Stability analysis for the incompressible Navier-Stokes equations with Navier boundary conditions

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Abstract

This paper concerns the instability and stability of the trivial steady states of the incompressible Navier-Stokes equations with Navier-slip boundary conditions in a slab domain in dimension two. The main results show that the stability (or instability) of this constant equilibrium depends crucially on whether the boundaries dissipate energy and the strengthen of the viscosity and slip length. It is shown that in the case that when all the boundaries are dissipative, then nonlinear asymptotic stability holds true, otherwise, there is a sharp critical viscosity, which distinguishes the nonlinear stability from instability.

Keywords: stability and instability, Navier-Stokes equations, Navier boundary conditions, critical viscosity.

AMS Subject Classification: 76N10, 35Q30, 35R35.

1 Formulation of the problem

As was proposed by L. Landau and E. Lifshitz in 1959 that flows occurring in the nature must not only obey the equations of fluid dynamics, but also be stable. The stability and instability problem is one of the most important topics in the studies and applications of Navier-Stokes equations.

There are several kinds of concept on stability, we refer the readers to [7, 8]. The most common concept is the Rayleigh-Taylor stability and instability due to heavier fluid on the upper forced by gravity, called RT stability and RT instability. RT stability and RT instability have been got wide studies, see [9–12, 15, 16, 26] and references therein.

It should be emphasized that, up to now, the researches for the stability problems are most subject to the no-slip boundary conditions, we refer, for instance, to [9–12] and the references

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therein. However, many other boundary conditions are more suitable to applications and realities, among which are the Navier-slip boundary conditions that we are going to consider in the present paper. In fact, to describe many phenomena which can be observed in nature, the slip boundary conditions are more appropriated. For instance, hurricanes and tornadoes, do slip along the ground, lose energy as they slip and do not penetrate the ground (see [3]).

Precisely, we will investigate the following equations in the 2 dimensional slab domain $\Omega = \mathbb{R} \times (0,1)$:

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \mu \Delta \mathbf{v} = 0, \text{ in } \mathbb{R} \times (0,1), t \geq 0, \\ \text{div}\mathbf{v} = 0, \text{ in } \mathbb{R} \times (0,1), t \geq 0; \end{cases} \tag{1.1}$$

with the Navier boundary conditions:

$$\mathbf{v} \cdot \mathbf{n} = 0, \text{ on } \{y = 0, 1\}, \tag{1.2}$$

$$[(-p\mathbb{I} + \mu(\nabla \mathbf{v} + \nabla^T \mathbf{v})) \cdot \mathbf{n}] \cdot \tau = k_1 \mathbf{v} \cdot \tau, \text{ on } \{y = 1\}, \tag{1.3}$$

$$[(-p\mathbb{I} + \mu(\nabla \mathbf{v} + \nabla^T \mathbf{v})) \cdot \mathbf{n}] \cdot \tau = k_0 \mathbf{v} \cdot \tau, \text{ on } \{y = 0\}. \tag{1.4}$$

where the superscript T means matrix transposition, $\mathbb{I}$ is the 2×2 identity matrix, $\mathbf{v}(x,y;t) = (v^1(x,y;t), v^2(x,y;t))$ and $p(x,y;t)$ are the velocity and pressure of the flow respectively, $\mathbf{n}$ is the outward unit normal vector and τ is the corresponding tangent vector of the boundary. In our consideration, $\mathbf{n} = (0,1)$ on $\{y = 1\}$ and $\mathbf{n} = (0,-1)$ on $\{y = 0\}$, while $\tau = (1,0)$ on both $\{y = 1\}$ and $\{y = 0\}$.

The viscosity μ is supposed to be strictly positive and the constants $|k_0|$, $|k_1|$ are the slip lengthes. k_0 and k_1 could be the friction coefficients, or the permeability of permeable materials, measurements of the roughness of rough boundaries, etc. In this paper, the coefficients k_0, k_1 do not have defined sign (as in Serrin [22], p.240). As it is well known, the case $k_i \leq 0$ $(i = 0, 1)$ correspond to the most studied case in the literatures, slip with friction. But, in this paper, we will also consider the cases where some of k_0 and k_1 are positive, the case which the boundary walls accelerate the fluid.

Such kinds of boundary conditions, called Navier boundary conditions or Navier-slip boundary conditions, were first introduced by Navier [20] and the first pioneer paper on the mathematical rigorous analysis of the Navier-Stokes equation with Navier boundary conditions should be due to Solonnikov and Ščadilov [23] for the linearized stationary equations, while the existence of the weak solutions and regularity for the nonlinear case are obtained by H. B. da Veiga [25] on half-space. Recently, [2] and the references therein give some more specified results on existence and regularity of the solutions for various domains. In addition, for results on the vanishing viscosity limit for the evolutionary case, see [29, 30] and the references given by these authors. In 2016, Hailiang Li and Xingwei Zhang in [19] obtained the nonlinear stability for Couette flow of three dimensional compressible Navier-Stokes equations with Navier boundary conditions on the lower flat boundary and moving condition on the upper flat boundary in which the friction coefficient on the lower boundary is restricted to be negative. For more physical applications and numerical analysis details, see [1, 4–6, 13, 14, 17, 21, 22].

Our main interest here is to study the linear and nonlinear stability and instability of the steady state solution, $\mathbf{v}_s(x,y) = (0,0)$, $p_s(x,y) = p_s$, where p_s is a constant, of this boundary value problem. To our knowledge, there are few literatures on such stability and instability problems, especially for the cases where k_0 and k_1 do not have defined sign. Our results show that the stability (or instability) of this equilibrium depends crucially on whether the boundaries dissipate energy and the strengthen of the viscosity and slip length. It is shown that in the case that all the boundaries are dissipative, then nonlinear asymptotic stability holds true. Otherwise, there is a sharp critical viscosity, which distinguishes the nonlinear stability from instability.

Denote the perturbation by

$$\mathbf{u} = \mathbf{v} - \mathbf{0},\ q = p - p_s.$$

Then $(\mathbf{u}, q)$ satisfies the perturbed equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q - \mu \Delta \mathbf{u} = 0, \\ \mathrm{div} \mathbf{u} = 0. \end{cases} \tag{1.5}$$

The boundary conditions, (1.2)-(1.4), are rewritten as follows:

$$u^2(x, 0) = u^2(x, 1) = 0,\ x \in \mathbb{R}, \tag{1.6}$$

$$\partial_y u^1(x, 1) = \frac{k_1}{\mu} u^1(x, 1),\ x \in \mathbb{R}, \tag{1.7}$$

$$\partial_y u^1(x, 0) = -\frac{k_0}{\mu} u^1(x, 0),\ x \in \mathbb{R}. \tag{1.8}$$

Linearizing (1.5) around the steady state $(\mathbf{0}, p_s)$ yields the linearized equations:

$$\begin{cases} \partial_t \mathbf{u} + \nabla q - \mu \Delta \mathbf{u} = 0, \\ \mathrm{div} \mathbf{u} = 0. \end{cases} \tag{1.9}$$

For convenience, we will use the following notations throughout this paper.

$$\Omega := \mathbb{R} \times (0, 1),\ L^p := L^p(\Omega),\ H^k := W^{2,k}(\Omega),\ \int := \int_\Omega,$$

$$H^1_\sigma := \{\mathbf{u} \in H^1 | \mathrm{div} \mathbf{u} = 0, u^2 = 0 \text{ on } \mathbb{R} \times \{0, 1\}\}.$$

$H^1_0(0, 1)$ and $H^2(0, 1)$ will be written as H^1_0 and H^2 respectively. In addition, a product space $(X)^2$ of vector functions is still denoted by X, for example, the vector function $\mathbf{u} \in (H^1)^2$ is denoted by $\mathbf{u} \in H^1$. $\mathbb{N}$ is the set of nonnegative integers.

First, we study the linear instability of the steady state $(\mathbf{0}, p_s)$. To this end, one looks for a growing mode solution to the linearized problem (1.6)-(1.9) in the form

$$\mathbf{v}(x, y; t) = \mathbf{w}(x, y) e^{\lambda t},\ q(x, y; t) = \tilde{p}(x, y) e^{\lambda t} \tag{1.10}$$

for some $\lambda > 0$. Putting this ansatz into (1.6)-(1.9) yields

$$\begin{cases} \lambda \mathbf{w} + \nabla \tilde{p} - \mu \Delta \mathbf{w} = 0,\ \text{in } \Omega, \\ \mathrm{div} \mathbf{w} = 0,\ \text{in } \Omega, \end{cases} \tag{1.11}$$

and the boundary conditions

$$w^2(x,0) = w^2(x,1) = 0,\ x \in \mathbb{R}; \tag{1.12}$$

$$\partial_y w^1(x,1) = \frac{k_1}{\mu} w^1(x,1),\ x \in \mathbb{R}; \tag{1.13}$$

$$\partial_y w^1(x,0) = -\frac{k_0}{\mu} w^1(x,0),\ x \in \mathbb{R}. \tag{1.14}$$

We will solve the problem, (1.11)-(1.14), by the standard normal mode analysis. That is, one can rewrite $\mathbf{w}$ and $\tilde{p}$ in terms of the new unknowns $\phi, \psi, \pi : (0,1) \to \mathbb{R}$ for each frequency ξ as:

$$w^1(x,y) = -i\phi(y)e^{ix\xi}, w^2(x,y) = \psi(y)e^{ix\xi}, \tilde{p}(x,y) = \pi(y)e^{ix\xi}. \tag{1.15}$$

For each fixed $\xi \neq 0$, this leads to the following system of ODEs

$$\begin{cases} -\lambda\phi + \xi\pi - \mu\xi^2\phi + \mu\phi'' = 0, \\ \lambda\psi + \pi' + \mu\xi^2\psi - \mu\psi'' = 0, \\ \xi\phi + \psi' = 0, \end{cases} \tag{1.16}$$

with boundary conditions

$$\psi(0) = \psi(1) = 0, \tag{1.17}$$

$$\phi'(1) = \frac{k_1}{\mu}\phi(1), \tag{1.18}$$

$$\phi'(0) = -\frac{k_0}{\mu}\phi(0). \tag{1.19}$$

Eliminating π from the second equation of (1.16) gives a fourth order ODE for ψ

$$-\lambda(\xi^2\psi - \psi'') = \mu(\psi^{(4)} - 2\xi^2\psi'' + \xi^4\psi), y \in (0,1) \tag{1.20}$$

with the boundary conditions

$$\psi(1) = \psi(0) = 0, \tag{1.21}$$

$$\psi''(1) = \frac{k_1}{\mu}\psi'(1), \tag{1.22}$$

$$\psi''(0) = -\frac{k_0}{\mu}\psi'(0). \tag{1.23}$$

If, for some frequency ξ, there exists a solution to (1.20)-(1.23) with positive λ, then the above steady state is said to be linearly instable. Since problem (1.20)-(1.23) has a natural variational structure, one may reach such an aim by solving the minimization problem

$$-\lambda = \inf_{H_0^1 \cap H^2} \frac{E(\psi)}{J(\psi)}, \tag{1.24}$$

where

$$E(\psi) = \frac{\mu}{2}\int_0^1 \left[(\psi'')^2 + 2\xi^2(\psi')^2 + \xi^4\psi^2\right] - \frac{k_1}{2}(\psi'(1))^2 - \frac{k_0}{2}(\psi'(0))^2 \tag{1.25}$$

and

$$J(\psi) = \frac{1}{2}\int_0^1 \left[\xi^2\psi^2 + (\psi')^2\right] \tag{1.26}$$

are both well-defined on the space $H_0^1 \cap H^2$.

In order to get a positive $\lambda(=\lambda(\xi^2))$ in the variational problem (1.24), we observe that if

$$\frac{\mu}{2}\int_0^1 (\psi'')^2 - \frac{k_1}{2}(\psi'(1))^2 - \frac{k_0}{2}(\psi'(0))^2$$

is negative for small viscosity μ, then $E(\psi)$ is negative for small ξ. This is a key observation which motivates us to define the critical viscosity by

$$\mu_c = \sup_{H_0^1\cap H^2, \int_0^1(\psi'')^2\neq 0} \frac{k_1(\psi'(1))^2 + k_0(\psi'(0))^2}{\int_0^1(\psi'')^2}. \tag{1.27}$$

It will be shown in next section that explicit values of the critical viscosity are

$$\mu_c = \begin{cases} 0, & k_1 \leq 0 \text{ and } k_0 \leq 0, \\ \dfrac{k}{6}, & k_1 = k_0 := k > 0, \\ \dfrac{(k_1+k_0) + \sqrt{k_1^2 + k_0^2 - k_1 k_0}}{6}, & \text{otherwise.} \end{cases} \tag{1.28}$$

Moreover, it will also be shown that this value of μ_c is a sharp threshold of the stability and instability. Precisely, we have the following main results.

The first result is on the linear instability.

Theorem 1.1. (Linear instability) *The steady state $(\mathbf{0}, p_s)$ is linearly unstable in H^k, for any $k \in \mathbb{N}$, in the sense that there are exponentially growing mode solutions to the linearized perturbed problem (1.6)-(1.9) in H^k if and only if $\mu_c > 0$ and $\mu \in (0, \mu_c)$.*

The nonlinear instability is stated in the following theorem.

