WITTEN DEFORMATION OF PRODUCT STRUCTURES 
ON DERHAM COMPLEX

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Abstract. Wedge product on deRham complex of a Riemannian manifold $M$ can be pulled back to $H^*(M)$ via explicit homotopy, constructed using Green’s operator, to give higher product structures. We prove Fukaya’s conjecture which suggests that Witten deformation of these higher product structures have semiclassical limits as operators defined by counting gradient flow trees with respect to Morse functions, which generalizes the remarkable Witten deformation of deRham differential from a statement concerning homology to one concerning rational homotopy type of $M$. Various applications of this conjecture to mirror symmetry are also suggested by Fukaya in [5].

1. Introduction

Let $f : M \rightarrow \mathbb{R}$ be a Morse function on an oriented compact Riemannian manifold $M$. Morse theory studies the homology of the manifold by the Morse complex $CM_f^*$, which is a finite dimensional vector space freely generated by critical points of $f$, equipped with the Morse differential $\delta$ defined by counting gradient flow lines of $f$. In an influential paper [13], Witten suggested a differential geometric approach toward Morse theory by deforming the exterior differential operator $d$ with

$$d_{f,h} := e^{-f/h}(h\Delta e^{f/h}) = h\Delta + df\wedge,$$

where $h \in \mathbb{R}^+$. We can obtain the formal adjoint of $d_{f,h}$ defined by

$$d_{f,h}^* := e^{f/h}(h\Delta^*)e^{-f/h} = h\Delta^* + \iota\nabla f$$

and the Laplace operator defined by

$$\Delta_{f,h} := d_{f,h}d_{f,h}^* + d_{f,h}^*d_{f,h}.$$  

Witten argued that if we consider eigenvalues of the operator $\Delta_{f,h}$ lying inside a small interval $[0, h^{3/2}]$, the sum of corresponding eigensubspaces $\Omega^*(M,h)_{sm} \subset \Omega^*(M,h)$ could be identified with the Morse complex $CM_f^*$

$$\phi = \phi(h) : CM_f^* \rightarrow \Omega^*(M,h)_{sm}.$$  

The eigenform corresponding to a critical point concentrated near that critical point, when $h$ small enough. Furthermore, the Witten differential has an asymptotic expansion $d_{f,h} \sim (\delta + O(h))$ under the above identification. Readers may see [14] for a detailed introduction. The complete proof can
be found in [7, 8, 9].

A natural question is whether the Witten’s approach can be extended to study the wedge product structure on differential forms, which will be an enhancement from a statement concerning homology to one concerning rational homotopy type, as these informations are captured in the differential graded algebra \((\Omega^*(M), d, \wedge)\) (if \(\pi_1(M) = 0\)) from [11, 12].

It is first conjectured by Fukaya in [5] that similar asymptotic expansions hold for higher products, which are combinations of \(d^*_f, \hbar, \wedge, \text{Green operator and projection to small eigenspaces}\), in the Witten twisted deRham theory. The leading order terms in the asymptotic expansions are conjectured to be operators defined by counting gradient trees, which are \(A_\infty\) products \(\{m^\text{Morse}_k\}_{k \in \mathbb{Z}^+}\) in the Morse category defined in [4], whose morphism space from \(f_i\) to \(f_j\) is the Morse complex \(CM^*_ij\) with respect to \(f_{ij} = f_j - f_i\).

To be more precise, we are forced to consider more than one Morse function in order to satisfy the Leibniz rule. This leads to the notation of the differential graded (dg) category \(DR^*_h(M)\), with objects being smooth functions on \(M\). The corresponding morphism complex relating \(f_i\) to \(f_j\) is given by the Witten twisted complex \(\Omega^*_ij(M, \hbar) = (\Omega^*(M), d_{ij} := e^{-f_{ij}/\hbar}(\hbar d) e^{f_{ij}/\hbar})\). The finite dimensional subcomplex \(\Omega^*_ij(M, \hbar)_sm \subset \Omega^*_ij(M, \hbar)\) is a homotopy retract under explicit homotopy involving Green operator. We can pull back the wedge product in the deRham category \(DR^*_h(M)\) via the homotopy, making use of homological perturbation lemma in [10], to give a deformed \(A_\infty\) category \(DR^*_h(M)_sm\) with \(A_\infty\) structure \(\{m_k(h)\}_{k \in \mathbb{Z}^+}\).

Fukaya’s conjecture says that the \(A_\infty\) structure \(\{m_k(h)\}_{k \in \mathbb{Z}^+}\), expressed explicitly in terms of Witten twisted Green operator and wedge product, has leading order given by \(\{m^\text{Morse}_k\}_{k \in \mathbb{Z}^+}\) defined by counting gradient flow trees, via the isomorphism \(\phi\).

**Conjecture** (Fukaya [5]). For generic sequence of functions \(\vec{f} = (f_0, \ldots, f_k)\), with corresponding sequence of points \(\vec{q} = (q_{01}, q_{12}, \ldots, q_{(k-1)k})\) such that \(q_{ij}\) is a critical point of \(f_{ij} = f_j - f_i\), we have

\[
(1.3) \quad m_k(h)(\phi(\vec{q})) = e^{-A(\vec{q})/h} (\phi(m^\text{Morse}_k(\vec{q}))) + O(h^{1/2}),
\]

where \(A(\vec{q}) = f_{0k}(q_{0k}) - f_{01}(q_{01}) - \cdots - f_{(k-1)k}(q_{(k-1)k})\).

We prove Fukaya’s conjecture in this paper.

**Theorem** (Main Theorem). Fukaya’s conjecture is true.
If we rescale \( \phi(q_{ij}) \) by multiplication of \( e^{-\frac{f_{ij}(q_{ij})}{\hbar}} \), the above statement simply reads
\[
\lim_{\hbar \to 0} m_k(\hbar) = m_k^{\text{Morse}}.
\]
As \( A_\infty \) relations of \( \{m_k(\hbar)\}_{k \in \mathbb{Z}_+} \) are obvious from their algebraic constructions while those of \( \{m_k^{\text{Morse}}\}_{k \in \mathbb{Z}_+} \) require studies for boundaries of moduli spaces of gradient flow trees (see e.g. [1, 4]), we obtain an alternative proof for \( A_\infty \) relations of \( \{m_k^{\text{Morse}}\}_{k \in \mathbb{Z}_+} \) as an corollary.

The original Witten-Morse theory is exactly the case \( k = 1 \), involving detailed estimate of operator \( d_{ij} \) along gradient flow lines. Starting from \( k \geq 3 \), our theorem involves the semi-classical analysis of the Witten twisted Green operator which is not included in the original theory.

Our Main Theorem for \( k = 2 \) involves three functions \( f_0, f_1, f_2 \), having \( q_{01}, q_{12}, q_{02} \) being critical points of \( f_{01}, f_{12}, f_{02} \) respectively, and can be proven using the analytical techniques in [7, 9]. We compute the leading order term in the matrix coefficients of \( m_2(\hbar) \), which is essentially the integral
\[
\int_M \langle m_2(\hbar)(\phi(q_{01}), \phi(q_{12})), \frac{\phi(q_{02})}{\|\phi(q_{02})\|^2} \rangle.
\]

First, we make use of the global a priori estimate of the form \( \phi(q_{ij}) \sim O(e^{\frac{\rho(q_{ij}, \cdot)}{\hbar}}) \) (lemma [16], with \( \rho \) being the Agmon distance defined in definition 9) to cut off the integrand to neighborhoods of gradient trees appeared in \( m_2^{\text{Morse}} \). After cutting off the integrand, we need to compute the leading order contribution from each gradient tree. The WKB approximation (lemma [25]) of the eigenforms \( \phi(q_{ij}) \) is used to compute the leading order contribution of (1.4).

When \( k \geq 3 \), what we need is an WKB approximation of \( G_{ij} \) along a gradient flow line of \( f_{ij} \) in §5. More precisely, we need to study the local behaviour of the inhomogeneous Witten Laplacian equation of the form
\[
\Delta_{ij} \zeta_E = d^*_{ij}(e^{-\frac{\psi_S}{\hbar}} \nu)
\]
along a gradient flow line segment of \( f_{ij} \) from \( x_S \) to \( x_E \), and obtain an approximation of \( \zeta_E \) of the form
\[
\zeta_E \sim e^{-\psi_E/\hbar} h^{-1/2}(\omega_{E,0} + \omega_{E,1} h^{1/2} + \ldots).
\]

The key step in our proof is to determine \( \psi_E \) from \( \psi_S \) and detailed construction is given in §5. A naive guess \( \tilde{\psi}_E(x) := \inf_y (\psi_S(y) + \rho(y, x)) \) captures the desired behaviors of \( \psi_E \) near \( x_E \) but is singular along a hypersurface \( U_S \) containing \( x_S \), which cannot be used to solve (1.5) iteratively. We solve the minimal configuration in variational problem associated to \( \inf_y (\psi_S(y) + \rho(y, x)) \) and find that the point \( y \) is forced to lie on \( U_S \), with a unique geodesic joining to \( x \) which realizes \( \rho(y, x) \), for those \( x \) closed enough
This family of geodesics \( \{ \gamma_y \}_{y \in U_S} \) gives a foliation of a neighborhood of the flow line segment. Therefore we can use \( \psi_E(\gamma_y(t)) = \psi_S(y) + t \) as an extension of \( \tilde{\psi}_E \) across \( U_S \). Provided the analytic results for \( G_{ij} \) \( (§4 \text{ and } §5) \), the proof of the general case is similar to the \( k = 3 \) case, but with more involved combinatorics.

This paper consists of two parts. The first part gives the basic setup and definitions in §2 and the proof modulo technical analysis in §3. The second part is a study of Witten twisted Green operator in §4 and §5 which is used in previous sections.

2. Setting

In this section, we introduce the definitions and notations we need and state our main theorem. We begin with the definition of deRham category.

2.1. deRham category. Given a compact oriented Riemannian manifold \( M \), we can construct the deRham category \( DR_\hbar(M) \) depending on a small real parameter \( \hbar \). Objects of the category are smooth functions \( f : M \to \mathbb{R} \).

For any two objects \( f_i \) and \( f_j \), we define the space of morphisms between them to be

\[
\text{Hom}^\bullet_{DR_\hbar(M)}(f_i, f_j) = \Omega^\bullet(M),
\]

with differential \( \hbar d + df_{ij} \land \), where \( f_{ij} : = f_j - f_i \). The composition of morphisms is defined to be the wedge product of differential forms on \( M \). This composition is associative and hence the resulted category is a dg category.

We denote the complex corresponding to \( \text{Hom}^\bullet_{DR_\hbar(M)}(f_i, f_j) \) by \( \Omega^\bullet_{ij}(M, \hbar) \) and the differential \( \hbar d + df_{ij} \) by \( d_{ij} \). We then consider the Morse category which is closely related to the deRham category.

2.2. Morse category. The Morse category \( Morse(M) \) has the same class of objects as the deRham category \( DR_\hbar(M) \), with the space of morphisms between two objects given by

\[
\text{Hom}^\bullet_{Morse(M)}(f_i, f_j) = CM^\bullet(f_{ij}) = \sum_{q \in \text{Crit}(f_{ij})} \mathbb{C} \cdot e_q.
\]

It is the Morse complex which is defined when \( f_{ij} \) is Morse. In this complex, \( e_q \)'s are declared to be an orthonormal basis and graded by the Morse index of corresponding critical point \( q \), which is the dimension of unstable submanifold \( V_q^- \). The Morse category \( Morse(M) \) is an \( A_\infty \)-category equipped with higher products \( m^{Morse}_k \) for every \( k \in \mathbb{Z}_+ \), or simply denoted by \( m_k \), which are given by counting gradient flow trees. To describe that, we first need some terminologies about directed trees.
2.2.1. Directed trees.

**Definition 1.** A trivalent directed $d$-leafed tree $T$ means an embedded tree in $\mathbb{R}^2$, together with the following data:

1. a finite set of vertices $V(T)$;
2. a set of internal edges $E(T)$;
3. a set of $d$ semi-infinite incoming edges $E_{in}(T)$;
4. a semi-infinite outgoing edge $e_{out}$.

Every vertex is required to be trivalent, having two incoming edges and one outgoing edge.

For simplicity, we will call it a $d$-tree. They are identified up to continuous map preserving the vertices and edges. Therefore, the topological class for $d$-trees will be finite.

Given a $d$-tree, by fixing the anticlockwise orientation of $\mathbb{R}^2$, we have cyclic ordering of all the semi-infinite edges. We can label the incoming edges by pairs of consecutive integers $(d-1)d, (d-1)(d-2), \ldots, 01$ and the outgoing edges by $0d$ such that the cyclic ordering $01, \ldots, (d-1)d, 0d$ agrees with the induced cyclic ordering of $\mathbb{R}^2$. Furthermore, we can extend this labeling to all the internal edges, by induction along the directed tree.

If we have an vertex $v$ with two incoming edges labelled $ij$ and $jk$, then we assign labeling $ik$ to the outgoing edge. For example, there are two different topological types for 3-tree, with corresponding labelings for their edges as shown in the following figure.

![Figure 1. two different types of 3-trees](image)

A pair $(e, v)$, with $e$ being an edge (either finite or semi-infinite) and $v$ being an adjacent vertex, is called a flag. The unique vertex attached to the outgoing semi-infinite edge is called the root vertex. The following figure shows different flags on the tree $T_1$.

For the purpose of Morse homology, we need the following notation of metric trees.

**Definition 2.** A metric $d$-tree $\tilde{T}$ is a $d$-tree together with a length function $l : E(T) \to (0, +\infty)$. 


Metric $d$-trees are identified up to homeomorphism preserving the length functions. The space of metric $d$-trees has finite number of components, with each component corresponding to a topological type $T$. The component corresponding to $T$, denoted by $\mathcal{S}(T)$, is a copy of $(0, +\infty)^{|E(T)|}$, where $|E(T)|$ is the number of internal edges and equals to $d - 2$. The space $\mathcal{S}(T)$ can be partially compactified to a manifold with corners $(0, +\infty)^{|E(T)|}$, by allowing the length of internal edges going to be infinity. In particular, it has codimension-1 boundary

$$\partial \mathcal{S}(T) = \bigsqcup_{T'\sqcup T''} \mathcal{S}(T') \times \mathcal{S}(T''),$$

where $\sqcup$ means joining the outgoing edge of $T'$ with one of the incoming edges of $T''$ to give an internal edge of infinite length.

### 2.2.2. Morse $A_\infty$ structure

We are going to describe the product $m_k$ of the Morse category. First of all, one may notice that the morphisms between two objects $f_i$ and $f_j$ is only defined when $f_{ij}$ is Morse. Therefore, when we consider a sequence of functions $f_0, \ldots, f_k$, we said the sequence is Morse if $f_{ij}$ are Morse for all $i \neq j$. Given a Morse sequence $\vec{f} = (f_0, \ldots, f_k)$, with a sequence of points $\vec{q} = (q_0, \ldots, q_{(k-1)k}, q_{0k})$ such that $q_{ij}$ is a critical point of $f_{ij}$, we have the following definition of gradient flow tree.

**Definition 3.** A gradient flow tree $\Gamma$ of $\vec{f}$ with endpoints at $\vec{q}$ is a continuous map $f : \tilde{T} \to M$ such that it is an upward gradient flow lines of $f_{ij}$ when restricted to the edge $ij$, the semi-infinite incoming edge $i(i + 1)$ begins at the critical point $q_{i(i+1)}$ and the semi-infinite outgoing edge $0k$ ends at the critical point $q_{0k}$.

We use $\mathcal{M}(\vec{f}, \vec{q})$ to denote the moduli space of gradient trees (in the case $k = 1$, the moduli of gradient flow line of a single Morse function has an
extra $\mathbb{R}$ symmetry given by translation in the domain. We will use this notation for the reduced moduli, that is the one after taking quotient by $\mathbb{R}$.

It has a decomposition according to topological types

$$\mathcal{M}(\tilde{f}, \tilde{q}) = \prod_T \mathcal{M}(\tilde{f}, \tilde{q})(T).$$

This space can be endowed with smooth manifold structure if we put generic assumption on the Morse sequence. For an incoming critical point $q_{i(i+1)}$, with corresponding stable submanifold $V^+_q$, we define a map

$$f_{T,i(i+1)} : V^+_q \times S(T) \to M.$$  

Fixing a point $x$ in $V^+_q$ together with a metric tree $\tilde{T}$, we need to determine a point in $M$. First, suppose $v$ is the vertex connected to the edge labelled $i(i+1)$, there is a unique path following the directed graph joining $v$ to the root vertex $v_r$. To determine the image of our function, we flow the point $x$ by gradient flow with respect to Morse function according to labeling of edges in the path, with time determined by the length of the edge.

The maps are then put together to give a map

$$(2.1) \quad f_T : V^-_{q_0} \times V^+_{q_{(k-1)k}} \times \cdots \times V^+_{q_0} \times S(T) \to \prod_{k+1} M,$$

where we use the embedding $\iota : V^-_{q_0} \to M$ for the first component. There is a generic assumption on $\tilde{f}$.

**Definition 4.** A Morse sequence $\tilde{f}$ is said to be generic if the image of $f_T$ intersect transversally with the diagonal submanifold $\Delta \cong M \hookrightarrow M^{k+1}$, for any sequence of critical point $\tilde{q}$ and any topological type $T$.

When the sequence is generic, the moduli space $\mathcal{M}(\tilde{f}, \tilde{q})$ is of dimension

$$\dim_{\mathbb{R}}(\mathcal{M}(\tilde{f}, \tilde{q})) = \deg(q_0) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2,$$

where $\deg(q_{ij})$ is the Morse index of the critical point. Therefore, we can define $m_k^{\text{Morse}}$, or simply denoted by $m_k$, using the signed count $\# \mathcal{M}(\tilde{f}, \tilde{q})$ of points in $\mathcal{M}(\tilde{f}, \tilde{q})$ when it is of dimension 0. In order to have a signed count, we have to get an orientation of the space $\mathcal{M}(\tilde{f}, \tilde{q})$. We will come to that later in definition 33.

We now give the definition of the higher products in the Morse category.

**Definition 5.** Given a generic Morse sequence $\tilde{f}$ with sequence of critical points $\tilde{q}$, we define

$$m_k : CM^*_k(k-1) \otimes \cdots \otimes CM^*_0 \to CM^*_0.$$
given by
\[ \langle m_k(q_{(k-1)k}, \ldots, q_{01}), q_{0k} \rangle = \#M(\tilde{f}, \tilde{q}), \]
when
\[ \deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2 = 0. \]
Otherwise, the \( m_k \) is defined to be zero.

One may notice \( m_k^{\text{Morse}} \) can only be defined when \( \tilde{f} \) is a Morse sequence satisfying the generic assumption in definition 4. The Morse category is indeed an \( A_\infty \) pre-category instead of an honest category. We will not go into detail about the algebraic problem on getting an honest category from this structures. For details about this, readers may see [1, 4].

2.3. From deRham to Morse. To relate \( DR_\hbar(M) \) and \( Morse(M) \), we need to apply homological perturbation to \( DR_\hbar(M) \). Fixing two functions \( f_i \) and \( f_j \), we consider the Witten Laplacian
\[ \Delta_{ij} = d_{ij}d_{ij}^* + d_{ij}^*d_{ij}, \]
where \( d_{ij}^* = \hbar d^* + \iota_{\nabla f_{ij}} \). We take the interval \( I(\hbar) = [0, \hbar^{3/2}] \) and denote the span of eigenspaces with eigenvalues contained in \( I(\hbar) \) by \( \Omega_{ij}^*(M, \hbar)_{sm} \).

By the result of [9], we have a map
\[ \phi = \phi_{ij}(\eta, \hbar) : CM^*(f_{ij}) \to \Omega_{ij}^*(M, \hbar)_{sm} \]
depending on \( \eta, \hbar \in \mathbb{R}_+ \) such that it is an isomorphism when \( \eta, \hbar \) are small enough. Here \( \eta \), which can be arbitrarily small, is the radius of some cut off function will be introduced in section 3.

Furthermore, under the identification \( \phi_{ij}(\eta, \hbar) \), we have the identification of differential \( d_{ij} \) and Morse differential \( m_1 \) from [9] as
\[ \langle d_{ij}\phi_{ij}(p), \phi_{ij}(q) \rangle = e^{-\frac{f_{ij}(q) - f_{ij}(p)}{\hbar}}(m_1(p), q)(1 + \mathcal{O}(\hbar)) \]
for \( \hbar \) small enough, if \( p, q \) are critical points of \( f_{ij} \). This is originally proposed by Witten to understand Morse theory using twisted deRham complex.

It is natural to ask whether the product structures of two categories are related via this identification, and the answer is definite. The first observation is that the Witten’s approach indeed produces an \( A_\infty \) category, denoted by \( DR_\hbar(M)_{sm} \), with \( A_\infty \) structure \( \{m_k(h)\}_{k \in \mathbb{Z}_+} \). It has the same class of objects as \( DR_\hbar(M) \). However, the space of morphisms between two objects \( f_i, f_j \) is taken to be \( \Omega_{ij}^*(M, \hbar)_{sm} \), with \( m_1(h) \) being the restriction of \( d_{ij} \) to the eigenspace \( \Omega_{ij}^*(M, \hbar)_{sm} \).
The natural way to define \( m_2(h) \) for any three objects \( f_0, f_1 \) and \( f_2 \) is the operation given by

\[
\Omega^*_{12}(M, h)_{sm} \otimes \Omega^*_{01}(M, h)_{sm} \xrightarrow{(\iota_{12}, \iota_{01})} \Omega^*_{12}(M, h) \otimes \Omega^*_{01}(M, h)
\]

\[
\downarrow \wedge
\]

\[
\Omega^*_{02}(M, h) \rightarrow \Omega^*_{02}(M, h)_{sm},
\]

where \( \iota_{12} \) and \( \iota_{01} \) are inclusion maps and \( P_{ij} : \Omega^*_{ij}(M, h) \rightarrow \Omega^*_{ij}(M, h)_{sm} \) is the orthogonal projection.

