\tilde{\Delta}_f r_\hbar = [\Delta,\chi]\zeta_E - \chi P_f d_f^*(\chi_S\zeta_S). \tag{5.27}$$

Therefore we have $r_\hbar = (\tilde{G}[\Delta,\chi]G - \tilde{G}\chi P_f)d_f^*(\chi_S\zeta_S)$. We consider it term by term to get estimate of $r_\hbar$. We have for any $\epsilon > 0$,

$$\tilde{G}[\Delta,\chi]G \sim \mathcal{O}_\epsilon\Big(\exp(\frac{-1}{\hbar}(\min_{z\in\mathrm{supp}(\nabla\chi)}(\rho(x,z) + \rho(z,y) - \epsilon)))\Big).$$

Using lemma 68, we can show there is some $\delta_0 > 0$ such that

$$\tilde{G}[\Delta,\chi]Gd_f^*(\chi_S\zeta_S) \sim \mathcal{O}(e^{-(\psi_E+\delta_0)/\hbar})$$

in K, for $\hbar$ small enough.

For the term $\tilde{G}\chi P_f$, we have

$$\tilde{G}\chi P_f \sim \mathcal{O}_\epsilon\Big(\sum_{q\in C_f^l}\exp(\frac{-1}{\hbar}(\rho(x,q) + \rho(q,y) - \epsilon))\Big),$$

where $l = deg(\zeta_S)$. Again, we can find a constant $\delta_1 > 0$ such that

$$\min_{x\in\mathrm{supp}(\chi_S)}(\psi_S(x) + \rho(x,q) + \rho(q,y)) \geq \psi_E(y) + 2\delta_1,$$

for $y \in K$. Similarly we have

$$\tilde{G}\chi P_f d_f^*(\chi_S \zeta_S) \sim \mathcal{O}(e^{-(\psi_E+\delta_1)/\hbar})$$

in K, for $\hbar$ small enough. Notice that the constant $\delta = \min\{\delta_0, \delta_1\}$ can chosen to be the same if we shrink $\mathrm{supp}(\chi_S)$ and K and keep $\tilde{W}$ and χ fixed. $\square$

Next, we obtain estimates for $\tilde{\zeta}_E$ similar to those in lemma 67 for ζ_E.

Lemma 70. *For any $j \in \mathbb{Z}_+$, there exists $\hbar_{j,0} > 0$ such that if $\hbar < \hbar_{j,0}$, we have*

$$\|e^{\psi_E/\hbar}\nabla^j \tilde{\zeta}_E\|^2_{L^\infty(\tilde{W})} \leq C_j \hbar^{-N_j}, \tag{5.28}$$

where N_j is an positive integer depending on j.

Proof. We consider the equation

$$\Delta_f \tilde{\zeta}_E = d_f^*(\chi_S \zeta_S) \tag{5.29}$$

in $\tilde{W}$ and divide the proof into steps:

Step 1: Without loss of generality, we assume there is a constant $C_0 > 0$ such that $C_0^{-1} \leq \psi_E \leq C_0$ and $C_0^{-1} \leq |df|^2 = |d\psi_E|^2 \leq C_0$ on $\tilde{W}$. We define the function

$$\Phi = \psi_E - C\hbar \log(\psi_E/\hbar), \tag{5.30}$$

with $C > 0$ to be chosen. Therefore we have

$$|df|^2 - |d\Phi|^2 \geq \frac{C\hbar |df|^2}{\psi_E} \geq \frac{C\hbar}{C_0^2}.$$