Theorem 1.2. (Nonlinear instability) *The steady state $(\mathbf{0}, p_s)$ is nonlinearly unstable in L^2 norm if and only if $\mu_c > 0$ and $\mu \in (0, \mu_c)$. More precisely, we have*

(i) Assume $\mu_c > 0$ and $\mu \in (0, \mu_c)$. Then there exists a constant $\varepsilon > 0$ and a function $\mathbf{u}_0$, $\|\mathbf{u}_0\|_{H^2} = 1$, such that for any δ: $0 < \delta < \varepsilon$, there exists a unique global strong solution $(\mathbf{u}^\delta, q)$ to the nonlinear perturbed problem (1.5)-(1.8) with the initial data $\mathbf{u}_0^\delta := \delta\mathbf{u}_0$, such that $\mathbf{u}^\delta \in C([0,T], H^2)$, $\nabla q \in L^2$ and

$$\|\mathbf{u}^\delta(T^\delta)\|_{L^2} \geq \varepsilon. \tag{1.29}$$

Here the escape time is $T^\delta := \frac{1}{\lambda_*}\ln(\varepsilon/\delta) \in (0, T)$, *where* λ_* *is defined in (4.54).*

(ii) Assume $\mu_c \geq 0$ *and* $\mu \in [\mu_c, +\infty)$. *Then there exists a unique global strong solution* $(\mathbf{u}, q)$ *to the nonlinear perturbed problem (1.5)-(1.8) with* $\mathbf{u} \in C([0,T], H^2)$ *and* $\nabla q \in L^2$, *such that*

$$\|\mathbf{u}\|_{L^2} \leq \|\mathbf{u}_0\|_{L^2} \tag{1.30}$$

where $\mathbf{u}_0$ *is the initial data of problem (1.5)-(1.8).*

Finally, we have the following stability theorem for $\mu > \mu_c$, where $\mu_c \geq 0$.

Theorem 1.3. (Nonlinear asymptotic stability) *The steady states* $(\mathbf{0}, p_s)$ *is nonlinear asymptotically stable globally provided that* $\mu > \mu_c \geq 0$, *that is, let* $(\mathbf{u}, q)$ *be a solution of nonlinear problem (1.5)-(1.8) with initial data* $\mathbf{u}_0$, *then the followings are true.*

(i) For general initial data $\mathbf{u}_0$, *there exist some constants* $\alpha > 0, C > 0$, *such that*

$$\sup_{0<t<+\infty} \|\mathbf{u}(t)\|_{H^2} \leq C\|\mathbf{u}_0\|_{H^2}, \lim_{t\to+\infty} \|\mathbf{u}(t)\|_{H^2} = 0, \text{ and } \|\mathbf{u}(t)\|_{H^1} \leq Ce^{-\alpha t}, \tag{1.31}$$

where $C = C(\mu, k_0, k_1, \|\mathbf{u}_0\|_{H^2})$ *is increasing with respect to* $\|\mathbf{u}_0\|_{H^2}$.

(ii) Moreover, if the initial data is small, then we have H^2*-norm decay estimates. That is, there exists a positive constant* β *such that*

$$\|\mathbf{u}(t)\|_{H^2} \leq Ce^{-\beta t}, \tag{1.32}$$

provided that the initial data $\mathbf{u}_0$ *satisfies* $\|\mathbf{u}_0\|_{H^2} \leq \bar{\sigma}$, *for some constant* $\bar{\sigma} > 0$.

Remark 1.4. *The focus of this paper is to investigate the effect of the boundary conditions on the stability of the trivial steady state of incompressible viscous fluid. Indeed, it is well known that in the case of homogeneous Dirichlet conditions (i.e. non-slip conditions), this steady state is asymptotically nonlinearly stable for any incompressible fluids with positive viscosity. However, in the present case (i.e. Navier-slip conditions), our main results here show that the stability of this steady state depends on the balance among the energies generated in the body and on the boundaries. In fact, we have given a sharp criteria in terms of the viscosity and the coefficients in the Navier boundary conditions. As we can see in (4.1) that the kinetic energy* $\mathcal{E}(t) := \frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega)}$ *satisfies the basic energy law*

$$\mathcal{E}(t) = \mathcal{E}(0) - \mu \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 dxdyds + \sum_{i=0}^{1} k_i \int_0^t \int_{\mathbb{R}} |u^1(x, i, s)|^2 dxds,$$

which and our conclusions imply the following facts

(1) If both of the boundaries dissipate energy, that is, $k_0 \leq 0$ *and* $k_1 \leq 0$, *then this steady state is nonlinearly stable for any positive viscosity.*

(2) If one of the boundaries absorbs energy, that is, $k_0 > 0$ *or* $k_1 > 0$, *then this steady state is linearly and nonlinearly stable if and only if the viscosity is large enough (for example,* $\mu > \mu_c$, *see above theorems).*

The rest of this paper is arranged as follows. First, we analyse in detail the problem (1.27) to determine the exact value of the critical viscosity. In Section 3 and Section 4, we prove the instability part of Theorem 1.1 and Theorem 1.2 respectively. The stability part of Theorem 1.1, Theorem 1.2 and Theorem 1.3 will be proved in Section 5.

2 The critical viscosity

In order to obtain the value of the critical viscosity, we consider the equivalent variational problem of (1.27) as

$$\sup_{\psi\in\mathcal{Y}} Z(\psi), \tag{2.1}$$

where

$$Z(\psi) = \frac{k_1}{2}(\psi'(1))^2 + \frac{k_0}{2}(\psi'(0))^2, \tag{2.2}$$

$$\mathcal{Y} = \left\{ \psi \in H_0^1 \cap H^2 \middle| \frac{1}{2}\int_0^1 (\psi'')^2 = 1 \right\}. \tag{2.3}$$

In what follows, we shall use the fact that, for any $f \in H_0^1 \cap H^2$, there holds

$$\|f'\|_{L^2}^2 \le \|f''\|_{L^2}^2, \tag{2.4}$$

where one can use Poincaré inequality and the fact $f \in H_0^1$ to prove (2.4).

Thus, for any $\psi \in \mathcal{Y}$,

$$\begin{aligned}
|Z(\psi)| =& \frac{1}{2}\left|\int_0^1 \left[((k_1+k_0)y - k_0)(\psi')^2\right]_y' dy\right| \\
=& \left|\frac{(k_1+k_0)}{2}\int_0^1 (\psi')^2 dy + \int_0^1 [(k_1+k_0)y - k_0]\psi'\psi'' dy\right| \\
\le& \frac{1}{2}\int_0^1 \left[C_1(\psi')^2 + C_2(\psi'')^2\right] dy \\
\le& \frac{1}{2}\int_0^1 C_3(\psi'')^2 dy = C_3,
\end{aligned} \tag{2.5}$$

for some positive constants C_1, C_2 and C_3, depending only on k_1 and k_0.

This shows that $\sup_{\psi\in\mathcal{Y}} Z(\psi)$ exists and is finite.

Set $\mu_c := \sup_{\psi\in\mathcal{Y}} Z(\psi)$. The exact values of μ_c will be given in different cases in the following two propositions.

Proposition 2.1. *Let* $\mu_c \in (-\infty, +\infty)$ *be defined in (1.27). Then* $\mu_c > 0$ *if* $\max\{k_0, k_1\} > 0$.

Proof. If both k_0 and k_1 are non-positive, then, clearly,

$$\mu_c = \sup_{\psi\in\mathcal{Y}} \left[\frac{k_1}{2}(\psi(1))^2 + \frac{k_0}{2}(\psi(0))^2\right] \le 0. \tag{2.6}$$

On the other hand, for a suitable choice of α,

$$\psi_1(x) := \begin{cases} 0, & x \in [0, \frac{1}{4}]; \\ \alpha \exp\{\frac{1}{(x-\frac{1}{2})^2 - \frac{1}{16}}\}, & x \in (\frac{1}{4}, \frac{3}{4}); \\ 0, & x \in [\frac{3}{4}, 1] \end{cases} \tag{2.7}$$

belongs to $\mathcal{Y}$. Moreover, $\psi_1 \in C_0^\infty([0,1])$ and $\psi_1'(1) = \psi_1'(0) = 0$, which implies $Z(\psi_1) = 0$. This, together with (2.6), implies that $\mu_c = 0$.

In the other case, without loss of generality, we suppose that $k_0 > 0$ and define

$$\psi_2(x) := \begin{cases} -\frac{2}{3}x(x - \frac{3\sqrt{2}-3}{2}), & x \in [0, \frac{3\sqrt{2}}{8}]; \\ \frac{2}{3}(x - \frac{3}{4})^2, & x \in (\frac{3\sqrt{2}}{8}, \frac{3}{4}); \\ 0, & x \in [\frac{3}{4}, 1]. \end{cases} \tag{2.8}$$

Then, one can verify that $\psi_2 \in \mathcal{Y}$ and $Z(\psi_2) > 0$, which means $\mu_c > 0$ in this case. □

To find the exact value of μ_c in the case of $\mu_c > 0$, we need the following Proposition.

Proposition 2.2. *Let μ_c be defined as in (1.27) and suppose that $\max\{k_0, k_1\} > 0$. Then*

$$\mu_c = \begin{cases} \dfrac{k}{6}, & k_1 = k_0 := k > 0, \\ \dfrac{(k_1 + k_0) + \sqrt{k_1^2 + k_0^2 - k_1 k_0}}{6}, & \text{otherwise.} \end{cases}$$

Proof. Let $\{\psi_n\}_{n=1}^\infty \in \mathcal{Y}$ be a maximizing sequence. It follows from (2.4) that

$$\|\psi_n\|_{H_0^1 \cap H^2} = \|\psi_n'\|_{L^2} + \|\psi_n''\|_{L^2} \le C_4$$

for some constant $C_4 > 0$. Therefore $\{\psi_n\}$ is bounded in $H_0^1 \cap H^2$. Hence, up to a subsequence if necessary, $\psi_n \rightharpoonup \psi$ weakly in H^2 and $\psi_n \to \psi$ strongly in H_0^1. This implies that

$$\begin{aligned} Z(\psi) &= \frac{k_1 + k_0}{2} \int_0^1 (\psi')^2 + \int_0^1 [(k_1 + k_0)y - k_0]\psi'\psi'' \\ &= \frac{k_1 + k_0}{2} \lim_{n\to\infty} \int_0^1 (\psi_n')^2 + \lim_{n\to\infty} \int_0^1 [(k_1 + k_0)y - k_0]\psi_n'\psi_n'' \\ &= \lim_{n\to\infty} Z(\psi_n) = \mu_c, \end{aligned} \tag{2.9}$$

and

$$\frac{1}{2}\|\psi''\|_{L^2}^2 \le \lim_{n\to\infty} \frac{1}{2}\|\psi_n''\|_{L^2}^2 = 1. \tag{2.10}$$

Now we claim that $\|\psi''\|_{L^2}^2 = 2$, i.e. $\psi \in \mathcal{Y}$.

Otherwise, one may assume that $\|\psi''\|_{L^2}^2 := 2r^2 < 2$, for some constant $0 \le r < 1$. Notice that if $r = 0$, then $\psi = 0$, which implies that $\mu_c = 0$. Thus, $0 < r < 1$ and $\tilde{\psi} = \psi/r \in \mathcal{Y}$. The definition of μ_c and (2.9) lead to

$$\mu_c \ge Z(\tilde{\psi}) = Z(\psi)/r^2 = \mu_c/r^2 > \mu_c,$$

which is a contradiction. Thus, $\psi \in \mathcal{Y}$ is a maximizer of the variational problem (2.1).

In what follows, we will find the exact expression of the unique maximizer ψ and then obtain the exact value of μ_c.

For any $\psi_0 \in H_0^1 \cap H^2$, $s, r \in \mathbb{R}$, define

$$I(s,r) = \frac{1}{2}\int_0^1 (\psi'' + s\psi_0'' + r\psi'')^2. \tag{2.11}$$

Notice that $I(s,r)$ is smooth and that

$$I(0,0) = 1, \tag{2.12}$$

$$\partial_s I(s,r)\big|_{(0,0)} = \int_0^1 (\psi'' + s\psi_0'' + r\psi'')\psi_0''\Big|_{(0,0)} = \int_0^1 \psi''\psi_0'', \tag{2.13}$$

$$\partial_r I(s,r)\big|_{(0,0)} = \int_0^1 (\psi'' + s\psi_0'' + r\psi'')\psi''\Big|_{(0,0)} = \int_0^1 (\psi'')^2 = 2 \neq 0. \tag{2.14}$$

By implicit function theorem, there exists a smooth function $r = r(s)$ defined near $s = 0$ such that $r(0) = 0,\ I(s, r(s)) \equiv 1$. It follows from this and the fact that ψ is a maximizer that

$$\begin{aligned} 0 &= \frac{d}{ds}\bigg|_{s=0} Z(\psi + s\psi_0 + r(s)\psi) \\ &= k_1\psi'(1)\psi_0'(1) + k_0\psi'(0)\psi_0'(0) + r'(0)(k_1(\psi'(1))^2 + k_0(\psi'(0))^2) \end{aligned} \tag{2.15}$$

for any test function $\psi_0 \in H_0^1 \cap H^2$.

