

REAL REPRESENTATION THEORY OF FINITE CATEGORICAL GROUPS

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ABSTRACT. We introduce the Real representation theory of finite categorical groups, thereby categorifying the Real representation theory of finite groups, as studied by Atiyah–Segal and Karoubi. We generalize the categorical character theory of Ganter–Kapranov and Bartlett to the Real setting. In particular, given a Real representation of a group G on a linear category, we associate a number (the secondary trace) to each graded commuting pair $(g, \omega) \in G \times \hat{G}$, where \hat{G} is the background Real structure on G . This collection of numbers defines the Real 2-character of the Real representation. We interpret results in Real categorical character theory in terms of geometric structures, namely gerbes, vector bundles and functions on iterated unoriented loop groupoids. This perspective naturally leads to connections with the representation category of thickened, or unoriented, versions of the twisted Drinfeld double of a finite group. We use our results to conjecture a generalized character theoretic description of the Real twisted Morava E -theory groups of classifying spaces of finite groups and the transfer maps between them.

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INTRODUCTION

Let G be a finite group. The complex representation theory of G is a classical and well-understood subject. In this paper we are interested in two variations of this theory. The first variation, also classical and well-understood, is the real representation theory of G . More generally, following Atiyah–Segal [1] and Karoubi [23], after fixing a short exact sequence of finite groups

$$1 \rightarrow G \rightarrow \hat{G} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 1$$

we can consider Real representations of G , by which we mean a complex vector space together with an action of \hat{G} in which elements of G act complex linearly while elements of $\hat{G} \setminus G$ act complex anti-linearly. The second, more recently studied variation is the 2-representation theory of G , in which elements of G act by autoequivalences of an object of a bicategory. Standard target bicategories include those of categories or of Kapranov–Voevodsky 2-vector spaces [21]. More generally, the group G itself can be categorified, for example by twisting by a cohomology class $\alpha \in H^3(BG, \mathbb{C}^\times)$, leading to the representation theory of finite categorical (or weak 2-) groups. The representation theory of categorical groups has been studied by many authors; most relevant to the present paper are the works of Elgueta [10], Ganter–Kapranov [15] and Bartlett [3]. In this paper we consider both of the above variations simultaneously, thereby introducing the Real representation theory of finite categorical groups. In order to systematically deal with the cohomological twisting data used to define categorical groups, we make use of the twisted loop transgression map introduced in [34]. Twisted loop transgression realizes a sort of dimensional reduction from Real 2-representation theory to the ordinary representation theory of unoriented loop groupoids. This perspective allows for natural geometric interpretations of our results.

Apart from its intrinsic importance, the representation theory of categorical (or higher) groups has also been studied because of its many interesting connections

to other seemingly-unrelated areas of mathematics and physics. For example, the work of Ganter–Kapranov [15] was largely motivated by equivariant homotopy theory while that of Bartlett [3] was motivated by oriented extended topological field theory. In both of these examples, the connections are further strengthened by considering the categorical character theory of 2-representations. For other related appearances of higher categorical traces in geometry and representation theory see, for example, the works of Ben-Zvi–Nadler [4] and Hoyois–Schrotozke–Sibilla [19]. Analogously, we expect Real 2-representations, and the resulting Real categorical character theory which we develop in this paper, to be related to certain \mathbb{Z}_2 -equivariant refinements of the connections appearing in 2-representation theory. In the above two examples, we have in mind Real equivariant homotopy theory and unoriented extended topological field theory.

In the remainder of this introduction we will explain our results and their expected applications in more detail. For simplicity, we restrict attention to Real 2-representations of finite groups on categories, leaving the general case to the body of the paper. A Real 2-representation of \mathbf{G} on a category \mathcal{C} is the data of autoequivalences

$$\rho(g) : \mathcal{C} \rightarrow \mathcal{C}, \quad g \in \mathbf{G}$$

and anti-autoequivalences

$$\rho(\omega) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}, \quad \omega \in \hat{\mathbf{G}} \backslash \mathbf{G}$$

together with coherence natural isomorphisms encoding their associativity. Here \mathcal{C}^{op} is the category opposite to \mathcal{C} . Retaining only the information attached to \mathbf{G} recovers the notion of a 2-representation of \mathbf{G} on \mathcal{C} . The above definition categorifies the algebraic, or Grothendieck–Witt, approach to the Real representation theory of \mathbf{G} . There is a variation of the above definition, in which the category \mathcal{C} is assumed to be \mathbb{C} -linear, elements of \mathbf{G} act by \mathbb{C} -linear autoequivalences and elements of $\hat{\mathbf{G}} \backslash \mathbf{G}$ act by \mathbb{C} -anti-linear autoequivalences, which categorifies the standard approach to the Real representation theory of \mathbf{G} . The drawback of this variation is that it is defined only in linear settings. In any case, it is a matter of preference which categorification one uses. All results of the paper hold in either of the above two approaches.

Associated to an ordinary 2-representation ρ of \mathbf{G} on \mathcal{C} is a collection of sets of natural transformations

$$\mathbb{T}\text{r}_\rho(g) = 2\text{Hom}_{\text{Cat}}(1_{\mathcal{C}}, \rho(g)), \quad g \in \mathbf{G}$$

called the categorical character of ρ . Ganter–Kapranov [15] and Bartlett [3] categorified the conjugation invariance of the character of a representation by constructing a system of compatible bijections

$$\beta_{g,h} : \mathbb{T}\text{r}_\rho(g) \xrightarrow{\sim} \mathbb{T}\text{r}_\rho(hgh^{-1}), \quad g, h \in \mathbf{G}.$$

Suppose now that ρ is a Real 2-representation of \mathbf{G} . In this setting, we prove that the conjugation invariance of the categorical character is enhanced to Real conjugation invariance, by which we mean a compatible system of bijections

$$\beta_{g,\omega} : \mathbb{T}\text{r}_\rho(g) \xrightarrow{\sim} \mathbb{T}\text{r}_\rho(\omega g^{\pi(\omega)} \omega^{-1}), \quad g \in \mathbf{G}, \omega \in \hat{\mathbf{G}}.$$

Note that this data, which we call the Real categorical character, contains strictly more information than the categorical character of the underlying 2-representation. This should be contrasted with the fact that the character of a Real representation

is a character subject to additional reality constraints. A pair $(g, \omega) \in \mathbf{G} \times \hat{\mathbf{G}}$ is called graded commuting if the equality

$$\omega g^{\pi(\omega)} = g\omega$$

holds. When the Real 2-representation is linear, Real conjugation invariance allows us to associate to each graded commuting pair (g, ω) a number

$$\chi_\rho(g, \omega) = \mathrm{tr}_{\mathrm{Tr}_\rho(g)}(\beta_{g, \omega}).$$

This collection of traces defines the Real 2-character of ρ . More geometrically, the Real categorical character can be interpreted as a flat vector bundle over the unoriented loop groupoid of the classifying stack $B\mathbf{G}$; see Theorem 5.4. Here we regard $B\mathbf{G}$ as the double cover $B\mathbf{G} \rightarrow B\hat{\mathbf{G}}$. The unoriented loop groupoid of $B\mathbf{G}$ is then the quotient of the loop groupoid of $B\mathbf{G}$ by the simultaneous action of deck transformations of $B\mathbf{G}$ and loop reflection. From this point of view, the Real 2-character of ρ is the holonomy of the Real categorical character; see Theorem 5.6. It is worth emphasizing that the Real categorical character is an ordinary, as opposed to Real, vector bundle; the Real information is entirely contained in the base of the vector bundle. This allows us to apply techniques from ordinary representation theory to study Real 2-representations.

As mentioned above, one of the motivations of Ganter and Kapranov [15] to develop their 2-character theory was to relate 2-representation theory to higher chromatic phenomena in equivariant homotopy theory. To explain this, denote by \mathbf{BG} a classifying space of \mathbf{G} . Hopkins, Kuhn and Ravenel [17] showed that the Borel equivariant Morava E -theory groups $E_n^\bullet(\mathbf{BG})$, $n \geq 1$, at a prime p admit a generalized character theoretic description. In this context, generalized characters are conjugation invariant functions on the set of commuting n -tuples in \mathbf{G} ; the precise values of such functions and the p^{th} order condition on the commuting elements will be ignored in this introduction. When $n = 1$ this recovers a p -completed version of the character theoretic description of $K^\bullet(\mathbf{BG})$ given by Atiyah and Segal [1]. When $n = 2$ this gives a generalized character theoretic description of the Borel equivariant elliptic cohomology group $E_2^\bullet(\mathbf{BG})$. Ganter and Kapranov showed that the 2-character theory of \mathbf{G} also leads to such generalized characters, although without the p^{th} order condition and with values in the ground field. To strengthen the analogy between 2-representation theory and equivariant E -theory, Ganter and Kapranov showed that 2-induction of 2-representations is given at the level of 2-characters by the same formula as transfer for Hopkins–Kuhn–Ravenel characters. This analogy persists in the twisted case, in which one considers representations of finite categorical groups and twisted Borel equivariant E_2^\bullet -theory. Twisted elliptic characters were used by Devoto [7] to define the twisted equivariant elliptic cohomology of \mathbf{BG} . The work of Ganter–Usher [16] and Willerton [33] shows that 2-characters of representations of finite categorical groups are twisted elliptic characters, while Corollary 7.2 describes 2-induction at the level of 2-characters. The analogy between $E_3^\bullet(\mathbf{BG})$ and 3-representations of \mathbf{G} was established by Wang [32].

In view of the work of Ganter and Kapranov, it is natural to expect that Real 2-representation theory can be used to shed light on Real versions of Morava E -theory at the prime $p = 2$. More precisely, we are interested in a \mathbb{Z}_2 -equivariant refinement ER_n^\bullet of E -theory applied to the double cover $\mathbf{BG} \rightarrow B\hat{\mathbf{G}}$. The real-oriented generalized cohomology theory ER_n^\bullet was constructed by Hu and Kriz [20]. As far as the

author is aware, a generalized character theoretic description of $E\mathbb{R}_n^\bullet(\mathbf{BG})$ is not known. At height 1, the group $E\mathbb{R}_1^0(\mathbf{BG})$ is Real K -theory $KR^0(\mathbf{BG})$, localized at the prime $p = 2$, which, again by the work of Atiyah–Segal [1], is known to admit a character theoretic description. Our Real 2-character theory suggests a generalized character theoretic description of the Real Borel equivariant elliptic cohomology group $E\mathbb{R}_2^0(\mathbf{BG})$. Explicitly, we are led to consider functions χ on the set of graded commuting pairs in $\mathbf{G} \times \hat{\mathbf{G}}$ which satisfy the diagonal (Real) conjugation invariance condition

$$\chi(g, \omega) = \chi(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}), \quad \sigma \in \hat{\mathbf{G}}.$$

Such functions, and their twisted generalizations, first appeared in [34], where the character theory of thickened, or unoriented, versions of twisted Drinfeld doubles of finite groups was studied. From the point of view of the present paper, this connection is explained by the fact that Real categorical characters are modules over thickened twisted Drinfeld doubles; see Corollary 5.5.

We also consider induction of Real 2-representations, of which there are two forms. The first categorifies the Realification (or hyperbolic) map from equivariant K - to KR -theory. The second is a categorification of induction internal to equivariant KR -theory. In both cases, we compute the result of 2-induction at the level of categorical and 2-characters; see Theorems 7.3, 7.6 and 7.7, 7.9. For example, given a subgroup $\hat{\mathbf{H}} \leq \hat{\mathbf{G}}$ compatible with the structure maps to \mathbb{Z}_2 and a Real 2-representation ρ of \mathbf{H} , the Real 2-character of the induced Real 2-representation $\mathrm{RInd}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}(\rho)$ is

$$\chi_{\mathrm{RInd}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}(\rho)}(g, \omega) = \frac{1}{2|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega)\sigma^{-1} \in \hat{\mathbf{H}}^2}} \chi_\rho(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}).$$

In view of our proposed character theoretic description of equivariant $E\mathbb{R}_n^0$ -theory, these results can be viewed as concrete predictions for the precise form of transfer for $E\mathbb{R}_2^0$. Our results on 2-induction are formulated in such a way so as to allow for immediate conjectural generalizations to both representations of higher groups and to higher heights in homotopy theory.

Finally, we mention some appearances of Real 2-representation theory in mathematical physics. Consider the categorical group \mathcal{G} determined by a finite group \mathbf{G} and a cocycle $\alpha \in Z^3(B\mathbf{G}, \mathbb{C}^\times)$. A Real structure $\hat{\mathbf{G}}$ on \mathbf{G} and a lift of α to $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, \mathbb{C}_\pi^\times)$, the coefficients twisted by the double cover $B\mathbf{G} \rightarrow B\hat{\mathbf{G}}$, determine a Real structure on \mathcal{G} , as defined in Section 3.3. On the other hand, such twisted cocycles $\hat{\alpha}$ are known to arise in unoriented topological gauge theory. For example, the pair $(\hat{\mathbf{G}}, \hat{\alpha})$ defines an unoriented variant of three dimensional α -twisted Dijkgraaf–Witten theory. The bicategory of Real 2-representations of \mathcal{G} is closely related to the value of this theory on a point. Not unrelated, Real 2-characters of \mathcal{G} recover expressions, originally found by Sharpe [28], for discrete torsion phase factors appearing in M -theory with orientifolds. See [34] for a slightly different perspective. We also mention that the relevance of twisted $E\mathbb{R}_n^\bullet$ -theory to string and M -theory with orientifolds has been conjectured by H. Sati. Similarly, Real structures on categorical groups appear in the theory of symmetry protected topological phases with symmetry categorical groups involving time reversal symmetry

and in unoriented field theory with higher gauge field symmetries. See the papers of Kapustin–Thorngren [22] and Sharpe [29] for the appearance of categorical groups in the corresponding oriented settings. Finally, the prominence of twisted loop transgression in this paper is particularly natural from the point of view of field theory, where it becomes an instance of the ‘quantization via cohomological push-pull’ procedure. Examples of this procedure in oriented settings can be found in the works of Freed [11] and Freed–Hopkins–Teleman [12].

A brief overview of the paper is as follows. In Section 1 we collect preliminary background material. Section 2 contains relevant results from the twisted Real representation theory of finite groups in a form which is convenient for categorification. In Section 3, after defining Real structures on finite categorical groups, we introduce the notion of a Real representation of a finite categorical group. In Section 4 we develop the basics of the Real categorical character theory of Real 2-representations. Section 5 then considers the general case of (linear) Real representations of finite categorical groups. We interpret the corresponding character theory geometrically in terms of vector bundles over gerbes on the unoriented loop groupoid of BG . Section 6, which serves as preparation for the following section, contains basic, but perhaps not widely available, material about Real and hyperbolic induction of twisted Real representations of finite groups. In Section 7 we introduce two forms of 2-induction of Real 2-representations. We compute the effect of 2-induction at the level of Real categorical and 2-characters. We also describe our conjectural applications to Real equivariant homotopy theory.

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1. BACKGROUND MATERIAL

1.1. Bicategories. We establish our notation for bicategories. For a detailed introduction to bicategories the reader is referred to [5].

A bicategory \mathcal{V} consists of the following data:

- (i) A class $\text{Obj}(\mathcal{V})$ of objects.
- (ii) For each pair $x, y \in \text{Obj}(\mathcal{V})$, a small category $1\text{Hom}_{\mathcal{V}}(x, y)$, objects and morphisms of which are called 1-morphisms and 2-morphisms, respectively. Composition of 2-morphisms within the same 1-morphism category is denoted by $- \circ_1 -$.
- (iii) For each triple $x, y, z \in \text{Obj}(\mathcal{V})$, a composition bifunctor

$$- \circ_0 - : 1\text{Hom}_{\mathcal{V}}(y, z) \times 1\text{Hom}_{\mathcal{V}}(x, y) \rightarrow 1\text{Hom}_{\mathcal{V}}(x, z).$$
- (iv) For each $x \in \text{Obj}(\mathcal{V})$, an identity 1-morphism $1_x : x \rightarrow x$.
- (v) For each triple of composable 1-morphisms f, g, h , an associator 2-isomorphism

$$\alpha_{f,g,h} : (f \circ_0 g) \circ_0 h \Longrightarrow f \circ_0 (g \circ_0 h).$$

- (vi) For each 1-morphism $f : x \rightarrow y$, a pair of unitor 2-isomorphisms

$$\lambda_f : 1_y \circ_0 f \Longrightarrow f, \quad \rho_f : f \circ_0 1_x \Longrightarrow f.$$

This data is subject to a number of coherence conditions which we do not recall.

When it will not lead to confusion we will write $- \circ -$ in place of $- \circ_0 -$ or $- \circ_1 -$. The set of 2-morphisms $\text{Hom}_{1\text{Hom}_{\mathcal{V}}(x,y)}(f, g)$ will be denoted by $2\text{Hom}_{\mathcal{V}}(f, g)$. Given 1-morphisms $f_1, f_2 : x \rightarrow y$ and $g : y \rightarrow z$ and a 2-morphism $u : f_1 \Rightarrow f_2$, the left whiskering of u by g , namely $1_g \circ_0 u : g \circ_0 f_1 \Rightarrow g \circ_0 f_2$, will be written $g \circ_0 u$. We adopt analogous notation for right whiskering.

A (strict) 2-category is a bicategory in which all associator 2-isomorphisms $\alpha_{f,g,h}$ and all unitor 2-isomorphisms λ_f, ρ_f are identity maps. Coherence for bicategories asserts that any bicategory is biequivalent to a 2-category.

Example. Small categories, their functors and their natural transformations form a 2-category Cat . For each field k , there is a sub-2-category $\text{Cat}_k \subset \text{Cat}$ of k -linear categories, k -linear functors and k -linear natural transformations. \triangleleft

Example. Let k be a field. Kapranov and Voevodsky defined in [21] the bicategory 2Vect_k of finite dimensional 2-vector spaces over k . As the name suggests, this is a 2-categorical analogue of the category Vect_k of vector spaces over k . One definition of 2Vect_k is as the bicategory of k -linear additive finitely semisimple categories, their k -linear functors and their k -linear natural transformations. We will use the following biequivalent model. Objects of 2Vect_k are non-negative integers $[n]$, $n \in \mathbb{Z}_{\geq 0}$. A 1-morphism $[n] \rightarrow [m]$ is an $m \times n$ matrix $A = (A_{ij})$ whose entries are finite dimensional vector spaces over k . The composition of the 1-morphisms $A : [m] \rightarrow [n]$ and $B : [n] \rightarrow [p]$ is defined by

$$(B \circ_0 A)_{ik} = \bigoplus_{j=1}^n B_{ij} \otimes_k A_{jk}.$$

Note that the composition $- \circ_0 -$ is not strictly associative. A 2-morphism $u : A \Rightarrow B$ is a collection of k -linear maps $(u_{ij} : A_{ij} \Rightarrow B_{ij})$. \triangleleft

1.2. Duality involutions on bicategories. In this section we introduce the categorical setting of Real 2-representation theory.

