

Boundary Layer Problems in the Viscosity-Diffusion Vanishing Limits for the Incompressible MHD Systems

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Abstract: In this paper, we study boundary layer problems for the incompressible MHD systems in the presence of physical boundaries with the standard Dirichlet boundary conditions with small generic viscosity and diffusion coefficients. We identify a non-trivial class of initial data for which we can establish the uniform stability of the Prandtl's type boundary layers and prove rigorously that the solutions to the viscous and diffusive incompressible MHD systems converges strongly to the superposition of the solution to the ideal MHD systems with a Prandtl's type boundary layer corrector. One of the main difficulties is to deal with the effect of the difference between viscosity and diffusion coefficients and to control the singular boundary layers resulting from the Dirichlet boundary conditions for both the viscosity and the magnetic fields. One key derivation here is that for the class of initial data we identify here, there exist cancelations between the boundary layers of the velocity field and that of the magnetic fields so that one can use an elaborate energy method to take advantage this special structure. In addition, in the case of fixed positive viscosity, we also establish the stability of diffusive boundary layer for the magnetic field and convergence of solutions in the limit of zero magnetic diffusion for general initial data.

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1 Introduction

We consider in this paper boundary layer problems, zero viscosity-diffusion vanishing inviscid limit and zero magnetic diffusion vanishing limit for the three/two-dimensional incompressible viscous and diffusive magnetohydrodynamic(MHD) systems with Dirichlet boundary (no-slip characteristic) boundary conditions

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon_1 \Delta u^\varepsilon = b^\varepsilon \cdot \nabla b^\varepsilon, \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$\partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - \varepsilon_2 \Delta b^\varepsilon = b^\varepsilon \cdot \nabla u^\varepsilon, \quad \text{in } \Omega \times (0, T), \quad (1.2)$$

$$\operatorname{div} u^\varepsilon = 0, \operatorname{div} b^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \quad (1.3)$$

$$u^\varepsilon = 0, b^\varepsilon = 0, \quad \text{in } \partial\Omega \times (0, T), \quad (1.4)$$

$$u^\varepsilon(t=0) = u_0^\varepsilon, b^\varepsilon(t=0) = b_0^\varepsilon, \quad \text{with } \operatorname{div} u_0^\varepsilon = \operatorname{div} b_0^\varepsilon = 0 \quad \text{on } \Omega, \quad (1.5)$$

where $\Omega = \omega \times [0, h]$ or $\Omega = \omega \times (0, \infty)$, and $\omega = \mathbb{T}^2$ or \mathbb{R}^2 in three dimensional case for MHD systems, and $\omega = \mathbb{T}^1$ or \mathbb{R}^1 in two dimensional case for MHD systems. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional or two-dimensional Laplace operator, $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$ is the viscosity coefficient and $\varepsilon_2 = \varepsilon_2(\varepsilon) > 0$ is the magnetic diffusion coefficient. The unknown functions $u^\varepsilon, p^\varepsilon, b^\varepsilon$ are the velocity, the pressure and the magnetic field of MHD.

The well-posedness, regularity and asymptotic limit problem on the incompressible viscous and diffusive MHD systems (1.1)-(1.3) in the whole space or with slip/no-slip boundary conditions have been studied extensively, see [2, 3, 4, 5, 9, 10, 19, 21, 22, 23, 25] and the references therein. When $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, MHD systems in the whole space and in the bounded domains with slip boundary conditions for the velocity and with no-slip boundary condition for the magnetic field has a unique global classical solution for smooth initial data when space dimension $d = 2$ and has a global weak solution for a class of initial data when $d = 3$, see [5, 19]. Some regularity criterions are also given by some authors, see [3, 4, 9, 21, 22] and therein references. Cao and Wu [3] obtain global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion for any smooth initial data. Wu [21] considers the inviscid limit problem of incompressible viscous MHD systems in the whole space. Xiao, Xin and Wu [25] investigate the solvability, regularity and vanishing viscosity limit of incompressible viscous MHD systems with slip without friction boundary conditions.

It should be noted that, as in the zero-viscosity vanishing limit for the Navier-Stokes equations, see[1, 3, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 20, 26] and related references, the zero-viscosity and diffusion vanishing limit for incompressible viscous and diffusive MHD system in a bounded domain, with Dirichlet boundary conditions at the boundary, is a challenging problem due to the possible appearance of boundary layers. Recently, [24] considers boundary layer problems and zero viscosity-diffusion vanishing limit of the incompressible MHD system with no-slip boundary conditions and the case $\varepsilon_1 = \varepsilon_2$ or the case of the the different horizontal and vertical viscosities and magnetic diffusions. In this

paper, we consider the boundary layer problems for the more general case $\varepsilon_1 \neq \varepsilon_2$ and also consider the zero magnetic diffusion limit.

Letting $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ in (1.1)-(1.5), one obtains formally the following inviscid MHD systems

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = b^0 \cdot \nabla b^0, \quad \text{in } \Omega \times (0, T), \quad (1.6)$$

$$\partial_t b^0 + u^0 \cdot \nabla b^0 = b^0 \cdot \nabla u^0, \quad \text{in } \Omega \times (0, T), \quad (1.7)$$

$$\operatorname{div} u^0 = 0, \operatorname{div} b^0 = 0, \quad \text{in } \Omega \times (0, T), \quad (1.8)$$

$$u^0 \cdot n = \pm u_3^0 = 0, b^0 \cdot n = \pm b_3^0 = 0, \quad \text{in } \partial\Omega \times (0, T), \quad (1.9)$$

$$u^0(t=0) = u_0^0, b^0(t=0) = b_0^0, \quad \text{with } \operatorname{div} u_0^0 = \operatorname{div} b_0^0 = 0 \quad \text{on } \Omega. \quad (1.10)$$

On the other hand, setting the magnetic diffusion coefficient $\varepsilon_2 \rightarrow 0$ in (1.1)-(1.5), one gets formally the following viscous MHD systems

$$\partial_t u^{\varepsilon_1, 0} + u^{\varepsilon_1, 0} \cdot \nabla u^{\varepsilon_1, 0} - \varepsilon_1 \Delta u^{\varepsilon_1, 0} + \nabla p^{\varepsilon_1, 0} = b^{\varepsilon_1, 0} \cdot \nabla b^{\varepsilon_1, 0}, \quad \text{in } \Omega \times (0, T), \quad (1.11)$$

$$\partial_t b^{\varepsilon_1, 0} + u^{\varepsilon_1, 0} \cdot \nabla b^{\varepsilon_1, 0} = b^{\varepsilon_1, 0} \cdot \nabla u^{\varepsilon_1, 0}, \quad \text{in } \Omega \times (0, T), \quad (1.12)$$

$$\operatorname{div} u^{\varepsilon_1, 0} = 0, \operatorname{div} b^{\varepsilon_1, 0} = 0, \quad \text{in } \Omega \times (0, T), \quad (1.13)$$

$$u^{\varepsilon_1, 0} = 0, b_3^{\varepsilon_1, 0} = 0, \quad \text{in } \partial\Omega \times (0, T), \quad (1.14)$$

$$u^{\varepsilon_1, 0}(t=0) = u_0^{\varepsilon_1, 0}, b^{\varepsilon_1, 0}(t=0) = b_0^{\varepsilon_1, 0}, \quad \text{with } \operatorname{div} u_0^{\varepsilon_1, 0} = \operatorname{div} b_0^{\varepsilon_1, 0} = 0 \quad \text{on } \Omega. \quad (1.15)$$

The purpose of this paper is to prove rigorously the above formal limits under some assumptions on initial data or viscosity and diffusive coefficients. Since we can recover the incompressible Navier-Stokes equations by taking $b^\varepsilon = 0$ in MHD systems (1.1)-(1.5), and, hence, the basic difficulties caused by such as the well-posedness of the Prandtl's type boundary layer equations, the thickness of the boundary layer and nonlocal pressure when dealing with the boundary layer problem and zero viscosity vanishing limit of the incompressible NS equations in domains with boundaries are kept here. The key ingredients here are that we will be able to identify a non-trivial class of initial data for which there exist cancelations between boundary layers of the velocity field and that of the magnetic field, which make the stability of the boundary layers and uniform convergence possible. Hence the method used here can not be extend directly to deal with the boundary layer problem for incompressible Navier-Stokes equations. Moreover, the zero magnetic diffusion limit for MHD systems in a domain with the boundary is also non-trivial due to the nonlinear coupling of the velocity field and the magnetic field. The boundary layer problem for the magnetic field can not also be obtained and it is remained open in general, see Remark 2.4.

The paper is organized as follows. In section 2 we give the main results of this paper. Section 3 is devoted to the proofs of Main results, including the constructs of the approximating boundary layer functions.

2 The main results

In this section, we state our main Theorems. For this, we first recall the following classical results on the existence of sufficiently regular solutions to the incompressible ideal MHD system (see [5, 19]).

Proposition 2.1 *Let (u_0^0, b_0^0) satisfy $u_0^0, b_0^0 \in H^s(\Omega)$, $s > \frac{3}{2} + 1$, $\operatorname{div} u_0^0 = \operatorname{div} b_0^0 = 0$ and $u_0^0 \cdot n|_{\partial\Omega} = b_0^0 \cdot n|_{\partial\Omega} = 0$. Then there exist $0 < T_* \leq \infty$, the maximal existence time, and a unique smooth solution (u^0, p^0, b^0) , also denoted by $(u^{0,0}, p^{0,0}, b^{0,0})$ below, of the incompressible ideal MHD equations (1.6)-(1.10) on $[0, T_*)$ satisfying, for any $T < T_*$,*

$$\sup_{0 \leq t \leq T} \left(\| (u^0, b^0) \|_{H^s} + \| (\partial_t u^0, \partial_t b^0) \|_{H^{s-1}(\Omega)} \right) \leq C(T).$$

Moreover, if (u_0^0, b_0^0) satisfies $u_0^0 = \pm b_0^0$, then there is a unique smooth solution (u^0, p^0, b^0) of incompressible inviscid MHD system satisfying $u^0(x, y, z, t) = \pm b^0(x, y, z, t)$ ($= u_0^0(x, y, z) = \pm b_0^0(x, y, z)$), $p^0(x, y, z, t) = 0$. Also, there exist the smooth solutions to the initial boundary problems for three/two dimensional incompressible ideal MHD systems for the smooth initial data, which maybe not belong to Sobolev space H^s in unbounded domain, for example, the shear flow.

Similarly, for the incompressible MHD with the viscosity (1.11)-(1.15), it is easy to get the following result on the existence of sufficiently regular solutions.

Proposition 2.2 *Assume that $\varepsilon_1 > 0$ be fixed. Let $(u_0^{\varepsilon_1,0}, b_0^{\varepsilon_1,0})$ satisfy $u_0^{\varepsilon_1,0}, b_0^{\varepsilon_1,0} \in H^s(\Omega)$, $s > \frac{3}{2} + 2$, $\operatorname{div} u_0^{\varepsilon_1,0} = \operatorname{div} b_0^{\varepsilon_1,0} = 0$. Then there exist $0 < T_* \leq \infty$, the maximal existence time, and a unique smooth solution $(u^{\varepsilon_1,0}, p^{\varepsilon_1,0}, b^{\varepsilon_1,0})$ of the incompressible MHD equations (1.11)-(1.15) on $[0, T_*)$ satisfying, for any $T < T_*$,*

$$\begin{aligned} \sup_{0 \leq t \leq T} \left(\| (u^{\varepsilon_1,0}, b^{\varepsilon_1,0}) \|_{H^s(\Omega)} + \| (\partial_t u^{\varepsilon_1,0}, \partial_t b^{\varepsilon_1,0}) \|_{H^{s-2}(\Omega)} \right) \\ + \varepsilon_1 \int_0^T \| \nabla u^{\varepsilon_1,0}(x, y, z, t) \|_{H^s(\Omega)}^2 dt \leq C(T) \end{aligned}$$

for some positive constant $C(T)$ independent of ε_2 . Moreover, if $b_0^{\varepsilon_1,0}|_{\partial\Omega} = 0$, then $b^{\varepsilon_1,0}(x, y, z, t)|_{\partial\Omega} = 0$.

Now we can state the main results of this paper.

For the MHD system (1.1)-(1.5), we have the following stability result of Prandtl's type boundary layer for a class of special initial data.