Differentiating the equation $I(s, r(s)) = 1$ yields that

$$I'(s, r(s)) = \partial_s I(s, r(s)) + \partial_r I(s, r(s))r'(s) \equiv 0, \tag{2.16}$$

which implies that

$$r'(0) = -\frac{1}{2}\int \psi''\psi_0''. \tag{2.17}$$

It follows from (2.17) and (2.15) that

$$\mu_c \int_0^1 \psi''\psi_0'' = k_1\psi'(1)\psi_0'(1) + k_0\psi'(0)\psi_0'(0). \tag{2.18}$$

First, choosing ψ_0 to be compactly supported in $(0,1)$ in (2.18) shows that

$$\mu_c\psi^{(4)} = 0, x \in (0,1), \tag{2.19}$$

in a weak sense. Standard bootstrapping arguments show that the solution ψ is smooth. Then (2.19) and (2.18) implies that

$$\begin{aligned} \mu_c\psi''(1) &= k_1\psi'(1), \\ \mu_c\psi''(0) &= -k_0\psi'(0). \end{aligned}$$

Therefore, the maximizers ψ must solve the following problem

$$\begin{cases} \mu_c \psi^{(4)} = 0, y \in (0,1); \\ \mu_c \psi''(1) = k_1 \psi'(1), \\ \mu_c \psi''(0) = -k_0 \psi'(0), \\ \psi(1) = \psi(0) = 0. \end{cases} \tag{2.20}$$

$(2.20)_1$ and $(2.20)_4$ imply that

$$\psi(x) = ax(x-1)(x-b), \tag{2.21}$$

with constants a and b to be determined and $a \neq 0$. Direct calculation yields

$$\psi'(1) = a - ab, \;\; \psi'(0) = ab,$$
$$\psi''(1) = 4a - 2ab, \;\; \psi''(0) = -2a - 2ab.$$

Substituting these into boundary conditions $(2.20)_2 - (2.20)_3$ yields

$$\begin{cases} 2a\mu_c(2-b) = ak_1(1-b), \\ 2a\mu_c(1+b) = ak_0 b. \end{cases} \tag{2.22}$$

Since $a \neq 0$, it follows that

$$\begin{cases} (k_1 - 2\mu_c)b = k_1 - 4\mu_c, \\ (k_0 - 2\mu_c)b = 2\mu_c. \end{cases} \tag{2.23}$$

Calculating $(2.23)_1 - (2.23)_2$ shows that

$$\mu_c = \frac{1}{6}(k_0 - k_1)b + \frac{k_1}{6}, \tag{2.24}$$

which together with $(2.23)_1$ implies that

$$(k_0 - k_1)b^2 - 2k_0 b + k_1 = 0. \tag{2.25}$$

If $k_1 = k_0 := k > 0$, then it follows from (2.24) and (2.25) that

$$\mu_c = \frac{k}{6}, \; b = \frac{1}{2}.$$

Otherwise, solving (2.25) gives

$$b = \frac{k_0 \pm \sqrt{k_1^2 + k_0^2 - k_1 k_0}}{k_0 - k_1},$$

which, together with (2.24), implies that

$$\mu_c = \frac{(k_1 + k_0) \pm \sqrt{k_1^2 + k_0^2 - k_1 k_0}}{6}. \tag{2.26}$$

We should notice that only the greater one in (2.26) is the critical viscosity because of the definition of maximum, that is,

$$\mu_c = \frac{(k_1 + k_0) + \sqrt{k_1^2 + k_0^2 - k_1 k_0}}{6}. \tag{2.27}$$

The unique maximizer is also given, with the coefficient a uniquely determined by using

$$\frac{1}{2}\int_0^1 (\psi'')^2 = 1.$$

□

3 The linear instability

3.1 Analysis for the variational problem (1.24)

In order to prove the instability part of Theorem 1.1 in this section, we will discuss the corresponding variational problem (1.24)-(1.26) with a fixed ξ by variational methods.

To rewrite this variational problem in an equivalent form, we define

$$\mathcal{A} = \{\psi \in H_0^1((0,1)) \cap H^2((0,1)) | J(\psi) = 1\} \tag{3.1}$$

where $J(\psi)$ is defined by (1.26).

The main task is to show that the minimum of $E(\psi)$ over $\mathcal{A}$ can be achieved and the minimizer solves the Euler-Lagrange equation, which is equivalent to (1.20) together with the corresponding boundary conditions (1.21)-(1.23). First, the existence of the minimizer is shown below.

Proposition 3.1. *$E(\psi)$ achieves its minimum on $\mathcal{A}$.*

Proof. Using the constraint on $J(\psi)$ and Cauchy inequality, we get

$$\begin{aligned}
E(\psi) &= \frac{\mu}{2}\int_0^1 \left[(\psi'')^2 + 2\xi^2(\psi')^2 + \xi^4\psi^2\right] - \frac{1}{2}\int_0^1 \left[((k_1+k_0)y - k_0)\,(\psi')^2\right]_y' dy \\
&= \frac{\mu}{2}\int_0^1 (\psi'')^2 + \frac{\mu\xi^2}{2}\int_0^1 \left[2(\psi')^2 + \xi^2\psi^2\right] \\
&\quad - \frac{1}{2}\int_0^1 \left[(k_1+k_0)(\psi')^2 + 2\psi'\psi''\left((k_1+k_0)y - k_0\right)\right] \\
&\geq \mu\xi^2 - \frac{1}{2}\int_0^1 \left[(k_1+k_0) + \mu^{-1}\left((k_1+k_0)y - k_0\right)^2\right](\psi')^2 \\
&\geq \mu\xi^2 - \frac{C_0}{2}\int_0^1 (\psi')^2 \geq \mu\xi^2 - C_0
\end{aligned} \tag{3.2}$$

for any fixed $\xi \in \mathbb{R}$, where

$$C_0 = \max_{0\leq y\leq 1}\left[|k_1 + k_0| + \mu^{-1}\left((k_1+k_0)y - k_0\right)^2\right]. \tag{3.3}$$

This means that E is bounded from below over $\mathcal{A}$, and thus $\inf_{\mathcal{A}} E(\psi)$ is well defined and finite.

Denote $-\lambda := \inf_{\mathcal{A}} E(\psi)$, and let $\{\psi_n\}_{n=1}^{\infty} \in \mathcal{A}$ be a minimizing sequence. Without lose of generality, one may assume that $E(\psi_n) \leq -\lambda + 1$. Then the constraint on $J(\psi_n)$ and the Poincaré inequality imply that ψ_n is uniformly bounded in H^1, which is independent of ξ^2. In addition, by the definition of E and the Cauchy inequality, one has

$$
\begin{aligned}
\mu \int_0^1 (\psi_n'')^2 &\leq 2E(\psi_n) + \int_0^1 \left[((k_1 + k_0)y - k_0)\, (\psi_n')^2 \right]' \\
&= 2E(\psi_n) + \int_0^1 \left[(k_1 + k_0)(\psi_n')^2 + 2\psi_n'\psi_n'' \left((k_1 + k_0)y - k_0 \right) \right] \\
&\leq 2E(\psi_n) + \frac{\mu}{2} \int_0^1 (\psi_n'')^2 + 2C_0 \int_0^1 (\psi_n')^2, \qquad (3.4)
\end{aligned}
$$

which implies that

$$
\int_0^1 (\psi_n'')^2 \leq 4\mu^{-1}(E(\psi_n) + C_0). \qquad (3.5)
$$

It follows that the sequence $\{\psi_n\}$ is bounded in $H_0^1 \cap H^2$, and thus, up to a subsequence if necessary, $\psi_n \rightharpoonup \psi$ weakly in H^2 and $\psi_n \to \psi$ strongly in H_0^1.

Rewrite E as

$$
E(\psi) = \frac{\mu}{2} \int_0^1 \left[(\psi'')^2 + \left(\xi^2 - \mu^{-1}(k_0 + k_1) \right) (\psi')^2 + \xi^4 \psi^2 \right] - \int_0^1 \psi'\psi'' \left((k_1 + k_0)y - k_0 \right). \qquad (3.6)
$$

It follows from the weak lower semi-continuity and weak convergence in H^2 and strong convergence in H_0^1 that

$$
\begin{aligned}
E(\psi) \leq &\frac{\mu}{2} \liminf_{n\to\infty} \int_0^1 (\psi_n'')^2 + \left(\mu\xi^2 - \frac{(k_0 + k_1)}{2} \right) \lim_{n\to\infty} \int_0^1 (\psi_n')^2 \\
&+ \frac{\mu\xi^4}{2} \lim_{n\to\infty} \int_0^1 \psi_n^2 - \lim_{n\to\infty} \int_0^1 \psi_n'\psi_n'' \left((k_1 + k_0)y - k_0 \right) \\
= &\liminf_{n\to\infty} E(\psi_n) = \inf_{\mathcal{A}} E(\psi). \qquad (3.7)
\end{aligned}
$$

Finally, the claim $J(\psi) = 1$ follows from the strong convergence in H_0^1. □

Remark 3.2. *Since the aim here is to look for growing mode solutions to the linearized equation (1.9) with boundary conditions (1.6)-(1.8), one should restrict the parameter ξ to stay in a specific range to guarantee that $\inf_{\mathcal{A}} E(\psi) = -\lambda < 0$. It requires, in view of (3.2), at least that*

$$
C_0 > 0 \text{ and } \xi^2 \leq \mu^{-1} C_0,
$$

so that it is possible to have a negative minimum for $E(\psi)$ over $\mathcal{A}$. This will be achieved later by finding a critical frequency.

Next we will show that the minimizer constructed above satisfies an Euler-Lagrangian equation equivalent to (1.20).

Proposition 3.3. *Let $\psi \in \mathcal{A}$ be the minimizer of E constructed in Proposition 3.1, and denote $-\lambda := E(\psi)$. Then ψ is smooth and satisfies*

$$-\lambda(\xi^2\psi - \psi'') = \mu(\psi^{(4)} - 2\xi^2\psi'' + \xi^4\psi), \tag{3.8}$$

along with the boundary conditions

$$\psi(1) = \psi(0) = 0, \tag{3.9}$$

$$\psi''(1) = \frac{k_1}{\mu}\psi'(1), \tag{3.10}$$

$$\psi''(0) = -\frac{k_0}{\mu}\psi'(0). \tag{3.11}$$

As a consequence, there exists a solution (ϕ, ψ, π) to the problem (1.16)-(1.19).

Proof. For any $\psi_0 \in H_0^1 \cap H^2,\ t, r \in \mathbb{R}$, let $\psi \in \mathcal{A}$ be a minimizer and define

$$j(t,r) := J(\psi + t\psi_0 + r\psi).$$

Then $j(t,r)$ is smooth and $j(0,0) = 1$,. Notice that

$$\partial_t j(0,0) = \int_0^1 (\xi^2\psi\psi_0 + \psi'\psi_0'), \text{ and } \partial_r j(0,0) = \int_0^1 (\xi^2\psi^2 + (\psi')^2) = 2 \neq 0. \tag{3.12}$$

Then, by implicit function theorem, there exists a smooth function $r = r(t)$ defined near 0 such that $r(0) = 0$ and $j(t, r(t)) = 1$.

Since ψ is a minimizer, it is clear that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} E(\psi + t\psi_0 + r(t)\psi) \\ &= \mu\int_0^1 \left(\psi''\psi_0'' + 2\xi^2\psi'\psi_0' + \xi^4\psi\psi_0\right) - k_1\psi'(1)\psi_0'(1) - k_0\psi'(0)\psi_0'(0) + 2r'(0)E(\psi). \end{aligned} \tag{3.13}$$

Now differentiating the equation $j(t, r(t)) = 1$ gives

$$r'(0) = -\frac{1}{2}\int_0^1 (\xi^2\psi\psi_0 + \psi'\psi_0'). \tag{3.14}$$

Substituting (3.14) into (3.13) yields

$$\begin{aligned} &\mu\int_0^1 \left(\psi''\psi_0'' + 2\xi^2\psi'\psi_0' + \xi^4\psi\psi_0\right) + \lambda\int_0^1 (\xi^2\psi\psi_0 + \psi'\psi_0') \\ =&k_1\psi'(1)\psi_0'(1) + k_0\psi'(0)\psi_0'(0). \end{aligned} \tag{3.15}$$

Choosing ψ_0 to be compactly supported in $(0,1)$ shows that $\psi_0'(1) = \psi_0'(0) = 0$. Substituting this into (3.15) yields that $\psi \in H^2$ solves (3.8) in a weak sense. Standard bootstrap arguments then show that the solution is smooth. Next, using equation (3.8) and integrating by part lead to

$$\mu \int_0^1 (\psi'' \psi_0')' = k_1 \psi'(1)\psi_0'(1) + k_0 \psi'(0)\psi_0'(0), \tag{3.16}$$

which is equivalent to

$$\left(\mu\psi''(1) - k_1\psi'(1)\right)\psi_0'(1) = \left(\mu\psi''(0) + k_0\psi'(0)\right)\psi_0'(0). \tag{3.17}$$

Since ψ_0 is arbitrarily chosen, it follows that

$$\psi''(1) = \frac{k_1}{\mu}\psi'(1) \text{ and } \psi''(0) = -\frac{k_0}{\mu}\psi'(0). \tag{3.18}$$

The Proposition follows. □

Remark 3.4. *It should be noted that for fixed ξ^2, the existence of the solutions (ϕ, ψ, π) and the corresponding eigenvalue λ of problem (1.16)-(1.19) are independent of the values of k_1, k_0 and $\mu > 0$. That is, for any fixed $\xi^2 \in [0, +\infty)$, the functions $(\phi(\xi^2, y), \psi(\xi^2, y), \pi(\xi^2, y))$ and eigenvalue $\lambda(\xi^2)$ are well-defined.*

To study the sign of $-\lambda(\xi^2)$, which determines the linearized stability of the steady states, we will study the relations among $k_1, k_0, \mu > 0$ and ξ^2 in details later.

3.2 Proof of the instability part of Theorem 1.1

It follows from the definition of μ_c that when $\mu_c > 0$ and $\mu \in (0, \mu_c)$, there exists $\tilde{\psi} \in H_0^1 \cap H^2$, such that

$$\mu \int_0^1 (\tilde{\psi}'')^2 - k_1(\tilde{\psi}'(1))^2 - k_0(\tilde{\psi}'(0))^2 < 0,$$

In order to prove the existence of growing mode solutions in this case, it suffices to prove that there is an eigenvalue $\lambda > 0$. To do this, since $E(\psi)$ is bounded from below over $\mathcal{A}$, one needs to prove that there exists a function $\tilde{\psi}$ belonging to $\mathcal{A}$ such that $E(\tilde{\psi}) < 0$.

