SCATTERING PHENOMENON IN SYMPLECTIC GEOMETRY

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Abstract. For any semi-flat symplectic Calabi-Yau manifold X_0 equipped with a Lagrangian torus fibration $p : X_0 \to B_0$, we introduce a differential graded Lie algebra (dgLa) which should govern "quantum" deformations of the symplectic structure on X_0. Given two non-parallel walls (equipped with wall-crossing factors) in the base B_0 intersecting transversally as inputs, we solve the Maurer-Cartan equation of the dgLa, and prove that the leading order terms of the solutions give rise to scattering diagrams, which have played a crucial role in important works of Fukaya, Kontsevich-Soibelman, and Gross-Siebert on the reconstruction problem in mirror symmetry. This realizes a key step in Fukaya's program on the understanding of quantum corrections in symplectic geometry.

Contents

1. Introduction

The celebrated Strominger-Yau-Zaslow (SYZ) conjecture [15] asserts that mirror symmetry is a T-duality. This gives a concrete description of the mirror of a Calabi-Yau manifold X as the fiberwise dual of a (special) Lagrangian torus fibration $p : X \to B$, leading to a beautiful geometric explanation for mirror symmetry. Strictly speaking, such a construction can be carried out directly only in the semi-flat case [12]; in general, due to the presence of singular fibers, non-trivial quantum corrections must be taken into account. The SYZ proposal suggests that such corrections are governed by holomorphic disks in X with boundary on the fibers of p. Intuitively, holomorphic disks can be glued to form holomorphic spheres, thus explaining why mirror symmetry is so powerful in making enumerative predictions.

The precise mechanism of how this works was first depicted by Fukaya in his program [8] on understanding quantum corrections in symplectic geometry. He described how quantum corrections arise near the large volume limit given by scaling of a symplectic structure on X, and put forward a conjecture claiming that holomorphic disks in X with boundary on fibers of p would collapse to Morse flow trees on the base B emanating from the locus of singular fibers $B^{sing} \subset B$. He also argued that after correcting by these data, the complex structure on the mirror would be extendable over the singular fibers – this is the so-called *reconstruction problem* which is a key step in understanding the geometry of mirror symmetry. Unfortunately, his arguments were only heuristical and the analysis involved to make them precise seemed intractable.

In [11], Kontsevich and Soibelman got around the analytical difficulty in Fukaya's approach by replacing holomorphic disks with tropical data on the base B and considering rigid analytic manifolds instead of complex manifolds. Starting from a 2-dimensional integral affine manifold B with 24 singular points of focus-focus type, they constructed a rigid analytic K3 surface by gluing together standard local pieces via automorphisms (or wall-crossing factors) attached to the tropical lines (or walls) on B. In [10], Gross and Siebert finally succeeded in constructing a degeneration of Calabi-Yau manifolds from an integral affine manifold with singularities in any dimension. This is one of the most important results in the Gross-Siebert program (which recasts the SYZ proposal in an algebro-geometric setting) and mirror symmetry in general.

During all these developments, a key role has been played by *scattering diagrams*, which first appeared as certain combinatorial objects arising from consistency conditions for automorphisms used to glue the local standard pieces when tropical lines collide and interact [11, 10]. The primary goal of this paper is to explain how scattering phenomena naturally occur in the study of quantum corrections on the symplectic side, or more precisely, in solving the Maurer-Cartan equation of the "quantum" deformation dgLa of symplectic structures, using semi-classical techniques motivated by Witten-Morse theory [16]. As Fukaya [8] has anticipated, holomorphic disks should degenerate to gradient flow lines of certain multi-valued Morse functions on the base when one goes near the large volume limit, and scattering diagrams, regarded as gradient flow trees, would describe how new holomorphic disks are being produced by gluing two holomorphic disks with a pair-of-pants. So understanding scattering phenomena is vital to the study of quantum corrections in symplectic geometry.

Let X be a Calabi-Yau manifold (regarded as a symplectic manifold), equipped with a Lagrangian torus fibration

$$\begin{tikzcd} (X,\omega,J) \arrow[r, "p"'] & B, \arrow[l, bend right, "\mathtt{s}"'] \end{tikzcd}$$

which admits a Lagrangian section $\mathtt{s}$, and whose discriminant locus is given by B^{sing}, where the affine structure develops singularities. If X degenerates to a large volume limit so that all singular fibers are pushed towards infinity, we obtain a semi-flat Calabi-Yau manifold $X_0 \hookrightarrow X$ which is a torus bundle over the smooth locus $B_0 = B \setminus B^{sing}$.

Now classical deformations of the symplectic structure on X_0 (0-th Fourier mode), which are rather trivial, are captured by the dgLa

$$(\Omega^*(B_0, TB_0), \nabla, [\cdot,\cdot])$$

on B_0. In order to recover the symplectic structures of X near the large volume limit, we need "quantum" deformations of the symplectic structure on X_0 (higher Fourier modes). There is as yet no definition of such deformations, but, in view of the SYZ conjecture, they should be determined by holomorphic disks in X with boundary on fibers of p and interior intersections with the singular fibers (at the singular points).

Motivated by Fukaya's ideas [8], and the relation between Witten-Morse theory and de Rham theory investigated in [6, 7], we define a differential graded Lie algebra (dgLa) as follows. First of all, let $\mathcal{M}$ be the space of fiberwise geodesic loops of the torus bundle $p : X_0 \to B_0$. We consider the complex

$$L_{X_0} = \Omega^*(\mathcal{M}, T\mathcal{M}^{\mathbb{C}})$$

equipped with the Witten differential locally defined by

$$d_W = e^{-f_n}\nabla e^{f_n},$$

where f_n is a function on $\mathcal{M}$ (or a multi-valued function on B_0) which records the loop $n \in \pi_1(p^{-1}(x), \mathtt{s}_0(x))$ that may shrink to a singular point in B and hence bound a holomorphic disk in X. Together with a natural Lie bracket $\{\cdot,\cdot\}$ (defined by combining the usual bracket on $T_{\mathcal{M}}$ with a convolution product), the triple

$$(L_{X_0}, d_W, \{\cdot,\cdot\})$$

forms a dgLa (see Proposition 2.13).

In view of the fact that the "0-th mode" of this dgLa is nothing but $(\Omega^*(B_0, TB_0), \nabla, [\cdot,\cdot])$ which captures the classical deformations of the symplectic structure on X_0, we believe that $(L_{X_0}, d_W, \{\cdot,\cdot\})$ captures the "quantum" deformations of the symplectic structure on X_0. Another evidence is provided by the fact that the Fourier (or SYZ) transform of this dgLa is precisely the Kodaira-Spencer dgLa on the mirror side (Proposition 2.15).

The idea that there should be Fourier-type transforms responsible for the interchange between symplectic-geometric data on one side and complex-geometric data on the mirror side can be traced back to the original SYZ paper [15], and this has already been applied successfully to the study of mirror symmetry for toric Fano manifolds [4, 5] and toric Calabi-Yau manifolds (and orbifolds) [3, 2] etc. Nevertheless, there is no scattering phenomenon in those cases. The main result of this paper shows that scattering diagrams naturally appear which one consider the asymptotic expansions of Maurer-Cartan elements of the dgLa $(L_{X_0}, d_W, \{\cdot,\cdot\})$ which governs quantum symplectic deformations, thereby realizing a key step in Fukaya's original program.

We work outside from the singular loci and also restrict ourselves to the 2-dimensional case (as in Fukaya's paper), so that we may assume that $B_0 = \mathbb{R}^2$. In this case, a scattering diagram can be viewed schematically as the process of creating new walls from two non-parallel walls supported on tropical lines in B_0 which intersect transversally. The combinatorics of this process is controlled by the algebra of the tropical vertex group [11, 9]. We will give a brief review of the definitions and notations in Section 3, following [9].

Suppose that we are given a wall $\mathbf{w}$ supported on a tropical line in B_0 and equipped with a wall-crossing factor $\check{\phi}$, which is an element in the tropical vertex group. From the Witten-Morse theory developed in [7], the shrinking of a fiberwise loop n towards the singular loci indicates the presence of a critical point of f_n in the singular loci, and the union of gradient flow lines emanating from the singular loci should be interpreted as a stable submanifold associated to the critical point. Furthermore, this codimension one stable submanifold should correspond to a differential 1-form concentrating on $\mathbf{w}$ (see [7]).

In view of these, corresponding to the wall $(\mathbf{w}, \check{\phi})$, we write down an ansatz $\check{\Xi}_{\mathbf{w}} \in L^1_{X_0}$ for a solution to the Maurer-Cartan equation of the dgLa $(L_{X_0}, d_W, \{\cdot,\cdot\})$ over B_0 (Section 4.1), and show by semi-classical analysis that it (or, more precisely, the corresponding gauge transformation) determines the wall-crossing factor $\check{\phi}$ by letting $\hbar \to 0$ (Proposition 4.20). The details are contained in Section 4.

The heart of this paper is Sections 5, where we study the case when two non-parallel walls $\mathbf{w}_1$, $\mathbf{w}_2$ supported on tropical lines in $B_0 = \mathbb{R}^2$ intersect transversally. In this case the sum $\check{\Xi} := \check{\Xi}_{\mathbf{w}_1} + \check{\Xi}_{\mathbf{w}_2} \in L^1_{X_0}$ does *not* solve the Maurer-Cartan equation of $(L_{X_0}, d_W, \{\cdot,\cdot\})$. Instead, by fixing the gauge using the homotopy operator in Definition 4.8, a solution Φ can be obtained by solving equation (5.2), and it is written as a sum over directed trivalent planar trees as in Definition 5.9 with input $\check{\Xi}$.

In general, to an element $\Psi \in L^1_{X_0}$ satisfying certain suitable assumptions on its asymptotic expansion in $\hbar$ (namely, having asymptotic support on an increasing set of subsets of rays $\{Ray(N_0)\}_{N_0 \in \mathbb{Z}_{>0}}$ as defined in Definition 5.19), one can associate a scattering diagram $\mathcal{D}(\Psi)$. One of the key observations of this paper is the following theorem which will be proved in Section 5.3.

Theorem 1.1. *If Ψ is any solution to the Maurer-Cartan equation* (2.2) *having asymptotic support on $\{Ray(N_0)\}_{N_0 \in \mathbb{Z}_{>0}}$ as in Definition 5.19, then the associated scattering diagram $\mathcal{D}(\Psi)$ is monodromy free.*

We analyze the above Maurer-Cartan element Φ, obtained by summing over trees as in Definition 5.9, again by semi-classical analysis and a careful estimate on the orders of the parameter $\hbar$ in its asymptotic expansion. Lemmas 5.10, 5.16 and 5.17 in Section 5.2 together say that Φ has asymptotic support on some set $\{Ray(N_0)\}_{N_0 \in \mathbb{Z}_{>0}}$ of subsets of rays; the proofs of these lemmas, which involve lengthy analysis on the asymptotic expansion of Φ, occupy the whole Section 5.4. Combining with Theorem 1.1 above, we arrive at the main result of this paper:

Theorem 1.2. *The Maurer-Cartan element Φ, obtained by summing over trees as in Definition 5.9, has asymptotic support on some increasing set of subsets of rays $\{Ray(N_0)\}_{N_0 \in \mathbb{Z}_{>0}}$. Furthermore, the associated scattering diagram $\mathcal{D}(\Phi)$ gives the unique (by passing to a minimal scattering diagram if necessary) monodromy free extension, determined by Kontsevich-Soibelman's Theorem 3.11, of the diagram consisting of two walls $\mathbf{w}_1$ and $\mathbf{w}_2$.*

See Section 5.3 for the details. The moral is that tropicalization should be equivalent to taking leading order terms of the asymptotic expansion of an analytic structure; in our case, a scattering diagram is on the one hand the tropicalization of holomorphic disks, and on the other hand the leading order terms of the asymptotic expansion of a Maurer-Cartan element.

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2. Fourier transform of deformation theories

2.1. General theory of dgLa's.

In this section, we review some basic definitions and properties of dgLa, and their deformation theory via Maurer-Cartan equations. We follow [13] and give a brief discussion on the parts we need. We will work over $\mathbb{C}$.

Definition 2.1. *A* differential graded Lie algebra (dgLa) *is a triple $(L, d, [\cdot,\cdot])$ where $L = \bigoplus_{i\in\mathbb{Z}} L^i$ is a graded vector space (we denote by $\bar{a}$ the degree of a homogeneous element a), $d : L \to L$ is a degree 1 differential satisfying $d^2 = 0$ and $[\cdot,\cdot] : L \times L \to L$ is a graded Lie bracket such that d and $[\cdot,\cdot]$ are compatible. More precisely, we require the following conditions to hold:*

(1) $[\cdot,\cdot]$ *is* homogeneous, *i.e.* $[L^i, L^j] \subset L^{i+j}$.
(2) $[\cdot,\cdot]$ *is* graded skewsymmetric, *i.e.*

$$[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$$

for any homogeneous elements a and b.
(3) $[\cdot,\cdot]$ *satisfies the* graded Jacobi identity, *meaning that*

$$[a, [b, c]] + (-1)^{\bar{a}\bar{b}+\bar{a}\bar{c}}[b, [c, a]] + (-1)^{\bar{a}\bar{c}+\bar{b}\bar{c}}[c, [a, b]] = 0$$

for any homogeneous elements a, b, c.
(4) $d(L^i) \subset L^{i+1}$ *and*

$$d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$$

for any homogeneous element a.

Definition 2.2. *A* morphism *between two dgLa's* $(L, d, [\cdot,\cdot])$ *and* $(M, d, [\cdot,\cdot])$ *is a homomorphism* $f : L^* \to M^*$ *between graded vector spaces such that* $f \circ d = d \circ f$ *and* $[\cdot,\cdot] = [f(\cdot), f(\cdot)]$. *We call* f *a* quasi-isomorphism *if it induces an isomorphism on cohomology.*

Notations 2.3. *We let* $R = \mathbb{C}[[t_1, \ldots, t_k]]$ *denote the ring of formal power series and* $\mathbf{m} = (t_1, \ldots t_k)$ *denote the maximal ideal generated by the formal variables. We consider only dgLas over formal power series ring* R *to avoid convergence issue.*

Given an element a in $L^0 \otimes_{\mathbb{C}} \mathbf{m}$, we have an isomorphism of $L \otimes_{\mathbb{C}} R$ (as dgLa over R) given by

$$e^a = \sum_{n\geq 0} \frac{ad_a^n}{n!},$$

where $ad_a(b) = [a, b]$. This defines the *exponential group*

$$\exp(L^0 \otimes \mathbf{m}) = \left\{ e^a \mid a \in L^0 \otimes \mathbf{m} \right\},$$

which acts on $L \otimes_{\mathbb{C}} R$ naturally.

Remark 2.4. *We have an explicit formula for the product* $e^a \cdot e^b = e^{a \bullet b}$, *where the product*

$$a \bullet b = a + b + \frac{1}{2}[a, b] + \ldots \tag{2.1}$$

is defined by the Baker-Campbell-Hausdorff formula*.*

We can deform a dgLa L formally over R by deforming the differential using an element $\xi \in L^1 \otimes \mathbf{m}$. The deformed differential $d_\xi = d + ad_\xi$ satisfies $d_\xi^2 = 0$ if and only if ξ satisfies the *Maurer-Cartan equation*

$$d\xi + \frac{1}{2}[\xi, \xi] = 0. \tag{2.2}$$

This suggests the following definition.

Definition 2.5. *The* Maurer-Cartan elements *of* L *over* R *is the set*

$$MC_L(R) = \left\{ \xi \in L^1 \otimes \mathbf{m} \mid d\xi + \frac{1}{2}[\xi, \xi] = 0 \right\}.$$

The set $MC_L(R)$ does not give the space of deformations since two deformed differentials $d + \xi$ and $d + \eta$ may be related by a gauge equivalence given by some $e^a \in \exp(L^0 \otimes \mathbf{m})$.

Definition 2.6. *We define the* gauge action *by the formal expansion*

$$\begin{aligned} e^a * \xi &= \sum_{n\geq 0} \frac{ad_a^n}{n!}\xi - \sum_{n\geq 0} \frac{ad_a^n}{(n+1)!} da \\ &= \xi - \sum_{n\geq 0} \frac{ad_a^n}{(n+1)!}(da + [\xi, a]), \end{aligned}$$

where $\xi \in MC_L(R)$ *and* $e^a \in \exp(L^0 \otimes \mathbf{m})$.

We see that $e^a * \xi = \xi$ if and only if $da + [\xi, a] = 0$ because

$$\sum_{n\geq 0} \frac{ad_a^n}{(n+1)!} = \frac{e^{ad_a} - I}{ad_a}$$

is invertible.

Remark 2.7. *We can also derive the gauge action from the adjoint action of a new dgLa $(L', d', [\cdot,\cdot]')$ constructed from L by setting*

$$(L')^i = \begin{cases} L^i & \text{for } i \neq 1, \\ L^1 \oplus \mathbb{C}\langle d \rangle & \text{for } i = 1, \end{cases}$$

where d is a formal symbol of degree 1. L' is equipped with the differential given by

$$d'(a + vd) = da,$$

and the Lie bracket given by

$$[a + vd, b + wd]' = [a, b] + vd(b) + (-1)^{\bar{a}} wd(a).$$

There is a natural inclusion $\iota : L \subset L'$ given by $\iota_{\pm}(x) = x \pm d$. Notice that the image of ι is stable under the adjoint action and we have the relation

$$e^a * \xi = \iota^{-1}(e^a \iota(\xi)) = e^a(\xi + d) - d.$$

Under the identification ι, the Maurer-Cartan equation $d\xi + \frac{1}{2}[\xi, \xi] = 0$ is equivalent to the condition $[\iota_+(\xi), \iota_-(\xi)]' = 0$.

Definition 2.8. *The* space of deformations *of L over R is defined by*

$$Def_L(R) = MC_L(R)/\exp(L^0 \otimes \mathbf{m}).$$

2.2. **Deformations of complex structures.** In this section, we review the deformation theory of complex structures via the Kodaira-Spencer dgLa.

Let $\check{X}_0$ be a complex manifold. The Kodaira-Spencer complex is defined by

$$KS_{\check{X}_0} = \Omega^{0,*}(\check{X}_0, T^{1,0}_{\check{X}_0}),$$

equipped with the differential $\bar{\partial}$. The Lie bracket structure is defined in local holomorphic coordinates $z_1, \ldots, z_n \in \check{X}_0$ by

$$[\phi d\bar{z}^I, \psi d\bar{z}^J] = [\phi, \psi] d\bar{z}^I \wedge d\bar{z}^J,$$

where $\phi, \psi \in \Gamma(T^{1,0}_{\check{X}_0})$. This is globally defined and gives a dgLa structure on $KS_{\check{X}_0}$.

An element $\xi \in \Omega^{0,1}(\check{X}_0, T^{1,0}_{\check{X}_0}) \otimes \mathbf{m}$ can be used to define a formal deformation of almost complex structures by letting

$$\tilde{T}^{0,1} = graph(\xi) = \left\{ u + u \lrcorner \xi \mid u \in T^{0,1}_{\check{X}_0} \right\}.$$

In local holomorphic coordinates $z_1, \ldots, z_n$ on $\check{X}_0$, $\tilde{T}^{0,1}$ is spanned by the frame

$$\left\{ \frac{\partial}{\partial \bar{z}^j} + \sum_k \xi_j^k \frac{\partial}{\partial z^k} \right\}_{j=1}^{n},$$

where $\xi = \sum_{j,k} \xi_j^k \frac{\partial}{\partial z^k} \otimes d\bar{z}^j$. An important fact is the following.

Proposition 2.9. $[\tilde{T}^{0,1}, \tilde{T}^{0,1}] \subset \tilde{T}^{0,1}$, *i.e. the almost complex structure defined by* ξ *is integrable if and only if* ξ *satisfies the Maurer-Cartan equation* $\bar{\partial}\xi + \frac{1}{2}[\xi, \xi] = 0$.

The exponential group $KS^0_{\check{X}_0} \otimes \mathbf{m}$ acting on $MC_{KS_{\check{X}_0}}(R)$ can be regarded as automorphism of the formal family of complex structures over R, so we make the following definition.

Definition 2.10. *The* deformations of complex structures *of* $\check{X}_0$ *over* R *is defined to be the space of deformations* $Def_{KS_{\check{X}_0}}(R)$ *of the dgLa* $KS_{\check{X}_0}$.

2.3. **A dgLa on the symplectic side.** We now turn to deformation theory on the A-model side. We restrict our attention to the deformation theory of a (non-compact) *semi-flat* symplectic manifold $(X_0, \omega + i\beta)$, which is constructed from an tropical affine manifold B_0 as follows.

We follow the definitions of affine manifolds in [1, Chapter 6]. Let

$$Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$$

be the group of affine transformation of $\mathbb{R}^n$, which is a map T of the form $T(x) = Ax + b$ with $A \in GL_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. We are particularly interested in the following subgroup of affine transformation

$$Aff_{\mathbb{R}}(\mathbb{Z}^n)_0 = \mathbb{R}^n \rtimes SL_n(\mathbb{Z}).$$

Definition 2.11. *An* n*-dimensional manifold* B *is called* tropical affine *if it admits an atlas* $\{(U_i, \psi_i)\}$ *of coordinate charts* $\psi_i : U_i \to \mathbb{R}^n$ *such that* $\psi_i \circ \psi_j^{-1} \in Aff_{\mathbb{R}}(\mathbb{Z}^n)_0$.

We consider the cotangent bundle T^*B_0, equipped with the canonical symplectic form $\omega_{can} = \sum_i dy_i \wedge dx^i$ where x^i's are affine coordinates on B_0 and y_i's are coordinates of the cotangent fibers with respect to the basis $dx^1, \dots, dx^n$. There is a lattice subbundle $\Lambda^* \leq T^*B_0$ generated by the covectors $dx^1, \dots, dx^n$. It is well defined since the transition functions lie in $Aff_{\mathbb{R}}(\mathbb{Z}^n)$. We put

$$X_0 = T^*B_0/\Lambda^*, \tag{2.3}$$

equipped with the symplectic form

$$\omega = \hbar^{-1} \sum_j dy_j \wedge dx^j$$

descended from $\hbar^{-1}\omega_{can}$. The natural projection map $p : X_0 \to B_0$ is a Lagrangian torus fibration. We can further consider *B-field* enriched symplectic structure $\omega + i\beta$ by a closed form $\beta = \sum_{i,j} \beta_i^j(x) dy_j \wedge dx^i$.

Now, let $\mathcal{M}$ be the space of fiberwise homotopy classes of loops with respect to the fibration $p : X_0 \to B_0$, i.e.

$$\mathcal{M} = \coprod_{x \in B_0} \pi_1(p^{-1}(x)).$$

There is a natural projection map $pr : \mathcal{M} \to B_0$ which is a local diffeomorphism. We will use $TB_0^{\mathbb{C}}$ to stand for the pullback of $TB_0^{\mathbb{C}}$ to $\mathcal{M}$ as well if there is no confusion. We consider the complex

$$L_{X_0} = \Omega^*(\mathcal{M}, TB_0^{\mathbb{C}}),$$

equipped with the *Witten differential*

$$d_W = \nabla - 2\pi s \lrcorner (\omega + i\beta).$$

Here ∇ is the pullback of the flat connection on TB_0, and $s([\gamma]) = \dot{\gamma} \in T(X_0/B_0)_{\gamma(0)}$ where γ is the unique affine loop in its homotopy class $[\gamma]$. For $X, Y \in \Gamma(\mathcal{M}, TB_0^{\mathbb{C}})$, we define the Lie bracket to be

$$\begin{aligned}\{X, Y\}_\gamma = \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ pr(\gamma_i)=pr(\gamma)}} & (\nabla_{X_{\gamma_1}} Y)_{\gamma_2} - (\nabla_{Y_{\gamma_2}} X)_{\gamma_1} \\ & + 2\pi \left((\omega - i\beta)(s(\gamma_2), X_{\gamma_1}) Y_{\gamma_2} - (\omega - i\beta)(s(\gamma_1), Y_{\gamma_2}) X_{\gamma_1} \right),\end{aligned}$$

at a point $\gamma \in \mathcal{M}$. It is then extended naturally to L_{X_0} to give a Lie bracket structure.