Notice that \( m_2(h) \) is not associative, and we need a \( m_3(h) \) to record the non-associativity. To do this, let us consider the Green’s operator \( G_{ij}^0 \) corresponding to Witten Laplacian \( \Delta_{ij} \). We let

\[
(2.4) \quad G_{ij} = (I - P_{ij}) G_{ij}^0
\]

and

\[
(2.5) \quad H_{ij} = d_{ij}^* G_{ij}.
\]

Then \( H_{ij} \) is a linear operator from \( \Omega^*_{ij}(M, h) \) to \( \Omega^*_{ij}(M, h)_{sm} \) and we have

\[
d_{ij} H_{ij} + H_{ij} d_{ij} = I - P_{ij},
\]

Namely \( \Omega^*_{ij}(M, h)_{sm} \) is a homotopy retract of \( \Omega^*_{ij}(M, h) \) with homotopy operator \( H_{ij} \). Suppose \( f_0, f_1, f_2 \) and \( f_3 \) are smooth functions on \( M \) and let \( \varphi_{ij} \in \Omega^*_{ij}(M, h)_{sm} \), the higher product

\[
m_3(h) : \Omega^*_{23}(M, h)_{sm} \otimes \Omega^*_{12}(M, h)_{sm} \otimes \Omega^*_{01}(M, h)_{sm} \rightarrow \Omega^*_{03}(M, h)_{sm}
\]

is defined by

\[
(2.6) \quad m_3(h)(\varphi_{23}, \varphi_{12}, \varphi_{01}) = \sum T m^T_k(h),
\]

where \( T \) runs over all topological types of \( k \)-trees.

In general, construction of \( m_k(h) \) can be described using \( k \)-tree. For \( k \geq 2 \), we decompose \( m_k(h) := \sum T m^T_k(h) \), where \( T \) runs over all topological types of \( k \)-trees.

\[
m^T_k(h) : \Omega^*_{(k-1)k}(M, h)_{sm} \otimes \cdots \otimes \Omega^*_{01}(M, h)_{sm} \rightarrow \Omega^*_{0k}(M, h)_{sm}
\]

is an operation defined along the directed tree \( T \) by

1. applying inclusion map \( \iota_{i(i+1)} : \Omega^*_{i(i+1)}(M, h)_{sm} \rightarrow \Omega^*_{i(i+1)}(M, h) \) at semi-infinite incoming edges;
2. applying wedge product \( \wedge \) to each interior vertex;
3. applying homotopy operator \( H_{ij} \) to each internal edge labelled \( ij \);
4. applying projection \( P_{0k} \) to the outgoing semi-infinite edge.
The following graph shows the operation associated to the unique 2-tree.

The higher products \( \{ m_k(\hbar) \}_{k \in \mathbb{Z}_+} \) satisfies the generalized associativity relation which is the so called \( A_\infty \) relation. One may treat the \( A_\infty \) product as a pullback of the wedge product under the homotopy retract \( P_{ij} : \Omega^*_{ij}(M, \hbar) \to \Omega^*_{ij}(M, \hbar)_{sm} \). This proceed is called the homological perturbation lemma. For details about this construction, readers may see [10].

As a result, we obtain an \( A_\infty \) pre-category \( DR_\hbar(M)_{sm} \).

Finally, we restate our Main Theorem with the notations from this section.

**Theorem 6 (Main Theorem).** Given \( f_0, \ldots, f_k \) satisfying generic assumption [4], with \( q_{ij} \in CM^*(f_{ij}) \) be corresponding critical points, there exist \( \eta_0, \hbar_0 > 0 \) and \( C_0 > 0 \), such that \( \phi_{ij}(\eta, \hbar) : CM^*(f_{ij}) \to \Omega^*_{ij}(M, \hbar)_{sm} \) are isomorphism for all \( i \neq j \) when \( \eta < \eta_0 \) and \( \hbar < \hbar_0 \). If we write \( \phi(q_{ij}) = \phi_{ij}(\eta, \hbar)(q_{ij}) \), then we have

\[
\langle m_k(\hbar)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01})), \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2} \rangle = \hbar^{2-k} e^{-A/\hbar}(\langle m_k^{Morse}(q_{(k-1)k}, \ldots, q_{01}), q_{0k} \rangle + R(\hbar)),
\]

with

\[ |R(\hbar)| \leq C_0 \hbar^{1/2} \]

and \( A = f_{0k}(q_{0k}) - f_{01}(q_{01}) - \cdots - f_{(k-1)k}(q_{(k-1)k}) \).

**Remark 7.** The constants \( \eta_0, C_0 \) and \( \hbar_0 \) depend on the functions \( f_0, \ldots, f_k \).

In general, we cannot choose fixed constants that the above statement holds true for all \( m_k(\hbar) \) and all sequences of functions.

**Remark 8.** The constant \( A \) has a geometric meaning. If we consider the cotangent bundle \( T^*M \) of a manifold \( M \) which equips the canonical symplectic form \( \omega_{can} \), and take \( L_i = \Gamma_{q_{ij}} \) to be the Lagrangian sections. Then \( q_{ij} \in L_i \cap L_j \) and \( A \) would be the symplectic area of a degenerated holomorphic disk passing through the intersection points \( q_{ij} \) and having boundary lying on \( L_i \). For details, one may consult [10].
3. Proof of Main Theorem

In the proof, we fix a generic sequence $\vec{f}$ of $k + 1$ functions, with corresponding sequence of critical points $\vec{q}$. First of all, we have

$$\deg(m_k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01}))) = k + 2,$$

so

$$\langle m_k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01})), \phi(q_{0k}) \rangle = \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) - k + 2 = \deg(q_{0k}).$$

holds, which is exactly the condition for $m_k^{Morse}$ in the Morse category to be non-trivial. We will therefore assume condition (3.1) and consider the integral

$$\langle m_k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01})), \phi(q_{0k}) \rangle = \int_M m_k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01})) \wedge^* \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2}.$$

Recall that each directed tree $T$ gives an operation $m^T_k(h)$ and $m_k(h) = \sum_T m^T_k(h)$ which is also the case in Morse category. Therefore, we just have to consider each $m^T_k(h)$ separately.

The first step uses the a priori estimate and resolvent estimate to show that there is an expected exponential decay $e^{-A/h}$ as described in the theorem. We can therefore drop out terms with faster exponential decay. It turns out that the integral localizes to gradient flow trees of corresponding Morse functions. The second step is to replace input eigenforms and the homotopy operators $H_{ij}$ by their WKB approximations. The WKB approximations are governed by ODEs which make computations possible. The final step is to carry out the explicit computations for the leading order term.

3.1. Results for a single Morse function. We start with stating the results on Witten deformation for a single Morse function. These results come from [9], with a few modifications to fit our content.

Definition 9. For a Morse function $f_{ij}$, the Agmon distance $\rho_{ij}$, or simply denoted by $\rho$, is the distance function with respect to the degenerated Riemannian metric $\langle \cdot, \cdot \rangle_{f_{ij}} = |df_{ij}|^2 \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the background metric.

Readers may see [6] for its basic properties. We denote the set of critical points by $C^i_{ij}$. For each $q \in C^i_{ij}$, we let

$$M_{q, \eta} = M \setminus \bigcup_{p \in C^i_{ij} \setminus \{q\}} B(p, \eta),$$
where $B(p, \eta)$ is the open ball centered at $p$ with radius $\eta$ with respect to the Agmon metric. $M_{q, \eta}$ is a manifold with boundary.

For each $q \in C^l$, we use $\Omega^{*}_{ij}(M_{q, \eta}, \hbar)_0$ to denote the space of differential forms with Dirichlet boundary condition, acting by Witten Laplacian $\Delta_{ij, q, 0}$. We have the following spectral gap lemma, saying that eigenvalues in the interval $I(\hbar)$ are well separated from the rest of the spectrum.

**Lemma 10.** For any $\epsilon, \eta > 0$ small enough, there is $\hbar_0 = h_0(\epsilon, \eta) > 0$ and $C_\epsilon > 0$ such that when $\hbar < \hbar_0$, we have

$$\text{Spec}(\Delta_{ij, q, 0}) \cap [\hbar^{3/2}, \hbar^{3/2} + C_\epsilon e^{-\epsilon/\hbar}] = \emptyset,$$

and also

$$\text{Spec}(\Delta_{ij}) \cap [\hbar^{3/2}, \hbar^{3/2} + C_\epsilon e^{-\epsilon/\hbar}] = \emptyset.$$

The eigenforms with corresponding eigenvalue in $I(\hbar)$ are what we concentrated on, and we have the following decay estimate for them.

**Lemma 11.** For any $\epsilon, \eta > 0$ small enough, we have $\hbar_0 = h_0(\epsilon, \eta) > 0$ such that when $\hbar < \hbar_0$, $\Delta_{ij, q, 0}$ has one dimensional eigenspace in $I(\hbar)$. If we let $\varphi_q \in \Omega^{*}_{ij}(M_{q, \eta}, \hbar)_0$ be the corresponding unit length eigenform, we have

$$\varphi_q = O_\epsilon(\hbar^{3/2}) \varphi_q,$$

where $O_\epsilon$ stands for $C^0$ bound with a constant depending on $\epsilon$. Same estimate holds for $d_{ij} \varphi_q$ and $d_{ij}^* \varphi_q$ as well.

We are now ready to give the definition of $\phi_{ij}(\eta, \hbar)$. For each critical point $p$, we take a cut off function $\theta_p$ such that $\theta_p \equiv 1$ in $\overline{B(p, \eta)}$ and compactly supported in $B(p, 2\eta)$. Given a critical point $q \in C^l$, we let

$$\chi_q = 1 - \sum_{p \in C^l \setminus \{q\}} \theta_p.$$

**Proposition 12.** For $\eta > 0$ small enough, there exists $\hbar_0 = h_0(\eta) > 0$, such that when $\hbar < \hbar_0$, we have a linear isomorphism

$$\hat{\phi}_{ij}(\eta, \hbar) : CM^*(f_{ij}) \to \Omega^{*}_{ij}(\hbar)_0 \text{sm}$$

defined by

$$\hat{\phi}_{ij}(\eta, \hbar)(q) = P_{ij} \chi_q \varphi_q,$$

where $P_{ij} : \Omega^{*}_{ij}(\hbar)_0 \to \Omega^{*}_{ij}(\hbar)_0 \text{sm}$ is the projection to the small eigenspace.

**Remark 13.** One may notice that $\varphi_q$ is defined only up to $\pm$ sign. Recall that in the definition of Morse category, we fix an orientation for unstable submanifold $V_q^-$ and stable submanifold $V_q^+$ at $q$. The sign of $\varphi_q$ is chosen such that it agrees with the orientation of $V_q^-$ at $q$. 
Definition 14. We renormalize \( \hat{\phi}_{ij}(\eta, h) \) to give a map \( \phi_{ij}(\eta, h) \) defined by
\[
(3.4) \quad \phi_{ij}(\eta, h)(q) = \frac{|\lambda_+|^{\frac{1}{2}}}{|\lambda_+|^{\frac{1}{2}}}(\frac{\pi h}{2})^{\frac{1}{2}(\frac{1}{2} - \text{deg}(q))} \hat{\phi}_{ij}(\eta, h)(q),
\]
where \( \lambda_+ \) and \( \lambda_- \) are products of positive and negative eigenvalues of \( \nabla^2 f \) at \( q \) respectively.

Remark 15. The meaning of the normalization is to get the following asymptotic expansion
\[
(3.5) \quad \int_{V_{\omega}} e^{f_{ij}/h} \phi_{ij}(\eta, h)(q) = 1 + O(h),
\]
which is the one appeared in [14].

From the estimate (3.2), we also have a similar estimate for \( \hat{\phi}_{ij}(q) \).

Lemma 16. For any \( \epsilon, \eta > 0 \) small enough, there exists \( h_0 = h_0(\epsilon, \eta) > 0 \) such that for \( h < h_0 \), we have
\[
(3.6) \quad \hat{\phi}_{ij}(q) = O_{\epsilon, \eta}(e^{-(\rho_{ij}(q, x) - \epsilon - 2\eta)/h}),
\]
and same estimate holds for \( d_{ij} \hat{\phi}_{ij}(q) \) and \( d_{ij}^* \hat{\phi}_{ij}(q) \).

The estimate for derivatives of \( \hat{\phi}_{ij}(q) \) can be strengthened. Notice that
\( d_{ij} \hat{\phi}_{ij}(q) \) is an eigenform of degree \( l + 1 \), similarly \( d_{ij}^* \hat{\phi}_{ij}(q) \) is an eigenform of degree \( l - 1 \). Making use of these, we have the following lemma.

Lemma 17. For any \( \epsilon, \eta > 0 \) small enough, there exists \( h_0 = h_0(\epsilon, \eta) > 0 \) such that for \( h < h_0(\epsilon, \eta) \), we have
\[
(3.7) \quad d_{ij} \hat{\phi}_{ij}(q) = O_{\epsilon, \eta}(e^{-(\alpha_q(x) - \epsilon - 4\eta)/h}),
\]
\[
(3.7) \quad d_{ij}^* \hat{\phi}_{ij}(q) = O_{\epsilon, \eta}(e^{-(\beta_q(x) - \epsilon - 4\eta)/h}),
\]
where
\[
\alpha_q(x) = \min_{p \in C^{l+1} \cup C^l \setminus \{q\}} (\rho_{ij}(q, p) + \rho_{ij}(p, x)),
\]
\[
\beta_q(x) = \min_{p \in C^{l+1} \cup C^l \setminus \{q\}} (\rho_{ij}(q, p) + \rho_{ij}(p, x)).
\]

Furthermore, we can compare the inner products on \( CM_{ij}^* \) and \( \Omega^*_{ij}(M, h)_{sm} \).

If we define a square matrix \( D \) with
\[
D_{pq} = \begin{cases} 0 & \text{if } p = q \\ e^{-(\rho_{ij}(p, q)/h)} & \text{if } p \neq q \end{cases},
\]
then we obtain the following estimates for inner product among eigenforms.

Lemma 18. For any \( \epsilon, \eta > 0 \) small enough, there exists \( h_0 = h_0(\epsilon, \eta) > 0 \) such that for \( h < h_0(\epsilon, \eta) \), we have
\[
(3.8) \quad \langle \hat{\phi}_{ij}(p), \hat{\phi}_{ij}(q) \rangle - \langle \chi_p \varphi_p, \chi_q \varphi_q \rangle = O_{\epsilon, \eta}(e^{(\epsilon + 4\eta)/h}(D^2 + D^3)_{pq}).
\]
Furthermore, we have an estimate
\begin{equation}
\langle \chi_p \varphi_p, \chi_q \varphi_q \rangle = \begin{cases} 
1 + O(\epsilon, \eta) & \text{if } p = q \\
O(\epsilon, \eta) \left( e^{-\rho_{ij}(x_0,y_0)-\epsilon}/h \right) & \text{if } p \neq q
\end{cases},
\end{equation}
where
\[ S_0 = \min_{p' \neq q' \in C^1} \rho_{ij}(p', q'). \]

Finally, we need the following resolvent estimate will be proved in section 4 for our operator \( G_{ij} \) defined in (2.4).

\textbf{Lemma 19.} For any \( j \in \mathbb{Z}_+ \) and \( \epsilon > 0 \), there is \( k_j \in \mathbb{Z}_+ \) and \( h_0 = h_0(\epsilon) > 0 \) such that for any two points \( x_0, y_0 \in M \), there exist neighborhoods \( V \) and \( U \) (depending on \( \epsilon \)) of \( x_0 \) and \( y_0 \) respectively, and \( C_{j, \epsilon} > 0 \) such that
\begin{equation}
\| \nabla_j (G_{ij} u) \|_{C^0(V)} \leq C_{j, \epsilon} e^{-\rho_{ij}(x_0, y_0)/h} \| u \|_{W^{k_j, 2}(U)},
\end{equation}
for all \( h < h_0 \) and \( u \in C^0_c(U) \), where \( W^{k,p} \) refers to the Sobolev norm.

\subsection*{3.2. A priori estimates.}
So far we have been considering a fixed Morse function \( f_{ij} \). From now on, we will consider a fixed generic sequence \( \vec{f} \) with corresponding sequence of critical points \( \vec{q} \) as in the beginning of section 3.

\textbf{Notations 20.} We use \( q_{ij} \) to denote a critical point of \( f_{ij} \). The eigenform \( \phi_{ij}(\eta, \hbar)(q_{ij}) \) associated to \( q_{ij} \) is abbreviated by \( \phi_{ij} \).

We will use the result in the previous section to localize the integral
\begin{equation}
\int_M m_k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01})) \wedge^{*} \frac{\phi(q_{0k})}{\| \phi(q_{0k}) \|^2}
\end{equation}
to gradient flow trees, when the degree condition (3.1) holds. We begin with the \( m_3(h) \) case which involves less combinatorics to illustrate the analytic argument.

\textbf{3.2.1.} \( m_3(h) \) case. There are two 3-leaved directed trees, which are denoted by \( T_1 \) and \( T_2 \). We simply consider \( m_3^{T_1}(h) \) for \( T_1 \) which is the tree shown in figure [2.2.1] and relate this operation to counting gradient trees of type \( T_1 \). \( T_1 \) has two interior vertices, which are denoted by \( v \) and \( v_r \) as in the figure. According to the combinatorics of \( T_1 \), we define \( \tilde{\rho}_{T_1} : M|^{V(T_1)} \to \mathbb{R}_+ \) which is given by
\[ \tilde{\rho}_{T_1}(x_v, x_{v_r}) = \rho_{13}(x_v, x_{v_r}) + \rho_{01}(x_{v_r}, q_{01}) + \rho_{12}(x_v, q_{12}) + \rho_{23}(x_v, q_{23}) + \rho_{03}(x_{v_r}, q_{03}). \]

Roughly speaking, it is the length of the geodesic tree of type \( T_1 \) with interior vertices \( x_v, x_{v_r} \) and end points of semi-infinite edges \( e_{ij} \)'s laying on \( q_{ij} \)'s as shown in the following figure.
Notice that \( \vec{\rho}_{T_1}(x_v, x_{v_r}) \geq A = f_{03}(q_{03}) - f_{01}(q_{01}) - f_{12}(q_{12}) - f_{23}(q_{23}) \) and the equality holds if and only if \((x_v, x_{v_r})\) are interior vertices of a gradient flow tree of the type \(T_1\).

The term
\[
(3.12) \quad \langle m_{T_1}^3(\varphi_{23}, \varphi_{12}, \varphi_{01}), \frac{\varphi_{03}}{\|\varphi_{03}\|^2} \rangle = \int_M H_{13}(\varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01} \wedge (\frac{\varphi_{03}}{\|\varphi_{03}\|^2}),
\]

can be controlled by \( \vec{\rho}_{T_1} \). More precisely, fixing two points \(x_v, x_{v_r} \in M\) and \(\epsilon\) small enough, lemma [19] holds for operator \(G_{13}\) and hence \(H_{13}\) with \(U\) and \(V\) being balls centering at \(x_v\) and \(x_{v_r}\) (with respect to \(\rho_{13}\)) of radius \(r_1\). If we have two cut off functions \(\chi\) and \(\chi_r\) supported in \(B(x_v, r_1)\) and \(B(x_{v_r}, r_1)\) respectively, then we have
\[
\|\chi_r H_{13}(\chi \varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01} \wedge \star \varphi_{03}\|_{L^\infty} \leq C_{\epsilon, \eta} e^{-((\vec{\rho}_{T_1}(x_v, x_{v_r})) - 4r_1 - 5\epsilon - 6\eta)/\hbar}
\]
for those small enough \(\hbar\). Here the decay factors \(\rho_{23}(q_{23}, x_v)\) and \(\rho_{12}(q_{12}, x_v)\) come from the a priori estimate in lemma [16] for the input forms \(\varphi_{23}\) and \(\varphi_{12}\) respectively, while the decay factor \(\rho_{13}(x_v, x_{v_r})\) comes from the resolvent estimate lemma [19]. Combining with the decay estimates for \(\varphi_{01}\) and \(\varphi_{03}\), we obtain
\[
\|\chi_r H_{13}(\chi \varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01} \wedge \star \varphi_{03}\|_{L^\infty} \leq C_{\epsilon, \eta} e^{-((\vec{\rho}_{T_1}(x_v, x_{v_r})) - 4r_1 - 5\epsilon - 6\eta)/\hbar}.
\]
We assume there are gradient trees \(\Gamma_1, \ldots \Gamma_l\) of the type \(T_1\). For each tree \(\Gamma_i\), we take open neighborhoods \(D_{\Gamma_i,v}\) and \(W_{\Gamma_i,v}\) of interiors vertices \(x_{\Gamma_i,v}\) with \(D_{\Gamma_i,v} \subset W_{\Gamma_i,v}\) and similarly \(D_{\Gamma_i,v_r}\) and \(W_{\Gamma_i,v_r}\) for \(x_{\Gamma_i,v_r}\). The following figure illustrates the situation.

We can assume there is a constant \(C\) such that \(\vec{\rho}_{T_1} \geq A + C\) in \(M^{V(T_1)} \setminus D_{\Gamma_i}\), where \(D_{\Gamma_i} = D_{\Gamma_i,v} \times D_{\Gamma_i,v_r}\). If \(\vec{B}(\vec{x}, r_1) = B(x_v, r_1) \times B(x_{v_r}, r_1)\) is away from
the $D_{\Gamma_i}$, we will have

$$\| \chi_{\Gamma_i} H_{13}(\chi \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge *\phi_{03} \|_{L^\infty} \leq C_{\epsilon, \eta} e^{-(A+\frac{\epsilon}{2})/\hbar}.$$ 

Therefore, we can take cut off functions $\chi_{\Gamma_i,v}$, $\chi_{\Gamma_i,v,r}$ associating to each tree $\Gamma_i$, with support in $W_{\Gamma_i,v}$, $W_{\Gamma_i,v,r}$ and equal to 1 on $D_{\Gamma_i,v}$, $D_{\Gamma_i,v,r}$ respectively, to get

$$\langle m_{T_3}^{\Gamma_i}(\phi_{23}, \phi_{12}, \phi_{01}), \frac{\phi_{03}}{\| \phi_{03} \|^2} \rangle = \sum \int_M \{ \chi_{\Gamma_i,v} H_{13}(\chi \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge *\phi_{03} \} + O(e^{-(A+\frac{\epsilon}{2})/\hbar}).$$

This localizes the integral computing $m_{T_3}^{\Gamma_i}$ to gradient trees of type $T_1$. Notice that the neighborhood $D_{\Gamma_i}$ and $W_{\Gamma_i}$ can be chosen to be arbitrarily small in the previous argument. Next, we will consider $m_k(\hbar)$ with arbitrary $k$ which has more complicated notations.