Theorem 2.3 (Stability of the Prandtl boundary Layer) *Let (u^0, p^0, b^0) be the solution to the incompressible ideal MHD system (1.6)-(1.10). Assume that $(u_0^\varepsilon, b_0^\varepsilon)$ strongly converges in $L^2(\Omega)$ to (u_0^0, b_0^0) , where $u_0^0, b_0^0 \in H^s(\Omega)$, $s > \frac{3}{2} + 1$ satisfies $\operatorname{div} u_0^0 = \operatorname{div} b_0^0 = 0$*

and $u_0^0 \cdot n|_{\partial\Omega} = b_0^0 \cdot n|_{\partial\Omega} = 0$. Assume that $u_0^0(x, y, z) = b_0^0(x, y, z)$ or $u_0^0(x, y, z) = -b_0^0(x, y, z)$. Furthermore, assume that $\varepsilon, \varepsilon_1, \varepsilon_2$ satisfy the following convergence:

$$\frac{\varepsilon_1 + \varepsilon_2}{\sqrt{\varepsilon}} \rightarrow 0, \quad \frac{(\varepsilon_1 - \varepsilon_2)^2}{\sqrt{\varepsilon}\varepsilon(\varepsilon_1 + \varepsilon_2)} \rightarrow 0, \quad \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)} \leq C \min\{\varepsilon_1, \varepsilon_2\} \quad (2.1)$$

for some constant $C > 0$, independent of $\varepsilon, \varepsilon_1, \varepsilon_2$, as $\varepsilon \rightarrow 0, \varepsilon_1 \rightarrow 0, \varepsilon_2 \rightarrow 0$. Then there exists a global Leray-Hopf weak solutions $(u^\varepsilon, p^\varepsilon, b^\varepsilon)$ of (1.1)-(1.5) such that

$$(u^\varepsilon - u^0, b^\varepsilon - b^0) \rightarrow (0, 0) \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \quad (2.2)$$

for any $T : 0 < T < \infty$, as viscosity coefficient $\varepsilon_1 \rightarrow 0$ and diffusion coefficient $\varepsilon_2 \rightarrow 0$.

Moreover, if

$$\|(u_0^\varepsilon - u_0^0, b_0^\varepsilon - b_0^0)\|_{L^2(\Omega)}^2 \leq C\varepsilon^\kappa, \quad \kappa > 1, \quad (2.3)$$

then there exists $C(T)$, independent of ε , such that, for $0 \leq t \leq T < \infty$,

$$\|(u^\varepsilon - u^0, b^\varepsilon - b^0)\|_{L^2(\Omega)}^2 \leq C(T)(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{\varepsilon}} + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}). \quad (2.4)$$

Furthermore, we have the following stronger L^∞ convergence results for the viscous and diffusive incompressible 2D MHD systems: Assume that $\omega = \mathbb{T}^1$ and $u_1^0(x, z = 0) = b_1^0(x, z = 0) = \text{const}$ (for example, the two-dimensional shear flow) and $\varepsilon_1 = \varepsilon_2$ or that $\varepsilon, \varepsilon_1, \varepsilon_2$ satisfy suitable relations (stated below). If, for some suitably large $\kappa > 2$,

$$\|(u_0^\varepsilon - u_0^0 - u_B^\varepsilon(t=0), b_0^\varepsilon - b_0^0 - b_B^\varepsilon(t=0))\|_{H^s(\Omega)}^2 \leq C\varepsilon^\kappa, \quad s > 3, \quad (2.5)$$

then there exists $C(T)$, independent of ε , such that, for $0 \leq t \leq T < \infty$,

$$\|(u^\varepsilon - u^0 - u_B^\varepsilon, b^\varepsilon - b^0 - b_B^\varepsilon)\|_{L^\infty(\Omega \times (0, T))} \leq C(T)\sqrt{\varepsilon} \quad \text{if } \varepsilon_1 = \varepsilon_2 = \varepsilon \quad (2.6)$$

$$\begin{aligned} & \|(u^\varepsilon - u^0 - u_B^\varepsilon, b^\varepsilon - b^0 - b_B^\varepsilon)\|_{L^\infty(\Omega \times (0, T))} \\ & \leq C(T) \left(\left(\frac{\beta_1(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}} \right)^{\frac{1}{4}} (\beta_2(\varepsilon))^{\frac{1}{4}} + (\beta_0(\varepsilon))^{\frac{1}{4}} \left(\frac{\beta_4(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}} \right)^{\frac{1}{4}} \right) \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0. \end{aligned} \quad (2.7)$$

Here $u_B^\varepsilon, b_B^\varepsilon, \beta_0(\varepsilon), \beta_1(\varepsilon), \beta_2(\varepsilon), \beta_4(\varepsilon)$ will be given more precisely in the next section.

Remark 2.1 If $\varepsilon_1 = \varepsilon_2$ or $\varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon + \varepsilon^{\alpha+1}$ with $\alpha > \frac{1}{2}$, then the assumption (2.1) holds. The boundary layers for the velocity field and the magnetic field in Theorem 2.3 occurs, which is the standard Prandtl boundary layer and will be given in the following section 3.1. The proof in establishing the stability of the Prandtl boundary layer here depends strongly upon the special structure of the solution (u^0, p^0, b^0) to the inviscid MHD system, i.e. $u^0 = \pm b^0$, which yields to that there exists the cancelation between the Prandtl boundary layer of the velocity and the one of the magnetic field. Also, the proof of Theorem ?? in the case $\varepsilon_1 \neq \varepsilon_2$ is more complex than that of the special case $\varepsilon_1 = \varepsilon_2 > 0$ discussed in the paper [24], and here we need some new techniques. Of course, if $u_0^0|_{\partial\Omega} = \pm b_0^0|_{\partial\Omega} = 0$ in Theorem 2.3, called well-prepared initial data, then no Prandtl's type boundary layer occurs.

Remark 2.2 For the shear flow $(u^0, p^0, b^0)(z, t) = (u_1^0(z), u_2^0(z), 0, 0, b_1^0(z), b_2^0(z), 0)$ of the incompressible inviscid MHD system, if $u^0 = \pm b^0$, then the Prandtl boundary layer for viscosity and diffusive MHD system is stable by using Theorem 2.3.

For the incompressible MHD system (1.1)-(1.5), we also have the following result on the zero magnetic diffusion limit when one fix the viscosity coefficient $\varepsilon_1 > 0$.

Theorem 2.4 (The Zero Magnetic Diffusion Limit) Let $\varepsilon_1 > 0$ be fixed. Let $\varepsilon = \varepsilon_2 \rightarrow 0$. Let us assume that $(u_0^\varepsilon, b_0^\varepsilon)$ strongly converges in $L^2(\Omega)$ to $(u_0^{\varepsilon_1, 0}, b_0^{\varepsilon_1, 0})$, where $(u_0^{\varepsilon_1, 0}, b_0^{\varepsilon_1, 0}) \in H^s(\Omega)$, $s > \frac{3}{2} + 2$. Assume also that $(u^{\varepsilon_1, 0}, p^{\varepsilon_1, 0}, b^{\varepsilon_1, 0})$ is the smooth solution to the system (1.11)-(1.15) defined on $[0, T^*)$ with $0 < T^* \leq \infty$, given by Proposition 2.2. Then there exists global Leray-Hopf weak solutions $(u^\varepsilon, p^\varepsilon, b^\varepsilon)$ of (1.1)-(1.5) such that

$$(u^\varepsilon, b^\varepsilon) - (u^{\varepsilon_1, 0}, b^{\varepsilon_1, 0}) \rightarrow (0, 0) \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \quad (2.8)$$

for any $T : 0 < T < T^*$, as $\varepsilon_2 \rightarrow 0$.

Moreover, if

$$\|(u_0^\varepsilon - u_0^{\varepsilon_1, 0}, b_0^\varepsilon - b_0^{\varepsilon_1, 0})\|_{L^2(\Omega)}^2 \leq C(\sqrt{\varepsilon_2})^{1-\tau} \quad (2.9)$$

for any given $0 \leq \tau < 1$, then there exists $C(T)$, independent of ε_2 , such that

$$\|(u^\varepsilon, b^\varepsilon) - (u^{\varepsilon_1, 0}, b^{\varepsilon_1, 0})\|_{L^2(\Omega)}^2 \leq C(T)(\sqrt{\varepsilon_2})^{1-\tau}. \quad (2.10)$$

Remark 2.3 This is also one boundary layer problem here. In fact, if $b_0^{\varepsilon_1, 0}|_{\partial\Omega} \neq 0$, then there occurs the boundary layer for the magnetic field due to the difference of the boundary conditions between b^ε and $b^{\varepsilon_1, 0}$ in the boundary $\partial\Omega$ of the domain.

Remark 2.4 When one replaces the viscosity term $\varepsilon_1 \Delta u^\varepsilon$ by $\varepsilon_1 \partial_z^2 u^\varepsilon$ in the system (1.1)-(1.5), similar zero magnetic diffusion limit result in Theorem 2.4 as $\varepsilon_2 \rightarrow 0$ holds. Note that the non-degeneration of the normal direction of the boundary plays a key role in establishing the stability of the boundary layer. In fact, the zero magnetic diffusion limit of the following MHD system

$$\begin{aligned} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon &= b^\varepsilon \cdot \nabla b^\varepsilon, \quad \text{in } \Omega \times (0, T), \\ \partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - \varepsilon_2 \Delta b^\varepsilon &= b^\varepsilon \cdot \nabla u^\varepsilon, \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} u^\varepsilon &= 0, \operatorname{div} b^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\ u_3^\varepsilon &= 0, b^\varepsilon = 0, \quad \text{in } \partial\Omega \times (0, T), \\ u^\varepsilon(t=0) &= u_0^\varepsilon, b^\varepsilon(t=0) = b_0^\varepsilon, \quad \text{with} \quad \operatorname{div} u_0^\varepsilon = \operatorname{div} b_0^\varepsilon = 0 \quad \text{on } \Omega \end{aligned}$$

or

$$\begin{aligned} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon_1 \Delta_{x,y} u^\varepsilon &= b^\varepsilon \cdot \nabla b^\varepsilon, \quad \text{in } \Omega \times (0, T), \\ \partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - \varepsilon_2 \Delta b^\varepsilon &= b^\varepsilon \cdot \nabla u^\varepsilon, \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} u^\varepsilon &= 0, \operatorname{div} b^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\ u_3^\varepsilon &= 0, b^\varepsilon = 0, \quad \text{in } \partial\Omega \times (0, T), \\ u^\varepsilon(t=0) &= u_0^\varepsilon, b^\varepsilon(t=0) = b_0^\varepsilon, \quad \text{with} \quad \operatorname{div} u_0^\varepsilon = \operatorname{div} b_0^\varepsilon = 0 \quad \text{on } \Omega \end{aligned}$$

is open if $b^\varepsilon|_{\partial\Omega} \neq 0$, which yields the appearance of the boundary layer. Here $\Delta_{x,y} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\varepsilon_2 = \varepsilon$.

Remark 2.5 *One of the main difficulties to establish the zero viscosity and diffusion limit is to deal with the terms related to the boundary layers in the error equations. However, the proof is elementary if there occurs no boundary layers. For example, it is easy to prove that there exists a $T > 0$, independent of ε , such that the solution $(u^\varepsilon, p^\varepsilon, b^\varepsilon)$ of the system*

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon_1 \Delta u^\varepsilon = b^\varepsilon \cdot \nabla b^\varepsilon, \quad \text{in } \Omega \times (0, T), \quad (2.11)$$

$$\partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - \varepsilon_2 \Delta b^\varepsilon = b^\varepsilon \cdot \nabla u^\varepsilon, \quad \text{in } \Omega \times (0, T), \quad (2.12)$$

$$\operatorname{div} u^\varepsilon = 0, \operatorname{div} b^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \quad (2.13)$$

$$u^\varepsilon = u^0, b^\varepsilon = b^0, \quad \text{in } \partial\Omega \times (0, T), \quad (2.14)$$

$$u^\varepsilon(t=0) = u_0^\varepsilon, b^\varepsilon(t=0) = b_0^\varepsilon, \quad \text{with } \operatorname{div} u_0^\varepsilon = \operatorname{div} b_0^\varepsilon = 0 \quad \text{on } \Omega \quad (2.15)$$

converges to the solution (u^0, p^0, b^0) of the ideal MHD system (1.6)-(1.10) in the interval $[0, T]$ in some kinds of norm, for example, in $L^2(\Omega)$, when $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$. Here the function (u^0, b^0) given in the boundary condition (2.14) of the system (2.11)-(2.15) is determined by the solution of the ideal MHD system (1.6)-(1.10).

3 The proofs of Theorems 2.3 and 2.4

We will prove Theorems 2.3 and 2.4 when $\Omega = \omega \times [0, h]$ for $0 < h < \infty$ and $\omega = \mathbb{T}^2$. The other cases (for example, when $h = \infty$) are similar. Our proof is based on the asymptotic analysis with multiple scales and the classical energy method. We will divide the proof into two cases. For well-prepared initial data $b^0|_{\partial\Omega} = b_0^0|_{\partial\Omega} = 0$ in Theorem 2.3 or $b_0^{\varepsilon_1, 0}|_{z=0} = 0$, there is no boundary layer for the magnetic field, and, hence, there is no boundary layer for the velocity in the proof of Theorem 2.3 for the case of well-prepared initial data due to $u_0^0(x, y, z) = b_0^0(x, y, z)$. For the general initial data, i.e., $b_0^0|_{\partial\Omega} = u_0^0|_{\partial\Omega} \neq 0$ or $b_0^{\varepsilon_1, 0}|_{z=0} \neq 0$, if one uses energy method to estimate the error function $(u^\varepsilon - u^0, b^\varepsilon - b^0)$ or $(u^\varepsilon - u^{\varepsilon_1, 0}, b^\varepsilon - b^{\varepsilon_1, 0})$, then integrations by parts introduce some terms which are difficult to control, because $u^\varepsilon - u^0, b^\varepsilon - b^0$ or $b^\varepsilon - b^{\varepsilon_1, 0}$ do not vanish at the boundary. So, for general initial data, one needs to construct the boundary layer correctors which allow one to recover zero Dirichlet boundary condition. When $0 < h < \infty$, we will construct the left and right boundary layers respectively. When $h = \infty$, we will construct only the left boundary layer by taking the right boundary layer to be zero. Note that there is no boundary layer for the velocity field in the case of Theorem 2.4.