Remark 2.12. *Both the Witten differential and Lie bracket can be written in a more explicit form in local coordinates. Let $U \subset B_0$ be a contractible open set with local coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$ for $p^{-1}(U)$. Then we can parametrize $pr^{-1}(U) \cong U \times \mathbb{Z}^n$ by (x, m) where $x \in U$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ representing an affine loop in the fiber $p^{-1}(x)$ with tangent vector $\sum_j m_j \frac{\partial}{\partial y_j}$. We denote the copy $U \times \{m\} \subset pr^{-1}(U)$ by U_m and the section s, when restricted to the copy U_m, by $s_m = \sum_{j=1}^n m_j \frac{\partial}{\partial y_j}$. Fixing a point $x_0 \in U$, we define a function $f_m = -2\pi \int_{x_0}^x s_m \lrcorner (\omega + i\beta)$ satisfying $df_m = -2\pi s_m \lrcorner (\omega + i\beta)$.*

We have a natural identification $\Omega^*(U_m, TB_0^{\mathbb{C}}) \cong \Omega^*(U, TB_0^{\mathbb{C}})$ via the projection pr and the relation

$$d_W = e^{-f_m} \nabla e^{f_m},$$

on $\Omega^*(U_m, TB_0^{\mathbb{C}})$ via the identification. For $X \in \Gamma(U_k, TB_0^{\mathbb{C}})$ and $Y \in \Gamma(U_l, TB_0^{\mathbb{C}})$, we have

$$\{X, Y\}_m = \begin{cases} e^{\bar{f}_{k+l}}[e^{-\bar{f}_k} X, e^{-\bar{f}_l} Y] & \text{for } m = k + l, \\ 0 & \text{for } m \neq k + l \end{cases}$$

via the identification, where $[\cdot, \cdot]$ is the usual Lie bracket on $\Gamma(U, TB_0^{\mathbb{C}})$.

Proposition 2.13. *$(L_{X_0}, d_W, \{\cdot, \cdot\})$ is a dgLa.*

Proof. It follows from its definition that $\{\cdot, \cdot\}$ is both homogeneous and graded skewsymmetric.

Now given vector fields $X, Y, Z \in \Gamma(pr^{-1}(U), TB_0^{\mathbb{C}})$ over a contractible open subset $U \subset B_0$, we have

$$\{\{X_k, Y_l\}, Z_m\} = e^{\bar{f}_{k+l+m}}[[e^{-\bar{f}_k} X_k, e^{-\bar{f}_l} Y_l], e^{-\bar{f}_m} Z_m]$$

by the usual one for Lie bracket on $\Gamma(U, TB_0^{\mathbb{C}})$, under the identification in Remark 2.12. The graded Jacobi identity follows.

For the compatibility between d_W and $\{\cdot,\cdot\}$, we take $\alpha \in \Omega^*(U_k, TB_0^{\mathbb{C}})$ and $\psi \in \Omega^*(U_l, TB_0^{\mathbb{C}})$. By direct computations, we have

$$\begin{aligned} d_W\{\alpha,\psi\} &= \left(e^{-f_{k+l}}\nabla e^{f_{k+l}}\right) e^{\bar f_{k+l}}[e^{-\bar f_k}\alpha, e^{-\bar f_l}\psi] \\ &= e^{\bar f_{k+l}}\left([\nabla(e^{-\bar f_k}\alpha), e^{-\bar f_l}\psi] + (-1)^{\bar\alpha}[e^{-\bar f_k}\alpha, \nabla(e^{-\bar f_l}\psi)]\right) \\ &\quad +2Re(df_{k+l})\wedge e^{\bar f_{k+l}}[e^{-\bar f_k}\alpha, e^{-\bar f_l}\psi] \\ &= e^{\bar f_{k+l}}\left([e^{-f_k-\bar f_k}\nabla(e^{f_k}\alpha), e^{-\bar f_l}\psi] + (-1)^{\bar\alpha}[e^{-\bar f_k}\alpha, e^{-f_l-\bar f_l}\nabla(e^{f_l}\psi)]\right) \\ &= \left(\{d_W\alpha,\psi\} + (-1)^{\bar\alpha}\{\alpha, d_W\psi\}\right). \end{aligned}$$

□

2.4. **Fourier transform of dgLa's.** Given a tropical affine manifold B_0 together with β, we have constructed a semi-flat symplectic manifold $(X_0, \omega + i\beta)$ in Section 2.3. We now consider the tangent bundle TB_0, equipped with the complex structure where the local complex coordinates are given by $y^j - \beta^j_k x^k + i\frac{x^j}{\hbar}$. Here y^j's are coordinates of the tangent fibers with respect to the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$, i.e. they are coordinates dual to $y_j's$ on T_0^B. The condition that $\beta^j_i(x)dx^i$ being closed, for each $j = 1, \dots, n$, is equivalent to integrability of the complex structure.

There is a well defined lattice subbundle $\Lambda \le TB_0$ generated locally by $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$. We set

$$\check{X}_0 = TB_0/\Lambda,$$

equipped with the complex structure $\check{J}$ descended from that of TB_0, so that the local complex coordinates can be written as $w^j = e^{-2\pi i(y^j - \beta^j_k x^k + i\frac{x^j}{\hbar})}$. The natural projection map $p : \check{X}_0 \to B_0$ is a torus fibration.

So we have dual torus fibrations:

$$\begin{tikzcd} X_0 \arrow[dr, "p"'] & & \check{X}_0 \arrow[dl, "\check{p}"] \\ & B_0 & \end{tikzcd}$$

and the mirror symmetry between the semi-flat pair X_0 and $\check{X}_0$ is a well-understood example of the SYZ construction [12] (cf. [1, Chapter 6]). The goal of this section is to relate the dgLa's $KS_{\check{X}_0}$ and L_{X_0} via a Fourier transform.

We define the Fourier transform

$$\hat{\mathcal{F}} : \Omega^{0,*}(\check{X}_0, T^{1,0}\check{X}_0) \to \Omega^*(\mathcal{M}, TB_0^{\mathbb{C}}), \tag{2.4}$$

as follows. We consider the fiber product $\mathcal{M} \times_{B_0} \check{X}_0$ and the projection maps

$$\begin{tikzcd} & \mathcal{M} \times_{B_0} \check{X}_0 \arrow[dl, "\pi"'] \arrow[dr, "\check{\pi}"] & \\ X_0 & & \check{X}_0. \end{tikzcd}$$

In a contractible open subset $U \subset B_0$ with coordinates as in Remark 2.12, we take the local holomorphic frame

$$\left\{ \frac{\partial}{\partial \log w^j} = \frac{i}{4\pi} \left(\frac{\partial}{\partial y^j} - i\hbar (\sum_k \beta_j^k \frac{\partial}{\partial y^k} + \frac{\partial}{\partial x^j}) \right) \right\}_{j=1}^n$$

of $T^{1,0}_{\check{X}_0}$. There is a natural identification $\check{p}^* TB_0^{\mathbb{C}} \cong T^{1,0}\check{X}_0$ given explicitly by

$$\frac{\partial}{\partial \log w^j} \mapsto \frac{\hbar}{4\pi} \frac{\partial}{\partial x^j}.$$

Given any $\alpha = \sum_k \sum_{j_1,\dots,j_m} \alpha^k_{j_1,\dots,j_m} d\log \bar{w}^{j_1} \dots d\log \bar{w}^{j_m} \otimes \frac{\partial}{\partial \log w^k}$, we define

$$\hat{\mathcal{F}}(\alpha) = (4\pi)^m \hbar^{-m} \sum_k \sum_{j_1,\dots,j_m} \int_\pi (\alpha^k_{j_1,\dots,j_m} e^{2\pi i (n,\check{y})}) dx^{j_1} \dots dx^{j_m} \otimes (\frac{\hbar}{4\pi} \frac{\partial}{\partial x^k}), \tag{2.5}$$

where $\check{y} = (y^1, \dots, y^n)$ and $\int_\pi$ denote the integration along fibers of π.

Remark 2.14. *The zero section of the fibration $p : X_0 \to B_0$ is a Lagrangian L_0, and our fiberwise loop space $\mathcal{M}$ is actually the fiberwise path space $M(L_0, L_0)$ introduced in* [6]. *The Fourier transform defined in* [6] *gives a transform*

$$\hat{\mathcal{F}} : \Omega^{0,*}(\check{X}_0) \to \Omega^*(\mathcal{M}). \tag{2.6}$$

Together with the natural isomorphism $T^{1,0}_{\check{X}_0} \to \check{p}^* TB_0^{\mathbb{C}}$ which is in fact given by composing the embedding $T^{1,0}_{\check{X}_0} \hookrightarrow T\check{X}_0^{\mathbb{C}}$ with the projection $T\check{X}_0^{\mathbb{C}} \twoheadrightarrow \check{p}^* TB_0^{\mathbb{C}}$, we get the Fourier transform introduced in (2.4).

Proposition 2.15. *The Fourier transform $\hat{\mathcal{F}} : KS_{\check{X}_0} \to L_{X_0}$ is an isomorphism of dgLa.*

Proof. By properties of Fourier series and direct computations using the local expression (2.4) for $\hat{\mathcal{F}}$, it is easy to see that $\hat{\mathcal{F}}(\bar{\partial}) = d_W$.

We then compare the Lie brackets on both sides. Without loss of generality, we will only prove this for 0-forms. Given $X = \sum_k X^k \frac{\partial}{\partial \log w^k}$ and $Y = \sum_l Y^l \frac{\partial}{\partial \log w^l}$, we have

$$\begin{aligned} \hat{\mathcal{F}}([X,Y]) &= \sum_{k,l} \hat{\mathcal{F}} \left(X^k \frac{\partial Y^l}{\partial \log w^k} \frac{\partial}{\partial \log w^l} - Y^l \frac{\partial X^k}{\partial \log w^l} \frac{\partial}{\partial \log w^k} \right) \\ &= \frac{\hbar}{4\pi} \sum_{k,l} \left(\hat{\mathcal{F}}(X^k) * \hat{\mathcal{F}}(\frac{\partial Y^l}{\partial \log w^k}) \frac{\partial}{\partial x^l} - \hat{\mathcal{F}}(Y^l) * \hat{\mathcal{F}}(\frac{\partial X^k}{\partial \log w^l}) \frac{\partial}{\partial x^k} \right). \end{aligned}$$

Here we have abused notations and used $\hat{\mathcal{F}}$ to stand for both Fourier transforms (2.4) and (2.6). We use the parametrization $U \times \mathbb{Z}^n$ for $pr^{-1}(U)$ as before. We write $\hat{X}^k_n \in \Omega^0(U_n)$ for the component of $\hat{\mathcal{F}}(X^k)$ on U_n, and similarly for Y^l and their differentials. We see that

$$\begin{aligned}
(\widehat{\frac{\partial Y^l}{\partial \log w^k}})_n &= \frac{1}{2}\left(n_k - i\hbar \sum_j \beta^j_k n_j\right) + \frac{\hbar}{4\pi}\frac{\partial \hat{Y}^l}{\partial x^k} \\
&= \frac{\hbar}{4\pi}\frac{\partial \hat{Y}^l}{\partial x^k} + 2\pi(\omega - i\beta)(s_n, \frac{\hbar}{4\pi}\frac{\partial}{\partial x^k}) \\
&= \nabla_{\frac{\hbar}{4\pi}\frac{\partial}{\partial x^k}} \hat{Y}^l + 2\pi(\omega - i\beta)(s_n, \frac{\hbar}{4\pi}\frac{\partial}{\partial x^k}) \\
&= e^{\bar{f}_n} \nabla_{\frac{\hbar}{4\pi}\frac{\partial}{\partial x^k}} (e^{-\bar{f}_n}\hat{Y}^l).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\hat{\mathcal{F}}([X,Y])_n &= \sum_{n_1+n_2=n} \sum_{k,l} \Bigg(\hat{X}^k_{n_1} e^{\bar{f}_{n_2}} \nabla_{\frac{\hbar}{4\pi}\frac{\partial}{\partial x^k}} \left(e^{-\bar{f}_{n_2}} (\frac{\hbar}{4\pi}\hat{Y}^l_{n_2}\frac{\partial}{\partial x^l}) \right) \\
&\qquad - \hat{Y}^l_{n_2} e^{\bar{f}_{n_1}} \nabla_{\frac{\hbar}{4\pi}\frac{\partial}{\partial x^l}} \left(e^{-\bar{f}_{n_1}} (\frac{\hbar}{4\pi}\hat{X}^k_{n_1}\frac{\partial}{\partial x^k}) \right) \Bigg) \\
&= \left\{ \hat{\mathcal{F}}(X), \hat{\mathcal{F}}(Y) \right\}_n .
\end{aligned}$$

These show that $\hat{\mathcal{F}}$ is a homomorphism of dgLa. To show that it is an isomorphism, we can define inverse Fourier transform $\hat{\mathcal{F}}^{-1} : L_{X_0} \to KS_{\check{X}_0}$ in a similar fashion and check that $\hat{\mathcal{F}} \circ \hat{\mathcal{F}}^{-1} = \mathrm{id} = \hat{\mathcal{F}}^{-1} \circ \hat{\mathcal{F}}$. Details are left to readers. $\square$

Remark 2.16. *Mirror symmetry predicts that deformations of structures on the A- and B-sides should match with each other. However, the deformation theory of complex structures is nonlinear while the deformation theory of semi-flat symplectic structures is linear. This suggests that we should look for "quantum" deformations of symplectic structures on the A-side. The above proposition indicates that our dgLa L_{X_0} is governing such quantum deformations.*

For later purposes, we also want to give Kähler structures on X_0 and $\check{X}_0$ by considering a metric g on B_0 of Hessian type:

Definition 2.17. *An Riemannian metric $g = (g_{ij})_{i,j}$ on B_0 is said to be* Hessian type *if it is locally given by $g = \sum_{i,j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} dx^i \otimes dx^j$ in affine coordinates $x^1, \dots, x^n$ for some convex function ϕ.*

Assuming first B-field $\beta = 0$, a Hessian type metric g on B_0 induces a metric on T^*B_0 which also descends to X_0. In local coordinates, the metric on X_0 is of the form

$$g_{X_0} = \sum_{i,j} \hbar^{-1}(g_{ij} dx^i \otimes dx^j + g^{ij} dy_i \otimes dy_j), \tag{2.7}$$

where $(g^{ij})_{i,j}$ is the inverse matrix of $(g_{ij})_{i,j}$. The metric g_{X_0} is compatible with ω and gives a complex structure J on X_0 with complex coordinates represented by a matrix

$$J = \begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}$$

with respect to the frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$, having $dz_j = dy_j + i\sum_{k=1}^n g_{jk}dx^k$. Then we have a natural holomorphic volume form which is

$$\Omega = dz_1 \wedge \cdots \wedge dz_n = \bigwedge_{j=1}^n (dy_j + i\sum_{k=1}^n g_{jk}dx^k).$$

The Kähler manifold (X_0, ω, J) is a Calabi-Yau manifold if and only if the potential ϕ satisfies the real Monge-Ampére equation

$$det(\frac{\partial^2\phi}{\partial x^i \partial x^j}) = const. \tag{2.8}$$

In such a case, $p : X_0 \to B_0$ is a special Lagrangian torus fibration.

On the other hand, there is a Riemannian metric on $\check{X}_0$ induced from g given by

$$g_{\check{X}_0} = \sum_{i,j}(\hbar^{-1}g_{ij}dx^i \otimes dx^j + \hbar g_{ij}dy^i \otimes dy^j).$$

It is compatible with the complex structure and gives a symplectic form

$$\check{\omega} = 2i\partial\bar{\partial}\phi = \sum_{i,j} g_{ij}dy^i \wedge dx^j.$$

Similarly, The potential ϕ satisfies the real Monge-Ampére equation (2.8) if and only if $(\check{X}_0, \check{\omega}, \check{J})$ is a Calabi-Yau manifold.

In the presence of β, we need further compatibility condition between β and g to obtain a Kähler structure. On X_0, we treat $\omega + i\beta$ as a complexified Kähler class on X_0 and require that $\beta \in \Omega^{1,1}(X_0)$ with respect to the complex structure J. This is same as saying

$$\sum_{i,j,k} \beta_i^j g_{jk} dx^i \wedge dx^k = 0, \tag{2.9}$$

if $\beta = \sum_{i,j} \beta_i^j(x)dy_j \wedge dx^i$ in local coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$.

On $\check{X}_0$, we treat β as an endomorphism of TB_0 represented by a matrix (i,j)-entry is given by $\beta(x)_i^j$ with respect to the frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. The complex structure we introduced before can be written as

$$\check{J}_\beta = \begin{pmatrix} -\hbar\beta & \hbar I \\ -\hbar^{-1}(I + \hbar^2\beta^2) & \hbar\beta \end{pmatrix} \tag{2.10}$$

with respect to the frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$. The corresponding holomorphic volume form is given by

$$\check{\Omega}_\beta = \bigwedge_{j=1}^n ((dy^j - \sum_k \beta_k^j dx^k) + i\hbar^{-1}dx^j). \tag{2.11}$$

The extra assumption (2.9) will be equivalent to the compatibility of $\check{J}_\beta$ with $\check{\omega}$. If we treat (g_{ij}) as a square matrix, we have the symplectic structure $\check{\omega}$ represented by

$$\check{\omega} = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}. \tag{2.12}$$

The compatibility condition

$$\begin{pmatrix} -\hbar\beta & \hbar I \\ -\hbar^{-1}(I+\hbar^2\beta^2) & \hbar\beta \end{pmatrix}^T \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix} \begin{pmatrix} -\hbar\beta & \hbar I \\ -\hbar^{-1}(I+\hbar^2\beta^2) & \hbar\beta \end{pmatrix} = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}$$

in terms of matrices is equivalent to $\beta g = g\beta$, which is the matrix form of (2.9). The metric tensor is represented by the matrix

$$g_{\check{X}_0} = \begin{pmatrix} \hbar^{-1}g(I+\hbar^2\beta^2) & -\hbar g\beta \\ -\hbar g\beta & \hbar g \end{pmatrix},$$

whose inverse is given by

$$g^{-1}_{\check{X}_0} = \begin{pmatrix} \hbar g^{-1} & \hbar\beta g^{-1} \\ \hbar\beta g^{-1} & \hbar^{-1}(I+\hbar^2\beta^2)g^{-1} \end{pmatrix}.$$

If we equip the tangent bundle $TB_0^{\mathbb{C}}$ with the metric $\frac{1}{2}\hbar^{-1}g$, we can define an L_2 inner product $\langle\langle\cdot,\cdot\rangle\rangle_{L_{X_0}}$ using the volume form

$$\det(g_{ij})dx^1 \dots dx^n.$$

Equipping $KS_{\check{X}_0}$ with the standard L_2 inner product induced from the Kähler structure on $\check{X}_0$, we have the following proposition.

Proposition 2.18. *The Fourier transform $\hat{\mathcal{F}} : KS_{\check{X}_0} \to L_{X_0}$ preserve the L_2 inner products on the two complexes.*

3. Scattering diagrams

In the section, we recall the scattering process in dimension 2 described in [9, 11]. We will adopt the setting and notations from [9] with slight modifications to fit into our context.

3.1. Sheaf of tropical vertex group.

We first give the definition of a tropical vertex group, which is a slight modification of that from [9]. As before, let B_0 be a tropical affine manifold, equipped with a Hessian type metric g and a B-field β.

We first embed the lattice bundle $\Lambda \hookrightarrow \check{p}_* T^{1,0}_{\check{X}_0}$ into the sheaf of holomorphic vector fields. In local coordinates, it is given by

$$n = (n_j) \mapsto \check{\partial}_n = \sum_j n_j \frac{\partial}{\partial \log w^j} = \frac{i}{4\pi}\sum_j n_j \left(\frac{\partial}{\partial y^j} - i\hbar \left(\sum_k \beta^k_j \frac{\partial}{\partial y^k} + \frac{\partial}{\partial x^j} \right) \right).$$

The embedding is globally defined, and we write $T^{1,0}_{B_0,\mathbb{Z}}$ to stand for its image.

Given a tropical affine manifold B_0, we can talk about the sheaf of integral affine functions on B_0.

Definition 3.1. *The* sheaf of integral affine functions $Aff^{\mathbb{Z}}_{B_0}$ *is a subsheaf of continuous functions on B_0 such that $m \in Aff^{\mathbb{Z}}_{B_0}(U)$ if and only if m can be expressed as*

$$m(x) = a_1 x^1 + \cdots + a_n x^n + b,$$

in small enough local affine coordinates of B_0, with $a_i \in \mathbb{Z}$ and $b \in \mathbb{R}$.

On the other hand, we consider the subsheaf of affine holomorphic functions $\mathcal{O}^{aff} \hookrightarrow \check{p}_*\mathcal{O}_{\check{X}_0}$ defined by an embedding $Aff^{\mathbb{Z}}_{B_0} \hookrightarrow \check{p}_*\mathcal{O}_{\check{X}_0}$:

Definition 3.2. *Given $m \in Aff^{\mathbb{Z}}_{B_0}(U)$, expressed locally as $m(x) = \sum_j a_k x^j + b$, we let*

$$w^m = e^{2\pi \frac{b}{\hbar}}(w^1)^{a_1} \dots (w^n)^{a_n} \in \mathcal{O}_{\check{X}_0, \check{q}},$$

where $w^j = e^{-2\pi i[(y^j - \sum_k \beta^j_k x^k) + i\hbar^{-1}x^j]}$. This gives an embedding

$$Aff^{\mathbb{Z}}_{B_0}(U) \hookrightarrow \mathcal{O}_{\check{X}_0}(\check{p}^{-1}(U)),$$

and we denote the image subsheaf by $\mathcal{O}^{aff}$.

Definition 3.3. *We let $\mathfrak{g} = \mathcal{O}^{aff} \otimes_{\mathbb{Z}} T^{1,0}_{B_0,\mathbb{Z}}$ and define a Lie bracket structure $[\cdot,\cdot]$ on $\mathfrak{g}$ by the restricting the usual Lie bracket on $\check{p}_*\mathcal{O}(T^{1,0}_{\check{X}_0})$ to $\mathfrak{g}$.*

This is well defined because we have the following fact.

Proposition 3.4. *$\mathfrak{g}$ is closed under the Lie bracket structure of $\check{p}_*\mathcal{O}(T^{1,0}_{\check{X}_0})$.*

Proof. Assuming we are in a small enough local affine chart, we compute the Lie bracket and obtain

$$\begin{aligned} [w^m \otimes \check{\partial}_n, w^{m'} \otimes \check{\partial}_{n'}] &= [w^m \otimes (\sum_j n_j \frac{\partial}{\partial \log w^j}), w^{m'} \otimes (\sum_k n'_k \frac{\partial}{\partial \log w^k})] \\ &= w^{m+m'}(\langle m', n\rangle \check{\partial}_{n'} - \langle m, n'\rangle \check{\partial}_n) \\ &= w^{m+m'}\check{\partial}_{\langle m',n\rangle n' - \langle m,n'\rangle n}. \end{aligned}$$

We see that $w^{m+m'}\check{\partial}_{\langle m',n\rangle n' - \langle m,n'\rangle n}$ is still a local section of $\mathfrak{g}$. $\square$

Remark 3.5. *There is an exact sequence of sheaves*

$$0 \to \underline{\mathbb{R}} \to Aff^{\mathbb{Z}}_{B_0} \to \Lambda^* \to 0,$$

where $\underline{\mathbb{R}}$ is the local constant sheaf of real numbers. The pairing $\langle m, n\rangle$ is the natural pairing for $m \in \Lambda^*_x$ and $n \in \Lambda_x$. Given a local section $n \in \Lambda(U)$, we let $n^\perp \subset Aff^{\mathbb{Z}}_{B_0}(U)$ be the subset which is perpendicular to n upon descending to $\Lambda^*(U)$.

Definition 3.6. *The subsheaf $\mathfrak{h} \hookrightarrow \mathfrak{g}$ consists of sections which lie in the image of the composition of maps*

$$\bigoplus_{n \in \Lambda(U)} n^\perp \otimes_{\mathbb{Z}} (\mathbb{Z}n) \to \mathcal{O}^{aff}(U) \otimes_{\mathbb{Z}} T^{1,0}_{B_0,\mathbb{Z}}(U) \to \mathfrak{g}(U),$$

locally in an affine coordinate chart U.