As a warm up, we recall some categorical notions. A category with duality is a category \mathcal{C} together with a functor $(-)^* : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ and a natural isomorphism $\Theta : 1_{\mathcal{C}} \Rightarrow (-)^* \circ ((-)^*)^{\text{op}}$ such that

$$\Theta_x^* \circ \Theta_{x^*} = 1_{x^*} \tag{1}$$

for all $x \in \text{Obj}(\mathcal{C})$. A morphism $(\mathcal{C}, (-)^*, \Theta) \rightarrow (\mathcal{D}, (-)^*, \Xi)$ of categories with duality is called a form functor and consists of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\varphi : F \circ (-)^* \Rightarrow (-)^* \circ F$ such that

$$\varphi_x^* \circ \Xi_{F(x)} = \varphi_{x^*} \circ F(\Theta_x)$$

for all $x \in \text{Obj}(\mathcal{C})$. Finally, a symmetric form in $(\mathcal{C}, (-)^*, \Theta)$ is an object $x \in \text{Obj}(\mathcal{C})$ together with an isomorphism $\psi : x^* \rightarrow x$ such that

$$\psi^* \circ \Theta_x = \psi.$$

Symmetric forms and their isometries form a category $\mathcal{C}^{h\mathbb{Z}_2}$. Form functors (equivalences) induce functors (equivalences) of homotopy fixed point categories.

We now categorify these categorical notions, in each case replacing the constraint with a new piece of data, which is then required to satisfy a new constraint. Let

then \mathcal{V} be a bicategory. The 2-cell dual \mathcal{V}° of \mathcal{V} is the bicategory obtained from \mathcal{V} by reversing its 2-cells. Hence, if

$$\begin{array}{ccc} & f & \\ x & \begin{array}{c} \curvearrowright \\ \Downarrow v \\ \curvearrowleft \end{array} & y \\ & h & \end{array}$$

is a 2-morphism in \mathcal{V} , then

$$\begin{array}{ccc} & f & \\ x & \begin{array}{c} \curvearrowleft \\ \Uparrow v \\ \curvearrowright \end{array} & y \\ & g & \end{array}$$

is a 2-morphism in \mathcal{V}° .

Definition ([30, Definition 2.1]). *A bicategory with weak duality involution is a bicategory \mathcal{V} together with*

- (i) a pseudofunctor $(-)^{\circ} : \mathcal{V}^{\circ} \rightarrow \mathcal{V}$,
- (ii) a pseudonatural adjoint equivalence $\eta : 1_{\mathcal{V}} \Rightarrow (-)^{\circ} \circ_0 ((-)^{\circ})^{\circ}$, and
- (iii) an invertible modification $\zeta : \eta \circ_0 (-)^{\circ} \Rightarrow (-)^{\circ} \circ_0 \eta^{\circ}$

such that, for each $x \in \text{Obj}(\mathcal{V})$, the equality

$$\zeta_{x^{\circ}} \circ_0 \eta_x = (\zeta_x^{\circ} \circ_0 \eta_x) \circ_1 \eta(x) \quad (2)$$

of 2-morphisms holds. Here $\eta(x) : \eta_{x^{\circ}} \circ_0 \eta_x \Rightarrow \eta_x^{\circ} \circ_0 \eta_x$ is a pseudonaturality constraint for η .

If \mathcal{V} is a 2-category, $(-)^{\circ}$ is a strict 2-functor and η and ζ are the identities, then the above data is said to define a strict duality involution on \mathcal{V} .

Definition ([30, Definition 2.2]). *A duality pseudofunctor $(\mathcal{V}, (-)^{\circ}, \eta) \rightarrow (\mathcal{W}, (-)^{\circ}, \lambda)$ between bicategories with weak duality involutions is a pseudofunctor $F : \mathcal{V} \rightarrow \mathcal{W}$ together with*

- (i) a pseudonatural adjoint equivalence $\mathbf{i} : (-)^{\circ} \circ_0 F^{\circ} \Rightarrow F \circ_0 (-)^{\circ}$, and
 - (ii) an invertible modification $\theta : (\mathbf{i} \circ_0 ((-)^{\circ})^{\circ}) \circ_1 ((-)^{\circ} \circ_0 \mathbf{i}^{\circ}) \circ_1 (\lambda \circ_0 F) \Rightarrow F \circ_0 \eta$
- such that, for each $x \in \text{Obj}(\mathcal{V})$ a coherence constraint, which we omit, is satisfied.

In much the same way that bicategories can be strictified, so too can bicategories with weak duality involutions. Indeed, any bicategory with weak duality involution is biequivalent via a duality pseudofunctor to a 2-category with strict duality involution [30, Theorem 2.3].

Examples of bicategories with duality involution can be found in [30, §2]. We restrict attention to the two examples which are most relevant to this paper.

Example. The strict 2-functor $(-)^{\text{op}} : \text{Cat}^{\circ} \rightarrow \text{Cat}$ which sends categories, functors and natural transformations to their opposites is a strict duality involution. The restriction of $(-)^{\text{op}}$ to Cat_k is again a strict duality involution. \triangleleft

Example. The bicategory 2Vect_k has a weak duality involution $(-)^{\vee}$ which is a 2-categorical analogue of the k -linear duality functor on Vect_k . On objects let $[n]^{\vee} = [n]$. On 1- and 2-morphisms let $(-)^{\vee}$ be given by k -linear duality. Explicitly, we have $(A_{ij})^{\vee} = (A_{ij}^{\vee})$ and $(u_{ij})^{\vee} = (u_{ij}^{\vee})$. The adjoint equivalence η is induced by ev , the canonical evaluation isomorphism from a finite dimensional vector space to its double dual. The modification ζ is the identity. \triangleleft

Next, we define homotopy fixed point objects.

Definition. A symmetric form in $(\mathcal{V}, (-)^\circ, \eta, \zeta)$ is the data of an object $x \in \text{Obj}(\mathcal{V})$ together with

- (i) an equivalence $\psi : x^\circ \rightarrow x$, and
- (ii) a 2-isomorphism $\mu : 1_x \Rightarrow \psi \circ_0 \psi^\circ \circ_0 \eta_x$

such that the 2-morphism

The diagram illustrates the relationships between objects x , x° , $x^{\circ\circ}$, and $x^{\circ\circ\circ}$. It features several 1-morphisms: $1_{x^\circ} : x^\circ \rightarrow x^\circ$, $\psi : x^\circ \rightarrow x$, $\eta_x : x^\circ \rightarrow x^{\circ\circ}$, $\eta_x^\circ : x^\circ \rightarrow x^{\circ\circ\circ}$, $\eta_x^{\circ\circ} : x^{\circ\circ} \rightarrow x^{\circ\circ\circ}$, $\eta_x^{\circ\circ\circ} : x^{\circ\circ\circ} \rightarrow x^{\circ\circ\circ\circ}$, $\eta_\psi : x \rightarrow x^{\circ\circ}$, and $1_x : x \rightarrow x$. There are also 2-morphisms: $\mu^\circ : 1_{x^\circ} \Rightarrow \psi \circ_0 \psi^\circ \circ_0 \eta_x$, $\theta : \eta_x \Rightarrow \eta_x^{\circ\circ}$, $\zeta_x : \eta_x^\circ \Rightarrow \eta_x^{\circ\circ}$, and η_ψ is a 2-morphism between η_x and $\eta_x^{\circ\circ}$.

is equal to 1_ψ .

Similar to the case of categories with duality, symmetric forms are the objects of a homotopy fixed point bicategory $\mathcal{V}^{h\mathbb{Z}_2}$. As we will not have the occasion to use 1- and 2-morphisms of this category, we omit their explicit definitions.

Given an object x of a bicategory with weak duality involution, define

$$\epsilon x = \begin{cases} x & \text{if } \epsilon = 1, \\ x^\circ & \text{if } \epsilon = -1. \end{cases} \quad (3)$$

Similar notation will be used for the action of $(-)^{\circ}$ on 1- and 2-morphisms.

Closely related to bicategories with duality involutions are bicategories with contravariance [30, §4]. Roughly speaking, these are bicategories which have both covariant and contravariant 1-morphisms. More precisely, a bicategory with contravariance consists of the following data:

- (i) A class $\text{Obj}(\mathcal{V})$ of objects.
- (ii) For each pair $x, y \in \text{Obj}(\mathcal{V})$ and each sign $\epsilon \in \{\pm 1\}$, a small category $1\text{Hom}_{\mathcal{V}}^\epsilon(x, y)$.
- (iii) For each triple $x, y, z \in \text{Obj}(\mathcal{V})$ and each pair $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, a composition bifunctor

$$- \circ_0 - : 1\text{Hom}_{\mathcal{V}}^{\epsilon_2}(y, z) \times {}^{\epsilon_2}1\text{Hom}_{\mathcal{V}}^{\epsilon_1}(x, y) \rightarrow 1\text{Hom}_{\mathcal{V}}^{\epsilon_2\epsilon_1}(x, z).$$

Here we apply equation (3) to $(\text{Cat}, (-)^{\text{op}})$, so that the left superscript ϵ_2 indicates whether or not we consider opposite categories.

- (iv) For each $x \in \text{Obj}(\mathcal{V})$, an identity 1-morphism $1_x \in \text{Hom}_{\mathcal{V}}^1(x, x)$.
- (v) Associator and unitor 2-isomorphisms.

This data is required to satisfy coherence constraints similar to those of a bicategory.

By keeping only the data associated to the sign $1 \in \{\pm 1\}$, each bicategory with contravariance \mathcal{V} defines a bicategory \mathcal{V}_1 .

A pseudofunctor preserving contravariance between bicategories with contravariance is defined as in the case of a pseudofunctor, with the additional requirement

that the sign $\epsilon \in \{\pm 1\}$ of 1-morphisms be preserved. Similarly, one defines pseudo-natural transformations respecting contravariance.

There is an obvious strictification of the above definition which leads to the notion of a 2-category with contravariance. Any bicategory with contravariance is biequivalent via a pseudofunctor preserving contravariance to a 2-category with contravariance [30, Theorems 8.1, 8.2].

Example. (1) A bicategory with weak duality involution \mathcal{V} defines a bicategory with contravariance $\underline{\mathcal{V}}$ by

$$\text{Obj}(\underline{\mathcal{V}}) = \text{Obj}(\mathcal{V}), \quad 1\text{Hom}_{\underline{\mathcal{V}}}^\epsilon(x, y) = 1\text{Hom}_{\mathcal{V}}(\epsilon x, y).$$

Composition of 1- and 2-morphisms is induced by the corresponding compositions in \mathcal{V} . See [30, Theorem 7.2] for details.

- (2) Applying the previous construction to $(\mathbf{Cat}, (-)^\text{op})$ yields a 2-category with contravariance whose objects are small categories and whose morphisms are covariant ($\epsilon = 1$) and contravariant ($\epsilon = -1$) functors.

◁

1.3. String diagrams. For a detailed introduction to string diagrams the reader is referred to [2, §4].

String diagrams, which will be used to perform calculations in 2-categories, are Poincaré dual to globular diagrams for 2-categories. Two dimensional regions of a string diagram are therefore labelled by objects of the 2-category while strings and nodes label 1- and 2-morphisms, respectively. Our conventions are such that string diagrams are read from right to left and from bottom to top. Below is a globular diagram (left) together with its corresponding string diagram (right):



The various compositions of 1- and 2-morphisms are represented by appropriate concatenations of string diagrams. Although arrows drawn on strings are redundant, we will often include them if they help to clarify diagrams. We sometimes omit labels of two dimensional regions and we do not draw identity 1-morphisms. For example, a 2-morphism $u : 1_x \Rightarrow f$ is depicted by the string diagram



With some additional effort, string diagrams can also be used to perform calculations in bicategories. One way to do so is to first specify a bracketing of the source and target 1-morphisms. However, if the bicategory is skeletal, as will be the case in all relevant examples considered below, then the choice of bracketing can be omitted at the expense of keeping track of associators.

1.4. \mathbb{Z}_2 -graded groups. Denote by \mathbb{Z}_2 the multiplicative group $\{\pm 1\}$. A group homomorphism $\pi : \hat{\mathbf{G}} \rightarrow \mathbb{Z}_2$ is called a \mathbb{Z}_2 -graded group. Morphisms of \mathbb{Z}_2 -graded groups are group homomorphisms which respect the structure maps to \mathbb{Z}_2 . We will always assume that a given \mathbb{Z}_2 -graded group is non-trivially graded in the sense

that the structure map is surjective. A non-trivially graded \mathbb{Z}_2 -graded group $\hat{\mathbf{G}}$ is necessarily an extension

$$1 \rightarrow \mathbf{G} \rightarrow \hat{\mathbf{G}} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 1. \quad (4)$$

The subgroup $\mathbf{G} = \ker(\pi)$ is called the ungraded subgroup of $\hat{\mathbf{G}}$. Similarly, if the group \mathbf{G} is given, then an extension of the form (4) is called a Real structure on \mathbf{G} .

Example. An involutive group homomorphism $\varsigma : \mathbf{G} \rightarrow \mathbf{G}$ defines a split Real structure $\hat{\mathbf{G}} = \mathbf{G} \rtimes_{\varsigma} \mathbb{Z}_2$ on \mathbf{G} . Atiyah and Segal restrict attention to such Real structures in their study of equivariant KR -theory [1]. \triangleleft

Let $\hat{\mathbf{G}}$ be a \mathbb{Z}_2 -graded group. Denote by $\text{Aut}_{\text{Grp}}^{\text{gen}}(\mathbf{G})$ the \mathbb{Z}_2 -graded group of automorphisms and anti-automorphisms of \mathbf{G} . The map $\varphi : \hat{\mathbf{G}} \rightarrow \text{Aut}_{\text{Grp}}^{\text{gen}}(\mathbf{G})$ given by $\varphi(\omega)(g) = \omega g^{\pi(\omega)} \omega^{-1}$ is a morphism of \mathbb{Z}_2 -graded groups. The induced action of $\hat{\mathbf{G}}$ on \mathbf{G} is called Real conjugation.

Example. Let $(\mathcal{C}, (-)^*, \Theta)$ be a category with duality. Given $x \in \text{Obj}(\mathcal{C})$, let $\text{Aut}_{\mathcal{C}}^{\text{gen}}(x)$ be the set of all automorphisms $x \rightarrow x$ and anti-automorphisms $x^* \rightarrow x$. For $f \in \text{Aut}_{\mathcal{C}}^{\text{gen}}(x)$, define $\pi(f) \in \mathbb{Z}_2$ so that $f : \pi(f)x \rightarrow x$, where we use notation similar to equation (3). Define product and inverse maps on $\text{Aut}_{\mathcal{C}}^{\text{gen}}(x)$ by

$$f_2 \cdot f_1 = f_2 \circ \pi(f_2) f_1^{\pi(f_2)} \circ \Theta_x^{\delta_{\pi(f_2), \pi(f_1)}, -1}$$

and

$$I(f) = \begin{cases} f^{-1} & \text{if } \pi(f) = 1, \\ \Theta_x^{-1} \circ f^* & \text{if } \pi(f) = -1 \end{cases}$$

where we have introduced the notation

$$\delta_{\epsilon_2, \epsilon_1, -1} = \begin{cases} 1 & \text{if } \epsilon_1 = \epsilon_2 = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\text{Aut}_{\mathcal{C}}^{\text{gen}}(x)$ is a \mathbb{Z}_2 -graded group with ungraded group $\text{Aut}_{\mathcal{C}}(x)$. For example, associativity for the composition of three anti-automorphisms follows from equation (1). \triangleleft

1.5. Loop groupoids. Recall that a groupoid is a category in which all morphisms are isomorphisms. A groupoid is called finite if it has only finitely many objects and morphisms.

Suppose that a group \mathbf{G} acts on a set X . The action groupoid $X//\mathbf{G}$ is the category with objects X and morphisms $\text{Hom}_{X//\mathbf{G}}(x, y) = \{g \in \mathbf{G} \mid gx = y\}$. The groupoid $\text{pt}//\mathbf{G}$ is denoted by $B\mathbf{G}$.

Definition. The loop groupoid of a finite groupoid \mathfrak{G} is the functor category

$$\Lambda\mathfrak{G} = 1\text{Hom}_{\text{Cat}}(B\mathbb{Z}_2, \mathfrak{G}).$$

Concretely, an object (x, γ) of $\Lambda\mathfrak{G}$ is a loop $\gamma : x \rightarrow x$ in \mathfrak{G} while a morphism $(x_1, \gamma_1) \rightarrow (x_2, \gamma_2)$ is a morphism $g : x_1 \rightarrow x_2$ which satisfies $\gamma_2 = g\gamma_1 g^{-1}$.

A finite groupoid over $B\mathbb{Z}_2$ is a functor $\pi : \hat{\mathfrak{G}} \rightarrow B\mathbb{Z}_2$ of finite groupoids. The functor π classifies an equivalence class of double covers $\pi : \mathfrak{G} \rightarrow \hat{\mathfrak{G}}$. Fix a choice of such a double cover. In the setting of finite groupoids over $B\mathbb{Z}_2$ there are a number of possible generalizations of the loop groupoid. The one relevant to the present paper is the following.

Definition ([34]). Let $\hat{\mathfrak{G}}$ be a finite groupoid over $B\mathbb{Z}_2$. The unoriented loop groupoid of \mathfrak{G} is the groupoid $\Lambda_\pi^{\text{ref}} \hat{\mathfrak{G}}$ with degree one loops in $\hat{\mathfrak{G}}$ as objects and morphisms $(x_1, \gamma_1) \rightarrow (x_2, \gamma_2)$ the morphisms $\omega : x_1 \rightarrow x_2$ which satisfy $\gamma_2 = \omega \gamma_1^{\pi(\omega)} \omega^{-1}$.

Here **ref** stands for ‘reflection’, since $\Lambda_\pi^{\text{ref}} \hat{\mathfrak{G}}$ is equivalent to the quotient of $\Lambda \hat{\mathfrak{G}}$ by the \mathbb{Z}_2 -action given by deck transformations of $\hat{\mathfrak{G}}$ and reflection of the circle.

Example. (i) Let G be a finite group. The loop groupoid ΛBG is equivalent to the conjugation action groupoid $G // G$.

(ii) Let \hat{G} be a finite \mathbb{Z}_2 -graded group. The canonical functor $B\hat{G} \rightarrow B\mathbb{Z}_2$ classifies the double cover $BG \rightarrow B\hat{G}$. The resulting unoriented loop groupoid $\Lambda_\pi^{\text{ref}} B\hat{G}$ is equivalent to the Real conjugation action groupoid $G //_\varphi \hat{G}$. \triangleleft

1.6. Twisted loop transgression. Loop transgression is a technique for producing an $(n-1)$ -cocycle on a free loop space from an n -cocycle on the original space. Loop transgression for finite groupoids was studied by Willerton [33]. We recall a version of loop transgression for finite groupoids over $B\mathbb{Z}_2$ [34].