We apply lemma 4.4 to $\tilde{\zeta}_E$ with the chosen Φ on $\tilde{W}$ and get

$$\begin{aligned} Re(\langle e^{2\Phi/\hbar} d_f^*(\chi_S\zeta_S), \tilde{\zeta}_E\rangle) &= \hbar^2(\|d(e^{\Phi/\hbar}\tilde{\zeta}_E)\|^2 + \|d^*(e^{\Phi/\hbar}\tilde{\zeta}_E)\|^2) \\ &\quad + \langle(|df|^2 - |d\Phi|^2 + \hbar M_f)e^{\Phi/\hbar}\tilde{\zeta}_E, e^{\Phi/\hbar}\tilde{\zeta}_E\rangle \end{aligned}$$

and if we choose $C > 0$ large enough to absorb the term $\langle \hbar M_f e^{\Phi/\hbar}\tilde{\zeta}_E, e^{\Phi/\hbar}\tilde{\zeta}_E\rangle$, we have

$$\begin{aligned} & \hbar^2(\|d(e^{\Phi/\hbar}\tilde{\zeta}_E)\|^2 + \|d^*(e^{\Phi/\hbar}\tilde{\zeta}_E)\|^2) + \frac{C\hbar}{2C_0^2}\|e^{\Phi/\hbar}\tilde{\zeta}_E\|^2 \\ \leq\; & C_1\|e^{\Phi/\hbar}d_f^*(\chi_S\zeta_S)\|^2 \leq C_1(\frac{\hbar}{\psi_E})^{2C}\|e^{\psi_E/\hbar}d_f^*(\chi_S\zeta_S)\|^2 \\ \leq\; & C_2(\frac{\hbar}{\psi_E})^{2C}. \end{aligned}$$

Therefore we get

$$\hbar^2(\|d(e^{\psi_E/\hbar}\tilde{\zeta}_E)\|^2 + \|d^*(e^{\psi_E/\hbar}\tilde{\zeta}_E)\|^2) + \hbar\|e^{\psi_E/\hbar}\tilde{\zeta}_E\|^2 \leq C_3,$$

and obtained $\|e^{\psi_E/\hbar}\tilde{\zeta}_E\|^2_{L^2(K)} \leq C_4\hbar^{-1}$, for $\hbar < \hbar_0$.

Step 2: We prove the L^2 estimate for derivatives of $\tilde{\zeta}_E$. We apply d_f and d_f^* to both sides of equation (5.29). We obtain

$$\Delta_f(d_f\tilde{\zeta}_E) = d_f d_f^*(\chi_S\zeta_S). \tag{5.31}$$

Applying the result in step 1 to $d_f\tilde{\zeta}_E$, we have

$$\|e^{\psi_E/\hbar} d_f\tilde{\zeta}_E\|^2_{L^2(K)} \leq C_4\hbar^{-1}.$$

Since $d_f = \hbar d + df\wedge$, we have

$$\|e^{\psi_E/\hbar} d\tilde{\zeta}_E\|^2_{L^2(K)} \leq C_5\hbar^{-3}.$$

Corresponding result for $d^*\tilde{\zeta}_E$ can be obtained by a similar argument. These combine to obtain result for $\nabla\tilde{\zeta}_E$. By applying ∇ successively, we obtain all higher derivatives' estimates in a similar fashion.

Step 3: Finally, we improve the estimate to L^∞ norm. Since we have L^2 norm estimate for all the derivatives of $\tilde{\zeta}_E$. We use the Sobolev embedding on $\tilde{W}$ to obtain the L^∞ norm estimate. Details are left to readers. □

Lemma 67 follows from lemma 69 and lemma 70 directly.

5.7. **WKB approximation.** Next, we consider the WKB approximation of ζ_E. From the WKB approximation (5.1) of ζ_S, we can take d_f^* on both side and obtain a WKB approximation of $d_f^*(\chi_S\zeta_S)$

$$d_f^*(\chi_S\zeta_S) \sim e^{-\psi_S/\hbar}(\hbar d^* + \iota_{2\nabla f} + \iota_{\nabla g_S})(\chi_S\omega_{S,0} + \chi_S\omega_{S,1}\hbar^{1/2} + \dots), \tag{5.32}$$

after grouping terms according to their orders of $\hbar$. We apply the iteration in the previous subsection 5.4 terms by terms to the above series and then group the terms according to orders of $\hbar$ of their L^2 norms. As a result, we obtain a WKB expansion

$$\zeta_E \sim e^{-\psi_E/\hbar}(\omega_{E,0}(\hbar) + \omega_{E,1}(\hbar) + \dots) \tag{5.33}$$