Note that $\mathfrak{h}$ is a sheaf of Lie subalgebras of $\mathfrak{g}$. Given a formal power series ring $R = \mathbb{C}[[t_1, \dots, t_k]]$, with maximal ideal $\mathbf{m} = (t_1, \dots, t_k)$, we write $\mathfrak{g}_R = \mathfrak{g} \otimes_{\mathbb{C}} R$ and $\mathfrak{h}_R = \mathfrak{h} \otimes_{\mathbb{C}} R$.

Definition 3.7. *The* sheaf of tropical vertex group *over R on B_0 is defined as the sheaf of exponential groups* $\exp(\mathfrak{h} \otimes \mathbf{m})$ *which acts as automorphisms on $\mathfrak{h}_R$ and $\mathfrak{g}_R$.*

3.2. **Scattering diagrams in dimension two.** Starting from this subsection, we fix once and for all a rank 2 lattice $M \cong \mathbb{Z} \cdot \mathsf{e}_1 \oplus \mathbb{Z} \cdot \mathsf{e}_2$, and its dual $N \cong \mathbb{Z} \cdot \check{\mathsf{e}}_1 \oplus \mathbb{Z} \cdot \check{\mathsf{e}}_2$. We take the integral affine manifold B_0 to be $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, equipped B_0 with coordinates $x^1\check{\mathsf{e}}_1 + x^2\check{\mathsf{e}}_2$, the standard metric $g = (dx^1)^2 + (dx^2)^2$ and a B-field β. We also write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have the identifications

$$X_0 \cong B_0 \times (M_{\mathbb{R}}/M), \quad \check{X}_0 \cong B_0 \times (N_{\mathbb{R}}/N).$$

There is also a natural identification $\mathcal{M} \cong B_0 \times M$. We will denote the connected component $B_0 \times \{m\}$ by $B_{0,m}$. We equip $\mathcal{M}$ as well as the bundle $TB_0^{\mathbb{C}}$ with the metric $\frac{1}{2}\hbar^{-1}g$ from B_0. We also denote the projection to B_0 by $pr : \mathcal{M} \to B_0$. Notice that when we make the identification $pr : B_{0,m} \cong B_0$, the metric is deferred by a factor of $\frac{1}{2}\hbar^{-1}$.

As in Remark 2.12, we can fix $x_0 = 0 \in B_{0,m} \cong B_0$ and define

$$f_m = -2\pi \int_0^x s_m \lrcorner (\omega + i\beta)$$

on each $B_{0,m}$. The gradient vector field $\nabla Re(f_m)$ on $B_{0,m}$ is independent of $\hbar$. Notice that $m \lrcorner g$ gives a vector field on B_0 and we have the identification $-4\pi m \lrcorner g = \nabla Re(f_m)$.

Definition 3.8. *A* wall $\mathbf{w}$ *is a triple* (m, ℓ, Θ) *where* m *lies in* $M \setminus \{0\}$ *and* ℓ *is a codimension one closed subset of* B_0 *of the form*

$$\ell = x_0 - \mathbb{R}_{\geq 0}(m \lrcorner g),$$

or

$$\ell = x_0 - \mathbb{R}(m \lrcorner g)$$

for some $x_0 \in B_0$. *If* ℓ *is a ray, we denote by* $Init(\mathbf{w}) = x_0$ *the initial point of* $\mathbf{w}$. *With a primitive* $n \in N \setminus \{0\}$ *determined by* $\langle m, n \rangle = 0$ *and the condition that* $(-m \lrcorner g, n)$ *is positively oriented,* Θ *is a section of the tropical vertex group restricting to* ℓ *of the form*

$$\Theta \in \Gamma\left(\ell, \exp\left((\mathbb{C}[w^m] \cdot w^m) \otimes_{\mathbb{Z}} (\mathbb{Z}\check{\partial}_n) \otimes \mathbf{m} \right)|_{\ell}\right) \hookrightarrow \Gamma(\ell, \exp(\mathfrak{h} \otimes \mathbf{m})|_{\ell}),$$

That means we can write $\Theta = \exp(\check{\phi} \otimes \check{\partial}_n)$ *for some*

$$\check{\phi} \in (\mathbb{C}[w^m] \cdot w^m) \otimes \mathbf{m}.$$

Definition 3.9. *A* scattering diagram $\mathcal{D}$ *is a set of walls* $\{(m_\alpha, \ell_\alpha, \Theta_\alpha)\}_\alpha$ *such that there are only finitely many* α*'s with* $\check{\phi}_\alpha \neq 0$ $(mod\ \mathbf{m}^k)$ *for every* $k \in \mathbb{Z}_+$, *where* $\Theta_\alpha = \exp(\check{\phi}_\alpha)$.

Given a scattering diagram $\mathcal{D}$, *we will define the* support *of* $\mathcal{D}$ *to be*

$$supp(\mathcal{D}) = \bigcup_{\mathbf{w} \in \mathcal{D}} \ell_{\mathbf{w}},$$

and the singular set *of* $\mathcal{D}$ *to be*

$$Sing(\mathcal{D}) = \bigcup_{\mathbf{w} \in \mathcal{D}} \partial \ell_{\mathbf{w}} \cup \bigcup_{\mathbf{w}_1 \pitchfork \mathbf{w}_2} \ell_{\mathbf{w}_1} \cap \ell_{\mathbf{w}_2},$$

where $\mathbf{w}_1 \pitchfork \mathbf{w}_2$ *means their intersection is* 0*-dimensional.*

3.3. **Analytic continuation along paths.** Given an embedded path

$$\gamma : [0,1] \to B_0 \setminus Sing(\mathcal{D}),$$

with $\gamma(0), \gamma(1) \notin supp(\mathcal{D})$ and which intersects all the walls in $\mathcal{D}$ transversally, we can define the *analytic continuation along* γ as in [9] (which was called the *path ordered product* there). It is a process of determining an element in the stalk

$$\Theta_{\gamma(1)} = \prod_{\mathbf{w}\in\mathcal{D}}^{\gamma} \Theta_{\mathbf{w}} \in \exp(\mathfrak{h}\otimes\mathbf{m})_{\gamma(1)},$$

if we prescribe the stalk at $\gamma(0)$ to be $\Theta_{\gamma(0)} = I \in \exp(\mathfrak{h}\otimes\mathbf{m})_{\gamma(0)}$ and "analytically continue" it to $\gamma(1)$. When γ crosses a wall $\mathbf{w}$, we require a "jumping" phenomenon given by multiplication by $\Theta_{\mathbf{w}}$.

More precisely, for each $k \in \mathbb{Z}_+$, we define $\Theta^k_{\gamma(1)} \in \exp(\mathfrak{h}\otimes(\mathbf{m}/\mathbf{m}^{k+1}))_{\gamma(1)}$ and let

$$\Theta_{\gamma(1)} = \lim_{k\to+\infty} \Theta^k_{\gamma(1)}.$$

For each k, there is a finite subset $\mathcal{D}^k \subset \mathcal{D}$ consisting of walls $\mathbf{w}$ with $\Theta_{\mathbf{w}} \neq I \pmod{\mathbf{m}^{k+1}}$. We then have a sequence of real numbers

$$0 = t_0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = 1$$

such that $\{\gamma(t_1), \dots, \gamma(t_s)\} = \gamma \cap supp(\mathcal{D}^k)$.

For each $0 < i < s+1$, there are walls $\mathbf{w}_{i,1}, \dots \mathbf{w}_{i,l_i}$ in $\mathcal{D}^k$ such that $\gamma(t_i) \in \ell_{i,j} = supp(\mathbf{w}_{i,j})$. Notice that we have $\dim(supp(\mathbf{w}_{i,j}) \cap supp(\mathbf{w}_{i,l})) = 1$ which follows from the assumption that γ is not hitting $Sing(\mathcal{D})$. We define the element in the stalk $\exp(\mathfrak{h}\otimes\mathbf{m})_{\gamma(t_i)}$ by

$$\Theta_{\gamma(t_i)} = \prod_{j=1}^{k} \Theta^{\sigma_j}_{\mathbf{w}_{i,j}},$$

where

$$\sigma_j = \begin{cases} 1 & \text{if } \langle \gamma'(t_i), n_{i,j}\rangle > 0, \\ -1 & \text{if } \langle \gamma'(t_i), n_{i,j}\rangle < 0. \end{cases}$$

This is well defined without prescribing the order since the elements $\Theta_{\mathbf{w}_{i,j}}$ are commuting with each other.

For each i, we define inductively a holomorphic section

$$\Theta_{[t_{i-1},t_i]} \in \Gamma\left([t_{i-1},t_i], \gamma^{-1}\exp(\mathfrak{h}\otimes\mathbf{m})|_{[t_{i-1},t_i]}\right)$$

by requiring $\Theta_{[0,t_1],0} = I$ and

$$\Theta_{[t_i,t_{i+1}],t_i} = \Theta_{\gamma(t_i)} \circ \Theta_{[t_{i-1},t_i],t_i}.$$

Finally, we let

$$\Theta^k_{\gamma(1)} = \Theta_{[t_s,1],1} \pmod{\mathbf{m}^{k+1}}.$$

Definition 3.10. *Two scattering diagrams $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are said to be* equivalent *if*

$$\Theta_{\gamma(1),\mathcal{D}} = \Theta_{\gamma(1),\tilde{\mathcal{D}}}$$

for any embedded curve γ such that analytic continuation is well defined for both $\mathcal{D}$ and $\tilde{\mathcal{D}}$.

Given a scattering diagram $\mathcal{D}$, there is a unique representative from its equivalent class which is minimal. First, we may remove those walls with trivial automorphism Θ as they do not contribute to the analytic continuation along paths. Second, if two walls $\mathbf{w}_1$ and $\mathbf{w}_2$ whose supports and directions coincide, we can simply take the multiplication $\Theta = \Theta_1 \circ \Theta_2$ and define a single wall $\mathbf{w}$. After doing so, we obtain a minimal scattering diagram equivalent to $\mathcal{D}$. From now on, unless specified otherwise, we will always assume that every scattering diagram is minimal.

The key combinatorial result concerning scattering diagrams is the following theorem from [11]; we state it as in [9].

Theorem 3.11 (Kontsevich and Soibelman [11])**.** *Given a scattering diagram $\mathcal{D}$, there exists a unique minimal scattering diagram $\mathcal{S}(\mathcal{D}) \supset \mathcal{D}$ given by adding walls supported on rays so that*

$$\Theta_{\gamma(1)} = I$$

for any closed loop γ such that analytic continuation along γ is well defined.

A scattering diagram having this property is said to be monodromy free*.*

In the rest of this paper, we will restrict ourselves to the case where $\mathcal{D}$ is a scattering diagram with walls $\mathbf{w}_1$ and $\mathbf{w}_2$ whose supports are lines passing through the origin. We have the following definition of a standard scattering diagram.

Definition 3.12. *A scattering diagram $\mathcal{D}$ is called* standard *if*

- *$\mathcal{D}$ consists of two walls $\{\mathbf{w}_i = (m_i, \ell_i, \Theta_i)\}_{i=1,2}$ whose supports ℓ_i are lines passing through the origin,*
- *the dual lattice vectors m_1 and m_2 are primitive, and*
- *for $i = 1, 2$,*
$$\check{\phi}_i \in (\mathbb{C}[w^{m_i}] \cdot w^{m_i}) \otimes_{\mathbb{C}} (\mathbb{C}[[t_i]] \cdot t_i),$$
i.e. t_i is the only formal variable in the series expansion of $\check{\phi}_i$. Here $\check{\phi}_i$ is defined by $\Theta_i = \exp(\check{\phi}_i \otimes \check{\partial}_{n_i})$ as in Definition 3.8.

When considering a standard scattering diagram, we can always restrict ourselves to the power series ring $R = \mathbb{C}[[t_1, t_2]]$. $\mathcal{S}(\mathcal{D})$ is obtained from $\mathcal{D}$ by adding walls supported on rays starting at the origin. Furthermore, each of the wall added will have its dual lattice vector m laying in the integral cone $\mathbb{Z}_{\geq 0} m_1 + \mathbb{Z}_{\geq 0} m_2$. We will end this section by giving two examples of standard scattering diagrams from [9].

Example 3.13. *We consider $\mathcal{D}$ with two walls $\mathbf{w}_i$ with wall crossing factors $\check{\phi}_i \otimes \check{\partial}_{n_i} = \log(1 + t_i (w^i)^{-1}) \otimes \check{\partial}_{n_i}$, $i = 1, 2$, as shown in Figure 1. The scattering diagram $\mathcal{S}(\mathcal{D})$ is obtained by adding one wall $\mathbf{w}_{1,1}$, which is colored in red, to the original diagram $\mathcal{D}$. If we look at the loop γ which is colored in blue in the figure, the analytic continuation along γ is given by*

$$\Theta_{\gamma(1), \mathcal{S}(\mathcal{D})} = e^{-\check{\phi}_1 \check{\partial}_{e_2}} \circ e^{\check{\phi}_2 \check{\partial}_{-e_1}} \circ e^{\check{\phi}_{1,1} \check{\partial}_{-e_1 + e_2}} \circ e^{\check{\phi}_1 \check{\partial}_{e_2}} \circ e^{-\check{\phi}_2 \check{\partial}_{-e_1}},$$

which is equal to the identity.

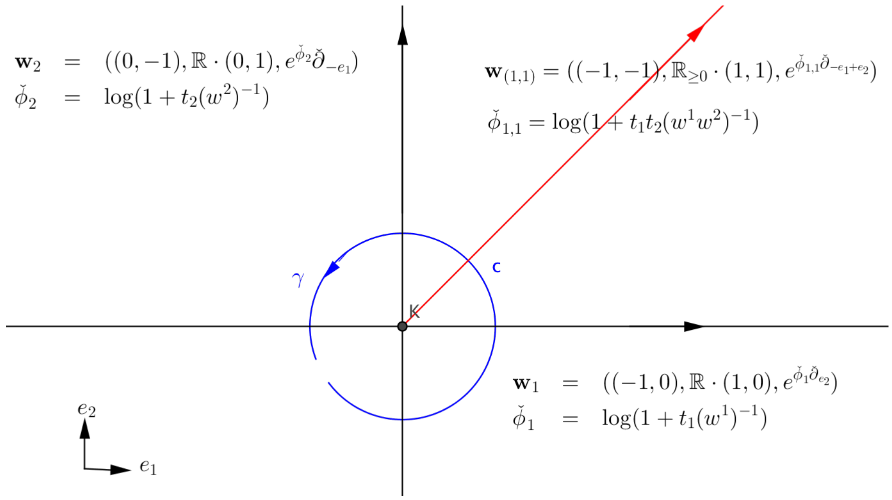

FIGURE 1.

Example 3.14. *In this example, we consider the diagram $\mathcal{D}$ with two walls $\mathbf{w}_i$ with the same support as above, but different wall crossing factors $\check{\phi}_i \otimes \check{\partial}_{n_i} = \log(1+t_i(w^i)^{-1})^{-2} \otimes \check{\partial}_{n_i}$ (see Figure 2). The diagram $\mathcal{S}(\mathcal{D})$ then has infinitely many walls. We have*

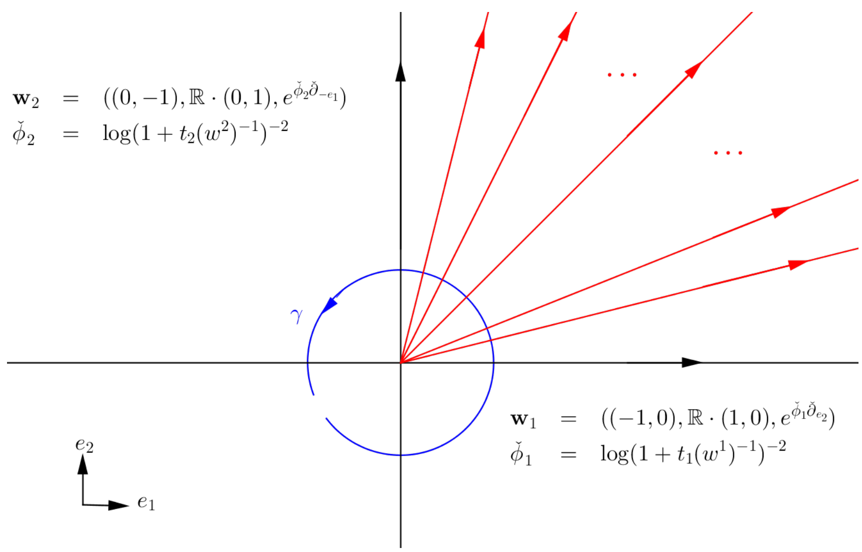

FIGURE 2.

$$\mathcal{S}(\mathcal{D}) \setminus \mathcal{D} = \bigcup_{k\in\mathbb{Z}_{>0}} \mathbf{w}_{k,k+1} \bigcup_{k\in\mathbb{Z}_{>0}} \mathbf{w}_{k+1,k} \cup \{\mathbf{w}_{1,1}\},$$

where the wall $\mathbf{w}_{k,k+1}$ has dual lattice vector $(k,k+1) \in M$ supported on a ray of slope $\frac{k+1}{k}$. The wall crossing factor $\check{\phi}_{k,k+1}$ is given by

$$\check{\phi}_{k,k+1} = 2\log(1+t_1^k t_2^{k+1}(w^1)^{-k}(w^2)^{-(k+1)}),$$

and similarly for $\check{\phi}_{k+1,k}$. The wall crossing factor associated to $\mathbf{w}_{1,1}$ is given by

$$\check{\phi}_{1,1} = -4\log(1 - t_1 t_2 (w^1 w^2)^{-1}).$$

Interesting relations between these wall crossing factors and relative Gromov-Witten invariants of certain weighted projective planes were established in [9]. Indeed it is expected that these automorphisms come from counting holomorphic disks on the mirror A-side, which was conjectured by Fukaya [8] to be closely related to Witten-Morse theory. In the rest of this paper, we will show how this can be made precise and proved.

4. Single wall diagrams as deformations

In this section, we simply take the ring $R = \mathbb{C}[[t]]$ and consider a scattering diagram with only one wall $\mathbf{w} = (-m, \ell, \Theta)$ where ℓ is a line passing through the origin. Writing $\Theta = \exp(\check{\phi} \otimes \check{\partial}_n)$, we let

$$\check{\phi} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} w^{-km} t^j, \tag{4.1}$$

where $a_{jk} \neq 0$ only for finitely many k's for each fixed j. The line ℓ divides the base $B_0 = \mathbb{R}^2$ into two half planes H_+ and H_-, with $\partial_{\pm n} \in H_{\pm}$. We are going to interpret $\check{\phi} \otimes \check{\partial}_n$ as a step function like (distributional) section $\check{\varphi}_0 \in \Omega^{0,0}(\check{X}_0 \setminus \check{p}^{-1}\ell, T^{1,0}_{\check{X}_0})[[t]]$ of the form

$$\check{\varphi}_0 = \begin{cases} \check{\phi} \otimes \check{\partial}_n & \text{on } H_+, \\ 0 & \text{on } H_-, \end{cases}$$

and write down an ansatz $e^{\check{\varphi}} * 0 = \check{\Xi} = \check{\Xi}_\hbar \in \Omega^{0,1}(\check{X}_0, T^{1,0})$ (we will often drop the $\hbar$ dependence in our notations) which represents a smoothing of $e^{\check{\varphi}_0} * 0$ (which is not well defined itself), and show that the leading order expansion of $\check{\varphi}$ is precisely $\check{\varphi}_0$ as $\hbar \to 0$.

4.1. Ansatz corresponding to a wall.

Suppose that we have a wall $\mathbf{w} = (-m, \ell, \Theta)$ as above, we are going to use the Fourier transform $\hat{\mathcal{F}} : KS_{\check{X}_0} \to L_{X_0}$ defined in Section 2.4 to obtain $\check{\Xi} \in \Omega^1(\mathcal{M}, T^{\mathbb{C}}_{B_0})[[t]]$, and perform all the computations on L_{X_0}. Via the identification $T^{1,0}_{\check{X}_0} \to \check{p}^* TB_0^{\mathbb{C}}$ given in Section 2.4, we use the notation ∂_n to stand for the vector $\frac{\hbar}{4\pi}\sum_j n_j \frac{\partial}{\partial x^j}$. As in Section 3.2, we write $\mathcal{M} = \coprod_{m\in M} B_{0,m}$ and define the Morse function f_m on each $B_{0,m}$.

Definition 4.1. *Fixing $-m \in M$, we use orthonormal coordinates $u^1\tilde{e}_1 + u^2\tilde{e}_2$ for B_0 with the properties that $\tilde{e}_1$ is parallel to $m \lrcorner g$ and $\tilde{e}_2$ is parallel to n, where $n \in N$ is the unique primitive element such that $\langle m, n\rangle = 0$ and $\{m \lrcorner g, n\}$ is positively oriented.*

We consider a function of the form

$$\delta_{-m} = \delta_{-m,\hbar} = \left(\frac{\lambda}{\hbar\pi}\right)^{\frac{1}{2}} e^{-\frac{\lambda (u^2)^2}{\hbar}} du^2, \tag{4.2}$$

for some $\lambda \in \mathbb{R}_+$, having the property that $\int_{u^1=c} \delta_{-m} \equiv 1$ for all c; this gives a smoothing of the delta function of ℓ. We fix a cut off function $\chi = \chi(u^2)$ satisfying $\chi \equiv 1$ on ℓ and which has compact support in $U^{-m} = \{-\epsilon \leq u^2 \leq \epsilon\}$ near ℓ. Then we can also use $\tilde{\delta}_{-m} = \chi\delta_{-m}$, which is supported near ℓ, as our smoothing of a delta function. The following Figure 3 illustrates the situation.

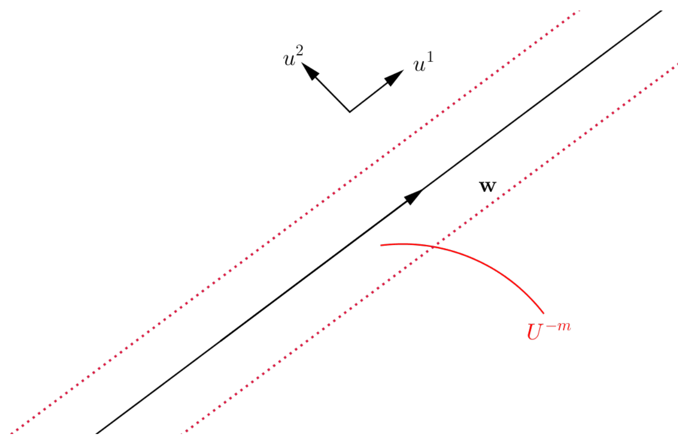

FIGURE 3.

Remark 4.2. *This is motivated by Witten-Morse theory where we regard the lines ℓ as stable submanifolds corresponding to the Morse function $Re(f_{-km})$ ($k \in \mathbb{Z}_+$) on $B_{0,-km}$ from a critical point of index 1 at infinity, and*

$$e^{-f_{-km}}\delta_{-m}$$

as the eigenform associated to that critical point. Adopting the notations from [7], we may write $g_{-m} = \lambda(u^2)^2$ and $\delta_{-m} = e^{-\frac{g_{-m}}{\hbar}}\mu_{-m}$ where $\mu_{-m} = (\frac{\lambda}{\hbar\pi})^{\frac{1}{2}}du^2$.

Definition 4.3. *Given a dual lattice vector $m \in M$, we let $\mathfrak{w}^m = \hat{\mathcal{F}}(w^m) \in \Omega^0(\mathcal{M}, TB_0^{\mathbb{C}})$ be defined by*

$$\mathfrak{w}^m = \begin{cases} e^{-f_m} & \text{on } B_{0,m}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

It follows from the definition of the Fourier transform that

$$\hat{\mathcal{F}}(w^m \otimes \check{\partial}_n) = \mathfrak{w}^m \otimes \partial_n$$

for any $n \in N$.

Treating $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$ as a module over $\Omega^*(B_0)$, we can multiple $\mathfrak{w}^{-m}$ by δ_{-m} to make it concentrated along ℓ to define delta function liked element $\breve{\Xi}$.

