Liouville type theorems on the steady Navier-Stokes equations in $\mathbf{R}^3$

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Abstract

In this paper we study the Liouville type properties for solutions to the steady incompressible Navier-Stoks equations in $\mathbf{R}^3$. It is shown that any solution to the steady Navier-Stokes equations in $\mathbf{R}^3$ with finite Dirichlet integral and vanishing velocity field at far fields must be trivial. This solves an open problem. The key ingredients of the proof include a Hodge decomposition of the energy-flux and the observation that the square of the deformation matrix lies in the local Hardy space. As a by-product, we also obtain a Liouville type theorem for the steady density-dependent Navier-Stokes equations.

1 Introduction

Consider the three-dimensional (3D) steady Navier-Stokes equations for incompressible (Newtonian) fluids:

$$\left\{\begin{array}{c} u\cdot\nabla u-\Delta u=-\nabla p \\ \operatorname{div} u=0, \end{array}\right. \tag{1}$$

where $u(x)=\left(u^1(x),u^2(x),u^3(x)\right)$ is the velocity, and $p(x)$ is the pressure. We are interested in the following problem: Is it possible to characterize all solutions of (1) in $\mathbf{R}^3$, which satisfy the following condition:

$$\int_{\mathbf{R}^3}|\nabla u|^2\,dx<+\infty,\ \text{ and }\ \lim_{|x|\to\infty}u(x)=0. \tag{2}$$

In particular one would like to know whether the problem (1)-(2) has only trivial solution $u\equiv 0$ and $p=$constant. This Liouville type problem is of significance not only in the theory for (1) itself, but also can serve as the first step

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in understanding the uniqueness and asymptotic structure of solutions to the nonhomogeneous problems for the Navier-Stokes equations (See [1], [2], [8], [10], [11], [13], [15], [17], [19], [21], [23], [25], [31], [33], [34]). For the detailed motivations and background of this problem we refer to [13] [28]. The corresponding problem in $\mathbf{R}^2$ had been solved positively by Gilbarg and Weinberger in 1978 [15] by a complex analysis approach, while for the case in $\mathbf{R}^n$ $(n \geq 4)$ the Liouville type theorem holds for smooth solutions as an easy consequence of the Sobolev's imbedding inequality (see Chapter XII [13]). So as discussed in many papers [3], [4], [5], [6], [13], [7], [19], [20], [21], [22], [18], [27], [28], the Liouville type problem remains open only in $\mathbf{R}^3$ though there have been many efforts to solve this open problem and related problems with interesting partial results recently (see [3], [4], [5], [6], [7], [13], [18], [19], [20], [21], [22], [27], [28] and the references therein). In particular, in [13], Galdi gave a positive answer to the open problem under the additional assumption that $u \in L^{\frac{9}{2}}\left(\mathbf{R}^3\right)$, which has been improved logarithmically in [6]. It was also shown that the Liouville type problem (1)-(2) has a positive answer under additional conditions that either $\Delta u \in L^{\frac{6}{5}}\left(\mathbf{R}^3\right)$ [3], or weak $L^{\frac{1}{2}}\left(\mathbf{R}^2\right)$ norm of u is small, or $\lim_{|x|\to 0} |x|^{\frac{5}{3}} |\operatorname{curl} u|$ is small [22]. Recently Seregin solved the open problem positively under the additional assumption that either $u \in BMO^{-1}\left(\mathbf{R}^3\right)$ [27] or the solution u belongs to some specific Morrey space for which (2) holds [28]. It should also be mentioned that the Liouville type problem (1)-(2) has a positive answer under the additional assumption that the flow is axially symmetric without swirl in [19] and [20], and the case with swirl has also been solved positively in [4]. For some other recent related results, we refer to references [19], [5], [31], [24] and [7]. As far as we know, the answer to the Liouville type problem (1)-(2) remains open for general cases.

On the other hand, it is well-known from the theory of minimal surface or elliptic systems that Liouville type properties are related to the issue of regularity of the solutions. This is also the case for the regularity problems for the 3D Navier-Stokes equation as shown by the recent progress that there are no Leray type self-similar finite time blow-up solutions to the 3D Navier-Stokes equations (see [24], [32], [19] and [3]). Thus one may expect that Liouville type results for the time-dependent 3D Navier-Stokes equations, or even the steady-state case (1) will share lights on the regularity problem for the 3D Navier-Stokes equations which is an outstanding open problem.

In this paper we prove that the condition (2) is sufficient to guarantee the solution of (1) to be trivial as conjectured by G. Seregin in [27] without any additional assumptions. Precisely we obtain the following theorem (main theorem):

Theorem 1 *Any weak solution to (1) satisfying condition (2) must be trivial.*

The major ingredients of the proof consist of the observation that for any weak solution, u, to (1) with finite Dirichlet integral and the decay property in (2) for the velocity field, $|\nabla u|^2$ is in fact in the local Hardy space $h^1\left(\mathbf{R}^3\right)$ and the Hodge decomposition of the energy flux (see (24)).

Similarly we also study Liouville type theorems for the steady density-dependent Navier-Stokes equations, it should be natural to ask the similar problem in this case. The steady density dependent Navier-Stokes equations read

$$\begin{cases} \operatorname{div}(\rho u) = 0 \\ \rho (u \cdot \nabla) u - \Delta u = -\nabla p \\ \operatorname{div} u = 0, \end{cases} \tag{3}$$

here $u = \left(u^1, u^2, u^3\right)$ is the velocity, $\rho(x) \geq 0$ denotes the density and $p(x)$ is the pressure. Then we are interested in under the condition

$$\int_{\mathbf{R}^3} |\nabla u|^2 \, dx < +\infty, \ \lim_{|x| \to \infty} u(x) = 0 \ \text{ and } \ \rho(x) \in L^\infty\left(\mathbf{R}^3\right) \tag{4}$$

whether the solution of (3) must be trivial ($u \equiv 0$, and ρ can be any positive function). As a by-product of our analysis in proving Theorem 1, we can prove the following Liouville type theorem.