in W, where $\omega_{E,i}(\hbar)$'s are functions also depending on $\hbar$. For every l and any compact subset $\tilde{K} \subset W$,

$$\|\omega_{E,l}(\hbar)\|^2_{L^\infty(\tilde{K})} \leq C_{l,\tilde{K}}\hbar^{2l-2\lfloor\frac{l}{2}\rfloor-1}$$

for those $\hbar < \hbar_{l,0}$, and also

$$\|e^{\psi_E/\hbar}(\Delta_f(e^{-\psi_E/\hbar}\sum_{i=0}^{N}\omega_{E,i}(\hbar)) - d_f^*(e^{-\psi_S/\hbar}\sum_{i=0}^{N}\omega_{S,i}))\|^2_{L^2(\tilde{K})} \leq C_{N,\tilde{K}}\hbar^{N+\frac{3}{2}},$$

for $\hbar < \hbar_{N,0}$. We need to argue that it is a good approximation, which is the main theorem in this section.

Theorem 71. *For any* $\operatorname{supp}(\chi_S)$ *and* K *small enough, and* N *large enough, there exists* $\hbar_{j,N,0} > 0$ *such that for* $\hbar < \hbar_{j,N,0}$ *we have*

$$\|e^{\psi_E/\hbar}\nabla^j\{\zeta_E - e^{-\psi_E/\hbar}(\sum_{i=0}^{N}\omega_{E,i}(\hbar))\}\|^2_{L^\infty(K)} \leq C_{j,N}\hbar^{N-2j}. \tag{5.34}$$

Proof. Making use of lemma 69, we can again consider the equation 5.29. It suffices to show that the approximation works for $\tilde{\zeta}_E$ on some small enough pre-compact neighborhood K of the flow line γ. We divide the proof into several steps.

Step 1: As $\omega_{E,i}(\hbar)$'s do not vanish on boundary of $\tilde{W}$, we first need to cut them off suitably for applying integration by part. $\omega_{E,i}(\hbar)$'s, being defined by integrating along flow of τ, have support as shown in the following figure.

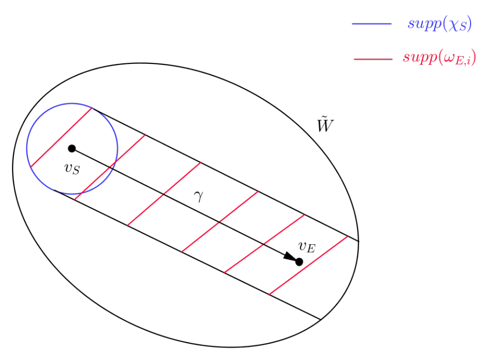


FIGURE 7. Support of $\omega_{E,i}$'s

Suppose we have $\tau_{\tilde{T}}(v_S) = v_E$, then we can choose $\tilde{\chi}$ only depending on variable t (using coordinate defined by τ) such that $\tilde{\chi} \equiv 1$ for $t \leq \tilde{T}$. The support of $\nabla\tilde{\chi}$ is shown in the following figure.

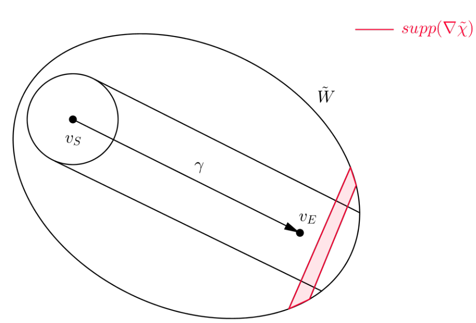


FIGURE 8. Support of $\nabla\tilde{\chi}$

By shrinking K and $\mathrm{supp}(\chi_S)$ if necessary, we obtain some $\epsilon > 0$ such that

$$\psi_E(y) + \rho(y,x) \geq \psi_E(x) + \epsilon \tag{5.35}$$

for $x \in K$ and $y \in \mathrm{supp}(\tilde{\chi})$. We define the function

$$\Phi_N = \min\{\Phi + N\hbar\log(1/\hbar), \min_{y\in\mathrm{supp}(\nabla\tilde{\chi})}(\Phi(y) + (1-\epsilon)\rho(x,y))\}, \tag{5.36}$$