Definition 4.4. *Given a wall $\mathbf{w} = (m, \ell, \Theta)$ with $\check{\phi}$ given as in (4.1), we let*

$$\breve{\Xi} = -\sum_{j,k\geq 1} a_{jk}\tilde{\delta}_{-m}(\mathfrak{w}^{-km} \otimes \partial_n)t^j \tag{4.4}$$

be the ansatz *corresponding to the wall crossing factor $\check{\phi}$.*

Proposition 4.5.
$$d_W \breve{\Xi} + \frac{1}{2}\left\{\breve{\Xi}, \breve{\Xi}\right\} = 0,$$
i.e. $\breve{\Xi}$ *satisfies the Maurer-Cartan (MC) equation of the dgLa* L_{X_0}.

Proof. Indeed we are going to show that both terms $d_W \breve{\Xi}$ and $\frac{1}{2}\left\{\breve{\Xi}, \breve{\Xi}\right\}$ vanish.

We first show that $d_W(\tilde{\delta}_{-m}\mathfrak{w}^{-km}\partial_n) = 0$. Since $d_W = e^{-f_{-km}}\nabla e^{f_{-km}}$, this is equivalent to showing that $d(\tilde{\delta}_{-m}) = 0$, but this is clear from the construction.

Next we show that $\left\{\tilde{\delta}_{-m}\mathfrak{w}^{-k_1 m}\partial_n, \tilde{\delta}_{-m}\mathfrak{w}^{-k_2 m}\partial_n\right\} = 0$ for any k_1, k_2. This is simply due to the fact that μ_{-m} is a covariant constant form (with respect to the affine connection) and hence
$$\left\{\tilde{\delta}_{-m}\mathfrak{w}^{-k_1 m}\partial_n, \tilde{\delta}_{-m}\mathfrak{w}^{-k_2 m}\partial_n\right\} = \mu_{-m} \wedge \mu_{-m}\left\{\chi e^{-\frac{g_{-m}}{\hbar}}\mathfrak{w}^{-k_1 m}\partial_n, \chi e^{-\frac{g_{-m}}{\hbar}}\mathfrak{w}^{-k_2 m}\partial_n\right\} = 0.$$
□

Taking the inverse Fourier transform $\mathcal{F}$, we see that $\mathcal{F}(\breve{\Xi})$ is a Maurer-Cartan element in the Kodaira-Spencer dgLa $KS_{\check{X}_0}$. Since $\check{X}_0 \cong (\mathbb{C}^*)^2$ has no non-trivial deformations, the element $\mathcal{F}(\breve{\Xi})$ must be gauge equivalent to 0. The same holds for $\breve{\Xi}$ since the Fourier transform gives an isomorphism of dgLa's.

4.2. **Relation with the wall crossing factor $\check{\phi}$.** We are going to write down a specific solution to the equation

$$e^{\varphi} * 0 = -\left(\frac{e^{ad_\varphi} - I}{ad_\varphi}\right) d_W \varphi = \breve{\Xi}, \tag{4.5}$$

for $\varphi \in \Omega^0(\mathcal{M}, TB_0^{\mathbb{C}})$. We use the coordinates u^1, u^2 on B_0 corresponding to the wall $\mathbf{w}$ described above and define a homotopy retract of $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$ to its cohomology. Since $TB_0^{\mathbb{C}}$ is a trivial flat bundle, it is enough to define a homotopy for $\Omega^*(\mathcal{M})$. Due to the fact the $\mathcal{M} = \coprod_{m \in \mathbb{Z}^2} B_{0,m}$, it is sufficient to define a homotopy for $\Omega^*(B_0)$, retracting to its cohomology $H^*(B_0) = \mathbb{C}$ which is generated by constant functions on B_0.

Definition 4.6. *Fixing a point* $(u_0^1, u_0^2) \in B_0$ *with* $u_0^1, u_0^2 << 0$ *and letting*
$$\varrho_{(u^1,u^2)}(t) = \begin{cases} (u_0^1, 2tu^2 + (1-2t)u_0^2) & \text{if } t \in [0, \frac{1}{2}], \\ ((2t-1)u^1 + (2-2t)u_0^1, u^2) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$
for any $(u^1, u^2) \in B_0$ *as shown in Figure 4, we define* $\tilde{H} : \Omega^*(B_0) \to \Omega^*(B_0)[-1]$ *by*
$$(\tilde{H}\alpha)(u^1, u^2) = \begin{cases} \int_{\varrho_{(u^1,u^2)}} \alpha & \text{for } \alpha \in \Omega^1(B_0), \\ \int_{u_0^1}^{u^1} \left(\iota_{\frac{\partial}{\partial u^1}}\alpha(s, u^2)\right) ds & \text{for } \alpha \in \Omega^2(B_0). \end{cases}$$

We also let $\tilde{P} : \Omega^*(B_0) \to H^*(B_0)$ *be the evaluation at the point* (u_0^1, u_0^2) *and* $\tilde{\iota} : H^*(B_0) \to \Omega^*(B_0)$ *be the embedding of constant functions on* B_0.

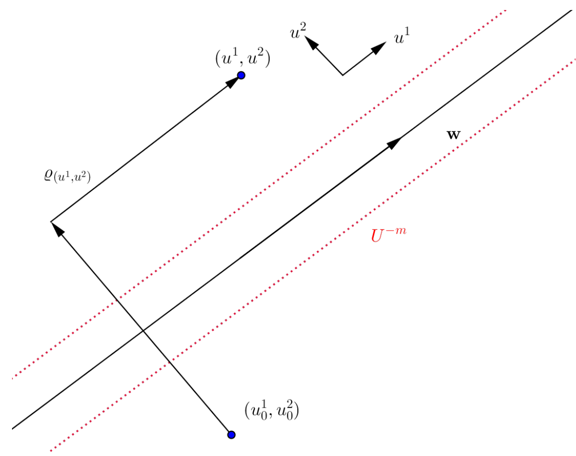

FIGURE 4.

Proposition 4.7. *$\tilde{H}$ is a homotopy retract of $\Omega^*(B_0)$ onto its cohomology, i.e. we have*

$$I - \tilde{\iota}\tilde{P} = d\tilde{H} + \tilde{H}d.$$

Definition 4.8. *We fix a base point $q_{m,0} \in B_{m,0}$ on each connected component $B_{0,m}$. We choose orthonormal coordinates (u^1, u^2) on $B_{0,m} \cong B_0$ according to the lattice $m \in M$ and we let $(u_0^1, u_0^2) = q_{m,0}$. We define the homotopy $H_m = e^{-f_m}\tilde{H}e^{f_m} : \Omega^*(B_{0,m}) \to \Omega^*(B_{0,m})[-1]$, projection $P_m = \tilde{P}e^{f_m} : \Omega^*(B_{0,m}) \to H^*(B_{0,m})$ and the inclusion $\iota_m = e^{-f_m}\tilde{\iota}$, where $\tilde{H}$, $\tilde{P}$ and $\tilde{\iota}$ are defined as in the above Definition 4.6 using the coordinates (u^1, u^2). They satisfy*

$$I - \iota_m \circ P_m = d_W H_m + H_m d_W$$

on $\Omega^*(B_{0,m})$.

The homotopy and projection are extended to $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$ by applying H_m and P_m on each component of $\mathcal{M}$, and they are denoted by H and P respectively.

Remark 4.9. *We should impose a rapid decay assumption on $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$: for $\alpha \in \Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$, we should have $\sup_{pr(\gamma)\in K} |\dot{\gamma}|^k |\alpha(\gamma)| \to 0$ as $|\dot{\gamma}| \to \infty$ for all $k \in \mathbb{Z}_{\geq 0}$ and compact $K \subset B_0$. Therefore $H^*(\mathcal{M}, TB_0^{\mathbb{C}})$ refers to those locally constant functions (i.e. constant on each connected component) satisfying the rapid decay assumption. Obviously the operators H, P and ι preserve this decay condition.*

We will fix $q_{m,0}$ to be the same point upon projecting to B_0 with $u_0^2 << 0$ such that the line $\{u^2 = u_0^2\}$ is far away from the support of χ. We impose the gauge fixing condition $P\varphi = 0$, or equivalently,

$$\varphi = H d_W \varphi$$

to solve the equation (4.5) order by order. This is possible because of the following lemma.

Lemma 4.10. *Among solutions of $e^{\varphi} * 0 = \breve{\Xi}$, there exists a unique one satisfying $P\varphi = 0$*

Proof. Notice that for any $\sigma = \sigma_1 + \sigma_2 + \cdots \in \Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})[[t]] \cdot (t)$ with $d\sigma = 0$, we have $e^{\sigma} * 0 = 0$, and hence $e^{\varphi \bullet \sigma} * 0 = \breve{\Xi}$ is still a solution for the same equation. With $\varphi \bullet \sigma$ given by the Baker-Campbell-Hausdorff formula as

$$\varphi \bullet \sigma = \varphi + \sigma + \frac{1}{2}\{\varphi, \sigma\} + \dots,$$

we solve the equation $P(\varphi \bullet \sigma) = 0$ order by order under the assumption that $d\sigma = 0$. □

Under the gauge fixing condition $P\varphi = 0$, we see that the unique solution to Equation (4.5) can be found iteratively using the homotopy H. Writing $\varphi = \varphi_1 + \varphi_2 + \dots$, the first equation to solve is $-d\varphi_1 = \breve{\Xi}_1$, which can be solved by taking

$$\varphi_1 = -H\breve{\Xi}.$$

The second equation is given by $-d_W\varphi_2 - \frac{1}{2}\{\varphi_1, d_W\varphi_1\} = \breve{\Xi}_2$, and we can use

$$\varphi_2 = H(-\breve{\Xi}_2 + \frac{1}{2}\{\varphi_1, \breve{\Xi}_1\})$$

to solve it. Suppose we have $\varphi^s = \varphi_1 + \varphi_2 + \dots + \varphi_s$ solving the equation

$$-\left(\frac{e^{ad_{\varphi^s}} - I}{ad_{\varphi^s}}\right) d_W\varphi^s = -\sum_{k\geq 0} \frac{ad_{\varphi^s}^k}{(k+1)!} d_W\varphi^s = \breve{\Xi} \pmod{t^{s+1}},$$

we would like to find φ_{s+1} such that $\varphi^{s+1} = \varphi_1 + \dots + \varphi_{s+1}$ solves the equation (4.5) (mod t^{s+2}). We set

$$\varphi_{s+1} = -H\left(\breve{\Xi} + \sum_{k\geq 0} \frac{ad_{\varphi^s}^k}{(k+1)!} d_W\varphi^s\right)_{s+1}, \tag{4.6}$$

where the subscript means the t^{s+1} coefficient in right-hand-side of the above equation.

Remark 4.11. *Notice that*

$$d_W\left(\breve{\Xi} + \sum_{k\geq 0} \frac{ad_{\varphi^s}^k}{(k+1)!} d_W\varphi^s\right)_{s+1} = 0$$

and hence the operator H defined by integration along paths is independent of the paths chosen upon applying to these terms.

Remark 4.12. *We also observe that φ_s's vanish on the components $B_{0,m'}$ for those $m' \neq -km$. Furthermore, we can see that $d_W\varphi_s$'s vanish outside the set $pr^{-1}(supp(\chi))$ inductively.*

We are going to analyze the behavior of φ as $\hbar \to 0$, showing that $\hat{\mathcal{F}}(\varphi)$ has an asymptotic expansion whose leading order term is exactly given by φ_0 on $\check{X}_0 \setminus \check{p}^{-1}(\ell)$.

4.2.1. *Semi-classical analysis for φ.* We will abbreviate δ_{-m} and $\tilde{\delta}_{-m}$ simply as δ and $\tilde{\delta}$ in this subsection if there is no confusion. As we mentioned in Remark 4.11, the operator $\tilde{H}$, when applying on closed 1-forms, can be replaced by the operator $\hat{H}$ given by

$$(\hat{H}\alpha)(u^1, u^2) = \begin{cases} \int_{\hat{\varrho}_{(u^1,u^2)}} \alpha & \text{for } \alpha \in \Omega^1(B_0), \\ \int_{u_0^2}^{u^2} \left(\iota_{\frac{\partial}{\partial u^2}} \alpha(u^1, s)\right) ds & \text{for } \alpha \in \Omega^2(B_0), \end{cases} \tag{4.7}$$

where $\hat{\varrho}_{(u^1,u^2)}$ is the path given by

$$\hat{\varrho}_{(u^1,u^2)}(t) = \begin{cases} ((1-2t)u_0^1 + 2tu^1, u_0^2) & \text{if } t \in [0, \frac{1}{2}], \\ (u^1, (2t-1)u^2 + (2-2t)u_0^2) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We can express φ_1 explicitly as

$$\varphi_1 = \sum_k a_{1k} \left(\int_{-c}^{u^2} \iota_{\frac{\partial}{\partial u^2}}(\tilde{\delta}) \right) \mathfrak{w}^{-km} \partial_n.$$

Since we have

$$\hat{H}(\tilde{\delta}) = \int_{-c}^{u^2} \iota_{\frac{\partial}{\partial u^2}}(\tilde{\delta}) = \int_{-c}^{u^2} (\frac{\lambda}{\hbar\pi})^{\frac{1}{2}} (\chi e^{-\frac{g}{\hbar}}) = \begin{cases} 1 + \mathcal{O}_{loc}(\hbar) & \text{on } H_+, \\ \mathcal{O}_{loc}(\hbar) & \text{on } H_-, \end{cases}$$

we see that $\hat{\mathcal{F}}(\varphi_1)$ have the desired asymptotic expansion with leading term given by $\varphi_{0,1}$, the coefficient of t^1 in φ_0.

Notations 4.13. *We say a function f on an open subset $U \subset B_0$ belongs to $\mathcal{O}_{loc}(\hbar^l)$ if it is bounded by $C_K \hbar^l$ for some constant (independent of $\hbar$) C_K on every compact subset $K \subset U$.*

Next we consider the second term φ_2. Notice that $\left\{ \mathfrak{w}^{-k_1 m} \partial_n, \mathfrak{w}^{-k_2 m} \partial_n \right\} = 0$ for all positive k_1, k_2. Therefore we have

$$\left\{ \varphi_1, \breve{\Xi}_1 \right\} = - \sum_{k_1,k_2} a_{1k_1} a_{1k_2} [(\hat{H}\tilde{\delta})(\nabla_{\partial_n} \tilde{\delta}) - \tilde{\delta}(\nabla_{\partial_n}(\hat{H}\tilde{\delta}))] \mathfrak{w}^{-(k_1+k_2)m} \partial_n. \tag{4.8}$$

We investigate terms by terms the order in $\hbar$. First, notice that $|\hat{H}\tilde{\delta}| \leq 2$ while

$$\begin{aligned} |\hat{H}(\hat{H}(\tilde{\delta})\nabla_{\partial_n}\tilde{\delta})| &= \frac{\hbar|n|}{4\pi} \Big| \int_{-c}^{u^2} (\hat{H}\tilde{\delta}) \iota_{\frac{\partial}{\partial u^2}} \nabla_{\frac{\partial}{\partial u^2}}(\tilde{\delta}) \Big| \\ &\leq C\hbar^{1/2} \Big| \int_{-c}^{u^2} (\nabla_{\frac{\partial}{\partial u^2}} e^{-\frac{g}{\hbar}}) \Big| \\ &\leq C\hbar^{1/2}. \end{aligned}$$

This follows from the fact that ∇g_{-m} vanishes along ℓ up to first order, which gives an extra order $\hbar^{1/2}$ of vanishing upon integrating against $e^{-\frac{g}{\hbar}}$. Similar, we can show that

$$|\hat{H}\left(\tilde{\delta}(\nabla_{\partial_n}(\hat{H}\tilde{\delta})) \right)| \leq C\hbar^{1/2}.$$

Therefore we have

$$\begin{aligned} \varphi_2 &= \sum_k a_{2k} (\hat{H}\tilde{\delta}) \mathfrak{w}^{-km} \partial_n + \sum_{k_1,k_2} a_{1k_1} a_{1k_2} \mathcal{O}_{loc}(\hbar^{1/2}) \mathfrak{w}^{-(k_1+k_2)m} \partial_n \\ &= \begin{cases} \sum_k a_{2k} \mathfrak{w}^{-km} \partial_n + \bigoplus_{k \geq 1} \mathcal{O}_{loc}(\hbar^{1/2}) \mathfrak{w}^{-km} \partial_n & \text{on } pr^{-1}(H_+), \\ \bigoplus_{k \geq 1} \mathcal{O}_{loc}(\hbar^{1/2}) \mathfrak{w}^{-km} \partial_n & \text{on } pr^{-1}(H_-). \end{cases} \end{aligned}$$

Here the notation $\bigoplus_{k \geq 1} \mathcal{O}_{loc}(\hbar^{1/2}) \mathfrak{w}^{-km} \partial_n$ stands for a finite sum of terms of the form $\phi \mathfrak{w}^{-km} \partial_n$ with $\phi \in \mathcal{O}_{loc}(\hbar^{1/2})$ on the corresponding open subsets. We are going to argue that this also holds true for general φ_s. To study the order of $\hbar$ in derivatives of the function $e^{-\frac{g}{\hbar}}$, we need the stationary phase approximation.

Lemma 4.14. *Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 with coordinates $x_1, \ldots, x_n$. Let $\varphi : U \to \mathbb{R}_{\geq 0}$ be a Morse function with unique minimum $\varphi(0) = 0$ in U. Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be a set of Morse coordinates near 0 such that*

$$\varphi(x) = \frac{1}{2}(\tilde{x}_1^2 + \cdots + \tilde{x}_n^2).$$

For every compact subset $K \subset U$, there exists a constant $C = C_{K,N}$ such that for every $u \in C^\infty(U)$ with $\operatorname{supp}(u) \subset K$, we have

$$\begin{aligned} &|(\int_K e^{-\varphi(x)/\hbar} u) - (2\pi\hbar)^{n/2}\Big(\sum_{k=0}^{N-1} \frac{\hbar^k}{2^k k!} \tilde{\Delta}^k (\frac{u}{\Im})(0)\Big)| \\ \leq \;& C\hbar^{n/2+N} \sum_{|\alpha| \leq 2N+n+1} \sup |\partial^\alpha u|, \end{aligned} \tag{4.9}$$

where

$$\tilde{\Delta} = \sum \frac{\partial^2}{\partial \tilde{x}_j^2}, \qquad \Im = \pm \det(\frac{d\tilde{x}}{dx}),$$

and $\Im(0) = (\det \nabla^2 \varphi(0))^{1/2}$.

In particular, if u vanishes at 0 up to order L, then we can take $N = \lceil L/2 \rceil$ and get

$$|\int_K e^{-\varphi(x)/\hbar} u| \leq C\hbar^{n/2 + \lceil L/2 \rceil}.$$

Lemma 4.15. *We have the norm estimate*

$$\left(\int_{u^1=a} |\nabla^j (e^{-\frac{g}{\hbar}})|^{2^k}\right)^{\frac{1}{2^k}} \leq C_{j,k} \hbar^{-\frac{j}{2} + \frac{1}{2^{k+1}}}$$

for any $j, k \in \mathbb{Z}_{\geq 0}$ and arbitrary $a \in \mathbb{R}$.

Proof. First we notice that $\nabla^j(e^{-g/\hbar})$ consists of terms of the form

$$\hbar^{-N} \left(\prod_{i=1}^N (\nabla^{s_i} g)\right) e^{-g/\hbar},$$

where $\sum_i s_i = j$. We see that

$$\nabla^l \left(\prod_{i=1}^N (\nabla^{s_i} g)\right)|_{\{u^2=0\}} \equiv 0,$$

for $l \leq \sum_{i=1}^N \max(0, 2 - s_i) =: L$. We observe that the terms with the lowest $\hbar$ power are either of the form $\hbar^{-\lfloor \frac{j+1}{2} \rfloor} \prod_{i=1}^{\lfloor \frac{j+1}{2} \rfloor} (\nabla^{s_i} g) e^{-g/\hbar}$ having $s_i \leq 2$, or of the form $\hbar^{-j} \prod_{i=1}^j (\nabla^{s_i} g) e^{-g/\hbar}$ having $s_i = 1$. In both cases, applying the stationary phase approximation in Lemma 4.14, we obtain

$$\left(\int_{u^1=a} |\nabla^j (e^{-g/\hbar})|^{2^k}\right)^{\frac{1}{2^k}} \leq C_{j,k} \hbar^{-\frac{j}{2} + \frac{1}{2^{k+1}}},$$

for arbitrary a. $\square$

Motivated by the above lemma, we consider a filtration

$$\cdots \subset F^{-s} \subset \dots F^{-1} \subset F^0 \subset F^1 \subset F^2 \subset \cdots \subset F^s \subset \cdots \subset C^\infty(B_0)$$

of the space of smooth functions on B_0 and derive some properties of these functions. Fixing a real number u_0^2, we have the following definition.

Definition 4.16. *A smooth function ϕ is in F^s if for any compact subset $K \subset B_0$, real number $a \in \mathbb{R}$ and any $j, k \in \mathbb{Z}_+$, we have $\phi = 0$ on $\mathbb{R} \times (u_0^2 - \epsilon, u_0^2 + \epsilon)$ and*

$$\|\nabla^j \phi\|_{L^{2^k}(K_a)} \leq C_{j,k,K} \hbar^{-\frac{j+s}{2} + \frac{1}{2^{k+1}}},$$

where $K_a = K \cap \{u^1 = a\}$.

So Lemma 4.15 says that $e^{-g/\hbar} \in F^0$.

Proposition 4.17. *We have $\nabla F^s \subset F^{s+1}$ and $F^s \cdot F^r \subset F^{r+s}$, where $\cdot$ denotes multiplication of functions.*

Proof. The first property is trivial. For the relation $F^s \cdot F^r \subset F^{r+s}$, we fix $j \in \mathbb{Z}_+$ and a compact subset K. For $\phi \in F^r$ and $\psi \in F^s$, we first observe that

$$\nabla^j(\phi\psi) = \sum_{k+l=j} (\nabla^k \phi) \otimes (\nabla^l \psi).$$

Then the Hölder inequality implies that

$$\begin{aligned} \|(\nabla^k \phi) \otimes (\nabla^l \psi)\|_{L^{2^k}(K_a)} &\leq C \|\nabla^k \phi\|_{L^{2^{k+1}}(K_a)} \|\nabla^l \psi\|_{L^{2^{k+1}}(K_a)} \\ &\leq C \hbar^{-\frac{k+s}{2} + \frac{1}{2^{k+2}}} \cdot \hbar^{-\frac{l+r}{2} + \frac{1}{2^{k+2}}} \\ &\leq C \hbar^{-\frac{j+r+s}{2} + \frac{1}{2^{k+1}}} \end{aligned}$$

and the result follows. $\square$

It is straightforward to extend the definition of $F^\bullet(\Omega^*(B_0))$ to differential forms $\Omega^*(B_0)$ and differential forms with values in $TB_0^{\mathbb{C}}$. Notice that we work on the base B_0 and use the metric $g = (dx^1)^2 + (dx^2)^2$ which is independent of $\hbar$ in defining these filtrations.

Lemma 4.18. *Given constant vector fields $v_1, v_2 \in \Gamma(B_0, TB_0^{\mathbb{C}})$, $\phi \in F^s(\Omega^1(B_0))$ and $\psi \in F^r(\Omega^1(B_0))$, we have*

$$[(\hat{H}\phi) \otimes v_1, \psi \otimes v_2] \in F^{r+s}(\Omega^1(B_0, TB_0^{\mathbb{C}})),$$

where $[\cdot,\cdot]$ is the Lie bracket structure on $\Omega^*(B_0, TB_0^{\mathbb{C}})$.