3.2.2. $m_k(\hbar)$ case. We fix a $k$-leafed tree $T$ and consider the operation corresponding to it, denoted by $m_T^k(\hbar)$. We try to relate this operation to counting of gradient trees of type $T$. We have the function $\tilde{\rho}_T : M^{|V(T)|} \rightarrow \mathbb{R}_+$ defined
according to the combinatorics of $T$ given by

$$
\tilde{\rho}_T(\vec{x}) = \sum_{e_{ij} \in E(T)} \rho_{ij}(x_{S}(e_{ij}), x_{E}(e_{ij})) + \sum_{i=0}^{k-1} \rho_{i(i+1)}(q_{i(i+1)}, x_{E}(e_{i(i+1)})) + \rho_{0k}(q_{0k}, x_{S}(e_{0k})).
$$

Here the variables $\vec{x}$ are labelled by the vertices of $T$. ($x_S(e)$ and $x_E(e)$ refer to the variables corresponding to vertices which are starting point and endpoint of the edge $e$ respectively.) Recall that $E(T)$ is the set of internal edges of $T$ and each interior edge $e$ has a unique label by two integers as $e_{ij}$, corresponding to the Morse function $f_{ij} = f_j - f_i$. The notation $\rho_{ij}$ refers to the Agmon distance corresponding to the Morse function $f_{ij}.$

$\tilde{\rho}_T(\vec{x})$ is the length function of a geodesic tree (may not be unique) with topological type $T$, with interior vertices $\vec{x}$ and semi-infinite edges ending on critical points $q_{ij}$. Similar to the case of $m_3(h)$, we have the following lemma.

**Lemma 21.** The function $\tilde{\rho}_T$ is bounded below by $A = f_{01}(q_{01}) + \cdots + f_{(k-1)k}(q_{(k-1)k}) - f_{0k}(q_{0k})$, and it attains minimum at $\vec{x}$ if and only if $\vec{x}$ is the vector consisting of interior vertices of a gradient flow tree of $\vec{f}$ of type $T$ ended at corresponding critical points $\vec{q}$.

**Proof.** The proof relies on the fact (see [9]) that we have

$$|f_{ij}(x) - f_{ij}(y)| \leq \rho_{ij}(x, y),$$

if $f_{ij}$ is a Morse function on $M$, and $\rho_{ij}(x, y)$ is the Agmon distance. Furthermore, the equality $f_{ij}(x) - f_{ij}(y) = \rho_{ij}(x, y)$ forces the geodesic from $y$ to $x$ to be a generalized integral curve of $\nabla f_{ij}$. We apply this fact to each term in (3.13) and the result follows. $\square$

Every gradient flow tree $\Gamma \in \mathcal{M}(\vec{f}, \vec{q})(T)$ is associated with a unique minimum point $\vec{x}_T \in M^{[V(T)]}$. For each tree, we take a covering $W_T$ of $\vec{x}_T$, given by a product $W_T = \prod_{v \in V(T)} W_{\Gamma_v,v}$, where each $W_{\Gamma_v,v}$ is an open subsets in $M$ containing $x_v$ such that all $W_{\Gamma_v,v}$’s are disjoint from each other. If we further take $D_T = \prod_{v \in V(T)} D_{\Gamma_v,v}$ such that $\overline{D_{\Gamma_v,v}} \subset W_{\Gamma_v,v}$, we have a constant $C > 0$ such that $\tilde{\rho}_T \geq A + C$ on $M^{[V(T)]} \setminus D_T$. We are going to show that the integral (3.11) can be localized.

We take a finite covering of $M$ with balls $\{B(x, r)\}_{B(x,r) \in \mathcal{J}}$ of radius $r$ centering at $x$, with a partition of unity $\{\chi_B\}_{B \in \mathcal{J}}$ subordinating to it. We choose a covering $\{B_r(\vec{x})\}_{B \in \mathcal{I}}$ of $M^{[V(T)]}$ given by product $B_r(\vec{x}) = \prod_{v \in V(T)} B(x_v, r)$, where $B(x_v, r) \in \mathcal{J}$. We decompose $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $B \in \mathcal{I}_2$ are those having empty intersection with $\overline{D_T}$, and $B \in \mathcal{I}_1$ satisfying $\overline{B} \subset W_T$. These can be achieved by choosing $r$ small enough.
We can take cut off functions subordinate to the covering \( \{ B \}_I \), given by product of functions \( \chi_B \) on \( M \). We write \( \tilde{\chi}_B = \prod_{i \in V(T)} \chi_B(x_i, r) \) for the function supported in \( B \). We will use \( \tilde{\chi}_B \) to cut off the following integral
\[
m_T^k(h)(\vec{q}) := \int_M m_T^k(h)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01}))) \wedge^* \frac{\phi(q_{0k})}{\| \phi(q_{0k}) \|^2}.
\]
Recall that the \( m_T^k(h) \) is defined using wedge product and the homotopy operator \( H_{ij} \), following the combinatorics of the tree \( T \). We cut off the operation \( m_T^k(h) \) using the function \( \chi_{B(x_v, r)} \) whenever taking wedge product at the vertex \( v \). We will write \( m_T^k(h, \vec{\chi}_B) \) for the integral after cutting off by \( \vec{\chi}_B \). Therefore we have
\[
m_T^k(h)(\vec{q}) = \sum_{B \in I_1} m_T^k(h, \tilde{\chi}_B)(\vec{q}) + \sum_{B \in I_2} m_T^k(h, \tilde{\chi}_B)(\vec{q}),
\]
where \( m_T^k(h, \tilde{\chi}_B)(\vec{q}) \) stand for the integral after cutting off by \( \tilde{\chi}_B \). Applying the resolvent estimate in section 4 and the estimate (16), we obtain the following lemma.

**Lemma 22.** For any \( \epsilon > 0 \), there exist positive \( r(\epsilon), \eta(\epsilon) \) and \( h(\epsilon) \) such that
\[
m_T^k(h, \tilde{\chi}_B) = O(r, \epsilon, \eta)(e^{-(A + C)/h})
\]
for \( h < h(\epsilon) \), if we take the covering of radius \( r < r(\epsilon) \) and \( \eta < \eta(\epsilon) \). Here \( \vec{\xi} \) is the center of the ball \( B \).

The proof is essentially the same as the case for \( m_3(h) \). Similarly, we can have
\[
\sum_{B \in I_2} m_T^k(h, \tilde{\chi}_B) = O(r, \epsilon, \eta)(e^{-(A + C)/h}),
\]
for \( h \) small enough. It follows from the fact that \( \tilde{\rho}_T(\vec{\xi}) \geq A + C \) for those covering in \( I_2 \). This result basically says that the integral \( m_T^k(h) \) can be localized to gradient flow tree using the cut off mentioned above. To summarize, we have the following proposition.

**Proposition 23.** For each gradient flow tree \( \Gamma \), there is a sequence of cutoff functions \( \{ \tilde{\chi}_\Gamma \} \) which is supported in \( W_\Gamma \) and satisfy \( \tilde{\chi}_\Gamma \equiv 1 \) on \( \overline{D_\Gamma} \) such that
\[
m_T^k(h, \tilde{\chi}_\Gamma)(\vec{q}) = \sum_{\Gamma \in M(\vec{f}, \vec{q})(T)} m_T^k(h, \tilde{\chi}_\Gamma)(\vec{q}) + O(e^{-(A + C)/h}),
\]
for \( h \) small enough.

**Remark 24.** In the above argument, the neighborhood \( W_\Gamma \) can be chosen to be arbitrary small. We will obtain a smaller constant \( C \) if we shrink the neighborhood \( W_\Gamma \).

After localizing the integral, we move on to the section concerning WKB approximation which helps to compute of the leading order contribution of \( m_T^k(h, \tilde{\chi}_\Gamma) \).
3.3. WKB method. In this section, we will state the results of WKB methods from section 5 and argue the WKB approximations can be used in the computation of leading order contribution of \(m_k(h)\). The WKB solutions are necessary for explicit computation as they are governed by ODEs instead of PDEs. We fix a gradient flow tree \(\Gamma\) to consider as the integrand in (3.11) is localized to gradient trees.

3.3.1. WKB method for Witten Laplace operators. We first state the result from [9].

Lemma 25. There is a WKB approximation of the eigenform \(\phi_{ij}\) of the form

\[
\phi_{ij} \sim e^{-\psi_{ij}/h^2} \left( \omega_{ij,0} + \omega_{ij,1}h^{1/2} + \ldots \right)
\]

in any small enough open set \(W\) containing \(V_{q_{ij}}^+ \cup V_{q_{ij}}^-\), where \(\psi_{ij} = \rho_{ij}(q_{ij}, \cdot)\) is the Agmon distance function from \(q_{ij}\).

Remark 26. The precise meaning of this WKB approximation is given in section 5.6. Roughly speaking, it is in the sense of \(C^\infty\) approximation on every compact subset of \(W\).

Remark 27. \(g_{ij}^+ = \psi_{ij} - (f_{ij} - f_{ij}(q_{ij}))\) is a nonnegative function which is Bott-Morse in a neighborhood of \(V^+_{q_{ij}}\) with zero set \(V^+_{q_{ij}}\).

Remark 28. There is also a similar approximation for \(*\phi_{ij}\), in a neighborhood of the unstable submanifold \(V^+_{q_{ij}} \cup V^-_{q_{ij}}\). In that case, \(g^-_{ij} = \psi_{ij} + (f_{ij} - f_{ij}(q_{ij}))\) is a nonnegative function which is Bott-Morse in a neighborhood of \(V^-_{q_{ij}}\) with zero set \(V^-_{q_{ij}}\).

3.3.2. WKB method for homotopy operators. Here we state a WKB method needed for the homotopy operators appearing in the higher products \(m_k(h)\). The proof can be found in section 5.

We begin by giving the setup of the lemma. Let \(\gamma(t)\) be a flow line of \(\nabla f_{ij}/|\nabla f_{ij}|_{\rho_{ij}}\) starts at \(\gamma(0) = x_S\) and \(\gamma(T) = x_E\) for a fixed \(T > 0\). We consider an input form \(\zeta_S\) defined in a neighborhood \(W_S\) of \(x_S\). Suppose we are given a WKB approximation of \(\zeta_S\) in \(W_S\), which is an approximation of \(\zeta_S\) according to order of \(h\) of the form

\[
\zeta_S \sim e^{-\psi_S/h^2} \left( \omega_{S,0} + \omega_{S,1}h^{1/2} + \omega_{S,2}h^1 + \ldots \right)
\]

(The precise meaning of this infinite series approximation can be found in section 5.6). We further assume that \(g_S = \psi_S - f_{ij}\) is a nonnegative Bott-Morse function in \(W_S\) with zero set \(V_S\). We consider the equation

\[
\Delta_{ij}\zeta_E = (I - P_{ij})d^*_{ij}(\chi_S\zeta_S),
\]

where \(\chi_S\) is a cutoff function compactly supported in \(W_S\), \(P_{ij} : \Omega^*_{ij}(M, h) \to \Omega^*_{ij}(M, h)_{\ast m}\) is the projection. We want to have a WKB approximation of \(\zeta_E = H_{ij}(\chi_S\zeta_S)\)
Lemma 29. For supp($\chi_S$) small enough, there is a WKB approximation of $\zeta_E$ in a small enough neighborhood $W_E$ of $E$, of the form

\[ \zeta_E \sim e^{-\psi_E/\hbar} \hbar^{-1/2} (\omega_E,0 + \omega_E,1 \hbar^{1/2} + \ldots). \]

Furthermore, the function $g_E := \psi_E - f_{ij}$ is a nonnegative function which is Bott-Morse in $W_E$ with zero set $V_E = (\bigcup_{-\infty < t < +\infty} \sigma_t(V_S)) \cap W_E$ which is closed in $W_E$, where $\sigma_t$ is the time $t$ flow of $\nabla f_{ij}/|\nabla f_{ij}|\rho_{ij}$.

Provided above lemmas, we are going to show that the WKB approximations can be used to compute the leading order contribution in $m^T_k(h, \bar{\chi}_\Gamma)$. We proceed in the same way as the last subsection by first considering the $m_3(h)$ case.

3.3.3. $m_3(h)$ case. We fix a gradient tree $\Gamma$ of type $T_1$ as in the subsection 3.2.1 with interior vertices $x_{\Gamma,v}$ and $x_{\Gamma,v'}$. Since the gradient tree $\Gamma$ is fixed, we tend to omit the dependence on $\Gamma$ in our notations. We take neighborhoods $W_v$ and $W_{v'}$ of $x_v$ and $x_{v'}$ respectively, with cut off functions $\chi_v$ and $\chi_{v'}$ supported in $W_v$ and $W_{v'}$ respectively as shown in the following figure.
As $x_v \in V_{q_{12}}^+ \cap V_{q_{23}}^+$, we can assume that the WKB approximations
\[ \phi_{12} \sim e^{-\psi_{12}/\hbar} \frac{\deg(q_{12})}{2} (\omega_{12,0} + \omega_{12,1} \hbar^{1/2} + \ldots), \]
and
\[ \phi_{23} \sim e^{-\psi_{23}/\hbar} \frac{\deg(q_{23})}{2} (\omega_{23,0} + \omega_{23,1} \hbar^{1/2} + \ldots) \]
hold in $W_v$, by taking a smaller $W_v$ if necessary.

We apply lemma 5.1 with Morse function $f_{13}$, input form $\zeta_S = \phi_{23} \wedge \phi_{12}$, starting vertex $x_S = x_v$, ending vertex $x_E = x_{v_r}$, with neighborhood $W_S = W_v$ and $W_E = W_{v_r}$ (This can be done by shrinking $W_v$ and $W_{v_r}$ if necessary).

As a result, we obtain the WKB approximation
\[ H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \sim e^{-\psi_{13}/\hbar} \frac{\deg(q_{23}) + \deg(q_{12}) + 1}{2} (\omega_{13,0} + \omega_{13,1} \hbar^{1/2} + \ldots), \]
by taking $\psi_E = \psi_{13}$ and $\omega_{E,i} = \omega_{13,i}$ in the lemma.

In order to compute
\[ m_{T_3}(\hbar, \tilde{\chi}^* \Gamma) = \int_M \chi_v H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge * \frac{\phi_{03}}{\|\phi_{03}\|^2} \]
up to an error of order $e^{-\frac{A}{\hbar}} \mathcal{O}(\hbar^{1/2})$, we can simply compute the integral
\[ \int_M \{ \chi_v (e^{-\psi_{13}/\hbar} \frac{\deg(q_{23}) + \deg(q_{12}) + 1}{2} \omega_{13,0}) \wedge (e^{-\psi_{01}/\hbar} \frac{\deg(q_{01})}{2} \omega_{01,0}) \wedge (\hbar \frac{\deg(q_{03})}{2} \frac{e^{-\psi_{03}/\hbar} \omega_{03,0}}{\|e^{-\psi_{03}/\hbar} \omega_{03,0}\|^2}) \} \]
\[ = \frac{\hbar^{-1}}{\|e^{-\psi_{03}/\hbar} \omega_{03,0}\|^2} \int_M \{ \chi_v (e^{-(\psi_{13} + \psi_{01} + \psi_{03})/\hbar} \omega_{13,0} \wedge \omega_{01,0} \wedge * \omega_{03,0}) \}. \]
What we obtained is an integral involving $\omega_{ij,0}$'s, which are governed by ODEs. This is easier for explicit computations. Next, we move on to show this also happens in the $m_k(\hbar)$ case for any $k$. 
3.3.4. $m_k(h)$ case. We consider a gradient tree $\Gamma$ of type $T$, with $k$ semi-infinite incoming edges. Recall in section 2.2.1 that each edge in $T$ is assigned with a label by two integer $ij$. We will use $ij$ to represent an edge in $T$ and denote the corresponding edge in the gradient tree $\Gamma$ by $e_{ij}$. The vertex in the gradient tree corresponding to $v$ in $T$ will be denoted by $x_v$. We again omit the dependence on $\Gamma$ in our notations as it is fixed. We are going to associate $\phi_{(ij,v)} \in \Omega^*_ij(M, h)$, together with its WKB approximation $\phi_{(ij,v)} \sim e^{-\psi_{(ij,v)}/h}h^{\nu_{(ij,v)}}(\omega_{(ij,v)},0 + \omega_{(ij,v)},1 + \ldots )$ in some neighborhood $W_v$ of $x_v$ to each flag $(ij,v)$ as shown in the following figure 4. We also fix cut off functions $\chi_v$’s supported in $W_v$ and restrict our attention to integral $m_k^T(h, \chi)(\vec{q})$, using the arguments in section 3.2.

We define the followings inductively.

1. for a semi-infinite incoming edge $i(i+1)$ and its ending vertex $v$, we take $\phi_{i(i+1),v}$ to be the input eigenform $\phi_{i(i+1)}$, with its the WKB approximation in $W_v$ as in lemma. We also let $g_{i(i+1),v} = \psi_{i(i+1),v} - (f_{i(i+1)} - f_{i(i+1)}(q))$. We choose $W_v$ small enough such that the WKB approximation of input eigenform $\phi_{i(i+1)}$ holds in $W_v$.

For example in the above figure 4 we require the WKB approximation of $\phi_{k-1,k}$ in $W_v$ and WKB approximations of $\phi_{k-2}(k-1), \phi_{k-3}(k-2)$ in $W_v$ hold;

2. for an internal edge $il$ with its starting vertex $v$ and assume $ij$ and $jl$ are two incoming edges meeting $il$ at $v$ as shown in figure 5 we
take $\phi(d,v) = \phi(jl,v) \wedge \phi(ij,v)$. The WKB expression of $\phi(d,v)$ comes from the expression of $\phi(jl,v) \wedge \phi(ij,v)$, which means

\[
\psi(d,v) = \psi(ij,v) + \psi(jl,v),
\]

\[
\omega(e_{d,v}, m) = \sum_{m' + m = n} \omega(jl,v),m \wedge \omega(ij,v),m',
\]

\[
r(d,v) = r(jl,v) + r(ij,v).
\]

We also let $g(d,v) = g(ij,v) + g(jl,v)$;

(3) for an internal edge $ij$ with its starting vertex $v_S$ and ending vertex $v_E$ as shown in figure 6, we take the WKB approximation in lemma 29 of $\phi(ij,v_E) = H_{ij}(\chi_{v_S} \phi(ij,v_S))$ in $W_{v_E}$ by taking supp($\chi_{v_S}$) and $W_{v_E}$ small enough if necessary. We also let $g(ij,v_E) = \psi(ij,v_E) - f_{ij} + \ldots$
\[ \sum_{i<m \leq j} f_{(m-1)m}(q_{(m-1)m}) \text{ and } r_{ij,v_E} = r_{ij,v_E} - \frac{1}{2}; \]

(4) for the semi-infinite outgoing edge \(0k\) with the root vertex \(v_r\), we take \(\phi_{0k,v_r}\) to be the eigenform \(\phi_{0k}\), with the WKB approximation of \(*\phi_{0k}\) in lemma 25 holds in \(W_{v_r}\). We also let \(g_{0k,v_r} = \psi_{0k,v_r} + (f_{0k,v_r} - f_{0k,v_r})\).

**Remark 30.** When we apply lemma 29 along an internal edge \(ij\) with starting vertex \(v_S\) and ending vertex \(v_E\), the size of \(\text{supp}(\chi_{v_S})\) and \(W_{v_E}\) depends on the input form \(\phi_{ij,v_S}\), or more precisely, properties of the function \(\psi_{ij,v_S}\).

From the definition of \(m^T_k(h, \Gamma)\), we see that

\[
\langle m^T_k(\phi_{(k-1)k}, \ldots, \phi_{01}), \frac{\phi_{0k}}{\|\phi_{0k}\|^2} \rangle = \int_M \phi_{jk,v_r} \wedge \phi_{0j,v_r} \wedge \frac{*\phi_{0k,v_r}}{\|\phi_{0k,v_r}\|^2},
\]

if three edges \(0j, jk\) and \(0k\) are meeting at the root vertex \(v_r\). Applying lemma 25 to input eigenforms and lemma 29 to homotopy operators \(H_{ij}\) along internal edges \(e_{ij}\)'s, we prove that each WKB approximation

\[
\phi_{ij,v} \sim e^{-\psi_{ij,v}/h} h^{r_{ij,v}} (\omega_{ij,v,0} + \omega_{ij,v,1} + \ldots)
\]

is an \(C^\infty\) approximation with error \(e^{-\psi_{ij,v}/h} \mathcal{O}(h^\infty)\). Therefore, we can replace \(\phi\)'s by first term in its WKB approximation for computing the leading order contribution. We obtain

\[
(3.23) \quad \langle m^T_k(\phi_{(k-1)k}, \ldots, \phi_{01}), \frac{\phi_{0k}}{\|\phi_{0k}\|^2} \rangle = \|h^{r_{jk,v_r}} + r_{0j,v_r} + r_{0k,v_r}\|_M \int e^{-\psi_{jk,v_r} + \psi_{0j,v_r} + \psi_{0k,v_r}}
\]

\[
\chi_{v_r} (\omega_{jk,v_r,0} \wedge \omega_{0j,v_r,0} \wedge \frac{\omega_{0k,v_r,0}}{\|\phi_{0k}\|^2}) (1 + \mathcal{O}(h^{1/2})).
\]

### 3.4. Explicit computations.

We are going to compute the leading order contribution of the above integral in this subsection. This is possible since \(\psi\)'s are explicit geometric functions and \(\omega\)'s are determined by ODEs. We again begin with the computation for \(m_3(h)\) case which involves less combinatorics.