where $\Phi := \psi_E - C\hbar\log(\psi_E/\hbar)$ is defined in (5.30), and the ϵ is chosen in lemma 68. We have

$$|df|^2 - |d\Phi_N|^2 \geq \frac{C\hbar|df|^2}{\psi_E} \geq \frac{C\hbar}{C_0^2},$$

for $\hbar$ small enough. Notice that we have $\Phi_N = \Phi + N\hbar\log(1/\hbar)$ in K for $\hbar$ small enough, and $\Phi_N = \Phi$ in $\mathrm{supp}(\nabla\tilde{\chi})$.

Step 2: Applying the lemma 43 to $r_k = \tilde{\chi}(\tilde{\zeta}_E - e^{-\psi_E/\hbar}(\sum_{i=0}^{k-1}\omega_{E,i}(\hbar)))$ and Φ_N, we get

$$\begin{aligned}
&\hbar^{2-2N}(\|d(e^{\Phi/\hbar}r_k)\|^2_{L^2(K)} + \|d^*(e^{\Phi/\hbar}r_k)\|^2_{L^2(K)}) + \frac{C\hbar^{1-2N}}{2C_0^2}\|e^{\Phi/\hbar}r_k\|^2_{L^2(K)}\\
\leq\ & D\|e^{\Phi_N/\hbar}d_f^*(\chi_S\zeta_S - e^{-\psi_S/\hbar}\sum_{i=0}^{k-1}\chi_S\omega_{S,i})\|^2_{L^2(\tilde{W})}\\
+\ & D\|e^{\Phi_N/\hbar}(d_f^*(e^{-\psi_S/\hbar}\sum_{i=0}^{k-1}\chi_S\omega_{S,i}) - \Delta_f(e^{-\psi_E/\hbar}\sum_{i=0}^{k-1}\omega_{E,i}(\hbar)))\|^2_{L^2(\tilde{W})}\\
+\ & D(\|e^{\Phi/\hbar}[\Delta,\tilde{\chi}]\tilde{\zeta}_E\|^2_{L^2(\tilde{W})} + \|e^{\Phi/\hbar}[\Delta,\tilde{\chi}](e^{-\psi_E/\hbar}\sum_{i=0}^{k-1}\omega_{E,i}(\hbar))\|^2_{L^2(\tilde{W})}).
\end{aligned}$$

We handle the right hand side term by term. First, we have

$$\|e^{\Phi_N/\hbar}d_f^*(\chi_S\zeta_S - e^{-\psi_S/\hbar}\sum_{i=0}^{k-1}\chi_S\omega_{S,i})\|^2 \leq C_k\hbar^{2C-2N+k}.$$

Second, we have

$$\|e^{\Phi_N/\hbar}(d_f^*(e^{-\psi_S/\hbar}\sum_{i=0}^{k-1}\chi_S\omega_{S,i}) - \Delta_f(e^{-\psi_E/\hbar}\sum_{i=0}^{k-1}\omega_{E,i}(\hbar)))\|^2 \leq C_k\hbar^{2C-2N+k}.$$

Third, we have

$$\|e^{\Phi/\hbar}[\Delta,\tilde{\chi}]\tilde{\zeta}_E\|^2 \leq D_1\hbar^{2C-N_0},$$

where N_0 is the integer in lemma 67. Finally, we have

$$\|e^{\Phi/\hbar}[\Delta,\tilde{\chi}](e^{-\psi_E/\hbar}\sum_{i=0}^{k-1}\omega_{E,i}(\hbar))\|^2 \leq C_k\hbar^{2C-N_0},$$

by choosing a larger N_0 independent of k, if necessary. Combining the above, by choosing $N = N_0 + k$, we have

$$\hbar^2(\|d(e^{\psi_E/\hbar}r_k)\|^2_{L^2(K)} + \|d^*(e^{\psi_E/\hbar}r_k)\|^2_{L^2(K)}) + \hbar\|e^{\psi_E/\hbar}r_k\|^2_{L^2(K)} \leq C_k\hbar^k,$$

which gives $\|e^{\psi_E/\hbar}r_k\|^2_{L^2(K)} \leq C_k\hbar^{k-1}$, for those $\hbar < \hbar_{k,0}$.