Proof. First of all, we have

$$[(\hat{H}\phi) \otimes v_1, \psi \otimes v_2] = (\hat{H}\phi)(\nabla_{v_1}\psi) \otimes v_2 - \psi \nabla_{v_2}(\hat{H}\phi) \otimes v_1,$$

so we may consider the functions $(\hat{H}\phi)(\nabla_{v_1}\psi)$ and $\psi\nabla_{v_2}(\hat{H}\phi)$ separately. Fixing $k, l \in \mathbb{Z}_+$ such that $k + l = j$, we consider terms of the form

$$\nabla^k_{\frac{\partial}{\partial u^1}}(\hat{H}\phi)\nabla^l_{\frac{\partial}{\partial u}}(\nabla_{v_1}\psi),$$

or terms of the form

$$\nabla^k_{\frac{\partial}{\partial u}}(\nabla_{\frac{\partial}{\partial u^2}}\hat{H}\phi)\nabla^l_{\frac{\partial}{\partial u}}(\nabla_{v_1}\psi),$$

with $k \geq 1$, where $\frac{\partial}{\partial u}$ can either be $\frac{\partial}{\partial u^1}$ or $\frac{\partial}{\partial u^2}$. Since we have $\nabla_{\frac{\partial}{\partial u^2}} \hat{H}\phi = \phi$ so the second case follows from the previous Lemma 4.15. For the first case, we have $\nabla^k_{\frac{\partial}{\partial u^1}}(\hat{H}\phi) = \hat{H}(\nabla^k_{\frac{\partial}{\partial u^1}}\phi)$ and hence

$$\begin{aligned}
\left(\int_{K_a} |\hat{H}(\nabla^k_{\frac{\partial}{\partial u^1}}\phi)\nabla^l_{\frac{\partial}{\partial u}}(\nabla_{v_1}\psi)|^{2^t}\right)^{\frac{1}{2^t}} &\leq \|\hat{H}(\nabla^k_{\frac{\partial}{\partial u^1}}\phi)\|_{L^\infty(K_a)}\left(\int_{K_a} |\nabla^l_{\frac{\partial}{\partial u}}(\nabla_{v_1}\psi)|^{2^t}\right)^{\frac{1}{2^t}} \\
&\leq C\hbar^{-\frac{k+s}{2}+\frac{1}{2}}\hbar^{-\frac{l+r+1}{2}+\frac{1}{2^{t+1}}} \\
&\leq C\hbar^{-\frac{j+s+r}{2}+\frac{1}{2^{t+1}}}.
\end{aligned}$$

The argument for the term $\psi\nabla_{v_2}(\hat{H}\phi)$ is similar. □

Now we go back to the semi-classical analysis of φ. We will inductively show that the term

$$H\left(\sum_{k\geq 0} \frac{ad^k_{\varphi^s}}{(k+1)!} d_W \varphi^s\right)_{s+1},$$

does not contribute to the leading $\hbar$ order term in the definition of φ_{s+1} given by Equation (4.6).

Lemma 4.19. *For $k > 0$, we have*

$$ad^k_{\varphi^s}(d_W\varphi^s) \in \bigoplus_{j,l\geq 1} F^0(\Omega^1(B_0)) \cdot (\mathfrak{w}^{-lm}\partial_n)t^j$$

for all s. Here $\bigoplus_{j,l\geq 1} F^0(\Omega^1(B_0)) \cdot (\mathfrak{w}^{-lm}\partial_n)t^j$ is a finite sum of terms defined by viewing $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$ as a module over $\Omega^*(B_0)$.

Proof. We proof the above statement by induction on s. The first case is when $s = 1$, which concerns the term $ad^k_{\varphi_1}(d_W\varphi_1) = -ad^k_{\varphi_1}(\breve{\Xi}_1)$. From the equation (4.8), the case for $k = 1$ is equivalent to

$$(\hat{H}\tilde{\delta})(\nabla_{\partial_n}\tilde{\delta}) - \tilde{\delta}(\nabla_{\partial_n}(\hat{H}\tilde{\delta})) \in F^0(\Omega^1(B_0)).$$

Notice that we have

$$\frac{\hbar|n|}{4\pi}[\hat{H}\tilde{\delta} \otimes \frac{\partial}{\partial u^2}, \tilde{\delta} \otimes \frac{\partial}{\partial u^2}] = (\hat{H}\tilde{\delta})(\nabla_{\partial_n}\tilde{\delta}) - \tilde{\delta}(\nabla_{\partial_n}(\hat{H}\tilde{\delta})),$$

where $[\cdot,\cdot]$ is the Lie bracket for B_0. Therefore the case $k = 1$ follows from Lemma 4.18 and the fact that $\tilde{\delta} \in F^1(\Omega^1(B_0))$. For $k \geq 2$, we can apply Lemma 4.18 and an induction on k.

Assuming it is true for all $l \leq s$, we first notice that

$$d_W\varphi_{s+1} = -\left(\breve{\Xi} + \sum_{k\geq 0} \frac{ad^k_{\varphi^s}}{(k+1)!} d_W\varphi^s\right)_{s+1} \in \bigoplus_{j,l\geq 1} F^1(\Omega^1(B_0)) \cdot (\mathfrak{w}^{-lm}\partial_n)t^j.$$

Therefore we have

$$\varphi^{s+1} \in \bigoplus_{j,l\geq 1} \hat{H}\left(F^1(\Omega^1(B_0))\right) \cdot (\mathfrak{w}^{-lm}\partial_n)t^j.$$

Applying Lemma 4.18 again, we obtain the desired result. □

From Lemma 4.19, we see that

$$\varphi_s = -\hat{H}\breve{\Xi}_s + \bigoplus_{l\geq 1} \hat{H}\left(F^0(\Omega^1(B_0))\right)\cdot(\mathfrak{w}^{-lm}\partial_n) \tag{4.10}$$

for all s. For $\phi \in F^0(\Omega^1(B_0))$, we have $\|\hat{H}(\phi)\|_{L^\infty(K)} \leq \hbar^{\frac{1}{2}}$ for an arbitrary compact subset $K \subset B_0$. Therefore only the term $-\hat{H}\breve{\Xi}_s$ contributes to the leading order in $\hbar$. As a conclusion, we have the following proposition.

Proposition 4.20. *For $\varphi = \varphi_1 + \varphi_2 + \dots$ defined by the equation (4.6), we have*

$$\varphi_s = \begin{cases} \sum_k a_{sk}\mathfrak{w}^{-km}\partial_n + \bigoplus_{k\geq 1}\mathcal{O}_{loc}(\hbar^{1/2})\mathfrak{w}^{-km}\partial_n & \text{on } pr^{-1}(H_+), \\ \bigoplus_{k\geq 1}\mathcal{O}_{loc}(\hbar^{1/2})\mathfrak{w}^{-km}\partial_n & \text{on } pr^{-1}(H_-), \end{cases}$$

or equivalently

$$\hat{\mathcal{F}}(\varphi) = \check{\varphi}_0 + \bigoplus_{k,j\geq 1}\mathcal{O}_{loc}(\hbar^{1/2})(w^{-km}\check{\partial}_n)t^j,$$

on $\check{X}_0 \setminus \check{p}^{-1}(\ell)$.

5. Maurer-Cartan solutions and scattering

In this section, we are going to interpret the scattering process, which produces the monodromy free diagram $\mathcal{S}(\mathcal{D})$ from a standard scattering diagram $\mathcal{D}$ consisting of two non-parallel walls in the affine manifold $B_0 = \mathbb{R}^2$, as solving a Maurer-Cartan (MC) equation. This is done by considering the MC equation in the Kodaira-Spencer dgLa $KS_{\check{X}_0}$, or equivalently the mirror dgLa L_{X_0}, and then letting $\check{X}_0$ degenerate to its large complex structure limit by sending the parameter $\hbar \to 0$ to obtain a semi-classical approximation for the solution.

5.1. Solving Maurer-Cartan equations in general.

Since we are concerned with solving the Maurer-Cartan equation (2.2) for a dgLa $(L, d, [\cdot,\cdot])$ over the formal power series ring R, we can solve the non-linear equation by solving linear equations inductively. We use Kuranishi's method which solves the MC equation with the help of a homotopy retracting L^* to its cohomology $H^*(L)$ that acts as gauge fixing; see e.g. [14].

Suppose we are given initial data

$$\breve{\Xi} = \breve{\Xi}_1 + \breve{\Xi}_2 + \dots$$

satisfying $d\breve{\Xi} = 0$, where $\breve{\Xi}_k \in L^2 \otimes (\mathbf{m}^k/\mathbf{m}^{k+1})$ is homogeneous of degree k. We attempt to find

$$\Xi = \Xi_2 + \Xi_3 + \dots$$

such that

$$\Phi = \Phi_1 + \Phi_2 + \dots,$$

defined by $\Phi_k = \breve{\Xi}_k + \Xi_k \in L^2 \otimes (\mathbf{m}^k/\mathbf{m}^{k+1})$, is a solution of the MC equation (2.2), i.e.

$$d\Phi + \frac{1}{2}[\Phi,\Phi] = 0.$$

This equation can be solved inductively. The first equation

$$d\Phi + \frac{1}{2}[\Phi,\Phi] = 0 \ (\mathrm{mod}\ \mathbf{m}^2) \Longleftrightarrow d\breve{\Xi}_1 = 0$$

is automatic. Writing $\Phi^k = \Phi_1 + \Phi_2 + \cdots + \Phi_k$ and suppose it satisfies

$$d\Phi^k + \frac{1}{2}[\Phi^k, \Phi^k] = 0 \ (\text{mod } \mathbf{m}^{k+1}),$$

we let

$$d\Phi^k + \frac{1}{2}[\Phi^k, \Phi^k] = \mathfrak{O}_{k+1} \ (\text{mod } \mathbf{m}^{k+2}).$$

Solving the MC equation for degree $k+1$ is same as solving

$$d\Xi_{k+1} + \mathfrak{O}_{k+1} = 0. \tag{5.1}$$

We first observe that $d\mathfrak{O}_{k+1} = 0$ since

$$\begin{aligned} d\mathfrak{O}_{k+1} &= [d\Phi^k, \Phi^k] \\ &= [\mathfrak{O}_{k+1}, \Phi^k] - \frac{1}{2}[[\Phi^k, \Phi^k], \Phi^k] = 0 \ (\text{mod } \mathbf{m}^{k+2}). \end{aligned}$$

Therefore we can solve the MC equation for degree $k+1$ if and only if the k-th obstruction class defined by $\mathfrak{O}_{k+1}$ in $H^1(L, d)$ vanishes.

The solutions Ξ_{k+1} for the equation (5.1) may differ by a d-closed element in L^1. This ambiguity can be fixed by choosing a homotopy retract of L to its cohomology $H^*(L)$. We assume that there are chain maps ι, P and homotopy H

$$\begin{tikzcd} H^*(L) \arrow[r, bend left, "\iota"] & L^* \arrow[l, bend left, "P"] \arrow[loop right, "H"] \end{tikzcd},$$

such that

$$\begin{aligned} P \circ \iota &= id, \\ id_L - \iota \circ P &= dH + Hd. \end{aligned}$$

Then, instead of the MC equation, we look for solutions Φ of the equation

$$\Phi = \breve{\Xi} - \frac{1}{2}H[\Phi, \Phi]. \tag{5.2}$$

Proposition 5.1. *Suppose that Φ satisfies the equation* (5.2)*. Then Φ satisfies the MC equation* (2.2) *if and only if $P[\Phi, \Phi] = 0$.*

Proof. Applying d on both sides of Equation (5.2), we obtain

$$d\Phi + \frac{1}{2}[\Phi, \Phi] = \frac{1}{2}(Hd[\Phi, \Phi] + \iota \circ P[\Phi, \Phi]).$$

Suppose that Φ satisfies the MC equation. Then we see that $d[\Phi, \Phi] = -[[\Phi, \Phi], \Phi] = 0$ and hence $P[\Phi, \Phi] = 0$.

For the converse, we let $\delta = d\Phi + \frac{1}{2}[\Phi, \Phi]$. It follows from the assumption $P[\Phi, \Phi] = 0$ that

$$\delta = H[d\Phi, \Phi] = H[\delta, \Phi] = (H \circ ad_\Phi)^m(\delta)$$

for any $m \in \mathbb{Z}_+$. Then by the fact that $\Phi \in L \otimes \mathbf{m}$, we have $\delta = 0$ (mod $\mathbf{m}^m$) by comparing the lowest order term. $\square$

Now we look at the equation (5.2)

$$\Xi + \frac{1}{2} H[\Phi, \Phi] = 0,$$

and try to solve it order by order. The first equation is simply $\Phi_1 = \breve{\Xi}_1$, and the second equation is

$$\Xi_2 + \frac{1}{2} H[\breve{\Xi}_1, \breve{\Xi}_1] = 0.$$

The k-th equation is

$$\Xi_k + \sum_{j+l=k} \frac{1}{2} H[\Phi_j, \Phi_l] = 0, \tag{5.3}$$

and Ξ_k is uniquely determined by the previous Ξ_j's.

Remark 5.2. *For a compact complex manifold $\check{X}$ equipped with a Hermitian metric, an explicit homotopy for its Kodaira-Spencer dgLa $KS_{\check{X}} = (\Omega^{0,*}(\check{X}, T^{1,0}_{\check{X}}), \bar{\partial}, [\cdot,\cdot])$ is given as follows. The Hermitian metric define $\bar{\partial}^*$ and the corresponding Green's operator G for $\Delta_{\bar{\partial}}$. We can then take $H^*(KS_{\check{X}}) = \mathcal{H}^{0,*}(\check{X}, T^{1,0}_{\check{X}})$ to be the space of harmonic forms, and let ι be the natural embedding and P be the harmonic projection. The homotopy operator is explicitly given by $H = \bar{\partial}^* G$.*

In the case $\breve{\Xi} = \breve{\Xi}_1 \in \mathcal{H}^{0,*}(\check{X}, T^{1,0}_{\check{X}})$ and $P[\Phi, \Phi] = 0$, we can solve the MC equation by iteratively solving (5.2)*. It can further be shown, using elliptic estimates, that the formal power series in t_i's we obtained indeed converges for small enough t_i's. This was originally due to M. Kuranishi; for details, we refer the reader to* [14].

There is also a combinatorial way to write down the solution Ξ from the input $\breve{\Xi}$ in terms of summing over trees.

Definition 5.3. *A* directed trivalent planar k-tree*, or just* k-tree*, T is an embedded trivalent (i.e. every vertex is trivalent, having two incoming edges and one out-going edge) tree in $\mathbb{R}^2$ together with the following data:*

- *a finite set of vertices $V(T)$,*
- *a set of internal edges $E(T)$,*
- *k semi-infinite incoming edges $E_{in}(T)$, and*
- *one semi-infinite outgoing edge e_{out}.*

Given a directed trivalent planar k-tree T, we define an operation

$$\mathfrak{l}_{k,T} : L^{\otimes k} \to L[1-k],$$

by

(1) aligning the inputs at the k semi-infinite incoming edges,
(2) applying the Lie bracket $[\cdot,\cdot]$ to each interior vertex, and
(3) applying the homotopy operator $-\frac{1}{2}H$ to each internal edge and the outgoing semi-infinite edge e_{out}.

We then let

$$\mathfrak{l}_k = \sum_T \mathfrak{l}_{k,T},$$

where the summation is over all directed trivalent planar k-trees. Finally if we define Φ by

$$\Phi = \sum_{k\geq 1} \mathfrak{l}_k(\breve{\Xi}, \ldots . \breve{\Xi}), \tag{5.4}$$

and Ξ by

$$\Xi = \sum_{k\geq 2} \mathfrak{l}_k(\breve{\Xi}, \ldots . \breve{\Xi}), \tag{5.5}$$

then $\Phi = \breve{\Xi} + \Xi$ is the unique solution to Equation (5.2).

5.2. **Solving the Maurer-Cartan equation with two walls.** Suppose that we have a standard scattering diagram $\mathscr{D}$ over $R = \mathbb{C}[[t_1, t_2]]$, consisting of two non-parallel walls in $\mathbb{R}^2$ intersecting transversally at the origin and equipped with the wall data $\mathbf{w}_i = (-m_i, \ell_i, \Theta_i)$ such that $\{m_1 \lrcorner g, m_2 \lrcorner g\}$ is positively oriented according to orientation of B_0. We abbreviate δ_{-m_i}, $\tilde{\delta}_{-m_i}$, g_{-m_i}, μ_{-m_i} and U_{-m_i} as δ_i, $\tilde{\delta}_i$, g_i, μ_i and U_i respectively, with $\tilde{\delta}_i$ compactly supported in $U_i \supset \ell_i$. Assume that the wall crossing factors $\breve{\phi}_i$ are of the form

$$\breve{\phi}_i = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}^{(i)} w^{-km_i} t_i^j, \ i = 1, 2.$$

We apply the ansatz

$$\breve{\Xi}^{(i)} = -\sum_{j,k\geq 1} a_{jk}^{(i)} \tilde{\delta}_i(\mathfrak{w}^{-km_i}\partial_{n_i}) t_i^j$$

as in Definition 4.4. Then we solve the MC equation of the dgLa $(L_{X_0}, d_W, \{\cdot,\cdot\})$ with the input data

$$\breve{\Xi} = \breve{\Xi}^{(1)} + \breve{\Xi}^{(2)},$$

by applying the process described in the previous Section 5.1. Notice that the two walls divide B_0 into 4 quadrants as shown in Figure 5, in order to define the homotopy, we choose

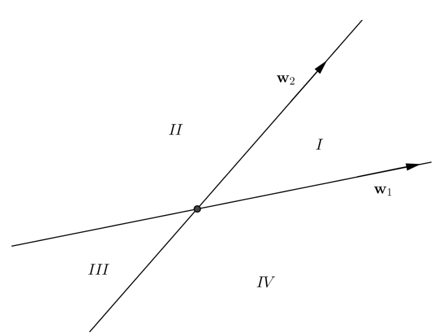

FIGURE 5.

the base point $q_{m,0}$ in Definition 4.8 to be lying in the third quadrant and sitting far away from the origin. Writing $\Phi = \mathit{\Xi} + \breve{\mathit{\Xi}}$ as before, the terms in $\mathit{\Xi}$ are determined iteratively by Equation (5.3).

Now, because of the fact that $\check{X}_0$ has no non-trivial deformations of complex structure, the solution Φ must be gauge equivalent to 0, i.e. there exists $\varphi \in \Omega^0(\mathcal{M}, TB_0^{\mathbb{C}}) \otimes \mathbf{m}$ solving the equation $e^{\varphi} * 0 = \Phi$. Our goal in this section is to show that the semi-classical limit of φ determines the scattering diagram $\mathcal{S}(\mathscr{D})$.

First of all, for the purpose of solving the MC equation with our specific input as above, we can restrict ourself to a differential graded Lie subalgebra of $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$.

Definition 5.4. *We let*

$$\mathcal{L}^*_{\mathscr{D}} = \bigoplus_{a \in (\mathbb{Z}_{\geq 0})^2_{prim}} \mathcal{L}^*_a \hookrightarrow \Omega^*(\mathcal{M}, TB_0^{\mathbb{C}}),$$

where $m_a = a^1 m_1 + a^2 m_2$ *for* $a = (a^1, a^2)$ *and*

$$\mathcal{L}^*_a = \bigoplus_{i=1,2} \bigoplus_{k \geq 1} \Omega^*(B_0) \cdot (\mathfrak{w}^{-k m_a} \partial_{n_i}),$$

defined using the module structure of $\Omega^*(\mathcal{M}, TB_0^{\mathbb{C}})$ *over* $\Omega^*(B_0)$. *Here* n_i *is the unique primitive vector normal to* $m_i \lrcorner g$ *such that* $\{m_i \lrcorner g, n_i\}$ *is positively oriented.*

Restricting to $\mathcal{L}^*_{\mathscr{D}}$, we find that the Lie bracket $\{\cdot,\cdot\}$ can be explicitly written in terms of the usual differentiation on the base B_0.

Definition 5.5. *For* $\alpha = \sum_j f_j \mathfrak{w}^{-k_j m_a} \partial_{n_j} \in \mathcal{L}^*_a$ *and* $\beta = \sum_l g_l \mathfrak{w}^{-k_l m_b} \partial_{n_l} \in \mathcal{L}^*_b$, *we can decompose the Lie bracket* $\{\cdot,\cdot\}$ *into three operators* $\natural$, $\sharp$ *and* $\flat$ *given by*

$$\begin{aligned}
\natural(\alpha,\beta) &= \sum_{j,l} f_j g_l \{\mathfrak{w}^{-k_j m_a} \partial_{n_j}, \mathfrak{w}^{-k_l m_b} \partial_{n_l}\},\\
\sharp(\alpha,\beta) &= \sum_{j,l} f_j (\nabla_{\partial_{n_j}} g_l) \mathfrak{w}^{-(k_j m_a + k_l m_b)} \partial_{n_l},\\
\flat(\alpha,\beta) &= (-1)^{\bar{f}_j + \bar{g}_l + 1} \sum_{j,l} g_l (\nabla_{\partial_{n_l}} f_j) \mathfrak{w}^{-(k_j m_a + k_l m_b)} \partial_{n_j}.
\end{aligned}$$

Definition 5.6. *We further consider an* $\Omega^*(B_0)$*-submodule* $\mathbf{h}^*_a$ *of* $\mathcal{L}^*_a$ *defined by*

$$\mathbf{h}^*_a = \bigoplus_{k \geq 1} \Omega^*(B_0) \cdot (\mathfrak{w}^{-k m_a} \partial_{n_a}) \hookrightarrow \mathcal{L}^*_a,$$

where $n_a \in m_a^{\perp} \subset N$ *is the unique normal vector determine by* m_a *and the orientation. We let*

$$\mathbf{h}^*_{\mathscr{D}} = \bigoplus_{a \in (\mathbb{Z}_{\geq 0})^2_{prim}} \mathbf{h}^*_a.$$

Note that $\mathbf{h}^*_a$ *(for any* a*) and* $\mathbf{h}^*_{\mathscr{D}}$ *are closed under the operation* $\natural$.

We will also decompose the operation $\mathfrak{l}_k$ as a summation over colored k-trees.

Definition 5.7. *A* colored k-tree $\mathcal{T}$ *is a* k*-tree together with a labeling of the internal vertices by* $\natural, \sharp$ *or* $\flat$.

Definition 5.8. *Given a colored k-tree $\mathcal{T}$, we define the corresponding operator*

$$\mathfrak{l}_{k,\mathcal{T}} : L^{\otimes k} \to L[1-k],$$

by

(1) *aligning the inputs at the k semi-infinite incoming edges,*
(2) *applying the operators $\natural$, $\sharp$ or $\flat$ to each interior vertex according to the color, and*
(3) *applying the homotopy operator $-\frac{1}{2}H$ to each internal edge and the outgoing semi-infinite edge.*

We separate the set of colored k-trees $\mathtt{C}^k$ into $\mathtt{C}^k_0$ and $\mathtt{C}^k_1$, where $\mathtt{C}^k_0$ consisting of trees having all vertices colored by $\natural$ and $\mathtt{C}^k_1 = \mathtt{C}^k \setminus \mathtt{C}^k_0$. We let $\mathfrak{l}_{k,0} = \sum_{\mathcal{T}\in\mathtt{C}^k_0} \mathfrak{l}_{k,\mathcal{T}}$ and $\mathfrak{l}_{k,1} = \sum_{\mathcal{T}\in\mathtt{C}^k_1} \mathfrak{l}_{k,\mathcal{T}}$. Therefore, we can decompose the solution to the MC equation in the following way.

Definition 5.9. *Given the input $\breve{\Xi} = \breve{\Xi}^{(1)} + \breve{\Xi}^{(2)}$, we define*

$$\varOmega = \sum_{k\geq 1} \mathfrak{l}_{k,0}(\breve{\Xi}, \dots, \breve{\Xi}) = \sum_{a\in(\mathbb{Z}_{\geq 0})^2_{prim}} \varOmega^{(a)},$$

with $\varOmega^{(a)} \in \mathbf{h}^*_a[[t_1,t_2]]$ and

$$\mathcal{R} = \sum_{k\geq 2} \mathfrak{l}_{k,1}(\breve{\Xi}, \dots, \breve{\Xi}) = \sum_{a\in(\mathbb{Z}_{\geq 0})^2_{prim}} \mathcal{R}^{(a)},$$

with $\mathcal{R}^{(a)} \in \mathcal{L}^*_a[[t_1,t_2]]$. Notice that we have $\varOmega^{((1,0))} = \breve{\Xi}^{(1)}$ and $\varOmega^{((0,1))} = \breve{\Xi}^{(2)}$, and also $\Phi = \varOmega + \mathcal{R}$.