3.1 The construction of the boundary layers

We only construct the boundary layer $(u_B^\varepsilon, b_B^\varepsilon)$ for the viscous and diffusive MHD system with $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$. In the case that $\varepsilon_1 > 0$ is a fixed given constant, the structure

and its properties of the boundary layer, denoted also by b_B^ε with $\varepsilon = \varepsilon_2$, are the same as b_B^ε by replacing the function b^0 by $b^{\varepsilon_1,0}$ and $\nu_2^* = (\theta\varepsilon_2)^{1+\tau}$ for any $0 \leq \tau < 1$ and for $\theta > 0$ sufficiently small.

Because the velocity u^ε and the magnetic field b^ε satisfy respectively the zero Dirichlet boundary condition at the boundary $z = 0, h$, but $u^0|_{z=0,h} \neq 0$ and $b^0|_{z=0,h} \neq 0$, we need respectively construct the correctors for u^0 and b^0 so as to recover the zero Dirichlet boundary condition for the error functions.

Now, we introduce the following exact boundary layers for the velocity field u^0 :

$$u_B^\varepsilon = u_{B+}^\varepsilon + u_{B-}^\varepsilon, \quad (3.1)$$

$$\begin{aligned} u_{B+}^\varepsilon &= \begin{pmatrix} U_{B+}^\varepsilon \\ \underline{u}_3^\varepsilon \end{pmatrix} = \begin{pmatrix} \underline{u}_1^\varepsilon \\ \underline{u}_2^\varepsilon \\ \underline{u}_3^\varepsilon \end{pmatrix} \\ &= \begin{pmatrix} -u_1^0(x, y, 0, t) e^{-\frac{z}{\sqrt{\nu_1^*}}} (\rho_1(z) - \rho_1'(z) \sqrt{\nu_1^*}) - u_1^0(x, y, 0, t) \rho_1'(z) \sqrt{\nu_1^*} \\ -u_2^0(x, y, 0, t) e^{-\frac{z}{\sqrt{\nu_1^*}}} (\rho_1(z) - \rho_1'(z) \sqrt{\nu_1^*}) - u_2^0(x, y, 0, t) \rho_1'(z) \sqrt{\nu_1^*} \\ \sqrt{\nu_1^*} \partial_z u_3^0(x, y, 0, t) \rho_1(z) (e^{-\frac{z}{\sqrt{\nu_1^*}}} - 1) \end{pmatrix} \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} u_{B-}^\varepsilon &= \begin{pmatrix} U_{B-}^\varepsilon \\ \bar{u}_3^\varepsilon \end{pmatrix} = \begin{pmatrix} \bar{u}_1^\varepsilon \\ \bar{u}_2^\varepsilon \\ \bar{u}_3^\varepsilon \end{pmatrix} \\ &= \begin{pmatrix} -u_1^0(x, y, h, t) e^{-\frac{h-z}{\sqrt{\nu_1^*}}} (\rho_2(z) + \rho_2'(z) \sqrt{\nu_1^*}) + u_1^0(x, y, h, t) \rho_2'(z) \sqrt{\nu_1^*} \\ -u_2^0(x, y, h, t) e^{-\frac{h-z}{\sqrt{\nu_1^*}}} (\rho_2(z) + \rho_2'(z) \sqrt{\nu_1^*}) + u_2^0(x, y, h, t) \rho_2'(z) \sqrt{\nu_1^*} \\ -\sqrt{\nu_1^*} \partial_z u_3^0(x, y, h, t) \rho_2(z) (e^{-\frac{h-z}{\sqrt{\nu_1^*}}} - 1) \end{pmatrix} \end{aligned} \quad (3.3)$$

where $\rho_1(z)$ and $\rho_2(z)$ satisfy

$$\rho_1(0) = 1, \rho_1'(0) = \rho_1''(0) = 0; \rho_1(z) \equiv 0 \text{ for } z \geq \frac{h}{4}; 0 \leq \rho_1(z) \leq 1 \text{ for } z \in [0, h] \quad (3.4)$$

and

$$\rho_2(h) = 1, \rho_2'(h) = \rho_2''(h) = 0; \rho_2(z) \equiv 0 \text{ for } z \leq \frac{3}{4}h; 0 \leq \rho_2(z) \leq 1 \text{ for } z \in [0, h]. \quad (3.5)$$

Here u_{B+}^ε and u_{B-}^ε are the left boundary layer at $z = 0$ and the right boundary layer at $z = h$ for u^0 respectively. When $h = \infty$, we can take $u_{B-}^\varepsilon = 0$.

Similarly, the exact boundary layers for the magnetic field b^0 can be exactly as:

$$b_B^\varepsilon = b_{B+}^\varepsilon + b_{B-}^\varepsilon, \quad (3.6)$$

$$\begin{aligned} b_{B+}^\varepsilon &= \begin{pmatrix} B_{B+}^\varepsilon \\ \underline{b}_3^\varepsilon \end{pmatrix} = \begin{pmatrix} \bar{b}_1^\varepsilon \\ \bar{b}_2^\varepsilon \\ \bar{b}_3^\varepsilon \end{pmatrix} \\ &= \begin{pmatrix} -b_1^0(x, y, 0, t) e^{-\frac{z}{\sqrt{\nu_2^*}}} (\rho_1(z) - \rho_1'(z) \sqrt{\nu_2^*}) - b_1^0(x, y, 0, t) \rho_1'(z) \sqrt{\nu_2^*} \\ -b_2^0(x, y, 0, t) e^{-\frac{z}{\sqrt{\nu_2^*}}} (\rho_1(z) - \rho_1'(z) \sqrt{\nu_2^*}) - b_2^0(x, y, 0, t) \rho_1'(z) \sqrt{\nu_2^*} \\ \sqrt{\nu_2^*} \partial_z b_3^0(x, y, 0, t) \rho_1(z) (e^{-\frac{z}{\sqrt{\nu_2^*}}} - 1) \end{pmatrix} \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} b_{B-}^\varepsilon &= \begin{pmatrix} B_{B-}^\varepsilon \\ \bar{\underline{b}}_3^\varepsilon \end{pmatrix} = \begin{pmatrix} \bar{\bar{b}}_1^\varepsilon \\ \bar{\bar{b}}_2^\varepsilon \\ \bar{\bar{b}}_3^\varepsilon \end{pmatrix} \\ &= \begin{pmatrix} -b_1^0(x, y, h, t) e^{-\frac{h-z}{\sqrt{\nu_2^*}}} (\rho_2(z) + \rho_2'(z) \sqrt{\nu_2^*}) + b_1^0(x, y, h, t) \rho_2'(z) \sqrt{\nu_2^*} \\ -b_2^0(x, y, h, t) e^{-\frac{h-z}{\sqrt{\nu_2^*}}} (\rho_2(z) + \rho_2'(z) \sqrt{\nu_2^*}) + b_2^0(x, y, h, t) \rho_2'(z) \sqrt{\nu_2^*} \\ -\sqrt{\nu_2^*} \partial_z b_3^0(x, y, h, t) \rho_2(z) (e^{-\frac{h-z}{\sqrt{\nu_2^*}}} - 1) \end{pmatrix}. \end{aligned} \quad (3.8)$$

Here b_{B+}^ε and b_{B-}^ε are the left boundary layer at $z = 0$ and the right boundary layer at $z = h$ for b^0 respectively. $\rho_1(z)$ and $\rho_2(z)$ are given by (3.4) and (3.5). When $h = \infty$, take $b_{B-}^\varepsilon = 0$. For well-prepared initial data, take $b_{B+}^\varepsilon = b_{B-}^\varepsilon = 0$.

It is easy to verify the following properties.

Lemma 3.1 *There is a positive constant C , depending upon $\|(u^0, b^0)\|_{H^s(\Omega)}$, $s > \frac{3}{2} + 1$,*

but independent of $\varepsilon, \varepsilon_1$ and ε_2 , such that

$$\begin{aligned}
& \|(u_B^\varepsilon, b_B^\varepsilon)\|_{L^\infty(\Omega)} \leq C; \quad \|(\partial_z u_{B3}^\varepsilon, \partial_z b_{B3}^\varepsilon)\|_{L^\infty(\Omega)} \leq C; \\
& \|(U_B^\varepsilon, \nabla_{x,y} U_B^\varepsilon)\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_1^*}}; \quad \|(B_B^\varepsilon, \nabla_{x,y} B_B^\varepsilon)\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_2^*}}; \\
& \|u_{B3}^\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\nu_1^*}; \quad \|b_{B3}^\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\nu_2^*}; \\
& \|\partial_z U_B^\varepsilon\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\sqrt{\nu_1^*}}}; \quad \|\partial_z B_B^\varepsilon\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\sqrt{\nu_2^*}}}; \\
& \|\partial_z u_{B3}^\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_1^*}}; \quad \|\partial_z b_{B3}^\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_2^*}}; \\
& \|(z\partial_z U_{B+}^\varepsilon, (h-z)\partial_z U_{B-}^\varepsilon)\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_1^*}}; \quad \|(z\partial_z B_{B+}^\varepsilon, (h-z)\partial_z B_{B-}^\varepsilon)\|_{L^2(\Omega)} \leq C\sqrt{\sqrt{\nu_2^*}}; \\
& \|(z^2\partial_z U_{B+}^\varepsilon, (h-z)^2\partial_z U_{B-}^\varepsilon)\|_{L^\infty(\Omega)} \leq C\sqrt{\nu_1^*}; \quad \|(z^2\partial_z B_{B+}^\varepsilon, z^2\partial_z B_{B-}^\varepsilon)\|_{L^\infty(\Omega)} \leq C\sqrt{\nu_2^*};
\end{aligned}$$

ν_1^*, ν_2^* will be chosen later. Here and in what follows we use $U, B, U_B, B_B, U_R, \dots$ or $u_{1,2}, b_{1,2}, \{u_B\}_{1,2}, \{b_B\}_{1,2}, \{u_R\}_{1,2}, \dots$ to denote the first and the second components of $u, b, u_B, b_B, u_R, \dots$ respectively. Also, $u_3, b_3, u_{B3}, b_{B3}, \dots$, denote the third components of the vectors u, b, u_B, b_B, \dots .

3.2 The proof of Theorem 2.3

The global existence of Leray-Hopf weak solutions to the three dimensional dissipative incompressible MHD system and the global existence of smooth solutions to the two dimensional dissipative MHD system can be proven as in [5, 19, 25] by Galerkin method, and also as for Navier-Stokes equations. We omit details here.

Let $(u^\varepsilon, b^\varepsilon)$ be Leray-Hopf weak solution of MHD system (1.1)-(1.5). Decompose $(u^\varepsilon, b^\varepsilon)$ as $(u^0 + u_B^\varepsilon + u_R^\varepsilon, b^0 + b_B^\varepsilon + b_R^\varepsilon)$. Taking $\nu_1^* = \nu_2^* = \varepsilon$ in the subsection 3.1 and using the system (1.6)-(1.10), we have

$$\begin{aligned}
& \partial_t u_B^\varepsilon + \partial_t u_R^\varepsilon + u^\varepsilon \cdot \nabla u_R^\varepsilon + u^0 \cdot \nabla u_B^\varepsilon + u_B^\varepsilon \cdot \nabla u_B^\varepsilon + u_R^\varepsilon \cdot \nabla u_B^\varepsilon + u_B^\varepsilon \cdot \nabla u^0 \\
& + u_R^\varepsilon \cdot \nabla u^0 - \varepsilon_1 \partial_z^2 u_R^\varepsilon - \varepsilon_1 \partial_z^2 u^0 - \varepsilon_1 \partial_z^2 u_B^\varepsilon - \varepsilon_1 \Delta_{x,y} u_R^\varepsilon - \varepsilon_1 \Delta_{x,y} u^0 - \varepsilon_1 \Delta_{x,y} u_B^\varepsilon \\
= & -\nabla(p^\varepsilon - p^0) + b^\varepsilon \cdot \nabla b_R^\varepsilon + b^0 \cdot \nabla b_B^\varepsilon + b_B^\varepsilon \cdot \nabla b_B^\varepsilon + b_R^\varepsilon \cdot \nabla b_B^\varepsilon \\
& + b_B^\varepsilon \cdot \nabla b^0 + b_R^\varepsilon \cdot \nabla b^0, \quad \text{in } \Omega \times (0, T), \tag{3.9}
\end{aligned}$$

$$\begin{aligned}
& \partial_t b_B^\varepsilon + \partial_t b_R^\varepsilon + u^\varepsilon \cdot \nabla b_R^\varepsilon + u^0 \cdot \nabla b_B^\varepsilon + u_B^\varepsilon \cdot \nabla b_B^\varepsilon + u_R^\varepsilon \cdot \nabla b_B^\varepsilon + u_B^\varepsilon \cdot \nabla b^0 \\
& + u_R^\varepsilon \cdot \nabla b^0 - \varepsilon_2 \partial_z^2 b_R^\varepsilon - \varepsilon_2 \partial_z^2 b^0 - \varepsilon_2 \partial_z^2 b_B^\varepsilon - \varepsilon_2 \Delta_{x,y} b_R^\varepsilon - \varepsilon_2 \Delta_{x,y} b^0 - \varepsilon_2 \Delta_{x,y} b_B^\varepsilon \\
= & b^\varepsilon \cdot \nabla u_R^\varepsilon + b^0 \cdot \nabla u_B^\varepsilon + b_B^\varepsilon \cdot \nabla u_B^\varepsilon + b_R^\varepsilon \cdot \nabla u_B^\varepsilon + b_B^\varepsilon \cdot \nabla u^0 \\
& + b_R^\varepsilon \cdot \nabla u^0, \quad \text{in } \Omega \times (0, T), \tag{3.10}
\end{aligned}$$

$$\operatorname{div} u^\varepsilon = \operatorname{div} b^\varepsilon = \operatorname{div} u_R^\varepsilon = \operatorname{div} b_R^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \tag{3.11}$$

$$u_R^\varepsilon = b_R^\varepsilon = 0, \quad \text{on } (x, y) \in \omega, z = 0, h, 0 \leq t \leq T \tag{3.12}$$

$$\begin{aligned}
& u_R^\varepsilon(t=0) = u^\varepsilon(0) - u^0(0) - u_B^\varepsilon(t=0), \\
& b_R^\varepsilon(t=0) = b^\varepsilon(0) - b^0(0) - b_B^\varepsilon(t=0), \quad \text{on } \Omega. \tag{3.13}
\end{aligned}$$