Theorem 2 *Let (u, ρ, p) be a weak solution to (3) satisfying condition (4), then u must be trivial and $\rho(x)$ be any L^∞ function.*

The paper is organized as follow: In section 2 we will prove the main Theorem 1. In this process, we need to achieve the crucial estimate (34) for the gradient part of the Hodge decomposition of the energy flux, for which some tools involving the Hardy space and the local Hardy space are needed. In section 3 we will give the proof of Theorem 2, for this we need also to prove an asymptotic result for solutions of (3). For readers' convenience, we will give a short explanation about the Hardy space and the local Hardy space in the Appendix.

We conclude this introduction by giving some conventions to be used in the sequel: $\mathbf{R}^3$ is the 3D Euclidean space with a fixed orthonormal basis. A physical point in $\mathbf{R}^3$ is denoted by $x = (x_1, x_2, x_3)$. As usual the summation convention over repeated indices is used in the rest of the paper. The standard Lebesgue spaces are denoted by L^p $(p \geq 1)$. $B_r(x)$ denotes the ball of radius $r > 0$ around the center $x \in \mathbf{R}^3$. The mean value of some function $f(x)$ over $B_r(x)$ is defined as

$$[f]_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(x).$$

Various constants arise in our paper; unless indicated otherwise, they are always absolute constants. The symbol C denotes a generic constant; its value may change from line to line.

2 Proof of main Theorem 1

Let us recall some results for the steady Navier-Stokes equations (1), which will be useful for estimating the solution. First, it is well known that any weak solution for the steady Navier-Stokes equations is smooth when dimension $n = 3$, or 4, see [14] and [12]. The second one is the following result proved by Galdi (see Theorem X.5.1 of [13] for a more general version).

Theorem 3 (Galdi) *Let $u(x)$ be a weak solution of (1) satisfying the condition (2) and $p(x)$ be the associated pressure, then there exists constant $p_1 \in R$ such that*

$$\lim_{x|\to\infty} |D^{\alpha}u(x)| + \lim_{x|\to\infty} |D^{\alpha}(p(x)-p_1)| = 0 \tag{5}$$

uniformly for all multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in [N \cup \{0\}]^3$.

The following well-known decomposition theorem will also be needed (Helmholtz-Weyl or Hodge theorem).

Theorem 4 (Helmholtz-Weyl) *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$ be either a domain of class C^2 or the whole space or a half-space. Then any vector function $u(x) \in L^p(\Omega)$, $p > 1$, can be decomposed uniquely as the sum*

$$u(x) = \nabla\varphi + F,$$

where $F(x)$ is a divergence free vector function, and the following estimate holds

$$\|\nabla\varphi\|_{L^p} + \|F\|_{L^p} \leq C\|u\|_{L^p}.$$

The following duality lemma on the whole space $\mathbf{R}^3$ is also needed. Here and in the sequel $\overset{\circ}{W}{}^{1,p}(\mathbf{R}^3)$ denote the homogeneous Sobolev space equipped with the norm $\|u\| = \left(\int_{\mathbf{R}^3} |\nabla u|^p\right)^{\frac{1}{p}}$.

Lemma 5 *Let $u \in \overset{\circ}{W}{}^{1,2}(\mathbf{R}^3)$. Then it holds that*

$$\|\nabla u\|_{L^{\frac{3}{2}}} \leq C \sup_{\substack{\varphi \in C_0^{\infty}(\mathbf{R}^3) \\ \|\nabla\varphi\|_{L^3} \leq 1}} \int_{\mathbf{R}^3} \nabla u \cdot \nabla\varphi. \tag{6}$$

Proof. It follows from the duality of the L^p space that

$$\begin{aligned} \|\nabla u\|_{L^{\frac{3}{2}}} &= \sup_{\|F\|_{L^3} \leq 1} \int_{\mathbf{R}^3} \nabla u \cdot F \\ &= \sup_{M>0} \sup_{\substack{\|F\|_{L^3} \leq 1 \\ \|F\|_{L^{\frac{6}{5}}} \leq M}} \int_{\mathbf{R}^3} \nabla u \cdot F. \end{aligned} \tag{7}$$

Assume first that the left hand side of (6) is finite. Then for any $\varepsilon > 0$ one can find $M_{\varepsilon} > 0$ sufficiently large, such that

$$\sup_{\substack{\|F\|_{L^3} \leq 1 \\ \|F\|_{L^{\frac{6}{5}}} \leq M_{\varepsilon}}} \int_{\mathbf{R}^3} \nabla u \cdot F \geq \|\nabla u\|_{L^{\frac{3}{2}}} - \varepsilon. \tag{8}$$

Hence the Hodge decomposition theorem yields that

$$F = \nabla \phi + G, \tag{9}$$

here $\operatorname{div} G = 0$, and satisfies

$$\begin{aligned} &\|\nabla \phi\|_{L^3} + \|G\|_{L^3} \leq C \|F\|_{L^3} \\ &\|\nabla \phi\|_{L^{\frac{6}{5}}} + \|G\|_{L^{\frac{6}{5}}} < +\infty. \end{aligned} \tag{10}$$

This, together with (8), yields

$$\|\nabla u\|_{L^{\frac{3}{2}}} \leq C \sup_{\|\nabla \phi\|_{L^3} \leq 1} \int_{\mathbf{R}^3} \nabla u \cdot \nabla \phi + \sup_{\substack{\|G\|_{L^3} \leq C \\ \|G\|_{L^{\frac{6}{5}}} \leq CM_\varepsilon}} \int_{\mathbf{R}^3} \nabla u \cdot G + \varepsilon. \tag{11}$$

Noticing that $uG \in L^1(\mathbf{R}^3)$ by the Sobolev and Hölder inequalities, then one can get

$$\lim_{R \to \infty} \int_{\partial B_R} |uG| \, d\sigma = 0, \tag{12}$$

and so

$$\begin{aligned} \int_{\mathbf{R}^3} \nabla u \cdot G &= \lim_{R \to \infty} \int_{B_R} \nabla u \cdot G \\ &= - \lim_{R \to \infty} \int_{\partial B_R} uG \cdot n d\sigma \\ &= 0. \end{aligned} \tag{13}$$

