

VANISHING VISCOSITY AND SURFACE TENSION LIMITS OF INCOMPRESSIBLE VISCOUS SURFACE WAVES

YANJIN WANG AND ZHOUPING XIN

ABSTRACT. We consider the dynamics of a layer of viscous incompressible fluid, subjected to gravity and surface tension forces. The upper boundary is a free boundary, and the lower boundary is a fixed boundary on which the Navier-slip boundary condition is imposed. It is proved that there is a uniform time interval on which the uniform estimates independent of both viscosity and surface tension coefficients can be established. These allow one to justify the vanishing viscosity and surface tension limits by the strong compactness argument. In particular, a unified local well-posedness of the free-surface incompressible Euler equations with or without surface tension can be obtained by the inviscid limits.

1. INTRODUCTION

1.1. Formulation. We consider the motion of an incompressible viscous fluid, subject to the influence of gravity and surface tension forces, in a moving domain

$$\Omega(t) = \{x \in \mathbb{R}^3 \mid -b < x_3 < h(t, x_1, x_2)\}. \quad (1.1)$$

The lower boundary of $\Omega(t)$ is assumed to be rigid and given with the constant $b > 0$, but the upper boundary is a free surface that is the graph of the unknown function $h : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$. The fluid is described by its velocity and pressure, which are given for each $t \geq 0$ by $u(t, \cdot) : \Omega(t) \rightarrow \mathbb{R}^3$ and $p(t, \cdot) : \Omega(t) \rightarrow \mathbb{R}$, respectively. For each $t > 0$, (u, p, h) solve the following problem for the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \varepsilon \Delta u = 0 & \text{in } \Omega(t) \\ \nabla \cdot u = 0 & \text{in } \Omega(t) \\ p\mathbf{n} - 2\varepsilon Su\mathbf{n} = gh\mathbf{n} - \sigma H\mathbf{n} & \text{on } \{x_3 = h(t, x_1, x_2)\} \\ \partial_t h = u \cdot \mathbf{N} & \text{on } \{x_3 = h(t, x_1, x_2)\} \\ u_3 = 0, \quad (Su(-e_3))_i = -\kappa u_i, \quad i = 1, 2 & \text{on } \{x_3 = -b\} \end{cases} \quad (1.2)$$

for $Su = \frac{1}{2}(\nabla u + \nabla u^t)$ the symmetric part of the gradient of u and $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ the outward unit normal of the free surface with $\mathbf{N} = (-\partial_1 h, -\partial_2 h, 1)^t$. $\varepsilon > 0$ is the viscosity, $g > 0$ is the strength of gravity, $\sigma > 0$ is the surface tension coefficient and κ is the friction coefficient. Finally, H is twice the mean curvature of the free surface given by the formula

$$H = \nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right). \quad (1.3)$$

The kinematic boundary condition, the fourth equation in (1.2), implies that the free surface is advected with the fluid, and the dynamic boundary condition, the third equation, states the balance of stress tensor on the free surface. We have imposed the Navier slip boundary condition on the lower fixed boundary. Note that in (1.2) we have shifted the gravitational

2000 *Mathematics Subject Classification.* Primary 35Q30, 35R35, 76D03; Secondary 35B40, 76E17.

Key words and phrases. Free boundary; Navier-Stokes equations; Euler equations; Incompressible fluids.

Y. J. Wang was supported by the National Natural Science Foundation of China (No. 11201389), A Foundation for the Author of National Excellent Doctoral Dissertation of PR China (No. 201418), and the Fujian Province Natural Science Funds for Distinguished Young Scholar (No. 2015J06001).

Z. P. Xin was partially supported by the Zheng Ge Ru Foundation, and Hong Kong RGC Earmarked Research Grants CUHK-4041/11P, CUHK-4048/13P, a Focus Area Grant from CUHK, a grant from Croucher Foundation, and a NSFC/RGC Joint Research Scheme.

forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm} , in the usual way by adjusting the actual pressure \bar{p} according to $p = \bar{p} + gx_3 - p_{atm}$.

The initial surface is given by the graph of the function $h(0) = h_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$, which yields the initial domain $\Omega(0)$ on which the initial velocity $u(0) = u_0 : \Omega(0) \rightarrow \mathbb{R}^3$ is specified. It will be assumed that $h_0 > -b$ and that (u_0, h_0) satisfy certain compatibility conditions.

The movement of the free boundary and the subsequent change of the domain create numerous mathematical difficulties. To circumvent these, as usual, we will transform the free boundary problem under consideration to a problem with a fixed domain and fixed boundary. Consider a family of diffeomorphism $\Phi(t, \cdot)$ of the form

$$\begin{aligned} \Phi(t, \cdot) : \Omega = \mathbb{R}^2 \times (-b, 0) &\rightarrow \Omega(t) \\ (y, z) &\rightarrow (y, \varphi(t, y, z)). \end{aligned} \quad (1.4)$$

φ is chosen so that $\partial_z \varphi > 0$ which ensures that $\Phi(t, \cdot)$ is a diffeomorphism. For the fluid domain under consideration, φ can be chosen as

$$\varphi(t, y, z) = z + \eta(t, y, z), \quad (1.5)$$

where η is a chosen extension of h onto $\{z \leq 0\}$ defined by

$$\hat{\eta}(t, \xi, z) = \left(1 + \frac{z}{b}\right) \exp(A|\xi|z) \hat{h}(t, \xi). \quad (1.6)$$

Here $\hat{\cdot}$ stands for the horizontal Fourier transform with respect to the y variable. It is verified in Proposition B.1 that for given h_0 if the number $A > 0$ is chosen sufficiently small, then

$$\partial_z \varphi(0, y, z) \geq c_0 > 0 \text{ in } \Omega. \quad (1.7)$$

Then one can reduce the problem into the fixed domain Ω by setting

$$v(t, y, z) = u(t, \Phi(t, y, z)), \quad q(t, y, z) = p(t, \Phi(t, y, z)) \text{ in } \Omega. \quad (1.8)$$

Set

$$\partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z, \quad i = t, 1, 2, \quad \partial_3^\varphi = \partial_z^\varphi = \frac{1}{\partial_z \varphi} \partial_z$$

such that

$$\partial_i u \circ \Phi(t, \cdot) = \partial_i^\varphi v, \quad i = t, 1, 2, 3.$$

Then by the change of coordinates (1.4), the problem (1.2) becomes

$$\begin{cases} \partial_t^\varphi v + v \cdot \nabla^\varphi v + \nabla^\varphi q - \varepsilon \Delta^\varphi v = 0 & \text{in } \Omega \\ \nabla^\varphi \cdot v = 0 & \text{in } \Omega \\ q \mathbf{n} - 2\varepsilon S^\varphi v \mathbf{n} = gh \mathbf{n} - \sigma H \mathbf{n} & \text{on } \{z = 0\} \\ \partial_t h = v \cdot \mathbf{N} & \text{on } \{z = 0\} \\ v_3 = 0, \quad (S^\varphi v e_3)_i = \kappa v_i, \quad i = 1, 2 & \text{on } \{z = -b\} \\ (v, h)|_{t=0} = (v_0, h_0). \end{cases} \quad (1.9)$$

Here we have naturally written $(\nabla^\varphi)_i = \partial_i^\varphi$, $\Delta^\varphi = \partial_i^\varphi \partial_i^\varphi$, $\nabla^\varphi \cdot v = \partial_i^\varphi v_i$ and $S^\varphi v = \frac{1}{2}(\nabla^\varphi v + (\nabla^\varphi v)^t)$. Note that $\nabla^\varphi \cdot S^\varphi v = \frac{1}{2} \Delta^\varphi v$ for vector fields satisfying $\nabla^\varphi \cdot v = 0$.

1.2. Previous works. Free boundary problems in fluid mechanics have been studied intensively in the mathematical community. There are a huge amount of mathematical works, and we only mention briefly some of them below. We may refer to the references cited in these works for more proper survey of the literature. For the incompressible Navier-Stokes equations, we refer to, for instance, Beale [6], Hataya [22], Guo and Tice [19, 20, 21] for the well-posedness without surface tension, and Beale [7], Tani [39], Tanaka and Tani [40] for the well-posedness with surface tension. Those well-posedness results are strongly based on the regularizing effect of the viscosity, and the solutions are shown to be global for the small initial data [22, 20, 21, 7, 40]. Note that the surface tension only has a regularizing effect on the free surface, and its effect serves to enhance the decay rate, see [20, 21] for more discussions. For the incompressible Euler equations, the problem becomes much more difficult. The early works were focused on

the irrotational fluids, which began with the pioneering work of Nalimov [34] of the local well-posedness without surface tension for the small initial data and was generalized to the general initial data by the breakthrough of Wu [41, 42] for the case without surface tension and by Beyer and Günther [9] for the case with surface tension. For the irrotational inviscid fluids, certain dispersive effects can be used to establish the global well-posedness for the small initial data; we refer to Wu [43, 44], Germain, Masmoudi and Shatah [16], Ionescu and Pusateri [25] and Alazard and Delort [1] for the case without surface tension, and Germain, Masmoudi and Shatah [17] and Ionescu and Pusateri [26] for the case with surface tension. For the general incompressible Euler equations without the irrotational assumption, only local well-posedness results could be found. The first local well-posedness in 3D was obtained by Lindblad [29] for the case without surface tension and by Coutand and Shkoller [11] for the case with (and without) surface tension. We also refer to the results of Shatah and Zeng [36] and Zhang and Zhang [47].

Various approaches are used to prove those well-posedness results mentioned above, depending on whether viscosity or surface tension is presented or not. It is then very natural and interesting to study the asymptotic behavior of vanishing these two parameters in the equations. The vanishing viscosity limit for the Navier-Stokes equations is a classical issue. When there is no boundary, the problem has been well studied; we refer to Swann [37], Kato [27], DiPerna and Majda [12, 13], Constantin [10] and Masmoudi [31] for example. However, in the presence of boundaries, the situation is more complicated and the problem becomes challenging due to the possible formation of boundary layers. In a fixed domain with the no-slip boundary condition, there is formation of boundary layers in the vicinity of the boundary and the solution u^ε of the Navier-Stokes equations is expected to behavior like $u^\varepsilon \sim u^0 + U(t, y, z/\sqrt{\varepsilon})$ (we assume the boundary is locally given by $z = 0$), where u^0 is the solution of the Euler equations satisfying only the impermeable boundary condition and U is some profile. In view of this small scale behavior, it is impossible in general to get uniform strong estimates in any Sobolev spaces containing normal derivatives. Consequently, the vanishing viscosity problem with the no-slip boundary condition is widely open except the works by Asano [5] and Sammartino and Caffisch [35] in the framework of analytic initial data and the work by Maekawa [30] for the initial vorticity located away from the boundary and a recent work by Guo and Nguyen [18] for a steady flow over a moving plane. However, when the no-slip boundary condition is replaced by the Navier slip boundary condition, the situation becomes better. Indeed, now the solution is expected to behavior like $u^\varepsilon \sim u^0 + \sqrt{\varepsilon}U(t, y, z/\sqrt{\varepsilon})$; the amplitude of the boundary layer is weaker, and one can hope to get an uniform estimates involving one normal derivative. In this case, the vanishing viscosity limit in 3D has been justified rigorously in Iftimie and Planas [23], Iftimie and Sueur [24], Masmoudi and Rousset [32] and Xiao and Xin [46]. Furthermore, for some special types of Navier boundary conditions or boundaires, uniform estimates in higher order Sobolev spaces can be obtained, see Xiao and Xin [45] and Beirão da Veiga and Crispo [8].

Going back to the free-surface incompressible Navier-Stokes equations, since the dynamic boundary condition can be viewed as the same type of slip boundary conditions, one has the hope to establish the vanishing viscosity limit. For the case without surface tension and there is no boundary below the fluid, Masmoudi and Rousset [33] justified the inviscid limit by using the framework of their earlier work [32] and some additional techniques. Later, Elgindi and Lee [14] discussed the same problem for the case with surface tension, however, some key points in their arguments are not clear to us. On the other hand, for the free boundary problems, it is also interesting to show the vanishing surface tension limit. This is supposed to be somewhat simpler than the inviscid limit problem since the equation on the free boundary is defined without boundary. Yet, one needs to develop the well-posedness which is uniform with respect to surface tension, and generally this is nontrivial. These have been done for the irrotational Euler equations, see [3, 4] and references therein; for the general Euler equations, a priori uniform estimates have been derived in [36]; while for the Navier-Stokes equations with small initial data, we refer to the recent work of Tan and Wang [38]. The purpose of this paper is to derive the uniform estimates of solutions of the system (1.2) (equivalently, (1.9)) on a time

interval independent of both viscosity and surface tension coefficients. These allow us to justify the vanishing viscosity and surface tension limits by the strong compactness argument. As a byproduct, we can get a unified local well-posedness of the incompressible free-surface Euler equations with or without surface tension by the inviscid limits.

2. MAIN RESULTS

2.1. Statement of the results. We shall use Sobolev conormal spaces on Ω as [32, 33]. Set

$$Z_i = \partial_i, \quad i = 1, 2, \quad Z_3 = z(z+b)\partial_z,$$

which are tangent to $\partial\Omega$. The Sobolev conormal space H_{co}^k is defined as

$$H_{co}^k(\Omega) = \{f \in L^2(\Omega), Z^\alpha f \in L^2(\Omega), \alpha \in \mathbb{N}^3, |\alpha| \leq k\},$$

where $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$, with norm defined as

$$\|f\|_k = \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| \leq k}} \|Z^\alpha f\|_{L^2}, \quad \|f\| = \|f\|_0 = \|f\|_{L^2}.$$

Similarly, $W_{co}^{k,\infty}$ is defined as

$$W_{co}^{k,\infty}(\Omega) = \{f \in L^\infty(\Omega), Z^\alpha f \in L^\infty(\Omega), \alpha \in \mathbb{N}^3, |\alpha| \leq k\}$$

with norm

$$\|f\|_{k,\infty} = \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| \leq k}} \|Z^\alpha f\|_{L^\infty}.$$

H^k and $W^{k,\infty}$ will denote for the usual Sobolev spaces on Ω , and $|\cdot|_s$ and $|\cdot|_{s,\infty}$ stand for the standard Sobolev norms on \mathbb{R}^2 . To estimate the time derivatives of the solution, as [14], we introduce also the *spatial-time* Sobolev conormal norms on Ω as:

$$\|f\|_{\mathbb{X}^{m,k}}^2 = \sum_{\ell=0}^m \left\| \partial_t^\ell f \right\|_{m+k-\ell}^2, \quad \|f\|_{\mathbb{X}^m} = \|f\|_{\mathbb{X}^{m,0}}, \quad \text{and} \quad \|f\|_{\mathbb{Y}^m} = \sum_{\ell=0}^m \left\| \partial_t^\ell f \right\|_{m-\ell,\infty}, \quad (2.1)$$

and the *spatial-time* Sobolev norms on \mathbb{R}^2 :

$$|f|_{\mathbb{X}^{m,s}} = \sum_{\ell=0}^m \left| \partial_t^\ell f \right|_{m+s-\ell}, \quad |f|_{\mathbb{X}^m} = |f|_{\mathbb{X}^{m,0}}, \quad \text{and} \quad |f|_{\mathbb{Y}^m} = \sum_{\ell=0}^m \left| \partial_t^\ell f \right|_{m-\ell,\infty}. \quad (2.2)$$

Note that in these definitions k and m are assumed to be non-negative integers, but s is allowed to be any real number, typically, halves. In the following we will abuse the notation $\frac{m}{2}$ for $[\frac{m}{2}]$.

The aim of this paper is to get a local well-posedness result for strong solutions of (1.9) in Sobolev conormal spaces which is valid on an interval of time independent of $\varepsilon, \sigma \in (0, 1]$. Note that such a result will also imply the local well-posedness for the Euler equation with or without surface tension. As it is well-known that when there is no surface tension a Taylor sign condition on the free boundary is needed to get local well-posedness for the Euler equation; when there is surface tension, no such condition is needed. By the change of coordinates (1.4), the Taylor sign condition reads as

$$-\partial_z^\varphi q + g \geq c_0 > 0 \text{ on } \{z = 0\}. \quad (2.3)$$

In the below, \mathbf{N} and \mathbf{n} are extended to $\bar{\Omega}$ by

$$\mathbf{N}(t, y, z) = (-\partial_1 \varphi(t, y, z), -\partial_2 \varphi(t, y, z), 1)^t \text{ and } \mathbf{n} = \mathbf{N}/|\mathbf{N}|. \quad (2.4)$$

Note that $\mathbf{N}(t, y, 0)$ is indeed the outward normal to the free surface defined before, and $\mathbf{N}(t, y, -b) = e_3$. Define $\Pi = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$, and let χ be a smooth function which takes the value zero in the vicinity of $\{z = 0\}$ and one in the vicinity of $\{z = -b\}$. Let $\|\cdot\|_{L^p_T X}$ be the norm of the space $L^p([0, T]; X)$. Then the main result in this paper is stated as follows.

Theorem 2.1. *Let $m \geq 14$. Assume that the initial data $(h_0^{\varepsilon,\sigma}, v_0^{\varepsilon,\sigma})$ is given such that*

$$\begin{aligned} & |h^{\varepsilon,\sigma}(0)|_{\mathbb{X}^m}^2 + \sigma |h^{\varepsilon,\sigma}(0)|_{\mathbb{X}^{m,1}}^2 + \varepsilon |h^{\varepsilon,\sigma}(0)|_{\mathbb{X}^{m,\frac{1}{2}}}^2 \\ & + \|v^{\varepsilon,\sigma}(0)\|_{\mathbb{X}^m}^2 + \|\partial_z v^{\varepsilon,\sigma}(0)\|_{\mathbb{X}^{m-1}}^2 + \|\partial_z v^{\varepsilon,\sigma}(0)\|_{\mathbb{Y}^{\frac{m}{2}+2}}^2 + \varepsilon \|\partial_{zz} v^{\varepsilon,\sigma}(0)\|_{L^\infty}^2 \leq R_0, \end{aligned} \quad (2.5)$$

and that initially the compatibility condition $\Pi^{\varepsilon,\sigma}(S^{\varphi^{\varepsilon,\sigma}} v^{\varepsilon,\sigma} \mathbf{n}^{\varepsilon,\sigma} - \kappa \chi v^{\varepsilon,\sigma}) = 0$ on $\{z = 0, -b\}$, the diffeomorphism (1.7) and the Taylor sign condition (2.3) on $\{z = 0\}$ hold. There exists $T > 0$ and $C > 0$ such that for every $\varepsilon, \sigma \in (0, 1]$, the unique solution $(v^{\varepsilon,\sigma}, h^{\varepsilon,\sigma})$ of (1.9) satisfies the estimate:

$$\begin{aligned} & |h^{\varepsilon,\sigma}|_{L_T^\infty \mathbb{X}^{m-1,1}}^2 + \sigma |h^{\varepsilon,\sigma}|_{L_T^\infty \mathbb{X}^{m-1,2}}^2 + \varepsilon |h^{\varepsilon,\sigma}|_{L_T^\infty \mathbb{X}^{m-1,\frac{3}{2}}}^2 \\ & + \|v^{\varepsilon,\sigma}\|_{L_T^\infty \mathbb{X}^{m-1,1}}^2 + \|\partial_z v^{\varepsilon,\sigma}\|_{L_T^\infty \mathbb{X}^{m-2}}^2 + \|\partial_z v^{\varepsilon,\sigma}\|_{L_T^\infty \mathbb{Y}^{\frac{m}{2}+2}}^2 + \varepsilon \|\partial_{zz} v^{\varepsilon,\sigma}\|_{L_T^\infty L^\infty}^2 \\ & + |\partial_t^m h^{\varepsilon,\sigma}|_{L_T^4 L^2}^2 + \sigma |\partial_t^m h^{\varepsilon,\sigma}|_{L_T^4 H^1}^2 + \varepsilon |\partial_t^m h^{\varepsilon,\sigma}|_{L_T^4 H^{\frac{1}{2}}}^2 + \|\partial_t^m v^{\varepsilon,\sigma}\|_{L_T^4 L^2}^2 + \|\partial_z v^{\varepsilon,\sigma}\|_{L_T^4 \mathbb{X}^{m-1}}^2 \\ & + \sigma^2 |h^{\varepsilon,\sigma}|_{L_T^2 \mathbb{X}^{m-1,\frac{5}{2}}}^2 + \varepsilon \|\nabla v^{\varepsilon,\sigma}\|_{L_T^2 \mathbb{X}^{m-1,1}}^2 + \varepsilon \|\nabla \partial_z v^{\varepsilon,\sigma}\|_{L_T^2 \mathbb{X}^{m-2}}^2 \leq C. \end{aligned} \quad (2.6)$$

It should be remarked that the reason for the L^4 -in-time estimates of $\|\partial_z v^{\varepsilon,\sigma}(t)\|_{\mathbb{X}^{m-1}}$ rather than in L^∞ stated in Theorem 2.1 is related to the boundary control of the vorticity for the Navier-Stokes system (*i.e.*, viscous boundary layers). Also, owing to the less regularity of the pressure as will be shown in our proof later, which is due to the presence of surface tension, we can only prove the L^4 -in-time estimates of the highest time derivative term $|\partial_t^m h^{\varepsilon,\sigma}(t)|_{L^2} + \sqrt{\sigma} |\partial_t^m h^{\varepsilon,\sigma}(t)|_{H^1} + \sqrt{\varepsilon} |\partial_t^m h^{\varepsilon,\sigma}(t)|_{H^{\frac{1}{2}}} + \|\partial_t^m v^{\varepsilon,\sigma}(t)\|_{L^2}$. As immediate consequences of the uniform estimates of Theorem 2.1, one establishes easily, by standard compactness arguments, the justification of the inviscid limit, vanishing surface tension limit, and any their combinations. In particular, we can obtain a local well-posedness of the free surface Euler equations, independent of surface tension, by the inviscid limit. Furthermore, for the Euler equations, by using the equation for the vorticity one can improve those L^4 -in-time estimates to be in L^∞ and also recover the standard Sobolev regularity.

2.2. Strategy of the proof. The main step in the proof of Theorem 2.1 is to derive the a priori uniform estimates on a time interval small but independent of $\varepsilon, \sigma \in (0, 1]$ for a sufficiently smooth solution of the equations (1.9). Our approach is strongly motivated by the strategy of Masmoudi and Rousset [33] where the vanishing viscosity limit was justified for the problem without surface tension. However, there are several new difficulties arising in the presence of surface tension. Indeed, it has been already known for the free-surface incompressible Euler equations that the problem with surface tension is more difficult than the problem without surface tension in certain sense, see Coutand and Shkoller [11] for some discussions. This is even so when one wants to show the inviscid limit for the free-surface incompressible Navier-Stokes equations. When showing the inviscid limit for each fixed $\sigma > 0$, one finds that surface tension only serves to provide the improved regularity of the free surface. Indeed, the presence of surface tension makes the problem more difficult: first, the required nonlinear estimates are more difficult to close due to the mean curvature term; second, the less regularity of the pressure makes the arguments much more involved. Note that the less regularity of the free surface for the problem without surface tension was get around in [33] by using Alinhac good unknowns [2]. In this paper we will still use Alinhac good unknowns also without taking advantage of the improved regularity of the free surface provided by any fixed surface tension; nevertheless, we will develop some ideas to overcome the difficulties caused by the presence of surface tension as illustrated below. As a consequence, we will be able to deal with both the vanishing viscosity and surface tension limits.

For notational convenience we shall suppress the subscripts ε and σ below. Let $\mathcal{N}(T)$ be the quantity appearing in the left hand side of (2.6), while $\mathcal{Q}(T)$ be the first two lines in (2.6). The crucial point is thus to get the closed a priori estimates of $\mathcal{N}(T)$ on a small time interval independent of ε and σ in terms of the initial data. We start with the basic physical energy

identity:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |v|^2 d\mathcal{V}_t + \int_{z=0} g|h|^2 + 2\sigma \left(\sqrt{1 + |\nabla_y h|^2} - 1 \right) dy \right) \\ + 2\varepsilon \int_{\Omega} |S^\varphi v|^2 d\mathcal{V}_t + 2\kappa\varepsilon \int_{z=-b} |v|^2 dy = 0. \end{aligned} \quad (2.7)$$

Here $d\mathcal{V}_t$ stands for the volume element induced by the change of variable (1.5): $d\mathcal{V}_t = \partial_z \varphi dy dz$.

To get the estimates for higher order conormal estimates, one then applies Z^α for $1 \leq |\alpha| \leq m$ to the equations (1.9). Since the operators ∂_i^φ involve $\nabla \varphi$, the estimate of the commutator between Z^α and ∂_i^φ needs a control of $\|Z^\alpha \nabla \varphi\| \lesssim |Z^\alpha h|_{1/2}$. In the absence of the surface tension this yields a loss of 1/2 derivative, and the difficulty was overcome in Masmoudi and Rousset [33] by using a crucial cancellation observed by Alinhac [2]. The idea is to use the good unknowns $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$ and $Q^\alpha = Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta$, and some cancellation occurs when considering the equations for V^α and Q^α which allows one to derive an L^2 type energy estimates similar to (2.7):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |V^\alpha|^2 d\mathcal{V}_t + \int_{z=0} (g - \partial_z^\varphi q) |Z^\alpha h|^2 + \sigma |\nabla_y Z^\alpha h|^2 dy \right) + 2\varepsilon \int_{\Omega} |S^\varphi V^\alpha|^2 d\mathcal{V}_t \\ = - \int_{z=0} \sigma Z^\alpha H \sum_{|\alpha'|=1} C_{\alpha'}^{\alpha'} Z^{\alpha'} \mathbf{N} \cdot Z^{\alpha-\alpha'} v dy - \int_{\Omega} \mathcal{C}^\alpha(d) Z^\alpha q d\mathcal{V}_t + \sum_0 + \sum_\sigma, \end{aligned} \quad (2.8)$$

where, using the symmetric commutator notation $[\cdot, \cdot, \cdot]$ defined by (A.3),

$$\partial_z \varphi \mathcal{C}^\alpha(d) = [Z^\alpha, \mathbf{N}, \cdot \partial_z v] + [Z^\alpha, \partial_z \eta, \partial_1 v_1 + \partial_2 v_2]. \quad (2.9)$$

Here \sum_0 denotes the terms that can be controlled in a similar way as the case without surface tension [33], and \sum_σ stands for the terms related to surface tension, after some delicate arguments, that can be controlled well with the improved energy estimates $\sigma |Z^\alpha h|_1^2$. The first two terms in the right hand side of (2.8) are singled out in order to indicate the main difficulties for the case with surface tension. Note that the regularity $\sigma |Z^\alpha h|_1^2$ in the energy is not enough to control the first term. Indeed, one needs $\sigma |Z^\alpha h|_{3/2}$, *i.e.*, there is a loss of 1/2 derivative. To improve the regularity of h , one then resorts to the ‘‘dissipation’’ estimate by using the dynamic boundary condition: $-\sigma H = q - gh - 2\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n}$ on $\{z = 0\}$, which requires a control of $|Z^\alpha q|_{-1/2}$. On the other hand, it seems that the only way to control the pressure q is through the elliptic equation with the Neumann boundary conditions

$$\begin{cases} \Delta^\varphi q = -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v) & \text{in } \Omega \\ \nabla^\varphi q \cdot \mathbf{N} = -\partial_t v \cdot \mathbf{N} - (v_y \cdot \nabla_y) v \cdot \mathbf{N} + \varepsilon \Delta^\varphi v \cdot \mathbf{N} & \text{on } \{z = 0\} \\ \nabla^\varphi q \cdot \mathbf{N} = \varepsilon \Delta^\varphi v \cdot \mathbf{N} & \text{on } \{z = -b\}. \end{cases} \quad (2.10)$$

It should be noted that on the boundary $\{z = 0\}$ one could not use the Dirichlet boundary condition $q = -\sigma H + gh + 2\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n}$ since there is no control for $-\sigma H$ yet, which is in contrast to the case $\sigma = 0$ where the Dirichlet boundary condition is successfully used in [33]. Note that the appearance of $\partial_t v$ in (2.10) forces one to include the time derivatives in Z^α . Then elliptic estimates for (2.10) provide a control of $\|\nabla q\|_{\mathbb{X}^{m-1}}$, which then yields a bound for $\sigma |h|_{\mathbb{X}^{m-1, \frac{5}{2}}}$. This implies in particular that we need to separate the estimates of (2.8) into two cases: $\alpha_0 \leq m - 1$ and $\alpha_0 = m$. For the former case, the main conclusion is that

$$\begin{aligned} \|v(t)\|_{\mathbb{X}^{m-1,1}}^2 + |h(t)|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h(t)|_{\mathbb{X}^{m-1,2}}^2 + \varepsilon |h(t)|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \\ \leq C_0 + \Lambda(\mathcal{Q}(T)) \left(t + \int_0^t |\partial_t^m h|_{-\frac{1}{2}}^2 + \sigma |\partial_t^m h|_1^2 + \|\partial_t^m v\|^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 \right). \end{aligned} \quad (2.11)$$

The crucial step is to treat (2.8) for the case $\alpha_0 = m$, which can be rewritten as

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |V^m|^2 d\mathcal{V}_t + \int_{z=0} (g - \partial_z^\varphi q) |\partial_t^m h|^2 + \sigma |\nabla_y \partial_t^m h|^2 dy \right) + 2\varepsilon \int_{\Omega} |S^\varphi V^m|^2 d\mathcal{V}_t \quad (2.12)$$

$$= - \int_{z=0} \sigma \partial_t^m H m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \, dy - \int_{\Omega} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} \partial_z v \partial_t^m q \, dy dz + \sum_0 + \sum_{\sigma} + \sum_q.$$

Here \sum_q stands for the terms related to q , which are easier to estimate after some delicate arguments than the second term on the right hand side above. The main difficulties now are that there are no any estimates for $\partial_t^m q$ so that it is difficult to bound the second term, and hence that the regularity $\sigma |\partial_t^m h|_1^2$ in the energy can no longer be improved through the dissipation so that it is difficult to bound the first term. Our approach to overcome these difficulties is to integrate by parts over t and z in an appropriate order to obtain a crucial cancelation. More precisely, we integrate by parts in z first and then in t the second term to obtain

$$\begin{aligned} & - \int_{\Omega} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} \partial_z v \partial_t^m q \, dy dz \\ &= - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q \, dy + \int_{\Omega} m \partial_z (\partial_t \mathbf{N} \partial_t^m q) \cdot \partial_t^{m-1} v \, dy dz \\ &= - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q \, dy + \frac{d}{dt} \int_{\Omega} m \partial_z (\partial_t \mathbf{N} \partial_t^{m-1} q) \cdot \partial_t^{m-1} v \, dy dz + \sum_0. \end{aligned} \quad (2.13)$$

Note that one can not integrate by parts in t for the first term in the last line of (2.13) since we can not control $\partial_t^m v$ on $\{z=0\}$. The crucial observation is that there is a cancelation between this term and the first term in the right hand side of (2.12). Indeed, by the kinematic boundary condition, it holds that

$$\begin{aligned} & - \int_{z=0} \sigma \partial_t^m H m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \, dy - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q \, dy \\ &= - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v (g \partial_t^m h + 2\varepsilon \partial_t^m (S^\varphi v \mathbf{n} \cdot \mathbf{n})) \, dy = \sum_0. \end{aligned} \quad (2.14)$$

The last difficulty in closing the conormal estimate for the case $\alpha_0 = m$ is due to the second term in the last line of (2.13): the control of $\|\nabla q\|_{\mathbb{X}^{m-1}}$ (especially involves the dissipation) is not pointwise in time but only L^2 . To get the estimates, we are forced to integrate in time twice and the main conclusion is that

$$\begin{aligned} & \left(\int_0^t \left(\|\partial_t^m v\|^2 + |\partial_t^m h|_0^2 + \sigma |\partial_t^m h|_1^2 + \varepsilon |\partial_t^m h|_{\frac{1}{2}}^2 \right) \right)^{\frac{1}{2}} \\ & \leq C_0 + \Lambda(\mathcal{Q}(T)) \left(t + \int_0^t |\partial_t^m h|_0^2 + \sigma |\partial_t^m h|_1^2 + \|\partial_t^m v\|^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon |\partial_t^m h|_{\frac{1}{2}}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.15)$$

The next step is to derive the conormal estimates for $\partial_z v$. As in [33], the main idea is to first introduce the equivalent quantity $S_{\mathbf{n}} = \Pi(S^\varphi v \mathbf{n} - \kappa \chi v)$, which satisfies a convection-diffusion type equation with the homogeneous Dirichlet boundary condition. The main difficulty is that $\mathcal{Q}(T)$ controls only $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$. Since we have included the time derivatives, so if one followed the arguments of [33], which only involves the spatial derivatives, to estimate the commutator resulting from the viscosity term, one would need to control $\sqrt{\varepsilon} \|\partial_{zz} v\|_{\mathbb{Y}^k}$ for some high $k \geq 1$. Our key observation here is that since in the vicinity of the boundary the solution behaves like $v(t, x) \sim v^0(t, x) + \sqrt{\varepsilon} U(t, y, z/\sqrt{\varepsilon})$, so it is possible to control $\varepsilon \partial_{zz} v$ (and even $\varepsilon \partial_{zzz} v$!). This can be done easily by using the first equation in (1.9). We can then perform the L^2 type energy estimates to conclude that

$$\|\partial_z v(t)\|_{\mathbb{X}^{m-2}}^2 \leq C_0 + \Lambda(\mathcal{Q}(T)) \left(t + \int_0^t \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 \right). \quad (2.16)$$

Note that the $m-2$ order estimate above cannot be improved to be $m-1$ due to the appearance of $(\nabla^\varphi)^2 q$ in the source term in the equation for $S_{\mathbf{n}}$, which is only in \mathbb{X}^{m-2} . To get a better estimate, following [33, 32, 45], one would proceed with the vorticity $\omega = \nabla^\varphi \times v$ instead of $S_{\mathbf{n}}$. ω again satisfies a convection-diffusion type equation, but the main difficulty is that it does not vanish on the boundary and the boundary value is at a low regularity. To split the difficulty, for $|\alpha| = m-1$ we set $Z^\alpha \omega = \omega_{nh}^\alpha + \omega_h^\alpha$, where ω_{nh}^α satisfies the nonhomogeneous

equation with the homogeneous boundary condition which can be handled by performing the L^2 type energy estimates and ω_{nh}^α satisfies the homogeneous equation with the nonhomogeneous boundary condition. Note that ω_{nh}^α solves exactly the same problem as [33] so that one can apply their estimates directly, though, the proof is quite involved. Combining these two estimates, one concludes that

$$\left(\int_0^t \|\partial_z v(t)\|_{\mathbb{X}^{m-1}}^4 \right)^{\frac{1}{2}} \leq C_0 + \Lambda(\mathcal{Q}(T)) \left(t + \int_0^t \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 \right). \quad (2.17)$$

The last step is to derive the L^∞ estimates of $\partial_z v$ and $\sqrt{\varepsilon} \partial_{zz} v$. In this step, one needs to estimate a low number of derivatives of v , say $\frac{m}{2} + 2$, while the boundary is H^m with m being as large as needed, it is convenient to use a normal geodesic coordinate system in the vicinity of the boundary so that the Laplacian has the simplest expression. Note that $\sqrt{\varepsilon} \partial_{zz} v$ can be controlled in the same way as [33], however, for $\partial_z v$ we will employ a different argument. Again, it is more convenient to estimate the equivalent quantity $S_{\mathbf{n}}$. After some computations, we find that ρ , an equivalent quantity of $S_{\mathbf{n}}$ in the new coordinates, solves

$$\partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = \mathcal{H} \quad (2.18)$$

for some source term \mathcal{H} and a vector field w with $w_3 = 0$ on the boundary. The main difficulty in the analysis of [33] is a commutator between Z^α and $\varepsilon \partial_{zz}$ which is difficult to control when applying the maximum principle. The main idea in [33] is to rewrite (2.18) into a one-dimensional Fokker-Planck type equation and then use the explicit representation of the solution. Our observation here is that since we have included the time derivatives in Z^α , it is easy to estimate the commutator by, roughly speaking, replacing the $\varepsilon \partial_{zz}$ with ∂_t by using the equations. Main conclusion in this step is that

$$\|\partial_z v(t)\|_{\mathbb{Y}^{\frac{m}{2}+2}}^2 + \varepsilon \|\partial_{zz} v(t)\|_{L^\infty}^2 \leq C_0 + \Lambda(\mathcal{Q}(T))t. \quad (2.19)$$

Combining the estimates in all these steps, we then derive the desired estimates $\mathcal{N}(T) \leq C_0$ for some T sufficiently small but independent of ε and σ . Note that the Taylor sign condition and the condition that $\Phi(t, \cdot)$ is a diffeomorphism can be easily justified due to our estimates of time derivatives. Finally we remark that for each fixed $\sigma > 0$, the Taylor sign condition is no longer needed for the inviscid limit problem. This can be seen from (2.7) and (2.8): even $g - \partial_z^\varphi q \leq 0$, one can use the Sobolev interpolation to get the estimates of h for each $\sigma > 0$.