Thanks to the fact that $u_0^0 = \pm b_0^0$, where $(u^0, p^0, b^0)(x, y, z, t) = (u_0^0, 0, b_0^0)$ is the special solution to the incompressible MHD equation, we have that $u_B^\varepsilon = \pm b_B^\varepsilon$, which shows that there exist cancelations between the boundary layers of the velocity and the magnetic field, and, hence, it follows from (3.9) and (3.10) that

$$\begin{aligned} & \partial_t(u_R^\varepsilon - b_R^\varepsilon) + u^\varepsilon \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) - \frac{\varepsilon_1 + \varepsilon_2}{2} \Delta(u_R^\varepsilon - b_R^\varepsilon) - \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_R^\varepsilon + b_R^\varepsilon) \\ = & -\nabla(p^\varepsilon - p^0) - b^\varepsilon \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_B^\varepsilon + b_B^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u^0 + b^0) \end{aligned} \quad (3.14)$$

or

$$\begin{aligned} & \partial_t(u_R^\varepsilon + b_R^\varepsilon) + u^\varepsilon \cdot \nabla(u_R^\varepsilon + b_R^\varepsilon) - \frac{\varepsilon_1 + \varepsilon_2}{2} \Delta(u_R^\varepsilon + b_R^\varepsilon) - \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_R^\varepsilon - b_R^\varepsilon) \\ = & -\nabla(p^\varepsilon - p^0) + b^\varepsilon \cdot \nabla(u_R^\varepsilon + b_R^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_B^\varepsilon - b_B^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u^0 - b^0) \end{aligned} \quad (3.15)$$

Here we has used the relation $\varepsilon_1 \Delta u_R^\varepsilon - \varepsilon_2 \Delta b_R^\varepsilon = \frac{\varepsilon_1 + \varepsilon_2}{2} \Delta(u_R^\varepsilon - b_R^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_R^\varepsilon + b_R^\varepsilon)$ or $\varepsilon_1 \Delta u_R^\varepsilon + \varepsilon_2 \Delta b_R^\varepsilon = \frac{\varepsilon_1 + \varepsilon_2}{2} \Delta(u_R^\varepsilon + b_R^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_R^\varepsilon - b_R^\varepsilon)$. Noting that there appears the term $-\frac{\varepsilon_1 - \varepsilon_2}{2} \Delta(u_R^\varepsilon + b_R^\varepsilon)$ in the system (3.14) due to the fact that $\varepsilon_1 \neq \varepsilon_2$, and, hence, one can not adopt the techniques in [24] here, so a new idea is needed to deal with the current case. Now, using (3.11)-(3.12) and taking the scalar product of (3.14) (or (3.15)) with $u_R^\varepsilon - b_R^\varepsilon$ (or $u_R^\varepsilon + b_R^\varepsilon$), we get, for the case of $u^0 = b^0$, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |u_R^\varepsilon - b_R^\varepsilon|^2 + \frac{\varepsilon_1 + \varepsilon_2}{2} \int |\nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\ = & -\frac{\varepsilon_1 - \varepsilon_2}{2} \int \nabla((u_R^\varepsilon + b_R^\varepsilon) + (u_B^\varepsilon + b_B^\varepsilon) + (u^0 + b^0)) \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) \end{aligned} \quad (3.16)$$

and for the case of $u^0 = -b^0$, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |u_R^\varepsilon + b_R^\varepsilon|^2 + \frac{\varepsilon_1 + \varepsilon_2}{2} \int |\nabla(u_R^\varepsilon + b_R^\varepsilon)|^2 \\ = & -\frac{\varepsilon_1 - \varepsilon_2}{2} \int \nabla((u_R^\varepsilon - b_R^\varepsilon) + (u_B^\varepsilon - b_B^\varepsilon) + (u^0 - b^0)) \cdot \nabla(u_R^\varepsilon + b_R^\varepsilon), \end{aligned} \quad (3.17)$$

which yields to, by using (3.13) and the properties of the boundary layer functions and with the help of the Young's inequality and the assumption (2.3) on initial data, that, for $0 \leq t \leq T < \infty$,

$$\begin{aligned} & \int |(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\ \leq & \int |(u_R^\varepsilon - b_R^\varepsilon)(t=0)|^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} \left[\int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + Ct + \frac{C}{\sqrt{\varepsilon}} t \right] \\ \leq & C\varepsilon^\kappa + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \end{aligned} \quad (3.18)$$

or

$$\begin{aligned} & \int |(u_R^\varepsilon + b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla(u_R^\varepsilon + b_R^\varepsilon)|^2 \\ & \leq C\varepsilon^\kappa + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \end{aligned} \quad (3.19)$$

for some constant $C > 0$ and $\delta > 0$ independent of $\varepsilon, \varepsilon_1, \varepsilon_2$, and $\kappa > 1$.

Now, by taking the scalar product of (3.14) with u_R^ε and the scalar product of (3.15) with b_R^ε , one can get that

$$\frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int |\nabla b_R^\varepsilon|^2 = \sum_{i=1}^{13} J_i, \quad (3.20)$$

where $J_i, i = 1, \dots, 13$, are given respectively as follows

$$\begin{aligned} J_1 &= - \int \nabla(p^\varepsilon - p^0)u_R^\varepsilon; \\ J_2 &= - \int \partial_t u_B^\varepsilon u_R^\varepsilon - \int \partial_t b_B^\varepsilon b_R^\varepsilon; \\ J_3 &= - \int u^\varepsilon \cdot \nabla u_R^\varepsilon u_R^\varepsilon - \int u^\varepsilon \cdot \nabla b_R^\varepsilon b_R^\varepsilon; \\ J_4 &= - \int u^0 \cdot \nabla u_B^\varepsilon u_R^\varepsilon - \int u^0 \cdot \nabla b_B^\varepsilon b_R^\varepsilon + \int b^0 \cdot \nabla b_B^\varepsilon u_R^\varepsilon + \int b^0 \cdot \nabla u_B^\varepsilon b_R^\varepsilon; \\ J_5 &= - \int u_B^\varepsilon \cdot \nabla u_B^\varepsilon u_R^\varepsilon - \int u_B^\varepsilon \cdot \nabla b_B^\varepsilon b_R^\varepsilon + \int b_B^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon + \int b_B^\varepsilon \cdot \nabla u_B^\varepsilon b_R^\varepsilon; \\ J_6 &= - \int u_R^\varepsilon \cdot \nabla u_B^\varepsilon u_R^\varepsilon - \int u_R^\varepsilon \cdot \nabla b_B^\varepsilon b_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla u_B^\varepsilon b_R^\varepsilon; \\ J_7 &= - \int u_B^\varepsilon \cdot \nabla u^0 u_R^\varepsilon - \int u_B^\varepsilon \cdot \nabla b^0 b_R^\varepsilon + \int b_B^\varepsilon \cdot \nabla b^0 u_R^\varepsilon + \int b_B^\varepsilon \cdot \nabla u^0 b_R^\varepsilon; \\ J_8 &= - \int u_R^\varepsilon \cdot \nabla u^0 u_R^\varepsilon - \int u_R^\varepsilon \cdot \nabla b^0 b_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla b^0 u_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla u^0 b_R^\varepsilon; \\ J_9 &= \int b^\varepsilon \cdot \nabla b_R^\varepsilon u_R^\varepsilon + \int b^\varepsilon \cdot \nabla u_R^\varepsilon b_R^\varepsilon; \\ J_{10} &= \varepsilon_1 \int \partial_z^2 u_B^\varepsilon u_R^\varepsilon + \varepsilon_2 \int \partial_z^2 b_B^\varepsilon b_R^\varepsilon; \quad J_{11} = \varepsilon_1 \int \partial_z^2 u^0 u_R^\varepsilon + \varepsilon_2 \int \partial_z^2 b^0 b_R^\varepsilon; \\ J_{12} &= \varepsilon_1 \int \Delta_{x,y} u_B^\varepsilon u_R^\varepsilon + \varepsilon_2 \int \Delta_{x,y} b_B^\varepsilon b_R^\varepsilon; \quad J_{13} = \varepsilon_1 \int \Delta_{x,y} u^0 u_R^\varepsilon + \varepsilon_2 \int \Delta_{x,y} b^0 b_R^\varepsilon. \end{aligned}$$

We now bound each of $J_i, i = 1, \dots, 13$. In the sequel, C denotes any constant depending only upon h . Also, in the following, we just consider the case of $u^0 = b^0$ in Theorem 2.3 because the case of $u^0 = -b^0$ can be treated similarly by using (3.19)

1) First, using the fact that $u^0 = b^0$, which is independent of the time t , and, hence, $u_B^\varepsilon = b_B^\varepsilon$, we get

$$J_2 = J_4 = J_5 = J_7 = 0.$$

2) Second, direct computation gives

$$J_1 = - \int \nabla(p^\varepsilon - p^0)u_R^\varepsilon = \int (p^\varepsilon - p^0)divu_R^\varepsilon = 0, \quad (3.21)$$

$$\begin{aligned} J_3 &= - \int u^\varepsilon \cdot \nabla u_R^\varepsilon u_R^\varepsilon - \int u^\varepsilon \cdot \nabla b_R^\varepsilon b_R^\varepsilon \\ &= \frac{1}{2} \int u^\varepsilon \cdot \nabla (|u_R^\varepsilon|^2) + \frac{1}{2} \int u^\varepsilon \cdot \nabla (|b_R^\varepsilon|^2) = 0, \end{aligned} \quad (3.22)$$

To estimate the term J_6 , noting that $u_B^\varepsilon = b_B^\varepsilon$ one gets by using the estimate (3.18) that

$$\begin{aligned} J_6 &= - \int (u_R^\varepsilon - b_R^\varepsilon) \cdot \nabla u_B^\varepsilon (u_R^\varepsilon + b_R^\varepsilon) \\ &= - \int (u_R^\varepsilon - b_R^\varepsilon)_{1,2} \cdot \nabla_{x,y} (u_B^\varepsilon)_{1,2} (u_R^\varepsilon + b_R^\varepsilon)_{1,2} - \int (u_R^\varepsilon - b_R^\varepsilon)_{1,2} \cdot \nabla_{x,y} (u_B^\varepsilon)_3 (u_R^\varepsilon + b_R^\varepsilon)_3 \\ &\quad - \int (u_R^\varepsilon - b_R^\varepsilon)_3 \cdot \nabla_z (u_B^\varepsilon)_{1,2} (u_R^\varepsilon + b_R^\varepsilon)_{1,2} - \int (u_R^\varepsilon - b_R^\varepsilon)_3 \cdot \nabla_z (u_B^\varepsilon)_3 (u_R^\varepsilon + b_R^\varepsilon)_3 \\ &\leq C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C \int \frac{|(u_R^\varepsilon - b_R^\varepsilon)_3|^2}{\varepsilon} \\ &\leq C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C\varepsilon^{\kappa-1} + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) \\ &\quad + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}, \end{aligned} \quad (3.23)$$

$$J_8 \leq 2 \max\{\|\nabla u^0\|_{L^\infty}, \|\nabla b^0\|_{L^\infty}\} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2), \quad (3.24)$$

$$J_9 = \int b^\varepsilon \cdot \nabla (u_R^\varepsilon \cdot b_R^\varepsilon) = \int div b^\varepsilon (u_R^\varepsilon \cdot b_R^\varepsilon) = 0, \quad (3.25)$$

$$\begin{aligned} J_{10} &= -\varepsilon_1 \int \partial_z u_B^\varepsilon \partial_z u_R^\varepsilon - \varepsilon_2 \int \partial_z b_B^\varepsilon \partial_z b_R^\varepsilon \\ &\leq \delta\varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + \delta\varepsilon_2 \int |\nabla b_R^\varepsilon|^2 + \frac{C(\varepsilon_1 + \varepsilon_2)}{\sqrt{\varepsilon}}, \end{aligned} \quad (3.26)$$

$$J_{11} + J_{12} + J_{13} \leq C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C(\varepsilon_1^2 + \varepsilon_2^2). \quad (3.27)$$

Combining (3.20) with (3.21)-(3.27) together and using the assumption (2.1) in Theorem 2.3, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + (1-\delta)\varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + (1-\delta)\varepsilon_2 \int |\nabla b_R^\varepsilon|^2 \\
\leq & C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) \\
& + C(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{\varepsilon}}) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \\
\leq & C \left[\int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla b_R^\varepsilon|^2 \right] \\
& + C(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{\varepsilon}}) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \tag{3.28}
\end{aligned}$$

It follows from (3.28) and by Gronwall's inequality that

$$\int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) \leq C(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{\varepsilon}} + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}). \tag{3.29}$$

Now, combining (3.18) and (3.29), we can get the estimate (2.4) in Theorem 2.3.