Let $\hat{\mathfrak{G}}$ be a finite groupoid over $B\mathbb{Z}_2$. The associated double cover $\pi : \mathfrak{G} \rightarrow \hat{\mathfrak{G}}$ can be used to twist local systems on (the simplicial complex associated to) $\hat{\mathfrak{G}}$. Let A be an abelian group, which we view as a \mathbb{Z}_2 -module by the inversion action. Let $C^\bullet(\hat{\mathfrak{G}}, A_\pi)$ be the complex of π -twisted simplicial cochains on $\hat{\mathfrak{G}}$. Denote by $[\omega_n | \cdots | \omega_1]$ the n -simplex of $\hat{\mathfrak{G}}$ determined by the diagram

$$x_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} x_{n+1}$$

in $\hat{\mathfrak{G}}$. In this notation, the differential $\hat{\beta} \in C^n(\hat{\mathfrak{G}}, A_\pi)$ is defined by

$$d\hat{\beta}([\omega_{n+1} | \cdots | \omega_1]) = \hat{\beta}([\omega_n | \cdots | \omega_1])^{\pi(\omega_{n+1})} \hat{\beta}([\omega_{n+1} | \cdots | \omega_2])^{(-1)^{n+1}} \times \prod_{i=1}^n \hat{\beta}([\omega_{n+1} | \cdots | \omega_{i+2} | \omega_{i+1} \omega_i | \omega_{i-1} | \cdots | \omega_1])^{(-1)^i}.$$

We use the notation $Z^\bullet \subset C^\bullet$ for the subgroup of cocycles. Without loss of generality, we will assume that all cochains are normalized in the sense that they are the identity when applied to chains in which one of the morphisms ω_i is an identity map. We also denote by $[\omega_n | \cdots | \omega_1] \gamma$ the n -simplex of $\Lambda_\pi^{\text{ref}} \hat{\mathfrak{G}}$ determined by the diagram

$$\gamma \xrightarrow{\omega_1} \omega_1 \gamma^{\pi(\omega_1)} \omega_1^{-1} \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_n} (\omega_n \cdots \omega_1) \gamma^{\pi(\omega_n \cdots \omega_1)} (\omega_n \cdots \omega_1)^{-1}.$$

Let k be a field. Reflection twisted loop transgression is a cochain map

$$\tau_\pi^{\text{ref}} : C^\bullet(\hat{\mathfrak{G}}, k_\pi^\times) \rightarrow C^{\bullet-1}(\Lambda_\pi^{\text{ref}} \hat{\mathfrak{G}}, k^\times).$$

As is standard for transgression maps, τ_π^{ref} is defined by a push-pull procedure. The main novelty of τ_π^{ref} is that it is defined using pushforward along an unoriented map, leading to the change in coefficient systems. We do not require a full description of τ_π^{ref} . Instead, we record that for a 2-cochain $\hat{\theta} \in C^2(\hat{\mathfrak{G}}, k_\pi^\times)$ we have

$$\tau_\pi^{\text{ref}}(\hat{\theta})([\omega] \gamma) = \hat{\theta}([\gamma^{-1} | \gamma])^{\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}([\omega \gamma^{\pi(\omega)} \omega^{-1} | \omega])}{\hat{\theta}([\omega | \gamma^{\pi(\omega)})}$$

while for a 3-cochain $\hat{\alpha} \in C^3(\hat{\mathfrak{G}}, k_\pi^\times)$ we have

$$\begin{aligned} \tau_\pi^{\text{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma) &= \hat{\alpha}([\gamma|\gamma^{-1}|\gamma])^{\delta_{\pi(\omega_2), \pi(\omega_1), -1}} \times \\ &\left(\frac{\hat{\alpha}([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1}|\omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1}|\omega_1])\hat{\alpha}([\omega_1|\gamma^{-\pi(\omega_1)}|\gamma^{\pi(\omega_1)})]}{\hat{\alpha}([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1}|\omega_1|\gamma^{\pi(\omega_1)})]} \right)^{\frac{\pi(\omega_2)-1}{2}} \times \\ &\frac{\hat{\alpha}([\omega_2|\omega_1|\gamma^{\pi(\omega_2\omega_1)}])\hat{\alpha}([\omega_2\omega_1\gamma^{\pi(\omega_2\omega_1)}(\omega_2\omega_1)^{-1}|\omega_2|\omega_1])}{\hat{\alpha}([\omega_2|\omega_1\gamma^{\pi(\omega_2\omega_1)}\omega_1^{-1}|\omega_1])}. \end{aligned}$$

If $\hat{\theta}$ above is in fact a 2-cocycle, then $\tau_\pi^{\text{ref}}(\hat{\theta})$ is a 1-cocycle, meaning the equality

$$\tau_\pi^{\text{ref}}(\hat{\theta})([\omega_2]\omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1})\tau_\pi^{\text{ref}}(\hat{\theta})([\omega_1]\gamma) = \tau_\pi^{\text{ref}}(\hat{\theta})([\omega_2\omega_1]\gamma) \quad (5)$$

holds for each 2-chain $[\omega_2|\omega_1]\gamma$. This follows from the general results of [34], but can also be verified directly. This and the corresponding statement for 3-cocycles are the only facts about τ_π^{ref} that will be assumed in this paper. In particular, the explicit expressions for $\tau_\pi^{\text{ref}}(\hat{\theta})$ and $\tau_\pi^{\text{ref}}(\hat{\alpha})$ will be derived from the point of view of Real (2-)representation theory.

When $\hat{\mathfrak{G}} = B\hat{\mathbf{G}}$ and $\mathfrak{G} = B\mathbf{G}$, the twisted transgression map τ_π^{ref} and Willerton's transgression map $\tau : C^\bullet(B\mathbf{G}, k^\times) \rightarrow C^{\bullet-1}(\Lambda B\mathbf{G}, k^\times)$ are compatible in the sense that τ is the restriction of τ_π^{ref} to untwisted cochains on $B\mathbf{G}$.

2. TWISTED REAL REPRESENTATION THEORY OF FINITE GROUPS

As motivation for the remainder of the paper, we recall some material about twisted (or projective) Real representations of finite groups. In the case of untwisted real representations, this material is standard [14]. Aspects of the untwisted Real case are treated in [1], [23]; a general reference is [34]. For twisted complex representations of finite group(oid)s, see [24], [33].

2.1. The anti-linear theory. Let k be a field of characteristic zero which contains all roots of unity. Suppose that k is a quadratic extension of a field k_0 , which we view as the fixed point set of a k_0 -linear Galois involution $k \rightarrow k$. A standard example is $k_0 = \mathbb{R} \subset k = \mathbb{C}$. A map $V \rightarrow W$ of vector spaces over k is called +1-linear (resp. -1-linear) if it is k -linear (resp. k -anti-linear with respect to the Galois involution).

Let \mathbf{G} be a finite group with Real structure $\hat{\mathbf{G}}$. Let $\hat{\theta} \in Z^2(B\hat{\mathbf{G}}, k_\pi^\times)$, where \mathbf{G} acts trivially on k^\times and $\hat{\mathbf{G}} \setminus \mathbf{G}$ acts by the Galois involution. Write $\theta \in Z^2(B\mathbf{G}, k^\times)$ for the restriction of $\hat{\theta}$ to $B\mathbf{G}$.

Definition. A $\hat{\theta}$ -twisted Real representation of \mathbf{G} is a finite dimensional vector space V over k together with $\pi(\omega)$ -linear maps $\rho(\omega) : V \rightarrow V$, $\omega \in \hat{\mathbf{G}}$, which satisfy $\rho(e) = 1_V$ and

$$\rho(\omega_2) \circ \rho(\omega_1) = \hat{\theta}([\omega_2|\omega_1])\rho(\omega_2\omega_1).$$

Twisted Real representations of \mathbf{G} and their $\hat{\mathbf{G}}$ -equivariant k -linear maps form a k_0 -linear category $\text{RRep}_{k_0}^{\hat{\theta}}(\mathbf{G})$. Denote by $KR^{0+\hat{\theta}}(B\mathbf{G})$ the Grothendieck group of $\text{RRep}_{k_0}^{\hat{\theta}}(\mathbf{G})$. Here we regard $B\mathbf{G}$ as a groupoid with involution determined by $\hat{\mathbf{G}}$. The notation KR is often reserved for split Real structures, as in [1], but we will use it in the non-split case as well.

The Real character of a $\hat{\theta}$ -twisted Real representation ρ is the function

$$\chi_\rho : \mathbf{G} \rightarrow k, \quad g \mapsto \mathrm{tr}_V(\rho(g)).$$

In other words, χ_ρ is the character of the underlying θ -twisted representation of \mathbf{G} . The new feature of Real characters is their Real conjugation equivariance,

$$\chi_\rho(\omega g^{\pi(\omega)} \omega^{-1}) = \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]g) \cdot \chi_\rho(g), \quad \omega \in \hat{\mathbf{G}} \quad (6)$$

which refines the conjugation equivariance of characters of θ -twisted representations. The Real character map extends to a k -linear map

$$\chi : KR^{0+\hat{\theta}}(B\mathbf{G}) \otimes_{\mathbb{Z}} k \rightarrow \Gamma_{\Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}}}(\tau_\pi^{\mathrm{ref}}(\hat{\theta})_k). \quad (7)$$

Following [33, §2.2], the right hand side is the space of flat sections of the transgressed line bundle $\tau_\pi^{\mathrm{ref}}(\hat{\theta})_k \rightarrow \Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}}$. Explicitly, this is the space of functions $\mathbf{G} \rightarrow k$ which satisfy the analogue of the condition (6). When $k = \mathbb{C}$ with complex conjugation as the involution, the map (7) is an isomorphism. See, for example, [34, Theorem 3.7].

Example. The real setting is $k = \mathbb{C}$ with $\pi : \hat{\mathbf{G}} = \mathbf{G} \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ the canonical projection and $\hat{\theta} = 1$. Then $\mathrm{RRep}_{\mathbb{C}}(\mathbf{G})$ is equivalent to $\mathrm{Rep}_{\mathbb{R}}(\mathbf{G})$ and $KR^0(B\mathbf{G}) \simeq RO(\mathbf{G})$. Equation (6) becomes the statement that characters of real representations are real valued class functions while the isomorphism (7) identifies $RO(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{C}$ with the space of functions on \mathbf{G} which are constant on conjugacy classes and their inverses. \triangleleft

2.2. The linear theory. We describe a linear analogue of the twisted Real representation theory of a finite group. The untwisted real case is treated in [35]. The approach of this section is the basis for our categorification in later sections.

We keep the notation from Section 2.1, although k is now an arbitrary field and $\hat{\mathbf{G}} \setminus \mathbf{G}$ acts on the coefficient system k_π^\times by inversion.

Lemma 2.1. *Let ρ be a θ -twisted representation of \mathbf{G} on V . For each $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$, the pair (ρ^ς, V^\vee) , where V^\vee is the k -linear dual of V and*

$$\rho^\varsigma(g) = \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1} \rho(\varsigma^{-1}g^{-1}\varsigma)^\vee, \quad g \in \mathbf{G}$$

is a θ -twisted representation of \mathbf{G} .

Proof. This can be verified by a direct calculation. The key point is the following identity, which is valid for all $g_1, g_2 \in \mathbf{G}$ and $\omega \in \hat{\mathbf{G}}$:

$$\frac{\hat{\theta}([\omega g_2 \omega^{-1} | \omega g_1 \omega^{-1}])}{\hat{\theta}([g_2 | g_1])^{\pi(\omega)}} = \frac{\hat{\theta}([\omega | \omega^{-1}]) \hat{\theta}([\omega g_2 g_1 | \omega^{-1}]) \hat{\theta}([\omega | g_2 g_1])}{\hat{\theta}([\omega | g_2]) \hat{\theta}([\omega g_2 | \omega^{-1}]) \hat{\theta}([\omega | g_1]) \hat{\theta}([\omega g_1 | \omega^{-1}])}. \quad (8)$$

□

Each element $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$ determines an exact duality structure $(P^\varsigma, \Theta^\varsigma)$ on $\mathrm{Rep}_k^\theta(\mathbf{G})$. The functor

$$P^\varsigma : \mathrm{Rep}_k^\theta(\mathbf{G})^{\mathrm{op}} \rightarrow \mathrm{Rep}_k^\theta(\mathbf{G})$$

sends a θ -twisted representation ρ to ρ^ς , as defined in Lemma 2.1. The natural isomorphism $\Theta^\varsigma : 1_{\mathrm{Rep}_k^\theta(\mathbf{G})} \Rightarrow P^\varsigma \circ (P^\varsigma)^{\mathrm{op}}$ has components

$$\Theta_V^\varsigma = \hat{\theta}([\varsigma^{-1} | \varsigma^{-1}]) \mathrm{ev}_V \circ \rho_V(\varsigma^{-2}).$$

Given a second element $\tilde{\varsigma} \in \hat{\mathbf{G}} \setminus \mathbf{G}$, the natural transformation $\nu^{\varsigma, \tilde{\varsigma}} : P^\varsigma \Rightarrow P^{\tilde{\varsigma}}$ with components $\nu_V^{\varsigma, \tilde{\varsigma}} = \rho_V(\varsigma^{-1}\tilde{\varsigma})^\vee$ lifts to a non-singular form functor

$$(\mathrm{Rep}_k^\theta(\mathbf{G}), P^\varsigma, \Theta^\varsigma) \xrightarrow{\sim} (\mathrm{Rep}_k^\theta(\mathbf{G}), P^{\tilde{\varsigma}}, \Theta^{\tilde{\varsigma}}).$$

In this way, a Real structure $\hat{\mathbf{G}}$ on \mathbf{G} and a lift $\hat{\theta}$ of θ determine a \mathbf{G} -torsor of exact duality structures on $\mathrm{Rep}_k^\theta(\mathbf{G})$.

We give two linear versions of the notion of a Real representation of \mathbf{G} . The first is slightly less natural, requiring the choice of an element $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$, but has the benefit that it fits into the setting of Grothendieck–Witt theory.

Definition. A $\hat{\theta}$ -twisted symmetric representation of \mathbf{G} is a symmetric form in $(\mathrm{Rep}_k^\theta(\mathbf{G}), P^\varsigma, \Theta^\varsigma)$.

Twisted symmetric representations form a (homotopy fixed point) category $\mathrm{Rep}_k^{\hat{\theta}, \varsigma}(\mathbf{G})$, a morphism $\phi : (N, \psi_N) \rightarrow (M, \psi_M)$ being a morphism of twisted representations which satisfies $P^\varsigma(\phi) \circ \psi_M \circ \phi = \psi_N$.

Example. When $\hat{\mathbf{G}} = \mathbf{G} \times \mathbb{Z}_2$ with ς the generator of \mathbb{Z}_2 , an untwisted symmetric representation is a representation together with a \mathbf{G} -invariant nondegenerate symmetric bilinear form. If instead $\hat{\theta}([\omega_2|\omega_1]) = (-1)^{\delta_{\pi(\omega_2), \pi(\omega_1)} - 1}$, then the bilinear form is required to be skew-symmetric \triangleleft

Definition. A $\hat{\theta}$ -twisted generalized symmetric representation of \mathbf{G} is a finite dimensional vector space N together with linear maps $\rho(\omega) : \pi(\omega)N \rightarrow N$, $\omega \in \hat{\mathbf{G}}$, which satisfy $\rho(e) = 1_N$ and

$$\rho(\omega_2) \circ \pi(\omega_2)\rho(\omega_1)\pi(\omega_2) = \hat{\theta}([\omega_2|\omega_1])\rho(\omega_2\omega_1)$$

where, for $\omega_1, \omega_2 \in \hat{\mathbf{G}} \setminus \mathbf{G}$, we identify N with $N^{\vee\vee}$ via ev_N .

Twisted generalized symmetric representations form a category $\mathrm{SRep}_k^{\hat{\theta}}(\mathbf{G})$, morphisms $\phi : N \rightarrow M$ being morphisms of twisted representations which satisfy $\phi^\vee \circ \rho_M(\omega) \circ \phi = \rho_N(\omega)$ for each $\omega \in \hat{\mathbf{G}} \setminus \mathbf{G}$.

Remark. More generally, a Real representation of \mathbf{G} on an object x of a category with duality $(\mathcal{C}, (-)^*, \Theta)$ is a \mathbb{Z}_2 -graded group morphism $\rho : \hat{\mathbf{G}} \rightarrow \mathrm{Aut}_{\mathcal{C}}^{\mathrm{gen}}(x)$. A $\hat{\theta}$ -twisted generalized symmetric representation of \mathbf{G} is then a Real representation of the \mathbb{Z}_2 -graded group ${}^{\hat{\theta}}\hat{\mathbf{G}}$, equal to $k^\times \times \hat{\mathbf{G}}$ as a set and with product

$$(z_2, \omega_2) \cdot (z_1, \omega_1) = (\hat{\theta}([\omega_2|\omega_1])z_2z_1^{\pi(\omega_2)}, \omega_2\omega_1),$$

on an object of $(\mathbf{Vect}_k, (-)^\vee, \mathrm{ev})$ in which the subgroup $k^\times \leq {}^{\hat{\theta}}\hat{\mathbf{G}}$ act via scalar multiples of the identity.

Proposition 2.2. The categories $\mathrm{Rep}_k^{\hat{\theta}, \varsigma}(\mathbf{G})$ and $\mathrm{SRep}_k^{\hat{\theta}}(\mathbf{G})$ are equivalent.

Proof. An equivalence $F^\varsigma : \mathrm{Rep}_k^{\hat{\theta}, \varsigma}(\mathbf{G}) \rightarrow \mathrm{SRep}_k^{\hat{\theta}}(\mathbf{G})$ is defined on objects by assigning to a twisted symmetric representation (N, ψ_N) the twisted generalized symmetric representation which is equal to N as a twisted representation and has

$$\rho(\omega) = \hat{\theta}([\omega|\varsigma^{-1}])\rho(\omega\varsigma^{-1}) \circ \psi_N, \quad \omega \in \hat{\mathbf{G}} \setminus \mathbf{G}.$$

On morphisms F^ς acts as the identity. \square

Let $\mathbf{GW}_0^{\hat{\theta}}(\mathbf{G})$ be the Grothendieck–Witt group of $(\mathbf{Rep}_k^{\hat{\theta}}(\mathbf{G}), P^{\varsigma}, \Theta^{\varsigma})$. Since non-singular form functors induce isomorphisms of Grothendieck–Witt groups, up to isomorphism, $\mathbf{GW}_0^{\hat{\theta}}(\mathbf{G})$ does not depend on the choice of $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$.

Characters of twisted (generalized) symmetric representations of \mathbf{G} are defined in the same way as Section 2.1. Real conjugation equivariance (6) continues to hold. When $k = \mathbb{C}$, the isomorphism (7) is replaced by the isomorphism

$$\chi : \mathbf{GW}_0^{\hat{\theta}}(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \Gamma_{\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}}(\tau_{\pi}^{\text{ref}}(\hat{\theta})_{\mathbb{C}}).$$

In fact, by picking a \mathbf{G} -invariant Hermitian metric on each twisted symmetric representation, we obtain an isomorphism of abelian groups

$$\mathbf{GW}_0^{\hat{\theta}}(\mathbf{G}) \rightarrow KR^{0+\hat{\theta}}(B\mathbf{G}).$$

So while the \mathbb{C} -linear and \mathbb{C} -anti-linear Real representation categories are not equivalent, the relevant Grothendieck(–Witt) groups are isomorphic.

3. REAL REPRESENTATIONS OF FINITE CATEGORICAL GROUPS

3.1. Categorical groups. The concept of a group can be categorified in a number of ways. A detailed discussion of these categorifications, and the relations between them, can be found in [2].