Combining (11) with (13) and noting the arbitrariness of ε, one obtains that

$$\|\nabla u\|_{L^{\frac{3}{2}}} \leq C \sup_{\|\nabla \varphi\|_{L^3} \leq 1} \int_{\mathbf{R}^3} \nabla u \cdot \nabla \varphi$$

which implies (6) immediately by the density of $C_0^\infty(\mathbf{R}^3)$ in $\overset{\circ}{W}{}^{1,3}(\mathbf{R}^3)$. Note that the above arguments assume that the left hand side of (6) is finite, the other case can be handled easily by modifying the above analysis. Then the proof of Lemma is completed. ■

The following proposition corresponds to the Proposition 1.92 in [26] with a different manner; in [26] S. Semmes uses a different version of a local Hardy space $\mathcal{H}^1_{loc}(\mathbf{R}^n)$ which is not a Banach space. By replacing $\mathcal{H}^1_{loc}(\mathbf{R}^n)$ with $h^1(\mathbf{R}^n)$ we obtain a similar result and furthermore get the estimate (14) below, which is a key step to prove the main theorem. The proof we give here is similar to Proposition 1.92 in [26], however we focus carefully on the estimate (14). Here and in the sequel we use $\mathcal{H}^1(\mathbf{R}^n)$ to denote the Hardy space and $h^1(\mathbf{R}^n)$ to denote the local Hardy space. More detailed discussions about the local Hardy space can be found in [16] or [9]. We will give the definitions and some basic properties of the Hardy space (including the local Hardy space) in the Appendix, for more properties about the Hardy space and the local Hardy space, we refer to [26], [27].

Proposition 6 *Let f lie in the local Hardy space, i.e., $f \in h^1(\mathbf{R}^n)$, $\theta_R(x)$ be a smooth cut-off function satisfying $\theta_R(x) = 1$, $x \in B_R(0)$; $\theta_R(x) = 0$, $x \in \mathbf{R}^n \diagdown B_{R+1}(0)$, and $\|\nabla\theta\|_{L^\infty} \le 1$. Then for any $R > 0$ there exists a number λ such that $\theta_R(x)(f-\lambda) \in \mathcal{H}^1(\mathbf{R}^n)$ and*

$$\|\theta_R(x)(f-\lambda)\|_{\mathcal{H}^1(\mathbf{R}^n)} \le C\|f\|_{h^1(\mathbf{R}^n)}, \tag{14}$$

with a constant C independent of R.

Remark 7 *Above conclusion would not be correct if the number λ is dropped in (14), this is due to the special demand of a Hardy space function. Actually one can always choose λ such that $\int_{\mathbf{R}^n}\theta_R(x)(f-\lambda) = 0$, so that $|\lambda| = \left|\frac{1}{\int_{\mathbf{R}^n}\theta_R(x)}\int_{\mathbf{R}^n}\theta_R(x)f(x)\right| \le \frac{C}{R^n}\int_{\mathbf{R}^n}|f(x)|$.*

Proof. Let $f^*(x)$ be the grand maximal function of $f(x)$ defined by

$$f^*(x) = \sup_{t>0}\sup_{\phi\in\mathcal{T}}\left|\int_{\mathbf{R}^n}\frac{1}{t^n}\phi\left(\frac{x-y}{t}\right)f(y)\,dy\right|,$$

and similarly define its local grand maximal function $f^*_{loc}(x)$ by

$$f^*_{loc}(x) = \sup_{0<t<1}\sup_{\phi\in\mathcal{T}}\left|\int_{\mathbf{R}^n}\frac{1}{t^n}\phi\left(\frac{x-y}{t}\right)f(y)\,dy\right|,$$

where

$$\mathcal{T} = \{\phi \in C_0^\infty(\mathbf{R}^n), \operatorname{supp}\phi \subseteq B_1(0) \text{ and } \|\nabla\phi\|_{L^\infty} \le 1\}.$$

Set $g(x) = \theta_R(x)(f(x)-\lambda)$. It holds that

$$\begin{aligned} &\|g^*(x)\|_{L^1(\mathbf{R}^n)} \\ &\le \max\left(\int_{\mathbf{R}^n} g^*_{loc}(x)\,dx, \int_{\mathbf{R}^n}\sup_{t\ge1}\sup_{\phi\in\mathcal{T}}\left|\int_{\mathbf{R}^n}\frac{1}{t^n}\phi\left(\frac{x-y}{t}\right)g(y)\,dy\right|dx\right) \\ &= \max(I_1, I_2). \end{aligned} \tag{15}$$

Now we estimate I_1 and I_2 respectively. For clarity we denote by $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$, then $g^*_{loc}(x) = \sup_{0<t<1}\sup_{\phi\in\mathcal{T}}|\phi_t * g|$, $*$ denotes the convolution. Note that

$$\phi_t * g = -\lambda\phi_t * \theta_R + \widetilde{\phi}_t * f,$$

here $\widetilde{\phi}(y) = \phi(y)\theta_R(x-ty)$. If $0<t<1$, it is easy to check that $\frac{1}{2}\widetilde{\phi}(2y) \in \mathcal{T}$

$$\begin{aligned} I_1 &= \int_{\mathbf{R}^n}\sup_{0<t<1}\sup_{\phi\in\mathcal{T}}|\phi_t * g|\,dx \\ &\le C\left[\left(\frac{1}{R^n}\int_{\mathbf{R}^n}|f(x)|\,dx\right)\int_{B_{R+1}(0)}1dx + \int_{\mathbf{R}^n}\sup_{0<t<1}\sup_{\phi\in\mathcal{T}}|\phi_t * f|\,dx\right] \\ &\le C\left(\|f\|_{L^1(\mathbf{R}^n)} + \|f\|_{h^1(\mathbf{R}^n)}\right). \end{aligned} \tag{16}$$

On the other hand, when $t \geq 1$ it is easy to see that $\phi_t * g(x) = 0$ for x such that $dist\,(x, B_{R+1}(0)) > t$; so

$$\begin{aligned} |\phi_t * g(x)| &= \left| \int_{\mathbf{R}^n} \phi_t (x-y)\, g(y)\, dy - \int_{\mathbf{R}^n} \phi_t(0)\, g(y)\, dy \right| \\ &\leq C \left\| \nabla \phi_t \right\|_{L^\infty} \left\| g \right\|_{L^1} \\ &\leq C \frac{1}{t^{n+1}} \left\| f \right\|_{L^1} \\ &\leq C \frac{1}{(1+|x|)^{n+1}} \left\| f \right\|_{L^1} \quad \text{as } |x| \geq R+2, \end{aligned} \tag{17}$$

here we have used the fact that $\int_{\mathbf{R}^n} g(x)\, dx = 0$.