For L^∞ convergence rate, we just consider the case of two-dimensional MHD system. For the three dimensional case, the regularity problem involved here is open. To complete our Theorem, we need to do higher order energy estimates. Of course, even though the basic ideas of doing the higher order energy estimates are the same as in L^2 estimates, but this is very complex. In the following, we just consider the case $u^0 = b^0$ and $0 < z < \infty$. The others are similar. First, to establish the convergence rate of high order derivatives, we need solve the exact Prandtl's type equations so as to obtain the more better convergence rate on the error functions. Second, limited to the length of paper, we just give estimates of some key terms which will appear when differentiating nonlinear terms of MHD system and which are required to obtain the estimates by using some kinds of different techniques from the previous steps.

Let the boundary functions $u_B^\varepsilon(x, z, t) = (u_{1B}^\varepsilon, u_{3B}^\varepsilon)$ and $b_B^\varepsilon(x, z, t) = (b_{1B}^\varepsilon, b_{3B}^\varepsilon)$ satisfy respectively the following Prandtl's type equations

$$\begin{aligned}
& \partial_t u_{1B}^\varepsilon = \varepsilon \partial_z^2 u_{1B}^\varepsilon \\
& \partial_1 u_{1B}^\varepsilon + \partial_3 u_{3B}^\varepsilon = 0 \\
& u_{1B}^\varepsilon(x, z = 0, t) = -u_1^0(x, z = 0), u_{1B}^\varepsilon(x, z = \infty, t) = 0
\end{aligned}$$

and

$$\begin{aligned}
& \partial_t b_{1B}^\varepsilon = \varepsilon \partial_z^2 b_{1B}^\varepsilon \\
& \partial_1 b_{1B}^\varepsilon + \partial_3 b_{3B}^\varepsilon = 0 \\
& b_{1B}^\varepsilon(x, z = 0, t) = -b_1^0(x, z = 0), b_{1B}^\varepsilon(x, z = \infty, t) = 0,
\end{aligned}$$

which can be solved as

$$\begin{aligned} u_{1B}^\varepsilon(x, z, t) &= -u_1^0(x, z = 0) \int_{\frac{z}{\sqrt{\varepsilon(t+s)}}}^{\infty} \frac{e^{-\xi^2/4}}{\sqrt{\pi}} d\xi \\ b_{1B}^\varepsilon(x, z, t) &= -b_1^0(x, z = 0) \int_{\frac{z}{\sqrt{\varepsilon(t+s)}}}^{\infty} \frac{e^{-\xi^2/4}}{\sqrt{\pi}} d\xi \end{aligned}$$

for any given constant $s > 0$ independent of ε . It is easy to verify that, due to the fact that $u_1^0(x, z = 0) = b_1^0(x, z = 0) = \text{const}$,

$$\partial_t u_{3B}^\varepsilon - \varepsilon \partial_z^2 u_{3B}^\varepsilon = 0, \partial_t b_{3B}^\varepsilon - \varepsilon \partial_z^2 b_{3B}^\varepsilon = 0$$

and the boundary functions $u_B^\varepsilon, b_B^\varepsilon$ have the similar properties as given in Lemma 3.1. When $u^0 = b^0$, $u_B^\varepsilon = b_B^\varepsilon$.

Now, replacing $(u^\varepsilon, b^\varepsilon)$ by $(u^0 + u_B^\varepsilon + u_R^\varepsilon, b^0 + b_B^\varepsilon + b_R^\varepsilon)$ in the MHD system (1.1)-(1.5) and using the system (1.6)-(1.10) in the two-dimensional case, we gets that

$$\begin{aligned} &\partial_t u_R^\varepsilon + u^\varepsilon \cdot \nabla u_R^\varepsilon + u_R^\varepsilon \cdot \nabla u_B^\varepsilon + u_R^\varepsilon \cdot \nabla u^0 - \varepsilon_1 \Delta u_R^\varepsilon - \varepsilon_1 \Delta u^0 - (\varepsilon_1 - \varepsilon) \partial_z^2 u_B^\varepsilon \\ &= -\nabla(p^\varepsilon - p^0) + b^\varepsilon \cdot \nabla b_R^\varepsilon + b_R^\varepsilon \cdot \nabla b_B^\varepsilon + b_R^\varepsilon \cdot \nabla b^0, \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (3.30)$$

$$\begin{aligned} &\partial_t b_R^\varepsilon + u^\varepsilon \cdot \nabla b_R^\varepsilon + u_R^\varepsilon \cdot \nabla b_B^\varepsilon + u_R^\varepsilon \cdot \nabla b^0 - \varepsilon_2 \Delta b_R^\varepsilon - \varepsilon_2 \Delta b^0 - (\varepsilon_2 - \varepsilon) \partial_z^2 b_B^\varepsilon \\ &= b^\varepsilon \cdot \nabla u_R^\varepsilon + b_R^\varepsilon \cdot \nabla u_B^\varepsilon + b_R^\varepsilon \cdot \nabla b^0, \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (3.31)$$

$$\text{div} u^\varepsilon = \text{div} b^\varepsilon = \text{div} u_R^\varepsilon = \text{div} b_R^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \quad (3.32)$$

$$u_R^\varepsilon = b_R^\varepsilon = 0, \quad \text{on } x \in \omega, z = 0, 0 \leq t \leq T \quad (3.33)$$

$$u_R^\varepsilon(t = 0) = u^\varepsilon(0) - u^0(0) - u_B^\varepsilon(t = 0),$$

$$b_R^\varepsilon(t = 0) = b^\varepsilon(0) - b^0(0) - b_B^\varepsilon(t = 0), \quad \text{on } \Omega. \quad (3.34)$$

As before, (3.30) and (3.31) imply that

$$\begin{aligned} &\partial_t (u_R^\varepsilon - b_R^\varepsilon) + u^\varepsilon \cdot \nabla (u_R^\varepsilon - b_R^\varepsilon) - \frac{\varepsilon_1 + \varepsilon_2}{2} \Delta (u_R^\varepsilon - b_R^\varepsilon) - \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta (u_R^\varepsilon + b_R^\varepsilon) \\ &= -\nabla(p^\varepsilon - p^0) - b^\varepsilon \cdot \nabla (u_R^\varepsilon - b_R^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta (u_B^\varepsilon + b_B^\varepsilon) + \frac{\varepsilon_1 - \varepsilon_2}{2} \Delta (u^0 + b^0) \end{aligned} \quad (3.35)$$

Now, using (3.32)-(3.33) and taking the scalar product of (3.35) with $u_R^\varepsilon - b_R^\varepsilon$, yield that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |u_R^\varepsilon - b_R^\varepsilon|^2 + \frac{\varepsilon_1 + \varepsilon_2}{2} \int |\nabla (u_R^\varepsilon - b_R^\varepsilon)|^2 \\ &= -\frac{\varepsilon_1 - \varepsilon_2}{2} \int \nabla \cdot ((u_R^\varepsilon + b_R^\varepsilon) + (u_B^\varepsilon + b_B^\varepsilon) + (u^0 + b^0)) \cdot \nabla (u_R^\varepsilon - b_R^\varepsilon), \end{aligned} \quad (3.36)$$

which imply, by using (3.34) and the properties of the boundary layer functions and with the help of the Young's inequality and the assumption (2.5) on initial data, that, for

$0 \leq t \leq T < \infty$,

$$\begin{aligned}
& \int |(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
& \leq \int |(u_R^\varepsilon - b_R^\varepsilon)(t=0)|^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} \left[\int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + Ct + \frac{C}{\sqrt{\varepsilon}}t \right] \\
& \leq C\varepsilon^\kappa + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \quad (3.37)
\end{aligned}$$

for some constant $C > 0$ and $\delta > 0$ independent of $\varepsilon, \varepsilon_1, \varepsilon_2$, and $\kappa > 2$.

Now, by taking the scalar product of (3.30) with u_R^ε and the scalar product of (3.31) with b_R^ε , we arrive at

$$\frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int |\nabla b_R^\varepsilon|^2 = \sum_{i=1}^7 K_i, \quad (3.38)$$

where $K_i, i = 1, \dots, 7$, are given respectively as follows

$$\begin{aligned}
K_1 &= - \int \nabla(p^\varepsilon - p^0)u_R^\varepsilon; \\
K_2 &= - \int u^\varepsilon \cdot \nabla u_R^\varepsilon u_R^\varepsilon - \int u^\varepsilon \cdot \nabla b_R^\varepsilon b_R^\varepsilon; \\
K_3 &= - \int u_R^\varepsilon \cdot \nabla u_B^\varepsilon u_R^\varepsilon - \int u_R^\varepsilon \cdot \nabla b_B^\varepsilon b_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla u_B^\varepsilon b_R^\varepsilon \\
K_4 &= - \int u_R^\varepsilon \cdot \nabla u^0 u_R^\varepsilon - \int u_R^\varepsilon \cdot \nabla b^0 b_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla b^0 u_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla u^0 b_R^\varepsilon; \\
K_5 &= \int b^\varepsilon \cdot \nabla b_R^\varepsilon u_R^\varepsilon + \int b^\varepsilon \cdot \nabla u_R^\varepsilon b_R^\varepsilon; \\
K_6 &= \varepsilon_1 \int \partial_z^2 u^0 u_R^\varepsilon + \varepsilon_2 \int \partial_z^2 b^0 b_R^\varepsilon; \quad K_7 = (\varepsilon_1 - \varepsilon) \int \partial_z^2 u_B^\varepsilon u_R^\varepsilon + (\varepsilon_2 - \varepsilon) \int \partial_z^2 b_B^\varepsilon b_R^\varepsilon.
\end{aligned}$$

We now bound each of $K_i, i = 1, \dots, 7$.

- 1) First, $K_i, i = 1, \dots, 6$ can be estimated as before for J_1, J_3, J_6, J_8, J_9 .
- 2) Second, we compute

$$\begin{aligned}
K_7 &= -(\varepsilon_1 - \varepsilon) \int \partial_z u_B^\varepsilon \partial_z u_R^\varepsilon - (\varepsilon_2 - \varepsilon) \int \partial_z b_B^\varepsilon \partial_z b_R^\varepsilon \\
&\leq \delta\varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + \delta\varepsilon_2 \int |\nabla b_R^\varepsilon|^2 + C \left(\frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1 \sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_2 \sqrt{\varepsilon}} \right), \quad (3.39)
\end{aligned}$$

Combining (3.38) with (3.21)-(3.25) for $K_i, i = 1, \dots, 6$ and (3.39) together and using the

assumptions (2.1) in Theorem 2.3, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + (1-\delta)\varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + (1-\delta)\varepsilon_2 \int |\nabla b_R^\varepsilon|^2 \\
\leq & C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon(\varepsilon_1 + \varepsilon_2)} \int_0^t \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) \\
& + C(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1\sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_2\sqrt{\varepsilon}}) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \\
\leq & C \left[\int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla b_R^\varepsilon|^2 \right] \\
& + C(\varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1\sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_2\sqrt{\varepsilon}}) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}. \tag{3.40}
\end{aligned}$$

It follows from (3.40) and by Gronwall's inequality that

$$\int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla b_R^\varepsilon|^2 \leq C\beta_0(\varepsilon). \tag{3.41}$$

Here

$$\beta_0(\varepsilon) = \varepsilon^{\kappa-1} + \varepsilon_1^2 + \varepsilon_2^2 + \frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1\sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_2\sqrt{\varepsilon}} + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}. \tag{3.42}$$

Then we have

$$\begin{aligned}
& \int |(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1-\delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
\leq & C \frac{(\varepsilon_1 - \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) + C\varepsilon^\kappa + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} = \bar{\beta}_0(\varepsilon) \tag{3.43}
\end{aligned}$$

Differentiate the equation (3.35) in time, multiply the resulting one by $\partial_t(u_R^\varepsilon - b_R^\varepsilon)$ and integrate over Ω . Notice that $\partial_t u_R^\varepsilon|_{t=0} = \varepsilon_1 \Delta u^0(t=0) + O(\varepsilon^\kappa + \varepsilon_1 \varepsilon^\kappa + \frac{\varepsilon_1 - \varepsilon}{\varepsilon})$, $\partial_t b_R^\varepsilon|_{t=0} =$