Step 3: We obtain L^2 estimate for all derivatives of r_k. We repeat the above argument for $d_f r_k$ and $d_f^* r_k$. For any $j, N \in \mathbb{Z}_+$, we can find a $k_{j,N}$ large enough such that for any $k > k_{j,N}$, we have

$$\|e^{\psi_E/\hbar}\nabla^j r_k\|^2_{L^2(K)} \leq C_{j,K,N}\hbar^N,$$

for $\hbar > \hbar_{j,k,N,0}$.

Step 4: We apply interior Sobolev embedding to improve the statement in step 3 into L^∞ norm, by further shrinking K if necessary. As a result, we have for N large enough, there exists $\hbar_{j,N,0} > 0$ and M_N such that we have

$$\|e^{\psi_E/\hbar}\nabla^j\{\tilde{\zeta}_E - e^{-\psi_E/\hbar}(\sum_{i=0}^{M_N}\omega_{E,i}(\hbar))\}\|^2_{L^\infty(K)} \leq C_{j,N}\hbar^{N-2j} \tag{5.37}$$

for $\hbar < \hbar_{j,N,0}$. Finally, we observe that $\|\nabla^j\omega_{E,i}(\hbar)\|^2_{L^\infty(K)} \leq C_{i,j}\hbar^{i-j-\frac{3}{2}}$ and hence obtain the result by dropping redundant terms in the approximation series.

□

Finally, we restrict our attention to a small enough neighborhood W_E of v_E. Since the operator I is given by an integral with an exponential decay $e^{\Psi/\hbar}$ along flow line, we can apply lemma 63 to obtain an expansion

$$\omega_{E,i}(\hbar) = \hbar^{-\frac{1}{2}}(\omega_{E,i,0} + \omega_{E,i,1}\hbar^1 + \omega_{E,i,2}\hbar^2 + \dots).$$

By regrouping terms according to their orders of $\hbar$, we obtain an expansion of the form given in equation (5.2).

5.8. **Relation between $\omega_{S,0}$ and $\omega_{E,0}$.** From section 5.4, we constructed a WKB approximation in W_E

$$\zeta_E = e^{-\psi_E/\hbar}(\omega_{E,0}(\hbar) + \omega_{E,1}(\hbar) + \cdots).$$

In particular, $\omega_{E,0}(\hbar)$ is given by

$$\omega_{E,0}(\hbar) = \frac{1}{2\hbar}(\int_{-\infty}^0 e^{\int_s^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})d\epsilon}\ \tau_s^*(e^{\Psi/\hbar}(\iota_{2\nabla f} + \iota_{\nabla g_S})\chi_S\omega_{S,0})ds). \tag{5.38}$$

In this section, we study the relation between integrals of $\omega_{S,0}$ and $\omega_{E,0}$ which is used in lemma 33. We begin by recalling lemma 31. Let M be a n-dimensional manifold and S be a k-dimensional submanifold in M, with a neighborhood B of S which can be identified as the normal bundle $\pi : NS \to S$. Suppose $\varphi : B \to \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set S, we have

Lemma 72. *Let $\beta \in \Omega^*(B)$ which is vertically compact support along the fiber of π. Then, we have*

$$\pi_*(e^{-\varphi(x)/\hbar}\beta) = (2\pi\hbar)^{(n-k)/2}(\iota_{\mathrm{vol}(\nabla^2\varphi)}\beta)|_V(1+\mathcal{O}(\hbar)),$$

where π_* is the integration along fiber. Here $\mathrm{vol}(\nabla^2\varphi)$ *stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2\varphi$ along fibers of π.*

We use the notations in section 5.1 and assume there is an identification of W_S and W_E with the normal bundle NV_S and NV_E of V_S and V_E respectively. We use π_S and π_E to stand for the bundle maps respectively. We have the following lemma which relates the integration of $\omega_{E,0}$ and $\omega_{S,0}$ along the fibers of π_E and π_S respectively.