For $x \in B_{R+2}(0)$, we claim that

$$\int_{B_{R+2}(0)} \sup_{t \geq 1} \sup_{\phi \in \mathcal{T}} \left| \int_{\mathbf{R}^n} \frac{1}{t^n} \phi\left(\frac{x-y}{t}\right) g(y)\, dy \right| dx \leq C \left\| f \right\|_{L^1(R^n)}. \tag{18}$$

Indeed for any $\varepsilon > 0$, let $1 \leq t_0 < \infty$, $\phi^0(x) \in \mathcal{T}$ such that

$$\left| \int_{\mathbf{R}^n} \frac{1}{t_0^n} \phi^0\left(\frac{x-y}{t_0}\right) g(y)\, dy \right| \geq \sup_{t \geq 1} \sup_{\phi \in \mathcal{T}} \left| \int_{\mathbf{R}^n} \frac{1}{t^n} \phi\left(\frac{x-y}{t}\right) g(y)\, dy \right| - \varepsilon. \tag{19}$$

Then it follows from the Fubini's theorem that

$$\begin{aligned} &\int_{B_{R+2}(0)} \sup_{t \geq 1} \sup_{\phi \in \mathcal{T}} \left| \int_{\mathbf{R}^n} \frac{1}{t^n} \phi\left(\frac{x-y}{t}\right) g(y)\, dy \right| dx \\ &\leq \int_{B_{R+2}(0)} \left| \int_{\mathbf{R}^n} \frac{1}{t_0^n} \phi^0\left(\frac{x-y}{t_0}\right) g(y)\, dy \right| dx + \varepsilon \omega_{n-1} (R+2)^n \\ &\leq \int_{\mathbf{R}^n} |g(y)| \int_{B_{R+2}(0)} \left| \frac{1}{t_0^n} \phi^0\left(\frac{x-y}{t_0}\right) \right| dx dy + \varepsilon \omega_{n-1} (R+2)^n \\ &\leq C \left\| g \right\|_{L^1(\mathbf{R}^n)} + \varepsilon \omega_{n-1} (R+2)^n \\ &\leq C \left\| f \right\|_{L^1(\mathbf{R}^n)} + \varepsilon \omega_{n-1} (R+2)^n, \end{aligned} \tag{20}$$

here ω_{n-1} denotes the volume of unit sphere. Due to the arbitrariness of ε, we obtain (18). Collecting (16), (17) and (18) together, we get the estimate (14) and this completes the proof. ■

To derive further estimate for the local Hardy norm of a function, we need the following nonhomogeneous div-curl result due to Galia Dafni (Theorem 3 in [9]).

Theorem 8 *Suppose that V and U are vector fields on $\mathbf{R}^n$ satisfying*

$$V \in L^p(\mathbf{R}^n)^n, U \in L^{p'}(\mathbf{R}^n)^n, 1 < p < \infty, p' = \frac{p}{p-1}$$

and

$$\operatorname{div} V = f \in L^p(\mathbf{R}^n), \quad \operatorname{curl} U = 0$$

in sense of distribution. Then $V \cdot U$ belongs to the local Hardy space $h^1(\mathbf{R}^n)$ with

$$\|V \cdot U\|_{h^1(\mathbf{R}^n)} \leq C \left(\|V\|_{L^p(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)} \right) \|U\|_{L^{p'}(\mathbf{R}^n)} . \tag{21}$$

Now we are in the position to prove the main Theorem 1.

Proof of Theorem 1. Note that the equations in (1) can be write as

$$\Delta u = u \cdot \left[\nabla u - (\nabla u)^T \right] + \nabla \left(p + \frac{1}{2} |u|^2 \right). \tag{22}$$

Taking inner product with u on both sides of (22) and noticing that $\left[\nabla u - (\nabla u)^T \right]$ is antisymmetric matrix and $\operatorname{div} u = 0$, we have

$$\Delta u \cdot u = \nabla \left(p + \frac{1}{2} |u|^2 \right) \cdot u$$

or

$$\Delta \left(\frac{1}{2} |u|^2 \right) - |\nabla u|^2 = \operatorname{div} \left[\left((p - p_1) + \frac{1}{2} |u|^2 \right) u \right], \tag{23}$$

here p_1 is constant in Theorem 3. It follows from the assumption (2) and the Sobolev embedding inequality that $u \in L^6(\mathbf{R}^3)$. To deal with the pressure term $p - p_1 = P$, one may recall that the pressure satisfies

$$-\Delta P = \partial_j \partial_k \left(u^j u^k \right) \quad \text{in } \mathbf{R}^3.$$

(Here we use the usual convention and sum over the repeated indices.) Then by the classical Calderón-Zygmund theorem [29] one can conclude that $P \in L^3(\mathbf{R}^3)$. It then follows from the Hölder inequality that $\left((p - p_1) + \frac{1}{2} |u|^2 \right) u \in L^2(\mathbf{R}^3)$. Then using the Helmholtz-Weyl decomposition Theorem 4 yields

$$\left((p - p_1) + \frac{1}{2} |u|^2 \right) u = \nabla G + F, \tag{24}$$

where G is a scalar function and F is a vector function such that

$$\|\nabla G\|_{L^2} + \|F\|_{L^2} \leq C \left\| \left((p - p_1) + \frac{1}{2} |u|^2 \right) u \right\|_{L^2}$$

and $\operatorname{div} F = 0$. Now we claim that

$$\nabla G \in L^{\frac{3}{2}}(\mathbf{R}^3), \tag{25}$$

this constitutes the main technical part for the proof.