We will set the conventions for notation to be used later. $\mathbb{N} = \{0, 1, 2, \dots\}$ stands for the collection of non-negative integers, $\mathbb{N}^{1+d} = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)\}$ is used to emphasize that the 0-index term is related to temporal derivatives, while \mathbb{N}^d is related for spatial derivatives. For $\alpha \in \mathbb{N}^{1+d}$, $Z^\alpha = \partial_t^{\alpha_0} Z_1^{\alpha_1} \dots Z_d^{\alpha_d}$. The Einstein convention of summing over repeated indices will be used. Throughout the paper $C > 0$ will denote a generic constant that does not depend on the data, the surface tension coefficient σ and the viscosity coefficient ε , but can depend on the other parameters of the problem, $g, \kappa, m \geq 14$ and Ω . We refer to such constants as ‘‘universal’’. Such constants are allowed to change from line to line. We will employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$. Throughout the paper, the notation $\Lambda(\cdot, \cdot)$ stands for a continuous increasing function in all its arguments, independent of ε and σ and that may change from line to line, and $\Lambda_0 = \Lambda(\frac{1}{c_0})$.

The rest of the paper is organized as follows. We collect some analytic tools related to Sobolev conormal spaces, the properties of Poisson extension and some geometric estimates in Appendixes A, B and C, respectively. In Section 3 we study the equations satisfied by $(Z^\alpha v, Z^\alpha q, Z^\alpha h)$ and present the estimates of the commutators. Section 4 is devoted to derive the pressure estimates using elliptic regularity in Sobolev conormal spaces, and Section 5 contains the smoothing regularity estimates of h due to viscosity and surface tension. In Section 6, the conormal estimates of the solution are derived, and the conormal estimates for normal derivatives are given in Section 7. In Section 8, we prove the needed L^∞ estimates for normal derivatives. Finally, the proof of Theorem 2.1 is given in Section 9.

3. EQUATIONS SATISFIED BY $(Z^\alpha v, Z^\alpha q, Z^\alpha h)$

3.1. A commutator estimate. In order to perform higher order conormal estimates, one needs to compute the equations satisfied by $(Z^\alpha v, Z^\alpha q, Z^\alpha h)$, which requires to commute Z^α with each term in the equations (1.9). It is thus useful to establish the following general expressions and estimates for commutators to be used often later.

We will not commute Z^α with ∂_t^φ directly. For $i = 1, 2, 3$, set

$$Z^\alpha \partial_i^\varphi f = \partial_i^\varphi Z^\alpha f - \partial_z^\varphi f \partial_i^\varphi Z^\alpha \eta + \mathcal{C}_i^\alpha(f), \quad (3.1)$$

where the commutator $\mathcal{C}_i^\alpha(f)$ is given for $\alpha \neq 0$ and $i \neq 3$ by

$$\mathcal{C}_i^\alpha(f) = \mathcal{C}_{i,1}^\alpha(f) + \mathcal{C}_{i,2}^\alpha(f) + \mathcal{C}_{i,3}^\alpha(f) \quad (3.2)$$

with

$$\mathcal{C}_{i,1}^\alpha = - \left[Z^\alpha, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f \right], \quad (3.3)$$

$$\mathcal{C}_{i,2}^\alpha = -\partial_z f \left[Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi} \right] - \partial_i \varphi \partial_z f \left[Z^{\alpha-\alpha'}, \frac{1}{(\partial_z \varphi)^2} \right] Z^{\alpha'} \partial_z \eta, \quad (3.4)$$

$$\mathcal{C}_{i,3}^\alpha = -\frac{\partial_i \varphi}{\partial_z \varphi} [Z^\alpha, \partial_z] f + \frac{\partial_i \varphi}{(\partial_z \varphi)^2} \partial_z f [Z^\alpha, \partial_z] \eta, \quad (3.5)$$

for any $|\alpha'| = 1$. Note that for $i = 1, 2$, $\partial_i \varphi = \partial_i \eta$ and that for $\alpha \neq 0$, $Z^\alpha \partial_z \varphi = Z^\alpha \partial_z \eta$. For $i = 3$, similar decomposition for the commutator holds (basically, it suffices to replace $\partial_i \varphi$ by 1 in the above expressions). Since ∂_i^φ and ∂_j^φ commute, it holds that

$$Z^\alpha \partial_i^\varphi f = \partial_i^\varphi (Z^\alpha f - \partial_z^\varphi f Z^\alpha \eta) + \partial_z^\varphi \partial_i^\varphi f Z^\alpha \eta + \mathcal{C}_i^\alpha(f). \quad (3.6)$$

It was first observed by Alinhac [2] that the highest order term of η (which is difficult to control for the case $\sigma = 0$) will be canceled when one uses the good unknown $Z^\alpha f - \partial_z^\varphi f Z^\alpha \eta$, which allows one to perform high order energy estimates.

Since the expressions as $f/\partial_z \varphi$ will appear often later, we shall first state a general estimate. It is assumed that $\partial_z \varphi \geq \frac{c_0}{2}$ and $|h|_{2,\infty} \leq \frac{1}{c_0}$.

Lemma 3.1. *For every $k \in \mathbb{N}$, it holds that*

$$\left\| \frac{f}{\partial_z \varphi} \right\|_{\mathbb{X}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|f\|_{\mathbb{Y}^{\frac{k}{2}}} \right) \left(|h|_{\mathbb{X}^{k,\frac{1}{2}}} + \|f\|_{\mathbb{X}^k} \right). \quad (3.7)$$

Proof. Since $\partial_z \varphi = 1 + \partial_z \eta$, so

$$\frac{f}{\partial_z \varphi} = f - f \frac{\partial_z \eta}{1 + \partial_z \eta} = f - f F(\partial_z \eta),$$

where $F(x) = x/(1+x)$ is smooth and bounded together with all its derivatives on $1+x \geq c_0 > 0$ and $F(0) = 0$. Consequently, the product estimate (A.1) implies that

$$\left\| \frac{f}{\partial_z \varphi} \right\|_{\mathbb{X}^k} \lesssim \|f\|_{\mathbb{X}^k} + \|f\|_{\mathbb{X}^k} \|F(\partial_z \eta)\|_{\mathbb{Y}^{\frac{k}{2}}} + \|f\|_{\mathbb{Y}^{\frac{k}{2}}} \|F(\partial_z \eta)\|_{\mathbb{X}^k}.$$

Notice that

$$\|F(\partial_z \eta)\|_{\mathbb{Y}^{\frac{k}{2}}} \leq \Lambda \left(\frac{1}{c_0}, \|\nabla \eta\|_{\mathbb{Y}^{\frac{k}{2}}} \right),$$

and (A.1) implies again,

$$\|F(\partial_z \eta)\|_{\mathbb{X}^k} \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla \eta\|_{\mathbb{Y}^{\frac{k}{2}}} \right) \|\partial_z \eta\|_{\mathbb{X}^k}.$$

Hence the estimate (3.7) follows from these, (B.7) and (B.6). \square

Next lemma deals with the estimates of the commutators $\mathcal{C}_i^\alpha(f)$.

Lemma 3.2. For $1 \leq |\alpha| \leq m$, $i = 1, 2, 3$, it holds that

$$\|\mathcal{C}_i^\alpha(f)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z f\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z f\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (3.8)$$

Proof. We only present the proof for $i = 1, 2$, and the last case is similar and slightly easier.

First, for $\mathcal{C}_{i,1}^\alpha$, it follows from the commutator estimate (A.4) that

$$\|\mathcal{C}_{i,1}^\alpha\| \lesssim \left\| Z \left(\frac{\partial_i \varphi}{\partial_z \varphi} \right) \right\|_{\mathbb{Y}^{\frac{m-2}{2}}} \|Z \partial_z f\|_{\mathbb{X}^{m-2}} + \|Z \partial_z f\|_{\mathbb{Y}^{\frac{m-2}{2}}} \left\| Z \left(\frac{\partial_i \varphi}{\partial_z \varphi} \right) \right\|_{\mathbb{X}^{m-2}}.$$

Consequently, (3.7) yields that

$$\|\mathcal{C}_{i,1}^\alpha\| \leq \Lambda \left(\frac{1}{c_0}, \|\nabla \varphi\|_{\mathbb{Y}^{\frac{m}{2}}} + |h|_{\mathbb{Y}^{\frac{m-1}{2}+1}} + \|\partial_z f\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z f\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\partial_i \eta\|_{\mathbb{X}^{m-1}} \right).$$

It then follows from (1.5), (B.7) and (B.6) that

$$\|\mathcal{C}_{i,1}^\alpha\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z f\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z f\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (3.9)$$

Next, for the first term in $\mathcal{C}_{i,2}^\alpha$, one can use similar arguments: (A.4) and (1.5) yield that

$$\left\| \partial_z f \left[Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi} \right] \right\| \lesssim \|\partial_z f\|_{L^\infty} \left(\|Z \partial_i \varphi\|_{\mathbb{X}^{m-2}} \left\| \frac{Z \partial_z \eta}{(\partial_z \varphi)^2} \right\|_{\mathbb{Y}^{\frac{m-2}{2}}} + \|Z \partial_i \varphi\|_{\mathbb{Y}^{\frac{m-2}{2}}} \left\| \frac{Z \partial_z \eta}{(\partial_z \varphi)^2} \right\|_{\mathbb{X}^{m-2}} \right)$$

and hence by using (3.7), (B.6) and (B.7), one can show that

$$\left\| \partial_z f \left[Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi} \right] \right\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z f\|_{L^\infty} \right) |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}.$$

By (A.2) instead of (A.4), the same estimate holds for the second term in $\mathcal{C}_{i,2}^\alpha$. Hence,

$$\|\mathcal{C}_{i,2}^\alpha\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z f\|_{L^\infty} \right) |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}. \quad (3.10)$$

It remains to estimate $\mathcal{C}_{i,3}^\alpha$. Notice that

$$[Z^\alpha, \partial_z] f = \sum_{|\beta| \leq m-1} c_\beta \partial_z (Z^\beta f) \quad (3.11)$$

for some harmless smooth bounded functions c_β . This yields, by using again (B.6),

$$\begin{aligned} \|\mathcal{C}_{i,3}^\alpha\| &\leq \Lambda \left(\frac{1}{c_0}, \|\partial_i \varphi\|_{L^\infty} + \|\partial_z f\|_{L^\infty} \right) \left(\|\partial_z f\|_{\mathbb{X}^{m-1}} + \|\partial_z \eta\|_{\mathbb{X}^{m-1}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{1, \infty} + \|\partial_z f\|_{L^\infty} \right) \left(\|\partial_z f\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \end{aligned} \quad (3.12)$$

Consequently, the estimate (3.8) follows by collecting (3.9), (3.10) and (3.12). \square

3.2. Interior equations. We shall now derive the equations in the domain Ω satisfied by the good unknowns $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$ and $Q^\alpha = Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta$.

Lemma 3.3. For $1 \leq |\alpha| \leq m$, it holds that

$$\partial_t^\varphi V^\alpha + v \cdot \nabla^\varphi V^\alpha + \nabla^\varphi Q^\alpha - 2\varepsilon \nabla^\varphi \cdot S^\varphi V^\alpha \quad (3.13)$$

$$= (\partial_z^\varphi v \cdot \nabla^\varphi v) Z^\alpha \eta - \mathcal{C}^\alpha(\mathcal{T}) - \mathcal{C}^\alpha(q) + \varepsilon \mathcal{D}^\alpha(S^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(v)),$$

$$\nabla^\varphi \cdot V^\alpha = -\mathcal{C}^\alpha(d), \quad (3.14)$$

where the commutators $\mathcal{C}^\alpha(q)$, $\mathcal{C}^\alpha(d)$, $\mathcal{E}^\alpha(v)$ and $\mathcal{C}^\alpha(\mathcal{T})$ satisfy the estimates:

$$\|\mathcal{C}^\alpha(q)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z q\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z q\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right), \quad (3.15)$$

$$\|\mathcal{C}^\alpha(d)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right), \quad (3.16)$$

$$\|\mathcal{E}^\alpha(v)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|\partial_z v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right), \quad (3.17)$$

$$\|\mathcal{C}^\alpha(\mathcal{T})\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|v\|_{\mathbb{X}^{m-1}} + \|\partial_z v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \right), \quad (3.18)$$

and $\mathcal{D}^\alpha(S^\varphi v)$ is given by

$$\mathcal{D}^\alpha(S^\varphi v)_{ij} = 2\mathcal{C}_j^\alpha(S^\varphi v)_{ij}.$$

It should be noted that the commutator $\mathcal{D}^\alpha(S^\varphi v)$ will be estimated later by using the integration by parts.

Proof. First, the equations (3.13)–(3.14) follows from applying Z^α to the equations (1.9). Indeed, (3.6) implies that

$$Z^\alpha \nabla^\varphi q = \nabla^\varphi Q^\alpha + \partial_z^\varphi \nabla^\varphi q Z^\alpha \eta + \mathcal{C}^\alpha(q), \quad (3.19)$$

where $\mathcal{C}^\alpha(q) = (\mathcal{C}_1^\alpha(q), \mathcal{C}_2^\alpha(q), \mathcal{C}_3^\alpha(q))^t$, and

$$Z^\alpha \nabla^\varphi \cdot v = \nabla^\varphi \cdot V^\alpha + \partial_z^\varphi \nabla^\varphi \cdot v Z^\alpha \eta + \mathcal{C}^\alpha(d), \quad (3.20)$$

where $\mathcal{C}^\alpha(d) = \sum_{i=1}^3 \mathcal{C}_i^\alpha(v_i)$.

Next, note that

$$\partial_t^\varphi + v \cdot \nabla^\varphi = \partial_t + v_y \cdot \nabla_y v + V_z \partial_z, \quad (3.21)$$

where V_z is defined by

$$V_z = \frac{1}{\partial_z \varphi} v_z \quad \text{with } v_z = v \cdot \mathbf{N} - \partial_t \varphi = v \cdot \mathbf{N} - \partial_t \eta. \quad (3.22)$$

By using (3.21), one can thus get that

$$\begin{aligned} Z^\alpha (\partial_t^\varphi + v \cdot \nabla^\varphi) v &= (\partial_t + v_y \cdot \nabla_y + V_z \partial_z) Z^\alpha v + (v \cdot Z^\alpha \mathbf{N} - \partial_t Z^\alpha \eta) \partial_z^\varphi v \\ &\quad - \partial_z^\varphi Z^\alpha \eta (v \cdot \mathbf{N} - \partial_t \eta) \partial_z^\varphi v + \mathcal{C}^\alpha(\mathcal{T}) \\ &= (\partial_t^\varphi + v \cdot \nabla^\varphi) Z^\alpha v - \partial_z^\varphi v (\partial_t^\varphi + v \cdot \nabla^\varphi) Z^\alpha \eta + \mathcal{C}^\alpha(\mathcal{T}) \\ &= (\partial_t^\varphi + v \cdot \nabla^\varphi) V^\alpha + \partial_z^\varphi (\partial_t^\varphi + v \cdot \nabla^\varphi) v Z^\alpha \eta - \partial_z^\varphi v \cdot \nabla^\varphi v Z^\alpha \eta + \mathcal{C}^\alpha(\mathcal{T}), \end{aligned} \quad (3.23)$$

where the commutator $\mathcal{C}^\alpha(\mathcal{T})$ is defined by

$$\begin{aligned} \mathcal{C}^\alpha(\mathcal{T}) &= [Z^\alpha, v_y] \partial_y v + [Z^\alpha, V_z, \partial_z v] + \left[Z^\alpha, v_z, \frac{1}{\partial_z \varphi} \right] \partial_z v + \frac{1}{\partial_z \varphi} [Z^\alpha, v] \cdot \mathbf{N} \partial_z v, \\ &\quad + v_z \partial_z v \left[Z^{\alpha-\alpha'}, \frac{1}{(\partial_z \varphi)^2} \right] Z^{\alpha'} \partial_z \eta + V_z [Z^\alpha, \partial_z] v + \frac{v_z \partial_z v}{(\partial_z \varphi)^2} [Z^\alpha, \partial_z] \eta \end{aligned} \quad (3.24)$$

for any $|\alpha'| = 1$.

It remains to compute $\varepsilon Z^\alpha \Delta^\varphi v = 2\varepsilon Z^\alpha \nabla^\varphi \cdot (S^\varphi v)$. Note that

$$2Z^\alpha \nabla^\varphi \cdot (S^\varphi v) = 2\nabla^\varphi \cdot (Z^\alpha S^\varphi v) - 2(\partial_z^\varphi S^\varphi v) \nabla^\varphi (Z^\alpha \eta) + \mathcal{D}^\alpha(S^\varphi v)$$

with $\mathcal{D}^\alpha(S^\varphi v)_i = 2\mathcal{C}_j^\alpha(S^\varphi v)_{ij}$, and

$$2Z^\alpha(S^\varphi v) = 2S^\varphi(Z^\alpha v) - \partial_z^\varphi v \otimes \nabla^\varphi Z^\alpha \eta - \nabla^\varphi Z^\alpha \eta \otimes \partial_z^\varphi v + \mathcal{E}^\alpha(v) \quad (3.25)$$

with $(\mathcal{E}^\alpha(v))_{ij} = \mathcal{C}_i^\alpha(v_j) + \mathcal{C}_j^\alpha(v_i)$. Hence one deduces that

$$\begin{aligned} \varepsilon Z^\alpha \Delta^\varphi v &= 2\varepsilon \nabla^\varphi \cdot S^\varphi(Z^\alpha v) - 2\varepsilon \nabla^\varphi \cdot \left(\partial_z^\varphi v \otimes \nabla^\varphi Z^\alpha \eta - \nabla^\varphi Z^\alpha \eta \otimes \partial_z^\varphi v \right) \\ &\quad - 2\varepsilon (\partial_z^\varphi S^\varphi v) \nabla^\varphi (Z^\alpha \eta) + \varepsilon \mathcal{D}^\alpha(S^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(v)) \\ &= 2\varepsilon \nabla^\varphi \cdot S^\varphi V^\alpha - 2\varepsilon (\partial_z^\varphi \nabla^\varphi \cdot S^\varphi v) Z^\alpha \eta + \varepsilon \mathcal{D}^\alpha(S^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(v)). \end{aligned} \quad (3.26)$$

Consequently, these and (1.9) imply (3.13)–(3.14).

Now we estimate these commutators. First, thanks to Lemma 3.2, the estimates (3.15)–(3.17) hold. To estimate the commutator $\mathcal{C}^\alpha(\mathcal{T})$ defined by (3.24), one needs to bound v_z and V_z . It follows from (B.7) that

$$\|v_z\|_{\mathbb{Y}^{\frac{m}{2}}} + \|V_z\|_{\mathbb{Y}^{\frac{m}{2}}} \leq \Lambda \left(\frac{1}{c_0}, \|v\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\nabla\eta\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\partial_t\eta\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \leq \Lambda \left(\frac{1}{c_0}, \|v\|_{\mathbb{Y}^{\frac{m}{2}}} + |h|_{\mathbb{Y}^{\frac{m}{2}+1}} \right). \quad (3.27)$$

And (A.1), (B.6), and (B.7) yield that

$$\begin{aligned} \|Zv_z\|_{\mathbb{X}^{m-2}} &\lesssim \|Z\partial_t\eta\|_{\mathbb{X}^{m-2}} + \left(1 + \|\nabla\eta\|_{\mathbb{Y}^{\frac{m-2}{2}}}\right) \|v\|_{\mathbb{X}^{m-1}} + \|v\|_{\mathbb{Y}^{\frac{m-2}{2}}} \|\nabla\eta\|_{\mathbb{X}^{m-1}} \\ &\leq \Lambda \left(\|v\|_{\mathbb{Y}^{\frac{m-2}{2}}} + |h|_{\mathbb{Y}^{\frac{m}{2}}} \right) \left(\|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \right). \end{aligned} \quad (3.28)$$

With (3.28) and (3.27) in hand, by using (A.1), (3.7), (B.6) and (B.7), one can obtain that

$$\begin{aligned} \|ZV_z\|_{\mathbb{X}^{m-2}} &\lesssim \left\| Z \left(\frac{1}{\partial_z\varphi} \right) v_z \right\|_{\mathbb{X}^{m-2}} + \left\| \frac{1}{\partial_z\varphi} Zv_z \right\|_{\mathbb{X}^{m-2}} \\ &\leq \Lambda \left(\frac{1}{c_0}, \|v\|_{\mathbb{Y}^{\frac{m}{2}}} + |h|_{\mathbb{Y}^{\frac{m}{2}+1}} \right) \left(\|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \right). \end{aligned} \quad (3.29)$$

Consequently, one may use (A.2), (A.4), (3.11) and (3.7) combined with (3.27)–(3.29) and also again (B.6), (B.7), similarly as in the proof of Lemma 3.2, to conclude the estimate (3.18). \square

3.3. Boundary conditions. We shall now also compute the boundary conditions satisfied by $(Z^\alpha v, Z^\alpha q, Z^\alpha h)$ when $\alpha_3 = 0$ (for $\alpha_3 \neq 0$, $Z^\alpha v = 0$ on the boundary). As a preliminary, one has one has the following.

Lemma 3.4. *For $k \in \mathbb{N}$:*

$$|\nabla v|_{\mathbb{X}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} \right) (|v|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^{k,1}}) \quad (3.30)$$

and

$$|\nabla v|_{\mathbb{X}^{k,s}} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) (|v|_{\mathbb{X}^{k,s+1}} + |h|_{\mathbb{X}^{k,s+1}}) \text{ for } s = -\frac{1}{2}, \frac{1}{2}. \quad (3.31)$$

Proof. Note that it suffices to prove the estimates for $\partial_z v$. Since $\nabla^\varphi \cdot v = 0$, thus

$$\partial_z\varphi (\partial_1 v_1 + \partial_2 v_2) + \partial_z v \cdot \mathbf{N} = 0. \quad (3.32)$$

Then for $s = -\frac{1}{2}, \frac{1}{2}$, (A.12) implies that

$$\begin{aligned} |\partial_z v \cdot \mathbf{n}|_{\mathbb{X}^{k,s}} &\leq \Lambda \left(\frac{1}{c_0}, |\nabla\eta|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\nabla_y v\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) (|v|_{\mathbb{X}^{k,s+1}} + |\nabla\eta|_{\mathbb{X}^{k,s}}) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|\nabla_y v\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) (|v|_{\mathbb{X}^{k,s+1}} + |h|_{\mathbb{X}^{k,s+1}}). \end{aligned} \quad (3.33)$$

where the second inequality follows from Lemma B.2 and the trace estimate (A.6).

To bound $\Pi\partial_z v$, we shall use the boundary conditions in (1.9) which yield

$$\Pi(S^\varphi v \mathbf{n} - \kappa \chi v) = 0. \quad (3.34)$$

To compute $\Pi(S^\varphi v \mathbf{n})$, one can use the local basis $(\partial_{y^1}, \partial_{y^2}, \partial_{y^3})$ in Ω_t induced by (1.4). The induced riemannian metric is given by $g_{ij} = \partial_{y^i} \cdot \partial_{y^j}$, whose inverse denoted by g^{ij} . It follows from the definition and (1.8) that $(\partial_{y^i} u)(t, \Phi(t, \cdot)) = \partial_i v$. Hence,

$$2S u \mathbf{n} = \mathbf{n} \cdot \nabla u + \nabla u_k \mathbf{n}_k = \partial_{\mathbf{n}} u + g^{ij} \partial_{y^j} u \cdot \mathbf{n} \partial_{y^i} = \partial_{\mathbf{n}} u + g^{ij} \partial_j v \cdot \mathbf{n} \partial_{y^i}. \quad (3.35)$$

Note also that

$$\partial_{\mathbf{n}} u = \frac{\mathbf{N}}{|\mathbf{N}|} \cdot \nabla^\varphi v = \frac{|\mathbf{N}|}{\partial_z\varphi} \partial_z v - \partial_1\varphi \partial_1 v - \partial_2\varphi \partial_2 v, \quad (3.36)$$

one then gets from (3.34) that

$$\Pi \partial_z v = \frac{\partial_z \varphi}{|\mathbf{N}|} (\partial_1 \varphi \Pi \partial_1 v + \partial_2 \varphi \Pi \partial_2 v - g^{ij} \partial_j v \cdot \mathbf{n} \Pi \partial_{y^i} - \kappa \chi \Pi v). \quad (3.37)$$

Hence, using (A.12) again shows that

$$\begin{aligned} |\Pi \partial_z v|_{\mathbb{X}^{k,s}} &\leq \Lambda \left(\frac{1}{c_0}, |\nabla \varphi|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) (|v|_{\mathbb{X}^{k,s+1}} + |h|_{\mathbb{X}^{k,s+1}} + |\partial_z v \cdot \mathbf{n}|_{\mathbb{X}^{k,s}}) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) (|v|_{\mathbb{X}^{k,s+1}} + |h|_{\mathbb{X}^{k,s+1}}). \end{aligned} \quad (3.38)$$

where in the second inequality (3.33) has been used.

Consequently, we conclude the estimate (3.31) by combining (3.33) and (3.38). And (3.30) follows similarly by using (A.1) instead of (A.12). \square

We now study the dynamic boundary condition on $\{z = 0\}$ and the Navier slip boundary condition on $\{z = -b\}$.

Lemma 3.5. *For $1 \leq |\alpha| \leq m$ such that $\alpha_3 = 0$, it holds that on $\{z = 0\}$*

$$\begin{aligned} 2\varepsilon S^\varphi V^\alpha \mathbf{N} - (Z^\alpha q - g Z^\alpha h + \sigma Z^\alpha H) \mathbf{N} \\ = -2\varepsilon S^\varphi v \Pi Z^\alpha \mathbf{N} - 2\varepsilon Z^\alpha h \partial_z^\varphi (S^\varphi v) \mathbf{N} + \varepsilon \mathcal{C}^\alpha(\mathcal{B}_\varepsilon), \end{aligned} \quad (3.39)$$

where the commutator $\mathcal{C}^\alpha(\mathcal{B}_\varepsilon)$ satisfies the estimate:

$$|\mathcal{C}^\alpha(\mathcal{B}_\varepsilon)|_0 \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}). \quad (3.40)$$

Similarly, on $\{z = -b\}$ one has that

$$V_3^\alpha = 0, \quad (S^\varphi V^\alpha e_3)_i = \kappa V_i^\alpha - \mathcal{E}^\alpha(v)_{i3}, \quad i = 1, 2, \quad (3.41)$$

and

$$|\mathcal{E}^\alpha(v)|_0 \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}). \quad (3.42)$$

Proof. Applying Z^α to the dynamic boundary condition and using (3.25), one gets

$$\begin{aligned} \varepsilon (2S^\varphi (Z^\alpha v) - \partial_z^\varphi v \otimes \nabla^\varphi Z^\alpha \varphi - \nabla^\varphi Z^\alpha v \otimes \partial_z^\varphi v + \varepsilon \mathcal{E}^\alpha(v)) \mathbf{N} - (Z^\alpha q - g Z^\alpha h + \sigma Z^\alpha H) \mathbf{N} \\ = - (2\varepsilon S^\varphi v - (q - gh + \sigma H) I) Z^\alpha \mathbf{N} - [Z^\alpha, 2\varepsilon S^\varphi v - (q - gh + \sigma H) I, \mathbf{N}]. \\ = -2\varepsilon (S^\varphi v - S^\varphi v \mathbf{n} \cdot \mathbf{n} I) Z^\alpha \mathbf{N} - 2\varepsilon [Z^\alpha, S^\varphi v - S^\varphi v \mathbf{n} \cdot \mathbf{n} I, \mathbf{N}] \\ \equiv -2\varepsilon S^\varphi v \Pi Z^\alpha \mathbf{N} - 2\varepsilon [Z^\alpha, S^\varphi v \Pi, \mathbf{N}]. \end{aligned} \quad (3.43)$$

This yields (3.39) with the commutator $\mathcal{C}^\alpha(\mathcal{B}_\varepsilon)$ defined by

$$\mathcal{C}^\alpha(\mathcal{B}_\varepsilon) = -\mathcal{E}^\alpha(v) \mathbf{N} - 2 [Z^\alpha, S^\varphi v \Pi, \mathbf{N}]. \quad (3.44)$$

Similarly, applying Z^α to the Navier slip boundary condition and noting that $V^\alpha = Z^\alpha v$, one shows (3.41).

Now, it follows from (A.4), (B.7) and Lemma 3.4 that

$$\begin{aligned} |[Z^\alpha, S^\varphi v \Pi, \mathbf{N}]|_0 &\lesssim |Z(S^\varphi v \Pi)|_{\mathbb{X}^{m-2}} |Z \mathbf{N}|_{\mathbb{Y}^{\frac{m-2}{2}}} + |Z(S^\varphi v \Pi)|_{\mathbb{Y}^{\frac{m-2}{2}}} |Z \mathbf{N}|_{\mathbb{X}^{m-2}} \\ &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla \eta\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|\nabla v|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1,1}}) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}). \end{aligned} \quad (3.45)$$

On the other hand, following the proof of Lemma 3.2 and using again Lemma 3.4, one has

$$\begin{aligned} |\mathcal{E}^\alpha(v)|_0 &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla \eta\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|\nabla v|_{\mathbb{X}^{m-1}} + |\nabla \eta|_{\mathbb{X}^{m-1}}) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}). \end{aligned} \quad (3.46)$$

Consequently, the estimates (3.42) and (3.40) follows. \square

Finally, we study the kinematic boundary condition on $\{z = 0\}$.

Lemma 3.6. *For $1 \leq |\alpha| \leq m$ such that $\alpha_3 = 0$, it holds that on $\{z = 0\}$*

$$\partial_t Z^\alpha h + v_y \cdot \nabla_y Z^\alpha h - V^\alpha \cdot \mathbf{N} = -\partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h + \mathcal{C}^\alpha(h) \quad (3.47)$$

where the commutator $\mathcal{C}^\alpha(h)$ satisfies

$$|\mathcal{C}^\alpha(h)|_0 \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1}}). \quad (3.48)$$

Moreover,

$$\mathcal{C}^\alpha(h) = \sum_{|\alpha'|=1} C_{\alpha'}^{\alpha'} Z^{\alpha-\alpha'} v_y \cdot \nabla_y Z^{\alpha'} h + \tilde{\mathcal{C}}^\alpha(h), \quad (3.49)$$

and $\tilde{\mathcal{C}}^\alpha(h)$ satisfies the estimate:

$$\left| \tilde{\mathcal{C}}^\alpha(h) \right|_1 \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+2}} + |v|_{\mathbb{Y}^{\frac{m}{2}+1}} \right) (|h|_{\mathbb{X}^{m-1,2}} + |v|_{\mathbb{X}^{m-2,1}}). \quad (3.50)$$

Proof. Applying Z^α to the kinematic boundary condition yields

$$\partial_t Z^\alpha h + v_y \cdot \nabla_y Z^\alpha h - Z^\alpha v \cdot \mathbf{N} = \mathcal{C}^\alpha(h) \quad (3.51)$$

where

$$\mathcal{C}^\alpha(h) = -[Z^\alpha, v_y, \nabla_y h]. \quad (3.52)$$

This yields (3.47), and the estimate (3.48) follows from (A.4). One may further single out the highest order derivative terms according to (3.49), where $\tilde{\mathcal{C}}^\alpha(h)$ is defined by

$$\tilde{\mathcal{C}}^\alpha(h) = - \sum_{\substack{\beta+\gamma=\alpha \\ \beta \neq 0, |\gamma| \geq 2}} C_{\beta,\gamma} Z^\beta v_y \cdot \nabla_y Z^\gamma h. \quad (3.53)$$

And the estimate (3.50) follows by using (A.1). \square

4. PRESSURE ESTIMATES

In view of the equation (3.13), one needs to estimate the pressure q . The first equation in (1.9) implies that

$$\Delta^\varphi q = -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v) \text{ in } \Omega. \quad (4.1)$$

Moreover, the dynamic boundary condition gives

$$q = 2\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n} + gh - \sigma H \text{ on } \{z = 0\}. \quad (4.2)$$

Projecting the first equation in (1.9) along \mathbf{N} onto $\{z = 0\}$ and $\{z = -b\}$ yields

$$\nabla^\varphi q \cdot \mathbf{N} = -\partial_t v \cdot \mathbf{N} - (v_y \cdot \nabla_y) v \cdot \mathbf{N} + \varepsilon \Delta^\varphi v \cdot \mathbf{N} \text{ on } \{z = 0\} \quad (4.3)$$

and

$$\nabla^\varphi q \cdot \mathbf{N} = \varepsilon \Delta^\varphi v \cdot \mathbf{N} \text{ on } \{z = -b\}. \quad (4.4)$$

Here in (4.4) one has used the fact that $\mathbf{N} = e_3$ and $v_3 = 0$ on $\{z = -b\}$.

Note that to solve the pressure, one has two choices of boundary conditions on $\{z = 0\}$, *i.e.*, (4.2) and (4.3). Without surface tension, one can use the elliptic problem (4.1), (4.2) and (4.4) to establish the regularity estimates for q . The subtlety lies in that the energy dissipation estimates of (1.9) in the case without surface tension provide the needed estimates for those boundary terms. When there is surface tension, however, the energy dissipation estimates do not provide enough estimates for the boundary term $-\sigma H$ (which is of one half regularity less). This would suggest that the elliptic problem (4.1), (4.2) and (4.4) is not the right choice for estimating the pressure q in the case with surface tension. Our way to get around this difficulty is to use instead the elliptic problem (4.1), (4.3) and (4.4). It is then noticed that this approach forces one to estimate the time derivatives of v , that is, one needs to perform energy estimates for the time derivatives of the solution. However, there is an essential difficulty arising: when doing energy estimates with time derivatives up to m order, we can only obtain the estimates

of time derivatives of q up to $m - 1$ order due to the presence of $\partial_t v$ in (4.3). Thus the energy estimates cannot be closed since it seems that m order time derivative of q is involved. We will explain this and our way to overcome it in more details in Section 6.3.