$\varepsilon_2 \Delta b^0(t=0) + O(\varepsilon^\kappa + \varepsilon_2 \varepsilon^\kappa + \frac{\varepsilon^{2-\varepsilon}}{\varepsilon})$ and

$$\begin{aligned}
& \left| \int \partial_t(u^\varepsilon + b^\varepsilon) \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t(u_R^\varepsilon - b_R^\varepsilon) \right| \\
\leq & \left| \int \partial_t(u_R^\varepsilon + b_R^\varepsilon) \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t(u_R^\varepsilon - b_R^\varepsilon) \right| \\
& + \left| \int \partial_t(u^0 + b^0 + u_B^\varepsilon + b_B^\varepsilon) \cdot \nabla(u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t(u_R^\varepsilon - b_R^\varepsilon) \right| \\
\leq & \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2} \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} + C \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^4}^2 + \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^4}^2) \\
& + C \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2} \|\nabla \partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2} + \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\nabla \partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}) \\
& + C \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \\
\leq & \delta(\varepsilon_1 + \varepsilon_2) \|\nabla \partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + \delta \varepsilon^2 \|\partial_t \nabla(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + C \frac{1}{\varepsilon_1 + \varepsilon_2} \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \frac{1}{\varepsilon^2} \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + C \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \frac{d}{dt} \int |\partial_t(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int |\partial_t \nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
\leq & C \frac{1}{\varepsilon_1 + \varepsilon_2} \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \\
& + C \frac{1}{\varepsilon^2} \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + C \|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \|\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \\
& + C \left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} + \delta \varepsilon^2 \right) \int (|\partial_t \nabla u_R^\varepsilon|^2 + |\partial_t \nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \quad (3.44)
\end{aligned}$$

or

$$\begin{aligned}
& \int |\partial_t(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\partial_t \nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
\leq & C \int_0^t \frac{\|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \|\partial_t(u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2)^2 \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) \\
& + C \frac{\varepsilon^\kappa}{\varepsilon_1 + \varepsilon_2} + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)^2} + C(\varepsilon_1 - \varepsilon_2)^2 \\
& + \left(\frac{C(\varepsilon_1 - \varepsilon_2)^2}{4\delta(\varepsilon_1 + \varepsilon_2)} + \delta \varepsilon^2 \right) \int_0^t \int (|\partial_t \nabla u_R^\varepsilon|^2 + |\partial_t \nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{4\delta\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \quad (3.45)
\end{aligned}$$

for some $\delta > 0$ independent of ε . Here we require $\frac{\beta_0(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}(\varepsilon_1 + \varepsilon_2)} \leq C$.

Differentiate the equations (3.30) and (3.31) in time, multiply the resulting ones respectively by $\partial_t u_R^\varepsilon$ and $\partial_t b_R^\varepsilon$ and integrate over Ω . Notice that

$$\begin{aligned}
& \left| \int (-\partial_t u^\varepsilon \cdot \nabla u_R^\varepsilon \partial_t u_R^\varepsilon + \partial_t b^\varepsilon \cdot \nabla b_R^\varepsilon \partial_t u_R^\varepsilon - \partial_t u^\varepsilon \cdot \nabla b_R^\varepsilon \partial_t b_R^\varepsilon + \partial_t b^\varepsilon \cdot \nabla u_R^\varepsilon \partial_t b_R^\varepsilon) \right| \\
\leq & \left| \int (-\partial_t u_R^\varepsilon \cdot \nabla u_R^\varepsilon \partial_t u_R^\varepsilon + \partial_t b_R^\varepsilon \cdot \nabla b_R^\varepsilon \partial_t u_R^\varepsilon - \partial_t u_R^\varepsilon \cdot \nabla b_R^\varepsilon \partial_t b_R^\varepsilon + \partial_t b_R^\varepsilon \cdot \nabla u_R^\varepsilon \partial_t b_R^\varepsilon) \right| \\
& + \left| \int (-\partial_t(u^0 + u_B^\varepsilon) \cdot \nabla u_R^\varepsilon \partial_t u_R^\varepsilon + \partial_t(b^0 + b_B^\varepsilon) \cdot \nabla b_R^\varepsilon \partial_t u_R^\varepsilon \right. \\
& \quad \left. - \partial_t(u^0 + u_B^\varepsilon) \cdot \nabla b_R^\varepsilon \partial_t b_R^\varepsilon + \partial_t(b^0 + b_B^\varepsilon) \cdot \nabla u_R^\varepsilon \partial_t b_R^\varepsilon) \right| \\
\leq & C \|\nabla u_R^\varepsilon\|_{L^2} (\|\partial_t u_R^\varepsilon\|_{L^4}^2 + \|\partial_t b_R^\varepsilon\|_{L^4}^2) + C \|\nabla b_R^\varepsilon\|_{L^2} \|\partial_t u_R^\varepsilon\|_{L^4}^2 + \|\partial_t b_R^\varepsilon\|_{L^4}^2 \\
& + C (\|\nabla u_R^\varepsilon\|_{L^2}^2 + \|\nabla b_R^\varepsilon\|_{L^2}^2) + C \|\partial_t u_R^\varepsilon\|_{L^2}^2 + C \|\partial_t b_R^\varepsilon\|_{L^2}^2 \\
\leq & C (\|\nabla u_R^\varepsilon\|_{L^2} + \|\nabla b_R^\varepsilon\|_{L^2}) (\|\partial_t u_R^\varepsilon\|_{L^2} \|\nabla \partial_t u_R^\varepsilon\|_{L^2} + \|\partial_t b_R^\varepsilon\|_{L^2} \|\nabla \partial_t b_R^\varepsilon\|_{L^2}) \\
& + C (\|\nabla u_R^\varepsilon\|_{L^2}^2 + \|\nabla b_R^\varepsilon\|_{L^2}^2) + C \|\partial_t u_R^\varepsilon\|_{L^2}^2 + C \|\partial_t b_R^\varepsilon\|_{L^2}^2 \\
\leq & \delta \varepsilon_1 \|\nabla \partial_t u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \|\partial_t u_R^\varepsilon\|_{L^2}^2 + \delta \varepsilon_2 \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \|\partial_t b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \|\partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \|\partial_t u_R^\varepsilon\|_{L^2}^2 \\
& + C (\|\nabla u_R^\varepsilon\|_{L^2}^2 + \|\nabla b_R^\varepsilon\|_{L^2}^2 + \|\partial_t u_R^\varepsilon\|_{L^2}^2 + \|\partial_t b_R^\varepsilon\|_{L^2}^2). \tag{3.46}
\end{aligned}$$

$$\begin{aligned}
& \left| \int (-\partial_t u_R^\varepsilon \cdot \nabla u_B^\varepsilon \partial_t u_R^\varepsilon + \partial_t b_R^\varepsilon \cdot \nabla b_B^\varepsilon \partial_t u_R^\varepsilon - \partial_t u_R^\varepsilon \nabla b_B^\varepsilon \partial_t b_R^\varepsilon + \partial_t b_R^\varepsilon \cdot \nabla u_B^\varepsilon \partial_t b_R^\varepsilon) \right| \\
= & \left| - \int \partial_t (u_R^\varepsilon - b_R^\varepsilon) \cdot \nabla u_B^\varepsilon \cdot \partial_t (u_R^\varepsilon + b_R^\varepsilon) \right| \\
\leq & C \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t u_B^\varepsilon|^2) + C \int \frac{|\partial_t (u_R^\varepsilon - b_R^\varepsilon)|^2}{\varepsilon}. \tag{3.47}
\end{aligned}$$

$$\begin{aligned}
& \left| \int (-u_R^\varepsilon \cdot \partial_t \nabla u_B^\varepsilon \partial_t u_R^\varepsilon + b_R^\varepsilon \cdot \nabla \partial_t b_B^\varepsilon \partial_t u_R^\varepsilon - u_R^\varepsilon \nabla \partial_t b_B^\varepsilon \partial_t b_R^\varepsilon + b_R^\varepsilon \cdot \nabla \partial_t u_B^\varepsilon \partial_t b_R^\varepsilon) \right| \\
= & \left| - \int (u_R^\varepsilon - b_R^\varepsilon) \cdot \nabla \partial_t u_B^\varepsilon \cdot \partial_t (u_R^\varepsilon + b_R^\varepsilon) \right| \\
\leq & C \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t u_B^\varepsilon|^2) + C \int \frac{|u_R^\varepsilon - b_R^\varepsilon|^2}{\varepsilon}. \tag{3.48}
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \frac{d}{dt} \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t b_R^\varepsilon|^2) + \varepsilon_1 \int |\nabla \partial_t u_R^\varepsilon|^2 + \varepsilon_2 \int |\nabla \partial_t b_R^\varepsilon|^2 \\
\leq & C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \int |\partial_t u_R^\varepsilon|^2 + C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \int |\partial_t b_R^\varepsilon|^2 \\
& + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \int |\partial_t b_R^\varepsilon|^2 + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \int |\partial_t u_R^\varepsilon|^2 \\
& + C \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + C \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t b_R^\varepsilon|^2) \\
& + C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C \int \frac{|\partial_t(u_R^\varepsilon - b_R^\varepsilon)|^2 + |(u_R^\varepsilon - b_R^\varepsilon)|^2}{\varepsilon} \\
& + C \left(\frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1 \sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_1 \sqrt{\varepsilon}} \right) \\
\leq & C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \int |\partial_t u_R^\varepsilon|^2 + C \frac{\|\nabla u_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \int |\partial_t b_R^\varepsilon|^2 \\
& + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_2} \int |\partial_t b_R^\varepsilon|^2 + C \frac{\|\nabla b_R^\varepsilon\|_{L^2}^2}{\varepsilon_1} \int |\partial_t u_R^\varepsilon|^2 \\
& + C \int (|\nabla u_R^\varepsilon|^2 + |\nabla b_R^\varepsilon|^2) + C \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t b_R^\varepsilon|^2) \\
& + C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C \int_0^t \frac{\|\nabla(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^3} \int (|\partial_t u_R^\varepsilon|^2 + |\partial_t b_R^\varepsilon|^2) \\
& + C \left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)} + \delta\varepsilon \right) \int_0^t \int (|\partial_t \nabla u_R^\varepsilon|^2 + |\partial_t \nabla b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon} + C\varepsilon^{\kappa-1} \\
& + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon \sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) + C \left(\frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1 \sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_1 \sqrt{\varepsilon}} \right) \\
& + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)^2 \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) + C \frac{\varepsilon^{\kappa-1}}{\varepsilon_1 + \varepsilon_2} \\
& + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon \sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)^2} + C(\varepsilon_1 - \varepsilon_2)^2. \tag{3.49}
\end{aligned}$$

It follows from Gronwall's inequality and (3.49), with the help of the estimates (3.41) and (3.43), that

$$\int (|\partial_t u_R^\varepsilon|^2 + |\partial_t b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla \partial_t u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla \partial_t b_R^\varepsilon|^2 \leq C\beta_1(\varepsilon). \tag{3.50}$$

Here

$$\begin{aligned} \beta_1(\varepsilon) = & \varepsilon_1^2 + \varepsilon_2^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)^2 \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) + \frac{\varepsilon^{\kappa-1}}{\varepsilon_1 + \varepsilon_2} \\ & + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)^2} + \left(\frac{(\varepsilon_1 - \varepsilon)^2}{\varepsilon_1\sqrt{\varepsilon}} + \frac{(\varepsilon_2 - \varepsilon)^2}{\varepsilon_1\sqrt{\varepsilon}} \right). \end{aligned} \quad (3.51)$$

Here we require $\frac{\bar{\beta}_0(\varepsilon)}{\varepsilon_1 + \varepsilon_2} \leq C$.

Then it follows from (3.44) and (3.50) that

$$\int |\partial_t(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (1 - \delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\partial_t \nabla(u_R^\varepsilon - b_R^\varepsilon)|^2 \leq C \bar{\beta}_1(\varepsilon),$$

$$\bar{\beta}_1(\varepsilon) = (\varepsilon_1 - \varepsilon_2)^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_1(\varepsilon) + \frac{\varepsilon^2}{\min\{\varepsilon_1, \varepsilon_2\}} \beta_1(\varepsilon) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}. \quad (3.52)$$

Similarly, we can obtain the estimates on tangential derivatives as follows:

$$\int (|\partial_x u_R^\varepsilon|^2 + |\partial_x b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla \partial_x u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla \partial_x b_R^\varepsilon|^2 \leq C \beta_2(\varepsilon). \quad (3.53)$$

Here

$$\beta_2(\varepsilon) = \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)^2 \min\{\varepsilon_1, \varepsilon_2\}} \beta_0(\varepsilon) + \varepsilon_1^2 + \varepsilon_2^2 + \frac{\varepsilon^{\kappa-1}}{\varepsilon_1 + \varepsilon_2} + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)^2}. \quad (3.54)$$

and

$$\int |\partial_x(u_R^\varepsilon - b_R^\varepsilon)(t)|^2 + (\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla \partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \leq C \bar{\beta}_2(\varepsilon), \quad (3.55)$$

$$\bar{\beta}_2(\varepsilon) = \varepsilon^\kappa + \frac{(\varepsilon_1 - \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_2(\varepsilon) + \frac{\varepsilon^2}{\min\{\varepsilon_1, \varepsilon_2\}} \beta_2(\varepsilon) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}. \quad (3.56)$$

Here $\|\partial_x(u_R^\varepsilon(t=0), b_R^\varepsilon(t=0))\|_{L^2}^2 = O(\varepsilon^\kappa)$ and $\partial_x u_B^\varepsilon = \partial_x b_B^\varepsilon = 0$.