Step 1. In this step, we intend to show that there exists $\tilde{\psi} \in H_0^1 \cap H^2$ such that $E(\tilde{\psi}) < 0$ for some frequency ξ, i.e.,

$$\xi^2 < \frac{k_1(\tilde{\psi}'(1))^2 + k_0(\tilde{\psi}'(0))^2 - \mu \int_0^1 (\tilde{\psi}'')^2}{\mu \int_0^1 \left(2(\tilde{\psi}')^2 + \xi^2 \tilde{\psi}^2\right)}. \tag{3.19}$$

The appearance of ξ^2 on the both sides of (3.19) makes it difficult to use variational techniques to express the critical value of ξ^2. In order to circumvent this difficulty, one can replace the ξ^2 on the right-hand side of (3.19) with an arbitrary parameter $s^2 \geq 0$. Precisely, we introduce a family of modified variational problems given by

$$\mathcal{N}^*(s^2) = \sup_{H_0^1 \cap H^2, \psi \neq 0} \mathcal{N}(\psi, s^2), \tag{3.20}$$

where

$$\mathcal{N}(\psi, s^2) := \frac{k_1(\psi'(1))^2 + k_0(\psi'(0))^2 - \mu \int_0^1 (\psi'')^2}{\mu \int_0^1 (2(\psi')^2 + s^2\psi^2)}, s^2 \in [0, +\infty). \tag{3.21}$$

Similar to the proof of Proposition 3.1, one can prove that $\mathcal{N}^*(s^2)$ is well-defined and the maximizer is achievable for any fixed $s^2 \in [0, +\infty)$. Moreover, if $\mu_c > 0$ and $\mu \in (0, \mu_c)$, then $\mathcal{N}^*(s^2) > 0$ for any $s^2 \in [0, +\infty)$.

To establish the continuity, boundedness and monotonicity for the function $\mathcal{N}^*(s^2)$, one sets, for convenience, that

$$\mathcal{N}_1(\psi) = k_1(\psi'(1))^2 + k_0(\psi'(0))^2 - \mu \int_0^1 (\psi'')^2, \tag{3.22}$$

$$\mathcal{N}_2(\psi, s^2) = \mu \int_0^1 \left(2(\psi')^2 + s^2\psi^2\right). \tag{3.23}$$

Proposition 3.5. *Let $\mathcal{N}^*(s^2) : [0, +\infty) \to \mathbb{R}^+$ be defined by (3.20)-(3.21). Then it holds that*
(i) $\mathcal{N}^*(s^2)$ is strictly decreasing,
(ii) $\mathcal{N}^*(s^2) \in C^{0,1}([0, +\infty))$, in particular, $\mathcal{N}^*(s^2) \in C^0([0, +\infty))$.

Proof. For any $s_1^2, s_2^2 \in [0, +\infty)$, define

$$\mathcal{N}^*(s_1^2) = \mathcal{N}(\psi_{s_1^2}, s_1^2), \mathcal{N}^*(s_2^2) = \mathcal{N}(\psi_{s_2^2}, s_2^2).$$

Then, for $s_1^2 < s_2^2$, the definition of supremum and the monotonicity of $\mathcal{N}$ with respect to s^2 give

$$\mathcal{N}^*(s_1^2) = \mathcal{N}(\psi_{s_1^2}, s_1^2) \geq \mathcal{N}(\psi_{s_2^2}, s_1^2), > \mathcal{N}(\psi_{s_2^2}, s_2^2) = \mathcal{N}^*(s_2^2) \tag{3.24}$$

which means that $\mathcal{N}^*$ is strictly decreasing with respect to s^2, this proves *(i)*.

Next, for any $s_1^2, s_2^2 \in [0, +\infty)$, by the definition of $\mathcal{N}^*(s^2)$, we have

$$\begin{aligned} \mathcal{N}^*(s_1^2) &= \frac{\mathcal{N}_1(\psi_{s_1^2})}{\mathcal{N}_2(\psi_{s_1^2}, s_1^2)} = \frac{\mathcal{N}_1(\psi_{s_1^2})}{\mathcal{N}_2(\psi_{s_1^2}, s_2^2)} + \frac{\mathcal{N}_1(\psi_{s_1^2})}{\mathcal{N}_2(\psi_{s_1^2}, s_1^2)} - \frac{\mathcal{N}_1(\psi_{s_1^2})}{\mathcal{N}_2(\psi_{s_1^2}, s_2^2)} \\ &\leq \mathcal{N}^*(s_2^2) + \frac{\mathcal{N}_1(\psi_{s_1^2})(s_2^2 - s_1^2) \int_0^1 \psi_{s_1^2}^2}{\mathcal{N}_2(\psi_{s_1^2}, s_1^2)\mathcal{N}_2(\psi_{s_1^2}, s_2^2)} \end{aligned} \tag{3.25}$$

In view of the fact that

$$0 < \mathcal{N}_1(\psi_{s_1^2}) = C_0 \int_0^1 (\psi_{s_1^2}')^2$$

and applying Poincaré inequality, one gets

$$\mathcal{N}^*(s_1^2) \le \mathcal{N}^*(s_2^2) + \frac{C_0|s_2^2 - s_1^2|}{4\mu^2} = \mathcal{N}^*(s_2^2) + K|s_2^2 - s_1^2|, \tag{3.26}$$

where $K := \frac{C_0}{4\mu^2}$. This implies that $\mathcal{N}^*(s^2) \in C^{0,1}([0,+\infty))$.

In addition, for any $s^2 \in [0,+\infty)$, it follows from a similar proof as for (3.26) that

$$0 < \mathcal{N}^*(s^2) = \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{\mathcal{N}_1(\psi)}{\mathcal{N}_2(\psi, s^2)} \le \frac{C_0}{2\mu}. \tag{3.27}$$

This verifies *(ii)* and thus the Proposition follows. □

Now define a function $\Phi : (0,+\infty) \to (0,+\infty)$ by

$$\Phi(s^2) = \frac{s^2}{\mathcal{N}^*(s^2)}.$$

It follows from the properties of $\mathcal{N}^*(s^2)$ that Φ is continuous and strictly increasing with respect to s^2. Since $\lim_{s^2 \to 0^+} \mathcal{N}^*(s^2) = \mathcal{N}^*(0) > 0$ and (3.27), thus

$$\lim_{s^2 \to 0^+} \Phi(s^2) = 0, \text{ and } \lim_{s^2 \to +\infty} \Phi(s^2) = +\infty.$$

Then by the mean value theorem, there exists $s_0^2 \in (0,+\infty)$ such that $\Phi(s_0^2) = 1$, i.e., $s_0^2 = \mathcal{N}^*(s_0^2)$. Taking $\xi_c^2 = s_0^2$ yields that

$$\xi_c^2 = \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{k_1(\psi'(1))^2 + k_0(\psi'(0))^2 - \mu \int_0^1 (\psi'')^2}{\mu \int_0^1 (2(\psi')^2 + \xi_c^2 \psi^2)}, \tag{3.28}$$

which implies that for any $\xi^2 \in [0, \xi_c^2)$, it holds that

$$\xi^2 < \xi_c^2 = \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{\mathcal{N}_1(\psi)}{\mathcal{N}_2(\psi, \xi_c^2)} < \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{\mathcal{N}_1(\psi)}{\mathcal{N}_2(\psi, \xi^2)} \tag{3.29}$$

By the definition of supremum, for any $\xi^2 \in [0, \xi_c^2)$, there exists $\tilde{\psi} \in H_0^1 \cap H^2$, so that

$$\xi^2 < \frac{\mathcal{N}_1(\tilde{\psi})}{\mathcal{N}_2(\tilde{\psi}, \xi^2)} = \frac{k_1(\tilde{\psi}'(1))^2 + k_0(\tilde{\psi}'(0))^2 - \mu \int_0^1 (\tilde{\psi}'')^2}{\mu \int_0^1 \left(2(\tilde{\psi}')^2 + \xi^2 \tilde{\psi}^2\right)}.$$

Thus, (3.19) is proved.

In order to emphasize the dependence on ξ^2, we will sometimes write

$$E(\psi, \xi^2) = E(\psi), J(\psi, \xi^2) = J(\psi), \text{ and } -\lambda(\xi^2) = \inf_{H_0^1 \cap H^2, \psi \neq 0} \frac{E(\psi, \xi^2)}{J(\psi, \xi^2)}.$$

Hence, if $\mu_c > 0$ and $\mu \in (0, \mu_c)$, for any $\xi^2 \in [0, \xi_c^2)$, it holds that $\lambda(\xi^2) > 0$.

Remark 3.6. *One should also notice that, under the assumptions that $\mu_c > 0$ and $\mu \in (0, \mu_c)$, for any $\xi^2 \geq \xi_c^2$, the fact*

$$\xi^2 \geq \xi_c^2 = \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{\mathcal{N}_1(\psi)}{\mathcal{N}_2(\psi, \xi_c^2)} \geq \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{\mathcal{N}_1(\psi)}{\mathcal{N}_2(\psi, \xi^2)}$$

leads to

$$\xi^2 \geq \frac{k_1(\psi'(1))^2 + k_0(\psi'(0))^2 - \mu \int_0^1 (\psi'')^2}{\mu \int_0^1 (2(\psi')^2 + \xi^2 \psi^2)}, \ \forall \psi \in H_0^1 \cap H^2,$$

which further implies that $\lambda(\xi^2) \leq 0$ with $\lambda(\xi^2) = 0$ if and only if $\xi^2 = \xi_c^2$.

In addition, if $\mu \geq \mu_c$, then $\lambda(\xi^2) \leq 0$ for any $\xi^2 \in [0, +\infty)$, and, $\lambda(\xi^2) = 0$ if and only if $\mu = \mu_c$ and $\xi^2 = 0$.

In fact, one can see that

$$\lambda(\xi^2) \leq \sup_{H_0^1 \cap H^2, \psi \neq 0} \frac{(\mu_c - \mu) \int_0^1 (\psi'')^2}{2J(\psi, \xi^2)} - \mu \xi^2.$$

Therefore for any $\mu > \mu_c$, we have $\lambda(\xi^2) \leq -\mu < 0$ if $\xi^2 \geq 1$. Moreover, for any $\mu > \mu_c$ and $0 \leq \xi^2 \leq 1$, one has

$$\lambda(\xi^2) \leq \frac{(\mu_c - \mu)\|\psi''\|_{L^2}^2}{\|\psi\|_{H^1}^2} \leq \mu_c - \mu < 0,$$

where (2.4) has been used.

In conclusion, one gets that $\lambda(\xi^2) < \mu_c - \mu < 0$ for any $\xi^2 \in [0, +\infty)$ provided that $\mu > \mu_c$.

Step 2. In this step, we show that λ is a bounded, continuous, strictly decreasing function with respect to ξ^2 on $[0, +\infty)$.

Proposition 3.7. *For $\mu_c > 0$ and $\mu \in (0, \mu_c)$, the function $\lambda : [0, +\infty) \to \mathbb{R}$ is continuous, strictly decreasing and satisfies*

$$\Lambda := \max_{\xi^2 \in [0, +\infty)} \lambda(\xi^2) = \lambda(0) \leq C_0. \tag{3.30}$$

where the constant C_0 is defined in (3.3), which is positive in this case.

Proof. In view of Remark 3.4 and similarly to Proposition 3.5, for any $\xi_1^2, \xi_2^2 \in [0, +\infty)$, we denote

$$\lambda(\xi_1^2) = \frac{-E(\psi_{\xi_1^2}, \xi_1^2)}{J(\psi_{\xi_1^2}, \xi_1^2)}, \ \lambda(\xi_2^2) = \frac{-E(\psi_{\xi_2^2}, \xi_2^2)}{J(\psi_{\xi_2^2}, \xi_2^2)}.$$

Notice that

$$\lambda(\xi_1^2) = \frac{-E(\psi_{\xi_1^2}, \xi_1^2)}{J(\psi_{\xi_1^2}, \xi_1^2)} = \frac{-E(\psi_{\xi_1^2}, \xi_2^2)}{J(\psi_{\xi_1^2}, \xi_2^2)} + \frac{-E(\psi_{\xi_1^2}, \xi_1^2)}{J(\psi_{\xi_1^2}, \xi_1^2)} - \frac{-E(\psi_{\xi_1^2}, \xi_2^2)}{J(\psi_{\xi_1^2}, \xi_2^2)}$$

$$
\begin{aligned}
&\leq \lambda(\xi_2^2) + \frac{\mathcal{N}_1(\psi_{\xi_1^2})(\xi_2^2-\xi_1^2)\int_0^1(\psi_{\xi_1^2})^2}{J(\psi_{\xi_1^2},\xi_1^2)J(\psi_{\xi_1^2},\xi_2^2)} + \mu(\xi_2^2-\xi_1^2) + \frac{\mu(\xi_2^2-\xi_1^2)\left(\int_0^1(\psi'_{\xi_1^2})^2\right)^2}{J(\psi_{\xi_1^2},\xi_1^2)J(\psi_{\xi_1^2},\xi_2^2)} \\
&\leq \lambda(\xi_2^2) + (C_0+2\mu)|\xi_2^2-\xi_1^2|, \qquad (3.31)
\end{aligned}
$$

where the fact that $\mathcal{N}_1(\psi_{\xi_1^2}) > 0$ has been used. The continuity of $\Lambda(\xi^2)$ then follows.