It follows from Lemma 5 that

$$\|\nabla G\|_{L^{\frac{3}{2}}(\mathbf{R}^3)} \leq C \sup_{\substack{\varphi \in C_0^\infty(\mathbf{R}^3) \\ \|\nabla \varphi\|_{L^3} \leq 1}} \int_{\mathbf{R}^3} \nabla G \cdot \nabla \varphi dx. \tag{26}$$

Then it follows from (23), (24) and by using integration by part that

$$\int_{\mathbf{R}^3} \nabla G \cdot \nabla \varphi dx = \int_{\mathbf{R}^3} |\nabla u|^2 \varphi(x)\, dx + \int_{\mathbf{R}^3} \nabla \varphi \cdot \nabla u \cdot u dx \tag{27}$$
$$= I_1 + I_2.$$

Now the terms I_1 and I_2 can be estimated respectively. By the Hölder inequality, I_2 can be estimated as

$$\int_{\mathbf{R}^3} \nabla \varphi \cdot \nabla u \cdot u dx \leq \left(\int_{\mathbf{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} |u|^6 \right)^{\frac{1}{6}} \left(\int_{\mathbf{R}^3} |\nabla \varphi|^3 \right)^{\frac{1}{3}} \tag{28}$$
$$\leq \left(\int_{\mathbf{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} |u|^6 \right)^{\frac{1}{6}}.$$

In order to estimate I_1, we will use the local Hardy space structure Theorem 8 to claim that the quantity $|\nabla u|^2$ belongs to the local Hardy space $h^1(\mathbf{R}^3)$. To this end one can choose $U = V = \nabla u^i \in L^2(\mathbf{R}^3)$, $i = 1, 2, 3$. Then $\operatorname{curl} U = 0$ and

$$\operatorname{div} V = \Delta u^i = u \cdot \nabla u^i + \partial_i p.$$

It is easy to see that $u \cdot \nabla u^i \in L^{\frac{3}{2}}(\mathbf{R}^3)$ by the Hölder inequality. Note that the equation for the pressure p reads

$$-\Delta P = \operatorname{div}(u \cdot \nabla u).$$

Then it follows from the Calderón-Zygmund inequality again that $\nabla P \in L^{\frac{3}{2}}(\mathbf{R}^3)$. Then $\operatorname{div} V \in L^{\frac{3}{2}}(\mathbf{R}^3)$. On the other hand, it follows from the known regularity Theorem 3 that any weak solution (u, p) satisfying (2) must be smooth and satisfies the asymptotic property (5). Hence

$$u \cdot \nabla u^i + \partial_i p \in L^\infty(\mathbf{R}^3).$$

Then it follows that $\operatorname{div} V \in L^2(\mathbf{R}^3)$. Now Theorem 8 yields that $|\nabla u|^2 \in h^1(\mathbf{R}^3)$ and

$$\left\| |\nabla u|^2 \right\|_{h^1(\mathbf{R}^3)} \leq C \|\nabla u\|^2_{L^2(\mathbf{R}^3)}. \tag{29}$$

Next we decompose I_1 as

$$
\begin{aligned}
I_1 &= \int_{\mathbf{R}^3} |\nabla u|^2 \varphi(x)\, dx \qquad (30)\\
&= \lim_{R\to\infty} \int_{\mathbf{R}^3} |\nabla u|^2 \theta_R(x) \varphi(x)\, dx\\
&= \lim_{R\to\infty} \int_{\mathbf{R}^3} \left(|\nabla u|^2 - \lambda\right) \theta_R(x) \varphi(x)\, dx\\
&+ \lim_{R\to\infty} \int_{\mathbf{R}^3} \lambda \theta_R(x) \left(\varphi(x) - [\varphi]_{B_{R+1}(0)}\right) dx\\
&+ \lim_{R\to\infty} \left([\varphi]_{B_{R+1}(0)} \int_{\mathbf{R}^3} \theta_R(x) |\nabla u|^2\, dx\right)\\
&= T_1 + T_2 + T_3,
\end{aligned}
$$

here we have used the same choice of $\theta_R(x)$ as in (14) and

$$
\lambda = \frac{1}{\int_{\mathbf{R}^n} \theta_R(x)} \int_{\mathbf{R}^n} \theta_R(x) |\nabla u|^2 .
$$

It follows from Proposition 6 that $\left(|\nabla u|^2 - \lambda\right) \theta_R(x) \in \mathcal{H}^1\left(\mathbf{R}^3\right)$. Note that $\varphi(x) \in C_0^\infty\left(\mathbf{R}^3\right)$ so it is easy to see that

$$
\begin{aligned}
T_3 &= \lim_{R\to\infty} \left([\varphi]_{B_{R+1}(0)} \int_{\mathbf{R}^3} \theta_R(x) |\nabla u|^2\, dx\right) \qquad (31)\\
&= 0.
\end{aligned}
$$

On the other hand the Poincáre inequality yields that $\varphi(x) \in BMO\left(\mathbf{R}^3\right)$, and $\|\varphi(x)\|_{BMO(\mathbf{R}^3)} \le C \|\nabla \varphi\|_{L^3(\mathbf{R}^3)}$. By using the duality of Hardy-BMO spaces, (14) and (29) we obtain

$$
\begin{aligned}
T_1 &\le \left\| \left(|\nabla u|^2 - \lambda\right) \theta_R(x) \right\|_{\mathcal{H}^1(\mathbf{R}^3)} \|\varphi(x)\|_{BMO(\mathbf{R}^3)} \qquad (32)\\
&\le C \|\nabla u\|^2_{L^2(\mathbf{R}^3)} < +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 &= \lim_{R\to\infty} \int_{\mathbf{R}^3} \lambda \theta_R(x) \left(\varphi(x) - [\varphi]_{B_{R+1}(0)}\right) dx \qquad (33)\\
&\le \sup_{R>0} \frac{1}{\int_{\mathbf{R}^3} \theta_R(x)} \int_{\mathbf{R}^3} \theta_R(x) |\nabla u|^2\, dx \int_{\mathbf{R}^3} \theta_R(x) \left|\varphi(x) - [\varphi]_{B_{R+1}(0)}\right| dx\\
&\le C \|\nabla u\|_{L^2(\mathbf{R}^3)} \sup_{R>0} \frac{1}{R^3} \int_{\boldsymbol{B}_{R+1}(0)} \left|\varphi(x) - [\varphi]_{B_{R+1}(0)}\right| dx\\
&\le C \|\nabla u\|_{L^2(\mathbf{R}^3)} \|\varphi(x)\|_{BMO(\mathbf{R}^3)} .
\end{aligned}
$$

Collecting (31), (33), (32), (30), (28) and (27) together, we prove the claim (25) which means

$$\|\nabla G\|_{L^{\frac{3}{2}}(\mathbf{R}^3)} \leq C < +\infty. \tag{34}$$

With estimate (34) in hand, we can proceed to prove the conclusion.