Finally, we apply $\partial_t \partial_x$ to the equations (3.30)-(3.33) and (3.35), multiply the resulting ones respectively by $\partial_t \partial_x u_R^\varepsilon$, $\partial_t \partial_x b_R^\varepsilon$ and $\partial_t \partial_x(u_R^\varepsilon - b_R^\varepsilon)$, and integrate over Ω . Notice that $\|\partial_t \partial_x(u_R^\varepsilon(t=0), b_R^\varepsilon(t=0))\|_{L^2}^2 = O(\varepsilon^\kappa)$, $\varepsilon_1 \partial_t \partial_x \Delta u^0 = 0 = \varepsilon_2 \partial_t \partial_x \Delta b^0$ and $\partial_t \partial_x u_B^\varepsilon =$

$\partial_t \partial_x b_B^\varepsilon = 0$. Also,

$$\begin{aligned}
& \left| \int \partial_t \partial_x (u^\varepsilon + b^\varepsilon) \cdot \nabla (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon) \right| \\
= & \left| \int \partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon) \cdot \nabla (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon) \right| \\
\leq & \|\nabla (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon) \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & C \|\nabla (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^4}^2 + \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^4}^2) \\
\leq & C \|\nabla (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \|\nabla \partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2) \\
\leq & \delta(\varepsilon_1 + \varepsilon_2) \|\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + \delta\varepsilon^2 \|\nabla \partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + C \frac{\|\nabla (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon_1 + \varepsilon_2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \frac{\|\nabla (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \|\nabla \partial_t \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int \partial_t (u^\varepsilon + b^\varepsilon) \cdot \nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon) \right| \\
\leq & \left| \int \partial_t (u_R^\varepsilon + b_R^\varepsilon) \cdot \nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon) \right| \\
& + \left| \int \partial_t (u^0 + b^0 + u_B^\varepsilon + b_B^\varepsilon) \cdot \nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon) \right| \\
\leq & \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t (u_R^\varepsilon + b_R^\varepsilon) \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} + C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t (u_R^\varepsilon + b_R^\varepsilon)\|_{L^4}^2 + \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^4}^2) \\
& + C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} (\|\partial_t (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \|\nabla \partial_t (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 + \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \|\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2) \\
& + C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2} \\
\leq & \delta(\varepsilon_1 + \varepsilon_2) \|\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \frac{\|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon_1 + \varepsilon_2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 \\
& + \delta\varepsilon^2 \|\nabla \partial_t (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 + C \frac{\|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \|\partial_t (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 \\
& + C \|\nabla \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2. \tag{3.58}
\end{aligned}$$

Similarly, $\int \partial_x (u^\varepsilon + b^\varepsilon) \cdot \nabla \partial_t (u_R^\varepsilon - b_R^\varepsilon) \cdot \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)$ can be controlled by $\delta(\varepsilon_1 + \varepsilon_2) \|\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + \delta\varepsilon^2 \|\nabla \partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2 + C \frac{\|\nabla \partial_t (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon_1 + \varepsilon_2} \|\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C \frac{\|\nabla \partial_t (u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \|\partial_x (u_R^\varepsilon + b_R^\varepsilon)\|_{L^2}^2$

$b_R^\varepsilon\|_{L^2}^2 + C\|\nabla\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2 + C\|\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2$. Thus, we have

$$\begin{aligned}
& \frac{d}{dt} \int |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 + (1-\delta)(\varepsilon_1 + \varepsilon_2) \int |\nabla\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
\leq & C\left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_1 + \varepsilon_2} + \delta\varepsilon^2\right) \int (|\nabla\partial_t\partial_x u_R^\varepsilon|^2 + |\nabla\partial_t\partial_x b_R^\varepsilon|^2) \\
& + C \int |\nabla\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 + C \int |\nabla\partial_t(u_R^\varepsilon - b_R^\varepsilon)|^2 + \int |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
& + C \frac{\|\nabla\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon_1 + \varepsilon_2} \int |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
& + C \frac{\|\nabla\partial_x(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon_1 + \varepsilon_2} \int |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
& + C \frac{\|\nabla\partial_x(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \int |\partial_t(u_R^\varepsilon + b_R^\varepsilon)|^2 \\
& + C \frac{\|\nabla\partial_t(u_R^\varepsilon - b_R^\varepsilon)\|_{L^2}^2}{\varepsilon^2} \int |\partial_x(u_R^\varepsilon + b_R^\varepsilon)|^2
\end{aligned} \tag{3.59}$$

or

$$\begin{aligned}
& \int |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 + (1-\delta)(\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 \\
\leq & C\left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_1 + \varepsilon_2} + \delta\varepsilon^2\right) \int_0^t \int (|\nabla\partial_t\partial_x u_R^\varepsilon|^2 + |\nabla\partial_t\partial_x b_R^\varepsilon|^2) + C\beta_3(\varepsilon).
\end{aligned} \tag{3.60}$$

Here

$$\beta_3(\varepsilon) = \varepsilon^\kappa + \frac{\beta_1(\varepsilon)\bar{\beta}_2(\varepsilon) + \beta_2(\varepsilon)\bar{\beta}_1(\varepsilon)}{\varepsilon^2(\varepsilon_1 + \varepsilon_2)}. \tag{3.61}$$

Here we require $\frac{\bar{\beta}_1(\varepsilon)}{(\varepsilon_1 + \varepsilon_2)^2} \leq C$.

On the other hand,

$$\begin{aligned}
& \left| - \int (\partial_x(u_R^\varepsilon - b_R^\varepsilon) \cdot \nabla\partial_t u_B^\varepsilon + \partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon) \cdot \nabla u_B^\varepsilon) \cdot \partial_t\partial_x(u_R^\varepsilon + b_R^\varepsilon) \right| \\
\leq & C \int (|\partial_t\partial_x u_R^\varepsilon|^2 + |\partial_t\partial_x b_R^\varepsilon|^2) + C \int \frac{|\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2 + |\partial_t\partial_x(u_R^\varepsilon - b_R^\varepsilon)|^2}{\varepsilon} \\
\leq & C \int (|\partial_t\partial_x u_R^\varepsilon|^2 + |\partial_t\partial_x b_R^\varepsilon|^2) + C\left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)} + \delta\varepsilon\right) \int_0^t \int (|\nabla\partial_t\partial_x u_R^\varepsilon|^2 + |\nabla\partial_t\partial_x b_R^\varepsilon|^2) \\
& + C\frac{\beta_3(\varepsilon)}{\varepsilon} + C\varepsilon^{\kappa-1} + C\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)\min\{\varepsilon_1, \varepsilon_2\}}\beta_2(\varepsilon) + C\frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon\sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}.
\end{aligned} \tag{3.62}$$

As in (3.57) and (3.58), one can get that

$$\begin{aligned}
& \int [-(\partial_t \partial_x u^\varepsilon \cdot \nabla u_R^\varepsilon + \partial_t u^\varepsilon \cdot \nabla \partial_x u_R^\varepsilon + \partial_x u^\varepsilon \cdot \nabla \partial_t u_R^\varepsilon) \\
& + (\partial_t \partial_x b^\varepsilon \cdot \nabla b_R^\varepsilon + \partial_t b^\varepsilon \cdot \nabla \partial_x b_R^\varepsilon + \partial_x b^\varepsilon \cdot \nabla \partial_t b_R^\varepsilon)] \cdot \partial_t \partial_x u_R^\varepsilon \\
\leq & \delta \varepsilon_1 \|\nabla \partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t u_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla \partial_t u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_x u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_t u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla \partial_x b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x b_R^\varepsilon\|_{L^2}^2 + C \|\nabla u_R^\varepsilon\|_{L^2}^2 + C \|\nabla b_R^\varepsilon\|_{L^2}^2. \tag{3.63}
\end{aligned}$$

$$\begin{aligned}
& \int [-(\partial_t \partial_x u^\varepsilon \cdot \nabla b_R^\varepsilon + \partial_t u^\varepsilon \cdot \nabla \partial_x b_R^\varepsilon + \partial_x u^\varepsilon \cdot \nabla \partial_t b_R^\varepsilon) \\
& + (\partial_t \partial_x b^\varepsilon \cdot \nabla u_R^\varepsilon + \partial_t b^\varepsilon \cdot \nabla \partial_x u_R^\varepsilon + \partial_x b^\varepsilon \cdot \nabla \partial_t u_R^\varepsilon)] \cdot \partial_t \partial_x b_R^\varepsilon \\
\leq & \delta \varepsilon_2 \|\nabla \partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t u_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla \partial_t b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_x b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla \partial_x u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x b_R^\varepsilon\|_{L^2}^2 + C \|\nabla u_R^\varepsilon\|_{L^2}^2 + C \|\nabla b_R^\varepsilon\|_{L^2}^2. \tag{3.64}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \frac{d}{dt} \int (|\partial_t \partial_x u_R^\varepsilon|^2 + |\partial_t \partial_x b_R^\varepsilon|^2) + (1-\delta)\varepsilon_1 \int |\nabla \partial_t \partial_x u_R^\varepsilon|^2 + (1-\delta)\varepsilon_2 \int |\nabla \partial_t \partial_x b_R^\varepsilon|^2 \\
\leq & C \int (|\partial_t \partial_x u_R^\varepsilon|^2 + |\partial_t \partial_x b_R^\varepsilon|^2) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2)} + \delta\varepsilon \int_0^t \int (|\nabla \partial_t \partial_x u_R^\varepsilon|^2 + |\nabla \partial_t \partial_x b_R^\varepsilon|^2) \\
& + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_x u_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_x u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla \partial_t u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \|\nabla \partial_t u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \|\nabla u_R^\varepsilon\|_{L^2}^2 + C \|\nabla b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\beta_3(\varepsilon)}{\varepsilon} + C \varepsilon^{\kappa-1} + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_2(\varepsilon) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon \sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \\
\leq & C \int (|\partial_t \partial_x u_R^\varepsilon|^2 + |\partial_t \partial_x b_R^\varepsilon|^2) + C \varepsilon_1 \int_0^t \int |\nabla \partial_t \partial_x u_R^\varepsilon|^2 + C \varepsilon_2 \int_0^t \int |\nabla \partial_t \partial_x b_R^\varepsilon|^2 \\
& + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_1} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla b_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x u_R^\varepsilon\|_{L^2}^2 + C \frac{\|\nabla u_R^\varepsilon\|^2}{\varepsilon_2} \|\partial_t \partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C (\|\nabla \partial_x u_R^\varepsilon\|^2 + \|\nabla \partial_x b_R^\varepsilon\|^2) \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) \beta_1(\varepsilon) + C (\|\nabla \partial_t u_R^\varepsilon\|^2 + \|\nabla \partial_t b_R^\varepsilon\|^2) \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) \beta_2(\varepsilon) \\
& + C \|\nabla \partial_t u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x u_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_t b_R^\varepsilon\|_{L^2}^2 + C \|\nabla \partial_x b_R^\varepsilon\|_{L^2}^2 \\
& + C \|\nabla u_R^\varepsilon\|_{L^2}^2 + C \|\nabla b_R^\varepsilon\|_{L^2}^2 \\
& + C \frac{\beta_3(\varepsilon)}{\varepsilon} + C \varepsilon^{\kappa-1} + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_2(\varepsilon) + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon \sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)}. \tag{3.65}
\end{aligned}$$

Using Gronwall's inequality into (3.65), one gets that

$$\int (|\partial_t \partial_x u_R^\varepsilon|^2 + |\partial_t \partial_x b_R^\varepsilon|^2) + \varepsilon_1 \int_0^t \int |\nabla \partial_t \partial_x u_R^\varepsilon|^2 + \varepsilon_2 \int_0^t \int |\nabla \partial_t \partial_x b_R^\varepsilon|^2 \leq C \beta_4(\varepsilon). \tag{3.66}$$

Here

$$\begin{aligned}
\beta_4(\varepsilon) = & \varepsilon^\kappa + \frac{\beta_3(\varepsilon)}{\varepsilon} + \varepsilon^{\kappa-1} + C \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} \beta_2(\varepsilon) + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon \sqrt{\varepsilon}(\varepsilon_1 + \varepsilon_2)} \\
& + \frac{1}{\min\{\varepsilon_1, \varepsilon_2\}} \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) \beta_1(\varepsilon) \beta_2(\varepsilon) + (\beta_1(\varepsilon) + \beta_2(\varepsilon)) \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) \\
& + \frac{\beta_0(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}}. \tag{3.67}
\end{aligned}$$

Here we require $\frac{\beta_0(\varepsilon)+\beta_1(\varepsilon)+\beta_2(\varepsilon)}{\min\{\varepsilon_1,\varepsilon_2\}}(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}) \leq C$.