It follows from the definition of ∂_z^φ that

$$\nabla^\varphi \cdot v = \frac{1}{\partial_z \varphi} \nabla \cdot (Pv), \quad \nabla^\varphi f = \frac{1}{\partial_z \varphi} P^* \nabla f, \quad P = \begin{pmatrix} \partial_z \varphi & 0 & 0 \\ 0 & \partial_z \varphi & 0 \\ -\partial_1 \varphi & -\partial_2 \varphi & 1 \end{pmatrix}. \quad (4.5)$$

And then Δ^φ can be expressed as

$$\Delta^\varphi f = \nabla^\varphi \cdot (\nabla^\varphi f) = \frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla f) \quad (4.6)$$

with the matrix E defined by

$$E = \frac{1}{\partial_z \varphi} P P^* \equiv \begin{pmatrix} \partial_z \varphi & 0 & -\partial_1 \varphi \\ 0 & \partial_z \varphi & -\partial_2 \varphi \\ -\partial_1 \varphi & -\partial_2 \varphi & \frac{1 + (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2}{\partial_z \varphi} \end{pmatrix}.$$

Note that E is symmetric positive and that if $\|\nabla_y \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ and $\partial_z \varphi \geq c_0 > 0$ then there exists $\delta(c_0) > 0$ such that

$$EX \cdot X \geq \delta |X|^2, \quad \forall X \in \mathbb{R}^3. \quad (4.7)$$

Moreover,

$$\|E\|_{\tilde{\mathbb{Y}}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right). \quad (4.8)$$

One can write

$$E = \text{Id} + \tilde{E}, \quad \tilde{E} = \begin{pmatrix} \partial_z \eta & 0 & -\partial_1 \eta \\ 0 & \partial_z \eta & -\partial_2 \eta \\ -\partial_1 \eta & -\partial_2 \eta & \frac{(\partial_1 \eta)^2 + (\partial_2 \eta)^2 - \partial_z \eta}{\partial_z \varphi} \end{pmatrix},$$

where

$$\|\tilde{E}\|_{\tilde{\mathbb{X}}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) |h|_{\mathbb{X}^{k, \frac{1}{2}}}. \quad (4.9)$$

Here $\tilde{\mathbb{X}}^k$ and $\tilde{\mathbb{Y}}^k$ are referred to the usual *spatial-time* Sobolev spaces as defined similarly as (2.1).

Since $\mathbf{N} = P^* e_3$ on $\{z = 0, -b\}$, the equations (4.1)–(4.4) can be rewritten as

$$-\nabla \cdot (E \nabla q) = F := \partial_z \varphi \nabla^\varphi v \cdot \nabla^\varphi v \quad \text{in } \Omega, \quad (4.10)$$

$$q = G^1 := 2\varepsilon S^\varphi v \cdot \mathbf{n} + gh - \sigma H \quad \text{on } \{z = 0\}, \quad (4.11)$$

$$E \nabla q \cdot e_3 = G^2 := -\partial_t v \cdot \mathbf{N} - (v_y \cdot \nabla_y) v \cdot \mathbf{N} + \varepsilon \Delta^\varphi v \cdot \mathbf{N} \quad \text{on } \{z = 0\}, \quad (4.12)$$

$$E \nabla q \cdot e_3 = G^3 := \varepsilon \Delta^\varphi v \cdot \mathbf{N} \quad \text{on } \{z = -b\}. \quad (4.13)$$

We shall now prove the estimates for the pressure q .

Proposition 4.1. *The following estimates hold:*

$$\|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} + \|\partial_{zz} q\|_{\mathbb{X}^{k-1}} \quad (4.14)$$

$$\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + |h|_{\mathbb{X}^{\frac{k+7}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{k+7}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+5}{2}}} \right) \\ \times \left(|h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{k, 2}} + \|v\|_{\mathbb{X}^{k+1}} + \|\nabla v\|_{\mathbb{X}^k} + \varepsilon |v|_{\mathbb{X}^{k, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right), \quad \text{for } k \geq 3;$$

$$\|q\|_{\mathbb{Y}^k} + \|\nabla q\|_{\mathbb{Y}^k} + \|\partial_{zz} q\|_{\mathbb{Y}^{k-1}} \quad (4.15)$$

$$\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+4}} + |h|_{\mathbb{X}^{k+4}} + \|v\|_{\mathbb{Y}^{\frac{k+5}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|v\|_{\mathbb{X}^{k+4}} + \|\nabla v\|_{\mathbb{X}^{k+3}} \right), \quad \text{for } k \geq 1.$$

Proof. Multiplying the equation (4.10) by q and then integrating by parts over Ω , using (4.12) and (4.13), one obtains

$$(E\nabla q, \nabla q)_\Omega = (F, q)_\Omega + (G^2, q)_{z=0} - (G^3, q)_{z=-b}. \quad (4.16)$$

Here $(\cdot, \cdot)_\Omega$, $(\cdot, \cdot)_{z=0}$ and $(\cdot, \cdot)_{z=-b}$ denote the L^2 inner products on Ω , $\{z=0\}$ and $\{z=-b\}$, respectively. It follows from the trace estimate $|q|_{\frac{1}{2}} \lesssim \|q\|_{H^1}$ and Cauchy's inequality that

$$\|\nabla q\|^2 \leq \frac{1}{\eta} \Lambda_0 \left(\|F\|^2 + |G^1|_0^2 + |G^2|_{-\frac{1}{2}}^2 + |G^3|_{-\frac{1}{2}}^2 \right) + \eta \|q\|_{H^1}^2 \quad (4.17)$$

for any $\eta > 0$. By the Poincaré inequality (A.8) and (4.11), one can get by taking η sufficiently small that

$$\|q\|_{H^1} \leq \Lambda_0 \left(\|F\| + |G^1|_0 + |G^2|_{-\frac{1}{2}} + |G^3|_{-\frac{1}{2}} \right). \quad (4.18)$$

Note that if one uses solely the problem (4.1), (4.2) and (4.4) to estimate p , then one needs $|G^1|_{\frac{1}{2}}$. This half less regularity requirement enables us to control the surface tension term by the energy dissipation estimates.

Next, applying Z^α with $|\alpha| = k$ to the equation (4.10) and using (4.12)–(4.13) lead to

$$(Z^\alpha(E \cdot \nabla q), \nabla Z^\alpha q)_\Omega = (Z^\alpha F, Z^\alpha q)_\Omega + (Z^\alpha G^2, Z^\alpha q)_{z=0} - (Z^\alpha G^3 Z^\alpha q)_{z=-b}. \quad (4.19)$$

Then as for (4.18), one can derive after using (4.11) that

$$\begin{aligned} \|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} &\leq \Lambda_0 \left(\|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}} \right. \\ &\quad \left. + \|E \cdot [Z^\alpha, \nabla] q\| + \|[Z^\alpha, E] \cdot \nabla q\| \right). \end{aligned} \quad (4.20)$$

We then estimate the commutators in the right hand side of (4.20). First, (3.11) implies that

$$\|E \cdot [Z^\alpha, \nabla] q\| \lesssim \|E\|_{L^\infty} \|\nabla q\|_{\mathbb{X}^{k-1}} \lesssim |h|_{\mathbb{Y}^1} \|\nabla q\|_{\mathbb{X}^{k-1}}. \quad (4.21)$$

However, the other commutator needs more attentions according to $1 \leq k \leq 3$ or $k \geq 4$. Indeed, for $1 \leq k \leq 3$, direct estimates by controlling the ∇q terms in L^2 and E terms in L^∞ yield

$$\|[Z^\alpha, E] \cdot \nabla q\| \lesssim \|E\|_{\tilde{\mathbb{Y}}^k} \|\nabla q\|_{\mathbb{X}^{k-1}} \lesssim |h|_{\mathbb{Y}^{k+1}} \|\nabla q\|_{\mathbb{X}^{k-1}}. \quad (4.22)$$

Plugging (4.21)–(4.22) into (4.20), by an induction argument and (4.18), one can deduce that

$$\begin{aligned} \|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} &\leq \Lambda \left(\frac{1}{c_0}, \|E\|_{\tilde{\mathbb{Y}}^3} \right) \left(\|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{k-1}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^4} \right) \left(\|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}} \right). \end{aligned} \quad (4.23)$$

Since the equation (4.10) gives

$$\partial_{zz} q = \frac{1}{E_{33}} \left(F - \partial_z \left(\sum_{j<3} E_{3,j} \partial_j q \right) - \sum_{i<3, j} \partial_i (E_{ij} \partial_j q) \right), \quad (4.24)$$

in a similar way and by (4.23), one can also obtain

$$\begin{aligned} \|\partial_{zz} q\|_{\mathbb{X}^{k-1}} &\leq \Lambda \left(\frac{1}{c_0}, \|E\|_{\tilde{\mathbb{Y}}^k} \right) \left(\|F\|_{\mathbb{X}^{k-1}} + \|\nabla q\|_{\mathbb{X}^k} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^4} \right) \left(\|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}} \right). \end{aligned} \quad (4.25)$$

It then follows from (4.23) and (4.25) that for $1 \leq k \leq 3$,

$$\begin{aligned} \|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} + \|\partial_{zz} q\|_{\mathbb{X}^{k-1}} \\ \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^4} + |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + |h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{1}{2}}} + \|F\|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^1|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^2|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} \right) \end{aligned} \quad (4.26)$$

$$\times \left(|h|_{\mathbb{X}^{k, \frac{1}{2}}} + \|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}} \right).$$

We now claim that (4.26) holds for all $k \geq 1$. This will be proved by induction. Assume that $k \geq 4$ and that (4.26) holds for $k - 1$. The commutator estimate (A.2) yields that

$$\begin{aligned} \|[Z^\alpha, E] \cdot \nabla q\| &\lesssim \|ZE\|_{\mathbb{X}^{k-1}} \|\nabla q\|_{\mathbb{Y}^{\frac{k-1}{2}}} + \|ZE\|_{\mathbb{Y}^{\frac{k-1}{2}}} \|\nabla q\|_{\mathbb{X}^{k-1}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla q\|_{\mathbb{Y}^{\frac{k-1}{2}}} \right) \left(|h|_{\mathbb{X}^{k, \frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{k-1}} \right). \end{aligned} \quad (4.27)$$

Plugging (4.27) and (4.21) into (4.20) and using (4.24) again lead to

$$\begin{aligned} \|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} + \|\partial_{zz}q\|_{\mathbb{X}^{k-1}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla q\|_{\mathbb{Y}^{\frac{k-1}{2}}} \right) \left(|h|_{\mathbb{X}^{k, \frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{k-1}} \right) \\ &\quad + \|F\|_{\mathbb{X}^k} + |G^1|_{\mathbb{X}^k} + |G^2|_{\mathbb{X}^{k, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{k, -\frac{1}{2}}}. \end{aligned} \quad (4.28)$$

To remove the dependence of Λ on $\|\nabla q\|_{\mathbb{Y}^{\frac{k-1}{2}}}$, one applies the anisotropic Sobolev embedding estimate (A.5) and the induction assumption to obtain

$$\begin{aligned} \|\nabla q\|_{\mathbb{Y}^{\frac{k-1}{2}}} &\lesssim \|\partial_z \nabla q\|_{\mathbb{X}^{\frac{k+1}{2}}} \|\nabla q\|_{\mathbb{X}^{\frac{k+3}{2}}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^4} + |h|_{\mathbb{Y}^{\frac{k+3}{2}+3}} + |h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{1}{2}}} + \|F\|_{\mathbb{X}^{\frac{k+3}{2}+3}} \right. \\ &\quad \left. + |G^1|_{\mathbb{X}^{\frac{k+3}{2}+3}} + |G^2|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} \right) \\ &\quad \times \left(|h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{1}{2}}} + \|F\|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^1|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^2|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^4} + |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + |h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{1}{2}}} + \|F\|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^1|_{\mathbb{X}^{\frac{k+3}{2}}} + |G^2|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} + |G^3|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} \right). \end{aligned} \quad (4.29)$$

Here one has used the fact that $\frac{k+3}{2} \leq k - 1$ since $k \geq 4$, which allows one to use the induction assumption. Plugging the estimate (4.29) into (4.28) and using the induction assumption to estimate $\|\nabla q\|_{\mathbb{X}^{k-1}}$, one thus concludes (4.26) for all $k \geq 4$.

We now estimate the right hand side of (4.26) for $k \geq 3$. It follows by the product estimate (A.1) that

$$\begin{aligned} \|F\|_{\mathbb{X}^k} &= \|\partial_z \varphi \nabla^\varphi v \cdot \nabla^\varphi v\|_{\mathbb{X}^k} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} \right) \left(|h|_{\mathbb{X}^{k, \frac{1}{2}}} + \|\nabla v\|_{\mathbb{X}^k} \right). \end{aligned} \quad (4.30)$$

Similarly, since $k \geq 3$,

$$\begin{aligned} \|F\|_{\mathbb{X}^{\frac{k+3}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{4}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+3}{4}}} \right) \left(|h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{1}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+3}{2}}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+1}} + |h|_{\mathbb{X}^{\frac{k}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+3}{2}}} \right). \end{aligned} \quad (4.31)$$

By (A.1) and Lemma 3.4,

$$\begin{aligned} |G^1|_{\mathbb{X}^k} &= |2\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n} + gh - \sigma H|_{\mathbb{X}^k} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} \right) (\varepsilon |\nabla v|_{\mathbb{X}^k} + |h|_{\mathbb{X}^k} + \sigma |h|_{\mathbb{X}^{k,2}}) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} \right) (\varepsilon |v|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^k} + \sigma |h|_{\mathbb{X}^{k,2}}). \end{aligned} \quad (4.32)$$

Similarly, since $k \geq 3$, the trace estimate implies that

$$|G^1|_{\mathbb{X}^{\frac{k+3}{2}}} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{4}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+3}{4}}} \right) (\varepsilon |v|_{\mathbb{X}^{\frac{k+3}{2},1}} + |h|_{\mathbb{X}^{\frac{k+3}{2}}} + \sigma |h|_{\mathbb{X}^{\frac{k+3}{2},2}}) \quad (4.33)$$

$$\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + |h|_{\mathbb{X}^{\frac{k+7}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}}} + \|v\|_{\mathbb{X}^{\frac{k}{2}+3}} + \|\nabla v\|_{\mathbb{X}^{\frac{k}{2}+2}} \right).$$

To estimate the most delicate term G^2 , we start with $\varepsilon \Delta^\varphi v \cdot \mathbf{N}$. Note that

$$\Delta^\varphi v \cdot \mathbf{N} = 2(\nabla^\varphi \cdot S^\varphi v) \cdot \mathbf{N} = 2\nabla^\varphi \cdot (S^\varphi v \mathbf{N}) - 2S^\varphi v : \nabla^\varphi \mathbf{N} \quad (4.34)$$

and

$$\nabla^\varphi \cdot (S^\varphi v \mathbf{N}) = \partial_1 (S^\varphi v \mathbf{N})_1 + \partial_2 (S^\varphi v \mathbf{N})_2 + \frac{1}{\partial_z \varphi} \partial_z (S^\varphi v \mathbf{N}) \cdot \mathbf{N}. \quad (4.35)$$

Hence by the estimate (A.12) with $s = -\frac{1}{2}, \frac{1}{2}$, one can get

$$\begin{aligned} |\Delta^\varphi v \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} &\lesssim \left| \frac{1}{\partial_z \varphi} \partial_z (S^\varphi v \mathbf{N}) \cdot \mathbf{N} \right|_{\mathbb{X}^{k, -\frac{1}{2}}} + |S^\varphi v \mathbf{N}|_{\mathbb{X}^{k, \frac{1}{2}}} + |S^\varphi v : \nabla^\varphi \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\partial_z (S^\varphi v \mathbf{N}) \cdot \mathbf{N}\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) \\ &\quad \times \left(|\partial_z (S^\varphi v \mathbf{N}) \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} + |\nabla v|_{\mathbb{X}^{k, \frac{1}{2}}} + |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right). \end{aligned} \quad (4.36)$$

But

$$\partial_z (S^\varphi v \mathbf{N}) \cdot \mathbf{N} = \partial_z (S^\varphi v \mathbf{N} \cdot \mathbf{N}) - S^\varphi v \mathbf{N} \cdot \partial_z \mathbf{N}, \quad (4.37)$$

and recalling the matrices P and E ,

$$\begin{aligned} \partial_z (S^\varphi v \mathbf{N} \cdot \mathbf{N}) &= \partial_z (\nabla^\varphi v \mathbf{N} \cdot \mathbf{N}) = \partial_z \left(\frac{1}{\partial_z \varphi} P^* \nabla v P^* e_3 \cdot P^* e_3 \right) \\ &= \nabla^\varphi (\partial_z v) \mathbf{N} \cdot \mathbf{N} - \partial_j v_i \partial_z (E_{3i} P_{3j}), \end{aligned} \quad (4.38)$$

one can then deduce from (4.36) that

$$\begin{aligned} |\Delta^\varphi v \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+1}} + \|\nabla^\varphi (\partial_z v) \mathbf{N} \cdot \mathbf{N}\|_{\mathbb{Y}^{\frac{k}{2}+1}} \right) \\ &\quad \times \left(|\nabla^\varphi (\partial_z v) \mathbf{N} \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} + |\nabla v|_{\mathbb{X}^{k, \frac{1}{2}}} + |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right). \end{aligned} \quad (4.39)$$

Note further that

$$\nabla^\varphi (\partial_z v) \mathbf{N} \cdot \mathbf{N} = \nabla^\varphi (\partial_z v \cdot \mathbf{N}) \cdot \mathbf{N} - (\mathbf{N} \cdot \nabla^\varphi) \mathbf{N} \cdot \partial_z v, \quad (4.40)$$

and one can compute by using $\nabla^\varphi \cdot v = 0$ that

$$\nabla^\varphi (\partial_z v \cdot \mathbf{N}) \cdot \mathbf{N} = -\nabla^\varphi (\partial_z \varphi (\partial_1 v_1 + \partial_2 v_2)) \cdot \mathbf{N}. \quad (4.41)$$

It follows from these, (4.39) and Lemma 3.4 that

$$\begin{aligned} |\Delta^\varphi v \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} \right) \left(|\nabla v|_{\mathbb{X}^{k, \frac{1}{2}}} + |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} \right) \left(|v|_{\mathbb{X}^{k, \frac{3}{2}}} + |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right). \end{aligned} \quad (4.42)$$

Next, one easily has

$$|(v_y \cdot \nabla_y) v \cdot \mathbf{N}|_{\mathbb{X}^{k, -\frac{1}{2}}} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|v\|_{\mathbb{Y}^{\frac{k}{2}+2}} \right) \left(|v|_{\mathbb{X}^{k, \frac{1}{2}}} + |h|_{\mathbb{X}^{k, \frac{1}{2}}} \right). \quad (4.43)$$

Finally, we estimate the remaining time derivative term $\partial_t v \cdot \mathbf{N}$. One first has

$$\begin{aligned} |\partial_t v \cdot \mathbf{N}|_{\mathbb{X}^{k-1, \frac{1}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, |\nabla h|_{\mathbb{Y}^{\frac{k-1}{2}+1}} + \|\partial_t v\|_{\mathbb{Y}^{\frac{k-1}{2}+1}} \right) \left(|\partial_t v|_{\mathbb{X}^{k-1, \frac{1}{2}}} + |h|_{\mathbb{X}^{k-1, \frac{3}{2}}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} \right) \left(|v|_{\mathbb{X}^{k, \frac{1}{2}}} + |h|_{\mathbb{X}^{k-1, \frac{3}{2}}} \right). \end{aligned} \quad (4.44)$$

It then suffices to estimate $|\partial_t^k (\partial_t v \cdot \mathbf{N})|_{-\frac{1}{2}}$. However, this will lead to some difficulties since k can be $m-1$, and energy estimates yield only $\partial_t^m v \in L^2(\Omega)$ which cannot ensure the control of

the $H^{-\frac{1}{2}}\{z=0\}$ norm of $\partial_t^m v$. The key observation is that $\partial_t^m v \cdot \mathbf{N}$ is indeed in $H^{-\frac{1}{2}}\{z=0\}$. The way of achieving this is to use the Alinhac good unknown, returning back to k , $V^{k+1} = \partial_t^{k+1} v - \partial_z^\varphi v \partial_t^{k+1} \eta$. Indeed, since $\nabla \varphi \cdot V^{k+1} = -C^{k+1}(d)$ with $C^{k+1}(d)$ defined as in (3.14) with $Z^\alpha = \partial_t^{k+1}$, by Lemma C.3, and using the estimate (3.16) and (B.6), one gets that

$$\begin{aligned} \left| V^{k+1} \cdot \mathbf{N} \right|_{-\frac{1}{2}} &\leq \Lambda \left(\frac{1}{c_0} \right) \left(\|V^{k+1}\| + \|\nabla \varphi \cdot V^{k+1}\| \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) \left(\|\partial_t^{k+1} v\| + \|\partial_t^{k+1} \eta\| + \|C^{k+1}(d)\| \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+1}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+1}{2}}} \right) \left(\|\partial_t^{k+1} v\| + |h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \|\nabla v\|_{\mathbb{X}^k} \right). \end{aligned} \quad (4.45)$$

This in turn implies

$$\begin{aligned} \left| \partial_t^{k+1} v \cdot \mathbf{N} \right|_{-\frac{1}{2}} &\leq \left| V^{k+1} \cdot \mathbf{N} \right|_{-\frac{1}{2}} + \left| \mathbf{N} \cdot \partial_z^\varphi v \partial_t^{k+1} \eta \right|_{-\frac{1}{2}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+1}{2}}} \right) \left(\|\partial_t^{k+1} v\| + |h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \|\nabla v\|_{\mathbb{X}^k} \right). \end{aligned} \quad (4.46)$$

Hence, it follows from (4.44), (4.46) and the trace estimate $|\cdot|_{-\frac{1}{2}} \leq |\cdot|_{\frac{1}{2}} \lesssim \|\cdot\|_{H^1}$ that

$$\begin{aligned} \left| \partial_t v \cdot \mathbf{N} \right|_{\mathbb{X}^k, -\frac{1}{2}} &\leq \left| \partial_t v \cdot \mathbf{N} \right|_{\mathbb{X}^{k-1, \frac{1}{2}}} + \left| \partial_t^k (\partial_t v \cdot \mathbf{N}) \right|_{-\frac{1}{2}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+1}{2}}} \right) \left(\|v\|_{\mathbb{X}^{k+1}} + \|\nabla v\|_{\mathbb{X}^k} + |h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} \right). \end{aligned} \quad (4.47)$$

Therefore, we conclude from (4.42), (4.43) and (4.47) that

$$\begin{aligned} |G^2|_{\mathbb{X}^k, -\frac{1}{2}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} \right) \\ &\quad \times \left(\|v\|_{\mathbb{X}^{k+1}} + \|\nabla v\|_{\mathbb{X}^k} + |h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \varepsilon |v|_{\mathbb{X}^k, \frac{3}{2}} + \varepsilon |h|_{\mathbb{X}^k, \frac{3}{2}} \right). \end{aligned} \quad (4.48)$$

Similarly, since $k \geq 3$, due to the trace estimates, one can get

$$\begin{aligned} |G^2|_{\mathbb{X}^{\frac{k+3}{2}, -\frac{1}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k+3}{4}+3}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}+3}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k+3}{4}+2}} \right) \\ &\quad \times \left(\|v\|_{\mathbb{X}^{\frac{k+3}{2}+1}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+3}{2}}} + |h|_{\mathbb{X}^{\frac{k+3}{2}+1, -\frac{1}{2}}} + \varepsilon |v|_{\mathbb{X}^{\frac{k+3}{2}, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{\frac{k+3}{2}, \frac{3}{2}}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + |h|_{\mathbb{X}^{\frac{k+7}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{k+7}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+5}{2}}} \right). \end{aligned} \quad (4.49)$$

Consequently, plugging the estimates (4.30)–(4.31), (4.32)–(4.33) and (4.48)–(4.49), along with doing the the same estimates for G^3 as for G^2 , into (4.26), we can obtain that for $k \geq 3$,

$$\begin{aligned} \|q\|_{\mathbb{X}^k} + \|\nabla q\|_{\mathbb{X}^k} + \|\partial_{zz} q\|_{\mathbb{X}^{k-1}} & \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+3}} + |h|_{\mathbb{X}^{\frac{k+7}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{k+7}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+5}{2}}} \right) \\ &\quad \times \left(|h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{k, 2}} + \|v\|_{\mathbb{X}^{k+1}} + \|\nabla v\|_{\mathbb{X}^k} + \varepsilon |v|_{\mathbb{X}^k, \frac{3}{2}} + \varepsilon |h|_{\mathbb{X}^k, \frac{3}{2}} \right). \end{aligned} \quad (4.50)$$

This proves the estimate (4.14).

To prove (4.15), one can use the anisotropic Sobolev embedding estimate (A.5) and the trace estimates to have that for $k \geq 1$, by (4.14),

$$\begin{aligned} \|q\|_{\mathbb{Y}^k} + \|\nabla q\|_{\mathbb{Y}^k} &\lesssim \|\nabla q\|_{\mathbb{X}^{k+2}} + \|\partial_{zz} q\|_{\mathbb{X}^{k+1}} \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+4}} + |h|_{\mathbb{X}^{\frac{k+9}{2}}} + \|v\|_{\mathbb{Y}^{\frac{k+5}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|v\|_{\mathbb{X}^{\frac{k+9}{2}}} + \|\nabla v\|_{\mathbb{X}^{\frac{k+7}{2}}} \right) \\ &\quad \times \left(|h|_{\mathbb{X}^{k+3, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{k+2, 2}} + \|v\|_{\mathbb{X}^{k+3}} + \|\nabla v\|_{\mathbb{X}^{k+2}} + \varepsilon |v|_{\mathbb{X}^{k+2, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{k+2, \frac{3}{2}}} \right) \end{aligned} \quad (4.51)$$

$$\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{k}{2}+4}} + |h|_{\mathbb{X}^{k+4}} + \|v\|_{\mathbb{Y}^{\frac{k+5}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{k}{2}+3}} + \|v\|_{\mathbb{X}^{k+4}} + \|\nabla v\|_{\mathbb{X}^{k+3}} \right).$$

Note that one has used the fact that $k+2 \geq 3$ so that (4.14) can be used with $k+2$. This, together with (4.24) again, proves (4.15). \square

5. SMOOTHING ESTIMATES OF h

We first show the smoothing regularity estimates of h coming from viscosity.

Proposition 5.1. *For every $m \in \mathbb{N}$, $\varepsilon \in (0, 1)$, it holds that*

$$\varepsilon |h(t)|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \leq \varepsilon |h(0)|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \int_0^t \Lambda \left(|h|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} \right) \left(\varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right) \quad (5.1)$$

and

$$\varepsilon |h(t)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \leq \varepsilon |h(0)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + \int_0^t \Lambda \left(|h|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} \right) \left(\varepsilon |v|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right). \quad (5.2)$$

Proof. We prove only the estimate (5.2), and the estimate (5.1) follows in the same way. Apply Z^α with $\alpha \in \mathbb{N}^{1+2}$, $|\alpha| \leq m$, to the kinematic boundary condition to get that on $\{z=0\}$:

$$\partial_t Z^\alpha h + v \cdot \nabla_y Z^\alpha h - Z^\alpha v_3 + [Z^\alpha, v_y] \cdot \nabla_y h = 0. \quad (5.3)$$

Then applying further $\Lambda^{\frac{1}{2}}$, the tangential Fourier multiplier, to (5.3) gives

$$\partial_t \Lambda^{\frac{1}{2}} Z^\alpha h + v \cdot \nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h - \Lambda^{\frac{1}{2}} Z^\alpha v_3 + \left[\Lambda^{\frac{1}{2}}, v_y \right] \cdot \nabla_y Z^\alpha h + \Lambda^{\frac{1}{2}} ([Z^\alpha, v_y] \cdot \nabla_y h) = 0. \quad (5.4)$$

A standard energy estimate on the equation (5.4) yields

$$\begin{aligned} \frac{d}{dt} |Z^\alpha h|_{\frac{1}{2}}^2 &\lesssim \|\nabla_y v\|_{L^\infty} |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \\ &\quad + \left(|v|_{\mathbb{X}^{m, \frac{1}{2}}} + \left| \left[\Lambda^{\frac{1}{2}}, v_y \right] \cdot \nabla_y Z^\alpha h \right|_0 + |[Z^\alpha, v_y] \cdot \nabla_y h|_{\frac{1}{2}} \right) |h|_{\mathbb{X}^{m, \frac{1}{2}}}. \end{aligned} \quad (5.5)$$

Due to the commutator estimate (A.10), one has

$$\left| \left[\Lambda^{\frac{1}{2}}, v_y \right] \cdot \nabla_y Z^\alpha h \right|_0 \lesssim \|\nabla_y v\|_{L^\infty} |h|_{\mathbb{X}^{m, \frac{1}{2}}} + \|\nabla_y h\|_{L^\infty} |v|_{\mathbb{X}^{m, \frac{1}{2}}}. \quad (5.6)$$

And the estimate (A.12) leads to

$$\begin{aligned} |[Z^\alpha, v_y] \cdot \nabla_y h|_{\frac{1}{2}} &\leq \sum_{\substack{|\beta|+|\gamma|=\alpha \\ |\beta'|=1}} \left| Z^{\beta-\beta'} Z^{\beta'} v_y \cdot \nabla_y Z^\gamma h \right|_{\frac{1}{2}} \\ &\lesssim \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} |h|_{\mathbb{X}^{m, \frac{1}{2}}} + |h|_{\mathbb{Y}^{\frac{m}{2}+2}} |v|_{\mathbb{X}^{m, \frac{1}{2}}}. \end{aligned} \quad (5.7)$$

Hence, plugging the estimates (5.6) and (5.7) into (5.5) and summing over $|\alpha| \leq m$, by Cauchy's inequality, one can deduce that

$$\frac{d}{dt} |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \lesssim \left(1 + |h|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} \right) \left(|v|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right). \quad (5.8)$$

Integrating the inequality (5.8) directly in time yields (5.2). \square

Next, we show the smoothing estimates of h due to surface tension.

Proposition 5.2. *For every $m \in \mathbb{N}$, $\varepsilon, \sigma \in (0, 1)$, it holds that*

$$\begin{aligned} \sigma^2 |h|_{\mathbb{X}^{m-1, \frac{5}{2}}}^2 &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m+1}{2}}} \right) \left(|q|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right. \\ &\quad \left. + \varepsilon^2 |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon^2 |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right). \end{aligned} \quad (5.9)$$

Proof. Apply Z^α with $\alpha \in \mathbb{N}^{1+2}$, $|\alpha| \leq m$, $\alpha_0 \leq m-1$, to the dynamic boundary condition to have

$$-\sigma Z^\alpha H = Z^\alpha q - 2\varepsilon Z^\alpha (S^\varphi v \mathbf{n} \cdot \mathbf{n}) - gZ^\alpha h. \quad (5.10)$$

Note that

$$\begin{aligned} Z^\alpha H &= \nabla_y \cdot \left(Z^\alpha \left(\frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}} \right) \right) \\ &= \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} + \nabla_y h Z^\alpha \left(\frac{1}{\sqrt{1 + |\nabla_y h|^2}} \right) + \left[Z^\alpha, \nabla_y h, \frac{1}{\sqrt{1 + |\nabla_y h|^2}} \right] \right) \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} Z^\alpha \left(\frac{1}{\sqrt{1 + |\nabla_y h|^2}} \right) &= Z^{\alpha-\alpha'} Z^{\alpha'} \left(\frac{1}{\sqrt{1 + |\nabla_y h|^2}} \right) = -Z^{\alpha-\alpha'} \left(\frac{\nabla_y h \cdot \nabla_y Z^{\alpha'} h}{\sqrt{1 + |\nabla_y h|^2}^3} \right) \\ &= -\frac{\nabla_y h \cdot \nabla_y Z^{\alpha'} h}{\sqrt{1 + |\nabla_y h|^2}^3} - \left[Z^{\alpha-\alpha'}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \right] \cdot \nabla_y Z^{\alpha'} h \end{aligned} \quad (5.12)$$

for any $|\alpha'| = 1$. Hence,

$$Z^\alpha H = \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h + \mathcal{C}(\mathcal{B}_\sigma^\alpha) \right), \quad (5.13)$$

where

$$\mathcal{C}(\mathcal{B}_\sigma^\alpha) = - \left[Z^{\alpha-\alpha'}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \right] \cdot \nabla_y Z^{\alpha'} h \nabla_y h + \left[Z^\alpha, \frac{1}{\sqrt{1 + |\nabla_y h|^2}}, \nabla_y h \right] \quad (5.14)$$

for any $|\alpha'| = 1$. It follows that

$$-\sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) = \sigma \nabla_y \cdot \mathcal{C}(\mathcal{B}_\sigma^\alpha) + Z^\alpha q - 2\varepsilon Z^\alpha (S^\varphi v \mathbf{n} \cdot \mathbf{n}) - gZ^\alpha h. \quad (5.15)$$

Then apply further $\Lambda^{\frac{1}{2}}$ to (5.15) to get

$$\begin{aligned} &-\sigma \nabla_y \cdot \left(\frac{\nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) \\ &= \sigma \nabla_y \cdot \left(\left[\Lambda^{\frac{1}{2}}, \frac{1}{\sqrt{1 + |\nabla_y h|^2}} - 1 \right] \nabla_y Z^\alpha h - \left[\Lambda^{\frac{1}{2}}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \cdot \right] \nabla_y Z^\alpha h \right) \\ &\quad + \Lambda^{\frac{1}{2}} (\sigma \nabla_y \cdot \mathcal{C}(\mathcal{B}_\sigma^\alpha) + Z^\alpha q - 2\varepsilon Z^\alpha (S^\varphi v \mathbf{n} \cdot \mathbf{n}) - gZ^\alpha h). \end{aligned} \quad (5.16)$$

It follows from a standard energy estimate for this elliptic equation that

$$\begin{aligned} &\sigma^2 \int_{z=0} \left(\frac{|\nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy \\ &= -\sigma^2 \left(\left[\Lambda^{\frac{1}{2}}, \frac{1}{\sqrt{1 + |\nabla_y h|^2}} - 1 \right] \nabla_y Z^\alpha h - \left[\Lambda^{\frac{1}{2}}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \cdot \right] \nabla_y Z^\alpha h, \nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h \right) \\ &\quad - \sigma^2 \left(\Lambda^{\frac{1}{2}} \mathcal{C}(\mathcal{B}_\sigma^\alpha), \nabla_y \Lambda^{\frac{1}{2}} Z^\alpha h \right) + \left(\Lambda^{\frac{1}{2}} (Z^\alpha q - 2\varepsilon Z^\alpha (S^\varphi v \mathbf{n} \cdot \mathbf{n}) - gZ^\alpha h), \sigma \Lambda^{\frac{1}{2}} Z^\alpha h \right). \end{aligned} \quad (5.17)$$

Since for any vector $\mathbf{a} \in \mathbb{R}^2$,

$$\frac{|\mathbf{a}|^2}{\sqrt{1 + |\nabla h|^2}} - \frac{|\nabla_y h \cdot \mathbf{a}|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \geq \frac{1}{\sqrt{1 + |\nabla_y h|^2}^3} |\mathbf{a}|^2, \quad (5.18)$$

so Cauchy's inequality, together with (A.11) and (A.12), since $|\alpha_1| + |\alpha_2| \geq 1$, give

$$\begin{aligned}
\sigma^2 |\nabla_y h|_{\mathbb{X}^{m-1, \frac{5}{2}}}^2 &\leq \Lambda_0 \left(|q|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + \varepsilon^2 |S^\varphi v \mathbf{n} \cdot \mathbf{n}|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right) \\
&\quad + \sigma^2 |\mathcal{C}(\mathcal{B}_\sigma^\alpha)|_{\frac{1}{2}}^2 + \sigma^2 \left| \left[\Lambda^{\frac{1}{2}}, \frac{1}{\sqrt{1 + |\nabla_y h|^2}} - 1 \right] \nabla_y Z^\alpha h \right|_0^2 \\
&\quad + \sigma^2 \left| \left[\Lambda^{\frac{1}{2}}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \cdot \right] \nabla_y Z^\alpha h \right|_0^2 \\
&\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m+3}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m+1}{2}}} \right) \left(|q|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right. \\
&\quad \left. + \sigma^2 |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon^2 |S^\varphi v \mathbf{n} \cdot \mathbf{n}|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right).
\end{aligned} \tag{5.19}$$

Using the similar arguments in the previous section, one can have

$$|S^\varphi v \mathbf{n} \cdot \mathbf{n}|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m-1}{2}+2}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m-1}{2}+1}} \right) \left(|v|_{\mathbb{X}^{m-1, \frac{3}{2}}} + |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right).$$

Then (5.9) follows from (5.19), the Sobolev interpolation and Young's inequality. \square

6. CONORMAL ESTIMATES

We shall derive a priori estimates on a time interval $[0, T^{\varepsilon, \sigma}]$ on which it is assumed that

$$\partial_z \varphi \geq \frac{c_0}{2}, \quad |h|_{2, \infty} \leq \frac{1}{c_0} \quad \text{and} \quad g - \partial_z^2 q \geq \frac{c_0}{2} \quad \text{on} \quad \{z = 0\}. \tag{6.1}$$

Note in particular that this will allow one to use Lemmas C.1 and C.2.