Let $\psi_R(x)$ be a smooth cut-off function such that $\psi_R(x) = 1$ as $x \in B_R(0)$, $\psi_R(x) = 0$ as $x \in \mathbf{R}^3 \diagdown B_{2R}(0)$ and $|\nabla \psi_R(x)| \leq \frac{C}{R}$. Then testing by $\psi_R(x)$ on both sides of (23) and using integration by part, one can show by the decomposition (24) that

$$\begin{aligned}\int_{\mathbf{R}^3} |\nabla u|^2 \psi_R(x)\, dx &= \int_{\mathbf{R}^3} \nabla G \cdot \nabla \psi_R(x)\, dx - \int_{\mathbf{R}^3} (\nabla \psi_R(x) \cdot \nabla)\, u \cdot u dx \qquad (35)\\ &\leq \frac{C}{R}\left(\int_{R\leq|x|\leq 2R} |\nabla G|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \left(\int_{|x|\leq 2R} 1 dx\right)^{\frac{1}{3}} \\ &\quad + \frac{C}{R}\left(\int_{R\leq|x|\leq 2R} |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_{R\leq|x|\leq 2R} |u|^6 dx\right)^{\frac{1}{6}} \cdot R \\ &\leq C\left(\int_{R\leq|x|\leq 2R} |\nabla G|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \\ &\quad + C\left(\int_{R\leq|x|\leq 2R} |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_{R\leq|x|\leq 2R} |u|^6 dx\right)^{\frac{1}{6}}.\end{aligned}$$

Taking $R \to \infty$ and using the finiteness of $\int_{\mathbf{R}^3} |\nabla G|^{\frac{3}{2}} dx$ (34) and $\int_{\mathbf{R}^3} |\nabla u|^2 dx$ (2), we conclude finally that

$$\int_{\mathbf{R}^3} |\nabla u|^2 dx = 0,$$

which implies that u must be a constant and then $u \equiv 0$. Thus the conclusion of Theorem 1 follows. ■

3 Proof of Theorem 2

In this section, we will obtain a similar Liouville type theorem for the steady density dependent Navier-Stokes equations (3). To this end, we first introduce the representation formula for the nonhomogeneous Stokes equations, which will be used to get the asymptotic behavior of solutions of (3). To our knowledge, for the case of the density dependent Navier-Stokes equations, this kind of results are unknown. However, one can prove the following asymptotic result.

Theorem 9 *Let (u, ρ, p) be a generalized solution to (3) and satisfy (4). Then $\nabla^2 u \in L^p(\mathbf{R}^3)$ for any $2 \leq p < \infty$, and $\nabla p \in BMO(\mathbf{R}^3)$. Furthermore, there is $p_1 \in \mathbf{R}^1$ such that:*

$$\lim_{|x|\to\infty} |p(x) - p_1| = 0 \tag{36}$$

and

$$\lim_{|x|\to\infty} |\nabla u(x)| = 0. \tag{37}$$

To prove this theorem, one needs a representation formula, whole proof can be found in [13]. Let $v(x)$ be a weak solution of the following linear Stokes equations

$$\begin{cases} \Delta v - \nabla p = f \\ \operatorname{div} v = 0, \end{cases}$$

then the following representation formula holds (Lemma V.3.1 in [13]).

$$D^\alpha v(x) = \int_{B_d(x)} U_{ij}^{(d)}(x-y)\, D^\alpha f_i(y)\, dy - \int_{\beta(x)} H_{ij}^{(d)}(x-y)\, D^\alpha v_i(y)\, dy. \tag{38}$$

Where $\beta(x) = B_d(x) - B_{\frac{d}{2}}(x)$ and $\left(U_{ij}^{(d)}(x), q_j^{(d)}(x)\right)$ denotes the Stokes-Fujita truncated fundamental solution, and

$$H_{ij}^{(d)}(x-y) = \begin{cases} 0 & \text{if } x = y \\ \delta_{ij}\Delta^2(\psi_d\Phi)(x-y) & \text{if } x \neq y, \end{cases}$$

here $\Phi(x)$ is the fundamental solution of the biharmonic equation and $\psi_d(x)$ is the truncation function. For details of the definition of (38), see [13]. Now we prove Theorem 9. The proof here is similar to that of Theorem X.5.1 in [13]. For completeness we sketch the proof, the details can be found in [13].
Proof of Theorem 9. Let $u(x)$ be a solution of (3) and satisfy (4). One can get the regularity property that $\nabla^2 u \in L^p(\mathbf{R}^3)$ for any $2 \leq p < \infty$ by using the standard bootstrap method for (3) and the pressure equation

$$\Delta p = -\operatorname{div}(\rho u \cdot \nabla u) = -\operatorname{div}\operatorname{div}(\rho u \otimes u). \tag{39}$$