For $\xi_1^2 < \xi_2^2$, by the definition of supremum, one has that

$$
\begin{aligned}
\lambda(\xi_1^2) = \frac{-E(\psi_{\xi_1^2},\xi_1^2)}{J(\psi_{\xi_1^2},\xi_1^2)} \geq \frac{-E(\psi_{\xi_2^2},\xi_1^2)}{J(\psi_{\xi_2^2},\xi_1^2)} &= \frac{\mathcal{N}_1(\psi_{\xi_2^2})}{J(\psi_{\xi_2^2},\xi_1^2)} - \mu\xi_1^2 - \frac{\mu\xi_1^2\int_0^1(\psi'_{\xi_2^2})^2}{J(\psi_{\xi_2^2},\xi_1^2)} \\
&> \frac{\mathcal{N}_1(\psi_{\xi_2^2})}{J(\psi_{\xi_2^2},\xi_2^2)} - \mu\xi_2^2 - \frac{\mu\xi_2^2\int_0^1(\psi'_{\xi_2^2})^2}{J(\psi_{\xi_2^2},\xi_2^2)} = \lambda(\xi_2^2), \qquad (3.32)
\end{aligned}
$$

where one has used the fact that $\mathcal{N}_1(\psi_{\xi_2^2}) > 0$. This yields the monotonicity of $\lambda(\xi^2)$.

Consequently, $\Lambda = \lambda(0)$. Moreover, by using the same technique as in (3.2), one can obtain that $\lambda(0) \leq C_0$. □

Step 3. In this step, we construct some growing mode solutions to (1.6)-(1.9) by using the results in *Step 1* and *Step 2*.

Proposition 3.8. *Let $f \in C_c^\infty(0,\xi_c^2)$ be a real-valued function and the real-valued functions $\phi(\xi^2,y)$, $\psi(\xi^2,y), \pi(\xi^2,y), \lambda(\xi^2)$ are the solutions, constructed in Proposition 3.1 and Proposition 3.3, to problem (1.16)-(1.19), where ξ_c^2 is the so called critical frequency which is positive and defined in (3.28). Define*

$$
u^1(x,y,t) = -\frac{1}{2\pi}\int_{\mathbb{R}} f(\xi^2)i\phi(\xi^2,y)e^{\lambda(\xi^2)t}e^{ix\xi}d\xi, \qquad (3.33)
$$

$$
u^2(x,y,t) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi^2)\psi(\xi^2,y)e^{\lambda(\xi^2)t}e^{ix\xi}d\xi, \qquad (3.34)
$$

$$
q(x,y,t) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi^2)\pi(\xi^2,y)e^{\lambda(\xi^2)t}e^{ix\xi}d\xi. \qquad (3.35)
$$

Then $(\mathbf{u} = (u^1,u^2), q)$ is a solution to linearized problem (1.6)-(1.9). Due to the smoothness of functions $\phi(y), \psi(y), \pi(y)$, we also have the estimates

$$
\|\mathbf{u}(0)\|_{H^k} + \|q(0)\|_{H^k} \leq \tilde{C}_k\left(\int_{\mathbb{R}}(1+\xi^2)^k|f(\xi)|^2\right)^{1/2} < +\infty, k\in\mathbb{N}, \qquad (3.36)
$$

where constant $\tilde{C}_k > 0$ depending on k_0, k_1, μ and k.

Moreover, for every $t > 0$, the boundedness of $\lambda(\xi^2)$ over $(0,\xi_c^2)$ implies that the solution $(\mathbf{u}(t), q(t)) \in H^k$ and satisfies

$$
e^{\lambda_f t}\|\mathbf{u}(0)\|_{H^k} \leq \|\mathbf{u}(t)\|_{H^k} \leq e^{\Lambda t}\|\mathbf{u}(0)\|_{H^k}, \qquad (3.37)
$$

$$
e^{\lambda_f t}\|q(0)\|_{H^k} \leq \|q(t)\|_{H^k} \leq e^{\Lambda t}\|q(0)\|_{H^k}, \qquad (3.38)
$$

where

$$\lambda_f := \inf_{\xi^2 \in \mathrm{supp}(f)} \lambda(\xi^2) > 0 \tag{3.39}$$

and Λ is a positive number defined in (3.30).

Proof. It follows from (1.16)-(1.19), Proposition 3.1, 3.3, Remark 3.6 and Proposition 3.7 that the solution given in (3.33)-(3.35) satisfies (3.36)-(3.38), and (3.39) holds. This verification is similar to the proof of Theorem 2.4 in [10], and thus is omitted.

This completes the proof of linear instability part of Theorem 1.1. □

4 The nonlinear instability

4.1 Global existence and nonlinear energy estimates

In this subsection, we prove that the nonlinear perturbed problem (1.5)-(1.8) admits at least one global strong solution.

The proof of local existence and uniqueness of strong solution is similar to that in section 4 of [27] (see also section 2 of [28]). Therefore, in order to get the global existence of strong solutions, it suffices to derive some global energy estimates. To this end, let $(\mathbf{u}, q)$ be a strong solution of the perturbed problem (1.5)-(1.8). In the sequel, for simplicity, C will denote a generic positive constant, which may depend on k_1, k_0 and μ, and $C(\alpha, \beta)$ denotes some constant also depending on parameters α and β.

Testing $(1.5)_1$ by $\mathbf{u}$, integrating by part over Ω and using $(1.5)_2$, boundary conditions (1.6)-(1.8), one has that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{u}(t)|^2 + \mu \int |\nabla \mathbf{u}(t)|^2 = \int_{\mathbb{R}} \left(k_1|u^1(x,1)|^2 + k_0|u^1(x,0)|^2\right) = I_1. \tag{4.1}$$

Notice that

$$\begin{aligned} I_1 &= \int_0^1 \frac{d}{dy}\left[\int_{\mathbb{R}} ((k_1+k_0)y - k_0)|u^1(x,y)|^2\right] \\ &\le \frac{\mu}{2}\int |\partial_y u^1(x,y)|^2 + 2C_0 \int |u^1(x,y)|^2. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1) yields

$$\frac{d}{dt}\int |\mathbf{u}(t)|^2 + \mu \int |\nabla \mathbf{u}(t)|^2 \le 4C_0 \int |\mathbf{u}(t)|^2, \tag{4.3}$$

which, together with Gronwall inequality, implies that for any fixed $T > 0$

$$\sup_{0\le t\le T} \|\mathbf{u}(t)\|_{L^2}^2 + \int_0^T \mu \|\nabla \mathbf{u}(t)\|_{L^2}^2 dt \le e^{4C_0 T}\|\mathbf{u}_0\|_{L^2}^2. \tag{4.4}$$

Similarly, one gets that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{u}_t(t)|^2 + \mu \int |\nabla \mathbf{u}_t(t)|^2 \le 4C_0\|\mathbf{u}_t\|_{L^2}^2 + \frac{\mu}{4}\|\nabla \mathbf{u}_t\|_{L^2}^2 - \int \mathbf{u}_t \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t$$

$$
\begin{aligned}
&\le \int |\mathbf{u}_t|^2|\nabla\mathbf{u}| + 4C_0\|\mathbf{u}_t\|_{L^2}^2 + \frac{\mu}{4}\|\nabla\mathbf{u}_t\|_{L^2}^2 \\
&\le 4C_0\|\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^4}^2 + \frac{\mu}{4}\|\nabla\mathbf{u}_t\|_{L^2}^2 \\
&\le \frac{\mu}{2}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C_1(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\mathbf{u}_t\|_{L^2}^2, \qquad (4.5)
\end{aligned}
$$

which implies that

$$
\frac{d}{dt}\int |\mathbf{u}_t(t)|^2 + \mu\int |\nabla\mathbf{u}_t(t)|^2 \le 2C_1(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\mathbf{u}_t\|_{L^2}^2. \qquad (4.6)
$$

Multiplying $(1.5)_1$ by $\mathbf{u}_t$, integrating in space and recalling $\mathrm{div}\mathbf{u}_t = 0$, one has

$$
\int |\mathbf{u}_t(t)|^2 = \int (\mu\Delta\mathbf{u}\cdot\mathbf{u}_t - \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t) \lesssim \int (|\mathbf{u}||\nabla\mathbf{u}| + |\Delta\mathbf{u}|)|\mathbf{u}_t|. \qquad (4.7)
$$

Using the Cauchy inequality, the Hölder inequality and the Sobolev embedding inequalities, we arrive at

$$
\begin{aligned}
\|\mathbf{u}_t(t)\|_{L^2}^2 &\lesssim \|\mathbf{u}(t)\|_{L^4}^2\|\nabla\mathbf{u}(t)\|_{L^4}^2 + \|\nabla^2\mathbf{u}(t)\|_{L^2}^2 \\
&\lesssim \|\mathbf{u}(t)\|_{L^2}^2\|\mathbf{u}(t)\|_{H^2}^2 + \|\mathbf{u}(t)\|_{H^2}^2 \\
&\le C(1+e^{4C_0t})\|\mathbf{u}(t)\|_{H^2}^2. \qquad (4.8)
\end{aligned}
$$

Taking $t \to 0^+$ in the above inequality yields

$$
\limsup_{t\to 0^+}\|\mathbf{u}_t(t)\|_{L^2}^2 \le C\|\mathbf{u}_0\|_{H^2}^2, \qquad (4.9)
$$

where $C > 0$ depends also on $\|\mathbf{u}_0\|_{L^2}^2$.

Therefore, applying Gronwall inequality to (4.6), we have

$$
\sup_{0\le t\le T}\|\mathbf{u}_t(t)\|_{L^2}^2 + \int_0^T \mu\|\nabla\mathbf{u}_t(t)\|_{L^2}^2 dt \le C(T, \|\mathbf{u}_0\|_{H^2}). \qquad (4.10)
$$

Testing $(1.5)_1$ by $\mathbf{u}_t$, integrating by part over Ω and using $(1.5)_2$, boundary conditions (1.6)-(1.8), we obtain

$$
\begin{aligned}
\frac{\mu}{2}\frac{d}{dt}\int |\nabla\mathbf{u}(t)|^2 + \int |\mathbf{u}_t(t)|^2 = &\int_{\mathbb{R}} \big(k_1u^1(x,1)u_t^1(x,1) + k_0u^1(x,0)u_t^1(x,0)\big) \\
&- \int \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t := I_2 + I_3. \qquad (4.11)
\end{aligned}
$$

Similar to (4.2), it holds that

$$
I_2 \le C\int (|\mathbf{u}||\mathbf{u}_t| + |\nabla\mathbf{u}||\mathbf{u}_t| + |\mathbf{u}||\nabla\mathbf{u}_t|) \le C(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\nabla\mathbf{u}_t\|_{L^2}^2), \qquad (4.12)
$$

and

$$I_3 = \int \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{u}_t \le \|\mathbf{u}\|_{L^4}^2 \|\nabla \mathbf{u}_t\|_{L^2} \le C(\|\mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2), \tag{4.13}$$

where the two-dimensional interpolation inequality

$$\|\mathbf{u}\|_{L^4}^2 \lesssim \|\mathbf{u}\|_{L^2} \|\mathbf{u}\|_{H^1}$$

has been used.

Substituting (4.12) and (4.13) into (4.11), and integrating over $[0,t] \subset [0,T]$, one gets

$$\sup_{0 \le t \le T} \|\nabla \mathbf{u}(t)\|_{L^2}^2 + \int_0^T \mu \|\mathbf{u}_t(t)\|_{L^2}^2 dt \le C(T, \|\mathbf{u}_0\|_{H^2}). \tag{4.14}$$

Finally, we recall that the pair $(\mathbf{u}, q)$ solves the Stokes equations

$$\begin{cases} -\mu \Delta \mathbf{u} + \nabla q = -\mathbf{u}_t - \mathbf{u} \cdot \nabla \mathbf{u}, \\ \mathrm{div} \mathbf{u} = 0. \end{cases} \tag{4.15}$$

By Stokes estimate (A.3) in the Appendix, it is clear that

$$\begin{aligned} \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 &\le C\|\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2 \\ &\le C\|\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^4}^2 + C\|\mathbf{u}\|_{L^2}^2 \\ &\le C\left(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^2\right) + C\|\mathbf{u}\|_{H^1}^3 \|\mathbf{u}\|_{H^2} \\ &\le C(T, \|\mathbf{u}_0\|_{H^2}) + \frac{1}{2}\|\nabla^2 \mathbf{u}\|_{L^2}^2, \end{aligned} \tag{4.16}$$

which implies

$$\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \le C(T, \|\mathbf{u}_0\|_{H^2}). \tag{4.17}$$

Summing up, we have obtained the global energy estimates to guarantee the global existence of strong solutions (see Proposition 4.1) as follows:

$$\begin{aligned} &\sup_{0 \le t \le T} (\|\mathbf{u}(t)\|_{H^2}^2 + \|\nabla q(t)\|_{L^2}^2 + \|\mathbf{u}_t(t)\|_{L^2}^2) + \int_0^T (\|\nabla \mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{u}_t(t)\|_{H^1}^2) dt \\ \le &C(T, \|\mathbf{u}_0\|_{H^2}). \end{aligned} \tag{4.18}$$

Proposition 4.1. *For any given $T > 0$ and initial data $\mathbf{u}_0 \in H^2$ satisfying the compatibility condition $\mathrm{div}\mathbf{u}_0 = 0$, there exists a strong solution $(\mathbf{u}, q) \in C([0,T]; H^2 \times H^1)$ to the perturbed problem (1.5)-(1.8). Moreover, there exists a constant $\bar{\sigma} \in (0,1]$, such that*

$$\begin{aligned} &\|\mathbf{u}(t)\|_{H^2}^2 + \|(\mathbf{u}_t, \nabla q)(t)\|_{L^2}^2 + \int_0^t \|(\nabla \mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}_t)(s)\|_{L^2}^2 ds \\ \le &C_1 \left(\|\mathbf{u}_0\|_{H^2}^2 + \int_0^t \|\mathbf{u}(s)\|_{L^2}^2 ds \right), \end{aligned} \tag{4.19}$$

provided that $\|\mathbf{u}(t)\|_{H^2}^2 \le \bar{\sigma}$ on $[0,T]$. Here the constant C_1 depends only on k_0, k_1, and μ.