To derive the higher order energy estimates, we shall use the good unknown $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$, $\alpha \neq 0$. A key point is that the control of V^α and $Z^\alpha h$ will yield a control of $Z^\alpha v$:

$$\|Z^\alpha v\| \lesssim \|V^\alpha\| + \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) |Z^\alpha h|_{-\frac{1}{2}}, \quad \|V^\alpha\| \lesssim \|Z^\alpha v\| + \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) |Z^\alpha h|_{-\frac{1}{2}}. \tag{6.2}$$

Define

$$\Lambda_\infty(t) = \Lambda \left(\frac{1}{c_0}, |h(t)|_{\mathbb{X}^{\frac{m}{2}+5}} + \|v(t)\|_{\mathbb{X}^{\frac{m}{2}+5}} + \|\partial_z v(t)\|_{\mathbb{X}^{\frac{m}{2}+4}} + \|\partial_z v(t)\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \varepsilon^{\frac{1}{2}} \|\partial_{zz} v(t)\|_{L^\infty} \right). \tag{6.3}$$

It will be shown then that those functions $\Lambda(\cdot, \cdot)$ defined in the previous three sections can be bounded by Λ_∞ for sufficiently large m , and also the elliptic estimates of q and the smoothing estimates of h will be restated along the way. First, taking $k = \frac{m}{2}$ for $m \geq 6$ in the estimate (4.15) yields

$$\begin{aligned}
&\|q\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\nabla q\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\partial_{zz} q\|_{\mathbb{Y}^{\frac{m}{2}-1}} \\
&\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{4}+4}} + |h|_{\mathbb{X}^{\frac{m}{2}+4}} + \|v\|_{\mathbb{Y}^{\frac{m}{4}+\frac{5}{2}}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{4}+3}} + \|v\|_{\mathbb{X}^{\frac{m}{2}+4}} + \|\nabla v\|_{\mathbb{X}^{\frac{m}{2}+3}} \right) \leq \Lambda_\infty,
\end{aligned} \tag{6.4}$$

while taking $k = m - 1$ in the estimate (4.14) gives

$$\begin{aligned}
&\|q\|_{\mathbb{X}^{m-1}} + \|\nabla q\|_{\mathbb{X}^{m-1}} + \|\partial_{zz} q\|_{\mathbb{X}^{m-2}} \\
&\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m+5}{2}}} + |h|_{\mathbb{X}^{\frac{m}{2}+3}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m+3}{2}}} + \|v\|_{\mathbb{X}^{\frac{m}{2}+3}} + \|\nabla v\|_{\mathbb{X}^{\frac{m}{2}+2}} \right) \\
&\quad \times \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-1, 2}} + \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^{m-1}} + \varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right) \\
&\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-1, 2}} + \|v\|_{\mathbb{X}^m} + \|\partial_z v\|_{\mathbb{X}^{m-1}} + \varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right).
\end{aligned} \tag{6.5}$$

Here one has required $m \geq 6$ so that by Sobolev's inequality

$$|h|_{\mathbb{Y}^{\frac{m}{4}+4}} \leq |h|_{\mathbb{Y}^{\frac{m+5}{2}}} \lesssim |h|_{\mathbb{X}^{\frac{m}{2}+5}},$$

and that by the anisotropic Sobolev embedding estimate (A.5)

$$\|v\|_{\mathbb{Y}^{\frac{m+3}{2}+1}} \lesssim \|v\|_{\mathbb{X}^{\frac{m}{2}+5}} + \|\partial_z v\|_{\mathbb{X}^{\frac{m}{2}+4}}.$$

It can be checked easily that all the functions $\Lambda(\cdot, \cdot)$ defined in Propositions 5.1 and 5.2 and Lemmas 3.3, 3.5 and 3.6 are bounded by Λ_∞ , due to (6.4). Moreover, Proposition 5.1 implies that

$$\varepsilon |h(t)|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \leq \varepsilon |h(0)|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \int_0^t \Lambda_\infty \left(\varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right) \quad (6.6)$$

and

$$\varepsilon |h(t)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \leq \varepsilon |h(0)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + \int_0^t \Lambda_\infty \left(\varepsilon |v|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right). \quad (6.7)$$

Proposition 5.2, together with (6.5) and the trace estimate $|q|_{\mathbb{X}^{m-1, \frac{1}{2}}} \lesssim \|q\|_{\mathbb{X}^{m-1}} + \|\nabla q\|_{\mathbb{X}^{m-1}}$, yields that

$$\sigma^2 |h|_{\mathbb{X}^{m-1, \frac{5}{2}}}^2 \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}}^2 + \sigma^2 |h|_{\mathbb{X}^{m-1, 2}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon^2 |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon^2 |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right). \quad (6.8)$$

Since $|h|_{\mathbb{X}^{m-1, 2}}^2 \leq |h|_{\mathbb{X}^{m-1}}^{\frac{1}{5}} |h|_{\mathbb{X}^{m-1, \frac{5}{2}}}^{\frac{4}{5}}$, one may improve (6.8) by using Young's inequality to get

$$\sigma^2 |h|_{\mathbb{X}^{m-1, \frac{5}{2}}}^2 \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon^2 |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon^2 |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right). \quad (6.9)$$

Note that in the following we will use frequently these L^∞ bounds involved in Λ_∞ .

6.1. Basic L^2 estimate. We start with the estimates of (v, h) itself, that is, the case $\alpha = 0$.

Proposition 6.1. *For any smooth solution of (1.9), it holds that*

$$\begin{aligned} & \|v(t)\|^2 + g |h(t)|_0^2 + \sigma |h(t)|_1^2 + \varepsilon \int_0^t \|\nabla v\|^2 \\ & \leq \Lambda_0 \left(\|v_0\|^2 + |h_0|_0^2 + \sigma |h_0|_1^2 + \int_0^t \|v\|^2 \right). \end{aligned} \quad (6.10)$$

Proof. Standard energy identity yields

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |v|^2 d\mathcal{V}_t + \varepsilon \int_\Omega |S^\varphi v|^2 d\mathcal{V}_t = \int_{z=0} (2\varepsilon S^\varphi v - q\mathbf{I}) \mathbf{N} \cdot v dy - \int_{z=-b} (2\varepsilon S^\varphi v - q\mathbf{I}) e_3 \cdot v dy. \quad (6.11)$$

The Navier slip boundary condition implies that

$$- \int_{z=-b} (2\varepsilon S^\varphi v - q\mathbf{I}) e_3 \cdot v dy = - \int_{z=-b} 2\varepsilon (S^\varphi v e_3)_i v_i dy = -2\kappa\varepsilon \int_{z=-b} |v|^2 dy. \quad (6.12)$$

While the dynamic boundary condition and the kinematic boundary condition give

$$\begin{aligned} \int_{z=0} (2\varepsilon S^\varphi v - q\mathbf{I}) \mathbf{N} \cdot v dy &= - \int_{z=0} (gh - \sigma H) \mathbf{N} \cdot v dy = - \int_{z=0} (gh - \sigma H) \partial_t h dy \\ &= - \frac{1}{2} \frac{d}{dt} \int_{z=0} g|h|^2 + 2\sigma \left(\sqrt{1 + |\nabla h|^2} - 1 \right) dy. \end{aligned} \quad (6.13)$$

Consequently,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int_\Omega |v|^2 d\mathcal{V}_t + \int_{z=0} g|h|^2 + 2\sigma \left(\sqrt{1 + |\nabla h|^2} - 1 \right) dy \right) \\ & + 2\varepsilon \int_\Omega |S^\varphi v|^2 d\mathcal{V}_t + 2\kappa\varepsilon \int_{z=-b} |v|^2 dy = 0. \end{aligned} \quad (6.14)$$

Note that

$$\sqrt{1 + |\nabla h|^2} - 1 \geq \frac{1}{2} \frac{1}{\sqrt{1 + \left(\frac{1}{c_0}\right)^2}} |\nabla h|^2$$

due to (6.1), and the trace estimate

$$|v|_{L^2(\{z=-b\})}^2 \lesssim \|\nabla v\| \|v\| + \|v\|^2.$$

Hence, (6.10) follows from (6.14), Lemma C.2 and Cauchy's inequality. \square

6.2. Estimate of $(Z^\alpha v, Z^\alpha h)$ for $\alpha_0 \leq m-1$. Next, we derive the energy estimates of $(Z^\alpha v, Z^\alpha h)$ for $1 \leq |\alpha| \leq m$ and $\alpha_0 \leq m-1$, that is, except the cases $\alpha = 0$ or $\alpha_0 = m$.

Proposition 6.2. *Any smooth solution of (1.9) satisfies the estimate*

$$\begin{aligned} & \|v(t)\|_{\mathbb{X}^{m-1,1}}^2 + |h(t)|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h(t)|_{\mathbb{X}^{m-1,2}}^2 + \varepsilon |h(t)|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 + \int_0^t \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1,1}}^2 + \sigma^2 |h|_{\mathbb{X}^{m-1,\frac{5}{2}}}^2 \\ & \leq \Lambda_0 \left(\|v(0)\|_{\mathbb{X}^{m-1,1}}^2 + |h(0)|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h(0)|_{\mathbb{X}^{m-1,2}}^2 + \varepsilon |h(0)|_{\mathbb{X}^{m,\frac{1}{2}}}^2 \right) \\ & + \int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m,-\frac{1}{2}}}^2 + \sigma |h|_{\mathbb{X}^{m,1}}^2 + \varepsilon |h|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 \right). \end{aligned} \quad (6.15)$$

Proof. The energy identity for the equations (3.13)–(3.14) yields

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |V^\alpha|^2 d\mathcal{V}_t + 2\varepsilon \int_\Omega |S^\varphi V^\alpha|^2 d\mathcal{V}_t = \mathcal{I}_0^\alpha + \mathcal{I}_b^\alpha + \mathcal{R}_C^\alpha + \mathcal{R}_S^\alpha, \quad (6.16)$$

where

$$\mathcal{I}_0^\alpha = \int_{z=0} (2\varepsilon S^\varphi V^\alpha - Q^\alpha \mathbf{I}) \mathbf{N} \cdot V^\alpha dy \quad (6.17)$$

$$\mathcal{I}_b^\alpha = - \int_{z=-b} (2\varepsilon S^\varphi V^\alpha - Q^\alpha \mathbf{I}) e_3 \cdot V^\alpha dy, \quad (6.18)$$

$$\mathcal{R}_C^\alpha = \int_\Omega ((\partial_z^\varphi v \cdot \nabla^\varphi v Z^\alpha \eta - C^\alpha(\mathcal{T}) - C^\alpha(q)) \cdot V^\alpha - C^\alpha(d)Q^\alpha) d\mathcal{V}_t, \quad (6.19)$$

$$\mathcal{R}_S^\alpha = \int_\Omega (\varepsilon \mathcal{D}^\alpha(S^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(v))) \cdot V^\alpha d\mathcal{V}_t. \quad (6.20)$$

We first estimate \mathcal{I}_0^α . The boundary condition (3.39) implies

$$\begin{aligned} \mathcal{I}_0^\alpha &= \int_{z=0} (2\varepsilon S^\varphi V^\alpha - Z^\alpha q \mathbf{I} + \partial_z^\varphi q Z^\alpha \eta \mathbf{I}) \mathbf{N} \cdot V^\alpha dy \\ &= \int_{z=0} -(g - \partial_z^\varphi q) Z^\alpha h \mathbf{N} \cdot V^\alpha dy + \int_{z=0} \sigma Z^\alpha H \mathbf{N} \cdot V^\alpha dy + \int_{z=0} \varepsilon C^\alpha(\mathcal{B}_\varepsilon) \cdot V^\alpha dy \\ &\quad - \int_{z=0} 2\varepsilon S^\varphi v \Pi Z^\alpha \mathbf{N} \cdot V^\alpha dy - \int_{z=0} 2\varepsilon Z^\alpha h \partial_z^\varphi (S^\varphi v) \mathbf{N} \cdot V^\alpha dy. \end{aligned} \quad (6.21)$$

By (3.40), the third term in the right hand side of (6.21) can be bounded by

$$\left| \int_{z=0} \varepsilon C^\alpha(\mathcal{B}_\varepsilon) \cdot V^\alpha dy \right| \leq \Lambda_\infty \varepsilon (|v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0. \quad (6.22)$$

Due to (A.11), it holds that

$$\begin{aligned} \left| \int_{z=0} 2\varepsilon S^\varphi v \Pi Z^\alpha \mathbf{N} \cdot V^\alpha dy \right| &\leq 2\varepsilon |Z^\alpha \nabla h|_{-\frac{1}{2}} |S^\varphi v \Pi V^\alpha|_{\frac{1}{2}} \\ &\leq \Lambda_\infty \varepsilon |h|_{\mathbb{X}^{m-1,\frac{3}{2}}} |V^\alpha|_{\frac{1}{2}}. \end{aligned} \quad (6.23)$$

Note that Λ_∞ involves $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$, one has

$$\begin{aligned} \left| \int_{z=0} 2\varepsilon Z^\alpha h \partial_z^\varphi (S^\varphi v) \mathbf{N} \cdot V^\alpha dy \right| &\leq 2\varepsilon |Z^\alpha h|_0 \|\partial_z^\varphi (S^\varphi v) \mathbf{N}\|_{L^\infty} |V^\alpha|_0 \\ &\leq \Lambda_\infty \varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1,1}} |V^\alpha|_0. \end{aligned} \quad (6.24)$$

For the first gravity term, one may use the boundary condition (3.47) to rewrite it as

$$\int_{z=0} -(g - \partial_z^\varphi q) Z^\alpha h \mathbf{N} \cdot V^\alpha dy \quad (6.25)$$

$$= \int_{z=0} - (g - \partial_z^\varphi q) Z^\alpha h (\partial_t Z^\alpha h + v_y \cdot \nabla_y Z^\alpha h + \partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h - \mathcal{C}^\alpha(h)) dy.$$

Integrating by parts in t and using (6.4) lead to

$$\int_{z=0} - (g - \partial_z^\varphi q) Z^\alpha h \partial_t Z^\alpha h dy \leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} (g - \partial_z^\varphi q) |Z^\alpha h|^2 dy + \Lambda_\infty |h|_{\mathbb{X}^{m-1,1}}^2. \quad (6.26)$$

The integration by parts in y gives

$$\left| \int_{z=0} (g - \partial_z^\varphi q) Z^\alpha h v_y \cdot \nabla_y Z^\alpha h dy \right| \leq \Lambda_\infty |h|_{\mathbb{X}^{m-1,1}}^2. \quad (6.27)$$

Due to (3.48), one has

$$\left| \int_{z=0} (g - \partial_z^\varphi q) Z^\alpha h (\partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h - \mathcal{C}^\alpha(h)) dy \right| \leq \Lambda_\infty |h|_{\mathbb{X}^{m-1,1}} (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1}}). \quad (6.28)$$

Hence, in light of the estimates (6.26)–(6.28), one may conclude from (6.25) that

$$\begin{aligned} & \int_{z=0} - (g - \partial_z^\varphi q) Z^\alpha h \mathbf{N} \cdot V^\alpha dy \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} (g - \partial_z^\varphi q) |Z^\alpha h|^2 dy + \Lambda_\infty (|h|_{\mathbb{X}^{m-1,1}}^2 + |v|_{\mathbb{X}^{m-1}}^2). \end{aligned} \quad (6.29)$$

To deal with the second term involving surface tension, one has by (5.13) that

$$\int_{z=0} \sigma Z^\alpha H \mathbf{N} \cdot V^\alpha dy = \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h + \mathcal{C}(\mathcal{B}_\sigma^\alpha) \right) \mathbf{N} \cdot V^\alpha dy, \quad (6.30)$$

where $\mathcal{C}(\mathcal{B}_\sigma^\alpha)$ is defined by (5.14). By Lemma 3.6, one may deduce

$$\begin{aligned} \int_{z=0} \sigma \nabla_y \cdot \mathcal{C}(\mathcal{B}_\sigma^\alpha) \mathbf{N} \cdot V^\alpha dy & \leq \sigma |\nabla_y \cdot \mathcal{C}(\mathcal{B}_\sigma^\alpha)|_0 |\partial_t Z^\alpha h + v_y \cdot \nabla_y Z^\alpha h + \partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h - \mathcal{C}^\alpha(h)|_0 \\ & \leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m-1,2}} (|h|_{\mathbb{X}^{m,1}} + |h|_{\mathbb{X}^{m-1,2}} + |h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1}}) \\ & \leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m-1,2}} (|h|_{\mathbb{X}^{m,1}} + |v|_{\mathbb{X}^{m-1}}). \end{aligned} \quad (6.31)$$

To study the other two terms, one rewrite it as, by using the boundary condition (3.47) again,

$$\begin{aligned} & \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) \mathbf{N} \cdot V^\alpha dy \\ & = \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) (\partial_t Z^\alpha h + v_y \cdot \nabla_y Z^\alpha h) dy + \mathcal{R}_{\mathcal{B}_\sigma^\alpha}^\alpha, \end{aligned} \quad (6.32)$$

where

$$\mathcal{R}_{\mathcal{B}_\sigma^\alpha}^\alpha = \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} + \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) (\partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h - \mathcal{C}^\alpha(h)) dy. \quad (6.33)$$

It follows from an integration by parts, (3.49) and (3.50) that

$$\begin{aligned} \mathcal{R}_{\mathcal{B}_\sigma^\alpha}^\alpha & \lesssim \sigma \left| \frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right|_0 \left| \nabla_y (\partial_z^\varphi v \cdot \mathbf{N} Z^\alpha h - \tilde{\mathcal{C}}^\alpha(h)) \right|_0 \\ & \quad + \sigma \left| \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) \right|_{-\frac{1}{2}} |Z^{\alpha-\alpha_1} v_y \cdot \nabla_y Z^{\alpha_1} h|_{\frac{1}{2}} \\ & \leq \Lambda_\infty \left(\sigma |h|_{\mathbb{X}^{m-1,2}} (|h|_{\mathbb{X}^{m-1,2}} + |v|_{\mathbb{X}^{m-2,1}}) + \sigma |h|_{\mathbb{X}^{m-1, \frac{5}{2}}} |v|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \end{aligned} \quad (6.34)$$

Integrating by parts in both y and t , one finds that

$$\begin{aligned} & \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla h|^2}^3} \nabla_y h \right) \partial_t Z^\alpha h \, dy \\ &= -\frac{1}{2} \frac{d}{dt} \int_{z=0} \sigma \left(\frac{|\nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy + \mathcal{R}_{\mathbb{B}_z^\alpha}^\alpha, \end{aligned} \quad (6.35)$$

where

$$\begin{aligned} \mathcal{R}_{\mathbb{B}_z^\alpha}^\alpha &= \frac{1}{2} \int_{z=0} \sigma \left(\partial_t \left(\frac{1}{\sqrt{1 + |\nabla_y h|^2}} \right) |\nabla_y Z^\alpha h|^2 - \partial_t \left(\frac{1}{\sqrt{1 + |\nabla_y h|^2}^3} \right) |\nabla_y h \cdot \nabla_y Z^\alpha h|^2 \right. \\ &\quad \left. - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y \partial_t h \cdot \nabla_y Z^\alpha h \right) dy \\ &\leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m-1,2}}^2. \end{aligned} \quad (6.36)$$

Similarly, the integration by parts twice yields

$$-\int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y Z^\alpha h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y Z^\alpha h}{\sqrt{1 + |\nabla h|^2}^3} \nabla_y h \right) v_y \cdot \nabla_y Z^\alpha h \, dy \leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m-1,2}}^2. \quad (6.37)$$

Hence, by the estimates (6.31), (6.34)–(6.37), one may conclude from (6.30) that

$$\begin{aligned} \int_{z=0} \sigma Z^\alpha H \mathbf{N} \cdot V^\alpha \, dy &\leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} \sigma \left(\frac{|\nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy \\ &\quad + \Lambda_\infty \left(\sigma |h|_{\mathbb{X}^{m-1,2}} |h|_{\mathbb{X}^{m,1}} + \sigma |h|_{\mathbb{X}^{m-1,\frac{5}{2}}} |v|_{\mathbb{X}^{m-1,\frac{1}{2}}} \right). \end{aligned} \quad (6.38)$$

Note also that Lemma A.4 implies that

$$|V^\alpha|_0 \lesssim |v|_{\mathbb{X}^{m-1,1}} + \Lambda_\infty |h|_{\mathbb{X}^{m-1,1}} \quad \text{and} \quad |V^\alpha|_{\frac{1}{2}} \lesssim |v|_{\mathbb{X}^{m-1,\frac{3}{2}}} + \Lambda_\infty |h|_{\mathbb{X}^{m-1,\frac{3}{2}}}. \quad (6.39)$$

Consequently, plugging the estimates (6.22)–(6.24), (6.29) and (6.38) into (6.21), by (6.39) and Cauchy's inequality, one may finish the estimates of \mathcal{I}_0^α as:

$$\begin{aligned} \mathcal{I}_0^\alpha &\leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} (g - \partial_z^\varphi q) |Z^\alpha h|^2 + \sigma \left(\frac{|\nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy \\ &\quad + \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,1}}^2 + |v|_{\mathbb{X}^{m-1}}^2 + \sigma |h|_{\mathbb{X}^{m-1,2}} |h|_{\mathbb{X}^{m,1}} + \sigma |h|_{\mathbb{X}^{m-1,\frac{5}{2}}} |v|_{\mathbb{X}^{m-1,\frac{1}{2}}} \right. \\ &\quad \left. + \varepsilon \left(|v|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 + |h|_{\mathbb{X}^{m-1,\frac{3}{2}}} |v|_{\mathbb{X}^{m-1,\frac{3}{2}}} \right) \right). \end{aligned} \quad (6.40)$$

It also follows from (3.42) and (6.39) that \mathcal{I}_b^α admits the following bound:

$$\begin{aligned} \mathcal{I}_b^\alpha &= -\int_{z=-b} (2\varepsilon S^\varphi V^\alpha e_3)_i \cdot V_i^\alpha \, dy = -\int_{z=-b} 2\varepsilon (\kappa V_i^\alpha - \mathcal{E}^\alpha(v)_{i3}) \cdot V_i^\alpha \, dy \\ &\leq \Lambda_\infty \varepsilon (|V^\alpha|_0 + |v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0 \\ &\leq \Lambda_\infty \varepsilon \left(|v|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1,1}}^2 \right). \end{aligned} \quad (6.41)$$

Next, the commutator \mathcal{R}_C is estimated by using (3.18), (3.15), (3.16), (6.2), (6.5) and (6.4) as

$$\begin{aligned} \mathcal{R}_C^\alpha &\leq \Lambda_\infty (\|Z^\alpha \eta\| + \|\mathcal{C}^\alpha(\mathcal{T})\| + \|\mathcal{C}^\alpha(q)\|) \|V^\alpha\| + \|\mathcal{C}^\alpha(d)\| \|Q^\alpha\| \\ &\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m,-\frac{1}{2}}} + \|v\|_{\mathbb{X}^{m-1}} + \|\nabla v\|_{\mathbb{X}^{m-1}} + \|\nabla q\|_{\mathbb{X}^{m-1}} \right) \left(\|v\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,\frac{1}{2}}} \right) \\ &\quad + \Lambda_\infty \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1,\frac{1}{2}}} \right) \left(\|q\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,\frac{1}{2}}} \right) \\ &\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m,-\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-1,2}} + \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^{m-1}} + \varepsilon |v|_{\mathbb{X}^{m-1,\frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1,\frac{3}{2}}} \right) \end{aligned} \quad (6.42)$$

$$\times \left(\|v\|_{\mathbb{X}^{m-1,1}} + \|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right).$$

It remains to estimate the commutator $\mathcal{R}_{S_i}^\alpha$. First, it follows from the integration by parts, (3.17) and (3.42) that

$$\begin{aligned} \int_{\Omega} \varepsilon \nabla^\varphi(\mathcal{E}^\alpha(v)) \cdot V^\alpha d\mathcal{V}_t &= - \int_{\Omega} \varepsilon \mathcal{E}^\alpha(v) \cdot \nabla V^\alpha d\mathcal{V}_t + \int_{z=0} \varepsilon \mathcal{E}^\alpha(v) \mathbf{N} \cdot V^\alpha dy \\ &\leq \Lambda_\infty \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\| + (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0. \end{aligned} \quad (6.43)$$

Next, for the first term, one actually has to estimate

$$\begin{aligned} \mathcal{R}_{S_i}^\alpha &= \varepsilon \int_{\Omega} \mathcal{C}_j^\alpha(S^\varphi v)_{ij} V_j^\alpha d\mathcal{V}_t \\ &= \varepsilon \int_{\Omega} \mathcal{C}_{j,1}^\alpha(S^\varphi v)_{ij} V_j^\alpha d\mathcal{V}_t + \varepsilon \int_{\Omega} \mathcal{C}_{j,2}^\alpha(S^\varphi v)_{ij} V_j^\alpha d\mathcal{V}_t + \varepsilon \int_{\Omega} \mathcal{C}_{j,3}^\alpha(S^\varphi v)_{ij} V_j^\alpha d\mathcal{V}_t \\ &:= \mathcal{R}_{S_i}^{\alpha,1} + \mathcal{R}_{S_i}^{\alpha,2} + \mathcal{R}_{S_i}^{\alpha,3} \end{aligned} \quad (6.44)$$

due to (3.2). For $\mathcal{R}_{S_i}^{\alpha,1}$, by (3.3), it suffices to estimate terms like

$$\varepsilon \int_{\Omega} Z^\beta \left(\frac{\partial_j \varphi}{\partial_z \varphi} \right) (Z^{\tilde{\gamma}} \partial_z(S^\varphi v)_{ij}) V_j^\alpha d\mathcal{V}_t,$$

where β and $\tilde{\gamma}$ are such that $\beta \neq 0$, $\tilde{\gamma} \neq 0$ and $|\beta| + |\tilde{\gamma}| = m$. By using (3.11), one can reduce the problem to the estimate of

$$\varepsilon \int_{\Omega} c_\gamma Z^\beta \left(\frac{\partial_j \varphi}{\partial_z \varphi} \right) \partial_z (Z^\gamma(S^\varphi v)_{ij}) V_j^\alpha d\mathcal{V}_t$$

with β as before (thus $|\beta| \leq m-1$) and $|\gamma| \leq |\tilde{\gamma}| \leq m-1$. The integration by parts shows that it suffices to estimate three types of terms:

$$\mathcal{I}_1 = \varepsilon \int_{\Omega} Z^\beta \left(\frac{\partial_j \varphi}{\partial_z \varphi} \right) Z^\gamma(S^\varphi v)_{ij} \partial_z V_j^\alpha d\mathcal{V}_t,$$

$$\mathcal{I}_2 = \varepsilon \int_{\Omega} \left(\partial_z Z^\beta \left(\frac{\partial_j \varphi}{\partial_z \varphi} \right) \right) Z^\gamma(S^\varphi v)_{ij} V_j^\alpha d\mathcal{V}_t,$$

and

$$\mathcal{I}_3 = \varepsilon \int_{z=0} Z^\beta \left(\frac{\partial_j \varphi}{\partial_z \varphi} \right) Z^\gamma(S^\varphi v)_{ij} V_j^\alpha dy.$$

For \mathcal{I}_1 and \mathcal{I}_2 , since $\beta \neq 0$, it follows from (A.1), (3.7) and Lemma B.2 that

$$|\mathcal{I}_1| \leq \Lambda_\infty \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\|$$

and

$$|\mathcal{I}_2| \leq \Lambda_\infty \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right) \|V^\alpha\|.$$

By (A.1), $\beta \neq 0$ and Lemma 3.4, it holds that

$$|\mathcal{I}_3| \leq \Lambda_\infty \varepsilon (|h|_{\mathbb{X}^{m-1,1}} + |\nabla v|_{\mathbb{X}^{m-1}}) |V^\alpha|_0 \leq \Lambda_\infty \varepsilon (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0.$$

Consequently, one can get from the previous three estimates that

$$\begin{aligned} \left| \mathcal{R}_{S_i}^{\alpha,1} \right| &\leq \Lambda_\infty \varepsilon \left(\left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\| + \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right) \|V^\alpha\| \right. \\ &\quad \left. + (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0 \right). \end{aligned} \quad (6.45)$$

The estimate of $\mathcal{R}_{S_i}^{\alpha,2}$ is straightforward, one gets from the definition (3.4) that

$$\left| \mathcal{R}_{S_i}^{\alpha,2} \right| \leq \Lambda_\infty \varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \|V^\alpha\|. \quad (6.46)$$

To estimate $\mathcal{R}_{S_i}^{\alpha,3}$, one derives from (3.5) and (3.11) that

$$\varepsilon \left| \int_{\Omega} \frac{\partial_i \varphi}{(\partial_z \varphi)^2} \partial_z(S^\varphi v) [Z^\alpha, \partial_z] \varphi V_j^\alpha d\mathcal{V}_t \right| \leq \Lambda_\infty \varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \|V^\alpha\|.$$

Note that one has used again in the previous two estimates the fact that Λ_∞ involves $\varepsilon^{\frac{1}{2}} \|\partial_{zz}v\|_{L^\infty}$. For the term

$$\varepsilon \int_{\Omega} \frac{\partial_i \varphi}{\partial_z \varphi} V^\alpha [Z^\alpha, \partial_z](S^\varphi v) d\mathcal{V}_t,$$

performing an integration by parts and using a similar arguments as for $\mathcal{R}_{S_i}^1$, show that

$$\left| \int_{\Omega} \frac{\partial_i \varphi}{\partial_z \varphi} V^\alpha [Z^\alpha, \partial_z](S^\varphi v) d\mathcal{V}_t \right| \leq \Lambda_\infty \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1}} (\|V^\alpha\| + \|\nabla V^\alpha\|).$$

Consequently,

$$\left| \mathcal{R}_{S_i}^{\alpha,3} \right| \leq \Lambda_\infty \left(\varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \|V^\alpha\| + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1}} (\|V^\alpha\| + \|\nabla V^\alpha\|) \right). \quad (6.47)$$

It then follows from (6.45)–(6.47) that

$$\begin{aligned} \varepsilon \left| \int_{\Omega} \mathcal{D}^\alpha(S^\varphi v) \cdot V^\alpha d\mathcal{V}_t \right| &\leq \Lambda_\infty \left(\left(\varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1}} \right) \|V^\alpha\| \right. \\ &\quad \left. + \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\| + (|h|_{\mathbb{X}^{m-1,1}} + |v|_{\mathbb{X}^{m-1,1}}) |V^\alpha|_0 \right). \end{aligned} \quad (6.48)$$

This, together with (6.43), (6.2) and (6.39), implies that

$$\begin{aligned} \mathcal{R}_S^\alpha &\leq \Lambda_\infty \left(\left(\varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1}} \right) \left(\|v\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \right. \\ &\quad \left. + \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\| + |h|_{\mathbb{X}^{m-1,1}}^2 + |v|_{\mathbb{X}^{m-1,1}}^2 \right). \end{aligned} \quad (6.49)$$

We can now finish the proof of the proposition. By the estimates (6.40)–(6.42) and (6.49), the trace estimates $|v|_{\mathbb{X}^{m-1, \frac{1}{2}}} \lesssim \|v\|_{\mathbb{X}^{m-1}} + \|\nabla v\|_{\mathbb{X}^{m-1}}$ and $|v|_{\mathbb{X}^{m-1,1}} \lesssim \|\nabla v\|_{\mathbb{X}^{m-1,1}}^{\frac{1}{2}} \|v\|_{\mathbb{X}^{m-1,1}}^{\frac{1}{2}} + \|v\|_{\mathbb{X}^{m-1,1}}$, using Cauchy's inequality, one may deduce from (6.16) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{E}^\alpha + 2\varepsilon \int_{\Omega} |S^\varphi V^\alpha|^2 d\mathcal{V}_t &\quad (6.50) \\ &\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h|_{\mathbb{X}^{m-1,2}} |h|_{\mathbb{X}^{m,1}} + \sigma |h|_{\mathbb{X}^{m-1, \frac{5}{2}}} (\|v\|_{\mathbb{X}^{m-1,1}} + \|\nabla v\|_{\mathbb{X}^{m-1}}) \right. \\ &\quad \left. + |h|_{\mathbb{X}^{m, -\frac{1}{2}}}^2 + \|\nabla v\|_{\mathbb{X}^{m-1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right. \\ &\quad \left. + \varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}} \left(\|v\|_{\mathbb{X}^{m-1,1}} + \|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right) \right. \\ &\quad \left. + \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^\alpha\| \right), \end{aligned}$$

where

$$\mathcal{E}^\alpha := \int_{\Omega} |V^\alpha|^2 d\mathcal{V}_t + \int_{z=0} (g - \partial_z^2 q) |Z^\alpha h|^2 + \sigma \left(\frac{|\nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y Z^\alpha h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy. \quad (6.51)$$

It follows from (6.2), the Taylor sign condition in (6.1) and (5.18) that

$$\|Z^\alpha v\|^2 + |Z^\alpha h|_0^2 + \sigma |Z^\alpha h|_1^2 \leq \Lambda \left(\frac{1}{c_0} \right) \mathcal{E}^\alpha.$$

One can use the Korn inequality of Lemma C.2 and (6.2) to get that

$$\|\nabla V^\alpha\|^2 \leq \Lambda \left(\frac{1}{c_0} \right) \left(\int_{\Omega} |S^\varphi V^\alpha|^2 d\mathcal{V}_t + \|v\|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right).$$

On the other hand, by the definition of V^α ,

$$\varepsilon \|\nabla Z^\alpha v\|^2 \leq \varepsilon \|\nabla V^\alpha\|^2 + \Lambda_\infty \left(\varepsilon \|h\|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \|h\|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 \right).$$

Then integrating (6.50) in time, using the trace estimate $|v|_{\mathbb{X}^{m-1, \frac{3}{2}}} \lesssim \|\nabla v\|_{\mathbb{X}^{m-1,1}} + \|v\|_{\mathbb{X}^{m-1,1}}$ and Cauchy's inequality, together with (6.10), one deduces that

$$\|v(t)\|_{\mathbb{X}^{m-1,1}}^2 + |h(t)|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h(t)|_{\mathbb{X}^{m-1,2}}^2 + \varepsilon \int_0^t \|\nabla v\|_{\mathbb{X}^{m-1,1}}^2 \quad (6.52)$$

$$\begin{aligned}
 &\leq \Lambda_0 \left(\|v(0)\|_{\mathbb{X}^{m-1,1}}^2 + |h(0)|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h(0)|_{\mathbb{X}^{m-1,2}}^2 \right) \\
 &\quad + \int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,1}}^2 + \sigma |h|_{\mathbb{X}^{m-1,2}} |h|_{\mathbb{X}^{m,1}} + \sigma |h|_{\mathbb{X}^{m-1,\frac{5}{2}}} (\|v\|_{\mathbb{X}^{m-1,1}} + \|\nabla v\|_{\mathbb{X}^{m-1}}) \right. \\
 &\quad \left. + |h|_{\mathbb{X}^{m,-\frac{1}{2}}}^2 + \|\nabla v\|_{\mathbb{X}^{m-1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \varepsilon |h|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 \right).
 \end{aligned}$$

This, together with (6.6), (6.9) and Cauchy's inequality, leads to (6.15). \square

6.3. Estimate of $(\partial_t^m v, \partial_t^m h)$. We now derive the energy estimates of $(\partial_t^m v, \partial_t^m h)$, that is, the case $\alpha_0 = m$.