Fixing an arbitrary $N_0 \in \mathbb{Z}_+$, we are going to show that $\Phi \pmod{\mathbf{m}^{N_0+1}}$ determines the scattering diagram $\mathcal{S}(\mathfrak{D}) \pmod{\mathbf{m}^{N_0+1}}$ (denoted by $\mathcal{S}_{N_0}(\mathfrak{D})$), by associating each $\varOmega^{(a)}$ to a wall $\mathbf{w}_a$ supported on the ray $\ell_a = \mathbb{R}_{\geq 0}\cdot(m_a \lrcorner g)$ and determining the wall crossing factor Θ_a (Θ_a may be trivial) from the asymptotic expansion of $\varOmega^{(a)}$. We prove this correspondence by showing that the Maurer-Cartan equation $\pmod{\mathbf{m}^{N_0+1}}$ implies that the scattering diagram $\{m_a, \ell_a, \Theta_a\}_a$ is a monodromy free scattering diagram and hence must be equivalent to $\mathcal{S}_{N_0}(\mathfrak{D})$ by the uniqueness Theorem 3.11.

To relate $\varOmega^{(a)}$ with the wall crossing factor Θ_a, we remove a closed ball $\overline{B(r_{N_0})}$ centered at the origin and consider the annulus $A = B_0 \setminus \overline{B(r_{N_0})}$ to study the monodromy around it. We use the polar coordinate (r,θ) on the universal cover $\tilde{A}$, which is isomorphic to a half plane. We fix once and for all θ_0 such that the ray $\mathfrak{R}_{\theta_0}$ is in the third quadrant (determined by the walls $\mathbf{w}_1, \mathbf{w}_2$), with a neighborhood defined by $\{\theta \mid \theta_0 - \epsilon_0 < \theta < \theta_0 + \epsilon_0\}$ away from all the possible walls $\mathbf{w}_a$ as shown in Figure 6. We will restrict ourself to the branch $\tilde{A}_0 = \{\theta \mid \theta_0 - \epsilon_0 < \theta < \theta_0 + 2\pi + \epsilon_0\}$ when we investigate the monodromy around the origin.

With the natural map $\mathtt{p} : \tilde{A} \to B_0 \setminus \overline{B(r_{N_0})}$, we consider the pullback dgLa $\tilde{L}^*_{X_0} = \Omega^*(\tilde{\mathcal{M}}, \mathtt{p}^*(TB_0^{\mathbb{C}}))$, where $\tilde{\mathcal{M}} = \mathcal{M}_{pr} \times_{\mathtt{p}} \tilde{A} \cong \tilde{A} \times M = \coprod_{m\in M} \tilde{A}_m$, equipped naturally with the pullback of the Witten differential d_W and the Lie bracket $\{\cdot,\cdot\}$.

Since $\mathtt{p} : \tilde{A} \to B_0 \setminus \overline{B(r_{N_0})}$ is a covering, we can pullback the element $\varOmega^{(a)}$ and $\mathcal{R}^{(a)}$ to $\tilde{A}$ as $\tilde{\varOmega}^{(a)}$ and $\tilde{\mathcal{R}}^{(a)}$ respectively. Choosing r_{N_0} large enough, we can divide $supp(\varOmega^{((1,0))})$ and

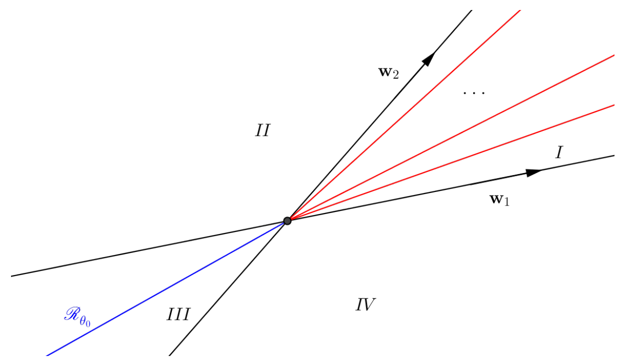

FIGURE 6.

$supp(\Omega^{((0,1))})$ which originally lie in small neighborhood of $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively, into two components in A. Therefore we can decompose $\mathtt{p}^*\Omega^{((1,0))}$ (resp. $\mathtt{p}^*\Omega^{((0,1))}$) as $\tilde{\Omega}^{((1,0))}+\tilde{\Omega}^{((-1,0))}$ (resp. $\tilde{\Omega}^{((0,1))}+\tilde{\Omega}^{((0,-1))}$), with $\tilde{\Omega}^{((1,0))}$ and $\tilde{\Omega}^{((-1,0))}$ supported near $\mathtt{p}^{-1}(\mathbb{R}_{\geq 0}\cdot m_1\lrcorner g)$ and $\mathtt{p}^{-1}(\mathbb{R}_{\geq 0}\cdot -m_1\lrcorner g)$ (similarly for $\tilde{\Omega}^{((0,1))}$ and $\tilde{\Omega}^{((0,-1))}$). We can therefore unify our notations by allowing $a\in(\mathbb{Z}_{\geq 0})^2_{prim}\cup\{(-1,0),(0,-1)\}$, and using the convention that $m_{(-1,0)}=m_{(1,0)}=m_1$ and $m_{(0,-1)}=m_{(0,1)}=m_2$.

Furthermore, we note that there are only finitely many a's involved when we modulo $\mathbf{m}^{N_0+1}$ and we denote the set of those a's by $Ray(N_0)$. For each $a\in Ray(N_0)$, we let $\theta_a\in\tilde{A}_0=\{\theta\mid\theta_0-\epsilon_0<\theta<\theta_0+2\pi+\epsilon_0\}$ be the direction of the ray determined by a. We will see that $\left\{\Omega^{(a)}+\mathcal{R}^{(a)},\Omega^{(a)}+\mathcal{R}^{(a)}\right\}=0$, and $d_W(\Omega^{(a)}+\mathcal{R}^{(a)})=0\pmod{\mathbf{m}^{N_0+1}}$ on $pr^{-1}(A)$ in the following Lemma 5.10, whose proof will be given in Section 5.4.

Lemma 5.10. *For each N_0, there exist r_{N_0} large enough and ϵ_{N_0} such that*

$$d_W(\Omega^{(a)}+\mathcal{R}^{(a)})=0=\{\Omega^{(a)}+\mathcal{R}^{(a)},\Omega^{(a)}+\mathcal{R}^{(a)}\}\ (mod\ \mathbf{m}^{N_0+1}),$$

on A. Furthermore, for $W_{a,k}=\{(r,\theta)\mid\theta_a-\epsilon_{N_0}+2k\pi<\theta_a<\theta_a+\epsilon_{N_0}+2k\pi\}$ defined for each ray $a\in Ray(N_0)$ and integer $k\in\mathbb{Z}$ which are disjoint from each others, we have

$$supp(\tilde{\Omega}^{(a)})\cup supp(\tilde{\mathcal{R}}^{(a)})\subset\bigcup_{k\in\mathbb{Z}}W_{a,k}\quad(mod\ \mathbf{m}^{N_0+1}),$$

Figure 7 shows the situation for an $a\in Ray(N_0)$ on the branch $\tilde{A}_0$.

This means that $\Omega^{(a)}+\mathcal{R}^{(a)}$ is itself a Maurer-Cartan element and hence is gauge equivalent to 0 via some gauge $\varphi_a\in\Omega^*(\tilde{\mathcal{M}},\mathtt{p}^*TB_0^{\mathbb{C}})\pmod{\mathbf{m}^{N_0+1}}$ on the universal cover $\tilde{A}$, i.e.

$$e^{\varphi_a}*0=\Omega^{(a)}+\mathcal{R}^{(a)}\pmod{\mathbf{m}^{N_0+1}}.$$

To find φ_a, we first define the homotopy operator $\mathcal{H}$ on $\tilde{L}_{X_0}$ similar to that in Definition 4.8. We fix a base point $(r_0,\theta_0)\in\tilde{A}$, and define a path $\hat{\varrho}_{(r,\theta)}$

$$\hat{\varrho}_{(r,\theta)}(t)=\begin{cases}((1-2t)r_0+2tr,\theta_0) & \text{if } t\in[0,\frac{1}{2}],\\ (r,(2t-1)\theta+(2-2t)\theta_0) & \text{if } t\in[\frac{1}{2},1]\end{cases}$$

for any given point $(r,\theta)\in\tilde{A}$. We define a homotopy on $\tilde{A}$ similar to that in Definition 4.6.

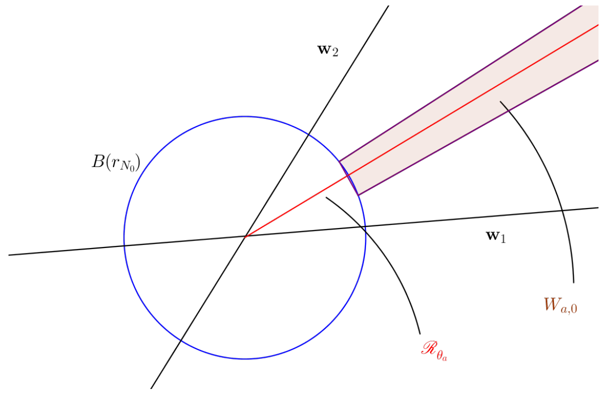

FIGURE 7.

Definition 5.11. *We define $\hat{\mathcal{H}} : \Omega^*(\tilde{A}) \to \Omega^*(\tilde{A})[-1]$ by*

$$(\hat{\mathcal{H}}\alpha)(r,\theta) = \begin{cases} \int_{\hat{\varrho}(r,\theta)} \alpha & \text{for } \alpha \in \Omega^1(\tilde{A}), \\ \int_{\theta_0}^{\theta} \left(\iota_{\frac{\partial}{\partial\theta}} \alpha(r,s)\right) ds & \text{for } \alpha \in \Omega^2(\tilde{A}). \end{cases} \tag{5.6}$$

We also let $\hat{\mathcal{P}} : \Omega^*(\tilde{A}) \to H^*(\tilde{A})$ be the evaluation at (r_0, θ_0) and $\hat{\iota} : H^*(\tilde{A}) \to \Omega^*(\tilde{A})$ embedding of constant functions on $\tilde{A}$.

Definition 5.12. *Fixing a point $q_{m,0} \in \tilde{A}_m$ mapping to (r_0, θ_0) via $\mathtt{p}$, and treating $f_m : \tilde{A}_m \to \mathbb{C}$, we define the homotopy $\hat{\mathcal{H}}_m = e^{-f_m}\hat{\mathcal{H}}e^{f_m} : \Omega^*(\tilde{A}_m) \to \Omega^*(\tilde{A}_m)[-1]$, projection $\hat{\mathcal{P}}_m = \hat{\mathcal{P}}e^{f_m} : \Omega^*(\tilde{A}_m) \to H^*(\tilde{A}_m)$ and the inclusion $\hat{\iota}_m = e^{-f_m}\hat{\iota}$. They are extended to $\Omega^*(\tilde{\mathcal{M}}, \mathtt{p}^*TB_0^{\mathbb{C}})$ as $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$ by applying $\hat{\mathcal{H}}_m$ and $\hat{\mathcal{P}}_m$ on each component of $\tilde{\mathcal{M}}$.*

Definition 5.13. *We let φ_a be the gauge associated to $\Omega^{(a)}$ by solving the following equation iteratively*

$$\varphi_a = -\hat{\mathcal{H}}\left(\frac{ad_{\varphi_a}}{e^{ad_{\varphi_a}} - I}\right)(\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)}) \pmod{\mathbf{m}^{N_0+1}},$$

under the gauge fixing condition $\mathcal{P}(\varphi_a) = 0 \pmod{\mathbf{m}^{N_0+1}}$.

Lemma 5.14. *We have*

$$\varphi_a \in \bigoplus_{i=1,2} \bigoplus_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} \mathbb{C}_\hbar \cdot (\mathfrak{w}^{-km_a}\partial_{n_i}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}$$

when restricted to $\tilde{A}_0 \setminus \bigcup_{k\in\mathbb{Z}} W_{a,k}$, i.e. $\varphi_a \pmod{\mathbf{m}^{N_0+1}}$ has locally constant coefficients (only depend on $\hbar$) away from $supp(\tilde{\Omega}^{(a)}) \cup supp(\tilde{\mathcal{R}}^{(a)})$. Here the notation $\mathbb{C}_\hbar$ stands for complex numbers depending only on $\hbar$.

Proof. Since $\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)} \pmod{\mathbf{m}^{N_0+1}}$ is a Maurer-Cartan element in $\tilde{L}_{X_0} \otimes (\mathbf{m}/\mathbf{m}^{N_0+1})$ which is known to be gauge equivalent to $0 \pmod{\mathbf{m}^{N_0+1}}$, we have

$$d_W\left(\frac{ad_{\varphi_a}}{e^{ad_{\varphi_a}} - I}\right)(\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)}) = 0 \pmod{\mathbf{m}^{N_0+1}},$$

under the gauge fixing condition. The homotopy operator $\hat{\mathcal{H}}$ defined by a path integral of 1-forms is independent of the path chosen when applying to solve for φ_a iteratively. For two point q, q' in the same component of $\tilde{A} \setminus \bigcup_{k\in\mathbb{Z}} W_{a,k}$, we can join them by a path $\varrho_{q,q'} \subset \tilde{A} \setminus \bigcup_{k\in\mathbb{Z}} W_{a,k}$ so that $\varrho_q^{-1} \circ \varrho_{q,q'} \circ \varrho_{q'}$ gives a closed loop in $\tilde{A}$. Therefore we see that the value of the path integral is the same at q and q' when applied to a closed form, since integral on $\varrho_{q,q'}$ gives zero. □

We will see later that $\tilde{\mathcal{R}}^{(a)}$ is in fact redundant and we can simply solve the linear equation

$$\varphi_a = -\hat{\mathcal{H}}(\tilde{\Omega}^{(a)}) \pmod{\mathbf{m}^{N_0+1}}$$

when we restrict our attention to the leading order terms of φ_a in $\hbar$. We will define a filtration

$$\cdots \subset F^{-s} \subset \ldots F^{-1} \subset F^0 \subset F^1 \subset F^2 \subset \cdots \subset F^s \subset \cdots \subset \Omega^*(\tilde{A})$$

of $\Omega^*(\tilde{A})$ similar to that in Definition 4.16.

Definition 5.15. *A smooth form $\alpha \in \Omega^*(\tilde{A})$ is in F^s if for any compact subset $K \subset \tilde{A}$, real number $a \in \mathbb{R}$ and any $j, k \in \mathbb{Z}_+$, it satisfies $\alpha = 0$ on $\{\theta \mid \theta_0 - \epsilon_0 < \theta < \theta_0 + \epsilon_0\}$ and*

$$\|\nabla^j \alpha\|_{L^{2^k}(K_a)} \leq C_{j,k,K} \hbar^{-\frac{j+s}{2}+\frac{1}{2^{k+1}}},$$

where $K_a = K \cap \{r = a\}$.

Lemma 5.16. *We have*

$$\tilde{\Omega}^{(a)} \in \bigoplus_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} F^1(\Omega^*(\tilde{A})) \cdot (\mathfrak{w}^{-km_a} \cdot \partial_{n_a}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}},$$

and

$$\tilde{\mathcal{R}}^{(a)} \in \bigoplus_{i=1,2} \bigoplus_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} F^0(\Omega^*(\tilde{A})) \cdot (\mathfrak{w}^{-km_a} \cdot \partial_{n_i}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}.$$

Therefore, we deduce that

$$\varphi_a = -\hat{\mathcal{H}}(\tilde{\Omega}^{(a)}) + \bigoplus_{i=1,2} \bigoplus_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} \hat{\mathcal{H}}\left(F^0(\Omega^*(\tilde{A}))\right) \cdot (\mathfrak{w}^{-km_a} \cdot \partial_{n_i}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}.$$

Furthermore, it can be shown that the asymptotic expansion of $-\hat{\mathcal{H}}(\tilde{\Omega}^{(a)})$, and hence φ_a takes a special form.

Lemma 5.17. *There exists constant $b_{jk}^{(a)} \in \mathbb{C}$ independent of $\hbar$, with $b_{jk}^{(a)} \neq 0$ for finitely many k's for each fixed j, such that*

$$-\hat{\mathcal{H}}(\tilde{\Omega}^{(a)}) = l\varphi_{a,0} + \bigoplus_{i=1,2} \bigoplus_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} \mathcal{O}_{loc}(\hbar^{1/2}) \cdot (\mathfrak{w}^{-km_a} \partial_{n_i}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}},$$

on the branch $(\theta_a + 2(l-1)\pi + \epsilon_{N_0}, \theta_a + 2l\pi - \epsilon_{N_0})$, where

$$\varphi_{a,0} = \sum_{\substack{k\geq 1 \\ j_1+j_2\leq N_0}} b_{j,k}^{(a)} (\mathfrak{w}^{-km_a} \partial_{n_a}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}.$$

The proofs of Lemmas 5.16 and 5.17, which involve careful estimates of the order of $\hbar$ in asymptotic expansions and is done by lengthy analysis, will be postponed to Section 5.4.

5.3. **Main theorems.** We are now ready to prove our main theorems, namely, Theorems 1.1 and 1.2. We first recall some notations in Section 5.2. We consider the annulus $A = B_0 \setminus \overline{B(r_{N_0})}$ for some r_{N_0}, together with its universal cover $\mathtt{p} : \tilde{A} \to A$ with coordinates (r, θ). We fix an angle θ_0 such that $\frac{\theta_0}{2\pi}$ is irrational and take the branch $\tilde{A}_0 = \{(r, \theta) \mid \theta_0 < \theta < \theta_0 + 2\pi\}$. We define $\tilde{\mathcal{M}} = \mathcal{M} \times_{B_0} \tilde{A}$ which is equipped with the natural map $\mathtt{p} : \tilde{\mathcal{M}} \to \mathcal{M}$ (by abuse the notation $\mathtt{p}$ if there is no confusion). As before, we consider the formal power series ring $\mathbb{C}[[t_1, t_2]]$ with its maximal ideal $\mathbf{m} = (t_1, t_2)$.

We use $Ray = (\mathbb{Z}^2)_{prim}$ to parametrize the rays with rational slopes emitted from the origin. Similar to Definitions 5.4 and 5.6, we define

$$\tilde{\mathcal{L}}^* = \bigoplus_{a \in (\mathbb{Z}^2)_{prim}} \tilde{\mathcal{L}}^*_a = \bigoplus_{a \in (\mathbb{Z}^2)_{prim}} \Big(\bigoplus_{n \in \mathbb{Z}^2} \bigoplus_{k \geq 1} \Omega^*(\tilde{A}) \cdot (\mathfrak{w}^{-k m_a} \partial_n) \Big)$$

and

$$\tilde{\mathbf{h}}^* = \bigoplus_{a \in (\mathbb{Z}^2)_{prim}} \tilde{\mathbf{h}}^*_a = \bigoplus_{a \in (\mathbb{Z}^2)_{prim}} \Big(\bigoplus_{k \geq 1} \Omega^*(\tilde{A}) \cdot (\mathfrak{w}^{-k m_a} \partial_{n_a}) \Big)$$

on $\tilde{\mathcal{M}}$ as modules over $\Omega^*(\tilde{A})$. Making use of the filtration F^* on $\Omega^*(\tilde{A})$, we can define a filtration

$$\cdots \subset F^{-s}(\tilde{\mathcal{L}}^*_a) \subset \ldots F^{-1}(\tilde{\mathcal{L}}^*_a) \subset F^0(\tilde{\mathcal{L}}^*_a) \subset F^1(\tilde{\mathcal{L}}^*_a) \subset F^2(\tilde{\mathcal{L}}^*_a) \subset \cdots \subset F^s(\tilde{\mathcal{L}}^*_a) \subset \cdots \subset \tilde{\mathcal{L}}^*_a$$

on $\tilde{\mathcal{L}}^*_a$, and a filtration

$$\cdots \subset F^{-s}(\tilde{\mathbf{h}}^*_a) \subset \ldots F^{-1}(\tilde{\mathbf{h}}^*_a) \subset F^0(\tilde{\mathbf{h}}^*_a) \subset F^1(\tilde{\mathbf{h}}^*_a) \subset F^2(\tilde{\mathbf{h}}^*_a) \subset \cdots \subset F^s(\tilde{\mathbf{h}}^*_a) \subset \cdots \subset \tilde{\mathbf{h}}^*_a$$

on $\tilde{\mathbf{h}}^*_a$. These filtrations will be used to describe the asymptotic order of $\hbar$.

Given a finite subset $Ray(N_0) \subset Ray$, we restrict our attention to those elements in $\Omega^1(\mathcal{M}, TB_0^{\mathbb{C}}) \otimes \mathbf{m}$ which have asymptotic support on $Ray(N_0)$. To give a precise definition to these elements, we consider disjoint open neighborhoods $W_{a,k}$ of each ray $\ell_a \in Ray(N_0)$, or more precisely, the pre-image $\mathtt{p}^{-1}(\ell_a)$ as in Lemma 5.10. Without loss of generality, we assume that the ray with angle θ_0 is not contained in any of $W_{a,k}$'s.

Definition 5.18. *An element $\Phi \in \Omega^1(\mathcal{M}, TB_0^{\mathbb{C}}) \otimes \mathbf{m}$ is said to have* asymptotic support on $Ray(N_0)$ *(mod $\mathbf{m}^{N_0+1}$) if we can find $r_{N_0} > 0$, a collection of small enough open neighborhoods $\{W_{a,k}\}$ and write*

$$\mathtt{p}^*(\Phi) = \sum_{a \in Ray(N_0)} \big(\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)} \big) \quad (mod\ \mathbf{m}^{N_0+1})$$

on $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_{r_{N_0}}$ with each individual summand $\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)}$ satisfying the Maurer-Cartan equation, such that

$$supp(\tilde{\Omega}^{(a)}) \cup supp(\tilde{\mathcal{R}}^{(a)}) \subset \bigcup_{k \in \mathbb{Z}} W_{a,k} \quad (mod\ \mathbf{m}^{N_0+1}),$$

$$\tilde{\Omega}^{(a)} \in F^1(\tilde{\mathbf{h}}^*_a), \quad \tilde{\mathcal{R}}^{(a)} \in F^0(\tilde{\mathcal{L}}^*_a) \quad (mod\ \mathbf{m}^{N_0+1}),$$

and the asymptotic expansion

$$-\hat{\mathcal{H}}(\tilde{\varOmega}^{(a)}) = \varphi_{a,0} + \bigoplus_{\substack{k\geq 1\\ j_1+j_2\leq N_0}} \mathcal{O}_{loc}(\hbar^{1/2})\cdot(\mathfrak{w}^{-km_a}\partial_{n_a})t_1^{j_1}t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}$$

holds on $\mathcal{M}\times_{B_0}\tilde{A}\setminus\bigcup_{k\in\mathbb{Z}}W_{a,k}$, with $\varphi_{a,0}\in\mathbf{h}_a^0$ independent of $\hbar$, where $n_a\perp m_a$ is defined in Definition 5.6 and $\hat{\mathcal{H}}$ is the homotopy operator in Definition 5.11 (we choose a based point (r_0,θ_0) by taking r_0 large enough).

Definition 5.19. *Given an increasing set of subsets of rays $\{Ray(N_0)\}_{N_0\in\mathbb{Z}_{>0}}$, we say that Φ has* asymptotic support on $\{Ray(N_0)\}_{N_0\in\mathbb{Z}_{>0}}$ *if it has asymptotic support on $Ray(N_0)$ for each N_0.*

Now it is not hard to see that a scattering diagram $\mathcal{D}(\Phi)$ with support on $\bigcup_{N_0}\bigcup_{\ell_a\in Ray(N_0)}\ell_a$ can be associated to those elements Φ having asymptotic support on $\{Ray(N_0)\}_{N_0\in\mathbb{Z}_{>0}}$. More precisely, this is done by defining $\mathcal{D}(\Phi) \pmod{\mathbf{m}^{N_0+1}}$ and letting $N_0\to\infty$. Fixing N_0 and given $\ell_a\in Ray(N_0)$, we use the angular coordinate θ_a to record the pre-image of the ray ℓ_a lying in the branch $\tilde{A}_0$. We use $W_{a,0}\subset\tilde{A}_0$ to denote the open set containing ℓ_a. From Definition 5.18, we can write

$$\varphi_{a,0} = \sum_{\substack{k\geq 1\\ j_1+j_2\leq N_0}} b_{j,k}^{(a)}(\mathfrak{w}^{-km_a}\partial_{n_a})t_1^{j_1}t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}}$$

on $\tilde{A}_0\setminus W_{a,0}$, for some constant $b_{j,k}^{(a)}$ independent of $\hbar$.