#### 3.4.1. \(m_3(h)\) case:

We have to compute the leading order contribution from the integral (3.22). We first take a look on the exponential decay factor
in the integral. We recall that $g_{13}$, $g_{01}^+$ and $g_{03}$ are defined by

$$\psi_{13} = g_{13} + f_{13} - f_{23}(q_{23}) - f_{12}(q_{12}),$$
$$\psi_{01} = g_{01}^+ + f_{01} - f_{01}(q_{01}),$$
$$\psi_{03} = g_{03}^+ - f_{03} + f_{03}(q_{03}).$$

Therefore we have

$$\psi_{13} + \psi_{01} + \psi_{03} = g_{13} + g_{01}^+ + g_{03}^+ + A,$$

where $A = f_{03}(q_{03}) - f_{01}(q_{01}) - f_{12}(q_{12}) - f_{23}(q_{23})$. Therefore, what we have to compute is the leading order term from the integral

$$\int_M \chi_{v_r} \left( e^{-\frac{g_{13} + g_{01}^+ + g_{03}^-}{k}} \omega_{13,0} \wedge \omega_{01,0} \wedge \ast \frac{\omega_{03,0}}{\|\phi_{03}\|^2} \right).$$

We claim that the exponential decay $e^{-\frac{g_{13} + g_{01}^+ + g_{03}^-}{k}}$ will limit to the delta distribution concentrating at the root vertex $x_{v_r}$ of the gradient tree $\Gamma$. This precisely means that $g_{13} + g_{01}^+ + g_{03}^- \geq 0$ is a Bott-Morse function in $W_{v_r}$ with zero at $x_{v_r}$.

We recall in remark 27 that $g_{01}^+$ and $g_{03}^-$ are Bott-Morse with absolute minimums on $V_{01}^+$ and $V_{03}^-$ respectively. We also recall from lemma 29 that $g_{13}$ is also Bott-Morse in $W_{v_r}$ with absolute minimum denoted by $V_{13}$ (colored red in the following figure), which is the submanifold $(\bigcup_{-\infty < t < +\infty} \sigma_t(V_{23}^+ \cap V_{12}^+) \cap W_{v_r}$ flowed out from $V_{23}^+ \cap V_{12}^+$ (colored blue in the following figure), under the flow of $\nabla f_{13}/|\nabla f_{13}|_{|_{\psi_{13}}}$ which is denoted by $\sigma_t$.

The definition of gradient tree indicates that \{x_{v_r}\} = $V_{13} \cap V_{01}^+ \cap V_{03}^-$ transversally which means $e^{-\frac{g_{13} + g_{01}^+ + g_{03}^-}{k}}$ concentrating at $x_{v_r}$. To deal with this situation, we have the following lemma which will be proven in section 5.8.
Lemma 31. Let $M$ be a $n$-dimensional manifold and $S$ be a $k$-dimensional submanifold in $M$, with a neighborhood $B$ of $S$ which can be identified as the normal bundle $\pi : NS \to S$. Suppose $\varphi : B \to \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set $S$ and $\beta \in \Omega^k(B)$ has vertical compactly support along the fiber of $\pi$, we have

$$
\pi^*(e^{-\varphi(x)/\hbar}\beta) = (2\pi \hbar)^{(n-k)/2}(\text{vol}(\nabla^2 \varphi)\beta)|_{V}(1 + O(\hbar)),
$$

where $\pi^*$ is the integration along fiber. Here $\text{vol}(\nabla^2 \varphi)$ stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2 \varphi$ along fibers of $\pi$.

We find from the above lemma that the leading order contribution in the above integral (3.24) depend only on values of $\omega_{13,0}, \omega_{01,0}$ and $*\omega_{03,0}$ at the point $v_r$. We will see later in lemma 33 that $\omega_{13,0}, \omega_{23,0}$ and $*\omega_{03,0}$ are forms in $\Lambda^{top}(NV_{13})^*, \Lambda^{top}(NV_{01})^*$ and $\Lambda^{top}(NV_{03})^*$ respectively. Therefore we have

$$
\int_M \chi_{v_r} e^{-\frac{g_{13} + g_{01} + g_{03}}{\hbar}} \omega_{13,0} \wedge \omega_{01,0} \wedge *\omega_{03,0}
= \pm \left( \int_{NV_{13},x_{v_r}} e^{-\frac{g_{13}}{\hbar}} \chi_{v_r} \omega_{13,0} \right) \left( \int_{NV_{01},x_{v_r}} e^{-\frac{g_{01}}{\hbar}} \chi_{v_r} \omega_{01,0} \right) \cdot \left( \int_{NV_{03},x_{v_r}} e^{-\frac{g_{03}}{\hbar}} \chi_{v_r} *\omega_{03,0} \right) (1 + O(\hbar)),
$$

where the sign depends on whether the orientations of $NV_{13} \oplus NV_{01} \oplus NV_{03}$ and $TM$ at the point $x_{v_r}$ match or not. We will compute the above three integrals one by one. We begin with equalities

$$
\hbar^{-\deg(q_{01})/2} \int_{NV_{01},x_{v_r}} e^{-\frac{g_{01}}{\hbar}} \chi_{v_r} \omega_{01,0} = 1 + O(\hbar)
$$

and

$$
\hbar^{-\deg(q_{03})/2} \int_{NV_{03},x_{v_r}} e^{-\frac{g_{03}}{\hbar}} \chi_{v_r} *\omega_{03,0} = 1 + O(\hbar).
$$

These come from [9] which is reformulated as the following lemma.

Lemma 32. For an eigenform $\phi_{ij}$ with its WKB approximation

$$
\phi_{ij} \sim e^{-\frac{\psi_{ij}}{\hbar}} \hbar^{-\deg(q_{ij})/2} (\omega_{ij,0} + \omega_{ij,1} \hbar^{1/2} + \ldots)
$$

in any small enough open set $W$ containing $V_{q_{ij}}^+ \cup V_{q_{ij}}^-$ as in lemma [25] we have

$$
\hbar^{-\deg(q_{ij})/2} \int_{NV_{q_{ij}},x} e^{-\frac{g_{ij}}{\hbar}} \chi_{\omega_{ij,0}} = 1 + O(\hbar).
$$
for any point \( x \in V_{q_{ij}}^+ \) and \( \chi \equiv 1 \) around \( x \) compactly supported in \( W \). Similarly, we also have
\[
\frac{\hbar^{-\deg(q_{ij})}}{\|\phi_{ij}\|^2} \int_{NV_{q_{ij},x}} e^{-\frac{g_{ij}}{\hbar}} \chi \omega_{ij,0} = 1 + O(\hbar),
\]
for any point \( x \in V_{q_{ij}}^- \) and \( \chi \) compactly supported in \( W \) and \( \chi \equiv 1 \) around \( x \).

Finally, we have an equality
\[
\frac{\hbar^{-\deg(q_{23})+\deg(q_{12})+1}}{2} \int_{N(V_{13}^+\cap V_{12}^+)} e^{-\frac{g_{23}+g_{12}}{\hbar}} \chi \omega_{23,0} \wedge \omega_{12,0} = 1 + O(\hbar^{1/2}).
\]
This depends on the fact that
\[
\frac{\hbar^{-\deg(q_{23})+\deg(q_{12})}}{2} \int_{N(V_{23}^+\cap V_{12}^+)^*} e^{-\frac{g_{23}+g_{12}}{\hbar}} \chi \omega_{23,0} \wedge \omega_{12,0}
\]
\( = \frac{\hbar^{-\deg(q_{23})}}{2} \int_{N(V_{23}^+)} e^{-\frac{g_{23}}{\hbar}} \chi \omega_{23,0} \left( \frac{\hbar^{-\deg(q_{12})}}{2} \int_{N(V_{12}^+)} e^{-\frac{g_{12}}{\hbar}} \chi \omega_{12,0} \right)(1 + O(\hbar))
\]
\( = 1 + O(\hbar),
\]
and the following lemma.

**Lemma 33.** Using same notations in lemma 29 and suppose \( \chi_S \) and \( \chi_E \) are cut off function supported in \( W_S \) and \( W_E \) respectively, then we have
\[
\int_{N(V_S^+)} e^{-\frac{g_S}{\hbar}} \hbar^{-1/2} \chi \omega_{S,0} = 1 + O(\hbar^{1/2}).
\]
Furthermore, suppose \( \omega_{S,0}(x_S) \in \wedge^{\text{top}} N(V_S)_{x_S}^* \), we have \( \omega_{E,0}(x_E) \in \wedge^{\text{top}} N(V_E)_{x_E}^* \).

Putting the above together, we get the following
\[
m_3^T(\hbar, \tilde{\chi}_T) = \pm \hbar^{-1} e^{-\frac{A}{\hbar}} (1 + O(\hbar^{1/2})),
\]
where the sign depends on matching the orientations of \( NV_{13}^+ \oplus NV_0^+ \oplus NV_0^- \) and \( TM \) at the point \( x_{v_r} \). The proof for \( m_3(h) \) is completed and we move on to the \( m_k(h) \) case for any \( k \).

**3.4.2.** \( m_k(h) \) case: The argument of the general case is similar to the case \( k = 3 \), with more combinatorics involved. As in section 3.3.4, we fix a gradient tree \( \Gamma \) of type \( T \). Similar to the previous section, we may drop the dependence of \( \Gamma \) in our notations. We are going to show that
\[
m_k^T(\hbar, \tilde{\chi}_T) = \pm \hbar^{2-k} e^{-\frac{A}{\hbar}} (1 + O(\hbar^{1/2})),
\]
where the sign agrees with that associated to the gradient tree \( \Gamma \) in Morse category. We begin with some notations associated to \( \Gamma \).
Notations 34. Given a gradient tree $\Gamma$, we inductively associate to each flag $(ij, v)$ an oriented closed submanifold $V_{(ij, v)} \subset W_v$ by specifying orientation of its normal bundle. We require:

1. For each semi-infinite incoming edge $i(i+1)$ with ending vertex $v$, we let $V_{i(i+1), v} := V_{q_{i(i+1)}}^+ \cap W_v$, where $V_{q_{i(i+1)}}^+$ is the stable submanifold of $f_{q_{i(i+1)}}$ from the critical point $q_{i(i+1)}$ with the chosen orientation $\nu_{i(i+1), v}$ equals to that in the Morse category;

2. For an internal edge $il$ with its starting vertex $v$ and assume $ij$ and $jl$ are two incoming edges meeting $e_{il}$ at $v$ as in the section 5. We let $V_{il, v} = V_{ij, v} \cap V_{jl, v}$ (the intersections is transversal from the generic assumption) and $\nu_{il, v} = \nu_{ij, v} \wedge \nu_{jl, v}$, if $\nu_{ij, v}$ and $\nu_{jl, v}$ are two corresponding orientation forms;

3. For an internal edge $ij$ with its starting vertex $v_S$ and ending vertex $v_E$, we define $V_{ij, v_E}$ to be $V_{ij, v_E}$ obtained from applying lemma 29 to the homotopy operator $H_{ij}$. The orientation form $\nu_{ij, v_E}$ is chosen such that $[\nu_{ij, v_E}] = [df_{ij} \wedge \nu_{ij, v_S}]$, under the identification by flow of $\nabla f_{ij}$;

4. For the semi-infinite incoming edge $0k$ with root vertex $v_r$, we let $V_{0k, v_r} := V_{q_{0k}}^- \cap W_{v_r}$, where $V_{q_{0k}}^-$ is the unstable submanifold of $f_{q_{0k}}$ from critical point $q_{0k}$ with the chosen orientation $\nu_{0k, v_r}$ equals to that in the Morse category;

We further choose an isomorphism and projection map for every flag $(ij, v)$

$$
W_v \xrightarrow{\cong} NV_{(ij, v)}
$$

(3.28)

by further shrinking $W_v$ suitably.

We can therefore assign a sign to the gradient tree $\Gamma$ in the following way.

Definition 35. For a generic sequence of Morse function $\vec{f}$ with corresponding critical points $q_{01}, \ldots, q_{(k-1)k}, q_{0k}$ satisfying the degree condition (3.1), with a gradient tree $\Gamma$, we define

$$
\text{sign}(\Gamma) = \text{sign}(\frac{\nu_{jk, v_r} \wedge \nu_{0j, v_r} \wedge \nu_{0k, v_r}}{\nu_M}),
$$

(3.29)

where $0j, jk$ and $0k$ are edges joining the root vertex $v_r$ as in section 5, $\nu_{ij, v}$ is the orientation of normal bundle defined in notation 34 and $\nu_M$ is the orientation of $M$.

We are going to argue that

$$
\int_{N(V_{(ij, v)})_{x_v}} (e^{-\frac{\nu_{ij, v}}{\hbar}} h^{r_{ij, v}} x_{v} \omega_{ij, v})_0 = h^{-r_v} (1 + O(h^{1/2})),
$$
for any flag \((ij, v)\) except the outgoing edge \(0k\), where \(r_v\) is the number of internal edge before the vertex \(v\). This can be seen inductively along the tree \(T\). We see that:

1. it is true for the semi-infinite incoming edge \(i(i + 1)\) by lemma \[\text{(32)}\];

2. for an internal edge \(il\) with its starting vertex \(v\) and assume \(ij\) and \(jl\) are two incoming edges meeting \(il\) at \(v\), we have

\[
h^{r(l,v)} \int_{N(V_{il,v})_{xv}} e^{-\frac{g_{il,v}}{\hbar}} \chi_v \omega_{il,v,0} = h^{r(l,v) + \tau(j,v)} \int_{N(V_{il,v} \cap V_{ij,v})_{xv}} e^{-\frac{g_{il,v} + g_{ij,v}}{\hbar}} \chi_v \omega_{il,v,0} \wedge \omega_{ij,v,0} = (h^{r(l,v)} \int_{N(V_{ij,v})_{xv}} e^{-\frac{g_{ij,v}}{\hbar}} \chi_v \omega_{ij,v,0} h^{\tau(j,v)} \int_{N(V_{ij,v})_{xv}} e^{-\frac{g_{ij,v}}{\hbar}} \chi_v \omega_{ij,v,0}) = 1,
\]

modulo an error of order \(O(h^{1/2})\);

3. for an internal edge \(ij\) with its starting vertex \(v_S\) and ending vertex \(v_E\), we make use of the lemma \[\text{(33)}\] to show that an extra \(h^{-1}\) is created.

Together with the fact that

\[
\frac{1}{\|\phi_{0k}\|^2} \int_{N(V_{0k,v_r})_{xv_r}} (e^{-\frac{g_{0k,v_r}}{\hbar}} h^{r(0k,v_r)} \chi_{v_r} \ast \omega_{0k,v_r,0}) = 1 + O(\hbar)
\]

for the outgoing edge \(0k\), we can now calculate the leading contribution from the integral \[\text{(3.23)}\]. Recall that

\[
\psi(0j,v_r) = g_{0j,v_r} + f_{0j} - \sum_{0 < m < j} f_{(m-1)m} (q_{(m-1)m})n,
\psi(jk,v_r) = g_{jk,v_r} + f_{jk} - \sum_{j < m < k} f_{(m-1)m} (q_{(m-1)m})n,
\psi(0k,v_r) = g_{0k,v_r} + f_{0k} (q_{0k}) - f_{0k}.
\]

Therefore we have

\[
(3.30) \quad \psi(0j,v_r) + \psi(jk,v_r) + \psi(0k,v_r) = g(0j,v_r) + g(jk,v_r) + g(0k,v_r) + A.
\]
Finally we have
\[
\int_M \left( \frac{h^r_0(\nu_0, \nu_r) + r(\nu_0, \nu_r) + r_0(\nu_0, \nu_r)}{e^{\nu_0(\nu_0, \nu_r) + \nu(\nu_0, \nu_r) + \nu_0(\nu_0, \nu_r)}} \right) \chi_{\nu}, \left( \omega(\nu_0, \nu_r), 0 \wedge \omega(\nu_0, \nu_r), 0 \wedge \omega_0(\nu_0, \nu_r), 0 \wedge \rho(\nu_0, \nu_r) \right) \right)
\]
\[
eq e^{-\frac{4}{\pi}} \left\{ \int_M \left( \frac{h^r_0(\nu_0, \nu_r) + r(\nu_0, \nu_r) + r_0(\nu_0, \nu_r)}{e^{\nu_0(\nu_0, \nu_r) + \nu(\nu_0, \nu_r) + \nu_0(\nu_0, \nu_r)}} \right) \chi_{\nu}, \left( \omega(\nu_0, \nu_r), 0 \wedge \omega(\nu_0, \nu_r), 0 \wedge \omega_0(\nu_0, \nu_r) \right) \right\}
\]
\[
= \pm e^{-\frac{4}{\pi}} \hbar^2 - k(1 + O(h^{1/2}))
\]
which means
\[
(3.31) \quad m_k^T(h, \chi_1) = \pm \hbar^2 - k e^{-\frac{4}{\pi}} (1 + O(h^{1/2})).
\]
The sign \pm comes from matching the orientation \([\nu(\nu_0, \nu_r) \wedge \nu(\nu_0, \nu_r)]\) against that of \(M\), which agrees with the sign in Morse category. This completes the proof of our Main Theorem.

4. RESOLVENT ESTIMATE

In this section, we obtain a rough estimate for the Green operator \(G_{ij}^0\) associated to Witten Laplacian \(\Delta_{ij}\), which is used in section 3.2. More precisely, we are looking at the twisted Green operator \(G_{ij} = (I - P_{ij})G_{ij}^0\) after projecting to the orthogonal complement of the space of small eigenforms. Roughly speaking, it is an estimate of the form
\[
(4.1) \quad G_{ij}(x, y) \sim O(e^{-\rho_{ij}(x, y)})
\]
for the kernel function on \(M \times M\). We first recall some of our setting from section 2. We fix \(ij\) from now on and consider the case of a single Morse function \(f = f_{ij} : M \rightarrow \mathbb{R}\). We define
\[
d_{f,h} := e^{-f/h}hde^{f/h} = hd + df \wedge, \quad \text{and}
\]
\[
d_{f,h}^* := e^{f/h}hd^*e^{-f/h} = hd^* + i\nabla f,
\]
where \(h\) is a small positive real number. The twisted Laplace operator can be defined as
\[
\Delta_{f,h} := d_{f,h}d_{f,h}^* + d_{f,h}^*d_{f,h}.
\]
A direct computation shows that
\[
\Delta_{f} = h^2\Delta + h(L_{\nabla f} + L_{\nabla f}^*) + \left| \nabla f \right|^2
\]
\[
(4.2) \quad = h^2\Delta + hM_j + \left| \nabla f \right|^2,
\]
where \(L_{\nabla f}\) is the Lie derivative and \(M_j = L_{\nabla f} + L_{\nabla f}^*\) is a tensor on \(M\).

The eigenvalues in the interval \(I(h) = [0, h^{3/2}]\) are called small eigenvalues. The direct sum of corresponding finite dimensional eigenspaces is denoted by \(\Omega(M, h)^{sm}\). We consider the twisted Green operator \(G_{f,h} = (I - P_{f,h})G_{f,h}^0\) which is the ordinary Green operator after projecting to the orthogonal complement of small eigenspace. The main result we have in this section is
Proposition 36. For any $\epsilon > 0$, we have $h_0 = h_0(\epsilon)$ such that
\[ G_{f,h} = \hat{O}_\epsilon(\epsilon^{-\rho_f(x,y)-\epsilon}/h) \]
for all $h < h_0$, where $\rho_f$ is the Agmon metric $f$ defined in [9].

To really explain what the above notation stands for, we recall from [8] the following

Definition 37. Let $A = A_h : L^2(M) \to W^{1,2}(M)$ be a family of bounded operators with $h \in (0, 1]$. We say the $A \in \hat{O}(e^{-f(x,y)/h})$ for a continuous $f \in C^0(\mathbb{M} \times \mathbb{M}; \mathbb{R})$, if for any $x_0, y_0 \in M$, there exist neighborhoods $V$ and $U$ in $M$ of $x_0$ and $y_0$ and a constant $C > 0$ such that
\[ \|Au\|_{W^{1,2}(V)} \leq Ce^{-f(x,y)/h}\|u\|_{L^2(U)} \]
for all $h$ small enough and $u \in L^2(M)$ such that $\text{supp } u \subset U$.

Remark 38. We will use subscript on $\hat{O}_\epsilon$, if we want to emphasize what the constant $C$ depends on.

Remark 39. If $A \in \hat{O}_\epsilon(e^{-f(x,y)/h})$ and $B \in \hat{O}_\epsilon(e^{-g(x,y)/h})$ for all $\epsilon$ small enough, then we have
\[ B \circ A \in \hat{O}_\epsilon(e^{-h(x,y)/h}) \]
for all $\epsilon$ small enough, where
\[ h(x, y) = \min_{z \in M}(g(x, z) + f(z, y)) \]

Remark 40. For convenience, we write $\hat{O}(e^{-f(x,y)/h} + e^{-g(x,y)/h})$ to stand for $\hat{O}(e^{-\min_{x,y}(f(x,y),g(x,y))/h})$.

Remark 41. We will use the same notation for a family of operators from $L^2(M)$ to $L^2(M)$ as well.

We can generalize proposition 36 easily to $L^\infty$ norm of all derivatives.

Proposition 42. For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $h_0 = h_0(\epsilon) > 0$ such that for any two point $x_0, y_0 \in M$, there exist neighborhoods $V$ and $U$ (depending on $\epsilon$) of $x_0$ and $y_0$ respectively, and $C_{j,\epsilon} > 0$ such that
\[ \|\nabla^j(G_{ij}u)\|_{L^\infty(V)} \leq C_{j,\epsilon}e^{-(\rho(x_0,y_0)-\epsilon)/h}\|u\|_{W^{k_j,p}(U)}, \]
for any $h < h_0$ and $u \in C^0_0(U)$. Here $W^{k_j,p}$ refers to the Sobolev norm.