A categorical group, called a weak 2-group in [2], is a weak monoidal groupoid $(\mathcal{G}, \otimes, \mathbf{1})$ in which every object admits a weak inverse. Explicitly, this means that for each object x of \mathcal{G} there exists an object y such that both $x \otimes y$ and $y \otimes x$ are equivalent to the monoidal unit $\mathbf{1}$. A morphism of categorical groups is a weak monoidal functor. By considering also monoidal natural transformations between monoidal functors, categorical groups assemble to a 2-category.

The monoidal structure \otimes gives the set of connected components $\pi_0(\mathcal{G})$ the structure of a group. The group $\text{Aut}_{\mathcal{G}}(\mathbf{1})$ of autoequivalences of $\mathbf{1}$ is denoted by $\pi_1(\mathcal{G})$. By an Eckmann–Hilton argument, $\pi_1(\mathcal{G})$ is abelian. As described in Section 3.2 below, the groups $\pi_0(\mathcal{G})$, $\pi_1(\mathcal{G})$, together with some additional data, determine the categorical group \mathcal{G} up to equivalence.

Example. Any group \mathbf{G} , considered as a discrete category with objects \mathbf{G} and monoidal structure determined (on objects) by its group law, defines a categorical group. By a slight abuse of notation, we denote this categorical group by \mathbf{G} . \triangleleft

Example. Let \mathbf{A} be an abelian group. The action groupoid $B\mathbf{A}$ is a categorical group, the monoidal structure determined (on morphisms) by the group law of \mathbf{A} . \triangleleft

Example. Let x be an object of a bicategory \mathcal{V} . Then $1\text{Aut}_{\mathcal{V}}(x)$, the groupoid of autoequivalences of x and the 2-isomorphisms between them, is a categorical group, called the weak automorphism 2-group of x [2, §8.1].

If \mathcal{V} is k -linear and we restrict attention to k -linear autoequivalences of x and their 2-isomorphisms, then we obtain the categorical group $\mathbf{GL}_k(x)$ of [13, §3.3.2]. \triangleleft

The categorical groups of interest in this paper satisfy the following finiteness condition.

Definition. A categorical group \mathcal{G} is called finite if $\pi_0(\mathcal{G})$ is finite.

3.2. Sinh's theorem. The following classification result indicates that categorical groups can be viewed as twisted extended versions of groups.

Theorem 3.1 ([31]; see also [2, §8.3]). *Categorical groups are classified up to equivalence by the following data:*

- (i) A group \mathbf{G} .
- (ii) An abelian group \mathbf{A} .
- (iii) A group homomorphism $\pi : \mathbf{G} \rightarrow \text{Aut}_{\text{Grp}}(\mathbf{A})$.
- (iv) A cohomology class $[\alpha] \in H^3(B\mathbf{G}, \mathbf{A}_\pi)$.

In the same way, equivalence classes of finite categorical groups are classified by the data (i)-(iv), with the additional condition that \mathbf{G} be finite.

Explicitly, the categorical group $\mathcal{G}(\mathbf{G}, \mathbf{A}, \pi, [\alpha])$ determined by Theorem 3.1 can be taken to be the skeletal groupoid with objects \mathbf{G} , a morphism $g \xrightarrow{a} g$ for each pair $(g, a) \in \mathbf{G} \times \mathbf{A}$ and composition law

$$(g \xrightarrow{a_2} g) \circ (g \xrightarrow{a_1} g) = (g \xrightarrow{a_1 \cdot a_2} g).$$

The monoidal bifunctor \otimes is determined on objects by the group law of \mathbf{G} and on morphisms by

$$(g \xrightarrow{a} g) \otimes (g' \xrightarrow{a'} g') = (gg' \xrightarrow{a \cdot \pi(g)(a')} gg').$$

The associator is given by the maps $g_3 g_2 g_1 \xrightarrow{\alpha([g_3|g_2|g_1])} g_3 g_2 g_1$, where $\alpha \in Z^3(B\mathbf{G}, \mathbf{A}_\pi)$ is a normalized representative of $[\alpha]$. Since α is normalized, the unitors can be taken to be identity maps.

Example. If \mathbf{A} is trivial, then $\mathcal{G}(\mathbf{G}, \mathbf{A}, \pi, \alpha)$ is the group \mathbf{G} , viewed as a categorical group. If \mathbf{A} is non-trivial but α is trivial, then $\mathcal{G}(\mathbf{G}, \mathbf{A}, \pi, \alpha)$ is the categorical group extension of \mathbf{G} by $B\mathbf{A}$ determined by π . \triangleleft

Example. Let k be a field. Let \mathbf{G} be a group and let $\pi : \mathbf{G} \rightarrow \text{Aut}_{\text{Grp}}(k^\times)$ be the trivial map. The associated categorical group, denoted simply by $\mathcal{G}(\mathbf{G}, \alpha)$, is an α -twisted categorical group extension of \mathbf{G} by Bk^\times . \triangleleft

3.3. \mathbb{Z}_2 -graded categorical groups. Before introducing Real representations of categorical groups, we categorify the notion of a Real structure on a group.

Definition. A homomorphism of categorical groups $\pi : \hat{\mathcal{G}} \rightarrow \mathbb{Z}_2$ is called a \mathbb{Z}_2 -graded categorical group.

We will always assume that a given \mathbb{Z}_2 -graded categorical group $\hat{\mathcal{G}}$ is non-trivially graded in the sense that π is non-trivial. The ungraded categorical group of $\hat{\mathcal{G}}$ is the full subcategory $\mathcal{G} \subset \hat{\mathcal{G}}$ on objects which map via π to $1 \in \mathbb{Z}_2$. We have morphisms of categorical groups

$$1 \rightarrow \mathcal{G} \xrightarrow{i} \hat{\mathcal{G}} \xrightarrow{\pi} \mathbb{Z}_2, \tag{9}$$

where i is an isomorphism onto its image and π is surjective on objects and full. Similarly, given a categorical group \mathcal{G} , a diagram of the form (9) is called a Real structure on \mathcal{G} .

Example. Let k be a field and let $\pi : \hat{\mathbf{G}} \rightarrow \mathbb{Z}_2$ be a \mathbb{Z}_2 -graded group. Denote also by $\pi : \hat{\mathbf{G}} \rightarrow \text{Aut}_{\text{Grp}}(k^\times)$ the map $\pi(\omega)(a) = a^{\pi(\omega)}$. Let $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_\pi^\times)$. Then the categorical group $\mathcal{G}(\hat{\mathbf{G}}, k^\times, \pi, \hat{\alpha})$ defined by Theorem 3.1, henceforth denoted by $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$,

is \mathbb{Z}_2 -graded with ungraded categorical group $\mathcal{G}(\mathbf{G}, \alpha)$, where $\alpha \in Z^3(B\mathbf{G}, k^\times)$ is the restriction of $\hat{\alpha}$ to $B\mathbf{G}$. \triangleleft

The next example extends weak automorphism 2-groups to the setting of bicategories with duality involutions.

Example. Let x be an object of a bicategory \mathcal{V} with weak duality involution. Then $1\text{Aut}_{\mathcal{V}}^{\text{gen}}(x)$, the collection of all autoequivalences $x \rightarrow x$ and anti-autoequivalences $x^\circ \rightarrow x$, together with the 2-isomorphisms between them, is a categorical group. The bifunctor \otimes is defined on objects by

$$f_2 \otimes f_1 = f_2 \circ_0 (\pi(f_2) f_1 \circ_0 \eta_x^{\delta_{\pi(f_2), \pi(f_1), -1}}).$$

Here $\pi(f) \in \mathbb{Z}_2$ is such that $f : \pi(f)x \rightarrow x$. The definition of \otimes on morphisms is similar. The associator for three anti-autoequivalences is

$$\begin{aligned} (f_3 \otimes f_2) \otimes f_1 &= (f_3 \circ (f_2^\circ \circ \eta_x)) \circ f_1 \xrightarrow{\alpha} f_3 \circ (f_2^\circ \circ (\eta_x \circ f_1)) \xrightarrow{\eta} \\ &f_3 \circ (f_2^\circ \circ (f_1^\circ \circ \eta_x^\circ)) \xrightarrow{\zeta_x} f_3 \circ (f_2^\circ \circ (f_1^\circ \circ \eta_x^\circ)) = f_3 \otimes (f_2 \otimes f_1) \end{aligned}$$

where α is a composition of associators for \mathcal{V} and the arrow labelled by η is a pseudo-naturality constraint for η . The remaining associators are similar, but do not use the modification ζ . Verification of the pentagon identity uses the constraint (2). If x has at least one anti-autoequivalence, then the morphism $\pi : 1\text{Aut}_{\mathcal{V}}^{\text{gen}}(x) \rightarrow \mathbb{Z}_2$ fits into an exact sequence of categorical groups:

$$1 \rightarrow 1\text{Aut}_{\mathcal{V}}(x) \rightarrow 1\text{Aut}_{\mathcal{V}}^{\text{gen}}(x) \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 1.$$

If \mathcal{V} is k -linear and we restrict attention to k -linear (anti-)autoequivalences and 2-isomorphisms, then we obtain a \mathbb{Z}_2 -graded categorical group $\text{GL}_k^{\text{gen}}(x)$ whose ungraded categorical group is $\text{GL}_k(x)$. \triangleleft

Example. The previous example has a variation in which the bicategory with duality involution is replaced by a bicategory \mathcal{V} with contravariance. In this way, for each object x of \mathcal{V} we obtain a \mathbb{Z}_2 -graded categorical group $1\text{Aut}_{\mathcal{V}}^{\text{gen}}(x)$ whose ungraded categorical group is $1\text{Aut}_{\mathcal{V}_1}(x)$. \triangleleft

A \mathbb{Z}_2 -graded categorical group $\pi : \hat{\mathcal{G}} \rightarrow \mathbb{Z}_2$ defines a bicategory with contravariance as follows. There is a single object pt. For a sign $\epsilon \in \mathbb{Z}_2$, the endomorphism category $\hat{\mathcal{G}}^\epsilon$ is the full subcategory of $\hat{\mathcal{G}}$ on objects which map via π to ϵ . The compositions $\hat{\mathcal{G}}^{\epsilon_2} \otimes \hat{\mathcal{G}}^{\epsilon_1} \rightarrow \hat{\mathcal{G}}^{\epsilon_2 \epsilon_1}$ and associators are induced by the monoidal structure of $\hat{\mathcal{G}}$.

3.4. Real representations of finite categorical groups. We introduce Real representations of finite categorical groups, categorifying the linear approach of Section 2.2. A categorification of the anti-linear approach can be found in Section 5.4.

Let \mathcal{G} be a finite categorical group with Real structure $\hat{\mathcal{G}}$.

Definition. A Real representation of \mathcal{G} on a bicategory \mathcal{V} with contravariance is a contravariance preserving pseudofunctor $\rho : \hat{\mathcal{G}} \rightarrow \mathcal{V}$. Here $\hat{\mathcal{G}}$ is viewed as a bicategory with contravariance as in Section 3.3.

Real representations of \mathcal{G} on \mathcal{V} assemble to a bicategory $\mathbf{RRep}_{\mathcal{V}}(\mathcal{G})$ whose 1- and 2-morphisms are pseudonatural transformations and modifications, respectively, which respect contravariance. More abstractly, we can define

$$\mathbf{RRep}_{\mathcal{V}}(\mathcal{G}) = 1\mathrm{Hom}_{\mathbf{Bicat}_{\mathrm{con}}}(\hat{\mathcal{G}}, \mathcal{V}),$$

where $\mathbf{Bicat}_{\mathrm{con}}$ is the tricategory of bicategories with contravariance; see [30]. If \mathcal{V} is in fact a 2-category, then so too is $\mathbf{RRep}_{\mathcal{V}}(\mathcal{G})$.

Let $\mathbf{Bicat}_{\mathrm{con}}^{\leq 1}$ be the category of bicategories with contravariance and their pseudo-functors preserving contravariance. Taking 1-morphism bicategories defines a functor

$$1\mathrm{Hom}_{\mathbf{Bicat}_{\mathrm{con}}}(-, -) : (\mathbf{Bicat}_{\mathrm{con}}^{\leq 1})^{\mathrm{op}} \times \mathbf{Bicat}_{\mathrm{con}}^{\leq 1} \rightarrow \mathbf{Bicat}^{\leq 1}.$$

Using this functor, it can be verified that if \mathcal{V} and \mathcal{V}' are biequivalent bicategories with contravariance and $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}'$ are equivalent \mathbb{Z}_2 -graded categorical groups, then $\mathbf{RRep}_{\mathcal{V}}(\mathcal{G})$ and $\mathbf{RRep}_{\mathcal{V}'}(\mathcal{G})$ are biequivalent. Compare [10, §3.5]. In particular, using Shulman's strictification of bicategories with contravariance, we may without loss of generality restrict attention to Real representations of \mathcal{G} on 2-categories with contravariance.

We will use the following interpretation of Real representations.

Lemma 3.2. *A Real representation of \mathcal{G} on a bicategory with contravariance \mathcal{V} is the data of an object $V \in \mathrm{Obj}(\mathcal{V})$ together with a morphism of \mathbb{Z}_2 -graded categorical groups $\rho : \hat{\mathcal{G}} \rightarrow 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(V)$.*

Motivated by Lemma 3.2, we define a Real representation of \mathcal{G} on a bicategory \mathcal{V} with weak duality involution to be an object V of \mathcal{V} together with a morphism of \mathbb{Z}_2 -graded categorical groups $\rho : \hat{\mathcal{G}} \rightarrow 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(V)$.

We briefly describe another point of view on Real representations, categorifying the homotopy fixed point perspective of Section 2.2. Choose an element $\varsigma \in \mathrm{Obj}(\hat{\mathcal{G}})$ such that $\pi(\varsigma) = -1$ together with a weak inverse $\bar{\varsigma}$. Conjugation by ς defines a biequivalence $F^{\varsigma} : \mathcal{G}^{\mathrm{co}} \rightarrow \mathcal{G}$. More precisely, F^{ς} assigns to $x : \mathrm{pt} \rightarrow \mathrm{pt}$ and $f : x \Rightarrow y$ in $\mathcal{G}^{\mathrm{co}}$ the 1- and 2-morphisms $\varsigma \otimes (x \otimes \bar{\varsigma})$ and $\varsigma \otimes (f^{-1} \otimes \bar{\varsigma})$ in \mathcal{G} , respectively. The biequivalence F^{ς} can be used to define a weak duality involution on $\mathbf{Rep}_{\mathcal{V}}(\mathcal{G})$ (cf. [30, Example 2.6]). The involution takes a pseudofunctor $\rho : \mathcal{G} \rightarrow \mathcal{V}$ to the composition

$$\mathcal{G} \xrightarrow{(F^{\varsigma})^{\mathrm{co}}} \mathcal{G}^{\mathrm{co}} \xrightarrow{\rho^{\mathrm{co}}} \mathcal{V}^{\mathrm{co}} \xrightarrow{(-)^{\circ}} \mathcal{V}.$$

The ρ^{th} component of the required adjoint equivalence $\tilde{\eta}$ assigns to pt the 1-morphism $\eta_V \circ_0 \rho(\bar{\varsigma}^2) : V \rightarrow V^{\circ}$.

Proposition 3.3. (1) *Up to duality biequivalence, the weak duality involution on $\mathbf{Rep}_{\mathcal{V}}(\mathcal{G})$ is independent of the choice of $\varsigma \in \mathrm{Obj}(\hat{\mathcal{G}})$.*

(2) *For any ς as above, there is a biequivalence $\mathbf{Rep}_{\mathcal{V}}(\mathcal{G})^{h\mathbb{Z}_2} \simeq \mathbf{RRep}_{\mathcal{V}}(\mathcal{G})$.*

Proof. Let $\varsigma_1, \varsigma_2 \in \mathrm{Obj}(\hat{\mathcal{G}})$ be as above with associated biequivalences $F^{\varsigma_1}, F^{\varsigma_2} : \mathcal{G}^{\mathrm{co}} \rightarrow \mathcal{G}$. Define a pseudonatural isomorphism $\nu^{\varsigma_1, \varsigma_2} : F^{\varsigma_1} \Rightarrow F^{\varsigma_2}$ to be conjugation by $\varsigma_2 \otimes \bar{\varsigma}_1$. The desired duality biequivalence is then induced by whiskering.

The second statement is proved in the same way as Proposition 2.2; we describe a biequivalence at the level of objects. Given a symmetric form (ρ, ψ, μ) in $\mathbf{Rep}_{\mathcal{V}}(\mathcal{G})$, with ρ a representation on $V \in \mathrm{Obj}(\mathcal{V})$, we obtain an equivalence $\psi_{\mathrm{pt}} : V^{\circ} \rightarrow V$. For $\omega \in \mathrm{Obj}(\hat{\mathcal{G}})$ with $\pi(\omega) = -1$ define $\rho(\omega)$ to be the composition $\rho(\omega \otimes \bar{\varsigma}) \circ_0 \psi_{\mathrm{pt}}$. The monoidal coherence morphisms $\psi_{\bullet, \bullet}$ are then induced by μ and $\tilde{\eta}$. \square

Finally, we state a k -linear version of the above definitions. We restrict to categorical groups of the form $\mathcal{G}(\mathbf{G}, \alpha)$ with Real structure $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$.

Definition. A linear Real representation of $\mathcal{G}(\mathbf{G}, \alpha)$ on a k -linear bicategory \mathcal{V} with contravariance is a contravariance preserving pseudofunctor $\rho : \mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha}) \rightarrow \mathcal{V}$ under which $\text{Aut}_{\hat{\mathcal{G}}}(\mathbf{1}) \simeq k^\times$ acts by scalar multiplication.

Linear Real representations of $\mathcal{G}(\mathbf{G}, \alpha)$ form a bicategory $\text{RRep}_{\mathcal{V}, k}(\mathcal{G})$. The obvious analogue of Lemma 3.2, with $1\text{Aut}_{\mathcal{V}}^{\text{gen}}(V)$ replaced by $\text{GL}_k^{\text{gen}}(V)$, holds.

Remark. (i) Upon restriction to the ungraded categorical group, the above definitions recover the previously known representation theory of finite categorical groups, as studied in [10], [15], [3], amongst other places.
(ii) While the above definitions make sense for categorical groups which are not finite, to be interesting in the continuous case they should be supplemented with additional topological coherence conditions.

4. REAL 2-REPRESENTATION THEORY OF FINITE GROUPS

We study the definitions of Section 3.4 when the categorical group is a finite group. The case of categorical groups, which is technically more involved, will be taken up in Section 5. We also define Real categorical and 2-characters.

4.1. Basic definition. Lemma 3.2 gives rise to an explicit description of a Real representation of the categorical group determined by a finite group; we state it as a new definition.

Definition. A Real 2-representation of a finite group \mathbf{G} on a 2-category \mathcal{V} with strict duality involution consists of the following data:

- (i) An object V of \mathcal{V} .
- (ii) For each $\omega \in \hat{\mathbf{G}}$, an equivalence $\rho(\omega) : \pi^{(\omega)}V \rightarrow V$.
- (iii) For each pair $\omega_1, \omega_2 \in \hat{\mathbf{G}}$, a 2-isomorphism

$$\psi_{\omega_2, \omega_1} : \rho(\omega_2) \circ \pi^{(\omega_2)}\rho(\omega_1) \Longrightarrow \rho(\omega_2\omega_1).$$

- (iv) A 2-isomorphism $\psi_e : \rho(e) \Rightarrow 1_V$.