Definition 5.20. *The wall $\mathbf{w}_a$ with wall crossing factor $\Theta_a=\exp(\check{\phi}_a\otimes\check{\partial}_{n_a})$ supported on ℓ_a is defined by the equation*

$$\check{\phi}_a\otimes\check{\partial}_{n_a} = \sum_{\substack{k\geq 1\\ j_1+j_2\leq N_0}} b_{j,k}^{(a)}(w^{-km_a}\check{\partial}_{n_a})t_1^{j_1}t_2^{j_2} = \hat{\mathcal{F}}(\varphi_{a,0}) \pmod{\mathbf{m}^{N_0+1}},$$

when N_0 runs through $\mathbb{Z}_{>0}$.

Similar to [9] or Section 3.3, we consider an anti-clockwise loop γ around the origin with $\gamma(0)=\gamma(1)=(1,\theta_0)$. The following theorem is one of the key observations of this paper, relating Maurer-Cartan elements to monodromy free scattering diagram.

Theorem 5.21 (=Theorem 1.1)**.** *If $\Phi\in\Omega^*(\mathcal{M},TB_0^{\mathbb{C}})\otimes\mathbf{m}$ is a solution to the Maurer-Cartan equation* (2.2) *having asymptotic support on $\{Ray(N_0)\}_{N_0\in\mathbb{Z}_{>0}}$, then the associated $\mathcal{D}(\Phi)$ is a monodromy free scattering diagram, i.e. the following identity*

$$\Theta_{\gamma(1),\mathcal{D}(\Phi)} = \prod_{\mathbf{w}_a\in\mathcal{D}(\Phi)}^{\gamma}\Theta_a = I,$$

holds for a path ordered product $\prod_{\mathbf{w}_a\in\mathcal{D}(\Phi)}^{\gamma}\Theta_a$ defined in Section 3.3.

Proof. It is enough to fix an integer N_0 and show that $\mathcal{D}(\Phi)$ is a monodromy free diagram for each N_0. From the fact that all deformations in $\Omega^*(\tilde{\mathcal{M}},\mathsf{p}^*TB_0^{\mathbb{C}})$ are trivial, we can write

$$e^{\varphi_a}*0 = \tilde{\varOmega}^{(a)} + \tilde{\mathcal{R}}^{(a)} \pmod{\mathbf{m}^{N_0+1}}$$

as in Definition 5.13, making use of the homotopy operators $\hat{\mathcal{H}}$ defined in Definition 5.11 and the corresponding projection $\hat{\mathcal{P}}$ (we choose a based point (r_0, θ_0) by taking r_0 large enough).

We first show that
$$\prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a} = id \pmod{\mathbf{m}^{N_0+1}}$$
which can simply be interpreted to be an ordered product, and then obtain our result by taking the leading order terms of the asymptotic expansion.

We first observe that
$$\prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a} * 0 = \sum_{a \in Ray(N_0)} (\tilde{\Omega}^{(a)} + \tilde{\mathcal{R}}^{(a)}) \pmod{\mathbf{m}^{N_0+1}},$$
on the branch $\tilde{A}_0$. Since $W_{a,0}$'s are disjoint from each others, the supports $supp(\tilde{\Omega}^{(a)}) \cup supp(\tilde{\mathcal{R}}^{(a)}) \pmod{\mathbf{m}^{N_0+1}}$ on the branch $\tilde{A}_0$ are disjoint for different a's, and $\varphi_a \equiv 0$ in the region $\{\theta \mid \theta < \theta_a - \epsilon_{N_0}\} \cap \tilde{A}_0$. Therefore we have
$$\left\{\varphi_{a_2}, \tilde{\Omega}^{(a_1)} + \tilde{\mathcal{R}}^{(a_1)}\right\} \equiv 0 \pmod{\mathbf{m}^{N_0+1}}$$
for $\theta_{a_1} < \theta_{a_2}$ in $\tilde{A}_0$, and hence
$$e^{\varphi_{a_2}} * (\tilde{\Omega}^{(a_1)} + \tilde{\mathcal{R}}^{(a_1)}) = \tilde{\Omega}^{(a_2)} + \tilde{\mathcal{R}}^{(a_2)} + \tilde{\Omega}^{(a_1)} + \tilde{\mathcal{R}}^{(a_1)} \pmod{\mathbf{m}^{N_0+1}}$$
in $\tilde{A}_0$. We obtain the above observation by applying this argument repeatedly according to the anti-clockwise ordering.

Since Φ is a Maurer-Cartan element in $L^*_{X_0}$, we have $\Phi = e^{\varphi} * 0$ as all deformations of $L^*_{X_0}$ are trivial. We can use the homotopy H in Definition 4.8 with base point $q_{m,0}$ satisfying $pr(q_{m,0}) = \mathtt{p}(r_0, \theta_0)$, to solve for the gauge $e^{\varphi} * 0 = \Phi$, subject to the condition that $P(\varphi) = \varphi(q_{m,0}) = 0$. We can further pullback φ to $\tilde{\mathcal{M}}$ using the covering map $\mathtt{p}$ so that it satisfies $\tilde{\mathcal{P}}(\mathtt{p}^*(\varphi)) = 0$ for $\tilde{\mathcal{P}}$ defined on $\tilde{\mathcal{M}}$ in a similar fashion. Therefore, we have the relation
$$\left(e^{-\mathtt{p}^*(\varphi)} \prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a}\right) * 0 = 0 \pmod{\mathbf{m}^{N_0+1}},$$
on $\tilde{A}_0$, which implies that $e^{\mathtt{p}^*(\varphi)} = \prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a}$ on $\tilde{A}_0 \pmod{\mathbf{m}^{N_0+1}}$. As a result, we see that the term $\prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a} \pmod{\mathbf{m}^{N_0+1}}$ is monodromy free.

Now we have the equation
$$\prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_a} = I \pmod{\mathbf{m}^{N_0+1}}$$
on $\tilde{A}_0 \cap \{(r,\theta) \mid \theta_0 + 2\pi - \epsilon_{N_0} < \theta\}$ disjoint from $\bigcup_{\ell_a \in Ray(N_0)} W_{a,0}$. From Definition 5.18, we can argue that φ_a depends only on $\hbar$ on $\tilde{A}_0 \setminus \bigcup_{\ell_a \in Ray(N_0)} W_{a,0}$ using similar arguments as in

Lemma 5.10. Writing $\varphi_a = \varphi_{a,0} + \sigma_a$ on $\tilde{A}_0 \setminus \bigcup_{\ell_a \in Ray(N_0)} W_{a,0}$, where

$$\sigma_a \in \bigoplus_{i=1,2} \bigoplus_{\substack{k\geq 1\\ j_1+j_2\leq N_0}} \mathcal{O}(\hbar^{1/2}) \cdot (\mathfrak{w}^{-km_a} \cdot \partial_{n_i}) t_1^{j_1} t_2^{j_2} \pmod{\mathbf{m}^{N_0+1}},$$

(Here the notation $\mathcal{O}(\hbar^{1/2})$ refers to those constants depending only on $\hbar$ which bounded by $\hbar^{1/2}$.) we can extract the leading order terms $\varphi_{a,0}$ by choosing $\hbar$ small enough to obtain

$$\prod_{\ell_a \in Ray(N_0)}^{\gamma} e^{\varphi_{a,0}} = I \pmod{\mathbf{m}^{N_0+1}}$$

as desired, using the explicit Baker-Campbell-Hausdorff formula. □

Combining Lemmas 5.10, 5.16, 5.17 and Theorem 5.21, we obtain our main result.

Theorem 5.22 (=Theorem 1.2)**.** *The Maurer-Cartan element Φ, defined by summing over trees as in Definition 5.9, has asymptotic support on some increasing set $\{Ray(N_0)\}_{N_0\in\mathbb{Z}_{>0}}$ of subsets of rays. Furthermore, the associated $\mathcal{D}(\Phi)$ gives the unique (by passing to a minimal scattering diagram if necessary) monodromy free extension of the diagram consisting of two walls $\mathbf{w}_i$'s determined by Theorem 3.11.*

Proof. We use notations from Section 5.2, parametrizing the rays in the first quadrant using $(\mathbb{Z}_{>0})^2_{prim}$ and the rays coming from the two initial walls $\mathbf{w}_i$'s using $\{(\pm 1, 0), (0, \pm 1)\}$. In view of Lemmas 5.10, 5.16, 5.17 and Theorem 5.21, we already know that $\mathcal{D}(\Phi)$ is a monodromy free scattering diagram consisting of rays parametrized by $(\mathbb{Z}_{>0})^2_{prim} \cup \{(\pm 1, 0), (0, \pm 1)\}$. It suffices to show the wall crossing factors associated to initial walls constructed from Φ agree with the given Θ_i's, i.e. to show the identities

$$\Theta_{(\pm 1,0)} = \Theta_1^{\pm 1}, \quad \Theta_{(0,\pm 1)} = \Theta_2^{\pm 1}.$$

This can be done as in the single wall case, using results from Section 4.2.1 with straightforward modifications. □

5.4. **Semi-classical analysis for integral operators associated to trees.** We will give the proofs of Lemmas 5.10, 5.16 and 5.17 in this section.

We need to investigate the operation $\mathfrak{l}_{k,\mathcal{T}}$ associated to a colored tree $\mathcal{T}$. Fixing a colored tree $\mathcal{T}$, we consider terms of the form

$$\mathfrak{l}_{k,\mathcal{T}}(\tilde{\delta}_{j_1} \mathfrak{w}^{-l_1 m_{j_1}} \partial_{n_{j_1}}, \ldots, \tilde{\delta}_{j_k} \mathfrak{w}^{-l_k m_{j_k}} \partial_{n_{j_k}}) \tag{5.7}$$

where $j_s = 0, 1$.

Notations 5.23. *We can attach an element $\rho_e \mathfrak{w}^{-m_e} \partial_{n_e}$ to each edge e inductively along the tree $\mathcal{T}$, with $n_e \in N$ in the following way. With two incoming edges e_1, e_2 and one outgoing edge e_3 at a vertex v such that e_1, e_2, e_3 being clockwise oriented, we let $\mathfrak{w}^{-m_{e_3}} = \mathfrak{w}^{-(m_{e_1}+m_{e_2})}$ and*

$$\partial_{n_{e_3}} = \begin{cases} \langle m_{e_1}, n_{e_2}\rangle \partial_{n_{e_1}} - \langle m_{e_2}, n_{e_1}\rangle \partial_{n_{e_2}} & \text{if } v \text{ is colored with } \natural, \\ \partial_{n_{e_2}} & \text{if } v \text{ is colored with } \sharp, \\ \partial_{n_{e_1}} & \text{if } v \text{ is colored with } \flat. \end{cases}$$

The form $\rho_{e_3} \in \Omega^*(B_0)$ is defined accordingly so that $\rho_{e_3}\mathfrak{w}^{-m_{e_3}}\partial_{n_{e_3}}$ is the term obtained after taking operation at the edge e_3.

We observe that the differential form ρ_e satisfies the property that $\iota_{m_e \lrcorner g}\rho_e = 0$ by the definition of the homotopy operator in Definition 4.8. Therefore if we have two incoming edges e_1, e_2 at v such that m_{e_1} is parallel to m_{e_2}, we can see that $\rho_{e_1} \wedge \rho_{e_2}$, $\rho_{e_1} \wedge \nabla\rho_{e_2}$ and $\nabla\rho_{e_1} \wedge \rho_{e_2}$ all vanish. Therefore we can always assume that m_{e_1} is not parallel to m_{e_2} to avoid trivial operations.

Assuming the non-trivial situation, we can then associate a ball $B(r_v)$ centered at origin to each vertex v, such that $supp(\rho_{e_1}) \cap supp(\rho_{e_2}) \subset B(r_v)$, where e_1, e_2 are the incoming edges at v. For the outgoing edge e_3 at v, we have $supp(\rho_{e_3}) \subset B(r_v) + \mathbb{R}_{\geq 0} \cdot m_{e_3} \lrcorner g$ since the homotopy operator applying to $f\mathfrak{w}^{-m_{e_3}}$ is defined by integration along the vector field $m_{e_3} \lrcorner g$ to its coefficient function $f \in C^\infty(B_0)$.

Proof of Lemma 5.10. For each of the terms (5.7) involved in order (mod $\mathbf{m}^{N_0+1}$), we consider the support of its coefficient ρ_e attached to an edge along a tree $\mathcal{T}$. Suppose v_r is the root vertex of the tree $\mathcal{T}$, with incoming edges e_1, e_2 and outgoing semi-infinite edge e_o, we have the ball $B(r_{v_o})$ satisfying $supp(\rho_{e_1}) \cap supp(\rho_{e_2}) \subset B(r_{v_o})$. Since ρ_{e_o} is essentially obtained from $\rho_{e_1} \wedge \rho_{e_2}$, $\rho_{e_1} \wedge \nabla\rho_{e_2}$ or $\nabla\rho_{e_1} \wedge \rho_{e_2}$ by apply $-\frac{1}{2}H_{-m_{e_o}}$ to them, we see that $supp(\rho_{e_o}) \subset B(r_{v_o}) + \mathbb{R}_{\geq 0} \cdot m_{e_o} \lrcorner g$ and $supp(d\rho_{e_o}) \subset B(r_{v_o})$. Since there are only finitely many trees and outputs ρ_{e_o} involved in order (mod $\mathbf{m}^{N_0+1}$), we can take r_{N_0} large enough such that lemma 5.10 holds. $\square$

Assumption 5.24. *As we are interested in those terms of the form* (5.7) *in $\varOmega$ and $\mathcal{R}$* (*mod* $\mathbf{m}^{N_0+1}$)*, from now on, we assume r_{N_0} large enough such that lemma 5.10 is satisfied and $supp(\rho_{e_1}) \cap supp(\rho_{e_2}) \subset B(r_{N_0})$ for any two incoming coming edges at a vertex v for any terms of the form* (5.7) *involved.*

5.4.1. *Asymptotic expansion of $\mathfrak{l}_{k,\mathcal{T}}$.* To prove Lemma 5.17, we need to obtain an asymptotic expansion for $\hat{\mathcal{H}}(\tilde{\varOmega}^{(a)})$. We restrict our attention to those terms of the form (5.7) associated to a colored tree $\mathcal{T}$ with all vertices labeled by $\natural$, which appear in $\tilde{\varOmega}^{(a)}$ (mod $\mathbf{m}^{N_0+1}$). According to Notations 5.23, what we need is an asymptotic expansion of ρ_{e_o} associated to the outgoing edge e_o of $\mathcal{T}$. We assume that r_{N_0} is large enough so that it satisfies Assumption 5.24.

We obtain an alternative way to describe the operation $\mathfrak{l}_{k,\mathcal{T}}$. We define a labeling m_e on edges of $\mathcal{T}$ as in Notations 5.23 to record the basis $\mathfrak{w}^{-m_e}$ on an edge e. We let $E(\mathcal{T})_0$ be the set of edges which are not semi-infinite incoming edges, and $\tau^e_s : B_0 \to B_0$ stand for the flow of $m_e \lrcorner g$ for time s associated to $e \in E(\mathcal{T})_0$.

Definition 5.25. *Given a sequence of edges $\mathfrak{e} = (e_0, e_1, \ldots, e_l)$ which is a path starting at $e = e_0$ following the directed tree $\mathcal{T}$, we can define a map*

$$\tau^{\mathfrak{e}} : \mathbb{R}^{|E(\mathcal{T})_{e,0}|} \times B_0 \to B_0,$$

by

$$\tau^{\mathfrak{e}}(\vec{s}, \vec{x}) = \tau^{e_1}_{s_1} \circ \cdots \circ \tau^{e_l}_{s_l}(\vec{x}),$$

where s_j is the time coordinate associated to e_j and $E(\mathcal{T})_{e,0} = \{e_1, \ldots, e_l\}$ is a subset of $E(\mathcal{T})_0$. It can also be extended naturally to a map

$$\hat{\tau}^{\mathfrak{e}} : \mathbb{R}^{|E(\mathcal{T})_0|} \times B_0 \to \mathbb{R}^{|E(\mathcal{T})^c_{e,0}|} \times B_0$$

by taking product of $\tau^{\mathfrak{e}}$ with the copy $\mathbb{R}^{|E(\mathcal{T})^c_{e,0}|}$, where $E(\mathcal{T})^c_{e,0} = E(\mathcal{T})_0 \setminus E(\mathcal{T})_{e,0}$.

Definition 5.26. *We define an orientation of $\mathbb{R}^{|E(\mathcal{T})_0|}$ inductively along the tree $\mathcal{T}$. We attach a differential form ν_e on $\mathbb{R}^{|E(\mathcal{T})_0|}$ to each $e \in E(\mathcal{T})_0$ satisfying:*

- *$\nu_e = ds_e$ for those e adjacent to an incoming edge of $\mathcal{T}$;*
- *$\nu_{e_3} = (-1)^{|\nu_{e_2}|}\nu_{e_1} \wedge \nu_{e_2} \wedge ds_{e_3}$ if v is an internal vertex with incoming edges $e_1, e_2 \in \mathcal{T}_0$ and outgoing edge e_3 such that e_1, e_2, e_3 is clockwise oriented.*

Let $e_1, \ldots, e_k$ be the incoming edges of $\mathcal{T}$ in clockwise ordering. We associate to each e_i a unique sequence $\mathfrak{e}_i$ joining e_i to the outgoing edge e_o. Writing the input at the edge e_i as $\rho_i \mathfrak{w}^{-m_{e_i}} \partial_{n_i}$ according to Notations 5.23 with $\rho_i = \tilde{\delta}_{j_i}$ $(j_i = 1, 2)$, we express the output form ρ_{e_o} defined by

$$\mathfrak{l}_{k,\mathcal{T}}(\rho_1 \mathfrak{w}^{-m_{e_1}} \partial_{n_1}, \ldots, \rho_k \mathfrak{w}^{-m_{e_k}} \partial_{n_k}) = \rho_{e_o} \mathfrak{l}_{k,\mathcal{T}}(\mathfrak{w}^{-m_{e_1}} \partial_{n_1}, \ldots, \mathfrak{w}^{-m_{e_k}} \partial_{n_k})$$

as an integral over the space $(-\infty, 0]^{|E(\mathcal{T})_0|}$ as follows.

Lemma 5.27. *We have the identity*

$$\rho_{e_o} = \int_{(-\infty,0]^{|E(\mathcal{T})_0|}} (\tau^{\mathfrak{e}_1})^*(\rho_1) \ldots (\tau^{\mathfrak{e}_k})^*(\rho_k),$$

with respect to the volume form defined by ν_{e_o} on $\mathbb{R}^{|E(\mathcal{T})_0|}$.

Proof. We proof by induction on the number of edges of $\mathcal{T}$. Taking the root vertex v_r of $\mathcal{T}$, we can split it into two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1, \mathcal{T}_2, e_o$ are clockwise oriented. We assume that the lemma holds for $\mathcal{T}_1$ and $\mathcal{T}_2$ with outgoing edges $\hat{e}_1$ and $\hat{e}_2$ respectively. Therefore we have

$$\rho_{\hat{e}_1} = \int_{(-\infty,0]^{|E(\mathcal{T}_1)_0|}} (\tau^{\tilde{\mathfrak{e}}_1})^*(\rho_1) \ldots (\tau^{\tilde{\mathfrak{e}}_l})^*(\rho_l),$$

and

$$\rho_{\hat{e}_2} = \int_{(-\infty,0]^{|E(\mathcal{T}_2)_0|}} (\tau^{\tilde{\mathfrak{e}}_{l+1}})^*(\rho_{l+1}) \ldots (\tau^{\tilde{\mathfrak{e}}_k})^*(\rho_k),$$

where $\tilde{\mathfrak{e}}_i$ is the sequence obtained from $\mathfrak{e}_i$ by removing the last edge e_o. We have

$$\begin{aligned}
\rho_{e_o} &= \int_{(-\infty,0]} (\tau^{e_o})^*(\rho_{\hat{e}_1} \wedge \rho_{\hat{e}_2}) \\
&= \int_{(-\infty,0]} (\tau^{e_o})^* \left(\int_{(-\infty,0]^{|E(\mathcal{T}_1)_0|}} (\tau^{\tilde{\mathfrak{e}}_1})^*(\rho_1) \ldots (\tau^{\tilde{\mathfrak{e}}_l})^*(\rho_l) \wedge \int_{(-\infty,0]^{|E(\mathcal{T}_2)_0|}} (\tau^{\tilde{\mathfrak{e}}_{l+1}})^*(\rho_{l+1}) \ldots (\tau^{\tilde{\mathfrak{e}}_k})^*(\rho_k) \right) \\
&= (-1)^{|E(\mathcal{T}_2)_0|} \int_{(-\infty,0]^{|E(\mathcal{T}_1)_0|} \times (-\infty,0]^{|E(\mathcal{T}_2)_0|} \times (-\infty,0]} (\tau^{e_o})^* \left((\tau^{\tilde{\mathfrak{e}}_1})^*(\rho_1) \ldots (\tau^{\tilde{\mathfrak{e}}_k})^*(\rho_k) \right) \\
&= \int_{(-\infty,0]^{|E(\mathcal{T})_0|}} (\tau^{\mathfrak{e}_1})^*(\rho_1) \ldots (\tau^{\mathfrak{e}_k})^*(\rho_k).
\end{aligned}$$

This completes the proof of the lemma. ∎

In order to compute the leading order expansion of $\hat{\mathcal{H}}$ acting on ρ_{e_o}, we essentially need to compute the integral

$$\int_{\xi_{e_o}=c} \rho_{e_o} = \int_{(-\infty,0]^{|E(\mathcal{T})_0|}\times\mathbb{R}_c} (\tau^{\mathfrak{e}_1})^*(\rho_1)\dots(\tau^{\mathfrak{e}_k})^*(\rho_k)$$

by restricting ρ_{e_o} to every line $\xi_{e_o} = c$, where we denote $\mathbb{R}_c = \{\xi_{e_o} = c\}$. We describe the change of coordinates which allows us to compute the integral, namely, the coordinates $(\tau^{\mathfrak{e}_1})^*(\eta_{e_1}), \dots, (\tau^{\mathfrak{e}_k})^*(\eta_{e_k})$. We consider $\mathbb{R}^k$ with coordinates η_{e_i}'s, and define

$$\vec{\tau} : \mathbb{R}^{|E(\mathcal{T})_0|} \times \mathbb{R}_c \to \mathbb{R}^k,$$

by $\eta_{e_i} = \eta_{e_i}(\tau^{\mathfrak{e}_i}(\vec{s}, \eta_{e_o}))$.

Lemma 5.28. *$\vec{\tau}$ is a linear isomorphism.*

Proof. It suffices to show that $\vec{\tau}^*(d\eta_{e_1}\dots d\eta_{e_k}) \neq 0$. We prove this again by induction on the number of edges of the tree $\mathcal{T}$. We split the tree at the root vertex v_r into two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ as in the proof of Lemma 5.27. By the induction hypothesis, both $(\tau^{\tilde{\mathfrak{e}}_1})^*(d\eta_{e_1})\dots(\tau^{\tilde{\mathfrak{e}}_l})^*(d\eta_{e_l})$ and $(\tau^{\tilde{\mathfrak{e}}_{l+1}})^*(d\eta_{e_{l+1}})\dots(\tau^{\tilde{\mathfrak{e}}_k})^*(d\eta_{e_k})$ are non-degenerate on $\mathbb{R}^{|E(\mathcal{T}_1)_0|} \times \mathbb{R}_{\xi_{\hat{e}_1}=c_1}$ and $\mathbb{R}^{|E(\mathcal{T}_2)_0|} \times \mathbb{R}_{\xi_{\hat{e}_2}=c_2}$ respectively. Therefore $(\tau^{\tilde{\mathfrak{e}}_1})^*(d\eta_{e_1})\dots(\tau^{\tilde{\mathfrak{e}}_k})^*(d\eta_{e_k})$ is non-degenerate on $\mathbb{R}^{|E(\mathcal{T}_1)_0|} \times \mathbb{R}^{|E(\mathcal{T}_1)_0|} \times B_0$ simply by taking the product. Finally we observe that $\tau^{e_o} : \mathbb{R} \times \mathbb{R}_{\xi_{e_o}=c} \to B_0$ is an linear isomorphism. $\square$

Letting $C(\vec{\tau})$ be the image of the standard cone $(-\infty, 0]^{|E(\mathcal{T})_0|} \times \mathbb{R}_c$, we have the identity

$$\int_{\xi_{e_o}=c} \rho_{e_o} = (\pi\hbar)^{-\frac{k}{2}} \int_{C(\vec{\tau})} (\prod_{i=1}^{k} \chi(\eta_{e_i}))e^{-\frac{\sum_{i=1}^k \eta_{e_i}^2}{\hbar}} d\eta_{e_1}\dots d\eta_{e_k}, \tag{5.8}$$

with respect to the orientation $\vec{\tau}_*(\nu_{e_o})$. Therefore, $\int_{\xi_{e_o}=c} \rho_{e_o}$ has an asymptotic expansion of the form

$$\int_{\xi_{e_o}=c} \rho_{e_o} = \frac{vol(C(\vec{\tau}) \cap B)}{vol(B)}(\pm 1 + C_N \mathcal{O}(\hbar^N)), \tag{5.9}$$

for arbitrary $N \in \mathbb{Z}_+$ by the stationary phase approximation in Lemma 4.14. Here B is the unit ball in $\mathbb{R}^k$ and vol is the Euclidean volume. Having the above expansion, we are ready to prove Lemma 5.17.