The rest of the section is devoted to the proof of proposition 36 and 42. In this section, we will drop the dependence of $h$ in our notations for simplicity, e.g. we will write $\Delta_f$ to denote $\Delta_{f,h}$. We will use skills from [8] with suitable modifications. We will make use of the following equality as a tool for various kinds of estimates.
Lemma 43. Let \( \Omega \subset M \) be a domain with smooth boundary. If \( v \in C^2(\Omega, \Lambda^*T^*_\Omega) \) with \( v|_{\partial \Omega} = 0 \) and \( \psi \in C^2(\Omega, \mathbb{R}) \), we have

\[
(4.4) \quad \text{Re}(e^{2\psi/h} \Delta_f v, v) = h^2\|d(e^{\psi/h}v)\|^2 + \|d^*(e^{\psi/h}v)\|^2 + \langle |df|^2 - |d\psi|^2 + hM_f e^{\psi/h}v, e^{\psi/h}v \rangle.
\]

Let \( C_f^* \) be the set of critical points of \( f \), we let \( B(p, \eta) \) be the open ball centered at \( p \) with radius \( \eta \) with respect to the Agmon metric. Then we define

\[
\hat{M}_p := M \backslash \bigcup_{q \in C_f^* \setminus \{p\}} B(q, \eta).
\]

Hence \( p \) is the only critical point of \( f \) in \( \hat{M}_p \). We further fix \( \delta > 0 \) and define

\[
\hat{M}_{p,0} := \hat{M}_p \backslash B(p, \delta),
\]

and so there is no critical point of \( f \) in \( \hat{M}_{p,0} \). We use \( G_{f,p} \) and \( G_{f,p,0} \) to stand for the twisted Green operators on \( \hat{M}_p \) and \( \hat{M}_{p,0} \) respectively, after projecting to the orthogonal complement of small eigenspaces. We first have the corresponding estimate for \( G_{f,p,0} \) which is

Lemma 44. For \( \epsilon > 0 \), there exists \( h_0 = h_0(\epsilon, \eta, \delta) > 0 \), such that

\[
G_{f,p,0} \in \mathcal{O}_{\epsilon, \eta, \delta}(e^{-\frac{\rho(x,p) - \epsilon}{h}}),
\]

for those \( h < h_0 \).

Proof. It follows from [8], since there is no small eigenvalue in \( \hat{M}_{p,0} \). \( \square \)

Next, we let \( \theta_p \) and \( \hat{\theta}_p \) be functions on \( M \) such that

\[
\theta_p \equiv 1 \text{ in a neighborhood of } \overline{B(p, \delta)} \text{ and } \text{supp } \theta_p \subset B(p, 2\delta),
\]

\[
\hat{\theta}_p \equiv 1 \text{ in a neighborhood of } \overline{B(p, 2\delta)} \text{ and } \text{supp } \hat{\theta}_p \subset B(p, 4\delta).
\]

We will use these functions to cut off the Green operator and consider its behavior near one critical point. First we have

Lemma 45. For any small \( \epsilon > 0 \), there exists \( h_0 = h_0(\epsilon, \eta, \delta) > 0 \), such that

\[
G_{f,p,0} \hat{\theta}_p \in \mathcal{O}_{\epsilon, \eta, \delta}(e^{-\frac{\rho(x,p) - \epsilon}{h}}),
\]

for any \( h < h_0 \).

Proof. Fixing two points \( x_0, y_0 \in \hat{M}_p \) with \( V = B(x_0, \epsilon) \) and \( U = B(y_0, \epsilon) \). We consider the relation \( v = G_{f,p,0} \hat{\theta}_p u \) with \( u \in C_0^\infty(\Omega^*(U)) \). Putting \( \psi(x) = (1 - \epsilon)\rho(x,p) \) into the equality (4.4), we get

\[
\text{Re}(\langle I - P_{f,p} \rangle \hat{\theta}_p u, e^{2\psi/h}v) = h^2\|d(e^{\psi/h}v)\|^2 + \|d^*(e^{\psi/h}v)\|^2 + h\langle M_fe^{\psi/h}v, e^{\psi/h}v \rangle + \langle |df|^2 e^{\psi/h}v, e^{\psi/h}v \rangle.
\]
There exists \( m = m(\epsilon, \eta, \delta) > 0 \) such that \( |df|^2 \geq m\epsilon \) in \( \tilde{M}_{p,0} \) and hence we have
\[
\langle (2 - \epsilon)\epsilon|df|^2 \rangle e^{\psi/h}v, e^{\psi/h}v \rangle \geq \frac{3}{2} m\epsilon^2 \|e^{\psi/h}v\|^2_{M_{p,0}}.
\]
Therefore, we get
\[
\Re\langle (I - P_{f,p})\hat{\theta}_pu, e^{2\psi/h}v \rangle \\
\geq \hbar^2 \|d(e^{\psi/h}v)\|^2 + \|d^*(e^{\psi/h}v)\|^2) + m\epsilon^2 \|e^{\psi/h}v\|^2_{M_{p,0}},
\]
since the term \( \hbar(M^J e^{\psi/h}v, e^{\psi/h}v) \) can be absorbed by taking \( \hbar \) small enough. Therefore, we have
\[
\frac{1}{2\epsilon} \|e^{\psi/h}(I - P_{f,p})\hat{\theta}_pu\|^2 + m\epsilon^2 \|e^{\psi/h}v\|^2_{B(p,\delta)}
\]
(4.7)
\[
\geq \hbar^2 \|d(e^{\psi/h}v)\|^2 + \|d^*(e^{\psi/h}v)\|^2) + \tilde{c}\|e^{\psi/h}v\|^2,
\]
for some \( \tilde{c} = \tilde{c}(\epsilon, \eta, \delta) > 0 \), if \( \hbar \) and \( c \) are small enough (both depending on \( \epsilon, \eta, \delta \)).

Since \( \|G_{f,p}\|_{L^2(L^2)} \leq C e^{\psi/h} \) for arbitrary \( \epsilon \) due to the fact that small eigenvalue are taken away, we have
\[
\|v\|^2_{B(p,\delta)} \leq C\epsilon e^{2\psi/h}\|\hat{\theta}_pu\|^2 \leq C\epsilon e^{\psi/h}\|u\|^2.
\]
As \( \psi(x) \leq (1 - \epsilon)\delta \) in \( B(p, \delta) \) and supp \( u \subset U \), we get
\[
me^2 \|e^{\psi/h}v\|^2_{B(p,\delta)} \leq C\epsilon e^{(2(1-\epsilon)\delta + 2\epsilon)/h}\|u\|^2_U
\]
(4.8)
by replacing another constant \( C = C(\epsilon, \eta, \delta) \). The next term to be controlled will be
\[
\|e^{\psi/h}(I - P_{f,p})\hat{\theta}_pu\|^2.
\]
As supp \( \hat{\theta} \subset B(p, 4\delta) \) and supp \( u \subset U \), we have
\[
\|e^{\psi/h}(I - P_{f,p})\hat{\theta}_pu\|^2 \leq \|e^{\psi/h}\hat{\theta}_pu\|^2_{B(p,4\delta)\cap U} + \|e^{\psi/h}(\hat{\theta}_pu, \varphi_p)\varphi_p\|^2 \\\n\leq \|e^{\psi/h}\hat{\theta}_pu\|^2_{B(p,4\delta)\cap U} + \|\hat{\theta}_pu\|^2 \|e^{\psi/h}\varphi_p\|^2,
\]
(4.9)
where \( \varphi_p \) is the unique small eigenform (as \( \tilde{M}_p \) has only one critical point \( p \)) corresponding to small eigenvalue of \( \Delta_{f,p} \). We make use of the fact about the eigenform \( \varphi_p \) in lemma 3.2 which says
\[
\varphi_p = O(e^{-\rho(p,x)-\epsilon}/h)
\]
for arbitrary \( \epsilon \) and get
\[
\|e^{\psi/h}(I - P_{f,p})\hat{\theta}_pu\|^2 \leq \tilde{C}e^{8(1-\epsilon)/h}\|u\|^2_U
\]
for some constant \( \tilde{C} \). Combining (4.7), (4.8), (4.9), we have
\[
\|e^{\psi/h}dv\| + \|e^{\psi/h}d^*v\| + \|e^{\psi/h}v\| \leq Ce^{(4\delta(1-\epsilon)+\epsilon)/h}\|u\|_U.
\]
Finally, we have an estimate for $\psi$ in $V$ and get
\[ e^{(1-\epsilon)(\rho(x_0,p)-\epsilon)/\hbar}\|v\|_{W^{1,2}(V)} \leq C e^{(\delta\delta(1-\epsilon)+\epsilon)/\hbar}\|u\|_{L^2(U)}, \]
which is the desired result by choosing a suitable $\epsilon$ to start with. \hfill \square

We have the decomposition from [8]
\[(4.10)\]
\[ G_{f,p} = (I - P_{f,p})(1 - \theta_p)G_{f,p,0}(1 - \hat{\theta}_p) + G_{f,p}\hat{\theta}_p + G_{f,p}\hat{\theta}_p[\Delta_f, \theta_p]G_{f,p,0}(1 - \hat{\theta}_p), \]
which can be verified by taking $\Delta_{f,p}$ to both sides of the equation. Combining lemmas [44, 45] and (4.10), we have

**Lemma 46.** For any $\epsilon > 0$, there exists $\hbar_0 = \hbar_0(\epsilon, \eta) > 0$, such that
\[ \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}) \]
for $\hbar < \hbar_0$.

**Proof.** We will estimate the right hand side of (4.10) term by term. From lemma [45] we have
\[ (1 - \theta_p)G_{f,p,0}(1 - \hat{\theta}_p) \in \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}). \]
Making use of lemma [3,2] we see that
\[ P_{f,p} \in \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}) \]
for small $\epsilon$. Using remark [39] and triangle inequality, we get the desired estimate for the first term. For the second term, recall from lemma [45] that
\[ G_{f,p}\hat{\theta}_p \in \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}). \]
For the operator to be non-trivial, we need the support of input to intersect supp($\hat{\theta}_p$). Therefore we can have
\[ G_{f,p}\hat{\theta}_p \in \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}). \]
Finally, we have
\[ \Delta_f, \theta]G_{f,p,0}(1 - \hat{\theta}_p) \in \hat{\Theta}_\epsilon,\eta,\delta(e^{-\rho(x,y)/\hbar}) \]
as an operator from $L^2(\hat{M}_p)$ to itself. We use remark [39] to obtain an estimate for the last term. To finish the proof, we may choose suitable $\delta$ and $\epsilon$ to obtain the desired statement. \hfill \square

Now, we move to the estimate of $G_f$ on the whole manifold $M$. We are going to use various cut off functions to relate it to $G_{f,p}$ on $\hat{M}_p$. We let $\vartheta_p$ be a function on $M$ such that
\[ \vartheta_p \equiv 1 \text{ in a neighborhood of } \overline{B(p,\eta)} \text{ and supp } \vartheta_p \subset B(p,2\eta). \]
We define two sets of cut off functions, one of them is given by
\[ \chi_p = 1 - \sum_{q \in C_f \setminus \{p\}} \vartheta_q. \]
Another one is \( \tilde{\chi}_p \in C_c^\infty(\text{int}(\hat{M}_p)) \), which is independent of how large is the ball \( B(p, \eta) \), and satisfies
\[
\sum_p \tilde{\chi}_p = 1.
\]

If \( \eta \) is small enough, we can assume
\[
\text{supp} \tilde{\chi}_p \cap \text{supp}(1 - \chi_p) = \emptyset.
\]

We first take an approximation \( G_{\text{app}} : L^2(\wedge^* T_M^*) \to W^{1,2}(\wedge^* T_M^*) \) defined by
\[
G_{\text{app}} = (I - P_f) \sum_p \chi_p G_{f,p} \tilde{\chi}_p.
\]

Then,
\[
\Delta_f G_{\text{app}} = (I - P_f) \left( \sum_p \Delta_f \chi_p G_{f,p} \tilde{\chi}_p \right)
\]
\[
= (I - P_f) \left( I - \sum_p \chi_p P_{f,p} \tilde{\chi}_p - \sum_{p \neq q} [\Delta_f, \vartheta_q] G_{f,p} \tilde{\chi}_p \right)
\]
\[
= (I - P_f) \left( I - (I - P_f) \sum_j \chi_p P_{f,p} \tilde{\chi}_p - \sum_{p \neq q} [\Delta_f, \vartheta_q] G_{f,p} \tilde{\chi}_p \right)
\]
\[
= \Delta_f G_f (I - L - K),
\]
where
\[
L = (I - P_f) \sum_p \chi_p P_{f,p} \tilde{\chi}_p \quad \text{and} \quad K = \sum_{p \neq q} [\Delta_f, \vartheta_q] G_{f,p} \tilde{\chi}_p.
\]

Notice that images of operators \( G_{\text{app}} \) and \( G_f(I - L - K) \) are orthogonal to the kernel of \( \Delta_f \), therefore we have
\[
G_{\text{app}} = G_f(I - L - K).
\]

We can express \( G_f \) in term of \( G_{\text{app}} \) if we are able to invert \( (I - L - K) \).

By checking the convergence of the series, we define
\[
(4.12) \quad \hat{K} = \sum_{j=1}^\infty K^j,
\]
\[
(4.13) \quad \hat{L} = \sum_{j=1}^\infty (L(I + \hat{K}))^j,
\]
and we can write
\[
(4.14) \quad G_f = G_{\text{app}}(I + \hat{K})(I + \hat{L}).
\]

We are going to estimate each term on the right hand side and obtain an estimate of \( G_f \). First, we have
\[
(4.15) \quad G_{\text{app}} \in \mathcal{O}_{\epsilon, \eta}(e^{-\frac{\rho(x,y)-\epsilon}{\kappa}}).
\]
Next, we look at the operator

\[ K = \sum_{p \neq q} [\Delta_f, \vartheta_q] G_{f,p} \tilde{x}_p : L^2(\wedge^* T^*_M) \to L^2(\wedge^* T^*_M). \]

Applying lemma 46 to \( G_{f,p} \), we obtain

\[ K \in \tilde{O}_{\epsilon, \eta}(e^{-\rho(x,y)/\hbar}). \]

If we look at the second term in \( \hat{K} \),

\[ K^2 = \sum_{p_0 \neq p_1, p_1 \neq p_2} [\Delta_f, \vartheta_{p_0}] \cdot (G_{f,p_1}[\Delta_f, \vartheta_{p_1}] \cdot G_{f,p_2} \tilde{x}_{p_2}, \]

by applying lemma 46 to \( G_{f,p_1} \) and \( G_{f,p_2} \) with \( U \) and \( V \) having diameters less than \( \epsilon \), we obtain

\[ K^2 \in \tilde{O}_{\epsilon, \eta}(e^{-\min\rho(x,z) + \rho(z,y)}/\hbar - 2\epsilon). \]

If we further take \( \text{supp}(\vartheta_{p_1}) \) into account, we have

\[ K^2 \in \tilde{O}_{\epsilon, \eta}(e^{-\min\rho(x,p_1) + \rho(p_1,y) - 2\epsilon}). \]

In general, for \( l \geq 2 \), we have

\[ K^l = \sum_{p_0 \neq p_1, \ldots, p_l-1 \neq p_l} [\Delta_f, \vartheta_{p_0}] \cdot (G_{f,p_1}[\Delta_f, \vartheta_{p_1}] \cdots (G_{f,p_{l-1}}[\Delta_f, \vartheta_{p_{l-1}}] \cdot G_{f,p_l} \tilde{x}_{p_l}, \]

and similarly we get

\[ K^l \in \tilde{O}_{\epsilon, \eta}
\left( \sum_{p_1 \neq p_2, \ldots, p_l-1 \neq p_{l-1}} \exp\left(-\frac{1}{\hbar} \left( \rho(x,p_1) + \rho(p_1,p_2) + \cdots + \rho(p_{l-1},y) - 4(l-1)\eta - (3l-2)\epsilon \right) \right) \right). \]

To summarize, we let \( D \) be the matrix defined by

\[ D_{p,q} = \begin{cases} 0 & \text{if } p \neq q, \\ e^{-\rho(p,q)/\hbar} & \text{if } p = q. \end{cases} \]

and the column vector \( T(x) \) with components given by

\[ T_p(x) = (e^{-\rho(p,x)/\hbar}). \]

Then we can rewrite (4.16) as

\[ K^l \in \tilde{O}_{\epsilon, \eta}(e^{-\rho(p,q)/\hbar}) T(x)^t \cdot D^{l-2} \cdot T(y). \]

Since there are only \( N \) critical points, for \( M \geq N \), we have an entry-wise inequality

\[ D^M \leq e^{-2T_0/\hbar} C_M (I + D + \cdots + D^{N-1}) \]
for some $C_M$, where $T_0 = \min_{p \neq q} \rho(p, q)$. We can absorb the error terms involving $e^{-4(l-1)/\eta}$ and $e^{-(3l-2)/\eta}$ into the decay term $e^{-2T_0/\hbar}$ to show convergence for $\tilde{K}$. Combining (4.12), (4.17) and (4.18), we obtain

\begin{equation}
\tilde{K} \in \mathcal{O}_\epsilon(e^{-\rho(x,y) - \epsilon \hbar})
\end{equation}

for any $\epsilon$ small enough.

Now, it comes to the estimate of the term $L$. We learn from [8] that

\begin{equation}
(I - P_f) \chi_p \varphi_p = \mathcal{O}_{\epsilon, \eta}(\sum_{q \neq p} e^{-\rho(x,q) + \rho(q,p) + \rho(p,y) - 4\eta - 3\epsilon}),
\end{equation}

where $\varphi_p$ is the unique eigenform of $\Delta_p$ on $\hat{M}_p$. Therefore, we get

\begin{equation}
L \in \mathcal{O}_{\epsilon, \eta}(\sum_{q \neq p} \exp(-1/\hbar (\rho(x,q) + \rho(q,p) + \rho(p,y) - 4\eta - 3\epsilon))).
\end{equation}

Similar to the $\tilde{K}$ case, we have the series in (4.13) converges and has

\begin{equation}
\hat{L} \in \mathcal{O}_\epsilon(e^{-\rho(x,y) - \epsilon \hbar}),
\end{equation}

by choosing suitable $\eta$ and $\epsilon$ to start with.

Finally, combining the estimates in (4.15), (4.19), (4.22) and applying formula (4.14), we conclude that

\begin{equation}
G_f \in \mathcal{O}_\epsilon(e^{-\rho(x,y) - \epsilon \hbar}).
\end{equation}

This completes the proof of proposition 36.

Proposition 42 can be proved by having higher derivatives estimate using the same argument as above, and applying Sobolev embedding.

5. WKB for Green operator

In the previous section, we have a rough estimate for the twisted Green operator, or the homotopy operator $H_f = d_f^*G_f(I - P_f)$, with an error of order $\mathcal{O}(\epsilon \hat{\chi})$. In a neighborhood of gradient flow line segment of $f$, we are going to improve the results in section 4 to estimate with error $\mathcal{O}(\hbar \infty)$. This is done by the WKB method for inhomogeneous Laplace equation (3.20).

We study the local behavior of the homotopy operator $H_f$ along a normalized gradient flow line segment

\begin{align*}
\gamma : [0, T] &\rightarrow M, \\
\frac{d\gamma}{dt} &= \frac{\nabla f}{|\nabla f|}, \\
\gamma(0) &= x_S, \quad \gamma(T) = x_E,
\end{align*}
as shown in the following figure. We consider the relation

\[ \zeta_E = H_f(\chi_S \zeta_S). \]

Suppose we have a WKB approximation of \( \zeta_S \) in \( W_S \) of the form

\[ (5.1) \quad \zeta_S \sim e^{-\psi_S/\hbar}(\omega_{S,0} + \omega_{S,1}\hbar^{1/2} + \omega_{S,2}\hbar^1 + \ldots), \]

we need to establish a similar expression

\[ (5.2) \quad \zeta_E \sim e^{-\psi_E/\hbar\hbar^{-1/2}}(\omega_{E,0} + \omega_{E,1}\hbar^{1/2} + \ldots), \]

of \( \zeta_E \) in a some open neighborhood \( W_E \) of \( x_E \).
The key step is to determine $\psi_E$, which is given in the following subsection. As a first trial, we consider the function
\[ \tilde{\psi}_E(x) := \inf_{y \in W_S} \{ \psi_S(y) + \rho_f(y, x) \}, \]
since $e^{-\tilde{\psi}_E}$ is the expected exponential decay suggested by the resolvent estimate in section 4.

$\tilde{\psi}_E$ is not the correct function since it is singular along a hypersurface $U_S$ through $x_S$, and cannot be used for the iteration process as we keep on differentiating it.

In the coming section 5.1, we will solve the minimal configuration in variational problem associated to $\inf_{y \in W_S} (\psi_S(y) + \rho_f(y, x))$ and find that the point $y$ is forced to lie on $U_S$, with a unique geodesic joining to $x$ which realizes $\rho(y, x)$, for those $x$ close enough to $x_E$. These family of geodesics $\{ \gamma_\epsilon \}_{\epsilon \in U_S}$ will give a foliation of a neighborhood of $\gamma$. Therefore we can use $\psi_E(\gamma_\epsilon(t)) = \psi_S(y) + t$ as an extension of $\tilde{\psi}_E$ across $U_S$. We then use $\psi_E$ in the iteration similar to classical WKB approximation to obtain the above expansion 5.2.

5.1. The phase function $\psi_E$. We apply variational method to study the function $\tilde{\psi}(x)$. Fixing $x \in M$, we take $\alpha(\epsilon, t) := \alpha_\epsilon(t) : (-\epsilon_0, \epsilon_0) \times [0, 1] \to M$ such that $\alpha_\epsilon(1) \equiv x$ for all $\epsilon$. To minimize the functional
\[ L(\epsilon) = \psi_S(\alpha_\epsilon(0)) + \int_0^1 |\partial_\epsilon \alpha_\epsilon|_f dt, \]
we take derivatives and get

**Lemma 47. (First variation formula)**

\[ \frac{dL}{d\epsilon} |_{\epsilon=0} = \langle \nabla \psi_S(\alpha_0), \partial_\epsilon \alpha_\epsilon \rangle |_{t=0} + \int_0^1 \frac{1}{|\partial_\epsilon \alpha_\epsilon|_f} \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_\epsilon \alpha \rangle_f dt. \]

Here $\tilde{\nabla}_t$ is the Levi-Civita connection corresponding to the Agmon metric $\langle \cdot, \cdot \rangle_f$ in definition 3.

If we assume $\alpha_0$ is a geodesic with $|\alpha_\epsilon'(t)| \equiv \text{const.}$, the Euler-Lagrange equation for $L(\epsilon)$ is
\[ \frac{dL}{d\epsilon} |_{\epsilon=0} = \langle \nabla \psi_S(\alpha_0) - \frac{\alpha_0'}{|\alpha_0'|} |\nabla f(\alpha_0)|, \partial_\epsilon \alpha \rangle |_{t=0} = 0. \]

Since $\partial_\epsilon \alpha(0, 0)$ can be chosen arbitrarily, we have
\[ \left( \nabla \psi_S(\alpha_0) - \frac{\alpha_0'}{|\alpha_0'|} |\nabla f(\alpha_0)| \right) |_{t=0} = 0. \]

Taking norm we obtain the equation
\[ |\nabla \psi_S| = |\nabla f|, \]
or equivalently, $|\tilde{\nabla}_t \psi_S|_f = 1$. 