Proof. One can follow the proof of section 4 of [27] (see also section 2 of [28]) to get the local existence and uniqueness of strong solution to the nonlinear perturbed problem (1.5)-(1.8). Then the global existence and uniqueness of the strong solution can be shown easily by using the above global *a priori* estimate (4.18).

It remains to prove (4.19). In view of the assumption that $\|\mathbf{u}(t)\|_{H^2}^2 \leq \bar{\sigma}$, one can estimate I_2, I_3 in (4.11) as follows.

$$I_2 \leq C \int (|\mathbf{u}||\mathbf{u}_t| + |\nabla \mathbf{u}||\mathbf{u}_t| + |\mathbf{u}||\nabla \mathbf{u}_t|) \leq \frac{1}{4}\|\mathbf{u}_t\|_{L^2}^2 + C_\epsilon \|\mathbf{u}\|_{H^1}^2 + \epsilon \|\nabla \mathbf{u}_t\|_{L^2}^2, \tag{4.20}$$

$$I_3 \leq \int |\mathbf{u}||\nabla \mathbf{u}||\mathbf{u}_t| \leq \|\mathbf{u}_t\|_{L^2} \|\nabla \mathbf{u}\|_{L^4} \|\mathbf{u}\|_{L^4} \leq \frac{1}{4}\|\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 \|\mathbf{u}\|_{L^2}^2. \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.11),we have

$$\frac{d}{dt}\int \mu |\nabla \mathbf{u}(t)|^2 + \int |\mathbf{u}_t|^2 \leq C_\epsilon \|\mathbf{u}\|_{H^1}^2 + \epsilon \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 \|\mathbf{u}\|_{L^2}^2. \tag{4.22}$$

Adding $K_1 \times (4.22), (4.6)$, and $K_2 \times (4.3)$ up with suitable large $K_1 > 0, K_2 > 0$ and taking $\epsilon > 0$ small enough, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\| \left(\mathbf{u}, \sqrt{\mu}\nabla \mathbf{u}, \mathbf{u}_t\right)(t)\|_{L^2}^2 + \| \left(\sqrt{\mu}\nabla \mathbf{u}, \mathbf{u}_t, \sqrt{\mu}\nabla \mathbf{u}_t\right)\|_{L^2}^2 \\
\leq& C\|\mathbf{u}\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 \left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{L^2}^2\right) \\
\leq& C\|\mathbf{u}\|_{L^2}^2 + C\bar{\sigma} \left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{L^2}^2\right)
\end{aligned} \tag{4.23}$$

provided that $\|\mathbf{u}\|_{H^2}^2 \leq \bar{\sigma}$.

Then, for suitably small $\bar{\sigma} \in (0, 1]$, one can get that

$$\frac{d}{dt}\| \left(\mathbf{u}, \sqrt{\mu}\nabla \mathbf{u}, \mathbf{u}_t\right)(t)\|_{L^2}^2 + \| \left(\sqrt{\mu}\nabla \mathbf{u}, \mathbf{u}_t, \sqrt{\mu}\nabla \mathbf{u}_t\right)\|_{L^2}^2 \leq C\|\mathbf{u}\|_{L^2}^2. \tag{4.24}$$

Thus, it follows from (4.24) that

$$\| \left(\mathbf{u}, \nabla \mathbf{u}, \mathbf{u}_t\right)(t)\|_{L^2}^2 + \int_0^t \| \left(\nabla \mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}_t\right)(s)\|_{L^2}^2 ds \leq C \left(\|\mathbf{u}_0\|_{H^2}^2 + \int_0^t \|\mathbf{u}(s)\|_{L^2}^2 ds\right) \tag{4.25}$$

Moreover, under the assumption $\|\mathbf{u}\|_{H^2}^2 \leq \bar{\sigma} \leq 1$ as in the proof of (4.16), one can get that

$$\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq C(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\cdot\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^2) \leq C\|(\mathbf{u}, \nabla \mathbf{u}, \mathbf{u}_t)\|_{L^2}^2, \tag{4.26}$$

which, together with (4.25), implies (4.19). □

4.2 Proof of Theorem 1.2 (i): nonlinear instability

In this subsection, we apply the bootstrap argument proposed by Y. Guo et al. in [2] to prove the nonlinear instability. More precisely, we shall show that there exists a constant $\varepsilon > 0$ such that for any $\delta > 0$, there exists a solution $\mathbf{u}^\delta(t)$ to the nonlinear problem (1.5)-(1.8) with initial data $\|\mathbf{u}_0^\delta\|_{H^2} = \delta$ and an escape time $T^\delta > 0$ such that $\|\mathbf{u}^\delta(T^\delta)\|_{H^2} > \varepsilon$.

To this end, we first give the following elementary inequality, which will be used in this section and in the next section.

Proposition 4.2. *Let $\mathbf{w} \in H^1_\sigma(\Omega) \cap H^2(\Omega)$, then it holds that*

$$-\mu \int |\nabla \mathbf{w}|^2 dxdy + k_1 \int_{\mathbb{R}} |w^1(x,1)|^2 dx + k_0 \int_{\mathbb{R}} |w^1(x,0)|^2 dx \le \Lambda \int |\mathbf{w}|^2 dxdy, \quad (4.27)$$

where Λ is defined in (3.30).

Proof. For any function $g \in L^2(\Omega)$, let

$$\hat{g}(\xi, y) = \int_{\mathbb{R}} g(x,y) e^{-i\xi x} dx, \ \xi \in \mathbb{R}.$$

Then it follows from *Fubini* theorem and *Parseval* equality that $\hat{g} \in L^2(\Omega)$ and

$$\int_\Omega |g(x,y)|^2 dxdy = \frac{1}{2\pi} \int_\Omega |\hat{g}(\xi,y)|^2 d\xi dy. \quad (4.28)$$

Hence,

$$\begin{aligned}
&-2\pi \int \mu |\nabla \mathbf{w}|^2 dxdy + 2\pi \int_{\mathbb{R}} k_1 |w^1(x,1)|^2 dx + 2\pi \int_{\mathbb{R}} k_0 |w^1(x,0)|^2 dx \\
=& -\int_{\mathbb{R}} \int_0^1 \mu \left[|i\xi \hat{w}^1(\xi,y)|^2 + |i\xi \hat{w}^2(\xi,y)|^2 + |\partial_y \hat{w}^1(\xi,y)|^2 + |\partial_y \hat{w}^2(\xi,y)|^2 \right] d\xi dy \\
&+ \int_{\mathbb{R}} \left[k_1 |\hat{w}^1(\xi,1)|^2 + k_0 |\hat{w}^1(\xi,0)|^2 \right] d\xi. \qquad (4.29)
\end{aligned}$$

For simplicity, denoting $\phi(y) = i\hat{w}^1(\xi,y), \psi(y) = \hat{w}^2(\xi,y)$ for fixed $\xi \neq 0$, then (4.29) becomes

$$\begin{aligned}
&-2\pi \int \mu |\nabla \mathbf{w}|^2 dxdy + 2\pi \int_{\mathbb{R}} k_1 |w^1(x,1)|^2 dx + 2\pi \int_{\mathbb{R}} k_0 |w^1(x,0)|^2 dx \\
=& -\int_{\mathbb{R}} \mu \int_0^1 (|\xi\phi|^2 + |\xi\psi|^2 + |\phi'|^2 + |\psi'|^2) dy - (k_1|\phi(1)|^2 + k_0|\phi(0)|^2) d\xi, \qquad (4.30)
\end{aligned}$$

where $' = \partial_y$. Set

$$Z(\phi,\psi;\xi) = -\mu \int_0^1 (|\xi\phi|^2 + |\xi\psi|^2 + |\phi'|^2 + |\psi'|^2) dy + k_1|\phi(1)|^2 + k_0|\phi(0)|^2.$$

Clearly,

$$Z(\phi,\psi;\xi)=Z(\Re\phi,\Re\psi;\xi)+Z(\Im\phi,\Im\psi;\xi).$$

Thus, it suffice to bound Z when ϕ,ψ are real-value functions.

Notice that $\mathrm{div}\mathbf{w}=0$, so $\xi\phi+\psi'=0$. Then, using (1.25), we may rewrite

$$Z(\phi,\psi;\xi)=-2E(\psi,\xi)/\xi^2,\ \xi\neq 0$$

and hence it follows from the definition and Proposition 3.7 that

$$Z(\phi,\psi;\xi)\leq\frac{2\lambda(\xi^2)}{\xi^2}J(\psi;\xi^2)=\frac{\lambda(\xi^2)}{\xi^2}\int_0^1(\xi^2|\psi|^2+|\psi'|^2)dy\leq\Lambda\int_0^1(|\psi|^2+|\phi|^2)dy. \tag{4.31}$$

Translating this inequality back to the original form yields that

$$\mu\int_0^1(|i\xi\hat{w}^1(\xi,y)|^2+|i\xi\hat{w}^2(\xi,y)|^2+|\partial_y\hat{w}^1(\xi,y)|^2+|\partial_y\hat{w}^2(\xi,y)|^2)dy$$
$$+k_1|\hat{w}^1(\xi,1)|^2+k_0|\hat{w}^1(\xi,0)|^2\leq\Lambda\int_0^1(|i\hat{w}^1|^2+|\hat{w}^2|^2)dy. \tag{4.32}$$

Then, integrating each side of this inequality over all $\xi\in\mathbb{R}$ and using (4.28), we obtain (4.27). The Proposition follows. □

Now we are on the position to prove the nonlinear instability.

By Theorem 1.1, one can construct a solution to the linear problem (1.6)-(1.9) in the form:

$$\bar{\mathbf{u}}(x,y,t)=\begin{cases}\bar{u}^1(x,y,t)=-\frac{1}{2\pi}\int_{\mathbb{R}}f(\xi^2)i\phi(\xi^2,y)e^{\lambda(\xi^2)t}e^{ix\xi}d\xi\\ \bar{u}^2(x,y,t)=\frac{1}{2\pi}\int_{\mathbb{R}}f(\xi^2)\psi(\xi^2,y)e^{\lambda(\xi^2)t}e^{ix\xi}d\xi\end{cases}\in H^2 \tag{4.33}$$

with initial data

$$\bar{\mathbf{u}}_0(x,y)=\begin{cases}\bar{u}^1(x,y,0)=-\frac{1}{2\pi}\int_{\mathbb{R}}f(\xi^2)i\phi(\xi^2,y)e^{ix\xi}d\xi\\ \bar{u}^2(x,y,0)=\frac{1}{2\pi}\int_{\mathbb{R}}f(\xi^2)\psi(\xi^2,y)e^{ix\xi}d\xi\end{cases}\in H^2 \tag{4.34}$$

satisfying $\mathrm{div}\bar{\mathbf{u}}_0=0$ and $\|\bar{\mathbf{u}}_0\|_{H^2}=1$.

Moreover, one can suitably choose the cut-off function $f\in C_0^\infty(0,\xi_c^2)$ such that

$$\lambda_*\leq\lambda_f<\Lambda, \tag{4.35}$$

where λ_f and Λ are defined in (3.39) and (3.30), and $\lambda_*>\frac{\Lambda}{2}$ will be determined later.

Denote $\mathbf{u}_0^\delta:=\delta\bar{\mathbf{u}}_0$ and $C_2:=\|\bar{\mathbf{u}}_0\|_{L^2}$. By Proposition 4.1, for any $\delta\in(0,\bar{\sigma})$, there exists a global strong solution $(\mathbf{u}^\delta,p^\delta)\in C([0,T];H^2\times H^1)$ to (1.5)-(1.8), with the initial data $\mathbf{u}_0^\delta$ satisfying $\|\mathbf{u}_0^\delta\|_{H^2}=\delta$.

Then, for any $\delta\in(0,\bar{\sigma})$ such that $\delta<\varepsilon_0$, define

$$T^\delta:=\frac{1}{\lambda_*}\ln\frac{\varepsilon_0}{\delta}\text{ i.e. }\delta e^{\lambda_*T^\delta}=\varepsilon_0, \tag{4.36}$$

where $\varepsilon_0 > 0$, independent of δ, is a small constant to be determined, and $\lambda_* = \lambda_*(\varepsilon_0, \delta)$ is the same parameter as in (4.35).