Lemma 73. *Assume $\omega_{S,0} \in \wedge^{top} NV_S^*$ on V_S, then*

$$\pi_{E*}(e^{-g_E/\hbar}\omega_{E,0}) = \frac{1}{\hbar}\varrho^*\pi_{S*}(e^{-g_S/\hbar}\omega_{S,0})(1+\mathcal{O}(\hbar^{1/2})),$$

where $\varrho : V_E \to V_S$ is the projection map using the identification $V_E \equiv (V_S \times \mathbb{R}) \cap W_E$ given by τ (flow of $\nabla\psi_E$). Furthermore, we have $\omega_{E,0} \in \wedge^{top} NV_E^*$ on V_E.

Proof. We use the coordinates $u_1, \ldots, u_{n-1}, t$ for W, where $u_1, \ldots, u_{n-1}$ are coordinates of U_S. We further assume that $\{u_{s+1} = 0, \ldots, u_{n-1} = 0\} = V_S$. From lemma 59, $\Psi \leq 0$ is a Bott-Morse function with zero set U_S. Applying lemma 72 to the equation (5.38), we have

$$\begin{aligned} & \omega_{E,0}(u,t) \\ \equiv\; & (\frac{\pi}{2\hbar})^{1/2}(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{-1/2}\Big(e^{\int_{-t}^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})\,d\epsilon}\ \tau_{-t}^*((\iota_{2\nabla f} + \iota_{\nabla g_S})\chi_S\omega_{S,0})\Big), \end{aligned}$$

modulo terms of $\mathcal{O}(\hbar^{1/2})$. From lemma 58, $g_E \geq 0$ is a Bott-Morse function with zero set V_E. Applying lemma 72 again, we get, modulo terms of $\mathcal{O}(\hbar^{1/2})$,

$$\begin{aligned} & \pi_{E*}(e^{-g_E/\hbar}\omega_{E,0})(u,t) \\ \equiv\; & (2\pi\hbar)^{(n-s-1)/2}\iota_{\mathrm{vol}(\nabla^2 g_E)}(\omega_{E,0}) \\ \equiv\; & \pi\Big((2\pi\hbar)^{(n-s-2)/2}\iota_{\mathrm{vol}(\nabla^2 g_E)}(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{-1/2}(e^{\int_{-t}^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})\,d\epsilon}\ \tau_{-t}^*(\iota_{2\nabla f}\omega_{S,0}))\Big), \end{aligned}$$

for those $(u,t) \in V_E$. The term involving $\iota_{\nabla g_S}$ is dropped as $\tau_{-t}^*(dg_S)$ vanishes for $(u,t) \in V_E$. To make further simplifications, we need the following lemma.

Lemma 74. *Fixing a point $(u,t) \in V_E$, we have*

$$e^{\int_{-t}^0 \frac{1}{2}\tau_\epsilon^*(M_{g_E})d\epsilon} = \Big(\frac{\det(\nabla^2 g_E)(u,t)}{\det(\nabla^2 g_E)(u,0)}\Big)^{1/2}$$

as operators on $\bigwedge^{top} NV_E^*$, where the right hand side acts as multiplication. Here $\nabla^2 g_E$ is treated as an operator acting on NV_E using the metric tensor.

From the fact that $\omega_{S,0} \in \bigwedge^{top} NV_S^*$ upon restricting to V_S, we have $\tau_{-t}^*(\iota_{\nabla f}\omega_{S,0}) \in \bigwedge^{top} NV_E^*$ for those $(u,t) \in V_E$ and

$$\begin{aligned} & \pi_{E*}(e^{-g_E/\hbar}\omega_{E,0})(u,t) \\ = \quad & 2\pi(2\pi\hbar)^{(n-s-2)/2}(\tfrac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{-1/2}\Big((\tfrac{\det(\nabla^2 g_E)(u,t)}{\det(\nabla^2 g_E)(u,0)})^{1/2}\iota_{\nabla f \wedge \mathrm{vol}(\nabla^2 g_E)}\tau_{-t}^*(\omega_{S,0})\Big). \end{aligned}$$