Proposition 6.3. *Any smooth solution of (1.9) satisfies the estimate*

$$\begin{aligned}
 &\int_0^t \left(\|\partial_t^m v\|^2 + |\partial_t^m h|_0^2 + \sigma |\partial_t^m h|_1^2 + \varepsilon |h|_{\mathbb{X}^{m,\frac{1}{2}}}^2 \right)^2 + \int_0^t \left(\varepsilon \int_0^s \|\nabla \partial_t^m v\|_0^2 \right)^2 \\
 &\leq t \Lambda_0 \left(\|\partial_t^m v(0)\|^2 + |\partial_t^m h(0)|_0^2 + \sigma |\partial_t^m h(0)|_1^2 + \varepsilon |h(0)|_{\mathbb{X}^{m,\frac{1}{2}}}^2 \right)^2 \\
 &\quad + t \left(\int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^m}^2 + \sigma |h|_{\mathbb{X}^{m,1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1,1}}^2 + \varepsilon |h|_{\mathbb{X}^{m,\frac{1}{2}}}^2 \right) \right)^2 \\
 &\quad + \int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,\frac{1}{2}}}^2 + \sigma |h|_{\mathbb{X}^{m-1,2}}^2 + \|v\|_{\mathbb{X}^{m-1,1}}^2 + \|\partial_z v\|_{\mathbb{X}^{m-2}}^2 \right) \\
 &\quad \times \left(|h|_{\mathbb{X}^{m,-\frac{1}{2}}}^2 + \sigma |h|_{\mathbb{X}^{m,1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon^2 |v|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 + \varepsilon^2 |h|_{\mathbb{X}^{m-1,\frac{3}{2}}}^2 \right).
 \end{aligned} \tag{6.53}$$

Proof. In the current case, (6.16) can be restated as:

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |V^m|^2 d\mathcal{V}_t + 2\varepsilon \int_\Omega |S^\varphi V^m|^2 d\mathcal{V}_t = \mathcal{I}_0^m + \mathcal{I}_b^m + \mathcal{R}_Q^m + \mathcal{R}_C^m + \mathcal{R}_S^m, \tag{6.54}$$

where

$$\mathcal{I}_0^m = \int_{z=0} (2\varepsilon S^\varphi V^m - Q^m \mathbf{I}) \mathbf{N} \cdot V^m dy, \tag{6.55}$$

$$\mathcal{I}_b^m = - \int_{z=0} (2\varepsilon S^\varphi V^m - Q^m \mathbf{I}) e_3 \cdot V^m dy, \tag{6.56}$$

$$\mathcal{R}_Q^m = - \int_\Omega \mathcal{C}^m(d) \partial_t^m q d\mathcal{V}_t, \tag{6.57}$$

$$\mathcal{R}_C^m = \int_\Omega ((\partial_z^\varphi v \cdot \nabla^\varphi v \partial_t^m \eta - \mathcal{C}^m(\mathcal{T}) - \mathcal{C}^m(q)) \cdot V^m + \mathcal{C}^m(d) \partial_z^\varphi q \partial_t^m \eta) d\mathcal{V}_t, \tag{6.58}$$

$$\mathcal{R}_S^m = \int_\Omega (\varepsilon \mathcal{D}^m(S^\varphi v) + \varepsilon \nabla^\varphi \cdot \mathcal{E}^m(v)) \cdot V^m d\mathcal{V}_t. \tag{6.59}$$

Here $V^m = \partial_t^m v - \partial_z^\varphi v \partial_t^m \eta$, $Q^m = \partial_t^m q - \partial_z^\varphi v \partial_t^m \eta$, and $\mathcal{C}^m(\cdot)$ are those commutators $\mathcal{C}^\alpha(\cdot)$ for the case $\alpha_0 = m$. Note that we have singled out the term \mathcal{R}_Q^m from \mathcal{R}_C^m .

We first estimate \mathcal{I}_0^m , which can be rewritten as (similar to (6.21))

$$\begin{aligned}
 \mathcal{I}_0^m &= \int_{z=0} -(g - \partial_z^\varphi q) \partial_t^m h \mathbf{N} \cdot V^m dy + \int_{z=0} \sigma \partial_t^m H \mathbf{N} \cdot V^m dy + \int_{z=0} \varepsilon \mathcal{C}^m(\mathcal{B}_\varepsilon) \cdot V^m dy \\
 &\quad - \int_{z=0} 2\varepsilon S^\varphi v \Pi \partial_t^m \mathbf{N} \cdot V^m dy - \int_{z=0} 2\varepsilon \partial_t^m h \partial_z^\varphi (S^\varphi v) \mathbf{N} \cdot V^m dy.
 \end{aligned} \tag{6.60}$$

Following the analysis in (6.22)–(6.24), one can bound the last three terms in (6.60) by

$$\Lambda_\infty \left(\varepsilon (\|v\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}) |V^m|_0 + \varepsilon |h|_{\mathbb{X}^{m,\frac{1}{2}}} |V^m|_{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^m} |V^m|_0 \right). \tag{6.61}$$

As (6.29), one deduces

$$\int_{z=0} -(g - \partial_z^\varphi q) \partial_t^m h \mathbf{N} \cdot V^m dy \tag{6.62}$$

$$\leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} (g - \partial_z^2 q) |\partial_t^m h|^2 dy + \Lambda_\infty \left(|h|_{\mathbb{X}^m}^2 + |v|_{\mathbb{X}^{m-1}}^2 \right).$$

However, as explained in Section 2, one can not use the arguments leading to (6.38) to estimate $\int_{z=0} \sigma \partial_t^m H \mathbf{N} \cdot V^m dy$ since there is one half regularity loss for $\partial_t^m h$ so that it is difficult to control the following term, after using the kinematic boundary condition,

$$-\int_{z=0} \sigma \partial_t^m H m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v dy. \quad (6.63)$$

The crucial observation here is that, this term will be cancelled out from estimating the term \mathcal{R}_Q^m defined by (6.57). So the estimates of this term will be postponed till we estimate \mathcal{R}_Q^m .

Similarly as (6.41), \mathcal{I}_b^m admits the bound

$$\mathcal{I}_b^m \leq \Lambda_\infty \varepsilon (|V^m|_0 + |v|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1,1}}) |V^m|_0. \quad (6.64)$$

We now estimate \mathcal{R}_Q^m . Note carefully that there is no any estimates of $\partial_t^m q$, so one needs to integrate by parts in t . To continue, one needs more explicit expression of $\mathcal{C}^m(d)$. Indeed, we will use a variant of (3.2). It follows from the divergence free condition that

$$\partial_z \varphi (\partial_1 v_1 + \partial_2 v_2) + \partial_z v \cdot \mathbf{N} = 0.$$

Applying ∂_t^m to the above and using the definition of $\mathcal{C}^m(d)$, one gets that

$$\partial_z \varphi \mathcal{C}^m(d) = [\partial_t^m, \mathbf{N}, \cdot \partial_z v] + [\partial_t^m, \partial_z \eta, \partial_1 v_1 + \partial_2 v_2]. \quad (6.65)$$

Moreover, to integrate by parts in t , one needs to single out in $\mathcal{C}^m(d)$ the highest $m-1$ order time derivatives terms and use the following decomposition

$$\partial_z \varphi \mathcal{C}^m(d) = \mathcal{C}^m(d)_1 + \mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3 + \mathcal{C}^m(d)_4 + \mathcal{C}^m(d)_5 \quad (6.66)$$

with

$$\mathcal{C}^m(d)_1 = m \partial_t \mathbf{N} \cdot \partial_t^{m-1} \partial_z v, \quad \mathcal{C}^m(d)_2 = m \partial_t \partial_z \eta \partial_t^{m-1} (\partial_1 v_1 + \partial_2 v_2), \quad (6.67)$$

$$\mathcal{C}^m(d)_3 = m \partial_t^{m-1} \mathbf{N} \cdot \partial_t \partial_z v, \quad \mathcal{C}^m(d)_4 = m \partial_t^{m-1} \partial_z \eta \partial_t (\partial_1 v_1 + \partial_2 v_2), \quad (6.68)$$

$$\mathcal{C}^m(d)_5 = \sum_{\ell=2}^{m-2} C_m^\ell \left(\partial_t^\ell \mathbf{N} \cdot \partial_t^{m-\ell} \partial_z v + \partial_t^\ell \partial_z \eta \cdot \partial_t^{m-\ell} (\partial_1 v_1 + \partial_2 v_2) \right). \quad (6.69)$$

Accordingly,

$$\mathcal{R}_Q^m = - \int_{\Omega} (\mathcal{C}^m(d)_1 + \mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3 + \mathcal{C}^m(d)_4 + \mathcal{C}^m(d)_5) \partial_t^m q dy dz. \quad (6.70)$$

The fifth term in (6.70) can be easily treated by the integration by parts in t as

$$- \int_{\Omega} \mathcal{C}^m(d)_5 \partial_t^m q dy dz = - \frac{d}{dt} \int_{\Omega} \mathcal{C}^m(d)_5 \partial_t^{m-1} q dy dz + \mathcal{R}_5^m, \quad (6.71)$$

with

$$\mathcal{R}_5^m = \int_{\Omega} \partial_t \mathcal{C}^m(d)_5 \partial_t^{m-1} q dy dz \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} \right) \|\partial_t^{m-1} q\|. \quad (6.72)$$

Integrate by parts in t to write the fourth term as

$$- \int_{\Omega} \mathcal{C}^m(d)_4 \partial_t^m q dy dz = - \frac{d}{dt} \int_{\Omega} \mathcal{C}^m(d)_4 \partial_t^{m-1} q dy dz + \mathcal{R}_4^m, \quad (6.73)$$

where, by further integrating by parts in z and the trace theory,

$$\begin{aligned} \mathcal{R}_4^m &= \int_{\Omega} m \left(\partial_t^m \partial_z \eta \partial_t (\partial_1 v_1 + \partial_2 v_2) + \partial_t^{m-1} \partial_z \eta \partial_t^2 (\partial_1 v_1 + \partial_2 v_2) \right) \partial_t^{m-1} q dy dz \\ &= \int_{z=0} m \partial_t^m h \partial_t (\partial_1 v_1 + \partial_2 v_2) \partial_t^{m-1} q dy - \int_{\Omega} m \partial_t^m \eta \partial_z \left((\partial_1 v_1 + \partial_2 v_2) \partial_t^{m-1} q \right) dy dz \\ &\quad + \int_{\Omega} m \partial_t^{m-1} \partial_z \eta \partial_t^2 (\partial_1 v_1 + \partial_2 v_2) \partial_t^{m-1} q dy dz \end{aligned} \quad (6.74)$$

$$\begin{aligned} &\leq \Lambda_\infty \left(|\partial_t^m h|_{-\frac{1}{2}} |\partial_t^{m-1} q|_{\frac{1}{2}} + \|\partial_t^m \eta\| \|\partial_t^{m-1} q\|_{H^1} + \|\partial_t^{m-1} \partial_z \eta\| \|\partial_t^{m-1} q\| \right) \\ &\leq \Lambda_\infty |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \|\partial_t^{m-1} q\|_{H^1}. \end{aligned}$$

Similarly, integrate by parts in both t and y to bound the second and third terms by

$$- \int_{\Omega} (\mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3) \partial_t^m q \, dydz = - \frac{d}{dt} \int_{\Omega} (\mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3) \partial_t^{m-1} q \, dydz + \mathcal{R}_{2,3}^m, \quad (6.75)$$

where

$$\begin{aligned} \mathcal{R}_{2,3}^m &= \int_{\Omega} \partial_t (\mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3) \partial_t^{m-1} q \, dydz \\ &\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^2} \right) \|\partial_t^{m-1} q\|_{H^1}. \end{aligned} \quad (6.76)$$

Finally, we turn to the most delicate term, the one involving $\mathcal{C}^m(d)_1$ in (6.70). Integrate by parts in z first to get

$$- \int_{\Omega} \mathcal{C}^m(d)_1 \partial_t^m q \, dydz = - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q \, dy + \int_{\Omega} m \partial_z (\partial_t^m q \partial_t \mathbf{N}) \cdot \partial_t^{m-1} v \, dydz. \quad (6.77)$$

Then integrate by parts in t to obtain

$$\int_{\Omega} m \partial_z (\partial_t^m q \partial_t \mathbf{N}) \cdot \partial_t^{m-1} v \, dydz = \frac{d}{dt} \int_{\Omega} m \partial_z (\partial_t^{m-1} q \partial_t \mathbf{N}) \cdot \partial_t^{m-1} v \, dydz + \mathcal{R}_1^m, \quad (6.78)$$

with

$$\begin{aligned} \mathcal{R}_1^m &= - \int_{\Omega} m \partial_z (\partial_t^{m-1} q \partial_t \mathbf{N}) \cdot \partial_t^m v + m \partial_z (\partial_t^{m-1} q \partial_t^2 \mathbf{N}) \cdot \partial_t^{m-1} v \, dydz \\ &\leq \Lambda_\infty \|v\|_{\mathbb{X}^m} \|\partial_z \partial_t^{m-1} q\|. \end{aligned} \quad (6.79)$$

Note carefully that we integrate by parts in z first rather than in t since there is no estimates of $\partial_t^m v$ on the boundary. This also indicates the difficulty in controlling the first term in the right hand side of (6.77) since one can no longer integrate by parts in t . Recall here that there was also one term out of control, that is, (6.63). Our crucial observation is that there is a cancelation between them since $q = gh - \sigma H + 2\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n}$ on $\{z = 0\}$. This motivates us to estimate together the first term in (6.77) and the second surface tension term in (6.60), by the kinematic boundary condition,

$$\begin{aligned} &\int_{z=0} \sigma \partial_t^m H \mathbf{N} \cdot V^m \, dy - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q \, dy \\ &= \int_{z=0} \sigma \partial_t^m H (\mathbf{N} \cdot V^m + m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v) \, dy \\ &\quad - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v (g \partial_t^m h + 2\varepsilon \partial_t^m (S^\varphi v \mathbf{n} \cdot \mathbf{n})) \, dy \\ &= \int_{z=0} \sigma \partial_t^m H \left(\partial_t^{m+1} h + v_y \cdot \nabla_y \partial_t^m h + \partial_z^\varphi v \cdot \mathbf{N} \partial_t^m h - \tilde{\mathcal{C}}^m(h) \right) \, dy \\ &\quad - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v (g \partial_t^m h + 2\varepsilon \partial_t^m (S^\varphi v \mathbf{n} \cdot \mathbf{n})) \, dy, \end{aligned} \quad (6.80)$$

where $\tilde{\mathcal{C}}^m(h)$ is the commutator $\tilde{\mathcal{C}}^\alpha(h)$ defined by (3.53) for the case $\alpha_0 = m$. Note that the last term in (6.80) can be estimated as follows, thanks to Lemma 3.4,

$$\begin{aligned} &- \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v (g \partial_t^m h + 2\varepsilon \partial_t^m (S^\varphi v \mathbf{n} \cdot \mathbf{n})) \, dy \\ &\leq \Lambda_\infty |\partial_t^{m-1} v|_{\frac{1}{2}} \left(|\partial_t^m h|_{-\frac{1}{2}} + \varepsilon |\partial_t^m (S^\varphi v \mathbf{n} \cdot \mathbf{n})|_{-\frac{1}{2}} \right) \\ &\leq \Lambda_\infty |\partial_t^{m-1} v|_{\frac{1}{2}} \left(|\partial_t^m h|_{-\frac{1}{2}} + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}} + \varepsilon |v|_{\mathbb{X}^{m, \frac{1}{2}}} \right). \end{aligned} \quad (6.81)$$

The integration by parts and (3.50) yield

$$\begin{aligned} & \int_{z=0} \sigma \partial_t^m H \left(\partial_z^\varphi v \cdot \mathbf{N} \partial_t^m h - \tilde{\mathcal{C}}^m(h) \right) dy \\ & \leq \sigma \left| \partial_t^m \left(\frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}} \right) \right|_0 \left| \nabla_y \left(\partial_z^\varphi v \cdot \mathbf{N} \partial_t^m h - \tilde{\mathcal{C}}^m(h) \right) \right|_0 \\ & \leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m,1}} (|\partial_t^m h|_1 + |h|_{\mathbb{X}^{m-1,2}} + |v|_{\mathbb{X}^{m-2,1}}). \end{aligned} \quad (6.82)$$

It follows from (5.13) and (5.14) that

$$\begin{aligned} & \int_{z=0} \sigma Z^\alpha H \left(\partial_t^{m+1} h + v_y \cdot \nabla_y \partial_t^m h \right) dy \\ & = \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y \partial_t^m h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y \partial_t^m h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h + \mathcal{C}(\mathcal{B}_\sigma^m) \right) \left(\partial_t^{m+1} h + v_y \cdot \nabla_y \partial_t^m h \right) dy, \end{aligned} \quad (6.83)$$

where

$$\mathcal{C}(\mathcal{B}_\sigma^m) = - \left[\partial_t^{m-1}, \frac{\nabla_y h}{\sqrt{1 + |\nabla_y h|^2}^3} \right] \cdot \nabla_y \partial_t h \nabla_y h + \left[\partial_t^m, \frac{1}{\sqrt{1 + |\nabla_y h|^2}}, \nabla_y h \right]. \quad (6.84)$$

Similarly as (6.35)–(6.37), one can deduce that

$$\begin{aligned} & \int_{z=0} \sigma \nabla_y \cdot \left(\frac{\nabla_y \partial_t^m h}{\sqrt{1 + |\nabla_y h|^2}} - \frac{\nabla_y h \cdot \nabla_y \partial_t^m h}{\sqrt{1 + |\nabla_y h|^2}^3} \nabla_y h \right) \left(\partial_t^{m+1} h + v_y \cdot \nabla_y \partial_t^m h \right) dy \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} \sigma \left(\frac{|\nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy + \Lambda_\infty \sigma |h|_{\mathbb{X}^{m,1}}^2. \end{aligned} \quad (6.85)$$

Integrate by parts in both t and y to have

$$\begin{aligned} & \int_{z=0} \sigma \nabla_y \cdot (\mathcal{C}(\mathcal{B}_\sigma^m)) \partial_t^{m+1} h dy \leq -\frac{d}{dt} \int_{z=0} \sigma \mathcal{C}(\mathcal{B}_\sigma^m) \cdot \nabla_y \partial_t^m h dy + \int_{z=0} \sigma \partial_t \mathcal{C}(\mathcal{B}_\sigma^m) \cdot \nabla_y \partial_t^m h dy \\ & \leq -\frac{d}{dt} \int_{z=0} \sigma \mathcal{C}(\mathcal{B}_\sigma^m) \cdot \nabla_y \partial_t^m h dy + \Lambda_\infty \sigma |h|_{\mathbb{X}^{m,1}}^2. \end{aligned} \quad (6.86)$$

One easily has

$$\int_{z=0} \sigma \nabla_y \cdot (\mathcal{C}(\mathcal{B}_\sigma^m)) v_y \cdot \nabla_y \partial_t^m h dy \leq \Lambda_\infty \sigma |h|_{\mathbb{X}^{m-1,2}} |h|_{\mathbb{X}^{m,1}}. \quad (6.87)$$

Hence, by the estimates (6.81), (6.82), (6.85)–(6.87), one may conclude from (6.80) that

$$\begin{aligned} & \int_{z=0} \sigma \partial_t^m H \mathbf{N} \cdot V^m dy - \int_{z=0} m \partial_t \mathbf{N} \cdot \partial_t^{m-1} v \partial_t^m q dy \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} \sigma \left(\frac{|\nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy - \frac{d}{dt} \int_{z=0} \sigma \mathcal{C}(\mathcal{B}_\sigma^m) \cdot \nabla_y \partial_t^m h dy \\ & \quad + \Lambda_\infty \left(\sigma |h|_{\mathbb{X}^{m,1}}^2 + \sigma |h|_{\mathbb{X}^{m,1}} |v|_{\mathbb{X}^{m-2,1}} + |\partial_t^{m-1} v|_{\frac{1}{2}} \left(|\partial_t^m h|_{-\frac{1}{2}} + \varepsilon |h|_{\mathbb{X}^{m,\frac{1}{2}}} + \varepsilon |v|_{\mathbb{X}^{m,\frac{1}{2}}} \right) \right). \end{aligned} \quad (6.88)$$

This in particular finishes the estimates of the second surface tension term in (6.60) and \mathcal{R}_Q^m , which can be stated as follows:

$$\begin{aligned} & \int_{z=0} \sigma \partial_t^m H \mathbf{N} \cdot V^m dy + \mathcal{R}_Q^m \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{z=0} \sigma \left(\frac{|\nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy - \frac{d}{dt} \mathcal{G}^m \\ & \quad + \Lambda_\infty \left(\sigma |h|_{\mathbb{X}^{m,1}}^2 + \sigma |h|_{\mathbb{X}^{m,1}} |v|_{\mathbb{X}^{m-2,1}} + |\partial_t^{m-1} v|_{\frac{1}{2}} \left(|\partial_t^m h|_{-\frac{1}{2}} + \varepsilon |h|_{\mathbb{X}^{m,\frac{1}{2}}} + \varepsilon |v|_{\mathbb{X}^{m,\frac{1}{2}}} \right) \right) \\ & \quad + \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1,\frac{1}{2}}} + \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^2} \right) \|\partial_t^{m-1} q\|_{H^1}, \end{aligned} \quad (6.89)$$

where

$$\begin{aligned} \mathcal{G}^m &= \int_{\Omega} (\mathcal{C}^m(d)_2 + \mathcal{C}^m(d)_3 + \mathcal{C}^m(d)_4 + \mathcal{C}^m(d)_5) \partial_t^{m-1} q \, dydz \\ &\quad - \int_{\Omega} m \partial_z (\partial_t^{m-1} q \partial_t \mathbf{N}) \cdot \partial_t^{m-1} v \, dydz + \int_{z=0} \sigma \mathcal{C}(\mathcal{B}_{\sigma}^m) \cdot \nabla_y \partial_t^m h \, dy. \end{aligned} \quad (6.90)$$

It remains to estimate the commutators \mathcal{R}_C^m and \mathcal{R}_S^m . It follows from (3.18), (3.15), (3.16), (6.2), (6.5) and (6.4) that

$$\begin{aligned} \mathcal{R}_C^m &\leq \Lambda_{\infty} (\|\partial_t^m \eta\| + \|\mathcal{C}^m(\mathcal{T})\| + \|\mathcal{C}^m(q)\|) \cdot V^m + \|\mathcal{C}^m(d)\| \|\partial_t^m \eta\| \\ &\leq \Lambda_{\infty} \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}} + \|v\|_{\mathbb{X}^{m-1}} + \|\nabla v\|_{\mathbb{X}^{m-1}} + \|\nabla q\|_{\mathbb{X}^{m-1}} \right) (\|V^m\| + \|\partial_t^m \eta\|) \\ &\leq \Lambda_{\infty} \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-1, 2}} + \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^{m-1}} + \varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right) \\ &\quad \times \left(\|v\|_{\mathbb{X}^m} + |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \right). \end{aligned} \quad (6.91)$$

Similarly as (6.49), it holds that

$$\begin{aligned} \mathcal{R}_S^m &\leq \Lambda_{\infty} \left(\varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1}} \right) \left(\|v\|_{\mathbb{X}^m} + |h|_{\mathbb{X}^{m, -\frac{1}{2}}} \right) \\ &\quad + \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^m\| + \left(|h|_{\mathbb{X}^{m-1, 1}} + |v|_{\mathbb{X}^{m-1, 1}} \right) \left(|h|_{\mathbb{X}^m} + |v|_{\mathbb{X}^m} \right). \end{aligned} \quad (6.92)$$

We can now finish the proof of the proposition. As a consequence of (6.61), (6.62), (6.64), (6.89), (6.91), (6.92) and Cauchy's inequality, one may deduce from (6.16) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{E}^m + \frac{d}{dt} \mathcal{G}^m + 2\varepsilon \int_{\Omega} |S^{\varphi} V^m|^2 \, d\mathcal{V}_t \\ \leq \Lambda_{\infty} \left(|h|_{\mathbb{X}^m}^2 + \sigma |h|_{\mathbb{X}^{m, 1}}^2 + \|\nabla v\|_{\mathbb{X}^{m-1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \varepsilon |v|_{\mathbb{X}^m}^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right. \\ \left. + \varepsilon |v|_{\mathbb{X}^{m, \frac{1}{2}}} \left(\|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^2} + |h|_{\mathbb{X}^{m, \frac{1}{2}}} \right) + \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) \|\nabla V^m\| \right), \end{aligned} \quad (6.93)$$

where

$$\mathcal{E}^m := \int_{\Omega} |V^m|^2 \, d\mathcal{V}_t + \int_{z=0} (g - \partial_z^{\varphi} q) |\partial_t^m h|^2 + \sigma \left(\frac{|\nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}} - \frac{|\nabla_y h \cdot \nabla_y \partial_t^m h|^2}{\sqrt{1 + |\nabla_y h|^2}^3} \right) dy. \quad (6.94)$$

Similarly as (6.52), by the trace estimates

$$|v|_{\mathbb{X}^{m, \frac{1}{2}}} \lesssim \|v\|_{\mathbb{X}^m} + \|\nabla v\|_{\mathbb{X}^m} \quad \text{and} \quad |v|_{\mathbb{X}^m} \lesssim \|\nabla v\|_{\mathbb{X}^m}^{\frac{1}{2}} \|v\|_{\mathbb{X}^m}^{\frac{1}{2}} + \|v\|_{\mathbb{X}^m},$$

using Cauchy's inequality and (6.7), one can then deduce that

$$\begin{aligned} \|\partial_t^m v(t)\|^2 + |\partial_t^m h(t)|_0^2 + \sigma |\partial_t^m h(t)|_1^2 + \varepsilon |h(t)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^m v\|_0^2 \\ \leq \Lambda_0 \left(\|\partial_t^m v(0)\|^2 + |\partial_t^m h(0)|_0^2 + \sigma |\partial_t^m h(0)|_1^2 + \varepsilon |h(0)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right) - \mathcal{G}^m \\ + \int_0^t \Lambda_{\infty} \left(|h|_{\mathbb{X}^m}^2 + \sigma |h|_{\mathbb{X}^{m, 1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1, 1}}^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right). \end{aligned} \quad (6.95)$$

Note that

$$-\mathcal{G}^m \leq \Lambda_{\infty} \left(\left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |v|_{\mathbb{X}^{m-1, 1}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} \right) \|\partial_t^{m-1} q\|_{H^1} + \sigma |h|_{\mathbb{X}^{m-1, 2}} |h|_{\mathbb{X}^{m, 1}} \right). \quad (6.96)$$

In contrast to the previous case, the difficulty here is that $\|\partial_t^{m-1} q\|_{H^1}$ and hence $-\mathcal{G}^m$ are not in $L^{\infty}([0, T])$ but only in $L^2([0, T])$. Our basic idea is to integrate in time twice. Indeed, we take the square and then integrate in time to have, by Cauchy's inequality,

$$\int_0^t \left(\|\partial_t^m v\|^2 + |\partial_t^m h|_0^2 + \sigma |\partial_t^m h|_1^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right)^2 + \int_0^t \left(\varepsilon \int_0^s \|\nabla \partial_t^m v\|_0^2 \right)^2 \quad (6.97)$$

$$\begin{aligned} &\leq t\Lambda_0 \left(\|\partial_t^m v(0)\|^2 + |\partial_t^m h(0)|_0^2 + \sigma |\partial_t^m h(0)|_1^2 + \varepsilon |h(0)|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right)^2 + \int_0^t |\mathcal{G}^m|^2 \\ &\quad + t \left(\int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^m}^2 + \sigma |h|_{\mathbb{X}^{m,1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-1,1}}^2 + \varepsilon |h|_{\mathbb{X}^{m, \frac{1}{2}}}^2 \right) \right)^2. \end{aligned}$$

It follows from (6.96) and (6.5) that

$$\begin{aligned} \int_0^t |\mathcal{G}^m|^2 &\leq \int_0^t \Lambda_\infty \left(\left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + |v|_{\mathbb{X}^{m-1,1}}^2 + \|\partial_z v\|_{\mathbb{X}^{m-2}}^2 \right) \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-1,2}} \right. \right. \\ &\quad \left. \left. + \|v\|_{\mathbb{X}^m} + \|\partial_z v\|_{\mathbb{X}^{m-1}} + \varepsilon |v|_{\mathbb{X}^{m-1, \frac{3}{2}}} + \varepsilon |h|_{\mathbb{X}^{m-1, \frac{3}{2}}} \right)^2 + \sigma^2 |h|_{\mathbb{X}^{m-1,2}}^2 |h|_{\mathbb{X}^{m,1}}^2 \right) \\ &\leq \int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + \sigma |h|_{\mathbb{X}^{m-1,2}}^2 + |v|_{\mathbb{X}^{m-1,1}}^2 + \|\partial_z v\|_{\mathbb{X}^{m-2}}^2 \right) \\ &\quad \times \left(|h|_{\mathbb{X}^{m, -\frac{1}{2}}}^2 + \sigma |h|_{\mathbb{X}^{m,1}}^2 + \|v\|_{\mathbb{X}^m}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \varepsilon^2 |v|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 + \varepsilon^2 |h|_{\mathbb{X}^{m-1, \frac{3}{2}}}^2 \right). \end{aligned} \quad (6.98)$$

We thus conclude (6.15) by plugging the estimate (6.98) into (6.97). \square

7. NORMAL DERIVATIVE ESTIMATES

In view of the conormal estimates in Propositions 6.2 and 6.3 in Section 6, the next main step is to estimate $\|\partial_z v\|_{\mathbb{X}^{m-1}}$.

Recall the definition of Λ_∞ from (6.3) and all the facts of the L^∞ controls elaborated in the beginning of Section 6. Note that Λ_∞ involves only $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$. For the case without surface tension [33] that involves only the spatial derivatives, this is sufficient for deriving the normal derivative estimates since in such situation applying the product or commutator estimates to control the commutators resulting from the viscosity term needs only the control of $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$. However, in the current case that involves the time derivatives, following the arguments of [33] would require the control of $\sqrt{\varepsilon} \|\partial_{zz} v\|_{\mathbb{Y}^k}$ for some $k \geq 1$. Recall from Proposition 9.8 in [33] that deriving the bound of $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$ requires a crucial use of the heat kernel and the first order compatibility condition $S_{\mathbf{n}}|_{t=0} = 0$ on the boundary. Hence, to control $\sqrt{\varepsilon} \|\partial_{zz} v\|_{\mathbb{Y}^k}$, it seems to involve much more delicate use of the various properties of the heat kernel for the time differentiated problems; furthermore, it should require more compatibility conditions of initial data.

Our key observation here is that since in the vicinity of the boundary the solution behaves as $v(t, x) \sim v^0(t, x) + \sqrt{\varepsilon} U(t, y, z/\sqrt{\varepsilon})$, it indicates that there may be better control of $\varepsilon \partial_{zz} v$ (and even $\varepsilon \partial_{zzz} v$!) in Sobolev conormal spaces. This is indeed the case as shown in the following lemma.

Lemma 7.1. *It holds that*

$$\varepsilon \|\partial_{zz} v\|_{\mathbb{Y}^{\frac{m}{2}+1}} + \varepsilon \|\partial_{zzz} v\|_{\mathbb{Y}^{\frac{m}{2}}} \leq \Lambda_\infty \quad (7.1)$$

and for $k \leq m-1$:

$$\varepsilon \|\partial_{zz} v\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\varepsilon \left(\|\partial_z v\|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^{k, \frac{3}{2}}} \right) + \|\partial_z v\|_{\mathbb{X}^k} + \|v\|_{\mathbb{X}^{k+1}} + |h|_{\mathbb{X}^{k+1, -\frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^k} \right) \quad (7.2)$$

and

$$\begin{aligned} \varepsilon \|\partial_{zzz} v\|_{\mathbb{X}^{k-1}} &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{k-1,2}} + |h|_{\mathbb{X}^{k-1, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{k, \frac{1}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^k} + \|v\|_{\mathbb{X}^{k,1}} + \|\nabla q\|_{\mathbb{X}^{k-1,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{k-1}} \right). \end{aligned} \quad (7.3)$$

Proof. It follows from the first equation in (1.9), (3.21) and (4.6) that

$$\varepsilon \partial_{zz} v = \frac{1}{E_{33}} \left(-\varepsilon \sum_{j < 3} \partial_z (E_{3,j} \partial_j v) - \varepsilon \sum_{i < 3, j} \partial_i (E_{ij} \partial_j v) + \partial_z \varphi (\partial_t v + v_y \cdot \nabla_y v) + v_z \partial_z v + \partial_z \varphi \nabla^\varphi q \right). \quad (7.4)$$

This implies that, since $v_z = 0$ on the boundary,

$$\begin{aligned} \varepsilon \|\partial_{zz}v\|_{\mathbb{Y}^{\frac{m}{2}+1}} &\leq \Lambda \left(\frac{1}{c_0}, \varepsilon \left(\|\nabla \nabla_y v\|_{\mathbb{Y}^{\frac{m}{2}+1}} + |h|_{\mathbb{Y}^{\frac{m}{2}+3}} \right) + |h|_{\mathbb{Y}^{\frac{m}{2}+2}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|\nabla q\|_{\mathbb{Y}^{\frac{m}{2}+1}} \right) \leq \Lambda_\infty. \end{aligned} \quad (7.5)$$

Applying ∂_z to (7.4), using the estimate (7.5), one deduces

$$\begin{aligned} \varepsilon \|\partial_{zzz}v\|_{\mathbb{Y}^{\frac{m}{2}}} &\leq \Lambda \left(\frac{1}{c_0}, \varepsilon \left(|h|_{\mathbb{Y}^{\frac{m}{2}+3}} + \|\nabla^2 \nabla_y v\|_{\mathbb{Y}^{\frac{m}{2}}} \right) + |h|_{\mathbb{Y}^{\frac{m}{2}+2}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}+1}} + \|v\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|\partial_z \nabla q\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\nabla q\|_{\mathbb{Y}^{\frac{m}{2}}} \right) \leq \Lambda_\infty, \end{aligned} \quad (7.6)$$

where one has used the fact that, since $v_z = 0$ on the boundary,

$$\begin{aligned} \|\partial_z(v_z \partial_z v)\|_{\mathbb{Y}^{\frac{m}{2}}} &\leq \|\partial_z v_z \partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} + \|v_z \partial_{zz}v\|_{\mathbb{Y}^{\frac{m}{2}}} = \|\partial_z v_z \partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} + \left\| \frac{1}{z(z+b)} v_z Z_3 \partial_z v \right\|_{\mathbb{Y}^{\frac{m}{2}}} \\ &\lesssim \|\partial_z v_z\|_{\mathbb{Y}^{\frac{m}{2}}} \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}}} + \|\partial_z v_z\|_{\mathbb{Y}^{\frac{m}{2}}} \|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}+1}}. \end{aligned} \quad (7.7)$$

Combining (7.5) and (7.7) proves the estimates (7.1).