Furthermore, define

$$T^* = \sup\{t \in (0, +\infty) \big| \|\mathbf{u}^\delta\|_{H^2} \le \bar{\sigma}\} \tag{4.37}$$

and

$$T^{**} = \sup\{t \in (0, +\infty) \big| \|\mathbf{u}^\delta\|_{L^2} \le 2C_2\delta e^{\lambda_* t}\}. \tag{4.38}$$

Obviously, $T^*, T^{**} > 0$ and

$$\|\mathbf{u}^\delta(T^*)\|_{H^2} = \bar{\sigma}, \text{ if } T^* < +\infty, \tag{4.39}$$

$$\|\mathbf{u}^\delta(T^{**})\|_{L^2} = 2C_2\delta e^{\lambda_* T^{**}}, \text{ if } T^{**} < +\infty. \tag{4.40}$$

For any $t \le \min\{T^*, T^{**}, T^\delta\}$, (4.19) implies that

$$\begin{aligned}\|\mathbf{u}^\delta(t)\|_{H^2}^2 + \|\mathbf{u}_t^\delta(t)\|_{L^2}^2 &\le C_1\|\mathbf{u}_0^\delta\|_{H^2}^2 + C_1\int_0^t (2C_2\delta e^{\lambda_* s})^2 ds \\ &\le C_1\delta^2 + 2C_1C_2^2\delta^2 e^{2\lambda_* t}/\lambda_* := C_3\delta^2 e^{2\lambda_* t},\end{aligned} \tag{4.41}$$

where C_3, independent of δ, is a positive constant.

Denote $\mathbf{u}^d = \mathbf{u}^\delta - \delta\bar{\mathbf{u}}$ and $\mathbf{u}_\delta^L = \delta\bar{\mathbf{u}}$. Note that $\mathbf{u}_\delta^L$ is also a strong solution to the linearized problem (1.6)-(1.9) with the initial data $\mathbf{u}_0^\delta \in H^2$. Thus $\mathbf{u}^d$ solves

$$\begin{cases}\mathbf{u}_t^d + \nabla p^d - \mu\Delta\mathbf{u}^d = -\mathbf{u}^\delta \cdot \nabla\mathbf{u}^\delta, \\ \operatorname{div}\mathbf{u}^d = 0,\end{cases} \tag{4.42}$$

with the boundary conditions

$$u^{d,2}(x,0) = u^{d,2}(x,1) = 0, \ x \in \mathbb{R}, \tag{4.43}$$

$$\partial_y u^{d,1}(x,1) = \frac{k_1}{\mu}u^{d,1}(x,1), \ x \in \mathbb{R}, \tag{4.44}$$

$$\partial_y u^{d,1}(x,0) = -\frac{k_0}{\mu}u^{d,1}(x,0), \ x \in \mathbb{R}, \tag{4.45}$$

where $u^{d,1}$ and $u^{d,2}$ stand for the first and second component of $\mathbf{u}^d$ respectively, and the initial condition $\mathbf{u}^d(0) = \mathbf{0}$.

Multiplying $(4.42)_1$ by $\mathbf{u}^d$ gives that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{u}^d|^2 = -\mu\int |\nabla\mathbf{u}^d|^2 + \sum_{i=0}^{1}\int_{\mathbb{R}} k_i|u^{d,1}(x,i)|^2 - \int \mathbf{u}^\delta \cdot \nabla\mathbf{u}^\delta \cdot \mathbf{u}^d. \tag{4.46}$$

Notice that

$$\int \mathbf{u}^\delta \cdot \nabla\mathbf{u}^\delta \cdot \mathbf{u}^d \le \int |\mathbf{u}^\delta \cdot \nabla\mathbf{u}^\delta||\mathbf{u}^d| \le \|\mathbf{u}^\delta \cdot \nabla\mathbf{u}^\delta\|_{L^2}\|\mathbf{u}^d\|_{L^2} \le C_4\|\mathbf{u}^\delta\|_{H^2}^2\|\mathbf{u}^d\|_{L^2}. \tag{4.47}$$

In addition, Proposition 4.2 implies that

$$-\mu\int|\nabla\mathbf{u}^d|^2+\sum_{i=0}^{1}\int_{\mathbb{R}}k_i|u^{d,1}(x,i)|^2\le\Lambda\int|\mathbf{u}^d|^2, \tag{4.48}$$

where $\Lambda>0$ is defined in (3.30).

Substituting (4.47) and (4.48) into (4.46) gives that

$$\frac{d}{dt}\|\mathbf{u}^d\|_{L^2}\le\Lambda\|\mathbf{u}^d\|_{L^2}+C_4\|\mathbf{u}^\delta\|_{H^2}^2. \tag{4.49}$$

Thus, it follows from the Gronwall inequality, (4.41) and (4.49) that

$$\|\mathbf{u}^d\|_{L^2}\le C_4e^{\Lambda t}\int_0^t e^{-\Lambda s}\|\mathbf{u}^\delta(s)\|_{H^2}^2ds\le C_3C_4\delta^2e^{\Lambda t}\int_0^t e^{(2\lambda_*-\Lambda)s}ds\le C_5\delta^2e^{2\lambda_*t}, \tag{4.50}$$

where the condition $2\lambda_*-\Lambda>0$ has been used.

Now we claim that

$$T^\delta=\min\{T^\delta,T^*,T^{**}\},\ \text{provided}\ \varepsilon_0=\min\{\frac{\bar\sigma}{2\sqrt{C_3}},\frac{C_2}{4C_5}\}. \tag{4.51}$$

Indeed, if $T^*=\min\{T^\delta,T^*,T^{**}\}$, then $T*<+\infty$. It follows from (4.41) and (4.36) that

$$\|\mathbf{u}^\delta(T^*)\|_{H^2}\le\sqrt{C_3}\delta e^{\lambda_*T^*}\le\sqrt{C_3}\delta e^{\lambda_*T^\delta}=\sqrt{C_3}\varepsilon_0<\bar\sigma, \tag{4.52}$$

which contradicts (4.39).

If $T^{**}=\min\{T^\delta,T^*,T^{**}\}$, then $T^{**}<+\infty$. In view of (3.37),(4.36) and (4.50), one obtains that

$$\begin{aligned}\|\mathbf{u}^\delta(T^{**})\|_{L^2}&\le\|\mathbf{u}^L_\delta(T^{**})\|_{L^2}+\|\mathbf{u}^d(T^{**})\|_{L^2}\le C_2\delta e^{\Lambda T^{**}}+C_5\delta^2e^{2\lambda_*T^{**}}\\&\le C_2\delta e^{\lambda_*T^{**}}\left(e^{(\Lambda-\lambda_*)T^\delta}+\frac{C_5}{C_2}\delta e^{\lambda_*T^\delta}\right)\le C_2\delta e^{\lambda_*T^{**}}\left[\left(\frac{\varepsilon_0}{\delta}\right)^{\frac{\Lambda}{\lambda_*}-1}+\frac{1}{4}\right].\end{aligned} \tag{4.53}$$

Take

$$\lambda_*=\Lambda\ln\left(\frac{2\varepsilon_0}{\delta}\right)/\ln\left(\frac{5\varepsilon_0}{2\delta}\right). \tag{4.54}$$

Then $\lambda_*>\Lambda/2$ since that $\varepsilon_0>\delta$. Therefore,

$$\|\mathbf{u}^\delta(T^{**})\|_{L^2}<2C_2\delta e^{\lambda_*T^{**}}, \tag{4.55}$$

which contradicts (4.40). Therefore, (4.51) holds.

Finally, we use (3.37),(4.36) and (4.50) to deduce that

$$\|\mathbf{u}^\delta(T^\delta)\|_{L^2}\ge\|\mathbf{u}^L_\delta(T^\delta)\|_{L^2}-\|\mathbf{u}^d(T^\delta)\|_{L^2}\ge C_2\delta e^{\lambda_*T^\delta}-C_5\delta^2e^{2\lambda_*T^\delta}>C_2\varepsilon_0/2, \tag{4.56}$$

which completes the proof of Theorem 1.2 (i) by defining $\varepsilon:=C_2\varepsilon_0/2$.

5 The linear and nonlinear stability

In the first subsection, we will prove Theorem 1.3, namely, asymptotic stability of the linear and nonlinear system under the assumption of $\mu > \mu_c \geq 0$. We will analyse for the case $\mu \geq \mu_c \geq 0$ in the second subsection to complete the proof of the stability part of Theorem 1.1 and the proof of Theorem 1.2 (ii).

It follows from Proposition 4.2 and Remark 3.6 that for any $\mathbf{u}(t) \in H^1_\sigma \cap H^2$, it holds that

$$\int_{\mathbb{R}} \left[k_1|u^1(x,1))|^2 + k_0|u^1(x,0)|^2\right] - \mu \int |\nabla \mathbf{u}|^2 \leq \Lambda \int |\mathbf{u}|^2, \tag{5.1}$$

where $\Lambda < 0$ provided $\mu > \mu_c$, while $\Lambda = 0$ for $\mu = \mu_c > 0$. This is crucial for the proof of the stability. In what follows, for simplicity, we denote by C a generic positive constant, which may depend on k_1, k_0 and μ.

5.1 Proof of Theorem 1.3.

Proof of Theorem 1.3 (i): general initial data.

Standard energy estimates and (5.1) yield

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2} = \sum_{i=0}^{1}\int_{\mathbb{R}} k_i|u^1(x,i)|^2 - \mu\|\nabla \mathbf{u}\|^2_{L^2} \leq \Lambda\|\mathbf{u}\|^2_{L^2}, \tag{5.2}$$

where $\Lambda < 0$. This implies that

$$\|\mathbf{u}(t)\|_{L^2} \leq e^{\Lambda t}\|\mathbf{u}_0\|_{L^2}. \tag{5.3}$$

In addition, one has

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2} + (\mu - \mu_c)\|\nabla \mathbf{u}\|^2_{L^2} = \sum_{i=0}^{1}\int_{\mathbb{R}} k_i|u^1(x,i)|^2 - \mu_c\|\nabla \mathbf{u}\|^2_{L^2} \leq 0, \tag{5.4}$$

which gives that

$$\|\mathbf{u}(t)\|^2_{L^2} + \int_0^t \|\nabla \mathbf{u}(t)\|^2_{L^2} \leq C\|\mathbf{u}_0\|^2_{L^2}. \tag{5.5}$$

Applying ∂_t to $(1.5)_1$, taking the inner product of the result with $\mathbf{u}_t$, and treating the boundary terms as in (5.4), one gets that for any $\epsilon > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2_{L^2} + (\mu - \mu_c)\|\nabla \mathbf{u}_t\|^2_{L^2} \lesssim \int |\mathbf{u}_t|^2|\nabla \mathbf{u}| \\ \lesssim &\|\mathbf{u}_t\|^2_{L^4}\|\nabla \mathbf{u}\|_{L^2} \lesssim \|\mathbf{u}_t\|_{L^2}\|\mathbf{u}_t\|_{H^1}\|\nabla \mathbf{u}\|_{L^2} \leq \epsilon\|\mathbf{u}_t\|^2_{H^1} + C_\epsilon\|\nabla \mathbf{u}\|^2_{L^2}\|\mathbf{u}_t\|^2_{L^2}, \end{aligned} \tag{5.6}$$

where Hölder inequality, Young inequality and Sobolev embedding theorems have been used.

It follows from (4.11) and a similar argument as for (5.6) that

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2 &\lesssim \int(|\mathbf{u}||\nabla\mathbf{u}||\mathbf{u}_t|+|\mathbf{u}_t||\mathbf{u}|+|\nabla\mathbf{u}_t||\mathbf{u}|+|\mathbf{u}_t||\nabla\mathbf{u}|)\\ &\lesssim \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^4}+\|\mathbf{u}\|_{H^1}\|\mathbf{u}_t\|_{H^1}\\ &\le \epsilon\|\mathbf{u}_t\|_{H^1}^2+C_\epsilon(\|\mathbf{u}\|_{H^1}^4+\|\mathbf{u}\|_{H^1}^2).\end{aligned}\tag{5.7}$$

Adding (5.6) and (5.7) and taking ϵ small enough yield

$$\frac{d}{dt}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2\right)+\|\mathbf{u}_t\|_{H^1}^2\le C\|\nabla\mathbf{u}\|_{L^2}^2\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2\right)+C\|\mathbf{u}\|_{H^1}^2.\tag{5.8}$$

Notice that

$$\int_0^t\|\mathbf{u}(s)\|_{H^1}^2\le\int_0^t e^{\Lambda s}\|\mathbf{u}_0\|_{L^2}^2ds+\int_0^t\|\nabla\mathbf{u}(s)\|_{L^2}^2ds\le C\|\mathbf{u}_0\|_{L^2}^2,\tag{5.9}$$

where (5.3) and (5.5) have been used.

Thus, by applying Gronwall inequality to (5.8), one gets

$$\|\nabla\mathbf{u}(t)\|_{L^2}^2+\|\mathbf{u}_t(t)\|_{L^2}^2+\int_0^t\|\mathbf{u}_t(s)\|_{H^1}^2ds\le C\|\mathbf{u}_0\|_{H^2}^2,\tag{5.10}$$

where (4.9) has been used.

By the Stokes estimate (A.2), we have

$$\begin{aligned}\|\mathbf{u}(t)\|_{H^2}+\|\nabla q(t)\|_{L^2} &\lesssim \|\mathbf{u}_t(t)\|_{L^2}+\|\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\|_{L^2}\\ &\lesssim \|\mathbf{u}_t(t)\|_{L^2}+\|\mathbf{u}(t)\|_{L^\infty}\|\nabla\mathbf{u}(t)\|_{L^2}\\ &\lesssim \|\mathbf{u}_t(t)\|_{L^2}+\|\mathbf{u}(t)\|_{L^2}^{1/2}\|\mathbf{u}(t)\|_{H^2}^{1/2}\|\nabla\mathbf{u}(t)\|_{L^2}\\ &\le \frac{1}{2}\|\mathbf{u}(t)\|_{H^2}+C\|\mathbf{u}_t(t)\|_{L^2}+C\|\mathbf{u}(t)\|_{L^2}\|\nabla\mathbf{u}(t)\|_{L^2}^2,\end{aligned}\tag{5.11}$$

which, together with (5.5) and (5.10), implies that

$$\|\mathbf{u}(t)\|_{H^2}\le C\|\mathbf{u}_0\|_{H^2}.\tag{5.12}$$

Furthermore, interpolation inequality implies that

$$\|\nabla\mathbf{u}(t)\|_{L^2}\le C\|\mathbf{u}(t)\|_{H^2}^{1/2}\|\mathbf{u}(t)\|_{L^2}^{1/2}\le Ce^{\Lambda t/2},\tag{5.13}$$

which, together with (5.3), yields the third inequality of (1.31) by taking $\alpha=-\Lambda/2$.