However we cannot expect more regularity on u due to the fact that ρ may not be continuous. Now we use formula (38) to derive (36) and (37). It follows from (3) and (38) that

$$\begin{aligned} D^k u_j(x) &= \int_{B_d(x)} D^k U_{ij}^{(d)}(x-y)\,[\rho(y)\, u \cdot \nabla u_i(y)]\, dy \\ &\quad - \int_{\beta(x)} D^k H_{ij}^{(d)}(x-y)\, u_i(y)\, dy \\ &= I_1 + I_2, \end{aligned} \tag{40}$$

here $k = 1, 2, 3$, and $D^k = \frac{\partial}{\partial x_k}$. By the properties of the fundamental solution Φ and the truncation function ψ_d (see [13]), one has that

$$\left| D^k U_{ij}^{(d)}(x-y) \right| \leq C\,|x-y|^{-2}, \;\; y \in B_d(x) \tag{41}$$

and

$$\left| D^k H_{ij}^{(d)}(x-y) \right| \leq C d^{-4}. \tag{42}$$

It follows from the regularity property and (4) that $u \in L^{\infty}\left(\mathbf{R}^3\right)$. Then it is easy to deduce that $\rho u \cdot \nabla u \in L^q\left(\mathbf{R}^3\right)$ for $\frac{3}{2} \leq q \leq 6$ from the interpolation theorem and the Sobolev inequality. Then one can get from the Hölder inequality that

$$\begin{aligned} |I_1| &= \left| \int_{B_d(x)} D^k U_{ij}^{(d)}(x-y)\left[\rho(y)\, u \cdot \nabla u_i(y)\right] dy \right| \\ &\leq C \left\| |x-y|^{-2} \right\|_{L^{\frac{q}{q-1}}(B_d(x))} \|u \cdot \nabla u\|_{L^q(B_d(x))}. \end{aligned} \tag{43}$$

Choose $q > 3$. Then it is easy to see that

$$I_1(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{44}$$

The Hölder inequality yields that

$$\begin{aligned} |I_2| &= \left| \int_{\beta(x)} D^k H_{ij}^{(d)}(x-y)\, u_i(y)\, dy \right| \\ &\leq C \|u\|_{L^6(B_d(x))}. \end{aligned} \tag{45}$$

Now the desired asymptotic property (37) follows from this inequality, (44) and (40). The asymptotic property (36) can be proved similarly by using Theorem II.9.1 in [13] (which can be also proved directly by using a Hardy inequality, details are omitted here). Note also that the asymptotic property (37) implies $\nabla u \in L^{\infty}\left(\mathbf{R}^3\right)$ and $u \in L^{\infty}\left(\mathbf{R}^3\right)$. It follows from (39) that $\nabla p \in BMO\left(\mathbf{R}^3\right)$ by using the Calderón-Zygmund Theorem (see [29]). This completes the proof. ■

With the help of Theorem 9, now we are in the position to prove the main Theorem 2. Most of arguments in the following are similar to that in Section 2 for the proof of Theorem 1.

Proof of Theorem 2. Note that $\operatorname{div}(\rho u) = 0$. So the momentum equations in (3) imply that

$$\Delta\left(\frac{1}{2}|u|^2\right) - |\nabla u|^2 = \operatorname{div}\left[\left((p-p_1) + \frac{1}{2}\rho|u|^2\right) u\right], \tag{46}$$

here p_1 is the constant in Theorem 9. By the assumption (4) and the Sobolev embedding inequality it holds that $u \in L^6\left(\mathbf{R}^3\right)$. Note that the pressure term $p - p_1 = P$ solves

$$-\Delta P = \partial_j \partial_k\left(\rho u^j u^k\right) \quad \text{in } \mathbf{R}^3.$$

Then it follows from the classical Calderón-Zygmund theorem [29] that $P \in L^3\left(\mathbf{R}^3\right)$. So the Hölder inequality shows that $\left((p-p_1) + \frac{1}{2}\rho|u|^2\right) u \in L^2\left(\mathbf{R}^3\right)$. Now, the Helmholtz-Weyl decomposition Theorem 4 yields that

$$\left((p-p_1) + \frac{1}{2}\rho|u|^2\right) u = \nabla G + F, \tag{47}$$

where G is a scalar function, F is a vector function such that $\nabla G \in L^2\left(\mathbf{R}^3\right)$, $F \in L^2\left(\mathbf{R}^3\right)$ and $\operatorname{div} F = 0$. Similar to (25), we claim that

$$\nabla G \in L^{\frac{3}{2}}\left(\mathbf{R}^3\right). \tag{48}$$

Again, Lemma 5 yields that

$$\|\nabla G\|_{L^{\frac{3}{2}}(\mathbf{R}^3)} \leq C \sup_{\substack{\varphi \in C_0^\infty\left(\mathbf{R}^3\right) \\ \|\nabla \varphi\|_{L^3} \leq 1}} \int_{\mathbf{R}^3} \nabla G \cdot \nabla \varphi dx.$$

It follows from (46) and integration by part that

$$\begin{aligned} \int_{\mathbf{R}^3} \nabla G \cdot \nabla \varphi dx &= \int_{\mathbf{R}^3} |\nabla u|^2 \varphi(x)\, dx + \int_{\mathbf{R}^3} \nabla \varphi \cdot \nabla u \cdot u dx \\ &= I_1 + I_2. \end{aligned} \tag{49}$$

I_1 and I_2 can be estimated in a similar way as in Section 2. It suffices to check whether the quantity $|\nabla u|^2$ belongs to the local Hardy space $h^1\left(\mathbf{R}^3\right)$. Similarly we choose $U = V = \nabla u^i \in L^2\left(\mathbf{R}^3\right)$, $i = 1, 2, 3$. Then $\operatorname{curl} U = 0$ and

$$\operatorname{div} V = \Delta u^i = \rho u \cdot \nabla u^i + \partial_i p.$$

It is easy to see that $u \cdot \nabla u^i \in L^{\frac{3}{2}}\left(\mathbf{R}^3\right)$, and $\nabla P \in L^{\frac{3}{2}}\left(\mathbf{R}^3\right)$ by using the Calderón-Zygmund inequality. Hence $\operatorname{div} V \in L^{\frac{3}{2}}\left(\mathbf{R}^3\right)$. On the other hand, Theorem 9 implies that $u \cdot \nabla u^i \in L^\infty\left(\mathbf{R}^3\right)$ and $\nabla p \in BMO\left(\mathbf{R}^3\right)$. Then the interpolation theorem shows that $\operatorname{div} V \in L^2\left(\mathbf{R}^3\right)$. By the Theorem 8 again, one get that $|\nabla u|^2 \in h^1\left(\mathbf{R}^3\right)$ and

$$\left\| |\nabla u|^2 \right\|_{h^1(\mathbf{R}^3)} \leq C \|\nabla u\|^2_{L^2(\mathbf{R}^3)}. \tag{50}$$

Then the desired estimate (48) follows from Theorem 8, Proposition 6 and the Hardy-BMO duality in the same way as for (25).