Proof of Lemma 5.17. Recall that we take the annuals $A = B_0 \setminus B(r_{N_0})$ with coordinates (r, θ), and we have to compute an asymptotic expansion of the integral $\int_{\theta_0}^{\theta} \rho_{e_o}$ and show that it has locally constant leading order terms on $\tilde{A} \setminus \bigcup_{k\in\mathbb{Z}} W_{a,k}$. We also recall from Lemma 5.10 that $supp(\rho_{e_o}) \subset \bigcup_{k\in\mathbb{Z}} W_{a,k}$ and we therefore consider the integral for those $(r, \theta) \notin \bigcup_{k\in\mathbb{Z}} W_{a,k}$. Since ρ_{e_o} is a closed form in $\tilde{A}$ and hence the integral we are interested in is independent of the path chosen. Therefore we can always choose a path γ which is homotopic to the arc from θ_0 to θ for a fixed $r > r_{N_0}$, such that γ is a part of line of the form $\xi_{e_o} = const.$ inside $\bigcup_{k\in\mathbb{Z}} W_{a,k}$ containing $supp(\rho_{e_o})$. The lemma then follows from the formula (5.8). $\square$

5.4.2. *Integral operators associated to generalized trees.* To prove Lemma 5.16, we consider the term $\rho_{e_o}\mathfrak{w}^{-m_{e_o}}\partial_{n_{e_o}}$ associated to the outgoing edge e_o of a colored tree $\mathcal{T}$, and investigate $\rho_o \in C^\infty(B_0)$ and its derivatives to obtain estimates for their orders in $\hbar$. To study the derivatives of ρ_{e_o}, we introduce generalized trees $\mathtt{T}$ and define the following integral operators associated to it, acting on a subspace of smooth functions $C^\infty_b(B_0) \subset C^\infty(B_0)$, defined in 5.33, in order to analysis their orders in $\hbar$.

We consider a tree T with k inputs and at most one output (possibly none), allowing higher valence vertices with more than two incoming edges and one unique outgoing edge joining to it.

Definition 5.29. $\mathfrak{m}$ *is a labeling of the edges by the lattice* $\mathbb{Z}_{\geq 0}m_1 + \mathbb{Z}_{\geq 0}m_2 \setminus \{0\}$ *satisfying*

- $\mathfrak{m}(e) = -lm_i$, *where* $l \geq 1$ *and* $i = 1, 2$, *for an incoming semi-infinite edge* e;
- *given incoming edges* $e_1, \ldots, e_s$ *and outgoing edge* e *intersecting at an internal vertex* v, *we require* $\mathfrak{m}(e) = \mathfrak{m}(e_1)+\cdots+\mathfrak{m}(e_s)$, *and at least two* $\mathfrak{m}(e_i)$, $\mathfrak{m}(e_j)$ *are non-parallel,*

ν *is a labeling on edges, and* μ *is a labeling on vertices such that* $\nu(e), \mu(v) \in (\mathbb{Z}_{\geq 0})^2$. *We call* $\mathtt{T} = (T, \mathfrak{m}, \nu, \mu)$ *a* generalized tree, *or simply a* tree *if there is no confusion.*

Definition 5.30. *Given a tree* $\mathtt{T} = (T, \mathfrak{m}, \nu, \mu)$ *with* k *inputs, we define*

$$\kappa(\mathtt{T}) = |E(\mathtt{T})| - 2k - \sum_{e \in E(T)} (\nu_1(e) + \nu_2(e)) + \sum_{v \in V(\mathtt{T})} (\mu_1(v) + \mu_2(v)),$$

as the order *of the tree.*

Definition 5.31. *We inductively define coordinates* (ξ_e, η_e) *of* B_0 *and a positive definite quadratic form* Q_e *associated to each edge* e *along the tree* $\mathtt{T}$.

- *For the incoming semi-infinite edge* e *with* $\mathfrak{m}(e) = -lm_i$, *we let* Q_e *be the quadratic form determined by the quadratic function* g_{-m_i} *(recall that* $\delta_i = e^{-\frac{g_i}{\hbar}}\mu_i$*), and* ξ_e, η_e *be the unique positively oriented coordinates such that* $Q_e = \xi_e^2 + \eta_e^2$ *where* $\frac{\partial}{\partial \xi_e}$ *is along the direction* $m_i \lrcorner g$.
- *For the outgoing edge* e *from a vertex* v *with incoming edges* $e_1, \ldots e_s$, *we let* $Q_e = \sum_{l=1}^s \eta_{e_l}^2$, *and* ξ_e, η_e *be the unique positively oriented coordinates such that* $Q_e = \xi_e^2 + \eta_e^2$ *where* $\frac{\partial}{\partial \xi_e}$ *is along the direction* $-\mathfrak{m}(e)\lrcorner g$.

We can also associate coordinates ξ_v, η_v *and a quadratic form* Q_v *for each vertex* v *by taking* $Q_v = \sum_l Q_{e_l}$, *where* $e_1, \ldots e_s$ *are the incoming edges at* v. *The coordinates* ξ_v, η_v *are positively oriented and chosen to satisfy* $Q_v = \xi_v^2 + \eta_v^2$ *where* $\frac{\partial}{\partial \xi_v}$ *is along the direction of* $-\sum_{l=1}^s \mathfrak{m}(e_l)\lrcorner g$, *i.e.* $Q_v = Q_e$ *if* v *has an outgoing edge* e.

Example 5.32. *We choose a basis which identifies* $M \cong \mathbb{Z}\cdot \mathtt{e}_1 \oplus \mathbb{Z}\cdot \mathtt{e}_2$ *and* $N \cong \mathbb{Z}\cdot \check{\mathtt{e}}_1 \oplus \mathbb{Z}\cdot \check{\mathtt{e}}_2$, *and use* $x^1\check{\mathtt{e}}_1 + x^2\check{\mathtt{e}}_2$ *as coordinates for* $B_0 \cong \mathbb{R}^2$. *We consider the tree* $\mathtt{T}$ *with three incoming semi-infinite edges* e_1, e_2, e_3 *and one semi-infinite outgoing edge* e_o *joining at a unique vertex* v_r, *with* $\mathfrak{m}(e_1) = \mathfrak{m}(e_2) = -\mathtt{e}_1$ *and* $\mathfrak{m}(e_3) = -\mathtt{e}_2$ *as shown in the Figure 8. We can take* $\eta_{e_o} = \frac{1}{\sqrt{3}}(-x^1 + 2x^2)$ *and* $\xi_{e_o} = \sqrt{\frac{2}{3}}(x^1 + x^2)$ *to write* $Q_{v_r} = Q_{e_o} = \xi_{e_o}^2 + \eta_{e_o}^2$ *such that* $\frac{\partial}{\partial \xi_{e_o}} = \frac{1}{\sqrt{6}}(2x^1 + x^2) = -\frac{1}{\sqrt{6}}(\mathfrak{m}(e_o)\lrcorner g)$.

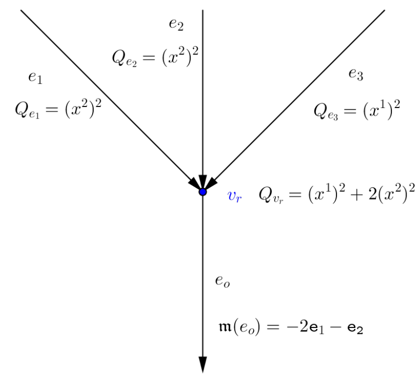

FIGURE 8.

Definition 5.33. *We denote by $C_b^\infty(B_0)$ the set of $f \in C^\infty(B_0)$ such that any $j \geq 0$ and any compact $K \subset B_0$, there is a constant (independent of $\hbar$) $C_{j,K}$, satisfying*

$$\|\nabla^j f\|_{L^\infty(K)} \leq C_{j,K}.$$

We can define an operation associated to $\mathtt{T}$ with inputs from $C_b^\infty(B_0)$ in the following way.

Definition 5.34. *Associated to $\mathtt{T} = (T, \mathfrak{m}, \nu, \mu)$, we define*

$$\mathfrak{I}_{k,\mathtt{T}} : C_b^\infty(B_0)^{\otimes k} \to C^\infty(B_0)$$

by an integral operation on $(f_{e_1}, \dots, f_{e_k}) \in C_b^\infty(B_0)^{\otimes k}$:

- *input $\hbar^{-\frac{1}{2}}(\nabla_{\nu(e)} e^{-\frac{g_i}{\hbar}}) f_e$ at an incoming semi-infinite edge e if $\mathfrak{m}(e) = -l m_i$ $(i = 1, 2)$ and $\nu(e) = (n_1, n_2)$, where $\nabla_{\nu(e)} = \nabla^{n_1}_{\frac{\partial}{\partial \xi_e}} \nabla^{n_2}_{\frac{\partial}{\partial \eta_e}}$;*
- *taking product of functions at an internal vertex, and then multiple by $\xi_v^{\mu_1(v)} \eta_v^{\mu_2(v)}$;*
- *for e being an internal edge or the outgoing edge, we take the operator $\nabla_{\nu(e)} I_e$, where $I_e(f)(p) = \int_{-\infty}^0 f(\hat{\tau}_s^e(p)) ds$ with $\hat{\tau}_s^e$ being flow of the vector field $\frac{\partial}{\partial \xi_e}$ (which is a constant multiple of $-\mathfrak{m}(e) \lrcorner g$) by time s.*

We will see that the operations $\mathfrak{I}_{k,\mathtt{T}}$ can be expressed as a linear combination of those associated to reduced trees.

Definition 5.35. *A tree $\mathtt{T} = (T, \mathfrak{m}, \nu)$ with two labelings is said to be* partially reduced *if $\nu(e) = (0, 0)$ for all e except the incoming edges. It is said to be* reduced *if $\nu(e) = 0$ for all edges and $\mu(v) = (\mu_1(v), 0)$ for all v except the root vertex.*

Proposition 5.36. *Given any tree $\mathtt{T} = (T, \mathfrak{m}, \nu)$ with at least one vertex and $\vec{f} \in C_b^\infty(B_0)^{\otimes k}$, there exist finitely many reduced trees $\tilde{\mathtt{T}}_1, \dots, \tilde{\mathtt{T}}_r$ and corresponding $\vec{f}_j \in C_b^\infty(B_0)^{\otimes k}$ such that*

$$\mathfrak{I}_{k,\mathtt{T}}(\vec{f}) = \sum_{j=1}^r c_j \hbar^{-s_j} \mathfrak{I}_{k,\tilde{\mathtt{T}}_j}(\vec{f}_j),$$

and

$$\kappa(\mathtt{T}) \leq \kappa(\mathtt{T}_j) - 2s_j,$$

for some $c_j \in \mathbb{C}$.

Proof. We first prove the same statement for partially reduced trees with $s_j = 0$ using an induction on the number of edges. Taking the root vertex v_r and splitting the tree $\mathtt{T}$ into several subtrees $\hat{\mathtt{T}}_{(1)}, \dots, \hat{\mathtt{T}}_{(N)}$ and the input function $\vec{f}$ as $\vec{f}_{(1)}, \dots, \vec{f}_{(N)}$. When v_r is joined to a unique outgoing edge e_o, we see that

$$\mathfrak{I}_{k,\mathtt{T}}(\vec{f}) = \nabla_{\nu(e_o)} I_{e_o} \left(\xi_{v_r}^{\mu_1(v_r)} \eta_{v_r}^{\mu_2(v_r)} \cdot \prod_{l=1}^{N} \mathfrak{l}_{k,\hat{\mathtt{T}}_{(l)}}(\vec{f}_{(l)}) \right).$$

We can first simplify the operator $\nabla_{\nu(e_o)} I_{e_o}$ to $I_{e_o} \nabla_{\nu(e_o)}$ if $\nu(e_o) = (0, \nu_2(e_o))$ by commuting with integral operator, or to $\nabla_{\nu(e_o)-(1,0)}$ by canceling differentiation with integration. In both cases, we can further use the commutator relation of operators

$$\begin{aligned} \nabla_{\frac{\partial}{\partial \eta_{e_o}}}(\xi_{v_r}) &= (\xi_{v_r}) \nabla_{\frac{\partial}{\partial \eta_{e_o}}} \\ \nabla_{\frac{\partial}{\partial \eta_{e_o}}}(\eta_{v_r}) &= (\eta_{v_r}) \nabla_{\frac{\partial}{\partial \eta_{e_o}}} + 1, \end{aligned}$$

to simplify the operation (notice that $\xi_{e_o} = \xi_{v_r}$ and $\eta_{e_o} = \eta_{v_r}$).

That means we can express $\mathfrak{I}_{k,\mathtt{T}}(\vec{f})$ as linear combinations

$$\mathfrak{I}_{k,\mathtt{T}}(\vec{f}) = \sum_s c_s I_{e_o} \left(\xi_{v_r}^{\mu_1^s(v_r)} \eta_{v_r}^{\mu_2^s(v_r)} \cdot \prod_{l=1}^{N} \mathfrak{l}_{k,\mathtt{T}^s_{(l)}}(\vec{f}_{(l)}) \right)$$

or

$$\mathfrak{I}_{k,\mathtt{T}}(\vec{f}) = \sum_s c_s \xi_{v_r}^{\mu_1^s(v_r)} \eta_{v_r}^{\mu_2^s(v_r)} \cdot \prod_{l=1}^{N} \mathfrak{l}_{k,\mathtt{T}^s_{(l)}}(\vec{f}_{(l)}),$$

where $\mathtt{T}^s_{(l)}$ are trees modified from $\hat{\mathtt{T}}_{(l)}$ by changing the label ν of the outgoing edge and $\mu^s(v_r)$ is some labeling on the root vertex v_r.

Fix each $\mathtt{T}^s_{(l)}$, there exist $\tilde{\mathtt{T}}^s_{(l),j}$'s together with $\vec{f}^s_{(l),j}$'s such that

$$\mathfrak{I}_{k,\mathtt{T}^s_{(l)}}(\vec{f}_{(l)}) = \sum_{j=1} c_{(l),j} \mathfrak{I}_{k,\tilde{\mathtt{T}}^s_{(l),j}}(\vec{f}^s_{(l),j}),$$

where each $\tilde{\mathtt{T}}_{(l),j}$ is partially reduced by the induction hypothesis. We also see that the order κ is not decreased when we expressed $\mathtt{T}$ as a linear combination of partially reduced trees.

To obtain the statement concerning reduced trees, we can assume that $\mathtt{T}$ is partially reduced with at least one vertex. We first observe that at an incoming edge e adjoining to a vertex v, the term $\nabla_{\nu(e)} e^{-\frac{g_i}{\hbar}}$ can be expressed as a linear combination of terms of the form $\hbar^{-s} \prod_{j=1}^{s} (\nabla_{\nu_j(e)} g_i) e^{-\frac{g_i}{\hbar}}$. Since the term $\nabla_{\nu_j(e)} g_i$ is either a constant or a linear function of the form $a\xi_v + b\eta_v$, we can express the $\mathtt{T}$ as a linear combination

$$\mathfrak{I}_{k,\mathtt{T}}(\vec{f}) = \sum_{j=1}^{r} c_j \hbar^{-s_j} \mathfrak{I}_{k,\tilde{\mathtt{T}}_j}(\vec{f}_j),$$

with $\nu(e) = 0$ for all edges in $\mathtt{T}_j$, with the inequality $\kappa(\mathtt{T}) \leq \kappa(\mathtt{T}_j) - 2s_j$. Finally, we simply observe that for a vertex v with outgoing edge e, The operator $\eta_v^{\mu_2(v)}$ commutes with I_e. Therefore we can rearrange the labeling $\mu(v)$ to obtain reduced trees. $\square$

5.4.3. *Estimate on Order of $\hbar$ in $\mathfrak{I}_{k,\mathtt{T}}$.* We are going to prove Lemma 5.16. We will restrict ourself to a reduced tree $\mathtt{T}$ with input $\vec{f}$ and count the order of $\hbar$ that the operation $\mathfrak{I}_{k,\mathtt{T}}$ carries. Without loss of generality, we first assume that there is an outgoing semi-infinite edge in $\mathtt{T}$. We can attach an element ζ_e to each edge e to stand for the function after taking the operation at the edge e similar to Notations 5.23.

Lemma 5.37. *For the unique outgoing edge e_o attaching to root vertex v_r, we have*

$$\zeta_{e_o} = \hbar^{\frac{\kappa(\mathtt{T})-\mu_2(v_r)}{2}}\psi\eta_{e_o}^{\mu_2(v_r)}e^{-\frac{\eta_{e_o}^2}{\hbar}},$$

such that $\|\psi\|_{L^\infty(B_0)} \leq C$ is a bounded function (C independent of $\hbar$).

Proof. For an edge e, we define the subtree $\mathtt{T}_e \leq \mathtt{T}$ by taking the part of $\mathtt{T}$ before e, with e being the outgoing edge of $\mathtt{T}_e$. We are going to show that

$$\zeta_e = \hbar^{\frac{\kappa(\mathtt{T}_e)}{2}}\psi_e e^{-\frac{\eta_e^2}{\hbar}}$$

for some bounded function ψ_e, inductively along the tree $\mathtt{T}$.

Suppose we have a vertex v with incoming edges $e_1, \dots, e_s$ and outgoing edge e such that the statement holds for ζ_{e_j}'s. We are going to show that it holds also for ζ_e. We see that

$$\zeta_e = \hbar^{(\sum_j \kappa(\mathtt{T}_{e_j}))/2} I_e(\xi_v^{\mu_1(v)}(\prod_{j=1}^s \psi_{e_j})e^{-\frac{Q_v}{\hbar}}).$$

Writing $\mu_1(v) = \mu_1$, $(\prod_{j=1}^s \psi_{e_j}) = \tilde{\psi}$ and $Q_v = \xi_e^s + \eta_e^2$, we have

$$\zeta_e = \hbar^{(\sum_j \kappa(\mathtt{T}_{e_j}))/2} e^{-\frac{\eta_e^2}{\hbar}}\left(\int_{-\infty}^0 \hat{\tau}_s^*(\xi_e^{\mu_1}\tilde{\psi}e^{-\frac{\xi_e^2}{\hbar}})ds\right).$$

Under the coordinates ξ_e, η_e, we can express the flow $\hat{\tau}_s$ explicitly as $\hat{\tau}_s(\xi_e, \eta_e) = (\xi_e + s, \eta_e)$ and therefore we have

$$\begin{aligned} |\int_{-\infty}^0 \hat{\tau}_s^*(\xi_e^{\mu_1}\tilde{\psi}e^{-\frac{\xi_e^2}{\hbar}})ds| &\leq \|\tilde{\psi}\| \int_{\mathbb{R}} |\xi_e|^{\mu_1} e^{-\frac{\xi_e^2}{\hbar}} d\xi_e \\ &\leq C\hbar^{\frac{\mu_1+1}{2}}. \end{aligned}$$

This complete the proof of the lemma. □

We can now prove Lemma 5.16 making use of Lemma 5.37.

Proof of Lemma 5.16. We consider the term $\nabla^j \rho_{e_o}$, where $\rho_{e_o}\mathfrak{w}^{-m_{e_o}}\partial_{n_{e_o}}$ is the output associated to a colored tree $\mathcal{T}$. Using Notations 5.23, we see that what we need to prove is $\mathtt{p}^*(\rho_{e_o}) \in F^1(\Omega^*(\tilde{A}))$ for $\mathcal{T} \in \mathtt{C}_0^k$, and $\mathtt{p}^*(\rho_{e_o}) \in F^0(\Omega^*(\tilde{A}))$ for $\mathcal{T} \in \mathtt{C}_1^k$. Therefore we need to consider $\left(\int_{K_a} |\nabla^j \mathtt{p}^*(\rho_{e_o})|^{2^k}\right)^{2^{-k}}$ for $K_a = K \cap \{\theta = a\}$ as in Definition 5.15.

We observe that $\nabla^j \rho_{e_o}$ can be expressed as $C\hbar^{d(\mathcal{T})}\mathfrak{I}_{k,\mathtt{T}}(\vec{\chi})d\eta_{e_o} = C\hbar^{d(\mathcal{T})}\zeta_{e_o}d\eta_{e_o}$, where C is some constant, $d(\mathcal{T})$ is the number of vertices in $\mathcal{T}$ having color $\sharp$ or $\flat$, and the inputs $\vec{\chi} = (\chi_{j_1}, \dots, \chi_{j_k})$ are cut off functions ($j_l = 1, 2$). $\mathtt{T}$ is obtained from from $\mathcal{T}$ by

- distributing the differentiation using Leibniz's rule in $\sharp$ or $\flat$ associated to a vertex to the incoming edges e_j's of the vertex, and label e_j with $\nu(e_j)$ accordingly, and

- putting a label $\nu(e_o)$ at the outgoing edge e_o according to the differentiation ∇^j.

Since $\mathcal{T}$ and hence $\mathtt{T}$ is a directed trivalent tree with one outgoing edge e_o, we compute and find that $\kappa(\mathtt{T}) = -1 - d(\mathcal{T}) - j$. Making use of Lemma 5.37, when $\mathtt{T}$ has an outgoing edge e_o, we see that

$$\nabla^j \rho_{e_o} = C\hbar^{d(\mathcal{T})}\zeta_{e_o} d\eta_{e_o} = \hbar^{\frac{d(\mathcal{T})-1-j-\lambda}{2}} \psi(\eta_{e_o}^{\lambda} e^{-\frac{\eta_{e_o}^2}{\hbar}}) d\eta_{e_o},$$

and hence

$$\begin{aligned} \left(\int_{\theta_1}^{\theta_2} |\nabla^j \rho_{e_o}|^{2^k} d\theta\right)^{2^{-k}} &\leq C\hbar^{\frac{d(\mathcal{T})-1-j-\lambda}{2}} \left(\int_{\theta_1}^{\theta_2} |\eta_{e_o}^{\lambda} e^{-\frac{\eta_{e_o}^2}{\hbar}}|^{2^k} d\theta\right)^{2^{-k}} \\ &\leq C\hbar^{\frac{d(\mathcal{T})-1-j}{2}+\frac{1}{2^{k+1}}} \end{aligned}$$

for a fixed $r > 0$ using the polar coordinates (r, θ). If $\mathtt{T}$ does not have an outgoing edge from the root vertex v_r, we have

$$\nabla^j \rho_{e_o} = \hbar^{\frac{\kappa(\mathtt{T})-\mu_1(v_r)-\mu_2(v_r)}{2}} \psi(\xi_{v_r}^{\mu_1(v_r)} \eta_{v_r}^{\mu_2(v_r)} e^{-\frac{\xi_v^2+\eta_v^2}{2}}) d\eta_{e_o},$$

which gives

$$\left(\int_{\theta_1}^{\theta_2} |\nabla^j \rho_{e_o}|^{2^k} d\theta\right)^{2^{-k}} \leq C_0 e^{-\frac{C_1}{\hbar}}$$

for any $r > 0$. Therefore we conclude that $\rho_{e_o} \in F^{1-d(\mathcal{T})}(\Omega^1(\tilde{A}))$. $\square$

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