Notice that (5.10) implies $\|\mathbf{u}_t(s)\|_{L^2}^2\to 0$ as $t\to+\infty$. Then one can see from (5.11) and (5.13) that $\|\mathbf{u}(t)\|_{H^2}\to 0$ as $t\to+\infty$.

Theorem 1.3 (i) follows.

Proof of Theorem 1.3 (ii): small initial data.

In fact, replacing $\mathbf{u}$ in (5.1) by $\mathbf{u}_t$ and integrating by part, one can re-estimate (5.6) as

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|_{L^2}^2+(-\Lambda)\|\mathbf{u}_t\|_{L^2}^2\le\int|\mathbf{u}_t||\nabla\mathbf{u}_t||\mathbf{u}|$$

$$\begin{aligned}&\lesssim \|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{L^\infty} \lesssim \|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{H^2}\\ &\le \epsilon\|\nabla \mathbf{u}_t\|_{L^2}^2 + C_\epsilon\|\mathbf{u}_t\|_{L^2}^2\|\mathbf{u}\|_{H^2}^2. \end{aligned} \tag{5.14}$$

Adding (5.14) and (5.6) with $\epsilon > 0$ small enough, we obtain that

$$\frac{d}{dt}\|\mathbf{u}_t\|_{L^2}^2 + \frac{(-3\Lambda)}{4}\|\mathbf{u}_t\|_{L^2}^2 \le C\|\mathbf{u}_t\|_{L^2}^2\|\mathbf{u}\|_{H^2}^2. \tag{5.15}$$

Taking $\|\mathbf{u}_0\|_{H^2}$ sufficiently small in (5.12), we have $C\|\mathbf{u}(t)\|_{H^2}^2 \le -\Lambda/4$. Then, we have

$$\frac{d}{dt}\|\mathbf{u}_t(t)\|_{L^2}^2 \le \Lambda/2\|\mathbf{u}_t(t)\|_{L^2}^2$$

which implies that

$$\|\mathbf{u}_t(t)\|_{L^2} \le Ce^{\Lambda t/2}. \tag{5.16}$$

(1.32) follows by (5.16),(5.11) and (5.13), where $\beta = -\Lambda/2$.

Theorem 1.3 (ii) is proved. □

5.2 Proof of the stability part of Theorem 1.1 and Theorem 1.2(ii)

In this subsection, we prove the stability part of Theorem 1.1. For Theorem 1.2(ii), we will give a remark at the end.

Proof of the stability part of Theorem 1.1.

Step 1. $\mu > \mu_c$: decay estimates.

In fact, one can see that in the linearized situation, by similar energy method as used in the proof of Theorem 1.3, it is easy to obtain the decay rate that

$$\|\mathbf{u}(t)\|_{H^2}^2 \le Ce^{\Lambda t/2}\|\mathbf{u}_0\|_{H^2}^2,$$

which automatically implies that $\|\mathbf{u}(t)\|_{H^2}^2 \to 0$ as $t \to 0$, since $\Lambda < 0$ provided $\mu > \mu_c$. It should be noticed that in the linearized situation, the initial data need not to be small for us to obtain this decay estimate.

Step 2. $\mu = \mu_c$: continuous dependence on initial data.

Similarly to (5.2), since in this case, $\Lambda = 0$, one only has

$$\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 = \int_{\mathbb{R}} \left[k_1|u^1(x,1)|^2 + k_0|u^1(x,0)|^2\right] - \mu_c\|\nabla \mathbf{u}\|_{L^2}^2 \le 0. \tag{5.17}$$

Multiplying $(1.9)_1$ by $\mathbf{u}_t$, using $(1.9)_2$ and the boundary conditions yield

$$\frac{1}{2}\frac{d}{dt}\left(\mu_c\|\nabla \mathbf{u}\|_{L^2}^2 - \sum_{i=0}^{1} k_i\int_{\mathbb{R}}|u^1(x,i)|^2\right) + \|\mathbf{u}_t\|_{L^2}^2 = 0. \tag{5.18}$$

Similar to (4.6) in Section 4, one has

$$\frac{d}{dt}\|\mathbf{u}_t(t)\|_{L^2}^2 + \mu\|\nabla \mathbf{u}_t(t)\|_{L^2}^2 \le 2C_0\|\mathbf{u}_t\|_{L^2}^2. \tag{5.19}$$

Adding up $K_4 \times$ (5.18), $K_5 \times$ (5.17) and (5.19) with suitably large $K_4 > 0$, we arrive at

$$\frac{d}{dt}\left(\|(\sqrt{K_5}\mathbf{u}, \sqrt{\mu_c K_4}\nabla \mathbf{u}, \mathbf{u}_t)(t)\|_{L^2}^2 - K_4 \sum_{i=0}^{1} k_i \int_{\mathbb{R}} |u^1(x,i,t)|^2 \right) \leq 0. \tag{5.20}$$

Finally, integrating (5.20) over $(0,t)$, we obtain

$$\|(\sqrt{K_5}\mathbf{u}, \sqrt{\mu_c K_4}\nabla \mathbf{u}, \mathbf{u}_t)(t)\|_{L^2}^2 - K_4 \sum_{i=0}^{1} k_i \int_{\mathbb{R}} |u^1(x,i,t)|^2 \leq C\|\mathbf{u}_0\|_{H^2}^2. \tag{5.21}$$

Taking $K_5 > 0$ large enough and applying the Stokes estimates (A.3) imply that

$$\|\mathbf{u}(t)\|_{H^2}^2 + \int_0^t \|\mathbf{u}_t(s)\|_{H^1}^2 ds \leq C\|\mathbf{u}_0\|_{H^2}^2. \tag{5.22}$$

The stability part of Theorem 1.1 is proved. □

Remark 5.1. *In this remark, we state a proof of Theorem 1.2 (ii). In fact, for the general case $\mu \geq \mu_c$, it follows from (5.3) that $\|\mathbf{u}(t)\|_{L^2}^2 \leq \|\mathbf{u}_0\|_{L^2}^2$, which completes the proof of Theorem 1.2 (ii).*

Appendix

A The Stokes estimates

Denote that $\Gamma_i := \mathbb{R} \times \{i\}, i = 0, 1$, and $\Omega := \mathbb{R} \times (0,1)$. Consider the following Stokes equations with Navier-slip boundary conditions,

$$\begin{cases} -\mu\Delta\mathbf{u} - \nabla p = \mathbf{F}, & \Omega, \\ \text{div}\mathbf{u} = 0, & \Omega, \\ u^2 = 0, & \Gamma_0 \cup \Gamma_1, \\ \mu\partial_y u^1 - k_1 u^1 = 0, & \Gamma_1, \\ \mu\partial_y u^1 + k_0 u^1 = 0, & \Gamma_0. \end{cases} \tag{A.1}$$

Theorem A.1. *Suppose that $\mu > 0$, $\mathbf{F} \in H^{m-1}$, $m \in \mathbb{N}$ and $(\mathbf{u}, p)$ solves (A.1). Then the following claims holds.*

(i) If $\mu > \mu_c > 0$, then

$$\|\mathbf{u}\|_{H^{m+1}}^2 + \|\nabla p\|_{H^{m-1}}^2 \leq C\|\mathbf{F}\|_{H^{m-1}}^2, \tag{A.2}$$

where μ_c is defined in (1.27) and C is a positive constant depending only on μ, k_1, k_0, m.

(ii) If $0 < \mu \leq \mu_c$, it holds that

$$\|\mathbf{u}\|_{H^{m+1}}^2 + \|\nabla p\|_{H^{m-1}}^2 \leq C\left(\|\mathbf{F}\|_{H^{m-1}}^2 + \|\mathbf{u}\|_{L^2}^2\right), \tag{A.3}$$

where C is a positive constant depending only on μ, k_1, k_0, m.

Proof of (i) ($\mu > \mu_c > 0$).

Step 1. Multiplying $(A.1)_1$ by $\mathbf{u}$ and integrating by part over Ω, we have

$$\mu \int |\nabla \mathbf{u}|^2 - k_1 \int_{\Gamma_1} |u^1|^2 - k_0 \int_{\Gamma_0} |u^1|^2 = \int \mathbf{F} \cdot \mathbf{u}.$$

Since $\mu > \mu_c$, one may choose $\delta > 0$ such that $\mu - \mu_c - \delta > 0$. Rewriting above equality as

$$(\mu - \mu_c - \delta) \int |\nabla \mathbf{u}|^2 + (\mu_c + \delta) \int |\nabla \mathbf{u}|^2 - k_1 \int_{\Gamma_1} |u^1|^2 - k_0 \int_{\Gamma_0} |u^1|^2 = \int \mathbf{F} \cdot \mathbf{u},$$

and using Proposition 4.2, one gets

$$(\mu - \mu_c - \delta) \int |\nabla \mathbf{u}|^2 - \Lambda \int |\mathbf{u}|^2 \leq \int \mathbf{F} \cdot \mathbf{u}$$

where $\Lambda < 0$ is a constant which implies

$$\|\mathbf{u}\|_{H^1} \leq C \|\mathbf{F}\|_{H^{-1}}. \tag{A.4}$$

Step 2. Applying horizontal differential operator ∇_x^m to (A.1), one has, similar to (A.4), that

$$\|\nabla_x^m \mathbf{u}\|_{H^1} \leq C \|\nabla_x^m \mathbf{F}\|_{H^{-1}} \leq C \|\mathbf{F}\|_{H^{m-1}}. \tag{A.5}$$

Step 3. Since that $\partial\Omega$ is horizontally flat, one has

$$\|\mathbf{u}\|^2_{H^m(\partial\Omega)} = \|\mathbf{u}\|^2_{L^2(\partial\Omega)} + \|\nabla_x^m \mathbf{u}\|^2_{L^2(\partial\Omega)}.$$

Then, the trace theorem and (A.4)-(A.5) yield

$$\begin{aligned} \|\mathbf{u}\|^2_{H^{m+\frac{1}{2}}(\partial\Omega)} &= \|\mathbf{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \|\nabla_x^m \mathbf{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C \left(\|\mathbf{u}\|^2_{H^1} + \|\nabla_x^m \mathbf{u}\|^2_{H^1} \right) \\ &\leq C \|\mathbf{F}\|^2_{H^{m-1}}. \end{aligned} \tag{A.6}$$

Step 4. By the regularity of $\mathbf{u}$ on the boundary, (A.6), one may use the classical estimates for the following problem

$$\begin{cases} -\mu \Delta \mathbf{u} - \nabla p = \mathbf{F}, & \Omega, \\ \mathrm{div} \mathbf{u} = 0, & \Omega, \\ \mathbf{u} = \mathbf{u}, & \partial\Omega \end{cases} \tag{A.7}$$

to obtain the following inequality (see [18, 24])

$$\|\mathbf{u}\|_{H^{m+1}} + \|\nabla p\|_{H^{m-1}} \leq C \left(\|\mathbf{F}\|_{H^{m-1}} + \|\mathbf{u}\|_{H^{m+\frac{1}{2}}(\partial\Omega)} \right) \leq C \|\mathbf{F}\|_{H^{m-1}} \tag{A.8}$$

where (A.6) has been used. Thus (i) is proved.

Proof of (ii) ($0 < \mu \leq \mu_c$).

In this case, we also have

$$\mu \int |\nabla \mathbf{u}|^2 - k_1 \int_{\Gamma_1} |u^1|^2 - k_0 \int_{\Gamma_0} |u^1|^2 = \int \mathbf{F} \cdot \mathbf{u}$$

which implies that

$$\begin{aligned} \mu \int |\nabla \mathbf{u}|^2 &\le |k_1| \int_{\Gamma_1} |u^1|^2 + |k_0| \int_{\Gamma_0} |u^1|^2 + \int \mathbf{F} \cdot \mathbf{u} \\ &= \int_{\mathbb{R}} dx \int_0^1 \left[((|k_0| + |k_1|)y - |k_0|) (u^1)^2 \right]_y' dy + \int \mathbf{F} \cdot \mathbf{u} \\ &\le \frac{\mu}{2} \int |\nabla \mathbf{u}|^2 + C \int |\mathbf{u}|^2 + C \|\mathbf{F}\|_{H^{-1}(\Omega)}^2. \end{aligned}$$

Therefore,

$$\|\mathbf{u}\|_{H^1} \le C \left(\|\mathbf{F}\|_{H^{-1}} + \|\mathbf{u}\|_{L^2} \right). \tag{A.9}$$

Now claim (ii) follows from (A.9) and the similar steps in the proof of (i).

Theorem A.1 follows. □

Acknowledgment

The authors would like to thank Professor Yanjin Wang and Professor Huanyao Wen for their constructive comments and discussions in the preparation of the paper.

Ding's reseach is supported by the National Natural Science Foundation of China (No.11071086, No.11371152, No.11128102 and No.11571117).

Xin's research is partially supported by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK-14305315 and CUHK4048/13P, NSFC/RGC Joint Research Scheme Grant N-CUHK 443-14, and a Focus Area Grant from the Chinese University of Hong Kong.

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