Definition 48. \( U_S := \{ |\tilde{\nabla}_S \psi_f| = 1 \} \cap W_S \).

If \( \alpha_0 \) is a local extrema of \( L \), it forces \( \alpha_0(0) \in U_S \). To obtain nice properties of \( U_S \), we are going to assume the following throughout the whole section.

Assumption 49. We assume \( g_S : W_S \to \mathbb{R}_{\geq 0} \), defined by \( g_S = \psi_S - f \), be a Bott-Morse function with zero set \( V_S \) such that \( v_S \in V_S \).

Lemma 50. \( U_S \) is a hypersurface containing \( V_S \), if we shrink \( W_S \) suitably.

Proof. Since we have \( \nabla g_S \equiv 0 \) on \( V_S \) and hence \( |\nabla_\psi S| = |\nabla f| \) on \( V_S \). This gives \( V_S \subset U_S \). Moreover, \( U_S \) can be defined by the equation \( \Phi(x) = 2(\nabla f(x), \nabla g_S(x)) + |\nabla g_S(x)|^2 = 0 \).

If \( v \in T_p M \) where \( p \in V_S \), then we have
\[
\nabla_v \Phi(p) = 2\nabla^2 f(p)(v, \nabla g_S(p)) + 2\nabla^2 g_S(p)(v, \nabla f(p)) + 2\nabla^2 g_S(p)(v, \nabla g_S(p))
\]
\[
= 2\nabla^2 g_S(p)(v, \nabla f(p)),
\]

since \( \nabla g_S(p) = 0 \) on \( V_S \). As \( g_S \) is a Bott-Morse function with critical set \( V_S \), \( \nabla^2 g_S(p) \) is nondegenerate when restricted to the orthogonal complement of \( T_p V_S \) in \( T_p M \). Therefore, there exists \( v \) such that \( \nabla_v \Phi(p) \neq 0 \). \( \square \)

We are going to parametrize a neighborhood of \( \gamma \) by \( U_S \times (-\delta, T+\delta) \) such that \( U_S \times \{0\} \to M \) is the embedding and \( \nu_S \times [0, T] \) is \( \gamma \). \( \psi_E \) is defined to be the coordinate function corresponding to the last variable.

Motivated from equation (5.4), we define a transversal vector field on \( U_S \) which is the initial tangent vector for minimizer of \( L \).

Definition 51. We define \( \nu \in \Gamma(U_S, T_M) \)
\[
(5.5) \quad \nu := \frac{\nabla \psi_S}{|\nabla \psi_S|} = \tilde{\nabla} \psi_S.
\]

Notice that \( \nu = \frac{\nabla f}{|\nabla f|} = \tilde{\nabla} f \) on \( V_S \).

It follows from the Euler-Lagrange equation (5.4) that any local extrema \( \alpha \) of \( L \) will have \( \alpha(0) \in U_S \) and \( \alpha'(0) = \nu(\alpha(0)) \). For convenience, we assume that \( \gamma \) is extended to gradient flow line defined on \( (a, b) \) containing \( [0, T] \).

Definition 52. We define a map
\[
(5.6) \quad \sigma : W_0 \subset U_S \times (a, b) \to M,
\]
given by
\[
\sigma(u, t) = \exp_u(tv),
\]
where \( W_0 \) is a suitable neighborhood of \( \gamma \) where the exponential map is well defined.
Lemma 53. Restricting to a small open neighborhood of \( \{x_S\} \times (a,b) \), \( \sigma \) is a diffeomorphism onto its image containing \( \gamma \).

This is achieved by showing there is no "conjugate point" along \( \gamma(t) \) for certain type of geodesic family, and using the fact that \( \gamma \) being a global minimizer of functional \( L \). Lemma 53 enable us to construct \( \psi_E \) needed for WKB approximation.

Definition 54. We define \( \psi_E \) on \( \sigma(U_S \times (a,b)) \) by
\[
(5.7) \quad \psi_E(\sigma(u,t)) = \psi_S(u) + t, 
\]
for \( (u,t) \in U_S \times (a,b) \).

5.2. Proof of lemma 53. We begin with the second variation formula of \( L \).
We assume \( \alpha : (-\epsilon_0, \epsilon_0) \times [0,l] \to M \) is a family such that \( \alpha_0(t) \) is arc-length parametrized geodesic satisfying the condition
\[
\left( \tilde{\nabla} \psi_S(\alpha) - \frac{\partial \alpha}{\partial t}\right)_{\epsilon=0} = 0. 
\]
From the first variation formula
\[
\frac{dL}{d\epsilon} = \langle \tilde{\nabla}_t \psi_S(\alpha_\epsilon(0)), \partial_t \alpha_\epsilon(0) \rangle_f + \int_0^l \langle \tilde{\nabla}_t \partial_t \alpha, \frac{\partial \alpha}{\partial t}\rangle_f \, dt, 
\]
we obtain

Lemma 55. (Second variation formula)
\[
\frac{d^2L}{d\epsilon^2}_{\epsilon=0} = \langle \tilde{\nabla}_t \tilde{\nabla}_t \psi_S, \partial_t \alpha \rangle_f \big|_{t=0} + \langle \tilde{\nabla}_t \psi_S, \tilde{\nabla}_t \partial_t \alpha \rangle_f \big|_{t=0} + \langle \tilde{\nabla}_t \partial_t \alpha, \partial_t \alpha \rangle_f \big|_{t=0} + \int_0^l \langle \tilde{\nabla}_t \partial_t \alpha, \tilde{\nabla}_t \partial_t \alpha \rangle_f + \langle \tilde{R}(\partial_t \alpha, \partial_t \alpha)\partial_t \alpha, \partial_t \alpha \rangle_f - \langle \tilde{\nabla}_t \partial_t \alpha, \partial_t \alpha \rangle_f^2 \, dt, 
\]
where the right hand side is evaluated at \( \epsilon = 0 \). Here \( \tilde{R} \) is the curvature tensor with respect to \( \langle \cdot, \cdot \rangle_f \).

If we further impose the condition that \( \alpha(\epsilon,0) \in U_S \) and \( \partial_\epsilon \alpha(\epsilon,l) \equiv 0 \) for all \( \epsilon \), we have
\[
\frac{d^2L}{d\epsilon^2}_{\epsilon=0} = \langle \tilde{\nabla}_t \tilde{\nabla}_t \psi_S, \partial_t \alpha \rangle_f \big|_{t=0} + \int_0^l \langle \tilde{\nabla}_t \partial_t \alpha, \tilde{\nabla}_t \partial_t \alpha \rangle_f + \langle \tilde{R}(\partial_t \alpha, \partial_t \alpha)\partial_t \alpha, \partial_t \alpha \rangle_f - \langle \tilde{\nabla}_t \partial_t \alpha, \partial_t \alpha \rangle_f^2 \, ds. 
\]
Therefore we consider the bilinear form \( I \) associated to the above quadratic form.
that depending on a small \( \eta > 0 \), the minimum of \( \sigma \) is a cut off function in \( [0) = 0, [l) \) and \( Z(l) = \bar{\nabla}_t \partial_t \alpha(0, t_0) \). We define a piecewise smooth vector field

\[
Y_\eta(t) := \begin{cases} 
\partial_t \alpha + \eta Z & \text{if } t \in [0, t_0], \\
\chi(\partial_t \alpha, \partial_t \alpha) \partial_t \alpha + \eta Z & \text{if } t \in [t_0, l],
\end{cases}
\]

where \( \chi \) is a cut off function in \( [t_0, l] \) with \( \chi(t_0) = 1 \) and \( \chi = 0 \) in a neighborhood of \( l \). Notice that \( \bar{\nabla}_t (\partial_t \alpha, \partial_t \alpha)_f = 0 \) from the fact that \( |\partial_t \alpha|_f \equiv 1 \). A direct computation shows

\[
I(Y_\eta, Y_\eta) = -2\eta |\bar{\nabla}_t \partial_t \alpha(0, t_0)|_f^2 + 2\eta^2 I(Z, Z).
\]

We have \( I(Y_\eta, Y_\eta) < 0 \) for \( \eta \) small enough.
By taking the family of curves $\beta_\epsilon$ corresponding to $Y_\eta$, we obtain
\[
\left. \frac{d^2 L_\beta}{d\epsilon^2} \right|_{\epsilon=0} < 0,
\]
where $L_\beta(\epsilon) = L(\beta(\epsilon))$. For small enough $\epsilon$, $\beta_\epsilon(t)$ will be a curve from $U_S$ to $\sigma(0,l)$ which gives a smaller value of $L$ comparing to $\beta_0 = \gamma$. This is impossible because we have
\[
L_\beta(\epsilon) \geq f(\sigma(0,l))
\]
and the lower bound is attained at $\gamma$.

For $t_0 \in (a,0]$, the argument is similar by considering the variational problem associated to the functional
\[
\tilde{L}(\epsilon) = \psi_S(\alpha_\epsilon(0)) - \int_0^1 |\partial_t \alpha_\epsilon| f dt.
\]
As a conclusion, we can show that $\sigma$ gives a local diffeomorphism onto its image by shrinking $W_0$ if necessary. Therefore it is injective in a contractible neighborhood of the gradient flow line $\gamma$. □

Under the identification $\sigma$, we use the coordinate $u_1, \ldots, u_{n-1}$ for $U_S$ and use $(u_1, \ldots, u_{n-1}, t)$, or simply $(u, t)$, as coordinate for image of $W_0$ under $\sigma$. By shrinking $W_0$ if necessary, we assume that $W_0$ is a coordinate chart through the map $\sigma$. This justifies the definition of $\psi_E$ as a smooth function on $\sigma(W_0) \subset M$.

5.3. Properties of $\psi_E$. We are going to study the first and second derivatives of $\psi_E$ which is necessary for having a WKB approximation for the equation (3.20). We define
\[
V_E := \sigma((V_S \times (a,b)) \cap W_0) \subset \sigma(W_0)
\]
as shown in the following picture.
Lemma 57. In $W_0$, we have
\[ \tilde{\nabla}_E \psi = d\sigma \frac{\partial}{\partial t}. \]
In particular, we have $\nabla_E \psi = \nabla f$ on $V_E$ and $|\nabla_E \psi| = |\nabla f|$. 

Proof. We construct a family of geodesics $\alpha_e$ using exponential map as before. Let $\beta(\epsilon)$ be a curve in $U_S$ such that $\beta(0) = u$. Also, we let $\alpha(\epsilon, t) = \exp_{\beta(\epsilon)}(t\nu) = \sigma(\beta(\epsilon), t)$. 

Applying the first variation together with the fact that $\alpha_{\epsilon}$ satisfies Euler-Lagrange equation for $L$, we have
\[ \langle \tilde{\nabla}_E \psi(\alpha(0, t)), \partial_{\epsilon} \alpha(0, t) \rangle = \frac{dL}{d\epsilon} \bigg|_{\epsilon=0} = \langle \partial_t \alpha(0, t), \partial_{\epsilon} \alpha(0, t) \rangle. \]

As $\partial_{\epsilon} \alpha(0, t)$ can be chosen arbitrarily, we get
\[ \tilde{\nabla}_E \psi(u, t) = \partial_t \alpha(0, t) = d\sigma(u, t) \frac{\partial}{\partial t}. \]
Furthermore, we have $|\tilde{\nabla}_E \psi(u, t)|^2 = |d\sigma(u, t) \frac{\partial}{\partial t}|^2 = 1$ which gives $|\nabla_E \psi(u, t)| = |\nabla f|$. As we know $\nabla_S \psi = \nabla f$ on $V_S$ and flow lines of $\nabla f$ are geodesic after reparametrizations, we get $\nabla_E \psi = \nabla f$ on $V_E$. □

We now consider second derivatives of $g_E = \psi_E - f$.

Lemma 58. By choosing a small enough $W_0$, we have
(1) $g_E \geq 0$ and 
(2) $g_E$ is a Bott-Morse function with critical set $V_E = \{g_E = 0\}$. 

Proof. The previous lemma implies that $\nabla g_E = 0$ on $V_E$. We are going to show $\nabla^2 g_E$ is positive definite in the normal bundle of $V_E$. Fixing any $t \in (a, b)$, we consider the submanifold $U_t = \sigma(U_S \times \{t\} \cap W_0)$. There is an isomorphism between the normal bundle of $V_t = \sigma(V_S \times \{t\} \cap W_0)$ in $U_t$ and normal bundle of $V_E$ in $W_0$. Therefore we restrict $g_E$ to $U_t$ and consider its Hessian.

We abuse the notations and write $u : W_0 \to U_S$ as the projection map. We take $h = g_E - g_S \circ u$. We have $h \geq 0$ on $U_t$ and $\nabla h = 0$ on $V_t$. Therefore we have $h$ is positive semi-definite on the normal bundle of $V_t$ in $U_t$. Moreover, we have $\nabla^2 (g_S \circ u) = (\nabla^2 g_S) \circ u$ on $V_S$ being positive definite in the normal bundle. This proves the lemma. □

Next, we consider the second order derivatives for $\Psi = \psi_E - \psi_S = g_E - g_S$ defined on $W_S$.

Lemma 59. By choosing small enough neighborhood $W_S$ of $v_S$ if necessary, we have
(1) $\Psi \leq 0$ on $W_S$ and 
(2) $\Psi$ is a Bott-Morse function with critical set $U_S = \{\Psi = 0\} \subset W_S$. 

Proof. We first have \( \nabla \Psi = 0 \) on \( U_S \) because \( \nabla \psi_E = \nabla \psi_S \) on \( U_S \). If we consider \( \nabla^2 \Psi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \) on \( V_S \), then we have \( \nabla^2 g_E(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0 \) and \( \nabla^2 g_S(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) > 0 \). Therefore, there exists an neighborhood \( U \) of \( V_S \) in \( H \) so that
\[
\nabla^2 \Psi(x)(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) < 0
\]
for all \( x \in U \). \( \square \)

**Remark 60.** We can extend the function \( \Psi \) from \( W_S \) to \( W_0 \) to be a non-negative function with critical set \( U_S \) which is also an absolute maximum. This is for our convenience in later arguments.

5.4. **The WKB iteration.** After knowing these properties of \( \psi_E \), we will describe the iteration procedure to define \( \omega_{E,i} \) inductively.

First, by lemma 57, we have \( |df|^2 = |d\psi_E|^2 \) and hence the expansion
\[
e^{\psi_E/h} \Delta_f e^{-\psi_E/h} = h^2 \Delta + hM_f + h(\mathcal{L}_\nabla \psi_E - \mathcal{L}_\nabla^* \psi_E)
\]
\[
= h^2 \Delta + h(2\mathcal{L}_\nabla \psi_E - M_{gE}),
\]
where \( M_{gE} = \mathcal{L}_\nabla g_E + \mathcal{L}_\nabla^* g_E \). Following [9], we let
\[
\mathcal{T} = 2\mathcal{L}_\nabla \psi_E - M_{gE},
\]
and consider the following equation
\[
(h^2 \Delta + \mathcal{T} h)(\mu_0(h) + \mu_1(h) + \cdots) = e^{\psi/h} \nu,
\]
order by order in \( h \) where \( \mu_i(h) \) is a function (depending on \( h \)). We often write \( \mu_i \) to simplify our notations. The first equation to be solved is
\[
(5.11) \quad hT \mu_0(h) = e^{\psi/h} \nu.
\]

In order to solve the above equation involving \( \mathcal{L}_\nabla \psi_E \), we need a map \( \tau \) describing the flow of \( \nabla \psi_E \). It is given by renormalising \( \sigma \) such that \( d\tau_s(\frac{\partial}{\partial t}) = \nabla \psi_E \) and is of the form
\[
(5.12) \quad \tau : W \subset U_S \times (-\infty, +\infty) \to M,
\]
with the same image as \( \sigma \). We use \( (u_1, \ldots, u_{n-1}, t) \) as coordinate of \( U_S \times \mathbb{R} \) as before and use it for coordinates of image as well from now on. We can assume that \( W \) also satisfies the property \( \{u\} \times [t_0, t_1] \subset W \) for any \( (u, t_0), (u, t_1) \in W \) under the identification by \( \tau \).

For the iteration process, we restrict our attention to
\[
\Omega^*_0(W) = \{ \beta \in \Omega^*(W) | \text{supp}(\beta) \cap (U_S \times (-\infty, t_0]) \text{ compact for all } t_0 \}.
\]
This allows us to define the iteration operator.

**Definition 61.** We let \( I : \Omega^*_0(W) \to \Omega^*_0(W) \) given by
\[
(5.13) \quad I(\phi) := \int_{-\infty}^{\tau} e^{\int_{s}^{\tau} \frac{1}{2} \tau_s^* (M_E) \text{ d}t} \tau_s^* (\phi) \text{ d}s,
\]
where \( \tau_s(u, t) = \tau(u, t + s) \) is the flow of \( \nabla \psi_E \) for time \( s \).
To solve \((5.11)\), we put
\[
\mu_0 = \frac{1}{2\hbar} I(e^{\Psi/\hbar}).
\]
Then it can be checked that \(\mu_0\) is the solution to \((5.11)\). The second equation to be solved is
\[
\hbar T \mu_1 = -\hbar^2 \Delta \mu_0.
\]
Again, we put
\[
\mu_1 = \frac{1}{2\hbar} I(-\hbar^2 \Delta \mu_0) = -\frac{\hbar}{2} I(\Delta \mu_0).
\]
In general, we have the transport equation for \(l \geq 0\)
\[
T \mu_{l+1} = \hbar \Delta \mu_l.
\]
This gives
\[
\mu_{l+1} = -\frac{\hbar}{2} I(\Delta \mu_l).
\]
as solutions in \(W\).

5.5. **Estimate of the WKB iteration.** In this section, we are going to obtain norm estimates for \(\mu_l\)’s. We consider terms appearing in the iteration which are essentially of the form

\[
I^j \left( e^{\Psi/\hbar} \prod_{\alpha} \partial_\alpha \Psi \right)^{\beta}
\]
with \(j \geq 0\) and \(\beta \in \Omega_0^j(W)\), where \(I^j\) is the composition of \(I\) for \(j\) times. Here each \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is a multi-index such that
\[
\partial_\alpha \Psi = \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-1}}}{\partial u_{n-1}^{\alpha_{n-1}}} \frac{\partial^{\alpha_n}}{\partial t^{\alpha_n}} \Psi.
\]
With
\[
m(\alpha) := \max\{0, 2 - \alpha_n\},
\]
we have
\[
\nabla^j \left( \prod_{\alpha} \partial_\alpha \Psi \right) |_{U_\alpha} = 0,
\]
for \(j \leq \sum \alpha m(\alpha)\) from lemma \([59]\).

Given any compact subset \(K\) of \(W\), we let \(\tilde{K} = (\bigcup_{(u,t) \in K} \{u\} \times (-\infty, t]) \cap W\) to be the union of backward flow line from \(K\) as shown in the following figure. We use the notations \((\{u\} \times \mathbb{R}) \cap K = K_u\) and \((\{u\} \times \mathbb{R}) \cap \tilde{K} = \tilde{K}_u\) to denote the intersection of \(K\) and \(\tilde{K}\) with the flow lines.
Lemma 62. For any $\beta \in \Omega^j_0(W)$ and $j \geq 0$,
\[
\int_{K_u} |I^j \left( e^{\Psi/h} \left( \prod_{\alpha} \partial_\alpha \Psi \right) \beta \right) |^2_{(u,t)} dt \leq C h^{1/2 + \sum_{\alpha} m(\alpha)} \|\beta\|_{L^\infty(\hat{K}_u)}^2,
\]
where $C$ depends on $j$, $\text{supp}(\beta) \cap \hat{K}$ and derivatives of $\Psi$ up to order $2 \sum_{\alpha} m(\alpha)$.

Proof. First of all, from the fact that $\beta \in \Omega^j_0(W)$ and $K$ being compact, we can reduce the lower limit in the integral from $-\infty$ to $-c$. If we denote the norm of $\wedge^* T^*_M(u,t)$ by $|\cdot|_{(u,t)}$, we have for any $\phi \in \Omega^j_0(W)$
\[
|I(\phi)|^2_{(u,t)} \leq \left( \int_{-c}^0 \left| e^{\int_0^s \frac{1}{2} \tau^*_s (M_{ge}) \, dt} \, \tau^*_s (\phi) \right|^2_{(u,t)} ds \right)^2 \leq c \int_{-c}^0 \left| e^{\int_0^s \frac{1}{2} \tau^*_s (M_{ge}) \, dt} \, \tau^*_s (\phi) \right|^2_{(u,t)} ds \leq c^2 \int_{-c}^0 \left| e^{\int_0^s \frac{1}{2} \tau^*_s (M_{ge}) \, dt} \right|^2_{L^\infty(\hat{K}_u)} \int_{-c}^0 \left| \tau^*_s (\phi) \right|^2_{(u,t)} ds \leq C \|\phi\|^2_{L^\infty(\hat{K}_u)}.
\]
Therefore, we conclude that $\|I(\phi)\|_{L^\infty(\hat{K}_u)}^2 \leq C \|\phi\|_{L^\infty(\hat{K}_u)}^2$.