This data is required to satisfy the following conditions:

- (a) For each triple $\omega_1, \omega_2, \omega_3 \in \hat{\mathbf{G}}$, the equality

$$\psi_{\omega_3\omega_2, \omega_1} \circ_1 \left(\psi_{\omega_3, \omega_2} \circ \pi^{(\omega_3\omega_2)}\rho(\omega_1) \right) = \psi_{\omega_3, \omega_2\omega_1} \circ_1 \left(\rho(\omega_3) \circ \pi^{(\omega_3)}\psi_{\omega_2, \omega_1}^{\pi(\omega_3)} \right) \quad (10)$$

of 2-isomorphisms $\rho(\omega_3) \circ \pi^{(\omega_3)}\rho(\omega_2) \circ \pi^{(\omega_3\omega_2)}\rho(\omega_1) \Longrightarrow \rho(\omega_3\omega_2\omega_1)$ holds.

- (b) For each $\omega \in \hat{\mathbf{G}}$, the equalities

$$\psi_{e, \omega} = \psi_e \circ \rho(\omega), \quad \psi_{\omega, e} = \rho(\omega) \circ \pi^{(\omega)}\psi_e \quad (11)$$

of 2-isomorphisms $\rho(\omega) \Rightarrow \rho(\omega)$ hold.

Denote by $\psi_{\omega_3, \omega_2, \omega_1}$ the 2-isomorphism defined by either side of equation (10).

4.2. Real conjugation invariance of categorical traces. In this section we study categorical traces, as introduced by Ganter–Kapranov [15] and Bartlett [3], in the presence of duality involutions. We apply these ideas to Real 2-representation theory in Sections 4.3 and 5.2. For simplicity, we will restrict attention to 2-categories.

Let x be an object of a 2-category \mathcal{V} . As in [15, §3.1], [3, §4.1], the categorical trace of a 1-endomorphism $f : x \rightarrow x$ is the set of 2-morphisms from the identity 1_x to f :

$$\mathrm{Tr}(x) = 2\mathrm{Hom}_{\mathcal{V}}(1_x, f).$$

Given a 2-morphism $u : f_1 \Rightarrow f_2$, define $\mathrm{Tr}(u) : \mathrm{Tr}(f_1) \rightarrow \mathrm{Tr}(f_2)$ to be $u \circ_1 (-)$. With these definitions, the categorical trace defines a functor

$$\mathrm{Tr} : 1\mathrm{End}_{\mathcal{V}}(x) \rightarrow \mathrm{Set}.$$

If \mathcal{V} is enriched in a category \mathcal{A} , then Tr takes values in \mathcal{A} . For example, when \mathcal{V} is k -linear the functor Tr is \mathbf{Vect}_k -valued.

In [15, §4.3] and [3, §4.3] a kind of conjugation invariance of categorical traces is established. We generalize this result in what follows, showing that categorical traces in 2-categories with duality involutions (or contravariance) enjoy what we call Real conjugation invariance.

Suppose then that \mathcal{V} is a 2-category with strict duality involution. Fix a sign $\epsilon \in \mathbb{Z}_2$. Let $f : x \rightarrow x$ be an autoequivalence. When $\epsilon = -1$ we also fix a quasi-inverse $\tilde{f} : x \rightarrow x$ of f and a 2-isomorphism $\mu : \tilde{f} \circ f \Rightarrow 1_x$. Write

$$f^\nu = \begin{cases} f & \text{if } \nu = 1, \\ \tilde{f} & \text{if } \nu = -1. \end{cases}$$

Let $h : {}^\epsilon x \rightarrow y$ be an equivalence with quasi-inverse $k : y \rightarrow {}^\epsilon x$ and 2-isomorphisms $u : 1_y \Rightarrow h \circ k$ and $v : 1_{{}^\epsilon x} \Rightarrow k \circ h$. This data can be used to define a map

$$\Psi(h, k, u, v; \mu) : \mathrm{Tr}(f) \rightarrow \mathrm{Tr}(h \circ {}^\epsilon f^\epsilon \circ k),$$

henceforth denoted by $\Psi(h)$. The map μ is required only when $\epsilon = -1$. Suppose that we are given $\phi \in \mathrm{Tr}(f)$. Interpret u as a 2-morphism

$$u : 1_y \Longrightarrow h \circ 1_{{}^\epsilon x} \circ k.$$

When $\epsilon = 1$ the map $\Psi(h)$ is defined by post-composing u with ϕ :

$$\Psi(h)(\phi) = (h \circ_0 \phi \circ_0 k) \circ_1 u.$$

This is the definition of [15], [3]. If instead $\epsilon = -1$, then we can form the composition

$$1_{x^\circ} \xrightarrow{\mu^\circ} \tilde{f}^\circ \circ f^\circ \xrightarrow{\tilde{f}^\circ \circ_0 \phi^\circ} \tilde{f}^\circ.$$

The map $\Psi(h)$ is then defined by further pre-composing with u :

$$\Psi(h)(\phi) = \left(h \circ_0 \left((\tilde{f}^\circ \circ_0 \phi^\circ) \circ_1 \mu^\circ \right) \circ_0 k \right) \circ_1 u.$$

The following result generalizes [15, Proposition 4.10] and [3, Proposition 4.3]. A further generalization is given in Theorem 5.4 below.

Proposition 4.1. *For each pair of equivalences $f : x \rightarrow x$ and $h : {}^\epsilon x \rightarrow y$ with quasi-inverse data as above, the map*

$$\Psi(h) : \mathrm{Tr}(f) \rightarrow \mathrm{Tr}(h \circ {}^\epsilon f^\epsilon \circ k)$$

is a bijection. Moreover, $\Psi(1_x) = 1_{\mathrm{Tr}(f)}$ and, given equivalences ${}^{\epsilon_1}x \xrightarrow{h_1} y_1$ and ${}^{\epsilon_2}y_1 \xrightarrow{h_2} y_2$ with quasi-inverse data, the equality

$$\Psi(h_2) \circ \Psi(h_1) = \Psi(h_2 \circ {}^{\epsilon_2}h_1)$$

holds.

Proof. That $\Psi(h)$ is a bijection follows from the assumption that h is an equivalence. The equality $\Psi(1_x) = 1_{\mathrm{Tr}(f)}$ is clear from the construction.

To explain the precise meaning of the final statement, we need to describe the quasi-inverse data implicit in the definition of $\Psi(h_2 \circ {}^{\epsilon_2}h_1)$. Since h_1 and h_2 are equivalences, so too is $h_2 \circ {}^{\epsilon_2}h_1$. We take ${}^{\epsilon_2}k_1 \circ k_2$ as the quasi-inverse of $h_2 \circ {}^{\epsilon_2}h_1$ with 2-isomorphisms

$$u : 1_{y_2} \xrightarrow{u_2} h_2 \circ k_2 \xrightarrow{h_2 \circ {}^{\epsilon_2}u_1 \circ {}^{\epsilon_2}k_2} h_2 \circ {}^{\epsilon_2}h_1 \circ {}^{\epsilon_2}k_1 \circ k_2$$

and

$$v : 1_{{}^{\epsilon_2}\epsilon_1 x} \xrightarrow{{}^{\epsilon_2}v_1} {}^{\epsilon_2}k_1 \circ {}^{\epsilon_2}h_1 \xrightarrow{{}^{\epsilon_2}k_1 \circ v_2 \circ {}^{\epsilon_2}h_1} {}^{\epsilon_2}k_1 \circ k_2 \circ h_2 \circ {}^{\epsilon_2}h_1.$$

When $\epsilon_1\epsilon_2 = 1$ no additional data is required to define $\Psi(h_2 \circ {}^{\epsilon_2}h_1)$. If $\epsilon_1 = -1$ and $\epsilon_2 = 1$, then we take for $\mu : \tilde{f} \circ f \Rightarrow 1_x$ the data used to define $\Psi(h_1)$. If instead $\epsilon_1 = 1$ and $\epsilon_2 = -1$, then part of the data used to define $\Psi(h_2)$ is a quasi-inverse \tilde{f}' of $f' = h_1 \circ f \circ k_1$ and a 2-isomorphism $\mu' : \tilde{f}' \circ f' \Rightarrow 1_{y_1}$. Set $\tilde{f} = k_1 \circ \tilde{f}' \circ h_1$ with 2-isomorphism $\mu : \tilde{f} \circ f \Rightarrow 1_x$ given by the composition

$$\tilde{f} \circ f \xrightarrow{\tilde{f} \circ f \circ v_1^{-1}} \tilde{f} \circ f \circ k_1 \circ h_1 = k_1 \circ \tilde{f}' \circ h_1 \circ f \circ k_1 \circ h_1 \xrightarrow{k_1 \circ \mu' \circ h_1} k_1 \circ h_1 \xrightarrow{v_1} 1_x.$$

Then $\Psi(h_2 \circ {}^{\epsilon_2}h_1)$ is defined to be $\Psi(h_2 \circ {}^{\epsilon_2}h_1, {}^{\epsilon_2}k_1 \circ k_2, u, v; \mu)$. From this point on verification of the equality $\Psi(h_2) \circ \Psi(h_1) = \Psi(h_2 \circ {}^{\epsilon_2}h_1)$ is straightforward. \square

Remark. While the categorical trace of an arbitrary 1-morphism $f : x \rightarrow x$ is defined, Proposition 4.1 requires that f be an equivalence.

Continuing with the above notation, let us now further assume that $x = y$ and that the 1-morphisms f and h graded commute in the sense that we are given a 2-isomorphism

$$\eta : h \circ {}^{\epsilon}f^{\epsilon} \Longrightarrow f \circ h.$$

In this setting, we define a map of sets

$$(h, \eta)_* : \mathrm{Tr}(f) \rightarrow \mathrm{Tr}(f)$$

by the composition

$$\mathrm{Tr}(f) \xrightarrow{\Psi(h)} \mathrm{Tr}(h \circ {}^{\epsilon}f^{\epsilon} \circ k) \xrightarrow{\mathrm{Tr}(\eta \circ k)} \mathrm{Tr}(f \circ h \circ k) \xrightarrow{\mathrm{Tr}(f \circ u^{-1})} \mathrm{Tr}(f).$$

When $\epsilon = 1$ this reduces to a construction of Ganter–Kapranov.

Note that if \mathcal{V} is enriched in \mathcal{A} , then $(h, \eta)_*$ is a morphism in \mathcal{A} . In particular, when $\mathcal{A} = \mathrm{Vect}_k$ we can make the following definition, generalizing that of [15, §3.6].

Definition. Let \mathcal{V} be a k -linear 2-category with strict duality involution and let $f : x \rightarrow x$ and $h : {}^{\epsilon}x \rightarrow x$ be graded commuting equivalences. Assuming that the vector space $\mathrm{Tr}(f)$ is finite dimensional, the joint trace of (f, h) is

$$\mathrm{tr}(f, h) = \mathrm{tr}((h, \eta)_* : \mathrm{Tr}(f) \rightarrow \mathrm{Tr}(f)).$$

4.3. Real categorical characters. We apply the results of Section 4.2 to Real 2-representations of finite groups.

Let ρ be a Real 2-representation of a finite group \mathbf{G} on a 2-category \mathcal{V} with strict duality involution. For each $g \in \mathbf{G}$, write $\mathrm{Tr}_\rho(g)$ for the set $\mathrm{Tr}(\rho(g))$. Fix $g \in \mathbf{G}$ and $\omega \in \hat{\mathbf{G}}$. By applying Proposition 4.1 to the equivalences

$$f = \rho(g), \quad \tilde{f} = \rho(g^{-1}), \quad h = \rho(\omega), \quad k = \pi(\omega)\rho(\omega^{-1}),$$

with the 2-isomorphisms

$$u = \psi_{\omega, \omega^{-1}}^{-1} \circ_0 \psi_e^{-1}, \quad \mu = \psi_e \circ_0 \psi_{g^{-1}, g}$$

we obtain a map

$$\mathrm{Tr}(\rho(g)) \rightarrow \mathrm{Tr}(\rho(\omega) \circ \pi(\omega)\rho(g^{\pi(\omega)}) \circ \pi(\omega)\rho(\omega^{-1})).$$

Post-composition with $\mathrm{Tr}(\psi_{\omega, g^{\pi(\omega)}, \omega^{-1}})$ gives a map $\beta_{g, \omega} : \mathrm{Tr}_\rho(g) \rightarrow \mathrm{Tr}_\rho(\omega g^{\pi(\omega)} \omega^{-1})$.

Definition. *The Real categorical character of ρ is the assignment*

$$g \mapsto \mathrm{Tr}_\rho(g) \in \mathrm{Obj}(\mathrm{Set}), \quad g \in \mathbf{G}$$

together with the bijections

$$\beta_{g, \omega} : \mathrm{Tr}_\rho(g) \rightarrow \mathrm{Tr}_\rho(\omega g^{\pi(\omega)} \omega^{-1}), \quad (g, \omega) \in \mathbf{G} \times \hat{\mathbf{G}}.$$

Note that the collection $\{\beta_{g_1, g_2}\}_{(g_1, g_2) \in \mathbf{G}^2}$ constitutes the categorical character of the underlying 2-representation of \mathbf{G} , as defined in [15], [3].

Proposition 4.2. *The Real categorical character of a Real 2-representation ρ of \mathbf{G} on \mathcal{V} defines a functor*

$$\mathrm{Tr}(\rho) : \Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}} \rightarrow \mathrm{Set}.$$

Moreover, if \mathcal{V} is enriched in \mathcal{A} , then the functor $\mathrm{Tr}(\rho)$ takes values in \mathcal{A} .

Proof. Degree one loops of $B\hat{\mathbf{G}}$, that is, objects of $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}}$, are identified with elements $g \in \mathbf{G}$. Set $\mathrm{Tr}(\rho)(g) = \mathrm{Tr}_\rho(g)$. Given a morphism $\omega : g \rightarrow \omega g^{\pi(\omega)} \omega^{-1}$ in $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}}$, define the map $\mathrm{Tr}(\rho)(\omega)$ to be $\beta_{g, \omega}$. That these assignments define a functor follows from Proposition 4.1. \square

For example, when \mathcal{V} is k -linear, Proposition 4.2 states that the Real categorical character of a linear Real 2-representation of \mathbf{G} is a vector bundle over $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathbf{G}}$.

5. TWISTED REAL 2-REPRESENTATION THEORY OF FINITE GROUPS

In this section we study linear Real representations of finite categorical groups. When the categorical group is a trivial extension of a finite group by Bk^\times this recovers the k -linear version of the results of Section 4.

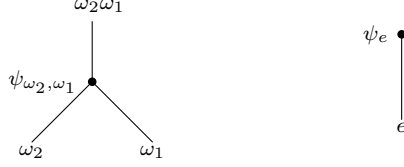
5.1. Basic definitions. Fix a field k . The following is a Real variant of definitions of Frenkel–Zhu [13, Definition 2.8] and Ganter–Usher [16, Definition 4.1].

Definition. *A twisted Real 2-representation of a finite group \mathbf{G} on a k -linear 2-category \mathcal{V} with strict duality involution consists of data $V \in \mathrm{Obj}(\mathcal{V})$, $\rho(\omega)$, $\psi_{\omega_2, \omega_1}$ and ψ_e as in Section 4.1, with the constraint (11) unchanged but with the constraint (10) replaced by the condition that*

$$\hat{\alpha}([\omega_3 | \omega_2 | \omega_1]) \cdot \psi_{\omega_3 \omega_2, \omega_1} \left(\psi_{\omega_3, \omega_2} \circ \pi(\omega_3 \omega_2) \rho(\omega_1) \right) = \psi_{\omega_3, \omega_2 \omega_1} \left(\rho(\omega_3) \circ \pi(\omega_3) \psi_{\omega_2, \omega_1}^{\pi(\omega_3)} \right) \quad (12)$$

for some map $\hat{\alpha} : \hat{\mathbf{G}} \times \hat{\mathbf{G}} \times \hat{\mathbf{G}} \rightarrow k^\times$.

We call $\hat{\alpha}$, which we regard as a 3-cochain on $B\hat{\mathbf{G}}$, the Real 2-Schur multiplier of ρ . In terms of string diagrams, the 2-isomorphisms $\psi_{\omega_2, \omega_1}$ and ψ_e become



and equation (12) is written as

(13)

The arrow in equation (13) indicates that the 2-morphism on the right is $\hat{\alpha}([\omega_3|\omega_2|\omega_1])$ times that on the left. Labels of 1-morphisms will often be omitted when they can be reconstructed from the labelled data in a string diagram. For example, the 2-morphism $\psi_e \circ \psi_{\omega, \omega}^{-1}$ will be drawn as

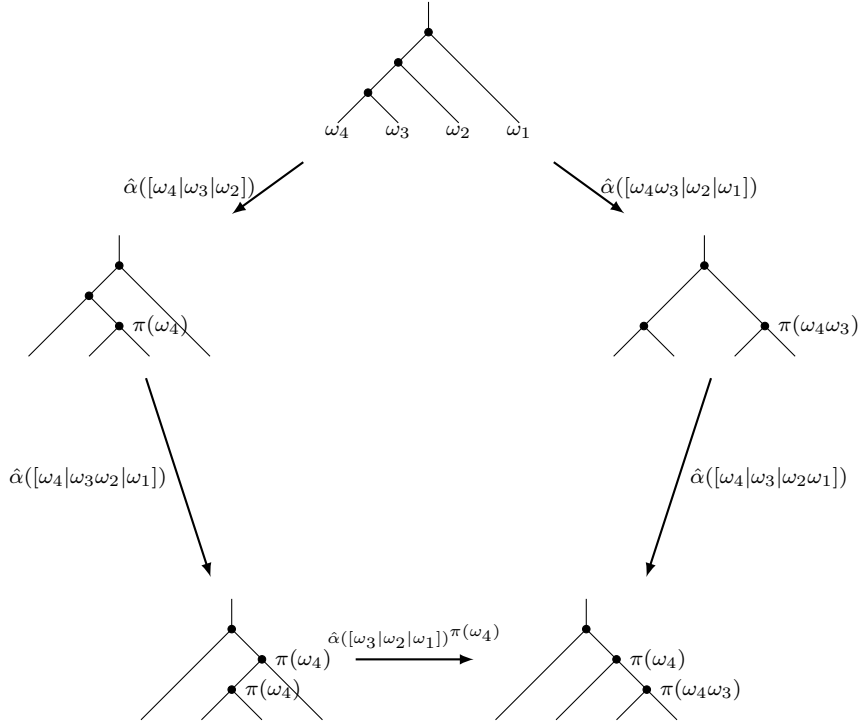
Lemma 5.1. *The Real 2-Schur multiplier is a twisted 3-cocycle $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_\pi^\times)$.*

Proof. We need to verify that the equality

$$\hat{\alpha}([\omega_4\omega_3|\omega_2|\omega_1])\hat{\alpha}([\omega_4|\omega_3|\omega_2\omega_1]) = \hat{\alpha}([\omega_3|\omega_2|\omega_1])^{\pi(\omega_4)}\hat{\alpha}([\omega_4|\omega_3\omega_2|\omega_1])\hat{\alpha}([\omega_4|\omega_3|\omega_2])$$

holds for all $\omega_1, \omega_2, \omega_3, \omega_4 \in \hat{\mathbf{G}}$. This can be proved using string diagrams, as in [16, Proposition 4.3], where the corresponding statement for twisted 2-representations was proved. Repeated application of equation (13) gives the following commutative

diagram of string diagrams:



A node labelled by -1 indicates that it is $\psi_{\bullet,\bullet}^{-\text{op}}$, instead of $\psi_{\bullet,\bullet}$, which is applied. For example, in the bottom right diagram the node labelled by $\pi(\omega_4)$ corresponds to the map $\pi(\omega_4)\psi_{\omega_3,\omega_2\omega_1}^{\pi(\omega_4)}$. The bottom arrow is multiplication by $\hat{\alpha}([\omega_3|\omega_2|\omega_1])^{\pi(\omega_4)}$, since it is the $\pi(\omega_4)^{\text{th}}$ power of equation (13) which is being applied. Commutativity of this diagram implies the desired cocycle condition. \square

By combining Theorem 3.1 and Lemma 5.1 we see that an $\hat{\alpha}$ -twisted Real 2-representation of \mathbf{G} determines a \mathbb{Z}_2 -graded finite categorical group $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$. The corresponding Real (2-)representation theories are closely related.