Notice that $\nabla f = \frac{\partial}{\partial t}$ when restricting on V_E, therefore we have

$$(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{1/2}\nabla f = \mathrm{vol}(\nabla_t^2(-\Psi)|_{t=0}),$$

where we view W as a $\mathbb{R}$-bundle over U_S and consider $\mathrm{vol}(\nabla_t^2(-\Psi)|_{t=0})$ as the volume vector field along its fibers. Furthermore, we have the relation

$$d\tau_{-t}^*((\frac{\det(\nabla^2 g_E)(u,t)}{\det(\nabla^2 g_E)(u,0)})^{1/2}\,\mathrm{vol}(\nabla^2 g_E)(u,t)) = \mathrm{vol}(\nabla^2 g_E)(u,0).$$

Combining the above, we have

$$\begin{aligned} & \pi_{E*}(e^{-g_E/\hbar}\omega_{E,0})(u,t) \\ = \quad & (2\pi)^{(n-s)/2}\hbar^{(n-s-2)/2}\Big(\tau_{-t}^*(\iota_{\mathrm{vol}(\nabla_t^2(-\Psi)|_{t=0})\wedge \mathrm{vol}(\nabla^2 g_E)|_{t=0}}\omega_{S,0})\Big). \end{aligned}$$

Finally, from the relation $\Psi = g_E - g_S$, we get

$$\mathrm{vol}(\nabla_t^2(-\Psi)) \wedge \mathrm{vol}(\nabla^2 g_E) = \mathrm{vol}(\nabla^2 g_S)$$

on V_S, where $\mathrm{vol}(\nabla^2 g_S)$ is the volume polyvector field along the fibers of π_S. Therefore, we have

$$\pi_{E*}(e^{-g_E/\hbar}\omega_{E,0})(u,t) \equiv \frac{1}{\hbar}\tau_{-t}^*(\pi_{S*}(e^{-g_S/\hbar}\omega_{S,0})(u,0))$$

modulo terms of $\mathcal{O}(\hbar^{1/2})$, for those $(u,t) \in V_E$. □

Proof of Lemma 74. First of all, we have the equality

$$\frac{1}{2}M_{g_E} = \nabla^2 g_E - \frac{1}{2}\,\mathrm{tr}(\nabla^2 g_E),$$

on the set $\{\nabla g_E = 0\}$. We can treat $\nabla^2 g_E$ as an operator acting on NV_E^* as g_E is Morse along V_S. Restricting to $\bigwedge^{top} NV_E^*$, it is just $\mathrm{tr}(\nabla^2 g_E)$. Therefore we have

$$\frac{1}{2}M_{g_E} = \frac{1}{2}\,\mathrm{tr}(\nabla^2 g_E),$$

acting on $\bigwedge^{top} NV_E^*$.
On V_E, we have

$$\nabla_t\Big(\int_0^t \frac{1}{2}\operatorname{tr}(\nabla^2 g_E)(u,\epsilon)\,d\epsilon) - \frac{1}{2}\log(\det(\nabla_u^2 g_E)(u,t))\Big) \tag{5.39}$$
$$= \frac{1}{2}\operatorname{tr}(\nabla^2 g_E)(u,t) - \frac{1}{2}\operatorname{tr}((\nabla^2 g_E(u,t))^{-1}\nabla_t(\nabla^2 g_E(u,t))).$$

We will show that the above expression vanish.