For $j > 0$, we have
\[
\|I^j \left( e^{\Psi/h} \left( \prod_{\alpha} \partial_\alpha \Psi \right) \beta \right) \|^2_{L^\infty(\hat{K}_u)} \leq C \|I^{j-1} \left( e^{\Psi/h} \left( \prod_{\alpha} \partial_\alpha \Psi \right) \beta \right) \|^2_{L^\infty(\hat{K}_u)} \leq C^{j-1} \left( \int_{\hat{K}_u} e^{2\Psi/h} \left( \prod_{\alpha} \partial_\alpha \Psi \right)^2 \right) \|\beta\|_{L^\infty(\hat{K}_u)}^2.
\]
By lemma 59, $\Psi \leq 0$ is a Morse function with zero set $U_S$, so using (5.19) and the following lemma 63, we have

$$\int_{K_u} e^{2\Psi/h}(\prod_\alpha \partial_\alpha \Psi)^2 ds \leq C'h^{1/2+\sum_\alpha m(\alpha)},$$

for some constant $C'$ depending on derivatives of $\Psi$ up to order $2 \sum_\alpha m(\alpha)$. Therefore we indeed have the $L^\infty$ norm estimate

$$\|I_j(e^{\Psi/h}(\prod_\alpha \partial_\alpha \Psi)^\beta)\|_{L^\infty(K_u)} \leq 2\tilde{C}h^{1/2+\sum_\alpha m(\alpha)}\|\beta\|_{L^\infty(\hat{K}_u)},$$

for $j > 0$. The case of $j = 0$ is obtained in a similar way. This is the only case that we have to consider the integration of $\|I_j(e^{\Psi/h}(\prod_\alpha \partial_\alpha \Psi)^\beta)\|_{L^\infty(K_u)}$ to obtain the estimate. □

The following semi-classical approximation lemma 63 used in the above proof, will also be used extensively in the rest of this section. Readers can refer to [3] for details.

**Lemma 63.** Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 with coordinates $x_1, \ldots, x_n$. Let $\varphi : U \to \mathbb{R}_{\geq 0}$ be a Morse function with unique minimum $\varphi(0) = 0$ in $U$. Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be a Morse coordinates near 0 such that

$$\varphi(x) = \frac{1}{2}(\tilde{x}_1^2 + \cdots + \tilde{x}_n^2).$$

For every compact subset $K \subset U$, there exists a constant $C = C_{K,N}$ such that for every $u \in C^\infty(U)$ with $\text{supp}(u) \subset K$, we have

$$|\langle \int_K e^{-\varphi(x)/h}u \rangle - (2\pi h)^{n/2}(\sum_{k=0}^{N-1} \frac{h^k}{2^kk!}\tilde{\Delta}^k(u)(0))| \leq C'h^{n/2+N}\sum_{|\alpha| \leq 2N+n+1} \sup |\partial^\alpha u|,$$

where

$$\tilde{\Delta} = \sum \frac{\partial^2}{\partial \tilde{x}_j^2}, \quad \Theta = \pm \det (\frac{d\tilde{x}}{dx}),$$

and $\Theta(0) = (\text{det} \nabla^2 \varphi(0))^{1/2}$.

In particular, if $u$ vanishes at 0 up to order $L$, then we can take $N = \lceil L/2 \rceil$ and get

$$|\langle \int_K e^{-\varphi(x)/h}u \rangle| \leq C'h^{n/2+[L/2]}.$$

**Remark 64.** The same argument holds true for terms of the form

$$I(F_1I(F_{-1}I(\cdots F_1I(e^{\Psi/h}(\prod_k \partial_{\alpha_k} \Psi)^\beta) \cdots))),$$

where $F_i$'s are tensors involving the function $g_E$ and the metric. The proof can be obtained simply by induction.
We now investigate the norm of $\mu_1$ concerning its order in $\hbar$.

**Proposition 65.** For $l \geq 0$, we have

$$\|\mu_1(h)\|^2_{L^\infty(K_u)} \leq C h^{2l-2[\frac{1}{2}]-1} \|\nu\|^2_{C^2(K_u)},$$

for some constant $C$ depending on derivatives of $g_E$ and the metric tensor up to order $2l + 2$.

**Proof.** W.L.O.G., we assume that the metric is standard under the coordinate $u_1, \ldots, u_{n-1}, t$ since the metric tensor as well as its derivatives are bounded.

We let $Q(u, t, s) = e_s^0 \frac{1}{2} \tau_s^* (M_{g_E}) \, ds$ and consider the effect of first order derivatives acting on $I(\phi/h, \nu) = \int_{-\infty}^0 Q(\tau_s^*(\phi/h, \nu)) \, ds$, we have

$$2\nabla_t I(\phi/h, \nu) = M_{g_E} I(\phi/h, \nu) + e^{\phi/h}$$

$$\nabla_{u_i} I(\phi/h, \nu) = \int_{-\infty}^0 (\nabla_{u_i} Q(\tau_s^*(\phi/h, \nu))) \, ds + h^{-1} I(\phi/h(\nabla_{u_i} \phi), \nu) + I(\phi/h \nabla_{u_i} \nu).$$

For second order derivatives acting on $I(\phi/h, \nu)$, we have

$$\nabla_t \nabla_t I(\phi/h, \nu) = \frac{1}{2} (\nabla_t (M_{g_E}) I(\phi/h, \nu) + \frac{1}{2} (M_{g_E})^2 I(\phi/h, \nu) + h^{-1} e^{\phi/h}(\nabla_t \phi) + e^{\phi/h} \nabla_t \nu)$$

$$\nabla_{u_i} \nabla_t I(\phi/h, \nu) = \frac{1}{2} \{(\nabla_{u_i} M_{g_E}) I(\phi/h, \nu) + h^{-1} M_{g_E} I(\phi/h(\nabla_{u_i} \phi), \nu) + M_{g_E} I(\phi/h(\nabla_{u_i} \nu))
+ M_{g_E} \left( \int_{-\infty}^0 \nabla_{u_i} Q(\tau_s^*(\phi/h, \nu)) \, ds \right) + h^{-1} e^{\phi/h}(\nabla_{u_i} \phi) + e^{\phi/h} \nabla_{u_i} \nu \}$$

$$\nabla_{u_i} \nabla_{u_j} I(\phi/h, \nu) = \int_{-\infty}^0 (\nabla_{u_i} \nabla_{u_j} Q(\tau_s^*(\phi/h, \nu))) \, ds + h^{-1} \int_{-\infty}^0 \nabla_{u_j} Q(\tau_s^*(\phi/h, \nabla_{u_i} \phi)) \, ds$$

$$+ \int_{-\infty}^0 \nabla_{u_j} Q(\tau_s^*(\phi/h, \nabla_{u_i} \phi)) \, ds + h^{-1} \int_{-\infty}^0 \nabla_{u_i} Q(\tau_s^*(\phi/h, \nabla_{u_j} \phi)) \, ds$$

$$+ \int_{-\infty}^0 \nabla_{u_i} Q(\tau_s^*(\phi/h, \nabla_{u_j} \phi)) \, ds + h^{-1} I(\phi/h(\nabla_{u_i} \phi)(\nabla_{u_j} \nu)) + h^{-1} I(\phi/h(\nabla_{u_j} \phi)(\nabla_{u_i} \nu))$$

$$+ h^{-2} I(\phi/h(\nabla_{u_i} \phi)(\nabla_{u_j} \phi)) + I(\phi/h \nabla_{u_i} \nabla_{u_j} \nu).$$

Using similar arguments as in the previous lemma, we obtain estimates

$$\|I(\nabla_t \nabla_t I(\phi/h, \nu))\|^2_{L^\infty(K_u)} = O(h) \|\nu\|^2_{C^1(K_u)},$$

$$\|I(\nabla_{u_i} \nabla_t I(\phi/h, \nu))\|^2_{L^\infty(K_u)} = O(h) \|\nu\|^2_{C^1(K_u)},$$

$$\|I(\nabla_{u_i} \nabla_{u_j} I(\phi/h, \nu))\|^2_{L^\infty(K_u)} = O(h) \|\nu\|^2_{C^2(K_u)}.$$

We conclude that $\|\mu_1\|^2_{L^\infty(K_u)} = \|\frac{1}{h} I(\Delta I(\phi/h, \nu))\|^2_{L^\infty(K_u)} = O(h) \|\nu\|^2_{C^2(K_u)}$.

In general, despite $\mu_1$ having a complicated expression with terms of the form as in remark 64, what we need to know is the vanishing order of the
integrand along $U$ as the integrand carries an exponential term $e^{\Phi/h}$. Notice that applying differentiation to the term $e^{\Phi/h}$ will give a term of the form $h^{-1}\nabla\Phi$. $\nabla_t\Phi$ and $\nabla_u\Phi$ vanish up to first order and second order, both contribute $h^2$ to the square norm.

In the iteration process, if we write $\Delta = \nabla_t\nabla_t + \sum_i \nabla_u\nabla_t + \sum_{i,j} \nabla_u\nabla_{u_j}$, we find that the terms involving $\nabla_t\nabla_t$ will give the lowest order in $h$. Therefore, we look at terms obtained from applying $I\nabla_t\nabla_t$ iteratively. For $\mu_2(h)$, that is the term $\frac{h}{8}I(\nabla_t\nabla_t)(\nabla_t\nabla_t)I(e^{\Phi/\nu})$. Since $\nabla_t$ essentially cancels with $I$, the lowest order term is $\frac{h}{8}I(h^{-1}(\nabla_t\nabla_t))e^{\Phi/\nu} = O(h^0)$.

For $\mu_k$, we consider the term $(I\nabla_t\nabla_t)^k I(e^{\Phi/\nu})$. When $k$ is even, the lowest order term becomes $\frac{h}{8(2k+1)}(\nabla_t\nabla_t)^{k/2} e^{\Phi/\nu}$, which has order $O(h^{k+1})$. When $k$ is odd, the lowest order term becomes $\frac{h}{8(2k+1)}(\nabla_t\nabla_t)^{(k-1)/2} e^{\Phi/\nu}$, which has order $O(h^{k+1})$. Combining these two cases, we obtain the statement in the proposition.

□

Remark 66. Using the same argument, we have similar estimates

$$\int_{K_u} |\nabla^j \mu_t(h)|^2 \leq C_j h^{2l-2(\frac{j+1}{2})-1} \|\nu\|^2_{C^{2l+j}(K_u)},$$

for all $j$. The integration along $t$ would be necessary for $j > 0$ as $\nabla_t$ may cancel with the operator $I$.

Making use of Sobolev embedding in dimension 1 along the $t$ direction, we obtain the estimate

$$\|\nabla^j \mu_t(h)\|^2_{L^\infty(K_u)} \leq C_j h^{2l-2(\frac{j+1}{2})-1} \|\nu\|^2_{C^{2l+j+1}(K_u)},$$

5.6. A priori estimate. We make use of the WKB iteration to construct the WKB expansion and prove that it does give a desired approximate to the solution in the rest of section 5. Before that, we obtain an a priori estimate for the solution in this subsection.

We consider the equation

$$\Delta_f \zeta_s = (I - P_f)d_f^*(\chi_S \zeta_S) \quad (5.21)$$

in $W$, where $\zeta_S \in \Omega^*(W_S)$ is the input form depending on $h$ and $\chi_S \in C_C^\infty(W_S)$ is some cut off function to be chosen later. We assume $\zeta_S$ has a WKB approximation on $W_S$ of the form

$$\zeta_S \sim e^{-\psi_S/h}(\omega_{S0} + \omega_{S1}h^{1/2} + \omega_{S2}h^1 + \ldots), \quad (5.22)$$
where $\omega_{S,i} \in \Omega^*(W_S)$ and $\psi_S = f + g_S$. It is an approximation in the sense that

$$\|e^{\psi_S/h}\zeta_S - \left( \sum_{i=0}^{N} \omega_{S,i} h^{i/2} \right)\|_{L^\infty(W_S)}^2 \leq C_N h^{N+1},$$

for $N$ large enough, where $C_N$ is a constant depending on $N$. We also require similar norm estimates for its derivatives

$$\|e^{\psi_S/h}\nabla^j (\zeta_S - e^{-\psi_S/h}(\sum_{i=0}^{N} \omega_{S,i} h^{i/2}))\|_{L^\infty(W_S)}^2 \leq C_{j,N} h^{N+1-2j},$$

with $C_{j,N}$ depending on $j, N$.

We want to get a similar expansion for $\zeta_E$, using the iteration defined in the section 5.4. We fix a cut off function $\chi$ supported in $W$ with $\chi \equiv 1$ on the flow line $\gamma$ from $x_S$ to $x_E$. We consider any small enough compact neighborhood $K \subset W$ of the flow line $\gamma$ with $\chi \equiv 1$ on $K$. $\chi_S$ is chosen so that $\text{supp}(\chi_S) \subset K$. The following figure illustrates the situation.

For small enough $K$, we have an a priori estimate of $\zeta_E$ in $K$, the technique is similar to the method used for eigenform in [7].

**Lemma 67.** For small enough $\text{supp}(\chi_S)$ and $K$, and any $j \in \mathbb{Z}_+$, there exists $h_{j,0} > 0$ such that for any $h < h_{j,0}$, we have

$$\|e^{\psi_E/h}\nabla^j \zeta_E\|_{L^\infty(K)}^2 \leq C_j h^{-N_j},$$

where $N_j$ is an positive integer depending on $j$.

In order to prove the above lemma, we need to know certain special properties about $\chi$ and our chosen compact set $K$. Letting $\tilde{\psi} := \inf_{y \in \text{supp}(\chi_S)} \{\psi_S + \rho_f(y, x)\}$, we have the following lemma.

**Lemma 68.** There exists $\epsilon > 0$ such that for all $K$ small enough, we have

$$\tilde{\psi}(x) + \rho(y, x) \geq \psi_E(y) + \epsilon,$$
for all \( y \in K \) and \( x \in \text{supp}(\nabla \chi) \).

Proof. Using the fact that \( \psi_E = f \) on \( V_E \), we have \( |\psi_E - f| \leq \epsilon \) on \( K \). We can simply prove

\[
\tilde{\psi}(x) + \rho(y, x) \geq f(y) + \epsilon,
\]

by choosing small enough \( K \) and \( \epsilon \). From properties of Agmon distance \( \rho \), we have

\[
\tilde{\psi}(x) \geq \min_{z \in \text{supp}(\chi_S)} (f(z) + f(x) - f(z)) = f(x),
\]

with equality holds only if \( z \in V_S \) and there is a generalized gradient line joining \( z \) to \( x \). Therefore, we have

\[
\tilde{\psi}(x) + \rho(y, x) \geq f(y) + \epsilon,
\]

with equality holds only if there is a generalized gradient line joining a point \( z \in V_S \) to \( x \in \text{supp}(\chi) \) and then to \( y \in K \). This is impossible by for our choices of \( \chi \) and \( K \). Hence we always have strict inequality and therefore we can find small \( \epsilon \) by compactness argument. \( \square \)

We consider a closed neighborhood \( \tilde{W} \) of \( \text{supp}(\chi) \) in \( W \) with smooth boundary. We let \( \tilde{G} \) to be the twisted Green’s operator on \( \tilde{W} \) using Dirichlet boundary condition. We first argue that \( \zeta_E \) can be replaced by \( \tilde{\zeta}_E = d_f^* \tilde{G} \chi_S \zeta_S \).

Lemma 69. There is a \( \delta > 0 \) such that

\[
\| e^{\frac{j}{h} \nabla j} (\chi \zeta_E - \tilde{\zeta}_E) \|_{L^\infty(K)} \leq C_j e^{-\frac{\delta}{h}},
\]

whenever \( \text{supp}(\chi_S) \) and \( K \) are chosen small enough and \( j \in \mathbb{Z}_+ \).

Proof. We let \( r_h = \chi \zeta_E - \tilde{\zeta}_E \). First, \( r_h \) satisfies the equation

\[
\tilde{\Delta} r_h = [\Delta, \chi] \zeta_E - \chi P_f d_f^*(\chi_S \zeta_S).
\]

Therefore we have \( r_h = (\tilde{G}[\Delta, \chi] G - \tilde{G} \chi_P f) d_f^*(\chi_S \zeta_S) \). We consider it term by term to get estimate of \( r_h \). We have for any \( \epsilon > 0 \),

\[
\tilde{G}[\Delta, \chi] G \sim \mathcal{O}_\epsilon \left( \exp \left( -\frac{1}{h} \left( \min_{z \in \text{supp}(\chi)} (\rho(x, z) + \rho(z, y) - \epsilon) \right) \right) \right).
\]

Using lemma 68 we can show there is some \( \delta_0 > 0 \) such that

\[
\tilde{G}[\Delta, \chi] G d_f^*(\chi_S \zeta_S) \sim \mathcal{O}(e^{-(\psi_E + \delta_0)/h})
\]

in \( K \), for \( h \) small enough.

For the term \( \tilde{G} \chi_P f \), we have

\[
\tilde{G} \chi_P f \sim \mathcal{O}_\epsilon \left( \sum_{q \in C_f^d} \exp \left( -\frac{1}{h} (\rho(x, q) + \rho(q, y) - \epsilon) \right) \right),
\]

where \( l = \text{deg}(\zeta_S) \). Again, we can find a constant \( \delta_1 > 0 \) such that

\[
\min_{x \in \text{supp}(\chi_S)} (\psi_S(x) + \rho(x, q) + \rho(q, y)) \geq \psi_E(y) + 2\delta_1,
\]
for \( y \in K \). Similarly we have

\[
\tilde{G}_\chi P_f d_f^j(\chi S \zeta S) \sim O(e^{-\langle \psi_E + \delta_1 \rangle / \hbar})
\]

in \( K \), for \( \hbar \) small enough. Notice that the constant \( \delta = \min\{\delta_0, \delta_1\} \) can chosen to be the same if we shrink \( \text{supp}(\chi S) \) and \( K \) and keep \( \tilde{W} \) and \( \chi \) fixed.

Next, we obtain estimates for \( \tilde{\zeta}_E \) similar to those in lemma \( \text{67} \) for \( \zeta_E \).

**Lemma 70.** For any \( j \in \mathbb{Z}_+ \), there exists \( h_{j,0} > 0 \) such that if \( h < h_{j,0} \), we have

\[
\|e^{\psi_E / \hbar} \nabla^j \tilde{\zeta}_E\|_{L^\infty(\tilde{W})}^2 \leq C_j h^{-N_j},
\]

where \( N_j \) is an positive integer depending on \( j \).

**Proof.** We consider the equation

\[
\Delta_j \tilde{\zeta}_E = d_f^j(\chi S \zeta S)
\]

in \( \tilde{W} \) and divide the proof into steps:

**Step 1:** Without loss of generality, we assume there is a constant \( C_0 > 0 \) such that \( C_0^{-1} \leq \psi_E \leq C_0 \) and \( C_0^{-1} \leq |df|^2 = |d\psi_E|^2 \leq C_0 \) on \( \tilde{W} \). We define the function

\[
\Phi = \psi_E - Ch \log(\psi_E / \hbar),
\]

with \( C > 0 \) to be chosen. Therefore we have

\[
|df|^2 - |d\Phi|^2 \geq \frac{Ch|df|^2}{\psi_E} \geq \frac{Ch}{C_0^2}.
\]

We apply lemma \( \text{4.4} \) to \( \tilde{\zeta}_E \) with the chosen \( \Phi \) on \( \tilde{W} \) and get

\[
\text{Re}(\langle e^{2\Phi / \hbar} d_f^j(\chi S \zeta S) \rangle) = \hbar^2 \langle \|d(e^{\Phi / \hbar} \tilde{\zeta}_E)\|^2 + \|d^* (e^{\Phi / \hbar} \tilde{\zeta}_E)\|^2 \rangle
\]

\[
+ \langle (|df|^2 - |d\Phi|^2 + \hbar M_f) e^{\Phi / \hbar} \tilde{\zeta}_E, e^{\Phi / \hbar} \tilde{\zeta}_E \rangle
\]

and if we choose \( C > 0 \) large enough to absorb the term \( \langle h M_f e^{\Phi / \hbar} \tilde{\zeta}_E, e^{\Phi / \hbar} \tilde{\zeta}_E \rangle \), we have

\[
\hbar^2 \langle \|d(e^{\Phi / \hbar} \tilde{\zeta}_E)\|^2 + \|d^* (e^{\Phi / \hbar} \tilde{\zeta}_E)\|^2 \rangle + \frac{Ch}{2C_0^2} \|e^{\Phi / \hbar} \tilde{\zeta}_E\|^2
\]

\[
\leq C_1 \|e^{\Phi / \hbar} d_f^j(\chi S \zeta S)\|^2 \leq C_1 \left( \frac{\hbar}{\psi_E} \right)^{2C} \|e^{\psi_E / \hbar} d_f^j(\chi S \zeta S)\|^2
\]

\[
\leq C_2 \left( \frac{h}{\psi_E} \right)^{2C}.
\]

Therefore we get

\[
\hbar^2 \langle \|d(e^{\psi_E / \hbar} \tilde{\zeta}_E)\|^2 + \|d^* (e^{\psi_E / \hbar} \tilde{\zeta}_E)\|^2 \rangle + \hbar \|e^{\psi_E / \hbar} \tilde{\zeta}_E\|^2 \leq C_3,
\]

and obtained \( \|e^{\psi_E / \hbar} \tilde{\zeta}_E\|^2_{L^2(K)} \leq C_4 h^{-1} \), for \( h < h_0 \).
Step 2: We prove the $L^2$ estimate for derivatives of $\tilde{\zeta}_E$. We apply $d_f$ and $d^*_f$ to both sides of equation (5.29). We obtain

\begin{equation}
\Delta_f(d_f\tilde{\zeta}_E) = d_fd^*_f(\chi_S\zeta_S).
\end{equation}

Applying the result in step 1 to $d_f\tilde{\zeta}_E$, we have

\begin{equation}
\|e^{\psi_E/\hbar}d_f\tilde{\zeta}_E\|_{L^2(K)}^2 \leq C_4\hbar^{-1}.
\end{equation}

Since $d_f = \hbar d + df \wedge$, we have

\begin{equation}
\|e^{\psi_E/\hbar}d_\zeta_E\|_{L^2(K)}^2 \leq C_5\hbar^{-3}.
\end{equation}

Corresponding result for $d^*_f\tilde{\zeta}_E$ can be obtained by a similar argument. These combine to obtain result for $\nabla\tilde{\zeta}_E$. By applying $\nabla$ successively, we obtain all higher derivatives’ estimates in a similar fashion.

Step 3: Finally, we improve the estimate to $L^\infty$ norm. Since we have $L^2$ norm estimate for all the derivatives of $\tilde{\zeta}_E$. We use the Sobolev embedding on $\tilde{\zeta}_E$ to obtain the $L^\infty$ norm estimate. Details are left to readers. □

Lemma 67 follows from lemma 69 and lemma 70 directly.