Proposition 5.2. *There is a canonical biequivalence between $\text{RRep}_{\mathcal{V},k}(\mathcal{G}(\mathbf{G}, \alpha))$, defined with respect to the Real structure $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$, and the bicategory of $\hat{\alpha}$ -twisted Real 2-representations of \mathbf{G} on \mathcal{V} .*

Proof. By construction, the cocycle $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_\pi^\times)$ determines the associator of $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$. After noting that equation (13) encodes the hexagon diagram for a monoidal functor $\mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha}) \rightarrow \text{GL}_k^{\text{gen}}(V)$ which is compatible with the structure functors to \mathbb{Z}_2 , the remainder of the proof is straightforward. \square

To end this section, we record some basic string diagram identities.

Lemma 5.3. *For all $g \in \mathbf{G}$ and $\omega_1, \omega_2 \in \hat{\mathbf{G}}$, the following identities hold:*

(i)

$$= \quad (14)$$

(ii)

$$= \quad (15)$$

(iii)

$$= \quad (16)$$

(iv)

$$= \quad (17)$$

(v)

$$= \quad (18)$$

Proof. The first two identities are obvious. For the remaining identities, see [16, Lemma 4.8], [16, Corollary 4.9] and [16, Corollary 4.10]. See also [3, §§3.2.1, 3.4] \square

5.2. Real 2-characters and 2-class functions. We extend the theory of Real categorical and 2-characters to twisted Real 2-representations. Instead of the direct approach Section 4.3, we use string diagrams. See [16, §4] for the ungraded twisted case.

We begin with some terminology.

Definition. Let \hat{G} be a finite \mathbb{Z}_2 -graded group. A pair $(g, \omega) \in \mathbf{G} \times \hat{G}$ is said to graded commute if $\omega g^{\pi(\omega)} = g\omega$.

The group \hat{G} acts on $\mathbf{G} \times \hat{G}$ by $\sigma \cdot (g, \omega) = (\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1})$. This action preserves the subset $\hat{G}^{(2)} \subset \mathbf{G} \times \hat{G}$ of graded commuting pairs.

Definition. The Real categorical character of a twisted Real 2-representation ρ of \mathbf{G} is the assignment

$$g \mapsto \mathrm{Tr}_\rho(g) \in \mathrm{Obj}(\mathrm{Vect}_k), \quad g \in \mathbf{G}$$

together with the collection of k -linear isomorphisms

$$\beta_{g,\omega} : \mathrm{Tr}_\rho(g) \rightarrow \mathrm{Tr}_\rho(\omega g \pi(\omega) \omega^{-1}), \quad (g, \omega) \in \mathbf{G} \times \hat{\mathbf{G}},$$

defined by the string diagrams

$$\phi \mapsto \begin{array}{c} \omega g \omega^{-1} \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \omega \end{array} \quad \begin{array}{c} \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \omega^{-1} \end{array} \quad \omega \in \mathbf{G} \quad (19)$$

and

$$\phi \mapsto \begin{array}{c} \omega g^{-1} \omega^{-1} \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \omega \end{array} \quad \begin{array}{c} \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \bullet \\ \nearrow \\ \omega^{-1} \end{array} \quad \omega \in \hat{\mathbf{G}} \setminus \mathbf{G}. \quad (20)$$

This definition recovers that of Section 4.3 in the case of trivial Real 2-Schur multiplier.

Definition. Assume that each vector space $\mathrm{Tr}_\rho(g)$, $g \in \mathbf{G}$, is finite dimensional. Then the Real 2-character of the twisted Real 2-representation ρ is the collection of traces

$$\chi_\rho(g, \omega) = \mathrm{tr}_{\mathrm{Tr}_\rho(g)}(\beta_{g,\omega}) \in k, \quad (g, \omega) \in \hat{\mathbf{G}}^{(2)}.$$

Example. In this example we assume basic familiarity with matrix factorizations. For background, see [26], [6].

Let k be a field of characteristic zero. Denote by \mathbf{LG}_k the bicategory of Landau–Ginzburg models over k , as in [6, §2.2]. Objects of \mathbf{LG}_k are pairs (R, W) consisting of a ring $R = k[[x_1, \dots, x_n]]$ for some $n \geq 0$ and a potential $W \in R$. The 1-morphism category $1\mathrm{Hom}_{\mathbf{LG}_k}((R_1, W_1), (R_2, W_2))$ is the triangulated category of finite rank matrix factorizations of $W_2 - W_1$:

$$1\mathrm{Hom}_{\mathbf{LG}_k}((R_1, W_1), (R_2, W_2)) = \mathrm{HMF}(R_1 \hat{\otimes}_k R_2, W_2 - W_1).$$

The composition $- \circ_0 -$ is defined using tensor product of matrix factorizations. For example, the identity 1-morphism $1_{(R,W)} : (R, W) \rightarrow (R, W)$ is the stabilized diagonal matrix factorization

$$\Delta_W = \bigwedge_{R^e} \left(\bigoplus_{i=1}^n R^e \cdot \theta_i \right), \quad d_{\Delta_W} = \sum_{i=1}^n (x_i - x'_i) \theta_i^\vee + \sum_{i=1}^n \partial_{[i]}^{x, x'} W \cdot \theta_i \wedge -$$

where $R^e = R \hat{\otimes}_k R \simeq k[[x_1, \dots, x_n, x'_1, \dots, x'_n]]$ and

$$\partial_{[i]}^{x, x'} W = \frac{W(x'_1, \dots, x'_{i-1}, x_i, \dots, x_n, x') - W(x'_1, \dots, x'_i, x_{i+1}, \dots, x_n, x')}{x_i - x'_i}.$$

Define a weak duality involution on \mathbf{LG}_k as follows. On objects set $(R, W)^\vee = (R, -W)$. On 1-morphism categories, $(-)^\vee$ acts as the linear dual of matrix factorizations. The adjoint equivalence η is induced by the canonical evaluation isomorphism from a finite rank free module to its double dual.

Suppose now that a finite \mathbb{Z}_2 -graded group $\hat{\mathbf{G}}$ acts on $R = k[[x_1, \dots, x_n]]$ by unital algebra automorphisms. Assume that $W \in R$ is a potential which satisfies

$$\omega(W) = \pi(\omega) \cdot W, \quad \omega \in \hat{\mathbf{G}}.$$

In words, \mathbf{G} and $\hat{\mathbf{G}} \setminus \mathbf{G}$ act by symmetries and anti-symmetries of W , respectively. While finite groups of symmetries of (R, W) appear in the theory of Landau–Ginzburg orbifolds, group actions of the above form are central to the theory of Landau–Ginzburg orientifolds [18]. For an explicit example, take $l, m \geq 1$ and let $W = x^{2l+1} + xy^{2m} \in k[[x, y]]$. Consider the exact sequence of multiplicative groups

$$1 \rightarrow \mathbb{Z}_m \rightarrow \mathbb{Z}_{2m} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 1.$$

Let ζ be a primitive $2m^{\text{th}}$ root of unity, which we assume to lie in k . Then a W -compatible action of \mathbb{Z}_{2m} on $k[[x, y]]$ is given by $\zeta \cdot (x, y) = (-x, \zeta y)$.

Define a Real representation ρ of \mathbf{G} on $(R, W) \in \text{Obj}(\mathbf{LG}_k)$ by letting $\omega \in \hat{\mathbf{G}}$ act by the 1-morphism $\omega \Delta_W \in \mathbf{HMF}(R \hat{\otimes}_k R, W - \pi(\omega) \cdot W)$ which is the pullback of Δ_W by $1 \otimes \omega$. Explicitly, we have

$$\omega \Delta_W = \bigwedge_{R^e} \left(\bigoplus_{i=1}^n R^e \cdot \theta_i \right), \quad d_{\omega \Delta_W} = \sum_{i=1}^n (\omega(x_i) - x'_i) \theta_i^\vee + \sum_{i=1}^n \partial_{[i]}^{\omega(x), x'} W \cdot \theta_i \wedge -.$$

The compatibility 2-isomorphisms $\psi_{\bullet, \bullet}$ are induced by the associators in \mathbf{LG}_k . One can also twist the maps $\psi_{\bullet, \bullet}$ by a cocycle $\hat{\theta} \in Z^2(B\hat{\mathbf{G}}, k_\pi^\times)$, thereby incorporating discrete torsion. Compare Section 5.3 below.

Using [26, Lemma 2.5.3], the Real categorical character of ρ is the Hochschild homology

$$\text{Tr}_\rho(g) \simeq HH_\bullet(\mathbf{MF}(R^g, W^g)), \quad g \in \mathbf{G}$$

together with the canonical linear isomorphisms

$$\beta_{g, \omega} : HH_\bullet(\mathbf{MF}(R^g, W^g)) \rightarrow HH_\bullet(\mathbf{MF}(R^{\omega g \pi(\omega)^{-1}}, W^{\omega g \pi(\omega)^{-1}})).$$

The pair (R^g, W^g) is defined by choosing coordinates x_1, \dots, x_n of R in which g acts linearly and such that $\text{Span}_k\{x_1, \dots, x_n\}^g = \text{Span}_k\{x_{t+1}, \dots, x_n\}$. Then $R^g = R/(x_1, \dots, x_t)$ and W^g is the image of W in R^g . By [9, §6.3], the Hochschild homology $HH_\bullet(\mathbf{MF}(R^g, W^g))$ is isomorphic to the Milnor algebra of W^g , supported in degree $n - t$.

As for Real 2-characters, we do not have a geometric interpretation of each number $\chi_\rho(g, \omega)$. However, it follows from [26, Theorem 2.5.4] that we have

$$\frac{1}{|\mathbf{G}|} \sum_{(g, h) \in \mathbf{G}^{(2)}} \chi_\rho(g, h) = \dim_k HH_\bullet^{\mathbf{G}}(\mathbf{MF}(R, W)).$$

Similarly, the Real 2-character computes the dimension of the \mathbf{G} -equivariant involutive Hochschild homology of $\mathbf{MF}(R, W)$:

$$\frac{1}{2|\mathbf{G}|} \sum_{(g, \omega) \in \hat{\mathbf{G}}^{(2)}} \chi_\rho(g, \omega) = \dim_k HH_\bullet^{\mathbf{G}, +}(\mathbf{MF}(R, W)).$$

The Real structure $\hat{\mathbf{G}}$ determines the involution required to define $HH_{\bullet}^{\mathbf{G},+}$. \triangleleft

Remark. The theory of 2-characters is weaker than its classical counterpart in that inequivalent 2-representations may have the same 2-character. For an example, see [25, §5]. Analogous statements hold for Real 2-characters.

The goal of the remainder of this section is to give geometric interpretations of $\mathrm{Tr}(\rho)$ and χ_{ρ} . We require some background material. Let \mathfrak{G} be a finite groupoid. Following Willerton [33, §2.3.1], a 2-cocycle $\theta \in Z^2(\mathfrak{G}, k^{\times})$ defines a k^{\times} -gerbe ${}^{\theta}\mathfrak{G}$ over \mathfrak{G} . Explicitly, ${}^{\theta}\mathfrak{G}$ is the category with

$$\mathrm{Obj}({}^{\theta}\mathfrak{G}) = \mathrm{Obj}(\mathfrak{G}), \quad \mathrm{Hom}_{{}^{\theta}\mathfrak{G}}(x_1, x_2) = k^{\times} \times \mathrm{Hom}_{\mathfrak{G}}(x_1, x_2)$$

and composition law

$$(z_2, g_2) \circ (z_1, g_1) = (\theta([g_2|g_1])z_2z_1, g_2g_1), \quad z_1, z_2 \in k^{\times}, \quad g_1, g_2 \in \mathrm{Mor}(\mathfrak{G}).$$

A vector bundle over ${}^{\theta}\mathfrak{G}$, also called a θ -twisted vector bundle over \mathfrak{G} , is a functor ${}^{\theta}\mathfrak{G} \rightarrow \mathbf{Vect}_k$ for which each $(Bk^{\times})|_x \subset {}^{\theta}\mathfrak{G}$ acts by scalar multiplication.

The following result generalizes Proposition 4.2 to finite categorical groups. The analogous result in the ungraded setting is [16, Theorem 4.17].

Theorem 5.4. *The Real categorical character of an $\hat{\alpha}$ -twisted Real 2-representation ρ of \mathbf{G} defines a $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ -twisted vector bundle over $\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathbf{G}}$*

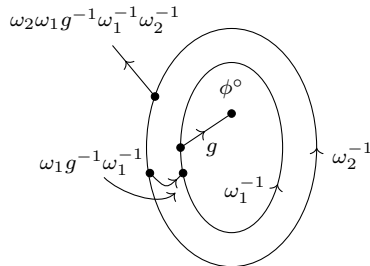
$$\mathrm{Tr}(\rho) : \tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathbf{G}} \rightarrow \mathbf{Vect}_k.$$

Proof. The theorem is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Tr}_{\rho}(g) & \xrightarrow{\beta_{g, \omega_2 \omega_1}} & \mathrm{Tr}_{\rho}(\omega_2 \omega_1 g^{\pi(\omega_2 \omega_1)} \omega_1^{-1} \omega_2^{-1}) \\ & \searrow \beta_{g, \omega_1} & \downarrow \tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\omega_2 | \omega_1]g) \\ & & \mathrm{Tr}_{\rho}(\omega_1 g^{\pi(\omega_2 \omega_1)} \omega_1^{-1}) \\ & & \nearrow \beta_{\omega_1 g^{\pi(\omega_1)} \omega_1^{-1}, \omega_2} \end{array}$$

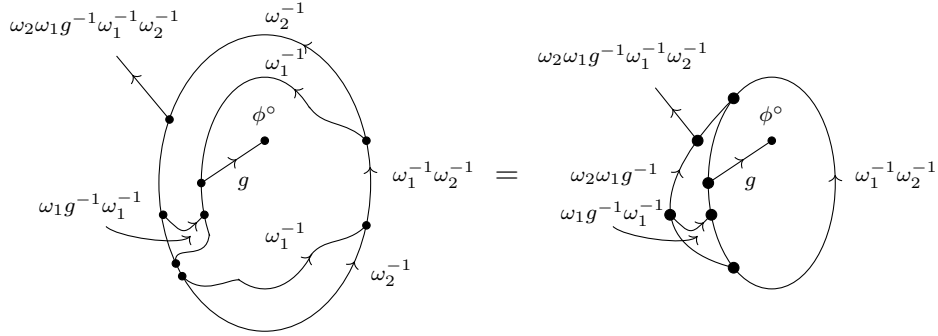
for all $g \in \mathbf{G}$ and $\omega_1, \omega_2 \in \hat{\mathbf{G}}$. The vertical arrow indicates multiplication by $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\omega_2 | \omega_1]g)$. To prove commutativity, suppose first that $\pi(\omega_2) = 1$. In this case the expression for $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\omega_2 | \omega_1]g)$ differs from that of $\tau(\alpha)([\omega_2 | \omega_1]g)$ only through the replacement of g with $g^{\pi(\omega_1)}$. The desired equality can therefore be verified by a straightforward modification of the arguments used to prove [16, Theorem 4.17].

Suppose then that $\pi(\omega_2) = -1$. Consider first the case $\pi(\omega_1) = 1$. Let $\phi \in \mathrm{Tr}_{\rho}(g)$. The composition $\beta_{\omega_1 g^{\pi(\omega_1)} \omega_1^{-1}, \omega_2}(\beta_{g, \omega_1}(\phi))$ is equal to the string diagram

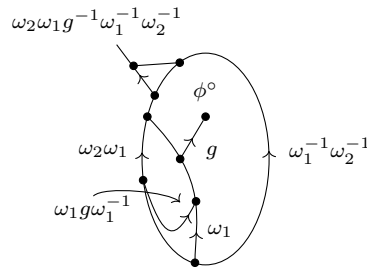


In this diagram, and those which follow, the exterior region is labelled by the category V while the interior regions are labelled by V^{op} . Using equations (15) and

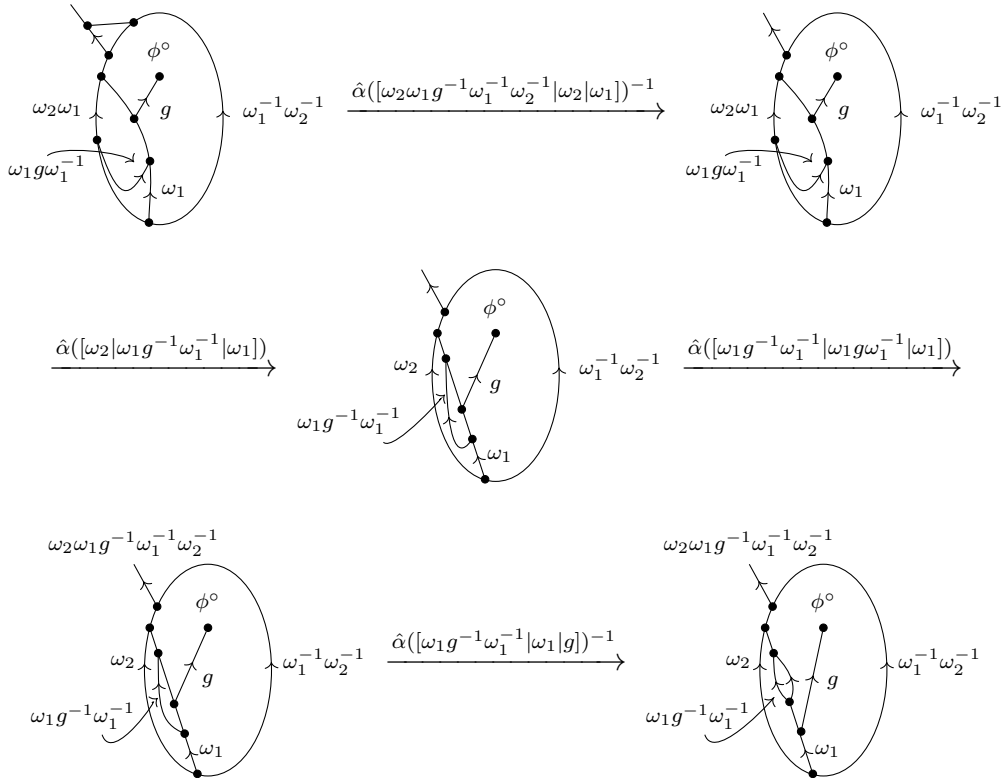
(18), the above string diagram is equal to



By equation (15), this is equal to



Repeatedly applying equation (13) gives



More precisely, the first two operations used the inverted form of equation (13) while in the last two operations used equation (13) but applied in the category V^{op} . This explains why we multiply by $\hat{\alpha}([\omega_1g^{-1}\omega_1^{-1}|\omega_1g\omega_1^{-1}|\omega_1])$ and $\hat{\alpha}([\omega_1g^{-1}\omega_1^{-1}|\omega_1|g])^{-1}$, rather than their inverses. By first removing the loop and then adding a different

loop (see equation (14)), the previous diagram becomes

Repeatedly applying equation (13) then gives

The first operation used equation (13) in V^{op} . By definition, the final string diagram computes $\beta_{g,\omega_2\omega_1}(\phi)$. The scalar introduced in the computation is

$$\frac{\hat{\alpha}([\omega_1|g^{-1}|g])\hat{\alpha}([\omega_1g^{-1}\omega_1^{-1}|\omega_1g\omega_1^{-1}|\omega_1])}{\hat{\alpha}([\omega_1g^{-1}\omega_1^{-1}|\omega_1|g])} \times \frac{\hat{\alpha}([\omega_2|\omega_1g^{-1}\omega_1^{-1}|\omega_1])}{\hat{\alpha}([\omega_2\omega_1g^{-1}\omega_1^{-1}\omega_2^{-1}|\omega_2|\omega_1])\hat{\alpha}([\omega_2|\omega_1|g^{-1}])},$$

which we recognize as $\tau_\pi^{\text{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)^{-1}$.