Restricting to the set $\{\nabla g_E = 0\}$, for any vector fields $X, Y \in TW$, we have

$$\begin{aligned}
\nabla_t(\nabla_u^2 g_E)(X,Y) &= \nabla_t(\nabla^2 g_E(X,Y)) - \nabla^2 g_E(\nabla_t X, Y) - \nabla^2 g_E(X, \nabla_t Y)\\
&= \nabla_t\langle X, \nabla_Y \nabla g_E\rangle - \langle \nabla_t X, \nabla_Y \nabla g_E\rangle - \langle \nabla_X \nabla g_E, \nabla_t Y\rangle\\
&= \langle X, \nabla_t \nabla_Y \nabla g_E\rangle + \langle \nabla_X \nabla g_E, [\partial_t, Y]\rangle + \langle \nabla_X \nabla g_E, \nabla_Y \partial_t\rangle\\
&= \langle X, \nabla_Y \nabla_t \nabla g_E\rangle + \langle (\nabla^2 t \nabla^2 g_E) X, Y\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\nabla^2(\nabla_t g_E)(X,Y) &= \langle \nabla_Y \nabla(\partial_t g_E), X\rangle\\
&= Y\langle \nabla(\partial_t g_E), X\rangle - \langle \nabla(\partial_t g_E), \nabla_Y X\rangle\\
&= Y\langle \nabla_X \nabla g_E, \partial_t\rangle + Y\langle \nabla g_E, \nabla_X \partial_t\rangle - \langle \nabla_{\nabla_Y X} \nabla g_E, \partial_t\rangle\\
&= Y\langle X, \nabla_t \nabla g_E\rangle + Y\langle \nabla g_E, \nabla_X \partial_t\rangle - \langle \nabla_Y X, \nabla_t \nabla g_E\rangle\\
&= \langle X, \nabla_Y \nabla_t \nabla g_E\rangle + (\nabla^2 g_E \nabla^2 t) X, Y\rangle.
\end{aligned}$$

Therefore, we have

$$\nabla_t(\nabla^2 g_E) - \nabla^2(\nabla_t g_E) = [\nabla^2 t, \nabla^2 g_E],$$

where the Hessians are treated as endomorphisms of TM. Restricting the above equation to the subspace NV_E and multipling by $(\nabla^2 g_E)^{-1}$, we have

$$\operatorname{tr}((\nabla^2 g_E)^{-1}(\nabla_t(\nabla^2 g_E)) = \operatorname{tr}((\nabla^2 g_E)^{-1}\nabla^2(\nabla_t g_E)).$$

Finally, from the equation $|\nabla \psi_E|^2 = |\nabla f|^2$, we obtain

$$\nabla_t g_E = \frac{1}{2}|\nabla g_E|^2.$$

Applying ∇^2 to both sides and restricting to V_E give

$$\nabla^2(\nabla_t g_E)(X,Y) = \langle \nabla^2 g_E(X), \nabla^2 g_E(Y)\rangle,$$

or simply

$$\nabla^2(\nabla_t g_E) = (\nabla^2 g_E)^2$$

if we treat both sides as operators on TM.

Substituting it back into equation (5.39), we find that the derivative in equation (5.39) vanish. Therefore we have

$$\begin{aligned} & \left(\int_0^t \frac{1}{2}\operatorname{tr}(\nabla^2 g_E)(u,\epsilon)\ d\epsilon\right) \\ = \quad & \frac{1}{2}\log(\det(\nabla^2 g_E)(u,t)) - \frac{1}{2}\log(\det(\nabla^2(g_E))(u,0)), \end{aligned}$$

which is the equation we needed. □

Therefore, we complete the proof of lemma 29 and 33 which are needed in the proof of our Main Theorem in section 3.

6. Conclusion

From the semi-classical analysis of the Witten twisted Green's operator in section 4 and 5, we obtain our main theorem **??** which can be viewed as an enhancement of the original Witten deformation of deRham complex, concerning cohomology of the manifold M, to one concerning its rational homotopy type by incorporating wedge product structures. In [5], Fukaya proposed a differential geometric approach to the Strominger-Yau-Zaslow (SYZ) by relating A-model holomorphic disks instantons of a Calabi-Yau manifold equipped with Lagrangian torus fibration, to certain Witten twisted differential constructed from the symplectic structure. Proving theorem **??** provides essential analytical technique for such an approach. For instance, the semi-classical analysis of Witten twisted Green's operator, can be applied to obtain a beautiful geometric interpretation of the complicated scattering diagram in [2].

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