The estimates (7.2)–(7.3) follow by a similar argument as that for (7.1). Indeed, it follows from (7.4) and (A.1) that for $k \geq m-1$,

$$\varepsilon \|\partial_{zz}v\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\varepsilon \left(\|\nabla \nabla_y v\|_{\mathbb{X}^k} + |h|_{\mathbb{X}^k, \frac{3}{2}} \right) + |h|_{\mathbb{X}^k, \frac{1}{2}} + \|\partial_z v\|_{\mathbb{X}^k} + \|v\|_{\mathbb{X}^{k+1}} + |h|_{\mathbb{X}^{k+1}, -\frac{1}{2}} + \|\nabla q\|_{\mathbb{X}^k} \right).$$

This yields (7.2). Now applying Z_i , $i = 1, 2, 3$ to (7.4) and then using (A.1) again lead to

$$\begin{aligned} \varepsilon \|\partial_{zz}v\|_{\mathbb{X}^{k-1,1}} &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla \nabla_y v\|_{\mathbb{X}^{k-1,1}} + |h|_{\mathbb{X}^{k-1, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{k-1, \frac{3}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^{k-1,1}} + \|v\|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^k, \frac{1}{2}} + \|\nabla q\|_{\mathbb{X}^{k-1,1}} \right). \end{aligned} \quad (7.8)$$

Then applying ∂_z to (7.4) and using the estimate (7.1) and the similar observation as in (7.7) give

$$\begin{aligned} \varepsilon \|\partial_{zzz}v\|_{\mathbb{X}^{k-1}} &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla^2 \nabla_y v\|_{\mathbb{X}^{k-1}} + |h|_{\mathbb{X}^{k-1, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{k-1, \frac{3}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^k} + \|v\|_{\mathbb{X}^{k-1,1}} + |h|_{\mathbb{X}^k, \frac{1}{2}} + \|\partial_z \nabla q\|_{\mathbb{X}^{k-1}} + \|\nabla q\|_{\mathbb{X}^{k-1}} \right). \end{aligned} \quad (7.9)$$

Combining (7.9) and (7.8) thus proves the estimate (7.3). \square

Lemma 7.1 then allows one to derive the normal derivative estimates.

7.1. Estimate of $\|\partial_z v\|_{\mathbb{X}^{m-2}}$. As in Section 8 of [33], one defines

$$S_{\mathbf{n}} = \Pi(S^\varphi \mathbf{v}\mathbf{n} - \kappa \chi v). \quad (7.10)$$

The main advantages of this quantity are that

$$S_{\mathbf{n}} = 0 \text{ on } \{z = 0, -b\} \quad (7.11)$$

and that the following estimates hold:

Lemma 7.2. *For every $k \geq 0$:*

$$\|\partial_z v\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\|S_{\mathbf{n}}\|_{\mathbb{X}^k} + |h|_{\mathbb{X}^k, \frac{1}{2}} + \|v\|_{\mathbb{X}^{k,1}} \right) \quad (7.12)$$

and

$$\|\partial_{zz}v\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\|\partial_z S_{\mathbf{n}}\|_{\mathbb{X}^k} + |h|_{\mathbb{X}^k, \frac{3}{2}} + \|v\|_{\mathbb{X}^{k,2}} \right). \quad (7.13)$$

Proof. We start with the estimate (7.12). The normal component of $\partial_z v$ is given by, due to the divergence free condition,

$$\partial_z v \cdot \mathbf{n} = -\frac{\partial_z \varphi}{|\mathbf{N}|} (\partial_1 v_1 + \partial_2 v_2). \quad (7.14)$$

Then it follows from (A.1) and Lemma B.2 that

$$\|\partial_z v \cdot \mathbf{n}\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^{k, \frac{1}{2}}} \right). \quad (7.15)$$

It thus suffices to estimate the tangential components of $\partial_z v$. Recall from the derivation of (3.37) that

$$\Pi \partial_z v = \frac{\partial_z \varphi}{|\mathbf{N}|} (\partial_1 \varphi \Pi \partial_1 v + \partial_2 \varphi \Pi \partial_2 v + 2S_{\mathbf{n}} - g^{ij} \partial_j v \cdot \mathbf{n} \Pi \partial_{y^i} - \kappa \chi \Pi v), \quad (7.16)$$

which follows from the derivation of (3.37). Hence, by (A.1) and Lemma B.2, one has

$$\|\Pi \partial_z v\|_{\mathbb{X}^k} \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{k,1}} + |h|_{\mathbb{X}^{k, \frac{1}{2}}} + \|S_{\mathbf{n}}\|_{\mathbb{X}^k} + \|\partial_z v \cdot \mathbf{n}\|_{\mathbb{X}^k} \right). \quad (7.17)$$

Thus the estimate (7.12) follows from combining (7.15) and (7.17).

The estimate (7.13) follows from applying ∂_z to (7.14) and (7.16), employing a similar argument as for (7.12) and using (7.12). \square

In light of Lemma 7.2, we then turn to estimate $S_{\mathbf{n}}$ instead of $\partial_z v$. It follows from the first equation in (1.9) that

$$\partial_t^\varphi \nabla^\varphi v + (v \cdot \nabla^\varphi) \nabla^\varphi v + (\nabla^\varphi v)^2 + (\nabla^\varphi)^2 q - \varepsilon \Delta^\varphi \nabla^\varphi v = 0. \quad (7.18)$$

Taking the symmetric part of (7.18) yields

$$\partial_t^\varphi S^\varphi v + (v \cdot \nabla^\varphi) S^\varphi v + \frac{1}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2) + (\nabla^\varphi)^2 q - \varepsilon \Delta^\varphi (S^\varphi v) = 0. \quad (7.19)$$

Consequently,

$$\partial_t^\varphi S_{\mathbf{n}} + (v \cdot \nabla^\varphi) S_{\mathbf{n}} - \varepsilon \Delta^\varphi (S_{\mathbf{n}}) = F_S \quad (7.20)$$

where

$$F_S = F_S^1 + F_S^2 + F_S^3 + F_S^4 \quad (7.21)$$

with

$$F_S^1 = -\frac{1}{2} \Pi ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2) \mathbf{n} + (\partial_t \Pi + v \cdot \nabla^\varphi \Pi) S^\varphi v \mathbf{n} + \Pi S^\varphi v (\partial_t \mathbf{n} + v \cdot \nabla^\varphi \mathbf{n}) + \kappa (\partial_t^\varphi + v \cdot \nabla^\varphi) (\chi \Pi v), \quad (7.22)$$

$$F_S^2 = -\Pi \left((\nabla^\varphi)^2 q \right) \mathbf{n}, \quad (7.23)$$

$$F_S^3 = -\varepsilon (\Delta^\varphi \Pi) S^\varphi v \mathbf{n} - \varepsilon \Pi S^\varphi v \Delta^\varphi \mathbf{n}, \quad (7.24)$$

$$F_S^4 = -2\varepsilon \partial_i^\varphi \Pi \partial_i^\varphi (S^\varphi v \mathbf{n}) - 2\varepsilon \Pi (\partial_i^\varphi (S^\varphi v) \partial_i^\varphi \mathbf{n}) - \varepsilon \kappa \Delta^\varphi (\chi \Pi v). \quad (7.25)$$

Note that the pressure estimates in (6.5) indicate that at most, one can estimate only $\|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}$ and hence $\|\partial_z v\|_{\mathbb{X}^{m-2}}$ at this stage. However, we shall prove a control of $\|\partial_z v\|_{\mathbb{X}^{m-1}}$ based on the vorticity equation in the next subsection.

Proposition 7.3. *Any smooth solution of (1.9) satisfies the estimate*

$$\begin{aligned} & \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}^2 + \varepsilon \int_0^t \|\nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}^2 \\ & \leq \Lambda_0 \|S_{\mathbf{n}}(0)\|_{\mathbb{X}^{m-2}}^2 + \int_0^t \Lambda_\infty \varepsilon^2 \left(\|\nabla v\|_{\mathbb{X}^{m-2,2}}^2 + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}}^2 \right) \\ & \quad + \int_0^t \Lambda_\infty |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + \sigma^2 |h|_{\mathbb{X}^{m-2,3}}^2 + \|v\|_{\mathbb{X}^{m-1,1}}^2 + \|\partial_z v\|_{\mathbb{X}^{m-2,1}}^2 + \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}^2. \end{aligned} \quad (7.26)$$

Proof. We start with the estimates of F_S . It follows from (A.1) and Lemma B.2 that

$$\|F_S^1\|_{\mathbb{X}^{m-2}} \leq \Lambda_\infty \left(\|\nabla v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|v\|_{\mathbb{X}^{m-2}} \right), \quad (7.27)$$

$$\|F_S^2\|_{\mathbb{X}^{m-2}} \leq \Lambda_\infty \|\nabla q\|_{\mathbb{X}^{m-2,1}}, \quad (7.28)$$

$$\|F_S^3\|_{\mathbb{X}^{m-2}} \leq \Lambda_\infty \varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right), \quad (7.29)$$

$$\begin{aligned} \|F_S^4\|_{\mathbb{X}^{m-2}} &\leq \Lambda_\infty \varepsilon \left(|h|_{\mathbb{Y}^{\frac{m-2}{2}+2}} \|\nabla^2 v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, \frac{3}{2}}} \|\nabla^2 v\|_{\mathbb{Y}^{\frac{m-2}{2}}} \right) \\ &\leq \Lambda_\infty \left(\varepsilon \|\nabla v\|_{\mathbb{X}^{m-2,1}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, -\frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, \frac{3}{2}}} \right). \end{aligned} \quad (7.30)$$

Here in the second inequality of (7.30), one has used (7.1) and (7.2) with $k = m - 2$. Hence,

$$\|F_S\|_{\mathbb{X}^{m-2}} \leq \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-2,1}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right) + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-2,1}} \right). \quad (7.31)$$

It follows from applying $\frac{1}{\partial_z \varphi} Z^\alpha (\partial_z \varphi \cdot)$ for $|\alpha| \leq m - 2$ to (7.20), (3.21) and (4.6) that

$$\partial_t^\varphi Z^\alpha S_{\mathbf{n}} + (v \cdot \nabla^\varphi) Z^\alpha S_{\mathbf{n}} - \varepsilon \Delta^\varphi Z^\alpha S_{\mathbf{n}} = Z^\alpha F_S + \mathcal{C}_S, \quad (7.32)$$

where the commutator is given by

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 \quad (7.33)$$

with

$$\begin{aligned} \mathcal{C}_S^1 &= \frac{1}{\partial_z \varphi} [Z^\alpha, \partial_z \varphi] (\partial_t + v_y \cdot \nabla_y) S_{\mathbf{n}} + [Z^\alpha, v_y] \cdot \nabla_y S_{\mathbf{n}} + [Z^\alpha, v_z] \partial_z S_{\mathbf{n}} \\ &:= \mathcal{C}_{S1}^1 + \mathcal{C}_{S2}^1 + \mathcal{C}_{S3}^1 \end{aligned} \quad (7.34)$$

and

$$\begin{aligned} \mathcal{C}_S^2 &= -\varepsilon \frac{1}{\partial_z \varphi} [Z^\alpha, \nabla] \cdot (E \nabla S_{\mathbf{n}}) - \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot ([Z^\alpha, E] \nabla S_{\mathbf{n}}) - \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot (E [Z^\alpha, \nabla] S_{\mathbf{n}}) \\ &:= \mathcal{C}_{S1}^2 + \mathcal{C}_{S2}^2 + \mathcal{C}_{S3}^2. \end{aligned} \quad (7.35)$$

Since $Z^\alpha S_{\mathbf{n}} = 0$ on the boundary, one can obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |Z^\alpha S_{\mathbf{n}}|^2 d\mathcal{V}_t + \varepsilon \int_\Omega |\nabla^\varphi Z^\alpha S_{\mathbf{n}}|^2 d\mathcal{V}_t = \int_\Omega (Z^\alpha F_S + \mathcal{C}_S) \cdot Z^\alpha S_{\mathbf{n}} d\mathcal{V}_t. \quad (7.36)$$

The right hand side of (7.36) can be estimated as follows. (7.31) implies immediately that

$$\begin{aligned} \int_\Omega Z^\alpha F_S \cdot Z^\alpha S_{\mathbf{n}} d\mathcal{V}_t &\leq \Lambda_\infty \left(\varepsilon \left(\|\partial_z v\|_{\mathbb{X}^{m-2,1}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right) + \|\partial_z v\|_{\mathbb{X}^{m-2}} \right. \\ &\quad \left. + \|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-2,1}} \right) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}. \end{aligned} \quad (7.37)$$

Next, we estimate the part involving \mathcal{C}_S^1 . (A.2) yields

$$\|\mathcal{C}_{S1}^1\| + \|\mathcal{C}_{S1}^2\| \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-2}} + \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \right). \quad (7.38)$$

Additional care is needed to estimate $\|\mathcal{C}_{S3}^1\|$, since $\partial_z S_{\mathbf{n}}$ can not be controlled. By expanding the commutator and using (3.11), one sees clearly that it suffices to estimate terms of the form

$$\left\| Z^\beta v_z \partial_z Z^\gamma S_{\mathbf{n}} \right\|$$

with $|\beta| + |\gamma| \leq m - 2$, $|\gamma| \leq m - 3$. Since

$$Z^\beta v_z \partial_z Z^\gamma S_{\mathbf{n}} = \frac{1}{z(z+b)} Z^\beta v_z Z_3 Z^\gamma S_{\mathbf{n}},$$

which can be further rewritten as a sum of terms of the form

$$c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1}{z(z+b)} v_z \right) Z_3 Z^\gamma S_{\mathbf{n}}, \quad (7.39)$$

where $c_{\tilde{\beta}}$ are harmless bounded functions and $|\tilde{\beta}| \leq |\beta|$. Indeed, this comes from the fact that $Z_3 \left(\frac{1}{z(z+b)} \right) = -\frac{2z+b}{z(z+b)}$. If $\tilde{\beta} = 0$, since (3.22) implies that $v_z = 0$ on the boundary, then

$$\left\| c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1}{z(z+b)} v_z \right) Z_3 Z^\gamma S_{\mathbf{n}} \right\| \lesssim \|\partial_z v_z\|_{L^\infty} \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}.$$

If $\tilde{\beta} \neq 0$, one can use (A.1) to obtain that

$$\begin{aligned} & \left\| c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1}{z(z+b)} v_z \right) Z_3 Z^\gamma S_{\mathbf{n}} \right\| \\ & \lesssim \left\| Z \left(\frac{1}{z(z+b)} v_z \right) \right\|_{\mathbb{Y}^{\frac{m-3}{2}}} \|Z_3 S_{\mathbf{n}}\|_{\mathbb{X}^{m-3}} + \left\| Z \left(\frac{1}{z(z+b)} v_z \right) \right\|_{\mathbb{X}^{m-3}} \|Z_3 S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m-3}{2}}}. \end{aligned}$$

Observe that by again that $v_z = 0$ on the boundary,

$$\left\| Z \left(\frac{1}{z(z+b)} v_z \right) \right\|_{\mathbb{Y}^{\frac{m-3}{2}}} \lesssim \|\partial_z v_z\|_{\mathbb{Y}^{\frac{m}{2}}}.$$

On the other hand, since

$$\left\| Z \left(\frac{1}{z(z+b)} v_z \right) \right\|_{\mathbb{X}^{m-3}} \lesssim \left\| \frac{1}{z(z+b)} Z v_z \right\|_{\mathbb{X}^{m-3}} + \left\| \frac{1}{z(z+b)} v_z \right\|_{\mathbb{X}^{m-3}},$$

it suffices to estimate

$$\left\| \frac{1}{z(z+b)} Z^\beta Z v_z \right\|, \quad \left\| \frac{1}{z(z+b)} Z^\beta v_z \right\|, \quad |\beta| \leq m-3.$$

Indeed, since $v_z = 0$ on the boundary, it follows from the Hardy inequality that

$$\left\| \frac{1}{z(z+b)} Z^\beta Z v_z \right\| \lesssim \left\| \partial_z Z^\beta Z v_z \right\| \quad \text{and} \quad \left\| \frac{1}{z(z+b)} Z^\beta v_z \right\| \lesssim \left\| \partial_z Z^\beta v_z \right\|.$$

We have thus proven that

$$\|C_{S3}^1\| \leq \|\partial_z v_z\|_{\mathbb{Y}^{\frac{m}{2}}} \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + \|S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m}{2}}} \|\partial_z v_z\|_{\mathbb{X}^{m-2}}.$$

By Lemma B.2, it holds that

$$\|\partial_z v_z\|_{\mathbb{Y}^{\frac{m}{2}}} \leq \Lambda_\infty \quad \text{and} \quad \|\partial_z v_z\|_{\mathbb{X}^{m-2}} \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-2}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right).$$

Hence,

$$\|C_{S3}^1\| \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-2}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} \right). \quad (7.40)$$

This together with (7.38) yields

$$\int_{\Omega} C_S^1 \cdot Z^\alpha S_{\mathbf{n}} \, d\mathcal{V}_t \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-2}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} \right) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}. \quad (7.41)$$

Next, we shall estimate the term involving C_S^2 . As mentioned in the beginning of this section, we need to employ a different argument from Proposition 8.3 in [33]. Due to (3.11) to handle the term involving C_{S1}^2 , it suffices to estimate

$$\varepsilon \int_{\Omega} Z^\beta \partial_z (E \nabla S_{\mathbf{n}}) \cdot Z^\alpha S_{\mathbf{n}} \, dydz$$

with $|\beta| \leq m-3$, which can be reduced to the estimate of

$$\varepsilon \int_{\Omega} \left(Z^{\beta'} \partial_z E Z^{\beta''} \nabla S_{\mathbf{n}} + Z^{\beta'} E Z^{\beta''} \partial_z \nabla S_{\mathbf{n}} \right) \cdot Z^\alpha S_{\mathbf{n}} \, dydz$$

with $\beta' + \beta'' = \beta$. It follows from (A.1), (7.1) and (7.2) with $k = m-3$ that

$$\begin{aligned} \varepsilon \left\| Z^{\beta'} \partial_z E Z^{\beta''} \nabla S_{\mathbf{n}} \right\| & \lesssim \|\partial_z E\|_{\mathbb{Y}^{\frac{m-3}{2}}} \varepsilon \|\nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-3}} + \|\partial_z E\|_{\mathbb{X}^{m-3}} \varepsilon \|\nabla S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m-3}{2}}} \\ & \leq \Lambda_\infty \left(\varepsilon \|\partial_z v\|_{\mathbb{X}^{m-3,1}} + \|\partial_z v\|_{\mathbb{X}^{m-3}} + \|v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, -\frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-3}} + |h|_{\mathbb{X}^{m-3, \frac{3}{2}}} \right), \end{aligned} \quad (7.42)$$

and (7.3) with $k = m - 2$ implies that

$$\begin{aligned} \varepsilon \left\| Z^{\beta'} E Z^{\beta''} \partial_z \nabla S_{\mathbf{n}} \right\| &\lesssim \|E\|_{\mathbb{Y}^{\frac{m-3}{2}}} \varepsilon \|\partial_z \nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-3}} + \|E\|_{\mathbb{X}^{m-3}} \varepsilon \|\partial_z \nabla S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m-3}{2}}} \\ &\leq \Lambda_{\infty} \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-2,1}} + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right). \end{aligned} \quad (7.43)$$

Thus,

$$\begin{aligned} \int_{\Omega} \mathcal{C}_{S_1}^2 \cdot Z^{\alpha} S_{\mathbf{n}} d\mathcal{V}_t &\leq \Lambda_{\infty} \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-2,1}} + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right). \end{aligned} \quad (7.44)$$

By expanding the second commutator $\mathcal{C}_{S_2}^2$, one sees that it suffices to estimate terms of the form

$$\varepsilon \int_{\Omega} \nabla \cdot \left(Z^{\beta} E Z^{\gamma} \nabla S_{\mathbf{n}} \right) \cdot Z^{\alpha} S_{\mathbf{n}} dydz$$

with $\beta + \gamma = \alpha$, $\beta \neq 0$. If $\gamma = 0$ and hence $\beta = \alpha$, since $Z^{\alpha} S_{\mathbf{n}} = 0$ on the boundary, one can then integrate by parts to get

$$\begin{aligned} \left| \varepsilon \int_{\Omega} \nabla \cdot \left(Z^{\alpha} E \nabla S_{\mathbf{n}} \right) \cdot Z^{\alpha} S_{\mathbf{n}} dydz \right| &= \left| \varepsilon \int_{\Omega} Z^{\alpha} E \nabla S_{\mathbf{n}} \cdot \nabla Z^{\alpha} S_{\mathbf{n}} dydz \right| \\ &\leq \varepsilon \|Z^{\alpha} E\| \|\nabla S_{\mathbf{n}}\|_{L^{\infty}} \|\nabla Z^{\alpha} S_{\mathbf{n}}\| \leq \Lambda_{\infty} |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \varepsilon^{\frac{1}{2}} \|\nabla Z^{\alpha} S_{\mathbf{n}}\|, \end{aligned} \quad (7.45)$$

where in the last inequality one has used the fact that Λ_{∞} involves $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^{\infty}}$. If $\gamma \neq 0$ and hence $1 \leq |\beta| \leq m - 3$, then one can expand $\nabla \cdot$, by (A.1) and Lemma 7.1, to estimate

$$\varepsilon \int_{\Omega} \left(Z^{\beta} \nabla E Z^{\gamma} \nabla S_{\mathbf{n}} + Z^{\beta} E Z^{\gamma} \nabla^2 S_{\mathbf{n}} \right) \cdot Z^{\alpha} S_{\mathbf{n}} dydz.$$

It follows from (A.1), (7.1) and (7.2) with $k = m - 3$ that

$$\begin{aligned} \varepsilon \left\| Z^{\beta} \nabla E Z^{\gamma} \nabla S_{\mathbf{n}} \right\| &\lesssim \|Z \nabla E\|_{\mathbb{Y}^{\frac{m-4}{2}}} \varepsilon \|Z \nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-4}} + \|Z \nabla E\|_{\mathbb{X}^{m-4}} \varepsilon \|Z \nabla S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m-4}{2}}} \\ &\leq \Lambda_{\infty} \left(\varepsilon \|\partial_z v\|_{\mathbb{X}^{m-3,1}} + \|\partial_z v\|_{\mathbb{X}^{m-3}} + \|v\|_{\mathbb{X}^{m-2}} + |h|_{\mathbb{X}^{m-2, -\frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-3}} + |h|_{\mathbb{X}^{m-3, \frac{3}{2}}} \right), \end{aligned} \quad (7.46)$$

and (7.3) with $k = m - 2$ that

$$\begin{aligned} \varepsilon \left\| Z^{\beta} E Z^{\gamma} \nabla^2 S_{\mathbf{n}} \right\| &\lesssim \|Z E\|_{\mathbb{Y}^{\frac{m-4}{2}}} \varepsilon \|Z \nabla^2 S_{\mathbf{n}}\|_{\mathbb{X}^{m-4}} + \|Z E\|_{\mathbb{X}^{m-4}} \varepsilon \|Z \nabla^2 S_{\mathbf{n}}\|_{\mathbb{Y}^{\frac{m-4}{2}}} \\ &\leq \Lambda_{\infty} \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \right. \\ &\quad \left. + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-2,1}} + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right). \end{aligned} \quad (7.47)$$

Hence,

$$\begin{aligned} \int_{\Omega} \mathcal{C}_{S_2}^2 \cdot Z^{\alpha} S_{\mathbf{n}} d\mathcal{V}_t &\leq \Lambda_{\infty} \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-2,1}} \right. \\ &\quad \left. + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + \Lambda_{\infty} |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \varepsilon^{\frac{1}{2}} \|\nabla Z^{\alpha} S_{\mathbf{n}}\|. \end{aligned} \quad (7.48)$$

To handle the term involving $\mathcal{C}_{S_3}^2$, due to (3.11), one needs to estimate

$$\varepsilon \int_{\Omega} \nabla \cdot \left(E Z^{\beta} \partial_z S_{\mathbf{n}} \right) \cdot Z^{\alpha} S_{\mathbf{n}} dydz$$

with $|\beta| \leq m - 3$, which can be bounded easily by using (7.2) with $k = m - 3$ and (7.3) with $k = m - 2$ so that

$$\int_{\Omega} \mathcal{C}_{S_3}^2 \cdot Z^{\alpha} S_{\mathbf{n}} d\mathcal{V}_t \leq \Lambda_{\infty} \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} \right) \quad (7.49)$$

$$+ \|v\|_{\mathbb{X}^{m-2,1}} + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}}) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}.$$

In view of (7.44), (7.48), (7.49), one has actually proven that

$$\begin{aligned} \int_{\Omega} \mathcal{C}_S^2 \cdot Z^\alpha S_{\mathbf{n}} d\mathcal{V}_t &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-3,2}} + |h|_{\mathbb{X}^{m-3, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-2,1}} \right. \\ &\quad \left. + \|\nabla q\|_{\mathbb{X}^{m-3,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + \Lambda_\infty |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \varepsilon^{\frac{1}{2}} \|\nabla Z^\alpha S_{\mathbf{n}}\|. \end{aligned} \quad (7.50)$$

Consequently, by (7.37), (7.41) and (7.50), one deduces from (7.36) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |Z^\alpha S_{\mathbf{n}}|^2 d\mathcal{V}_t + \varepsilon \int_{\Omega} |\nabla^\varphi Z^\alpha S_{\mathbf{n}}|^2 d\mathcal{V}_t &\quad (7.51) \\ &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-2,1}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right) + \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + \|\partial_z v\|_{\mathbb{X}^{m-2}} + \|v\|_{\mathbb{X}^{m-1}} \right. \\ &\quad \left. + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\nabla q\|_{\mathbb{X}^{m-2,1}} + \|\partial_z \nabla q\|_{\mathbb{X}^{m-3}} \right) \|S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}} + \Lambda_\infty |h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \varepsilon^{\frac{1}{2}} \|\nabla Z^\alpha S_{\mathbf{n}}\|. \end{aligned}$$

To conclude, one can use Lemma C.1 to replace $\|\nabla^\varphi S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}$ by $\|\nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}$ in the left hand side and then use Cauchy's inequality to absorb the last term. On the other hand, one can follow the derivation of (6.5) to get that

$$\begin{aligned} &\|q\|_{\mathbb{X}^{m-2,1}} + \|\nabla q\|_{\mathbb{X}^{m-2,1}} + \|\partial_{zz} q\|_{\mathbb{X}^{m-2}} \\ &\leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \sigma |h|_{\mathbb{X}^{m-2,3}} + \|v\|_{\mathbb{X}^{m-1,1}} + \|\nabla v\|_{\mathbb{X}^{m-2,1}} + \varepsilon \|\nabla v\|_{\mathbb{X}^{m-2,2}} + \varepsilon |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right), \end{aligned}$$

which reduces the order of time derivatives to the spatial derivatives. Finally, we integrate the resulting inequality in time to obtain (7.26). \square

7.2. Estimate of $\|\partial_z v\|_{\mathbb{X}^{m-1}}$. Note that Proposition 7.3 only provides the control of $\|\partial_z v\|_{\mathbb{X}^{m-2}}$, and this is due to the appearance of $(\nabla^\varphi)^2 q$ in the source term in the equation of $S_{\mathbf{n}}$. To get around this, a natural way is to use the vorticity instead of $S_{\mathbf{n}}$.

Set $\omega = \nabla^\varphi \times v$. Then

$$\omega \times \mathbf{n} = \frac{1}{2} (\nabla^\varphi v \mathbf{n} - (\nabla^\varphi v)^t \mathbf{n}) = S^\varphi v \mathbf{n} - (\nabla^\varphi v)^t \mathbf{n}, \quad (7.52)$$

and hence by (3.35),

$$\omega \times \mathbf{n} = \frac{1}{2} \partial_{\mathbf{n}} u - g^{ij} (\partial_j v \cdot \mathbf{n}) \partial_{y^i}.$$

Consequently, as in Lemma 7.2 one can get that

$$\|\partial_z v\|_{\mathbb{X}^{m-1}} \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\omega\|_{\mathbb{X}^{m-1}} \right). \quad (7.53)$$

To estimate $\|\omega\|_{\mathbb{X}^{m-1}}$, one notes that the first equation in (1.9) implies

$$\partial_t^\varphi \omega + v \cdot \nabla^\varphi \omega - \omega \cdot \nabla^\varphi v = \varepsilon \Delta^\varphi \omega. \quad (7.54)$$

As in the previous subsection, applying $\frac{1}{\partial_z \varphi} Z^\alpha (\partial_z \varphi \cdot)$ for $|\alpha| \leq m-1$ to (7.54) yields

$$\partial_t^\varphi Z^\alpha \omega + (v \cdot \nabla^\varphi) Z^\alpha \omega - \varepsilon \Delta Z^\alpha \omega = F. \quad (7.55)$$

Here F is given by

$$F = Z^\alpha (\omega \cdot \nabla^\varphi v) + \mathcal{C}_S, \quad (7.56)$$

where \mathcal{C}_S is given, as in (7.33), by

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 \quad (7.57)$$

with

$$\begin{aligned} \mathcal{C}_S^1 &= \frac{1}{\partial_z \varphi} [Z^\alpha, \partial_z \varphi] (\partial_t + v_y \cdot \nabla_y) \omega + [Z^\alpha, v_y] \cdot \nabla_y \omega + [Z^\alpha, v_z] \partial_z \omega \\ &:= \mathcal{C}_{S1}^1 + \mathcal{C}_{S2}^1 + \mathcal{C}_{S3}^1 \end{aligned} \quad (7.58)$$

and

$$\begin{aligned} \mathcal{C}_S^2 &= -\varepsilon \frac{1}{\partial_z \varphi} [Z^\alpha, \nabla] \cdot (E \nabla \omega) - \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot ([Z^\alpha, E] \nabla \omega) - \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot (E [Z^\alpha, \nabla] \omega) \\ &:= \mathcal{C}_{S1}^2 + \mathcal{C}_{S2}^2 + \mathcal{C}_{S3}^2. \end{aligned} \quad (7.59)$$

The main difficulty lies in that the vorticity does not vanish on the boundary. Thus, set

$$Z^\alpha \omega = \omega_{nh}^\alpha + \omega_h^\alpha \quad (7.60)$$

where ω_{nh}^α solves

$$\begin{cases} \partial_t^\varphi \omega_{nh}^\alpha + (v \cdot \nabla^\varphi) \omega_{nh}^\alpha - \varepsilon \Delta^\varphi \omega_{nh}^\alpha = F & \text{in } \Omega \\ \omega_{nh}^\alpha = 0 & \text{on } \{z = 0, -b\} \\ \omega_{nh}^\alpha|_{t=0} = \omega(0) \end{cases} \quad (7.61)$$

and ω_h^α solves

$$\begin{cases} \partial_t^\varphi \omega_h^\alpha + (v \cdot \nabla^\varphi) \omega_h^\alpha - \varepsilon \Delta^\varphi \omega_h^\alpha = 0 & \text{in } \Omega \\ \omega_h^\alpha = Z^\alpha \omega & \text{on } \{z = 0, -b\} \\ \omega_h^\alpha|_{t=0} = 0. \end{cases} \quad (7.62)$$

The estimate of ω_{nh}^α is given as follows.

Proposition 7.4. *For $|\alpha| \leq m - 1$, the solution ω_{nh}^α of (7.61) satisfies the estimate*

$$\begin{aligned} \|\omega_{nh}^\alpha(t)\|^2 + \varepsilon \int_0^t \|\nabla \omega_{nh}^\alpha\|^2 &\leq \Lambda_0 \|\omega(0)\|_{\mathbb{X}^{m-1}}^2 + \int_0^t \Lambda_\infty \varepsilon^2 \left(\|\nabla v\|_{\mathbb{X}^{m-2,2}}^2 + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}}^2 \right) \\ &+ \int_0^t \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}}^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2 + \|v\|_{\mathbb{X}^{m-1,1}}^2 + \|\omega_{nh}^\alpha\|^2 \right). \end{aligned} \quad (7.63)$$

Proof. Since $\omega_{nh}^\alpha = 0$ on the boundary, it follows from (7.61) that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega_{nh}^\alpha|^2 d\mathcal{V}_t + \varepsilon \int_\Omega |\nabla^\varphi \omega_{nh}^\alpha|^2 d\mathcal{V}_t = \int_\Omega F \cdot \omega_{nh}^\alpha d\mathcal{V}_t. \quad (7.64)$$

We now estimate the right hand side of (7.64). (A.1) implies that

$$\|Z^\alpha (\omega \cdot \nabla^\varphi v)\| \leq \Lambda_\infty (\|\omega\|_{\mathbb{X}^{m-1}} + \|\nabla^\varphi v\|_{\mathbb{X}^{m-1}}) \leq \Lambda_\infty \left(\|\omega\|_{\mathbb{X}^{m-1}} + \|\nabla v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (7.65)$$

To estimate the part involving \mathcal{C}_S , one first can change S_n into ω and m into $m + 1$ in (7.50) to get

$$\begin{aligned} \int_\Omega \mathcal{C}_S^2 \cdot \omega_{nh}^\alpha d\mathcal{V}_t &\leq \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-2,2}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right) + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right. \\ &\left. + \|\partial_z v\|_{\mathbb{X}^{m-1}} + \|v\|_{\mathbb{X}^{m-1,1}} \right) \|\omega_{nh}^\alpha\| + \Lambda_\infty |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \varepsilon^{\frac{1}{2}} \|\nabla \omega_{nh}^\alpha\|. \end{aligned} \quad (7.66)$$

For the part involving \mathcal{C}_{S1}^1 , one can change S_n into ω and m into $m + 1$ in (7.38) to obtain

$$\|\mathcal{C}_{S1}^1\| + \|\mathcal{C}_{S1}^2\| \leq \Lambda_\infty \left(\|v\|_{\mathbb{X}^{m-1}} + \|\omega\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (7.67)$$

The commutator \mathcal{C}_{S3}^1 requires much more care. Indeed, one can not change m into $m + 1$ in (7.40) since it would involve $|h|_{\mathbb{X}^{m, \frac{1}{2}}}$. As in (7.39), this commutator can be expanded into a sum of terms of the form

$$c_\beta Z^\beta \left(\frac{1}{z(z+b)} v_z \right) Z_3 Z^\gamma \omega$$

such that $|\beta| + |\gamma| \leq m - 1$, $|\gamma| \leq m - 2$. As the arguments in previous sections, by (A.1) and $v_z = 0$ on the boundary that, one deduces

$$\left\| Z^\beta \left(\frac{1}{z(z+b)} v_z \right) Z_3 Z^\gamma \omega \right\| \leq \Lambda_\infty \left(\|\omega\|_{\mathbb{X}^{m-2,1}} + \left\| Z \left(\frac{1}{z(z+b)} v_z \right) \right\|_{\mathbb{X}^{m-2}} \right) \quad (7.68)$$

$$\leq \Lambda_\infty \left(\|\omega\|_{\mathbb{X}^{m-2,1}} + \left\| \frac{1}{z(z+b)} v_z \right\|_{\mathbb{X}^{m-1}} \right).$$

It follows from $v_z = 0$ on the boundary and the Hardy inequality that, since $v_z = v \cdot \mathbf{N} - \partial_t \eta$,

$$\begin{aligned} \left\| \frac{1}{z(z+b)} v_z \right\|_{\mathbb{X}^{m-1}} &\lesssim \|\partial_z v_z\|_{\mathbb{X}^{m-1}} \\ &\leq \Lambda_\infty \left(\|\partial_z v\|_{\mathbb{X}^{m-1}} + \|v\|_{\mathbb{X}^{m-1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right) + \sum_{|\alpha|=m-1} \|v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \eta\|, \end{aligned} \quad (7.69)$$

where the last term requires furthermore analysis. Due to (1.6), it holds that

$$|Z_3 \hat{\eta}| \lesssim |\tilde{\chi}(|\xi|z) \hat{h}|$$

where $\tilde{\chi}$ has a slightly bigger support than χ . Hence Z_3 acts as a zero order operator:

$$\|\nabla Z_3 \eta\| \lesssim |h|_{\frac{1}{2}}. \quad (7.70)$$

This yields that if $\alpha_3 \neq 0$, one gains at least one derivative, and thus,

$$\|v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \eta\| \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-2, \frac{3}{2}}} + |\partial_t h|_{\mathbb{X}^{m-2, \frac{1}{2}}} \right) \leq \Lambda_\infty |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}. \quad (7.71)$$