A similar calculation can be performed when $\pi(\omega_1) = -1$. The key difference is that, since both ω_1 and ω_2 are of degree -1 , at the end of the calculation we produce a scalar multiple of the string diagram

This gives the additional factor of $\hat{\alpha}([g|g^{-1}|g])$ appearing in $\tau_\pi^{\text{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)$ when both ω_1 and ω_2 are of degree -1 . This completes the proof. \square

Let $\alpha \in Z^3(BG, k^\times)$. In [33, §3.1] Willerton showed that the α -twisted Drinfeld double of G is isomorphic (as an algebra) to the $\tau(\alpha)$ -twisted groupoid algebra of ΛBG :

$$D^\alpha(G) \simeq k^{\tau(\alpha)}[\Lambda BG].$$

Motivated by this, for a twisted cocycle $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_\pi^\times)$ the $\hat{\alpha}$ -twisted thickened Drinfeld double of \mathbf{G} was defined in [34, §4.2] to be the $\tau_\pi^{\text{ref}}(\hat{\alpha})$ -twisted groupoid algebra of $\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$:

$$D^{\hat{\alpha}}(\mathbf{G}) = k^{\tau_\pi^{\text{ref}}(\hat{\alpha})}[\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}].$$

The inclusion $\mathbf{G} \hookrightarrow \hat{\mathbf{G}}$ defines a faithful functor $\Lambda B\mathbf{G} \rightarrow \Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$ under which $\tau_\pi^{\text{ref}}(\hat{\alpha})$ restricts to $\tau(\alpha)$. This induces a natural algebra embedding $D^\alpha(\mathbf{G}) \hookrightarrow D^{\hat{\alpha}}(\mathbf{G})$.

Corollary 5.5. *The Real categorical character of an $\hat{\alpha}$ -twisted Real 2-representation of \mathbf{G} is a module over the $\hat{\alpha}$ -twisted thickened Drinfeld double of \mathbf{G} .*

Proof. The category of vector bundles over $\tau_\pi^{\text{ref}}(\hat{\alpha})\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$ is equivalent to the category of $\tau_\pi^{\text{ref}}(\hat{\alpha})$ -twisted $k[\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}]$ -modules which is in turn equivalent to the category of $k^{\tau_\pi^{\text{ref}}(\hat{\alpha})}[\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}]$ -modules. The statement now follows from Theorem 5.4. \square

The next result describes the equivariance properties of Real 2-characters.

Theorem 5.6. *The Real 2-character of an $\hat{\alpha}$ -twisted Real 2-representation ρ of \mathbf{G} is a flat section of the line bundle $\tau_\pi^{\text{ref}}(\hat{\alpha})_k \rightarrow \Lambda\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$. Equivalently, the equality*

$$\chi_\rho(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}) = \frac{\tau_\pi^{\text{ref}}(\hat{\alpha})([\sigma \omega \sigma^{-1} | \sigma]g)}{\tau_\pi^{\text{ref}}(\hat{\alpha})([\sigma | \omega]g)} \cdot \chi_\rho(g, \omega)$$

holds for all $(g, \omega) \in \hat{\mathbf{G}}^{(2)}$ and $\sigma \in \hat{\mathbf{G}}$.

Proof. By Theorem 5.4, the Real categorical character $\text{Tr}(\rho)$ is a $\tau_\pi^{\text{ref}}(\hat{\alpha})$ -twisted vector bundle over $\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$. By the results of [33, §2.3.3], the holonomy of $\text{Tr}(\rho)$ is a flat section of the transgressed line bundle $\tau_\pi^{\text{ref}}(\hat{\alpha})_k \rightarrow \Lambda\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$. On the other hand, the holonomy of $\text{Tr}(\rho)$ is the Real 2-character of ρ . Combining these results gives the desired statement.

The explicit description of the $\hat{\mathbf{G}}$ -equivariance of χ_ρ follows from Willerton's formula for the loop transgression of an untwisted 2-cocycle [33, §1.3.3]. \square

Flat sections of the line bundle $\tau_\pi^{\text{ref}}(\hat{\alpha})_k \rightarrow \Lambda\Lambda_\pi^{\text{ref}} B\hat{\mathbf{G}}$ were first studied in [34, §4.4] (with $k = \mathbb{C}$), where they were shown to describe the complexified representation ring of $D^{\hat{\alpha}}(\mathbf{G})$. Theorem 5.6 gives a second interpretation of such sections, namely, as Real 2-class functions for $\hat{\alpha}$ -twisted Real 2-representations of \mathbf{G} . Corollary 5.5 explains the relationship between Real 2-representations and thickened Drinfeld doubles.

5.3. Real 2-representations on 2Vect_k . We study twisted Real 2-representations on 2Vect_k . The ungraded case is treated in [10]; see also [15, §§5.1-2], [25].

Consider 2Vect_k with its weak duality involution $(-)^{\vee}$ from Section 1.2. We begin with a cohomological classification of linear Real 2-representations on 2Vect_k . The underlying object $[n] \in \text{Obj}(2\text{Vect}_k)$ of such a representation is called its dimension.

Proposition 5.7. *Equivalence classes of linear Real 2-representations of \mathbf{G} on 2Vect_k of dimension n are in bijection with the data of*

- (i) a group homomorphism $\rho_0 : \hat{\mathbf{G}} \rightarrow \mathfrak{S}_n$, and
- (ii) a class $[\hat{\theta}] \in H^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$, where the $\hat{\mathbf{G}}$ -module $(k_\pi^\times)_{\rho_0}^n$ is defined by

$$\omega \cdot (a_1, \dots, a_n) = (a_{\rho_0(\omega)^{-1}(1)}^{\pi(\omega)}, \dots, a_{\rho_0(\omega)^{-1}(n)}^{\pi(\omega)}).$$

Two such data are identified if they differ by the action of \mathfrak{S}_n on $\text{Hom}_{\text{Grp}}(\hat{\mathbf{G}}, \mathfrak{S}_n)$.

Proof. The proof is a modification of the classification in the ungraded case; see [10, Theorem 5.5], [25, Proposition 4].

Let ρ be a linear Real 2-representation of dimension n . For each $\omega \in \hat{\mathbf{G}}$, the 1-morphism $\rho(\omega) : [n] \rightarrow [n]$ is an equivalence and hence is a permutation 2-matrix (see [15, Lemma 5.3]). After noting that $(-)^{\vee}$ fixes the isomorphism class of a permutation 2-matrix, the existence of a 2-isomorphism $\rho(\omega_2) \circ^{\pi(\omega_2)} \rho(\omega_1) \simeq \rho(\omega_2\omega_1)$ implies that ρ defines a group homomorphism $\rho_0 : \hat{\mathbf{G}} \rightarrow \mathfrak{S}_n$. Fixing a basis of each vector space of each 1-morphism $\rho(\omega)$. Then the 2-isomorphism $\psi_{\omega_2, \omega_1}$ itself is given by an n -tuple $(\hat{\theta}_i([\omega_2|\omega_1]))_{i=1}^n \in (k^\times)^n$, the i^{th} component being the isomorphism between the unique one dimensional vector spaces of the i^{th} rows of $\rho(\omega_2) \circ^{\pi(\omega_2)} \rho(\omega_1)$ and $\rho(\omega_2\omega_1)$. By the associativity¹ constraint (10), the $\hat{\theta}_i$ assemble to a 2-cocycle $\hat{\theta} \in Z^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$. A change of basis defines leads to a cohomologous 2-cocycle. A similar argument shows that a contravariance respecting pseudonatural isomorphism $u : \rho \Rightarrow \rho'$ defines a 1-cochain $\hat{u} \in C^1(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$ with the property that $\hat{\theta} = \hat{\theta}' d\hat{u}$. In this way, each equivalence class of linear Real 2-representations of dimension n defines a class in $H^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$.

Reversing the above construction associates to a 2-cocycle $\hat{\theta} \in Z^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$ an n -dimensional linear Real 2-representation, the entries of each 1-morphism $\rho(\omega)$ being either trivial or a copy of k . Moreover, this association is quasi-inverse to the construction of the previous paragraph. \square

The next result is a Real version of [25, Theorem 10].

Proposition 5.8. *The Real 2-character of the Real 2-representation $\rho_{[\hat{\theta}]}$ of \mathbf{G} determined by $[\hat{\theta}] \in H^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$ is*

$$\chi_{\rho_{[\hat{\theta}]}}(g, \omega) = \sum_{\substack{i \in \{1, \dots, n\} \\ \rho_0(g)(i) = \rho_0(\omega)(i)}} \hat{\theta}_i([g^{-1}|g])^{-\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}_i([\omega|g^{\pi(\omega)}])}{\hat{\theta}_i([g|\omega])}.$$

Proof. Fix a normalized representative $\hat{\theta}$ of $[\hat{\theta}]$. A 2-morphism $\phi : 1_{[n]} \Rightarrow \rho_{\hat{\theta}}(g)$, $g \in \mathbf{G}$, is an $n \times n$ matrix which has non-zero entries only at those diagonals for which $\rho_{\hat{\theta}}(g)$ is non-zero. Using this observation, the computation of $\chi_{\rho_{\hat{\theta}}}$ reduces to the one dimensional case. In this case, direct inspection of the string diagrams (19) and (20) shows that $\beta_{g, \omega} : k \rightarrow k$ is multiplication by

$$\hat{\theta}([g^{-1}|g])^{-\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}([\omega|g^{\pi(\omega)}])}{\hat{\theta}([\omega g^{\pi(\omega)} \omega^{-1}|\omega])}.$$

Restriction to graded commuting pairs gives the claimed formula. \square

A geometric version of Proposition 5.7 is as follows. Instead of a matrix model of $2\mathbf{Vect}_k$ we work with the equivalent model given by k -linear additive finitely semisimple categories. The object $[n] \in \text{Obj}(2\mathbf{Vect}_k)$ is modelled by the category of vector bundles $\mathbf{Vect}_k(X)$ for any set X of cardinality n . Suppose that $\hat{\mathbf{G}}$ acts

¹As $2\mathbf{Vect}_k$ is not a 2-category, the non-trivial associator 2-isomorphism must be included in equation (10) in the obvious way.

on X . Fix a 2-cocycle $\hat{\theta} \in Z^2(X//\hat{\mathbf{G}}, k_\pi^\times)$ with coefficient system k_π^\times twisted by the double cover $\pi : X//\mathbf{G} \rightarrow X//\hat{\mathbf{G}}$. Write $\hat{\theta}_x([\omega_2|\omega_1])$ for the value of $\hat{\theta}$ on the 2-chain $[\omega_2|\omega_1]x$. A linear Real 2-representation $\rho_{\hat{\theta}}$ is defined on $\mathbf{Vect}_k(X)$ by setting $\rho_{\hat{\theta}}(g) = (g^{-1})^*$ for $g \in \mathbf{G}$ and $\rho_{\hat{\theta}}(\omega) = (-)^\vee \circ (\omega^{-1})^*$ for $\omega \in \hat{\mathbf{G}} \setminus \mathbf{G}$. The 2-isomorphism $\psi_{\omega_2, \omega_1}$ multiplies the fibre over $x \in X$ by $\hat{\theta}_x([\omega_2|\omega_1])$, pre-composing with ev if $\omega_1, \omega_2 \in \hat{\mathbf{G}} \setminus \mathbf{G}$. In this language, Proposition 5.7 becomes the statement that all linear Real 2-representations in 2-vector spaces arise in this way while Proposition 5.8 becomes the fixed point formula

$$\chi_{\rho_{\hat{\theta}}}(g, \omega) = \sum_{x \in X^{g, \omega}} \tau_\pi^{\text{ref}}(\hat{\theta}_x)([\omega]g)^{-1}.$$

To end this section, we mention the $\hat{\alpha}$ -twisted generalization of Proposition 5.7. An $\hat{\alpha}$ -twisted Real 2-representation of \mathbf{G} on $2\mathbf{Vect}_k$ determines the data of

- (i) a group homomorphism $\rho_0 : \hat{\mathbf{G}} \rightarrow \mathfrak{S}_n$,
- (ii) a morphism $\rho_1 : k_\pi^\times \rightarrow (k_\pi^\times)_{\rho_0}^n$ of $\hat{\mathbf{G}}$ -modules, and
- (iii) a normalized 2-cochain $\hat{\theta} \in C^2(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$

such that the equality $d\hat{\theta} = \rho_1(\hat{\alpha})$ holds in $Z^3(B\hat{\mathbf{G}}, (k_\pi^\times)_{\rho_0}^n)$. This data, up to equivalence as in Proposition 5.7, classifies equivalence classes of twisted Real 2-representations on $2\mathbf{Vect}_k$. We omit the details, which are a straightforward combination of those of [10, Theorem 5.5] and Proposition 5.7. Proposition 5.8 is unchanged.

5.4. The anti-linear theory. We briefly explain a categorification of the standard (anti-linear) theory of Real representations of a finite group from Section 2.1.

Let k be a field which is a quadratic extension of a subfield k_0 . Galois conjugation defines a k_0 -linear strict involution $\overline{(-)} : \mathbf{Vect}_k \rightarrow \mathbf{Vect}_k$.

Given a k -linear category \mathcal{C} , denote by $\overline{\mathcal{C}}$ the k -linear category with

$$\text{Obj}(\overline{\mathcal{C}}) = \text{Obj}(\mathcal{C}), \quad \text{Hom}_{\overline{\mathcal{C}}}(x, y) = \overline{\text{Hom}_{\mathcal{C}}(x, y)}.$$

The assignment $\mathcal{C} \mapsto \overline{\mathcal{C}}$ extends to a strict involutive 2-functor $\overline{(-)} : \mathbf{Cat}_k \rightarrow \mathbf{Cat}_k$. For each k -linear bicategory \mathcal{V} we obtain an involutive pseudofunctor $\overline{(-)} : \mathcal{V} \rightarrow \mathcal{V}$, acting trivially on objects and by conjugation on 1-morphism categories. In particular, the action of $\overline{(-)}$ on 2-morphisms is anti-linear.

We can now formulate the anti-linear approach to Real 2-representation theory.

Definition. A linear Real 2-representation of \mathbf{G} on a k -linear 2-category \mathcal{V} (in the anti-linear approach) consists of data $V \in \text{Obj}(\mathcal{V})$, $\rho(\omega)$, $\psi_{\omega_2, \omega_1}$ and ψ_e as in Section 4.1, with the constraint (11) unchanged but with the constraint (10) replaced by the constraint

$$\psi_{\omega_3 \omega_2, \omega_1} \circ \left(\psi_{\omega_3, \omega_2} \circ \pi^{(\omega_3 \omega_2)} \rho(\omega_1) \right) = \psi_{\omega_3, \omega_2 \omega_1} \circ \left(\rho(\omega_3) \circ \pi^{(\omega_3)} \psi_{\omega_2, \omega_1} \right). \quad (21)$$

Left superscripts now determine whether or not the 2-functor $\overline{(-)}$ is applied.

The above definition admits an obvious twisted generalization. All results of Section 5 and Section 7 below continue to hold in the anti-linear approach with essentially the same proofs. The key point is that while $\pi^{(\omega_3)} \psi_{\omega_2, \omega_1}$ instead of $\pi^{(\omega_3)} \psi_{\omega_2, \omega_1}^{\pi(\omega_3)}$ appears in equation (21), the pseudofunctor $\overline{(-)} : \mathcal{V} \rightarrow \mathcal{V}$ is anti-linear on 2-morphisms, instead of linear but direction reversing. For example, in the twisted

case we again find that the Real 2-Schur multiplier is an element of $H^3(B\hat{\mathbf{G}}, k^\times)$, where now $\hat{\mathbf{G}} \backslash \mathbf{G}$ acts on k^\times via the Galois involution.

6. TWISTED REAL INDUCTION

This section, which contains motivation and background material for that which follows, describes relevant aspects of the theory of twisted (Real) induction. All statements are formulated so as to facilitate their categorification. In the complex case the results are standard, but we were unable to find a general discussion of the Real case in the literature.

6.1. Complex induction. Let \mathbf{G} be a finite group. Fix a cocycle $\theta \in Z^2(B\mathbf{G}, k^\times)$. For later use, note that a θ -twisted representation ρ of \mathbf{G} satisfies

$$\rho(g)^{-1} = \theta([g|g^{-1}])^{-1} \rho(g^{-1}), \quad g \in \mathbf{G}. \quad (22)$$

The character of ρ satisfies (*cf.* equation (6))

$$\chi_\rho(hgh^{-1}) = \tau(\theta)([h]g) \chi_\rho(g), \quad g, h \in \mathbf{G}. \quad (23)$$

Let $\mathbf{H} \leq \mathbf{G}$ be a subgroup. Define the twisted induction functor $\text{Ind}_{\mathbf{H}}^{\mathbf{G}} : \text{Rep}_k^{\theta|_{\mathbf{H}}}(\mathbf{H}) \rightarrow \text{Rep}_k^{\theta}(\mathbf{G})$ as follows. Fix a complete set of representatives $\{r_1, \dots, r_p\}$ of the left cosets \mathbf{G}/\mathbf{H} . Given a $\theta|_{\mathbf{H}}$ -twisted representation ρ of \mathbf{H} on V , set

$$\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho) = \bigoplus_{i=1}^p r_i \cdot V.$$

Here $r_i \cdot V$ is an isomorphic copy of V . For each $g \in \mathbf{G}$ and $i, j \in \{1, \dots, p\}$, set

$$\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho)(g)_{r_j r_i} = \begin{cases} \frac{\theta([g|r_i])}{\theta([r_j|r_j^{-1}gr_i])} \rho(r_j^{-1}gr_i) & \text{if } r_j^{-1}gr_i \in \mathbf{H}, \\ 0 & \text{else.} \end{cases}$$

Here we view $\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho)(g)$ as a $p \times p$ matrix. Up to isomorphism, $\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho)$ is independent of the choice of representatives of \mathbf{G}/\mathbf{H} .