Then by multiplying a cut-off function $\psi_R(x)$ on both sides of (39) and taking $R \to \infty$, we conclude

$$\int_{\mathbf{R}^3} |\nabla u|^2 dx = 0,$$

which implies $u \equiv 0$. ■

We would like to mention a different version of above result by adding a remark.

Remark 10 *We can also study a similar problem for (3) by replacing the condition (4) with*

$$\int_{\mathbf{R}^3} |\nabla u|^2 dx < +\infty, \ \lim_{|x| \to \infty} u(x) = u_0 \quad \text{and} \quad \rho(x) \in L^\infty\left(\mathbf{R}^3\right),$$

here $u_0 \neq 0$ is a constant vector. In this case the result will change slightly. In a similar way, we can prove that $u \equiv u_0$ and $\rho(x)$ is any L^∞ function depends only on two variables which are perpendicular to u_0. The conclusion about $\rho(x)$ can be deduced by equation $\operatorname{div}(\rho u) = 0$ *and the assumption $\rho \in L^\infty$.*

4 Appendix

4.1 Hardy space and local Hardy space

Define a class $\mathcal{T}$ of normalized test functions on $\mathbf{R}^n$ by

$$\mathcal{T} = \{\phi \in C_0^\infty(\mathbf{R}^n) : \operatorname{supp}\phi \subseteq B_1(0) \text{ and } \|\nabla\phi\| \leq 1\}.$$

Then define the grand maximal function $f^*(x)$ of a distribution f by

$$f^*(x) = \sup_{t>0} \sup_{\phi \in \mathcal{T}} |\phi_t * f(x)|,$$

here $\phi_t(y) = \frac{1}{t^n}\phi\left(\frac{y}{t}\right)$.

Definition 11 *A function $f \in L^1(\mathbf{R}^n)$ lies in the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ if $f^*(x) \in L^1(\mathbf{R}^n)$, and the Hardy space norm is defined by*

$$\|f\|_{\mathcal{H}^1(\mathbf{R}^n)} = \|f^*\|_{L^1(\mathbf{R}^n)} + \|f\|_{L^1(\mathbf{R}^n)}.$$

There is another definition that is equivalent and simpler but some time not easy to verify. Let ψ be a given C_0^∞ function and satisfy $\int_{\mathbf{R}^n} \psi = 1$, and also $\psi_t(y) = \frac{1}{t^n}\psi\left(\frac{y}{t}\right)$.

Definition 12 *A function $f \in L^1(\mathbf{R}^n)$ lies in the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ if $\sup_{t>0} |\psi_t * f| \in L^1(\mathbf{R}^n)$, and the Hardy space norm is defined by*

$$\|f\|_{\mathcal{H}^1(\mathbf{R}^n)} = \left\|\sup_{t>0} |\psi_t * f|\right\|_{L^1(\mathbf{R}^n)} + \|f\|_{L^1(\mathbf{R}^n)}.$$

The most important result about Hardy space is its dual space $\left(\mathcal{H}^1(\mathbf{R}^n)\right)^* = BMO(\mathbf{R}^n)$, where $BMO(\mathbf{R}^n) = \left\{g \in L^1_{loc}(\mathbf{R}^n) : \sup_{B \subset \mathbf{R}^n} \frac{1}{|B|}\int_B |g - [g]_B| < \infty\right\}$. For convenience we would like to list also two useful facts about the Hardy space:

(1), If $f \geq 0$ and $f \in \mathcal{H}^1(\mathbf{R}^n)$, then $f \equiv 0$.

(2), If $f \in L^p(\mathbf{R}^n)$, $p > 1$ (or $f \in L\log L(\mathbf{R}^n)$ Zygmund class), $\operatorname{supp} f$ is compact, and $\int_{\mathbf{R}^n} f = 0$, then $f \in \mathcal{H}^1(\mathbf{R}^n)$.

Related to $\mathcal{H}^1(\mathbf{R}^n)$ is the so called local Hardy space $h^1(\mathbf{R}^n)$. Let ψ be a given C_0^∞ function as above.

Definition 13 *A function $f \in L^1(\mathbf{R}^n)$ lies in the local Hardy space $h^1(\mathbf{R}^n)$ if $\sup_{0<t<1} |\psi_t * f| \in L^1(\mathbf{R}^n)$, and the h^1 norm is defined by*

$$\|f\|_{h^1(\mathbf{R}^n)} = \left\|\sup_{0<t<1} |\psi_t * f|\right\|_{L^1(\mathbf{R}^n)} + \|f\|_{L^1(\mathbf{R}^n)}.$$

Or equivalently $h^1(\mathbf{R}^n)$ can be defined by using the normalized class $\mathcal{T}$. Define the local maximal function $f^*_{loc}(x)$ of a distribution f by

$$f^*_{loc}(x) = \sup_{0<t<1} \sup_{\phi \in \mathcal{T}} |\phi_t * f(x)|,$$

then we can also define $f \in h^1(\mathbf{R}^n)$ if and only if $f^*_{loc}(x) \in L^1(\mathbf{R}^n)$. The equivalence of these two definitions can be found in [30].

The space $\mathcal{H}^1(\mathbf{R}^n)$ is not stable by multiplications of smooth functions. However for the local Hardy space $h^1(\mathbf{R}^n)$, as long as the multiplier function is sufficiently regular, then there is a stability. For instance if $f \in h^1(\mathbf{R}^n)$ and $g \in C^{0,\alpha}(\mathbf{R}^n)$, then $g \cdot f \in h^1(\mathbf{R}^n)$ and

$$\|g \cdot f\|_{h^1} \leq C(\alpha) \|g\|_{C^{0,\alpha}} \|f\|_{h^1}.$$

More detailed properties of the local Hardy space can be found in [16].

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