Hence it suffices to estimate this term for the case with $|\alpha| = m-1$ and $\alpha_3 = 0$. Note that (1.6) implies that

$$\eta = \left(1 + \frac{z}{b}\right) \frac{1}{z^2} \psi \left(\frac{\cdot}{z}\right) \star_y h := \left(1 + \frac{z}{b}\right) \psi_z \star_y h, \quad (7.72)$$

where \star_y stands for the convolution in the y variable and ψ is in $L^1(\mathbb{R}^2)$. Then

$$\begin{aligned} v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \eta &= -\frac{1}{b} (v_y \cdot \psi_z \star_y \nabla_y Z^\alpha h + \psi_z \star_y \partial_t Z^\alpha h) \\ &\quad - \left(1 + \frac{z}{b}\right) \partial_z (v_y \cdot \psi_z \star_y \nabla_y Z^\alpha h + \psi_z \star_y \partial_t Z^\alpha h) := \mathcal{T}_\alpha = \mathcal{T}_{\alpha 1} + \mathcal{T}_{\alpha 2}. \end{aligned} \quad (7.73)$$

It is clear that

$$\|\mathcal{T}_{\alpha 1}\| \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |\partial_t h|_{\mathbb{X}^{m-1, -\frac{1}{2}}} \right) \leq \Lambda_\infty |h|_{\mathbb{X}^{m, -\frac{1}{2}}}. \quad (7.74)$$

For the second term, one can separate it into two cases. For $-b < z \leq -\frac{b}{2}$, then

$$\mathcal{T}_{\alpha 2} = -\frac{1}{bz} Z_3 (v_y \cdot \psi_z \star_y \nabla_y Z^\alpha h + \psi_z \star_y \partial_t Z^\alpha h).$$

It then follows from (7.70) that

$$\|\mathcal{T}_{\alpha 2}\|_{L^2(\{-b \leq z \leq -\frac{b}{2}\})} \leq \Lambda_\infty \left(|h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |\partial_t h|_{\mathbb{X}^{m-1, -\frac{1}{2}}} \right) \leq \Lambda_\infty |h|_{\mathbb{X}^{m, -\frac{1}{2}}}. \quad (7.75)$$

For $-\frac{b}{2} \leq z < 0$, $\mathcal{T}_{\alpha 2}$ can be written as

$$\mathcal{T}_{\alpha 2} = -\left(1 + \frac{z}{b}\right) (v_y(t, y, 0) \cdot \partial_z (\psi_z \star_y \nabla_y Z^\alpha h) + \partial_z (\psi_z \star_y \partial_t Z^\alpha h)) + \mathcal{R}, \quad (7.76)$$

where

$$|\mathcal{R}| \leq \Lambda_\infty |z(z+b)| |\partial_z (\psi_z \star Z^\alpha \nabla h)| \leq \Lambda_\infty |Z_3 (\psi_z \star Z^\alpha \nabla h)|.$$

By using (7.70) again, one can get that

$$\|\mathcal{R}\|_{L^2(\Omega \cap \{-\frac{b}{2} \leq z < 0\})} \leq \Lambda_\infty |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}. \quad (7.77)$$

To estimate the first term in (7.76), one notes that

$$\begin{aligned} v_y(t, y, 0) \cdot \partial_z (\psi_z \star_y \nabla_y Z^\alpha h) + \partial_z (\psi_z \star_y \partial_t Z^\alpha h) &= \partial_z (\psi_z \star_y (v_y(t, y, 0) \cdot \nabla_y Z^\alpha h + \partial_t Z^\alpha h)) \\ &\quad + \partial_z \left(\int_{\mathbb{R}^2} (v_y(t, y, 0) - v_y(t, y', 0)) \cdot \psi_z(y - y') \nabla_y Z^\alpha h(t, y') \right). \end{aligned}$$

For the second term in the right hand side of the above, one can employ the Taylor formula for v_y to get that

$$\left| (v_y(t, y, 0) - v_y(t, y', 0)) \partial_z \psi_z(y - y') \right| \leq \Lambda_\infty \frac{1}{z^2} \tilde{\psi} \left(\frac{y - y'}{z} \right),$$

where $\tilde{\psi}$ is still an L^1 function. This yields that

$$\sup_{z \in (-\frac{b}{2}, 0)} \left| \partial_z \left(\int_{\mathbb{R}^2} (v_y(t, y, 0) - v_y(t, y', 0)) \cdot \psi_z(y - y') \nabla_y Z^\alpha h(t, y') \right) \right|_0 \leq \Lambda_\infty |h|_{\mathbb{X}^{m-1, \frac{1}{2}}}. \quad (7.78)$$

For the first term, one shall use (3.47) to get

$$v_y \cdot \nabla_y Z^\alpha h + \partial_t Z^\alpha h = -\partial_z^\alpha v \cdot \mathbf{N} Z^\alpha h + \mathcal{C}^\alpha(h) + V^\alpha \cdot \mathbf{N},$$

which implies, since $|\alpha| = m - 1$,

$$|v_y \cdot \nabla_y Z^\alpha h + \partial_t Z^\alpha h|_{\frac{1}{2}} \leq \Lambda_\infty \left(|v|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right).$$

It thus holds that

$$\begin{aligned} \|\partial_z (\psi_z \star_y (v_y(t, y, 0) \cdot \nabla_y Z^\alpha h - \partial_t Z^\alpha h))\| &\lesssim |v_y \cdot \nabla_y Z^\alpha h - \partial_t Z^\alpha h|_{\frac{1}{2}} \\ &\leq \Lambda_\infty \left(|v|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right), \end{aligned} \quad (7.79)$$

Collecting all the estimates (7.77)–(7.79), one deduces from (7.76) that

$$\|\mathcal{T}_\alpha\|_{L^2(\Omega \cap \{z \geq -\frac{b}{2}\})} \leq \Lambda_\infty \left(|v|_{\mathbb{X}^{m-1, \frac{1}{2}}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (7.80)$$

In view of the estimates (7.68), (7.69), (7.71), (7.74), (7.75) and (7.80), by the trace estimates, one finally gets

$$\|\mathcal{C}_{53}^1\| \leq \Lambda_\infty \left(\|\partial_z v\|_{\mathbb{X}^{m-1}} + \|v\|_{\mathbb{X}^{m-1,1}} + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \right). \quad (7.81)$$

As a consequence of (7.65), (7.66), (7.67), (7.81) and Lemma C.1, we deduce from (7.64) that

$$\begin{aligned} \|\omega_h^\alpha(t)\|^2 + \varepsilon \int_0^t \|\nabla \omega_h^\alpha\|^2 &\leq \Lambda_0 \|\omega(0)\|_{\mathbb{X}^{m-1}}^2 + \int_0^t \Lambda_\infty \left(\varepsilon \left(\|\nabla v\|_{\mathbb{X}^{m-2,2}} + |h|_{\mathbb{X}^{m-2, \frac{5}{2}}} \right) \right. \\ &\quad \left. + |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} + \|\partial_z v\|_{\mathbb{X}^{m-1}} + \|v\|_{\mathbb{X}^{m-1,1}} \right) \|\omega_h^\alpha\| + \Lambda_\infty \varepsilon^{\frac{1}{2}} |h|_{\mathbb{X}^{m-1, \frac{1}{2}}} \|\nabla \omega_h^\alpha\|. \end{aligned} \quad (7.82)$$

We then conclude the proposition by using Cauchy's inequality. \square

It remains to estimate ω_h^α of (7.62). Note that Lemma 3.4 and the trace estimate imply that

$$\begin{aligned} \sqrt{\varepsilon} \int_0^t |Z^\alpha \omega|_0^2 &\leq \sqrt{\varepsilon} \int_0^t \Lambda_\infty \left(|v|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1,1}}^2 \right) \\ &\leq \sqrt{\varepsilon} \int_0^t \Lambda_\infty \left(\|\nabla v\|_{\mathbb{X}^{m-1,1}} \|v\|_{\mathbb{X}^{m-1,1}} + \|v\|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1,1}}^2 \right). \end{aligned} \quad (7.83)$$

Thus the main difficulty will be to handle the non-homogeneous boundary value problem, (7.62), whose boundary value is at a low level of regularity (it is $L^2_{t,y}$ and no more) due to (7.83). This creates two difficulties: the first is that one cannot lift the boundary condition easily and perform a standard energy estimate; the second one is that due to the lower boundary estimate, one cannot expect an estimate of $\|\omega_h^\alpha\|$ in $L^\infty(0, T)$ independent of ε , as was well explained in Section 10.2 of [33]: in using the model of the heat equation, one may expect a control in $H^{\frac{1}{4}}(0, T) \subset L^4(0, T)$.

Proposition 7.5. *Assume that $\Lambda_\infty(t) + \int_0^t \|\partial_{zz} v\|_{1,\infty} \leq M$ for $M > 0$. Then for $|\alpha| \leq m - 1$, the solution ω_h^α of (7.62) satisfies the estimate*

$$\|\omega_h^\alpha\|_{L^4_T L^2}^2 \leq \Lambda(M) \int_0^T \left(\|v\|_{\mathbb{X}^{m-1,1}}^2 + |h|_{\mathbb{X}^{m-1,1}}^2 \right) + \varepsilon \int_0^T \|\nabla v\|_{\mathbb{X}^{m-1,1}}^2. \quad (7.84)$$

Proof. Since the situation here is similar to Section 10.2 in [33], thus this proposition is a restatement of Proposition 10.4 in [33]. However, for completeness, we will sketch the main idea and some steps of the proof and state our estimates with a slight modification.

First, it is convenient to eliminate the convection term in (7.62) by considering Lagrangian coordinates. Define a parametrization of $\Omega(t)$, $X(t, \cdot) : \Omega \mapsto \Omega(t)$, by

$$\begin{cases} \partial_t X(t, x) = u(t, X(t, x)) = v(t, \Phi(t, \cdot)^{-1} \circ X) \\ X(0, x) = \Phi(0, x), \end{cases} \quad (7.85)$$

where the map $\Phi(t, \cdot)$ is defined by (1.4). Denote $J(t, x) = |\det \nabla X(t, x)|$ for the Jacobian of the change of coordinates. Then $J(t, x) = J_0(x)$ by the divergence free condition. Define

$$\Omega^\alpha = e^{-\gamma t} \omega_h^\alpha(t, \Phi^{-1} \circ X), \quad (7.86)$$

where $\gamma > 0$ is a large parameter to be chosen. Then Ω^α solves

$$a_0(\partial_t \Omega^\alpha + \gamma \Omega^\alpha) - \varepsilon \partial_i (a_{ij} \partial_j \Omega^\alpha) = 0 \quad \text{in } \Omega. \quad (7.87)$$

Here $a_0 = |J_0|^{\frac{1}{2}}$ and the matrix (a_{ij}) is defined by

$$(a_{ij}) = |J_0|^{\frac{1}{2}} P^{-1} \quad \text{with } P_{ij} = \partial_i X \cdot \partial_j X.$$

Due to the assumption $\Lambda_\infty(t) + \int_0^t \|\partial_{zz} v\|_{1,\infty} \leq M$, Lemma 10.5 in [33] holds. Then the following estimates hold:

$$\|J\|_{W^{1,\infty}} + \|J^{-1}\|_{W^{1,\infty}} \leq \Lambda_0, \quad (7.88)$$

$$\|\nabla X\|_{L^\infty} + \|\partial_t \nabla X\|_{L^\infty} \leq \Lambda(M) e^{t\Lambda(M)}, \quad (7.89)$$

$$\|\nabla X\|_{1,\infty} + \|\partial_t \nabla X\|_{1,\infty} \leq \Lambda(M) e^{t\Lambda(M)}, \quad (7.90)$$

$$\sqrt{\varepsilon} \|\nabla^2 X\|_{1,\infty} + \sqrt{\varepsilon} \|\partial_t \nabla^2 X\|_{L^\infty} \leq \Lambda(M)(1+t) e^{t\Lambda(M)}. \quad (7.91)$$

Indeed, the estimate (7.88) follows directly by $J(t, x) = J_0(x)$. Next, (7.85) implies that

$$\partial_t \nabla X = \nabla v \nabla \Phi^{-1} \nabla X \quad (7.92)$$

and hence

$$\|\nabla X\|_{L^\infty} \leq \|\nabla X(0)\|_{L^\infty} + \Lambda_0 \int_0^t \|\nabla v\|_{L^\infty} \|\nabla X\|_{L^\infty} \leq \Lambda_0 + \Lambda(M) \int_0^t \|\nabla X\|_{L^\infty}.$$

This yields the first part of (7.89) by the Gronwall inequality. Next, applying one spatial conormal derivative to (7.92), one can get

$$\|\nabla X\|_{1,\infty} \leq \|\nabla X(0)\|_{1,\infty} + \Lambda_0 \int_0^t \|\nabla v\|_{1,\infty} \|\nabla X\|_{1,\infty} \leq \Lambda_0 + \Lambda(M) \int_0^t \|\nabla X\|_{1,\infty}.$$

and hence the first part of (7.90) follows from the Gronwall inequality. The estimates hold also for the time derivative in (7.89) and (7.90) by using again the equation (7.92). To prove (7.91), one applies $\sqrt{\varepsilon} \partial_{x_i}$ to (7.92) to find that

$$\partial_t \sqrt{\varepsilon} \partial_{x_i} \nabla X = \sqrt{\varepsilon} (\partial_{x_i} \nabla v \nabla \Phi^{-1} \nabla X + \nabla v \partial_{x_i} \nabla \Phi^{-1} \nabla X + \nabla v \nabla \Phi^{-1} \partial_{x_i} \nabla X). \quad (7.93)$$

It follows from (7.90) that

$$\sqrt{\varepsilon} \|\nabla^2 X\|_{1,\infty} \leq \Lambda(M)(1+t) + \Lambda(M) \int_0^t \sqrt{\varepsilon} \|\nabla^2 X\|_{1,\infty} + \Lambda(M) e^{\Lambda(M)t} \int_0^t \sqrt{\varepsilon} \|\nabla^2 v\|_{1,\infty}$$

and hence, by using the assumption $\int_0^t \|\partial_{zz} v\|_{1,\infty} \leq M$,

$$\sqrt{\varepsilon} \|\nabla^2 X\|_{1,\infty} \leq \Lambda(M)(1+t) e^{\Lambda(M)t} + \Lambda(M) \int_0^t \sqrt{\varepsilon} \|\nabla^2 X\|_{1,\infty}$$

and so the first part of (7.91) follows from the Gronwall inequality. For the second part of (7.91), it follows by using again (7.93) and the fact that Λ_∞ involves the control of $\sqrt{\varepsilon} \|\nabla^2 v\|_{L^\infty}$.

As consequences of the estimates (7.88)–(7.91), one gets

$$a_0 \geq m, \quad a_{3,3} \geq m, \quad (a_{ij}) \geq c_0 \text{Id} \quad (7.94)$$

by a suitable choice of m and c_0 depending on M for $t \in [0, 1]$ and that

$$\|(a_0, a_{ij})\|_{L^\infty} + \sqrt{\varepsilon} \|\partial_z(a_0, a_{ij})\|_{L^\infty} \leq \Lambda(M), \quad (7.95)$$

$$\|\partial_{t,y}(a_0, a_{ij})\|_{L^\infty} + \sqrt{\varepsilon} \|\partial_{t,y} \nabla a_{ij}\|_{L^\infty} \leq \Lambda(M). \quad (7.96)$$

Note that these coefficients are lack of uniform regularity with respect to normal variables. To get the estimates for solutions of (7.87), the authors in [33] use the paradifferential calculus to prove Theorem 10.6 in [33], which yields that there exists γ_0 depending only on M such that for $\gamma \geq \gamma_0$, the solution of (7.87) satisfies the estimate

$$\|\Omega^\alpha\|_{H_T^{\frac{1}{4}} L^2}^2 \leq \sqrt{\varepsilon} \Lambda(M) \int_0^T |\Omega^\alpha|_0^2. \quad (7.97)$$

By the Minkowski inequality and the one-dimensional Sobolev embedding $H^{\frac{1}{4}} \subset L^4$,

$$\|\Omega^\alpha\|_{L_T^4 L^2}^2 \lesssim \|\Omega^\alpha\|_{L^2 L_T^4}^2 \lesssim \|\Omega^\alpha\|_{L^2 H_T^{\frac{1}{4}}}^2 = \|\Omega^\alpha\|_{H_T^{\frac{1}{4}} L^2}^2,$$

one then has

$$\|\Omega^\alpha\|_{L_T^4 L^2}^2 \leq \sqrt{\varepsilon} \Lambda(M) \int_0^T |\Omega^\alpha|_0^2. \quad (7.98)$$

This and (7.86) show that

$$\|\omega_h^\alpha\|_{L_T^4 L^2}^2 \leq \Lambda(M) \sqrt{\varepsilon} \int_0^T |Z^\alpha \omega|_0^2.$$

Thus (7.84) follows from (7.83) and Cauchy's inequality. \square

8. L^∞ ESTIMATES

In order to close the a priori estimates, we shall now estimate the L^∞ norms contained in Λ_∞ , $\|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}+2}}$ and $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$, and $\int_0^t \sqrt{\varepsilon} \|\partial_{zz} v\|_{1,\infty}$ that was used in Proposition 7.5. We will prove that they can be bounded in terms of the quantities in the left hand side of the estimates of Propositions 7.3 and 6.2, which will then be shown to be in L^∞ in time.

The key point is to use again the quantity $S_{\mathbf{n}}$, and one has the following:

Lemma 8.1. *For any $k \in \mathbb{N}$:*

$$\|\partial_z v\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right) (\|S_{\mathbf{n}}\|_{\mathbb{Y}^k} + \|v\|_{\mathbb{Y}^{k+1}}). \quad (8.1)$$

Also,

$$\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty} \leq \Lambda_0 \left(\sqrt{\varepsilon} \|\partial_z S_{\mathbf{n}}\|_{L^\infty} + \|S_{\mathbf{n}}\|_{1,\infty} + \|v\|_{2,\infty} \right) \quad (8.2)$$

and

$$\sqrt{\varepsilon} \|\partial_{zz} v\|_{1,\infty} \leq \Lambda_0 \left(\sqrt{\varepsilon} \|\partial_z S_{\mathbf{n}}\|_{1,\infty} + \|S_{\mathbf{n}}\|_{2,\infty} + \|v\|_{3,\infty} \right). \quad (8.3)$$

Proof. This follows again from (7.14) and (7.16). \square

As a consequence of Lemma 8.1, it suffices to estimate $S_{\mathbf{n}}$. Before proceeding further, one has the following observation.

Lemma 8.2. *For any smooth cut-off function χ such that $\chi = 0$ in a vicinity of $\partial\Omega = \{z = 0, -b\}$, we have for $m > k + 3/2$:*

$$\|\chi f\|_{W^{k,\infty}} \lesssim \|f\|_m. \quad (8.4)$$

Proof. It follows directly by the Sobolev embedding $H^k(\Omega) \subset L^\infty(\Omega)$ for $k > 3/2$ and the fact that away from the boundary $\partial\Omega$ the conormal Sobolev norm $\|\cdot\|_k$ is equivalent to the H^k norm. \square

Hence, one needs only to estimate $\chi S_{\mathbf{n}}$ for $\chi(z)$ compactly supported and such that $\chi = 1$ in a vicinity of $\partial\Omega$. In this step, as in [32, 33], it is convenient to use a coordinate system where the Laplacian has the simplest expression. We shall thus use a normal geodesic coordinate system in the vicinity of $\partial\Omega$. Note that this coordinate system has not been used before since it requires more regularity for the boundary: to get an H^m (or C^m) coordinate system, the boundary needs to be H^{m+1} (or C^{m+1}). Nevertheless, at this step, this is not a problem since

one needs only to estimate a low number of derivatives of the velocity, say $\frac{m}{2} + 2$, while the boundary is H^m and m can be as large as needed. We shall choose the cut-off function χ in order to get a well defined coordinate system in the vicinity of $\partial\Omega$. For sake of brevity, we consider only the estimates near $\{z = 0\}$, and the estimates near $\{z = -b\}$ may follow in the same way and a bit simpler.

Define a different parametrization of the vicinity of $\{z = 0\}$ by

$$\begin{aligned} \Psi(t, \cdot) : \Omega &= \mathbb{R}^2 \times (-\infty, 0) \rightarrow \Omega(t) \\ (y, z) &\mapsto \begin{pmatrix} y \\ h(t, y) \end{pmatrix} + z\mathbf{n}^b(t, y), \end{aligned} \quad (8.5)$$

where $\mathbf{n}^b = \mathbf{N}^b/|\mathbf{N}^b|$ is the unit exterior normal with $\mathbf{N}^b = (-\partial_1 h, -\partial_2 h, 1)$. Note that $\Psi(t, \cdot)$ is a diffeomorphism from $\mathbb{R}^2 \times (-\delta, 0)$ to a vicinity of $\partial\Omega(t)$ for some δ which depends only on c_0 , and for every $t \in [0, T^{\varepsilon, \sigma}]$ thanks to (6.1). By this parametrization, the induced Riemannian metric has the block structure

$$g(t, y, z) = \begin{pmatrix} \tilde{g}(t, y, z) & 0 \\ 0 & 1 \end{pmatrix}. \quad (8.6)$$

Hence, the Laplacian in this coordinate system reads:

$$\Delta_g f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f, \quad (8.7)$$

where $|g|$ denotes the determinant of the matrix g and $\Delta_{\tilde{g}}$ is defined by

$$\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1 \leq i, j \leq 2} \partial_{y^i} (\tilde{g}^{ij} |\tilde{g}|^{\frac{1}{2}} \partial_{y^j} f). \quad (8.8)$$

To use this coordinate system, one shall first localize the equation for $S^\varphi v$ in a vicinity of $\{z = 0\}$. Set

$$S^X = \chi(z) S^\varphi v, \quad (8.9)$$

where $\chi(z) \in [0, 1]$ is smooth compactly supported near $\{z = 0\}$ and takes the value 1 in a vicinity of $\{z = 0\}$. (7.19) yields that

$$\partial_t^\varphi S^X + (v \cdot \nabla^\varphi) S^X - \varepsilon \Delta^\varphi S^X = F_{S^X} \quad \text{in } \Omega, \quad (8.10)$$

where

$$F_{S^X} = F^X + F_v \quad (8.11)$$

with

$$F^X = (V_z \partial_z \chi) S^\varphi v - \varepsilon \nabla^\varphi \chi \cdot \nabla^\varphi S^\varphi v - \varepsilon \Delta^\varphi \chi S^\varphi v, \quad (8.12)$$

$$F_v = -\chi (D^\varphi)^2 q - \frac{\chi}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2). \quad (8.13)$$

Next, define implicitly \tilde{S} in $\Omega(t)$ by $\tilde{S}(t, \Phi(t, y, z)) = S^X(t, y, z)$. Then (8.10) yields

$$\partial_t \tilde{S} + u \cdot \nabla \tilde{S} - \varepsilon \Delta \tilde{S} = F_{S^X}(t, \Phi(t, \cdot)^{-1}) \quad \text{in } \Omega(t). \quad (8.14)$$

Finally, define S^Ψ in Ω by $S^\Psi(t, y, z) = \tilde{S}(t, \Psi(t, y, z)) \equiv S^X(t, \Phi(t, \cdot)^{-1} \circ \Psi)$. It then follows from (8.14) and (8.7) that

$$\partial_t S^\Psi + w \cdot \nabla S^\Psi - \varepsilon (\partial_{zz} S^\Psi + \frac{1}{2} \partial_z (\ln |g|) \partial_z S^\Psi + \Delta_{\tilde{g}} S^\Psi) = F_{S^X}(t, \Phi^{-1} \circ \Psi(t, \cdot)) \quad \text{in } \Omega, \quad (8.15)$$

where the vector field w is given by

$$w = \bar{\chi} (\nabla \Psi)^{-1} (u(t, \Psi) - \partial_t \Psi) \equiv \bar{\chi} (\nabla \Psi)^{-1} (v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi). \quad (8.16)$$

Note that S^Ψ is compactly supported in z in a vicinity of $\{z = 0\}$. The function $\bar{\chi}(z)$ is a function with a slightly larger support such that $\bar{\chi} S^\Psi = S^\Psi$. The introduction of this function

allows to have w also supported in a vicinity of $\{z = 0\}$. Note that on $\{z = 0\}$, $w_3 = 0$. Indeed, since

$$\nabla\Psi(t, y, 0) = \begin{pmatrix} 1 & 0 & \mathbf{n}_1^b \\ 0 & 1 & \mathbf{n}_2^b \\ \partial_1 h & \partial_2 h & \mathbf{n}_3^b \end{pmatrix}, \quad (8.17)$$

thus for any $Y \in \mathbb{R}^3$,

$$((\nabla\Psi(t, y, 0))^{-1}Y)_3 = Y \cdot \mathbf{n}^b. \quad (8.18)$$

Hence, (8.16) implies that on $\{z = 0\}$,

$$w_3 = v \cdot \mathbf{n}^b - \partial_t h \mathbf{n}_3^b = \frac{1}{|\mathbf{N}^b|} (v \cdot \mathbf{N}^b - \partial_t h) = 0 \quad (8.19)$$

thanks to the kinematic boundary condition.

Now, set

$$S_{\mathbf{n}}^\Psi(t, y, z) = \Pi^b(t, y) S^\Psi(t, y, z) \mathbf{n}^b(t, y) \quad (8.20)$$

with $\Pi^b = \text{Id} - \mathbf{n}^b \otimes \mathbf{n}^b$. Note that Π^b and \mathbf{n}^b are independent of z . Moreover, since the equation (8.15) is compactly supported in z in a vicinity of $\{z = 0\}$, this yields that $S_{\mathbf{n}}^\Psi$ solves

$$\partial_t S_{\mathbf{n}}^\Psi + w \cdot \nabla S_{\mathbf{n}}^\Psi - \varepsilon (\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z) S_{\mathbf{n}}^\Psi = F_{\mathbf{n}}^\Psi \quad \text{in } \{z < 0\}, \quad (8.21)$$

where

$$F_{\mathbf{n}}^\Psi = \Pi^b F_{S^\Psi} \mathbf{n}^b + F_{\mathbf{n}}^{\Psi,1} + F_{\mathbf{n}}^{\Psi,2} \quad (8.22)$$

with

$$F_{\mathbf{n}}^{\Psi,1} = ((\partial_t + w_y \cdot \nabla_y) \Pi^b) S^\Psi \mathbf{n}^b + \Pi^b S^\Psi (\partial_t + w_y \cdot \nabla_y) \mathbf{n}^b, \quad (8.23)$$

$$F_{\mathbf{n}}^{\Psi,2} = -\varepsilon \Pi^b (\Delta_{\tilde{g}} S^\Psi) \mathbf{n}^b. \quad (8.24)$$

On $\{z = 0\}$, $S_{\mathbf{n}}^\Psi = S_{\mathbf{n}} = 0$. Furthermore, it is convenient to eliminate the term $\varepsilon \partial_z \ln |g| \partial_z$ in the equation (8.21). Set

$$\rho = |g|^{\frac{1}{4}} S_{\mathbf{n}}^\Psi, \quad (8.25)$$

then

$$\partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = \mathcal{H} \quad \text{in } \{z < 0\}, \quad (8.26)$$

where

$$\mathcal{H} = |g|^{\frac{1}{4}} (F_{\mathbf{n}}^\Psi + F_g) \quad \text{with } F_g = \rho |g|^{-\frac{1}{2}} (\partial_t + w \cdot \nabla - \varepsilon \partial_{zz}) |g|^{\frac{1}{4}}. \quad (8.27)$$

Trivially, on $\{z = 0\}$, $\rho = 0$.

8.1. Estimate of $\|\partial_z v\|_{\mathbb{Y}^{\frac{m}{2}+2}}$. We now establish the first L^∞ estimate. Note that it is equivalent to estimate $S_{\mathbf{n}}$ or $S_{\mathbf{n}}^\Psi$ or ρ . Indeed, by the definition (8.20), using the chain rule and the fact that Z is tangent to $\{z = 0\}$, one has

$$\|S_{\mathbf{n}}^\Psi\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right) \left\| \Pi^b S^\Psi v \mathbf{n}^b \right\|_{\mathbb{Y}^k}.$$

Since $|\Pi - \Pi^b| + |\mathbf{n} - \mathbf{n}^b| = \mathcal{O}(z)$ in the vicinity of $\{z = 0\}$, thus

$$\|S_{\mathbf{n}}^\Psi\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right) (\|S_{\mathbf{n}}\|_{\mathbb{Y}^k} + \|v\|_{\mathbb{Y}^{k+1}}). \quad (8.28)$$

Similar arguments show that

$$\|S_{\mathbf{n}}\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right) (\|S_{\mathbf{n}}^\Psi\|_{\mathbb{Y}^k} + \|v\|_{\mathbb{Y}^{k+1}}). \quad (8.29)$$

On the other hand, it is easy to see that it is equivalent to estimate ρ or $S_{\mathbf{n}}^\Psi$. By (8.1), it thus suffices to estimate ρ .

Set

$$\tilde{\Lambda}_\infty(t) = \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{X}^{\frac{m}{2}+6}} \right). \quad (8.30)$$

Proposition 8.3. *For $m \geq 14$, it holds that*

$$\|\rho(t)\|_{\mathbb{Y}^{\frac{m}{2}+2}} \lesssim \|\rho(0)\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \int_0^t \tilde{\Lambda}_\infty (1 + \varepsilon \|\nabla S_{\mathbf{n}}\|_{\mathbb{X}^{m-2}}). \quad (8.31)$$

Proof. Apply Z^α , for $\alpha \in \mathbb{N}^{1+3}$ with $|\alpha| = k \leq \frac{m}{2} + 2$, to (8.26) to obtain

$$\partial_t Z^\alpha \rho + w \cdot \nabla Z^\alpha \rho - \varepsilon \partial_{zz} Z^\alpha \rho = Z^\alpha \mathcal{H} + \mathcal{C}_S \quad \text{in } \{z < 0\}, \quad (8.32)$$

where

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 \quad (8.33)$$

with

$$\mathcal{C}_S^1 = [Z^\alpha, w_y] \cdot \nabla_y \rho + [Z^\alpha, w_3] \cdot \partial_z \rho := \mathcal{C}_{S_y} + \mathcal{C}_{S_z} \quad \text{and} \quad \mathcal{C}_S^2 = -\varepsilon [Z^\alpha, \partial_{zz}] \rho. \quad (8.34)$$

The maximum principle on (8.32) yields that, since $Z^\alpha \rho = 0$ on $\{z = 0\}$,

$$\|Z^\alpha \rho\|_{L^\infty} \leq \|Z^\alpha \rho(0)\|_{L^\infty} + \int_0^t (\|Z^\alpha \mathcal{H}\|_{L^\infty} + \|\mathcal{C}_S\|_{L^\infty}). \quad (8.35)$$

The right hand side of (8.35) can be estimated as follows. For the commutator \mathcal{C}_S^1 , the direct estimates yield

$$\|\mathcal{C}_{S_y}\|_{L^\infty} \lesssim \|w_y\|_{\mathbb{Y}^k} \|\rho\|_{\mathbb{Y}^k}. \quad (8.36)$$

To estimate \mathcal{C}_{S_z} , by expanding the commutator and using (3.11), one needs to estimate terms of the form

$$\left\| Z^\beta w_3 \partial_z Z^\gamma \rho \right\|_{L^\infty}$$

with $\beta + \gamma \leq \alpha$ and $|\gamma| \leq |\alpha| - 1$. Since $w_3 = 0$ on $\{z = 0\}$, so

$$\left\| Z^\beta w_3 \partial_z Z^\gamma \rho \right\|_{L^\infty} = \left\| \frac{Z^\beta w_3}{z(z+b)} Z^\gamma \rho \right\|_{L^\infty} \lesssim \|\partial_z w_3\|_{\mathbb{Y}^k} \|\rho\|_{\mathbb{Y}^k}. \quad (8.37)$$

For the commutator \mathcal{C}_S^2 , using (3.11) repeatedly leads to

$$\begin{aligned} -[Z^\alpha, \partial_{zz}] \rho &= \partial_z ([Z^\alpha, \partial_z] \rho) + [Z^\alpha, \partial_z] \partial_z \rho \\ &= \sum_{|\beta| \leq k-1} \partial_z (c_\beta \partial_z (Z^\beta \rho)) + \sum_{|\beta| \leq k-1} c_\beta \partial_z (Z^\beta \partial_z \rho) \\ &= \sum_{|\beta'| \leq k-1} c_{\beta'} Z^{\beta'} \partial_z \rho + \sum_{|\beta'| \leq k-1} \tilde{c}_{\beta'} Z^{\beta'} \partial_{zz} \rho. \end{aligned} \quad (8.38)$$

Hence,

$$\|\mathcal{C}_S^2\|_{L^\infty} \lesssim \varepsilon \|\partial_z \rho\|_{\mathbb{Y}^{k-1}} + \varepsilon \|\partial_{zz} \rho\|_{\mathbb{Y}^{k-1}}. \quad (8.39)$$

Note that using the equation (8.26) implies

$$\varepsilon \|\partial_{zz} \rho\|_{\mathbb{Y}^{k-1}} \leq \|\partial_t \rho + w \cdot \nabla \rho - \mathcal{H}\|_{\mathbb{Y}^{k-1}} \leq \|\rho\|_{\mathbb{Y}^k} + \|w \cdot \nabla \rho\|_{\mathbb{Y}^{k-1}} + \|\mathcal{H}\|_{\mathbb{Y}^{k-1}}. \quad (8.40)$$

Recall from (7.1) that

$$\varepsilon \|\partial_z \rho\|_{\mathbb{Y}^{k-1}} \leq \varepsilon \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+1}} \right) \|\nabla^2 v\|_{\mathbb{Y}^{k-1}} \leq \Lambda_\infty. \quad (8.41)$$

As (8.36) and (8.37),

$$\|w \cdot \nabla \rho\|_{\mathbb{Y}^{k-1}} \lesssim (\|w_y\|_{\mathbb{Y}^k} + \|\partial_z w_3\|_{\mathbb{Y}^k}) \|\rho\|_{\mathbb{Y}^k}. \quad (8.42)$$

Thus,

$$\|\mathcal{C}_S^2\|_{L^\infty} \lesssim (1 + \|w_y\|_{\mathbb{Y}^k} + \|\partial_z w_3\|_{\mathbb{Y}^k}) \|\rho\|_{\mathbb{Y}^k} + \|\mathcal{H}\|_{\mathbb{Y}^{k-1}} + \Lambda_\infty. \quad (8.43)$$

Consequently, in light of the estimates (8.36), (8.37) and (8.43) into (8.35), one obtains

$$\|\rho\|_{\mathbb{Y}^k} \lesssim \|\rho(0)\|_{\mathbb{Y}^k} + \int_0^t \|\mathcal{H}\|_{\mathbb{Y}^k} + (1 + \|w_y\|_{\mathbb{Y}^k} + \|\partial_z w_3\|_{\mathbb{Y}^k}) \|\rho\|_{\mathbb{Y}^k} + \Lambda_\infty. \quad (8.44)$$

Now we estimate \mathcal{H} . Note first that

$$\left\| |g|^{\frac{1}{4}} F_g \right\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, \|\rho\|_{\mathbb{Y}^k} + \|w\|_{\mathbb{Y}^k} + |h|_{\mathbb{Y}^{k+3}} \right).$$

Next, it follows from (8.23) and (8.24) that

$$\left\| |g|^{\frac{1}{4}} F_{\mathbf{n}}^{\Psi,1} \right\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+2}} + \|w\|_{\mathbb{Y}^k} + \|\nabla v\|_{\mathbb{Y}^k} \right)$$

and

$$\left\| |g|^{\frac{1}{4}} F_{\mathbf{n}}^{\Psi,2} \right\|_{\mathbb{Y}^k} \leq \varepsilon \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+3}} \right) \|\nabla v\|_{\mathbb{Y}^{k+2}}.$$

Using (8.12) and (8.13), the fact that F^χ is supported away from $\{z = 0\}$ and Lemma 8.2, one gets

$$\left\| |g|^{\frac{1}{4}} \Pi^b F^\chi \mathbf{n}^b \right\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+2}} + \|v\|_{\mathbb{X}^{k+4}} \right)$$

and

$$\left\| |g|^{\frac{1}{4}} \Pi^b F_v \mathbf{n}^b \right\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+2}} + \|\nabla v\|_{\mathbb{Y}^k} \right) (1 + \|\nabla^2 q\|_{\mathbb{Y}^k}).$$

Recalling (8.27), (8.22) and (8.11), and collecting these estimates, one arrives at

$$\|\mathcal{H}\|_{\mathbb{Y}^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{k+3}} + \|v\|_{\mathbb{X}^{k+4}} + \|\nabla v\|_{\mathbb{Y}^k} \right) (1 + \varepsilon \|\nabla v\|_{\mathbb{Y}^{k+2}} + \|\nabla^2 q\|_{\mathbb{Y}^k}). \quad (8.45)$$

Recall from (4.15) that

$$\begin{aligned} \|\nabla^2 q\|_{\mathbb{Y}^{\frac{m}{2}+2}} &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m+6}{4}+4}} + |h|_{\mathbb{X}^{\frac{m+6}{2}+4}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m+6}{4}+3}} + \|v\|_{\mathbb{X}^{\frac{m+6}{2}+4}} + \|\nabla v\|_{\mathbb{X}^{\frac{m+6}{2}+3}} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{Y}^{\frac{m}{2}+5}} + |h|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{X}^{\frac{m}{2}+6}} \right) \end{aligned} \quad (8.46)$$

if $m \geq 10$. Hence, (8.44) implies

$$\begin{aligned} \|\rho\|_{\mathbb{Y}^{\frac{m}{2}+2}} &\lesssim \|\rho_0\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \int_0^t \Lambda \left(\frac{1}{c_0}, |h|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}+2}} + \|v\|_{\mathbb{X}^{\frac{m}{2}+7}} + \|\nabla v\|_{\mathbb{X}^{\frac{m}{2}+6}} \right) \\ &\quad \times \left(1 + \varepsilon \|\nabla v\|_{\mathbb{Y}^{\frac{m}{2}+4}} \right). \end{aligned} \quad (8.47)$$

Then the desired estimates (8.29) follows for $m \geq 14$ so that $\frac{m}{2} + 4 + 1 \leq m - 2$. \square

8.2. Estimate of $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$. The next L^∞ estimate is the only place where one needs to use the compatibility condition on the initial data. As in the previous subsection, by (8.2), one can reduce the problem to the estimate of $\sqrt{\varepsilon} \|\partial_z \rho\|_{L^\infty}$.