5.7. WKB approximation. Next, we consider the WKB approximation of $\zeta_E$. From the WKB approximation (5.1) of $\zeta_S$, we can take $d^*_f$ on both side and obtain a WKB approximation of $d^*_f(\chi_S\zeta_S)$

\begin{equation}
d^*_f(\chi_S\zeta_S) \sim e^{-\psi_E/\hbar}(hd^* + i2\nabla_f + i\nabla_{gs})(\chi_S\omega_{S,0} + \chi_S\omega_{S,1}\hbar^{1/2} + \ldots),
\end{equation}

after grouping terms according to their orders of $\hbar$. We apply the iteration in the previous subsection 5.4 terms by terms to the above series and then group the terms according to orders of $\hbar$ of their $L^2$ norms. As a result, we obtain a WKB expansion

\begin{equation}
\zeta_E \sim e^{-\psi_E/\hbar}(\omega_{E,0}(h) + \omega_{E,1}(h) + \ldots)
\end{equation}

in $W$, where $\omega_{E,i}(h)$’s are functions also depending on $\hbar$. For every $l$ and any compact subset $\tilde{K} \subset W$,

\begin{equation}
\|\omega_{E,l}(h)\|^2_{L^\infty(\tilde{K})} \leq C_{l,\tilde{K}}\hbar^{2l-2\left\lfloor\frac{l}{2}\right\rfloor-1}
\end{equation}

for those $h < h_{l,0}$, and also

\begin{equation}
\|e^{\psi_E/\hbar}(\Delta_f(e^{-\psi_E/\hbar}\sum_{i=0}^{N} \omega_{E,i}(h)) - d^*_f(e^{-\psi_S/\hbar}\sum_{i=0}^{N} \omega_{S,i})\|_{L^2(\tilde{K})}^2 \leq C_{N,\tilde{K}}\hbar^{N+\frac{3}{2}}.
\end{equation}

for $h < h_{N,0}$. We need to argue that it is a good approximation, which is the main theorem in this section.
**Theorem 71.** For any $\text{supp}(\chi_S)$ and $K$ small enough, and $N$ large enough, there exists $h_{j,N,0} > 0$ such that for $h < h_{j,N,0}$ we have

$$
\| e^{\psi_E/h} \nabla j \{ \zeta_E - e^{-\psi_E/h} \left( \sum_{i=0}^{N} \omega_{E,i}(h) \right) \} \|^2_{L^\infty(K)} \leq C_{j,N} h^{N-2j}.
$$

**Proof.** Making use of lemma 69 we can again consider the equation 5.29. It suffices to show that the approximation works for $\tilde{\zeta}_E$ on some small enough pre-compact neighborhood $K$ of the flow line $\gamma$. We divide the proof into several steps.

**Step 1:** As $\omega_{E,i}(h)$’s do not vanish on boundary of $\tilde{W}$, we first need to cut them off suitably for applying integration by part. $\omega_{E,i}(h)$’s, being defined by integrating along flow of $\tau$, have support as shown in the following figure.

![Figure 7. Support of $\omega_{E,i}$’s](image)

Suppose we have $\tau_{\tilde{T}}(v_S) = v_E$, then we can choose $\tilde{\chi}$ only depending on variable $t$ (using coordinate defined by $\tau$) such that $\tilde{\chi} \equiv 1$ for $t \leq \tilde{T}$. The support of $\nabla \tilde{\chi}$ is shown in the following figure.

![Figure 8. Support of $\nabla \tilde{\chi}$](image)
By shrinking $K$ and $\text{supp}(\chi_S)$ if necessary, we obtain some $\epsilon > 0$ such that
\begin{equation}
\psi_E(y) + \rho(y, x) \geq \psi_E(x) + \epsilon
\end{equation}
for $x \in K$ and $y \in \text{supp}(\tilde{\chi})$. We define the function
\begin{equation}
\Phi_N = \min\{\Phi + Nh\log(1/h), \min_{y \in \text{supp}(\nabla \tilde{\chi})} (\Phi(y) + (1 - \epsilon)\rho(x, y))\},
\end{equation}
where $\Phi := \psi_E - Ch\log(\psi_E/h)$ is defined in (5.30), and the $\epsilon$ is chosen in lemma 68. We have
\[|df|^2 - |d\Phi_N|^2 \geq \frac{Ch|df|^2}{\psi_E} \geq \frac{Ch}{C_0^2},\]
for $h$ small enough. Notice that we have $\Phi_N = \Phi + Nh\log(1/h)$ in $K$ for $h$ small enough, and $\Phi_N = \Phi$ in $\text{supp}(\nabla \tilde{\chi})$.

**Step 2:** Applying the lemma 43 to $r_k = \tilde{\chi}(\zeta_E - e^{-\psi_E/h}(\sum_{i=0}^{k-1} \omega_{E,i}(h)))$ and $\Phi_N$, we get
\[h^{2-2N} (\|d(\Phi_N/h)_k\|_{L^2(K)}^2 + \|d^*(\Phi_N/h)_k\|_{L^2(K)}^2) + \frac{Ch^{1-2N}}{2C_0^2} \|\Phi_N/h_k\|_{L^2(K)}^2 \leq D\|\Phi_N/h d_f^*(\chi_S \zeta_E - e^{-\psi_S/h} \sum_{i=0}^{k-1} \chi_S \omega_{S,i})\|_{L^2(W)}^2,
\]
\[+ \quad D\|\Phi_N/h (d_f^*(e^{-\psi_S/h} \sum_{i=0}^{k-1} \chi_S \omega_{S,i}) - \Delta_f (e^{-\psi_E/h} \sum_{i=0}^{k-1} \omega_{E,i}(h)))\|_{L^2(W)}^2 \]
\[+ \quad D(\|\Phi/h [\Delta, \tilde{\chi}] \zeta_E\|_{L^2(W)}^2 + \|\Phi/h [\Delta, \tilde{\chi}] (e^{-\psi_E/h} \sum_{i=0}^{k-1} \omega_{E,i}(h)))\|_{L^2(W)}^2).
\]
We handle the right hand side term by term. First, we have
\[\|\Phi_N/h d_f^*(\chi_S \zeta_E - e^{-\psi_S/h} \sum_{i=0}^{k-1} \chi_S \omega_{S,i})\|_{L^2(W)}^2 \leq C_k h^{2C-2N+k}.
\]
Second, we have
\[\|\Phi_N/h (d_f^*(e^{-\psi_S/h} \sum_{i=0}^{k-1} \chi_S \omega_{S,i}) - \Delta_f (e^{-\psi_E/h} \sum_{i=0}^{k-1} \omega_{E,i}(h)))\|_{L^2(W)}^2 \leq C_k h^{2C-2N+k}.
\]
Third, we have
\[\|\Phi/h [\Delta, \tilde{\chi}] \zeta_E\|_{L^2(W)}^2 \leq D_1 h^{2C-N_0},
\]
where $N_0$ is the integer in lemma 67. Finally, we have
\[\|\Phi/h [\Delta, \tilde{\chi}] (e^{-\psi_E/h} \sum_{i=0}^{k-1} \omega_{E,i}(h)))\|_{L^2(W)}^2 \leq C_k h^{2C-N_0},
\]
by choosing a larger $N_0$ independent of $k$, if necessary. Combining the above, by choosing $N = N_0 + k$, we have
\[ h^2(\|d(e^{\psi_E/h} r_k)\|_{L^2(K)}^2 + \|d^* (e^{\psi_E/h} r_k)\|_{L^2(K)}^2 + h\|e^{\psi_E/h} r_k\|_{L^2(K)} \leq C_k h^k, \]
which gives $\|e^{\psi_E/h} r_k\|_{L^2(K)}^2 \leq C_k h^{k-1}$, for those $h < h_{k,0}$.

**Step 3:** We obtain $L^2$ estimate for all derivatives of $r_k$. We repeat the above argument for $d f r_k$ and $d^* r_k$. For any $j, N \in \mathbb{Z}_+$, we can find a $k_{j,N}$ large enough such that for any $k > k_{j,N}$, we have
\[ \|e^{\psi_E/h} \nabla^j r_k\|_{L^2(K)}^2 \leq C_{j,N,k} h^{N}, \]
for $h > h_{j,N,0}$.

**Step 4:** We apply interior Sobolev embedding to improve the statement in step 3 into $L^\infty$ norm, by further shrinking $K$ if necessary. As a result, we have $\|d f r_k\|_{L^\infty(K)}$ for $N$ large enough, there exists $h_{j,N,0} > 0$ and $M_N$ such that we have
\[ (5.37) \quad \|\operatorname{e}^{\psi_E/h} \nabla^j \{\hat{\omega}_E - \operatorname{e}^{-\psi_E/h}(\sum_{i=0}^{M_N} \omega_{E,i}(h))\}\|_{L^\infty(K)} \leq C_{j,N,k} h^{N-2j} \]
for $h < h_{j,N,0}$. Finally, we observe that $\|\nabla^j \omega_{E,i}(h)\|_{L^\infty(K)} \leq C_{i,j} h^{-j-\frac{N}{2}}$ and hence obtain the result by dropping redundant terms in the approximation series.

Finally, we restrict our attention to a small enough neighborhood $W_E$ of $v_E$. Since the operator $I$ is given by an integral with an exponential decay $e^{\Psi/h}$ along flow line, we can apply lemma 33 to obtain an expansion
\[ \omega_{E,i}(h) = h^{-\frac{N}{2}} (\omega_{E,i,0} + \omega_{E,i,1} h^1 + \omega_{E,i,2} h^2 + \ldots). \]
By regrouping terms according to their orders of $h$, we obtain an expansion of the form given in equation (5.2).

**5.8. Relation between $\omega_{S,0}$ and $\omega_{E,0}$:** From section 5.4, we constructed a WKB approximation in $W_E$
\[ \hat{\omega}_E = \operatorname{e}^{-\psi_E/h}(\omega_{E,0}(h) + \omega_{E,1}(h) + \ldots). \]
In particular, $\omega_{E,0}(h)$ is given by
\[ (5.38) \quad \omega_{E,0}(h) = \frac{1}{2h} \left( \int_{-\infty}^{0} e^{\frac{1}{2} \tau^2(M_{g_E})} d\tau \mathcal{A}_E^*(\operatorname{e}^{\Psi/h}(\nu_2 \nabla f + \nu \nabla g_s)_{x} \chi_{S,0}) ds \right). \]
In this section, we study the relation between integrals of $\omega_{S,0}$ and $\omega_{E,0}$ which is used in lemma 33. We begin by recalling lemma 31. Let $M$ be a $n$-dimensional manifold and $S$ be a $k$-dimensional submanifold in $M$, with a neighborhood $B$ of $S$ which can be identified as the normal bundle $\pi : NS \to S$. Suppose $\varphi : B \to \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set $S$, we have
Lemma 72. Let $\beta \in \Omega^*(B)$ which is vertically compact support along the fiber of $\pi$. Then, we have
\[
\pi_*(e^{-\varphi(x)/h} \beta) = (2\pi h)^{(n-k)/2} (\tau_{\text{vol}(\nabla^2 \varphi)} \beta)|_{\mathcal{V}} (1 + O(h)),
\]
where $\pi_*$ is the integration along fiber. Here $\text{vol}(\nabla^2 \varphi)$ stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2 \varphi$ along fibers of $\pi$.

We use the notations in section 5.1 and assume there is an identification of $W_S$ and $W_E$ with the normal bundle $NV_S$ and $NV_E$ of $V_S$ and $V_E$ respectively. We use $\pi_S$ and $\pi_E$ to stand for the bundle maps respectively. We have the following lemma which relates the integration of $\omega_{E,0}$ and $\omega_{S,0}$ along the fibers of $\pi_E$ and $\pi_S$ respectively.

**Lemma 73.** Assume $\omega_{S,0} \in \wedge^{\text{top}} NV^*_S$ on $V_S$, then
\[
\pi_{E*}(e^{-g_E/h} \omega_{E,0}) = \frac{1}{h} \theta^* \pi_S*(e^{-g_S/h} \omega_{S,0})(1 + O(h^{1/2})),
\]
where $\theta : V_E \to V_S$ is the projection map using the identification $V_E \equiv (V_S \times \mathbb{R}) \cap W_E$ given by $\tau$ (flow of $\nabla \psi_E$). Furthermore, we have $\omega_{E,0} \in \wedge^{\text{top}} NV^*_E$ on $V_E$.

**Proof.** We use the coordinates $u_1, \ldots, u_{n-1}, t$ for $W$, where $u_1, \ldots, u_{n-1}$ are coordinates of $U_S$. We further assume that $\{u_{s+1} = 0, \ldots, u_{n-1} = 0\} = V_S$. From lemma 59, $\Psi \leq 0$ is a Bott-Morse function with zero set $U_S$. Applying lemma 72 to the equation (5.38), we have
\[
\omega_{E,0}(u, t)
\equiv \frac{\pi}{2h} \frac{1}{1/2} (\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{-1/2} \left( e^{\int_{t_0}^{0} \frac{1}{2} \tau^* (M_{g_E}) de} t_{-t}(\tau_{\text{vol}}(\nabla^2 \varphi) \chi_S \omega_{S,0}) \right),
\]
modulo terms of $O(h^{1/2})$. From lemma 58, $g_E \geq 0$ is a Bott-Morse function with zero set $V_E$. Applying lemma 72 again, we get, modulo terms of $O(h^{1/2})$,
\[
\pi_{E*}(e^{-g_E/h} \omega_{E,0})(u, t)
\equiv (2\pi h)^{(n-s-1)/2} (\tau_{\text{vol}}(\nabla^2 g_E)(\omega_{E,0})
\equiv \pi \left( (2\pi h)^{(n-s-2)/2} (\tau_{\text{vol}}(\nabla^2 g_E)(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0})^{-1/2} (e^{\int_{t_0}^{0} \frac{1}{2} \tau^* (M_{g_E}) de} t_{-t}(\tau_{\text{vol}}(\nabla^2 g_E)(\chi_S \omega_{S,0})) \right),
\]
for those $(u, t) \in V_E$. The term involving $\tau_{\text{vol}}$ is dropped as $\tau^*(d\gamma_S)$ vanishes for $(u, t) \in V_E$. To make further simplifications, we need the following lemma.

**Lemma 74.** Fixing a point $(u, t) \in V_E$, we have
\[
e^{\int_{t_0}^{0} \frac{1}{2} \tau^* (M_{g_E}) de} = \left( \frac{\text{det}(\nabla^2 g_E)(u, t)}{\text{det}(\nabla^2 g_E)(u, 0)} \right)^{1/2}
\]
as operators on $\bigwedge^{\text{top}} NV_E^*$, where the right hand side acts as multiplication. Here $\nabla^2 g_E$ is treated as an operator acting on $NV_E$ using the metric tensor.

From the fact that $\omega_{S,0} \in \bigwedge^{\text{top}} NV_S^*$ upon restricting to $V_S$, we have $\tau^*_E(t \nabla_f \omega_{S,0}) \in \bigwedge^{\text{top}} NV_E^*$ for those $(u, t) \in V_E$ and

$$\pi_{E*}(e^{-g_E/h} \omega_{E,0})(u, t) = 2\pi (2\pi h)^{(n-s-2)/2} \left( \frac{\partial^2}{\partial t^2} (-\Psi)_{t=0} \right)^{-1/2} \left( \frac{\det(\nabla^2 g_E)(u, t)}{\det(\nabla^2 g_E)(u, 0)} \right)^{1/2} t \nabla_f \bigwedge \tau^*_E (\omega_{S,0}).$$

Notice that $\nabla f = \frac{\partial}{\partial t}$ when restricting on $V_E$, therefore we have

$$\left( \frac{\partial^2}{\partial t^2} (-\Psi)_{t=0} \right)^{1/2} \nabla f = \text{vol}(\nabla^2 g_E((-\Psi)_{t=0}),$$

where we view $W$ as a $\mathbb{R}$-bundle over $U_S$ and consider $\text{vol}(\nabla^2 (-\Psi)_{t=0})$ as the volume vector field along its fibers. Furthermore, we have the relation

$$d\tau^*_E \left( \frac{\det(\nabla^2 g_E)(u, t)}{\det(\nabla^2 g_E)(u, 0)} \right)^{1/2} \text{vol}(\nabla^2 g_E)(u, t) = \text{vol}(\nabla^2 g_E)(u, 0).$$

Combining the above, we have

$$\pi_{E*}(e^{-g_E/h} \omega_{E,0})(u, t) = (2\pi)^{(n-s)/2} h^{(n-s-2)/2} \left( \tau^*_E (t \text{vol}(\nabla^2 (-\Psi)_{t=0}) \bigwedge \nabla^2 g_E)_{t=0} \omega_{S,0} \right).$$

Finally, from the relation $\Psi = g_E - g_S$, we get

$$\text{vol}(\nabla^2 (-\Psi)) \bigwedge \text{vol}(\nabla^2 g_E) = \text{vol}(\nabla^2 g_S)$$

on $V_S$, where $\text{vol}(\nabla^2 g_S)$ is the volume polyvector field along the fibers of $\pi_S$. Therefore, we have

$$\pi_{E*}(e^{-g_E/h} \omega_{E,0})(u, t) \equiv \frac{1}{h} \tau^*_E(\pi_{S*}(e^{-g_S/h} \omega_{S,0}))(u, 0)$$

modulo terms of $O(h^{1/2})$, for those $(u, t) \in V_E$. \hfill \Box

**Proof of Lemma 74** First of all, we have the equality

$$\frac{1}{2} M_{g_E} = \nabla^2 g_E - \frac{1}{2} \text{tr}(\nabla^2 g_E),$$

on the set $\{ \nabla g_E = 0 \}$. We can treat $\nabla^2 g_E$ as an operator acting on $NV_E^*$ as $g_E$ is Morse along $V_S$. Restricting to $\bigwedge^{\text{top}} NV_E^*$, it is just $\text{tr}(\nabla^2 g_E)$. Therefore we have

$$\frac{1}{2} M_{g_E} = \frac{1}{2} \text{tr}(\nabla^2 g_E),$$
acting on $\Lambda^{top} NV^*_E$.

On $V_E$, we have

\begin{equation}
\nabla_t \left( \int_0^t \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, \epsilon) \, d\epsilon - \frac{1}{2} \log(\text{det}(\nabla^2 g_E)(u, t)) \right) = \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, t) - \frac{1}{2} \text{tr}((\nabla^2 g_E(u, t))^{-1} \nabla_t(\nabla^2 g_E(u, t))).
\end{equation}

We will show that the above expression vanish.

Restricting to the set $\{\nabla g_E = 0\}$, for any vector fields $X, Y \in TW$, we have

$$\nabla_t(\nabla^2 g_E)(X, Y) = \nabla_t(\nabla^2 g_E(X, Y)) - \nabla^2 g_E(\nabla_t X, Y) - \nabla^2 g_E(X, \nabla_t Y)$$

$$= \nabla_t(X, \nabla Y \nabla g_E) - \langle \nabla_t X, \nabla Y \nabla g_E \rangle - \langle \nabla X \nabla g_E, \nabla_t Y \rangle$$

$$= \langle X, \nabla_t \nabla Y \nabla g_E \rangle + \langle \nabla X \nabla g_E, [\partial_t, Y] \rangle + \langle \nabla X \nabla g_E, \nabla Y \partial_t \rangle$$

$$= \langle X, \nabla Y \nabla \nabla g_E \rangle + \langle \nabla^2(\nabla^2 g_E) X, Y \rangle,$$

and

$$\nabla^2(\nabla_t g_E)(X, Y) = \langle \nabla Y \nabla (\partial_t g_E), X \rangle$$

$$= Y \langle \nabla (\partial_t g_E), X \rangle - \langle \nabla (\partial_t g_E), \nabla Y X \rangle$$

$$= Y \langle \nabla (\partial_t g_E), X \rangle - \langle \nabla X \nabla g_E, \partial_t \rangle - \langle \nabla X, \nabla g_E, \partial_t \rangle$$

$$= Y \langle \nabla X \nabla (\partial_t g_E), X \rangle - \langle \nabla Y X, \nabla_t g_E \rangle$$

$$= \langle X, \nabla Y \nabla \nabla g_E \rangle + \langle (\nabla^2 g_E)^2, X, Y \rangle.$$

Therefore, we have

$$\nabla_t(\nabla^2 g_E) - \nabla^2(\nabla_t g_E) = [\nabla^2 t, \nabla^2 g_E],$$

where the Hessians are treated as endomorphisms of $TM$. Restricting the above equation to the subspace $NV_E$ and multiplying by $(\nabla^2 g_E)^{-1}$, we have

$$\text{tr}((\nabla^2 g_E)^{-1}(\nabla_t(\nabla^2 g_E))) = \text{tr}((\nabla^2 g_E)^{-1} \nabla^2(\nabla_t g_E)).$$

Finally, from the equation $|\nabla \psi_E|^2 = |\nabla f|^2$, we obtain

$$\nabla_t g_E = \frac{1}{2} |\nabla g_E|^2.$$

Applying $\nabla^2$ to both sides and restricting to $V_E$ give

$$\nabla^2(\nabla_t g_E)(X, Y) = \langle \nabla^2 g_E(X), \nabla^2 g_E(Y) \rangle,$$

or simply

$$\nabla^2(\nabla_t g_E) = (\nabla^2 g_E)^2$$

if we treat both sides as operators on $TM$. 
Substituting it back into equation (5.39), we find that the derivative in equation (5.39) vanish. Therefore we have
\[
\left( \int_0^t \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, \epsilon) \, d\epsilon \right) = \frac{1}{2} \log(\det(\nabla^2 g_E)(u, t)) - \frac{1}{2} \log(\det(\nabla^2 g_E)(u, 0)),
\]
which is the equation we needed. □

Therefore, we complete the proof of lemma 29 and 33 which are needed in the proof of our Main Theorem in section 3.

6. Conclusion

From the semi-classical analysis of the Witten twisted Green’s operator in section 4 and 5 we obtain our main theorem ?? which can be viewed as an enhancement of the original Witten deformation of deRham complex, concerning cohomology of the manifold $M$, to one concerning its rational homotopy type by incorporating wedge product structures. In [5], Fukaya proposed a differential geometric approach to the Strominger-Yau-Zaslow (SYZ) by relating A-model holomorphic disks instantons of a Calabi-Yau manifold equipped with Lagrangian torus fibration, to certain Witten twisted differential constructed from the symplectic structure. Proving theorem ?? provides essential analytical technique for such an approach. For instance, the semi-classical analysis of Witten twisted Green’s operator, can be applied to obtain a beautiful geometric interpretation of the complicated scattering diagram in [2].

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