Hence

$$\begin{aligned} & \int |\partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)|^2 + (\varepsilon_1 + \varepsilon_2) \int_0^t \int |\nabla \partial_t \partial_x (u_R^\varepsilon - b_R^\varepsilon)|^2 \\ & \leq C \left(\frac{(\varepsilon_1 - \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2) \min\{\varepsilon_1, \varepsilon_2\}} + \frac{\varepsilon^2}{\min\{\varepsilon_1, \varepsilon_2\}} \right) \beta_4(\varepsilon) + \beta_3(\varepsilon). \end{aligned} \quad (3.68)$$

Now we apply the anisotropic Sobolev imbedding inequality [20] and we get

$$\begin{aligned} \|(u_R^\varepsilon, b_R^\varepsilon)\|_{L^\infty(\Omega \times (0, T))} & \leq C(\|(u_R^\varepsilon, b_R^\varepsilon)\|_{L^\infty(0, T; H)}^{\frac{1}{2}} \|\partial_x(u_R^\varepsilon, b_R^\varepsilon)\|_{L^\infty(0, T; L^2)}^{\frac{1}{2}} \\ & \quad + \|(u_R^\varepsilon, b_R^\varepsilon)\|_{L^\infty(0, T; L^2)}^{\frac{1}{2}} \|\partial_x \partial_z(u_R^\varepsilon, b_R^\varepsilon)\|_{L^\infty(0, T; L^2)}^{\frac{1}{2}}) \\ & \leq C \left(\frac{\beta_1(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}} \right)^{\frac{1}{4}} (\beta_2(\varepsilon))^{\frac{1}{4}} + C(\beta_0(\varepsilon))^{\frac{1}{4}} \left(\frac{\beta_4(\varepsilon)}{\min\{\varepsilon_1, \varepsilon_2\}} \right)^{\frac{1}{4}} \rightarrow 0 \text{ when } \varepsilon \rightarrow 0. \end{aligned}$$

This completes the proof of estimates (2.6) and (2.7) in Theorem 2.3.

3.3 The Proof of Theorem 2.4

Let $(u^\varepsilon, b^\varepsilon)$ be the Leray-Hopf weak solutions to MHD systems (1.1)-(1.5). Decompose the solution as $(u^\varepsilon, b^\varepsilon) = (u^{\varepsilon_1, 0} + u_R^\varepsilon, b^{\varepsilon_1, 0} + b_B^\varepsilon + b_R^\varepsilon)$. Taking $\nu_2^* = (\theta \varepsilon_2)^{1+\tau}$ with $\tau \in [0, 1)$ and using the system (1.11)-(1.15), we have

$$\begin{aligned} & \partial_t u_R^\varepsilon + u^\varepsilon \cdot \nabla u_R^\varepsilon + u_R^\varepsilon \cdot \nabla u^{\varepsilon_1, 0} - \varepsilon_1 \Delta u_R^\varepsilon \\ & = -\nabla(p^\varepsilon - p^{\varepsilon_1, 0}) + b^\varepsilon \cdot \nabla b_R^\varepsilon + b^{\varepsilon_1, 0} \cdot \nabla b_B^\varepsilon + b_B^\varepsilon \cdot \nabla b_B^\varepsilon + b_R^\varepsilon \cdot \nabla b_B^\varepsilon \\ & \quad + b_B^\varepsilon \cdot \nabla b^{\varepsilon_1, 0} + b_R^\varepsilon \cdot \nabla b^{\varepsilon_1, 0}, \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (3.69)$$

$$\begin{aligned} & \partial_t b_B^\varepsilon + \partial_t b_R^\varepsilon + u^\varepsilon \cdot \nabla b_R^\varepsilon + u^{\varepsilon_1, 0} \cdot \nabla b_B^\varepsilon + u_R^\varepsilon \cdot \nabla b_B^\varepsilon \\ & \quad + u_R^\varepsilon \cdot \nabla b^{\varepsilon_1, 0} - \varepsilon_2 \Delta b_R^\varepsilon - \varepsilon_2 \Delta b^{\varepsilon_1, 0} - \varepsilon_2 \Delta b_B^\varepsilon \\ & = b^\varepsilon \cdot \nabla u_R^\varepsilon + b_B^\varepsilon \cdot \nabla u^{\varepsilon_1, 0} + b_R^\varepsilon \cdot \nabla u^{\varepsilon_1, 0}, \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (3.70)$$

$$\operatorname{div} u^\varepsilon = \operatorname{div} b^\varepsilon = \operatorname{div} u_R^\varepsilon = \operatorname{div} b_R^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \quad (3.71)$$

$$u_R^\varepsilon = b_R^\varepsilon = 0, \quad \text{on } (x, y) \in \omega, z = 0, h, \quad 0 \leq t \leq T \quad (3.72)$$

$$\begin{aligned} & u_R^\varepsilon(t=0) = u^\varepsilon(0) - u^{\varepsilon_1, 0}(0), \\ & b_R^\varepsilon(t=0) = b^\varepsilon(0) - b^{\varepsilon_1, 0}(0) - b_B^\varepsilon(t=0), \quad \text{on } \Omega. \end{aligned} \quad (3.73)$$

By taking the scalar product of (3.69) with u_R^ε and the scalar product of (3.70) with b_R^ε , we have

$$\frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + \varepsilon_1 \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 \int |\nabla b_R^\varepsilon|^2 = \sum_{i=1}^{11} I_i, \quad (3.74)$$

where $I_i, i = 1, \dots, 11$, are given respectively as follows

$$\begin{aligned}
I_1 &= - \int \nabla(p^\varepsilon - p^{\varepsilon_1,0})u_R^\varepsilon; & I_2 &= \int \partial_t b_B^\varepsilon b_R^\varepsilon; \\
I_3 &= - \int u^\varepsilon \cdot \nabla u_R^\varepsilon u_R^\varepsilon - \int u^\varepsilon \cdot \nabla b_R^\varepsilon b_R^\varepsilon; \\
I_4 &= \int b^{\varepsilon_1,0} \cdot \nabla b_B^\varepsilon u_R^\varepsilon - \int u^{\varepsilon_1,0} \cdot \nabla b_B^\varepsilon b_R^\varepsilon \\
I_5 &= \int b_B^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon; \\
I_6 &= - \int u_R^\varepsilon \cdot \nabla b_B^\varepsilon b_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon \\
I_7 &= \int b_B^\varepsilon \cdot \nabla b^{\varepsilon_1,0} u_R^\varepsilon + \int b_B^\varepsilon \cdot \nabla u^{\varepsilon_1,0} b_R^\varepsilon; \\
I_8 &= - \int u_R^\varepsilon \cdot \nabla u^{\varepsilon_1,0} u_R^\varepsilon - \int u_R^\varepsilon \cdot \nabla b^{\varepsilon_1,0} b_R^\varepsilon \\
&\quad + \int b_R^\varepsilon \cdot \nabla b^{\varepsilon_1,0} u_R^\varepsilon + \int b_R^\varepsilon \cdot \nabla u^{\varepsilon_1,0} b_R^\varepsilon; \\
I_9 &= \int b^\varepsilon \cdot \nabla b_R^\varepsilon u_R^\varepsilon + \int b^\varepsilon \cdot \nabla u_R^\varepsilon b_R^\varepsilon; \\
I_{10} &= \varepsilon_2 \int \Delta b^{\varepsilon_1,0} b_R^\varepsilon; & I_{11} &= \varepsilon_2 \int \Delta b_B^\varepsilon b_R^\varepsilon;
\end{aligned}$$

We now bound each of $I_i, i = 1, \dots, 11$.

First, similar to the estimates of $J_1, \dots, J_5, J_7, \dots, J_{13}$, we can estimate $I_1, \dots, I_5, I_7, \dots, I_{11}$ to get

$$I_1 + \dots + I_5 + I_7 + \dots + I_{11} \leq C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C \left(\frac{(\sqrt{\varepsilon_2})^{1-\tau}}{\sqrt{\theta}} + (\sqrt{\theta\varepsilon_2})^{1+\tau} \right). \quad (3.75)$$

Secondly, we estimate I_6 as follows:

$$I_6 = I_{61} + I_{62},$$

where

$$I_{61} = - \int u_R^\varepsilon \cdot \nabla b_B^\varepsilon b_R^\varepsilon; \quad I_{62} = \int b_R^\varepsilon \cdot \nabla b_B^\varepsilon u_R^\varepsilon.$$

Now we estimate I_{61} , which is split into four parts:

$$\begin{aligned}
I_{61} &= - \int U_R^\varepsilon \cdot \nabla_{x,y} B_B^\varepsilon B_R^\varepsilon - \int U_R^\varepsilon \cdot \nabla_{x,y} b_{B3}^\varepsilon b_{R3}^\varepsilon \\
&\quad - \int u_{R3}^\varepsilon \partial_z b_{B3}^\varepsilon b_{R3}^\varepsilon - \int u_{R3}^\varepsilon \partial_z B_B^\varepsilon B_R^\varepsilon \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

The integrals K_1, K_2, K_3 can be easily bounded by

$$K_1 + K_2 + K_3 \leq C \|(u^{\varepsilon_1, 0}, b^{\varepsilon_1, 0})\|_{H^s} \|(u_R^\varepsilon, b_R^\varepsilon)\|_{L^2}^2, \quad s > \frac{5}{2}.$$

For K_4 , we have

$$\begin{aligned} K_4 &= - \int_{\omega} \int_0^{\frac{h}{4}} u_{R3}^\varepsilon \cdot \partial_z B_{B+}^\varepsilon B_R^\varepsilon - \int_{\omega} \int_{\frac{3h}{4}}^h u_{R3}^\varepsilon \cdot \partial_z B_{B-}^\varepsilon B_R^\varepsilon \\ &= - \int_{\omega} \int_0^{\frac{h}{4}} \frac{u_{R3}^\varepsilon}{z} z^2 \partial_z B_{B+}^\varepsilon \frac{B_R^\varepsilon}{z} - \int_{\omega} \int_{\frac{3h}{4}}^h \frac{u_{R3}^\varepsilon}{h-z} (h-z)^2 \partial_z B_{B-}^\varepsilon \frac{B_R^\varepsilon}{h-z} \\ &\leq \left\| \frac{u_{R3}^\varepsilon}{z} \right\|_{L^2} \|z^2 \partial_z B_{B+}^\varepsilon\|_{L^\infty} \left\| \frac{B_R^\varepsilon}{z} \right\|_{L^2} + \left\| \frac{u_{R3}^\varepsilon}{h-z} \right\|_{L^2} \|(h-z)^2 \partial_z B_{B-}^\varepsilon\|_{L^\infty} \left\| \frac{B_R^\varepsilon}{h-z} \right\|_{L^2} \\ &\leq C \|(u^{\varepsilon_1, 0}, b^{\varepsilon_1, 0})\|_{H^s} \sqrt{(\theta \varepsilon_2)^{1+\tau}} \|\partial_z u_{R3}^\varepsilon\|_{L^2} \|\partial_z B_R^\varepsilon\|_{L^2} \\ &\leq \theta^{1+\tau} (\varepsilon_2)^{1+\frac{\tau}{2}} \|\partial_z b_R^\varepsilon\|_{L^2}^2 + C \theta^{1+\tau} (\varepsilon_2)^{\frac{\tau}{2}} \|(u^{\varepsilon_1, 0}, b^{\varepsilon_1, 0})\|_{H^s}^2 \|\partial_z u_R^\varepsilon\|_{L^2}^2 \\ &\leq \theta^{1+\tau} \varepsilon_2 \|\partial_z b_R^\varepsilon\|_{L^2}^2 + C \theta^{1+\tau} \|\partial_z u_R^\varepsilon\|_{L^2}^2 \end{aligned}$$

for some $\theta > 0$ sufficiently small if $\varepsilon_2 \rightarrow 0$. Here we used the Hardy's inequality $\|(\frac{f(z)}{z}, \frac{f(z)}{h-z})\|_{L^2(0,1)} \leq C \|\partial_z f(z)\|_{L^2(0,1)}$ when $f(0) = 0$.

Hence, we have

$$I_{61} \leq \theta^{1+\tau} \varepsilon_2 \|\partial_z b_R^\varepsilon\|_{L^2}^2 + C \theta^{1+\tau} \|\partial_z u_R^\varepsilon\|_{L^2}^2 + C(\|u_R^\varepsilon\|_{L^2}^2 + \|b_R^\varepsilon\|_{L^2}^2).$$

Similarly, I_{62} can be bounded by

$$I_{62} \leq \theta^{1+\tau} \varepsilon_2 \|\partial_z b_R^\varepsilon\|_{L^2}^2 + C \theta^{1+\tau} \|\partial_z u_R^\varepsilon\|_{L^2}^2 + C(\|u_R^\varepsilon\|_{L^2}^2 + \|b_R^\varepsilon\|_{L^2}^2).$$

Thus, we have the estimate on I_6

$$I_6 \leq \theta^{1+\tau} \varepsilon_2 \|\partial_z b_R^\varepsilon\|_{L^2}^2 + C \theta^{1+\tau} \|\partial_z u_R^\varepsilon\|_{L^2}^2 + C(\|u_R^\varepsilon\|_{L^2}^2 + \|b_R^\varepsilon\|_{L^2}^2). \quad (3.76)$$

Putting (3.75)-(3.76) into (3.74), one have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + (\varepsilon_1 - C \theta^{1+\tau}) \int |\nabla u_R^\varepsilon|^2 + \varepsilon_2 (1 - \theta^{1+\tau}) \int |\nabla b_R^\varepsilon|^2 \\ &\leq C \int (|u_R^\varepsilon|^2 + |b_R^\varepsilon|^2) + C \left(\frac{(\sqrt{\varepsilon_2})^{1-\tau}}{\sqrt{\theta}} + (\sqrt{\theta \varepsilon_2})^{1+\tau} \right). \end{aligned} \quad (3.77)$$

Taking $\theta > 0$ to be sufficiently small and to be independent of ε_2 and applying Gronwall's inequality to (3.77) and using the assumption (2.9) on initial data, we can get the estimate rate (2.10) in Theorem 2.4.

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