Proposition 8.4. *Assume that the initial data satisfies the compatibility condition $S_{\mathbf{n}}(0) = 0$ on $\{z = 0, -b\}$. Then it holds that for $m \geq 6$,*

$$\sqrt{\varepsilon} \|\partial_z \rho(t)\|_{L^\infty} \lesssim \sqrt{\varepsilon} \|\partial_z \rho_0\|_{L^\infty} + \int_0^t \frac{\Lambda_\infty}{\sqrt{t-\tau}}. \quad (8.48)$$

Proof. The proof follows the spirit of the proof of Proposition 9.8 in [33]. Recall that ρ satisfies (8.26) in $\{z < 0\}$ with $\rho = 0$ on $\{z = 0\}$. Note that one can not apply $\sqrt{\varepsilon} \partial_z$ to (8.26) and then use the maximum principle due to boundary condition. We shall use a precise description of the solution of (8.26). Indeed, one can use the one-dimensional heat kernel of $\{z < 0\}$:

$$G(t, z, z') = \frac{1}{\sqrt{4\pi\varepsilon t}} \left(e^{-\frac{(z-z')^2}{4\varepsilon t}} - e^{-\frac{(z+z')^2}{4\varepsilon t}} \right) \quad (8.49)$$

to write that

$$\begin{aligned} \sqrt{\varepsilon} \partial_z \rho(t, y, z) &= \int_0^{+\infty} \sqrt{\varepsilon} \partial_z G(t, z, z') \rho_0(y, z') dz' \\ &\quad + \int_0^t \sqrt{\varepsilon} \partial_z G(t-\tau, z, z') (\mathcal{H}(\tau, y, z') - w \cdot \nabla \rho) dz' d\tau. \end{aligned} \quad (8.50)$$

Since $\rho_0 = 0$ on $\{z = 0\}$, thanks to the compatibility condition, one can integrate by parts the first term to obtain

$$\sqrt{\varepsilon} \|\partial_z \rho(t)\|_{L^\infty} \lesssim \sqrt{\varepsilon} \|\partial_z \rho_0\|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-\tau}} (\|\mathcal{H}\|_{L^\infty} + \|w \cdot \nabla \rho\|_{L^\infty}). \quad (8.51)$$

Next, it follows from (8.45) with $k = 0$ that

$$\|\mathcal{H}\|_{L^\infty} \leq \Lambda_\infty (1 + \|\nabla^2 q\|_{L^\infty}) \leq \Lambda_\infty. \quad (8.52)$$

On the other hand, as (8.42),

$$\|w \cdot \nabla \rho\|_{L^\infty} \leq (\|w_y\|_{L^\infty} + \|\partial_z w_3\|_{L^\infty}) \|\rho\|_{1,\infty} \leq \Lambda_\infty. \quad (8.53)$$

Consequently, plugging (8.52) and (8.53) into (8.51) yields that

$$\sqrt{\varepsilon} \|\partial_z \rho(t)\|_{L^\infty} \lesssim \sqrt{\varepsilon} \|\partial_z \rho_0\|_{L^\infty} + \int_0^t \frac{\Lambda_\infty}{\sqrt{t-\tau}}, \quad (8.54)$$

which completes the proof. \square

8.3. Estimate of $\sqrt{\varepsilon} \|\partial_{zz} v\|_{1,\infty}$. The last L^∞ estimate is the one that was used in Proposition 7.5. By (8.3), one can again reduce the problem to the estimate of $\sqrt{\varepsilon} \|\partial_z \rho\|_{1,\infty}$.

Proposition 8.5. *For $m \geq 6$, it holds that:*

$$\int_0^t \sqrt{\varepsilon} \|\partial_z \rho\|_{1,\infty} \lesssim \sqrt{t} \|\rho_0\|_{1,\infty} + t \int_0^t \frac{\Lambda_\infty}{\sqrt{t-\tau}}. \quad (8.55)$$

Proof. We will use a different argument from the proof of Lemma 9.9 in [33]. A direct use of the Duhamel formula (8.50) yields

$$\sqrt{\varepsilon} \|\partial_z \rho\|_{1,\infty} \lesssim \frac{1}{\sqrt{t}} \|\rho_0\|_{1,\infty} + \int_0^t \frac{1}{\sqrt{t-\tau}} (\|\mathcal{H}\|_{1,\infty} + \|w \cdot \nabla \rho\|_{1,\infty}). \quad (8.56)$$

Next, it follows as the previous arguments that

$$\|\mathcal{H}\|_{1,\infty} \leq \Lambda_\infty (1 + \|\nabla^2 q\|_{1,\infty}) \leq \Lambda_\infty \quad (8.57)$$

and

$$\|w \cdot \nabla \rho\|_{1,\infty} \leq \Lambda_\infty (\|w_y\|_{2,\infty} + \|\partial_z w_3\|_{2,\infty}) \|\rho\|_{2,\infty} \leq \Lambda_\infty. \quad (8.58)$$

Consequently,

$$\sqrt{\varepsilon} \|\partial_z \rho\|_{1,\infty} \lesssim \frac{1}{\sqrt{t}} \|\rho_0\|_{1,\infty} + \int_0^t \frac{\Lambda_\infty}{\sqrt{t-\tau}}. \quad (8.59)$$

Integration in time yields

$$\int_0^t \sqrt{\varepsilon} \|\partial_z \rho\|_{1,\infty} \lesssim \sqrt{t} \|\rho_0\|_{1,\infty} + t \int_0^t \frac{\Lambda_\infty}{\sqrt{t-\tau}}. \quad (8.60)$$

which completes the proof of the proposition. \square

9. PROOF OF THEOREM 2.1

Collecting the estimates obtained in Sections 6–8, one can prove Theorem 2.1 in the similar way as that for Theorem 1.1 of [33] with slight modifications.

Recall $\mathcal{N}(T)$ and $\mathcal{Q}(T)$. For two parameters R and c_0 to be chosen $1/c_0 \ll R$, define

$$T_*^{\varepsilon,\delta} = \sup_{T>0} \left\{ T \in [0, 1] \mid \mathcal{N}(t) \leq R, \ |h(t)|_{2,\infty} \leq \frac{1}{c_0}, \ \partial_z \varphi(t) \geq \frac{c_0}{2} \right. \\ \left. \text{and } g - \partial_z^2 q(t) \geq \frac{c_0}{2} \text{ on } \{z = 0\}, \ \forall t \in [0, T]. \right\}. \quad (9.1)$$

Proposition 8.5 yields

$$\int_0^T \|\partial_{zz} v\|_{1,\infty} \leq \Lambda(R). \quad (9.2)$$

This allows one to use Proposition 7.5, which together with Propositions 6.2, 6.3, 7.3, 8.3 and 8.4 implies that, by a suitable linear combination,

$$\mathcal{N}(t) \leq \Lambda \left(\frac{1}{c_0}, \mathcal{N}(0) \right) + \Lambda(R) \left(T^{\frac{1}{2}} + \mathcal{W}(T) + \mathcal{W}(T)^{\frac{1}{2}} \right), \quad (9.3)$$

where

$$\mathcal{W}(T) := \int_0^T |\partial_t^m h|_0^2 + \sigma |\partial_t^m h|_1^2 + \|\partial_t^m v\|^2 + \|\partial_z v\|_{\mathbb{X}^{m-1}}^2.$$

It follows from the Cauchy-Schwarz inequality that

$$\mathcal{W}(T) \leq \mathcal{N}(T)T^{\frac{1}{2}} \leq RT^{\frac{1}{2}}. \quad (9.4)$$

Hence, one deduces from (9.3) that

$$\mathcal{N}(t) \leq \Lambda \left(\frac{1}{c_0}, \mathcal{N}(0) \right) + \Lambda(R)T^{\frac{1}{4}}. \quad (9.5)$$

On the other hand, since $\mathcal{N}(T)$ involves time derivatives, one gets easily that

$$|h(t)|_{2,\infty} \leq |h(0)|_{2,\infty} + \Lambda(R)T, \quad (9.6)$$

$$\partial_z \varphi(t) \geq \partial_z \varphi(0) - \Lambda(R)T \quad (9.7)$$

and

$$g - \partial_z^\varphi q(t) \geq g - \partial_z^\varphi q(0) - \Lambda(R)T. \quad (9.8)$$

Consequently, one can choose c_0 so that $|h(0)|_{2,\infty} \leq \frac{1}{2c_0}$ and then $R = 2\Lambda \left(\frac{1}{c_0}, \mathcal{N}(0) \right)$, then there exists T_* which depends only on R so that for $T \leq \text{Min} (T_*, T_*^{\varepsilon,\delta})$,

$$\mathcal{N}(t) \leq \frac{3R}{4}, \quad \partial_z \varphi(t) \geq \frac{3c_0}{4}, \quad |h(t)|_{2,\infty} \leq \frac{3}{4c_0} \quad \text{and} \quad g - \partial_z^\varphi q(t) \geq \frac{3c_0}{4} \quad \text{on} \quad \{z = 0\}, \quad \forall t \in [0, T].$$

This yields $T_*^{\varepsilon,\delta} \geq T_*$ by the definition (9.1) and also the estimate (2.6). The proof of Theorem 2.1 is thus completed. \square

APPENDIX A. SOBOLEV CONORMAL SPACES

We recall the Sobolev conormal spaces \mathbb{X}^m and \mathbb{Y}^m from (2.1).

Lemma A.1. *The following product and commutator estimates hold.*

(i) For $|\alpha| + |\beta| = k \geq 0$:

$$\|Z^\alpha f Z^\beta g\| \lesssim \|f\|_{\mathbb{X}^k} \|g\|_{\mathbb{Y}^{\frac{k}{2}}} + \|f\|_{\mathbb{Y}^{\frac{k}{2}}} \|g\|_{\mathbb{X}^k}. \quad (A.1)$$

(ii) For $|\alpha| = k \geq 1$:

$$\|[Z^\alpha, f]g\| \lesssim \|Zf\|_{\mathbb{X}^{k-1}} \|g\|_{\mathbb{Y}^{\frac{k-1}{2}}} + \|Zf\|_{\mathbb{Y}^{\frac{k-1}{2}}} \|g\|_{\mathbb{X}^{k-1}} \quad (A.2)$$

(iii) For $|\alpha| = k \geq 2$, define the symmetric commutator

$$[Z^\alpha, f, g] = Z^\alpha(fg) - Z^\alpha f g - f Z^\alpha g. \quad (A.3)$$

Then

$$\|[Z^\alpha, f, g]\| \lesssim \|Zf\|_{\mathbb{X}^{k-2}} \|Zg\|_{\mathbb{Y}^{\frac{k-2}{2}}} + \|Zf\|_{\mathbb{Y}^{\frac{k-2}{2}}} \|Zg\|_{\mathbb{X}^{k-2}}. \quad (A.4)$$

Proof. The product estimate (A.1) follows by controlling the product with the lower order derivative term in L^∞ and the higher order derivative term in L^2 . To prove the commutator estimate (A.2), one uses the Leibnitz formula to expand

$$[Z^\alpha, f]g = \sum_{\substack{\beta+\gamma=\alpha \\ \beta \neq 0}} C_{\beta,\gamma} Z^\beta f Z^\gamma g.$$

Since $\beta \neq 0$, one can write $Z^\beta = Z^{\beta-\beta'} Z^{\beta'}$ with $|\beta'| = 1$. Then (A.1) yields

$$\|Z^{\beta-\beta'} Z^{\beta'} f Z^\gamma g\| \lesssim \|Z^{\beta'} f\|_{\mathbb{X}^{k-1}} \|g\|_{\mathbb{Y}^{\frac{k-1}{2}}} + \|Z^{\beta'} f\|_{\mathbb{Y}^{\frac{k-1}{2}}} \|g\|_{\mathbb{X}^{k-1}}.$$

This proves (A.2). The commutator estimate (A.4) can be proved in the same way. \square

We shall also use the Sobolev tangential spaces defined by

$$H_{tan}^s(\Omega) = \left\{ f \in L^2(\Omega), \|f\|_{H_{tan}^s} = \|\Lambda^s f\|_{L^2} < \infty \right\}, \quad s \in \mathbb{R},$$

where Λ^s is the tangential Fourier multiplier by $(1 + |\xi|^2)^{\frac{s}{2}}$, $\xi \in \mathbb{R}^2$. Note that

$$\|f\|_{H_{tan}^s} \lesssim \|f\|_k \text{ for } s \leq k, k \in \mathbb{N}.$$

Lemma A.2. *The following anisotropic Sobolev embedding and trace estimates hold.*

(i) For $s_1 + s_2 > 2$, $s_3 + s_4 > 2$:

$$\|f\|_{L^\infty} \lesssim \|\partial_z f\|_{H_{tan}^{s_1}}^{\frac{1}{2}} \|f\|_{H_{tan}^{s_2}}^{\frac{1}{2}} + \|f\|_{H_{tan}^{s_3}}^{\frac{1}{2}} \|f\|_{H_{tan}^{s_4}}^{\frac{1}{2}}. \quad (\text{A.5})$$

(ii) For $s_1 + s_2 = s_3 + s_4 = 2s$:

$$|f|_s \lesssim \|\partial_z f\|_{H_{tan}^{s_1}}^{\frac{1}{2}} \|f\|_{H_{tan}^{s_2}}^{\frac{1}{2}} + \|f\|_{H_{tan}^{s_3}}^{\frac{1}{2}} \|f\|_{H_{tan}^{s_4}}^{\frac{1}{2}}. \quad (\text{A.6})$$

Proof. We need to modify the proof of Proposition 2.2 in [33] since our domain here is of finite depth. To get the anisotropic Sobolev embedding estimate (A.5), one first notes that

$$|\hat{f}(\xi, z)| \leq |\hat{f}(\xi, z')| + \left(\int_{-z'}^z 2 \left| \partial_z \hat{f}(\xi, x_3) \right| |\hat{f}(\xi, x_3)| dx_3 \right)^{\frac{1}{2}}.$$

Integrating the inequality above with respect to $z' \in (-b, 0)$ yields

$$|\hat{f}(\xi, z)| \lesssim \int_{-b}^0 |\hat{f}(\xi, z')| dz' + \left(\int_{-b}^0 \left| \partial_z \hat{f}(\xi, x_3) \right| |\hat{f}(\xi, x_3)| dx_3 \right)^{\frac{1}{2}}.$$

Hence, it follows from the Cauchy-Schwarz inequality and the fact that $s_1 + s_2 > 2$, $s_3 + s_4 > 2$ that

$$\begin{aligned} \|f\|_{L^\infty} &\leq \sup_{z \in (-b, 0)} \int_{\mathbb{R}_\xi^2} |\hat{f}(\xi, z)| d\xi \\ &\lesssim \int_{\mathbb{R}_\xi^2} \int_{-b}^0 |\hat{f}(\xi, z')| d\xi dz' + b \int_{\mathbb{R}_\xi^2} \left(\int_{-b}^0 \left| \partial_z \hat{f}(\xi, x_3) \right| |\hat{f}(\xi, x_3)| dx_3 \right)^{\frac{1}{2}} d\xi \\ &\lesssim \left(\int_{-b}^0 \int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{s_3+s_4} |\hat{f}(\xi, z)|^2 d\xi dz \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{s_1+s_2} \int_{-b}^0 \left| \partial_z \hat{f}(\xi, z) \right| |\hat{f}(\xi, z)| dz d\xi \right)^{\frac{1}{2}} \\ &\lesssim \|\Lambda^{s_3} f\|_{L^2}^{\frac{1}{2}} \|\Lambda^{s_4} f\|_{L^2}^{\frac{1}{2}} + \|\partial_z \Lambda^{s_1} f\|_{L^2}^{\frac{1}{2}} \|\Lambda^{s_2} f\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

To prove the trace estimate (A.6), since $s_1 + s_2 = 2s$, one may write

$$\begin{aligned} |f(\cdot, 0)|_{H^s}^2 &= |f(\cdot, z')|_{H^s}^2 + \int_{\mathbb{R}_y^2} \int_{z'}^0 2\partial_z \Lambda^s f(z, y) \Lambda^s f(z, y) dz dy \\ &= |f(\cdot, z')|_{H^s}^2 + \int_{\mathbb{R}_y^2} \int_{z'}^0 2\partial_z \Lambda^{s_1} f(z, y) \Lambda^{s_2} f(z, y) dz dy \end{aligned} \quad (\text{A.7})$$

Integrating the equality above with respect to $z' \in (-b, 0)$ and using the Cauchy-Schwarz inequality give the desired estimate. \square

Following similar arguments, we also have the following Poincaré inequality.

Lemma A.3. *It holds that*

$$\|f\| \lesssim |f|_0 + \|\partial_z f\|. \quad (\text{A.8})$$

Proof. The proof of the estimate (A.6) with $s_1 = s_2 = s = 0$ also leads to

$$\|f\| \lesssim |f|_0 + \|\partial_z f\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}}.$$

Then the Poincaré inequality (A.8) follows by Cauchy's inequality. \square

We also recall the classical product and commutator estimates in \mathbb{R}^2 :

Lemma A.4. *The followings hold.*

$$|\Lambda^s(fg)|_{L^2} \lesssim |f|_{L^\infty} |\Lambda^s g|_{L^2} + |g|_{L^\infty} |\Lambda^s f|_{L^2} \text{ for } s \geq 0, \quad (\text{A.9})$$

$$|[\Lambda^s, f] \nabla_y g|_{L^2} \lesssim |\nabla_y f|_{L^\infty} |\Lambda^s g|_{L^2} + |\nabla_y g|_{L^\infty} |\Lambda^s f|_{L^2} \text{ for } s \geq 0, \quad (\text{A.10})$$

$$|fg|_{\frac{1}{2}} \lesssim |f|_{1,\infty} |g|_{\frac{1}{2}} \text{ and } |fg|_{-\frac{1}{2}} \lesssim |f|_{1,\infty} |g|_{-\frac{1}{2}}. \quad (\text{A.11})$$

Proof. These estimates (A.9) and (A.10) are classical, see [28] for example. Note that

$$|fg|_0 \lesssim |f|_{L^\infty} |g|_0 \text{ and } |fg|_1 \lesssim |f|_{1,\infty} |g|_1,$$

the estimate (A.11) follows by the theory of interpolation and duality. \square

Note that Lemma A.1 also holds on \mathbb{R}^2 , while we also need the following for half regularities.

Lemma A.5. *For $|\alpha| + |\beta| = k \geq 0$:*

$$\left| Z^\alpha f Z^\beta g \right|_s \lesssim |f|_{\mathbb{X}^{k,s}} |g|_{\mathbb{Y}^{\frac{k}{2}+1}} + |f|_{\mathbb{Y}^{\frac{k}{2}+1}} |g|_{\mathbb{X}^{k,s}}, \quad s = -\frac{1}{2}, \frac{1}{2}. \quad (\text{A.12})$$

Proof. The estimate (A.12) follows by using (A.11) to control the product with the higher order derivative term in H^s and the lower order derivative term in $W^{1,\infty}$. \square

APPENDIX B. POISSON EXTENSION

We recall the extension η of h onto $\{z \leq 0\}$ defined by (1.6) with parameter $A > 0$ in the following form

$$\eta(y, z) = \left(1 + \frac{z}{b}\right) \zeta(y, z) \text{ with } \hat{\zeta}(\xi, z) = \exp(A|\xi|z) \hat{h}(\xi). \quad (\text{B.1})$$

We first verify that φ defined by (1.5) is a diffeomorphism.

Proposition B.1. *Assume that $h_0 \in H^s(\{z = 0\})$, $s > 5/2$ and $h_0 > -b$. Then there exists sufficiently small $A > 0$ such that*

$$\partial_z \varphi_0 \geq \frac{1}{2} \left(1 + \frac{1}{b} h_0\right) > 0 \text{ in } \Omega. \quad (\text{B.2})$$

Proof. Note that

$$\left| \partial_z \hat{\zeta}(\xi, z) \right| \leq \left| \partial_z \hat{\zeta}(\xi, z') \right| + \int_{z'}^z \left| \partial_z^2 \hat{\zeta}(\xi, x_3) \right| dx_3. \quad (\text{B.3})$$

Integrating the inequality above with respect to $z' \in (-b, 0)$, one can deduce

$$b \|\partial_z \zeta\|_{L^\infty} \leq b \sup_{z \in [-b, 0]} \int_{\mathbb{R}_\xi^2} \left| \partial_z \hat{\zeta}(\xi, z) \right| d\xi \leq \int_{-b}^0 \int_{\mathbb{R}_\xi^2} \left| \partial_z \hat{\zeta}(\xi, z) \right| dz + \int_{-b}^0 \int_{\mathbb{R}_\xi^2} \left| \partial_z^2 \hat{\zeta}(\xi, z) \right| dz.$$

For $s > 1$, it then follows from the Cauchy-Schwarz inequality and the definition (B.1) that

$$\begin{aligned} \int_{-b}^0 \int_{\mathbb{R}_\xi^2} \left| \partial_z \hat{\zeta}(\xi, z) \right| dz &\lesssim \left(\int_{-b}^0 \int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{2s} \left| \partial_z \hat{\zeta}(\xi, z) \right|^2 dz \right)^{\frac{1}{2}} \\ &= A \left(\int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{2s} |\xi|^2 \left| \hat{h}(\xi) \right|^2 \int_{-b}^0 \exp(2A|\xi|z) dz \right)^{\frac{1}{2}} \\ &= A \left(\int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{2s} |\xi|^2 \left| \hat{h}(\xi) \right|^2 \frac{1 - \exp(2A|\xi|z)}{2A|\xi|} dz \right)^{\frac{1}{2}} \lesssim A^{\frac{1}{2}} |h|_{s+\frac{1}{2}} \end{aligned}$$

and

$$\int_{-b}^0 \int_{\mathbb{R}_\xi^2} \left| \partial_z \hat{\zeta}(\xi, z) \right| dz \lesssim \left(\int_{-b}^0 \int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{2s} \left| \partial_z \hat{\zeta}(\xi, z) \right|^2 dz \right)^{\frac{1}{2}}$$

$$= A^2 \left(\int_{\mathbb{R}_\xi^2} (1 + |\xi|)^{2s} |\xi|^4 |\hat{h}(\xi)|^2 \int_{-b}^0 \exp(2A|\xi|z) dz \right)^{\frac{1}{2}} \lesssim A^{\frac{3}{2}} |h|_{s+\frac{3}{2}}.$$

We thus deduce

$$\|\partial_z \zeta\|_{L^\infty} \lesssim A^{\frac{1}{2}} |h|_{s+\frac{3}{2}}. \quad (\text{B.4})$$

Now we prove (B.2). It follows from the definitions (1.5) and (B.1) that

$$\partial_z \varphi_0 = 1 + \frac{1}{b} \zeta_0 + \left(1 + \frac{1}{b}\right) \partial_z \zeta_0. \quad (\text{B.5})$$

By (B.4), this yields that for $s > \frac{5}{2}$,

$$\begin{aligned} \partial_z \varphi_0 &\geq 1 + \frac{1}{b} h_0 + \frac{1}{b} (\zeta_0 - h_0) + \left(1 + \frac{1}{b}\right) \partial_z \zeta_0 \\ &\geq 1 + \frac{1}{b} h_0 - \left(2 + \frac{1}{b}\right) \|\partial_z \zeta_0\|_{L^\infty} \geq 1 + \frac{1}{b} h_0 - A^{\frac{1}{2}} \left(2 + \frac{1}{b}\right) |h_0|_{\frac{5}{2}} \geq \frac{1}{2} \left(1 + \frac{1}{b} h_0\right) \end{aligned}$$

if A has been chosen sufficiently small. \square

We also have the following well-known estimates for η .

Lemma B.2. *For $s \in \mathbb{R}$:*

$$\|\eta\|_{H^s} \lesssim |h|_{s-\frac{1}{2}}. \quad (\text{B.6})$$

For $k \in \mathbb{N}$:

$$\|\eta\|_{W^{k,\infty}} \lesssim |h|_{k,\infty}. \quad (\text{B.7})$$

Proof. One deduces in the same way as Proposition 3.1 in [33] that

$$\|\nabla \zeta\|_{H^s} \lesssim |h|_{s+\frac{1}{2}}, \quad s \in \mathbb{R} \text{ and } \|\eta\|_{W^{k,\infty}} \lesssim |h|_{k,\infty}, \quad k \in \mathbb{N}.$$

Then the estimates (B.6)–(B.7) follow by noting that $\eta = (1 + \frac{z}{b})\zeta$ for $z \in [-b, 0]$. \square

APPENDIX C. SOME GEOMETRIC ESTIMATES

We recall that the control of quantities like $\int_\Omega |\nabla^\varphi f|^2 d\mathcal{V}_t$ yields a control of the standard H^1 norm of f .

Lemma C.1. *Assume that $\partial_z \varphi \geq c_0$ and $\|\nabla \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then*

$$\|\nabla f\|^2 \leq \Lambda_0 \int_\Omega |\nabla^\varphi f|^2 d\mathcal{V}_t. \quad (\text{C.1})$$

Proof. We refer to Lemma 2.8 in [33]. \square

We also need the Korn type inequality to control the energy dissipation term.

Lemma C.2. *Assume that $\partial_z \varphi \geq c_0$ and $\|\nabla \varphi\|_{L^\infty} + \|\nabla^2 \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then*

$$\|\nabla v\|^2 \leq \Lambda_0 \left(\int_\Omega |S^\varphi v|^2 d\mathcal{V}_t + \|v\|^2 \right). \quad (\text{C.2})$$

Proof. We refer to Proposition 2.9 in [33]. \square

Finally, we will also need the following $H^{-1/2}$ boundary estimates for functions satisfying $v \in L^2$ and $\nabla^\varphi \cdot v \in L^2$.

Lemma C.3. *If $\|\nabla \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then*

$$|v \cdot \mathbf{N}|_{-\frac{1}{2}} \leq \Lambda_0 (\|v\| + \|\nabla^\varphi \cdot v\|). \quad (\text{C.3})$$

Proof. We adapt the proof of Lemma 3.3 in [19]. We will only prove the result on $\{z = 0\}$. Let $\psi \in H^{1/2}$, and let $\tilde{\psi} \in H^1(\Omega)$ be a bounded extension. Then

$$\begin{aligned} \int_{z=0} \psi v \cdot \mathbf{N} &= \int_{\Omega} \nabla^{\varphi} \cdot (\tilde{\psi} v) d\mathcal{V}_t = \int_{\Omega} \left(\nabla^{\varphi} \tilde{\psi} \cdot v + \tilde{\psi} \nabla^{\varphi} \cdot v \right) d\mathcal{V}_t \\ &\leq \Lambda_0 \left(\|\tilde{\psi}\| \|\nabla^{\varphi} \cdot v\| + \|\nabla^{\varphi} \tilde{\psi}\| \|v\| \right) \leq \Lambda_0 |\psi|_{\frac{1}{2}} (\|v\| + \|\nabla^{\varphi} \cdot v\|). \end{aligned}$$

Then the estimate (C.3) follows from this inequality above by taking the supremum over all ψ so that $|\psi|_{\frac{1}{2}} \leq 1$. \square

REFERENCES

- [1] T. Alazard, J. M. Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. Preprint (2013), arXiv:1305.4090.
- [2] S. Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. (French. English summary) [Existence of rarefaction waves for multidimensional hyperbolic quasilinear systems] *Comm. Partial Differential Equations* **14** (1989), no. 2, 173–230.
- [3] D. M. Ambrose, N. Masmoudi. The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.* **58** (2005), no. 10, 1287–1315.
- [4] D. M. Ambrose, N. Masmoudi. The zero surface tension limit of three-dimensional water waves. *Indiana Univ. Math. J.* **58** (2009), no. 2, 479–521.
- [5] K. Asano. Zero-viscosity limit of the incompressible Navier-Stokes equation. II. Mathematical analysis of fluid and plasma dynamics, I (Kyoto, 1986). *Sûrikaiseikikenkyûsho, Kôkyûroku No. 656*, (1988), 105–128.
- [6] J. Beale. The initial value problem for the Navier-Stokes equations with a free surface. *Comm. Pure Appl. Math.* **34** (1981), no. 3, 359–392.
- [7] J. Beale. Large-time regularity of viscous surface waves. *Arch. Ration. Mech. Anal.* **84** (1983/84), no. 4, 307–352.
- [8] H. Beirão da Veiga, F. Crispo. Concerning the $W^{k,p}$ -inviscid limit for 3-D flows under a slip boundary condition. *J. Math. Fluid Mech.* **13** (2011), no. 1, 117–135.
- [9] K. Beyler, M. Günther, M. On the Cauchy problem for a capillary drop. I. Irrotational motion. *Math. Methods Appl. Sci.* **21** (1998), no. 12, 1149–1183.
- [10] P. Constantin. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. *Comm. Math. Phys.* **104** (1986), no. 2, 311–326.
- [11] D. Coutand, S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.* **20** (2007), no. 3, 829–930.
- [12] R. J. DiPerna, A. J. Majda. Concentrations in regularizations for 2-D incompressible flow. *Comm. Pure Appl. Math.* **40** (1987), no. 3, 301–345.
- [13] R. J. DiPerna, A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.* **108** (1987), no. 4, 667–689.
- [14] T. Elgindi, D. Lee. Uniform regularity for free-boundary Navier-Stokes equations with surface tension. Preprint (2014), arXiv:1403.0980.
- [15] D. Gérard-Varet, E. Dormy. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.* **23** (2010), no. 2, 591–609.
- [16] P. Germain, N. Masmoudi, J. Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)* **175** (2012), no. 2, 691–754.
- [17] P. Germain, N. Masmoudi, J. Shatah. Global solutions for capillary waves equation. *Comm. Pure Appl. Math.* **68** (2015), no. 4, 625–687.
- [18] Y. Guo, T. Nguyen. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. Preprint (2014), arXiv:1411.6984.
- [19] Y. Guo, I. Tice. Local well-posedness of the viscous surface wave problem without surface tension. *Anal. PDE* **6** (2013), no. 2, 287–369.
- [20] Y. Guo, I. Tice. Almost exponential decay of periodic viscous surface waves without surface tension. *Arch. Ration. Mech. Anal.* **207** (2013), no. 2, 459–531.
- [21] Y. Guo, I. Tice. Decay of viscous surface waves without surface tension in horizontally infinite domains. *Anal. PDE* **6** (2013), no. 6, 1429–1533.
- [22] Y. Hataya. Decaying solution of a Navier-Stokes flow without surface tension. *J. Math. Kyoto Univ.* **49** (2009), no. 4, 691–717.
- [23] D. Iftimie, G. Planas. Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions. *Nonlinearity* **19** (2006), no. 4, 899–918.
- [24] D. Iftimie, F. Sueur. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Anal.* **199** (2011), no. 1, 145–175.

- [25] A. Ionescu, F. Pusateri. Global solutions for the gravity water waves system in 2D. *Invent. Math.* **199** (2015), no. 3, 653–804.
- [26] A. Ionescu, F. Pusateri. Global regularity for 2D water waves with surface tension. Preprint (2014), arXiv:1408.4428.
- [27] T. Kato. Nonstationary flows of viscous and ideal fluids in \mathbb{R}^3 . *J. Funct. Anal.* **9** (1972), no. 3, 296–305.
- [28] T. Kato, G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.* **41** (1988), no. 7, 891–907.
- [29] H. Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)* **162** (2005), no. 1, 109–194.
- [30] Y. Maekawa. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* **67** (2014), no. 7, 1045–1128.
- [31] N. Masmoudi. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.* **270** (2007), no. 3, 777–788.
- [32] N. Masmoudi, F. Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.* **203** (2012), no. 2, 529–575.
- [33] N. Masmoudi, F. Rousset. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. Preprint (2012), arXiv:1202.0657.
- [34] V. I. Nalimov. The Cauchy-Poisson problem. (Russian) *Dinamika Splošn. Sredy Vyp. 18 Dinamika Židkost. so Svobod. Granicami.* **254** (1974), 104–210.
- [35] M. Sammartino, R. E. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. II. *Comm. Math. Phys.* **192** (1998), no. 2, 433–461, 463–491.
- [36] J. Shatah, C. Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.* **61** (2008), no. 5, 698–744.
- [37] H. S. G. Swann. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in \mathbb{R}^3 . *Trans. Amer. Math. Soc.* **157** (1971), 373–397.
- [38] Z. Tan, Y. J. Wang. Zero surface tension limit of viscous surface waves. *Comm. Math. Phys.* **328** (2014), no. 2, 733–807.
- [39] A. Tani. Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. *Arch. Ration. Mech. Anal.* **133** (1996), no. 4, 299–331.
- [40] A. Tani, N. Tanaka. Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. *Arch. Ration. Mech. Anal.* **130** (1995), no. 4, 303–314.
- [41] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* **130** (1997), no. 1, 39–72.
- [42] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.* **12** (1999), no. 2, 445–495.
- [43] S. Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.* **177** (2009), no. 1, 45–135.
- [44] S. Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.* **184** (2011), no. 1, 125–220.
- [45] Y. Xiao, Z. Xin. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. *Comm. Pure Appl. Math.* **60** (2007), no. 7, 1027–1055.
- [46] Y. Xiao, Z. Xin. On the inviscid limit of the 3D Navier-Stokes equations with generalized Navier-slip boundary conditions. *Commun. Math. Stat.* **1** (2013), no. 3, 259–279.
- [47] P. Zhang, Z. Zhang. On the free boundary problem of three-dimensional incompressible Euler equations. *Comm. Pure Appl. Math.* **61** (2008), no. 7, 877–940.

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, FUJIAN 361005, CHINA

AND

THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, NT, HONG KONG

E-mail address, Y. J. Wang: yanjin_wang@xmu.edu.cn

THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, NT, HONG KONG

E-mail address, Z. P. Xin: zpxin@ims.cuhk.edu.hk