Proposition 6.1. *The character of $\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho)$ is*

$$\chi_{\text{Ind}_{\mathbf{H}}^{\mathbf{G}}(\rho)}(g) = \frac{1}{|\mathbf{H}|} \sum_{\substack{r \in \mathbf{G} \\ rgr^{-1} \in \mathbf{H}}} \tau(\theta)([r]g)^{-1} \chi_\rho(rgr^{-1}).$$

Proof. See, for example, [24, Proposition 4.1]. \square

Proposition 6.1 admits the following generalization to finite groupoids.

Proposition 6.2. *Let $f : \mathfrak{H} \rightarrow \mathfrak{G}$ be a faithful functor of finite groupoids. Fix $\theta \in Z^2(\mathfrak{G}, k^\times)$. For each $\theta|_{\mathfrak{H}}$ -twisted representation ρ of \mathfrak{H} and loop (x, γ) in \mathfrak{G} , we have*

$$\chi_{\text{Ind}_{\mathfrak{H}}^{\mathfrak{G}}(\rho)}(x \xrightarrow{\gamma} x) = \sum_{y \in \text{Obj}(\mathfrak{H})} \frac{1}{|\mathcal{O}_{\mathfrak{H}}(y)| |\text{Aut}_{\mathfrak{H}}(y)|} \sum_{\substack{(x \xrightarrow{s} y) \in \mathfrak{G} \\ s\gamma s^{-1} \in \text{Aut}_{\mathfrak{H}}(y)}} \tau(\theta)([s]\gamma)^{-1} \chi_\rho(y \xrightarrow{s\gamma s^{-1}} y),$$

where $\mathcal{O}_{\mathfrak{H}}(y)$ denotes the orbit of y in \mathfrak{H} .

Proof. The proof is a twisted generalization of [15, Proposition 6.11]. Let $\{y_1, \dots, y_n\}$ be a complete set of representatives for the \mathfrak{H} -isomorphism classes of objects which map to $\mathcal{O}_{\mathfrak{G}}(x)$ via f . For each $j \in \{1, \dots, n\}$, fix a morphism $x \xrightarrow{s_j} f(y_j)$. We compute

$$\begin{aligned}
\chi_{\text{Ind}_{\mathfrak{H}}^{\mathfrak{G}}(\rho)}(x \xrightarrow{\gamma} x) &= \sum_{j=1}^n \chi_{\text{Ind}_{B\text{Aut}_{\mathfrak{H}}(y_j)}^{\mathfrak{G}}(\rho)}(x \xrightarrow{\gamma} x) \\
&\stackrel{\text{Eq. (23)}}{=} \sum_{j=1}^n \tau(\theta)([s_j]\gamma)^{-1} \chi_{\text{Ind}_{B\text{Aut}_{\mathfrak{H}}(y_j)}^{\mathfrak{G}}(\rho)}(f(y_j) \xrightarrow{s_j \gamma s_j^{-1}} f(y_j)) \\
&\stackrel{\text{Prop. 6.1}}{=} \sum_{j=1}^n \frac{1}{|\text{Aut}_{\mathfrak{H}}(y_j)|} \times \\
&\quad \sum_{\substack{x \xrightarrow{s} y_j \\ s\gamma s^{-1} \in \text{Aut}_{\mathfrak{H}}(y_j)}} \tau(\theta)([s_j]\gamma)^{-1} \tau(\theta)([s s_j^{-1}] s_j \gamma s_j^{-1})^{-1} \chi_{\rho}(s\gamma s^{-1}) \\
&\stackrel{\text{Eq. (5)}}{=} \sum_{j=1}^n \frac{1}{|\text{Aut}_{\mathfrak{H}}(y_j)|} \sum_{\substack{x \xrightarrow{s} y_j \\ s\gamma s^{-1} \in \text{Aut}_{\mathfrak{H}}(y_j)}} \tau(\theta)([s]\gamma)^{-1} \chi_{\rho}(s\gamma s^{-1})
\end{aligned}$$

which is easily seen to equal the desired expression. \square

6.2. Real induction. Let $\hat{\mathfrak{H}} \leq \hat{\mathfrak{G}}$ be a \mathbb{Z}_2 -graded subgroup. Fix $\hat{\theta} \in Z^2(B\hat{\mathfrak{G}}, k_{\pi}^{\times})$. In this section we interpret Real representations as generalized symmetric representations. Define the twisted Real induction functor

$$\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}} : \text{RRep}_k^{\hat{\theta}|_{\hat{\mathfrak{H}}}}(\mathfrak{H}) \rightarrow \text{RRep}_{\hat{\theta}}^{\hat{\mathfrak{G}}}(\mathfrak{G})$$

as follows. Fix a complete set $\mathcal{S} = \{\sigma_1, \dots, \sigma_q\}$ of representatives of $\hat{\mathfrak{G}}/\hat{\mathfrak{H}}$. Let ρ be a $\hat{\theta}|_{\hat{\mathfrak{H}}}$ -twisted Real representation of \mathfrak{H} on V . As a vector space, $\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}}(\rho)$ is

$$\bigoplus_{i=1}^q \sigma_i \cdot \pi(\sigma_i) V.$$

For each $\omega \in \hat{\mathfrak{G}}$ and $i, j \in \{1, \dots, q\}$, set

$$\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}}(\rho)(\omega)_{\sigma_j \sigma_i} = \begin{cases} \frac{\hat{\theta}([\omega|\sigma_i])}{\hat{\theta}([\sigma_j|\sigma_j^{-1}\omega\sigma_i])} \cdot \pi(\sigma_j) \rho(\sigma_j^{-1}\omega\sigma_i)^{\pi(\sigma_j)} & \text{if } \sigma_j^{-1}\omega\sigma_i \in \hat{\mathfrak{H}}, \\ 0 & \text{else.} \end{cases}$$

Given a morphism $\phi : \rho_1 \rightarrow \rho_2$ of $\hat{\theta}|_{\hat{\mathfrak{H}}}$ -twisted Real representations, the value of $\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}}(\phi)$ on the i^{th} summand is ϕ if $\pi(\sigma_i) = 1$ and is $\rho_2(\sigma_i)^{-1} \circ \phi \circ \rho_1(\sigma_i)$ if $\pi(\sigma_i) = -1$.

There natural isomorphism of functors

$$\text{Res}_{\mathfrak{G}}^{\hat{\mathfrak{G}}} \circ \text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}} \simeq \text{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \circ \text{Res}_{\mathfrak{H}}^{\hat{\mathfrak{H}}}$$

can be used to compute $\chi_{\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}}(\rho)}$. However, for comparison with the case of Real 2-representations it is useful to compute $\chi_{\text{RInd}_{\hat{\mathfrak{H}}}^{\hat{\mathfrak{G}}}(\rho)}$ directly.

Proposition 6.3. *The Real character of $\mathrm{RInd}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}(\rho)$ is*

$$\chi_{\mathrm{RInd}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}(\rho)}(g) = \frac{1}{2|\mathbf{H}|} \sum_{\substack{\omega \in \hat{\mathbf{G}} \\ \omega g^{\pi(\omega)} \omega^{-1} \in \mathbf{H}}} \tau_{\pi}^{\mathrm{ref}}(\hat{\theta})([\omega]g)^{-1} \chi_{\rho}(\omega g^{\pi(\omega)} \omega^{-1}).$$

Proof. For each $g \in \mathbf{G}$, we compute

$$\begin{aligned} \chi_{\mathrm{RInd}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}(\rho)}(g) &= \sum_{\substack{i \in \{1, \dots, q\} \\ \sigma_i^{-1} g \sigma_i \in \mathbf{H}}} \frac{\hat{\theta}([g|\sigma_i])}{\hat{\theta}([\sigma_i|\sigma_i^{-1}g\sigma_i])} \mathrm{tr}_{\pi(\sigma_i)V}^{\pi(\sigma_i)} \rho(\sigma_i^{-1}g\sigma_i)^{\pi(\sigma_i)} \\ &= \frac{1}{2|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma g^{\pi(\sigma)} \sigma^{-1} \in \mathbf{H}}} \frac{\hat{\theta}([g|\sigma^{-1}])}{\hat{\theta}([\sigma^{-1}|\sigma g \sigma^{-1}])} \mathrm{tr}_V \rho(\sigma^{-1}g\sigma)^{\pi(\sigma)}. \end{aligned}$$

After using equation (22), this is seen to equal

$$\frac{1}{2|\hat{\mathbf{H}}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma g^{\pi(\sigma)} \sigma^{-1} \in \mathbf{H}}} \frac{\hat{\theta}([g|\sigma^{-1}])}{\hat{\theta}([\sigma^{-1}|\sigma g \sigma^{-1}])} \hat{\theta}([\sigma g \sigma^{-1}|\sigma g^{-1}\sigma^{-1}])^{\frac{\pi(\sigma)-1}{2}} \chi_{\rho}(\sigma g^{\pi(\sigma)} \sigma^{-1}).$$

That the coefficient of $\chi_{\rho}(\sigma g^{\pi(\sigma)} \sigma^{-1})$ is equal to $\tau_{\pi}^{\mathrm{ref}}(\hat{\theta})([\sigma]g)^{-1}$ can be verified using equation (8). \square

6.3. Hyperbolic induction. A second form of Real induction, different from that of Section 6.2, is the hyperbolic construction. Let $\hat{\mathbf{G}}$ be a finite \mathbb{Z}_2 -graded group. Fix $\hat{\theta} \in Z^2(B\hat{\mathbf{G}}, k_{\pi}^{\times})$. The hyperbolic induction functor

$$\mathrm{HInd}_{\hat{\mathbf{G}}}^{\hat{\theta}} : \mathrm{Rep}_{k}^{\theta}(\mathbf{G}) \rightarrow \mathrm{RRep}_{k}^{\hat{\theta}}(\mathbf{G})$$

admits the following explicit description. Fix an element $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$. Given a θ -twisted representation ρ of \mathbf{G} on V , let $\mathrm{HInd}_{\hat{\mathbf{G}}}^{\hat{\theta}}(\rho)$ be $V \oplus V^{\vee}$ as a vector space. By Lemma 2.1 and Proposition 2.2, the $\hat{\mathbf{G}}$ -action is given by

$$\mathrm{HInd}_{\hat{\mathbf{G}}}^{\hat{\theta}}(\rho)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \tau_{\pi}^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1} \rho(\varsigma^{-1}g^{-1}\varsigma)^{\vee} \end{pmatrix}, \quad g \in \mathbf{G}$$

and

$$\mathrm{HInd}_{\hat{\mathbf{G}}}^{\hat{\theta}}(\rho)(\omega) = \begin{pmatrix} 0 & \frac{\hat{\theta}([\omega|\varsigma])}{\hat{\theta}([\varsigma^{-1}|\varsigma])} \rho(\omega\varsigma) \\ \frac{\hat{\theta}([\omega|\omega^{-1}])\hat{\theta}([\omega^{-1}|\varsigma])}{\hat{\theta}([\varsigma^{-1}|\varsigma])} \rho(\omega^{-1}\varsigma)^{\vee} & 0 \end{pmatrix}, \quad \omega \in \hat{\mathbf{G}} \setminus \mathbf{G}.$$

More generally, for a subgroup $\mathbf{H} \leq \mathbf{G}$ the functor $\mathrm{HInd}_{\hat{\mathbf{H}}}^{\hat{\theta}} : \mathrm{Rep}_{k}^{\theta}(\mathbf{H}) \rightarrow \mathrm{RRep}_{k}^{\hat{\theta}}(\mathbf{G})$ is defined to be the composition $\mathrm{HInd}_{\hat{\mathbf{G}}}^{\hat{\theta}} \circ \mathrm{Ind}_{\hat{\mathbf{H}}}^{\hat{\mathbf{G}}}$.

Proposition 6.4. *The Real character of $\mathrm{HInd}_{\hat{\mathbf{H}}}^{\hat{\theta}}(\rho)$ is*

$$\chi_{\mathrm{HInd}_{\hat{\mathbf{H}}}^{\hat{\theta}}(\rho)}(g) = \frac{1}{|\mathbf{H}|} \sum_{\substack{\omega \in \hat{\mathbf{G}} \\ \omega g^{\pi(\omega)} \omega^{-1} \in \mathbf{H}}} \tau_{\pi}^{\mathrm{ref}}(\hat{\theta})([\omega]g)^{-1} \chi_{\rho}(\omega g^{\pi(\omega)} \omega^{-1}).$$

Proof. Suppose first that $\mathbf{H} = \mathbf{G}$. For each $g \in \mathbf{G}$, we compute

$$\begin{aligned} \chi_{\mathbf{H}\text{Ind}_{\hat{\mathbf{G}}}(\rho)}(g) &= \chi_{\rho}(g) + \tau_{\pi}^{\text{ref}}(\hat{\theta})([\zeta^{-1}]g)^{-1}\chi_{\rho}(\zeta^{-1}g^{-1}\zeta) \\ &\stackrel{\text{Eq. (23)}}{=} \frac{1}{|\mathbf{G}|} \sum_{s \in \mathbf{G}} \tau(\theta)([s]g)^{-1}\chi_{\rho}(sgs^{-1}) + \\ &\quad \frac{1}{|\mathbf{G}|} \sum_{s \in \mathbf{G}} \tau_{\pi}^{\text{ref}}(\hat{\theta})([\zeta^{-1}]g)^{-1}\tau(\theta)([s]\zeta^{-1}g^{-1}\zeta)^{-1}\chi_{\rho}(s\zeta^{-1}g^{-1}\zeta s^{-1}) \\ &\stackrel{\text{Eq. (5)}}{=} \frac{1}{|\mathbf{G}|} \sum_{\omega \in \hat{\mathbf{G}}} \tau_{\pi}^{\text{ref}}(\hat{\theta})([\omega]g)^{-1}\chi_{\rho}(\omega g^{\pi(\omega)}\omega^{-1}). \end{aligned}$$

The case of an arbitrary subgroup follows by combining the previous case with equation (5) and Proposition 6.1. \square

7. TWISTED REAL 2-INDUCTION

We study various forms of induction for linear Real representations of finite categorical groups. We use our results to propose a character theoretic description of twisted Borel equivariant Real Morava E -theory.

7.1. Twisted 2-induction. Let $\mathbf{H} \leq \mathbf{G}$ be finite groups. Fix $\alpha \in Z^3(B\mathbf{G}, k^{\times})$. Let ρ be a linear representation of $\mathcal{H} = \mathcal{G}(\mathbf{H}, \alpha|_{\mathbf{H}})$ on a category V . The induced representation $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)$ of $\mathcal{G} = \mathcal{G}(\mathbf{G}, \alpha)$ was constructed in [16, Proposition 5.6]. An explicit construction, which generalizes that of [15, §7.1] in the untwisted case, is as follows. Keeping the notation from Section 6.1, as a category let

$$\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho) = \prod_{i=1}^p r_i \cdot V.$$

An element $g \in \mathbf{G}$ acts via the $p \times p$ matrix whose $(j, i)^{\text{th}}$ entry is

$$\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g)_{r_j r_i} = \begin{cases} \rho(h) & \text{if } gr_i = r_j h \text{ for some } h \in \mathbf{H}, \\ 0 & \text{else.} \end{cases}$$

The $(k, i)^{\text{th}}$ entry of $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g_2) \circ \text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g_1)$ is $\rho(h_2) \circ \rho(h_1)$ if $g_1 r_i = r_j h_1$ and $g_2 r_j = r_k h_2$ for some $h_1, h_2 \in \mathbf{H}$ and is zero otherwise. If non-trivial, the component of the 2-isomorphism $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\psi)_{g_2, g_1}$ at this entry is defined to be

$$\frac{\alpha([g_2|r_j|h_1])}{\alpha([g_2|g_1|r_i])\alpha([r_k|h_2|h_1])} \psi_{h_2, h_1}.$$

Theorem 7.1. *There is an isomorphism*

$$\text{Tr}(\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)) \simeq \text{Ind}_{\Lambda\mathbf{B}\mathbf{H}}^{\Lambda\mathbf{B}\mathbf{G}}(\text{Tr}(\rho))$$

of $\tau(\alpha)$ -twisted representations of $\Lambda\mathbf{B}\mathbf{G}$.

Here we view $\text{Tr}(\rho)$ as a $\tau(\alpha|_{\mathbf{H}})$ -twisted representation of $\Lambda\mathbf{B}\mathbf{H}$ and $\text{Ind}_{\Lambda\mathbf{B}\mathbf{H}}^{\Lambda\mathbf{B}\mathbf{G}}$ is twisted induction for groupoids. The untwisted version of Theorem 7.1 was proved by Ganter–Kapranov [15, Theorem 7.5]; the twisted case can be handled by an elaboration of their argument. We omit the argument, as a further elaboration will be used to prove Theorem 7.3 below.

Corollary 7.2 (cf. [15, Corollary 7.6]). *The 2-character of $\text{Ind}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)$ is given by*

$$\chi_{\text{Ind}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)}(g_1, g_2) = \frac{1}{|\mathbf{H}|} \sum_{\substack{s \in \mathbf{G} \\ s(g_1, g_2)s^{-1} \in \mathbf{H}^2}} \tau^2(\alpha)([s]g_1 \xrightarrow{g_2} g_1)^{-1} \cdot \chi_{\rho}(sg_1s^{-1}, sg_2s^{-1}).$$

Proof. By Theorem 7.1, it suffices to compute the character of $\text{Ind}_{\Lambda B\mathbf{H}}^{\Lambda B\hat{\mathcal{G}}}(\text{Tr}(\rho))$. As the canonical functor $\Lambda B\mathbf{H} \rightarrow \Lambda B\mathbf{G}$ is faithful, this character can be computed by Proposition 6.2, giving the desired result. \square

7.2. Real 2-induction. Consider again a \mathbb{Z}_2 -graded subgroup $\hat{\mathbf{H}} \leq \hat{\mathbf{G}}$. Let $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_{\pi}^{\times})$ and let ρ be a linear Real representation of $\mathcal{H} = \mathcal{G}(\mathbf{H}, \alpha|_{\mathbf{H}})$ on a category V . The Real structure is $\hat{\mathcal{H}} = \mathcal{G}(\hat{\mathbf{H}}, \hat{\alpha}|_{\hat{\mathbf{H}}})$. Define a Real 2-representation $\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)$ of $\mathcal{G} = \mathcal{G}(\mathbf{G}, \alpha)$ with Real structure $\hat{\mathcal{G}} = \mathcal{G}(\hat{\mathbf{G}}, \hat{\alpha})$ as follows. As a category, set

$$\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho) = \prod_{i=1}^q \sigma_i \cdot \pi(\sigma_i)V$$

with $\mathcal{S} = \{\sigma_1, \dots, \sigma_q\}$ as in Section 6.2. An element $\omega \in \hat{\mathbf{G}}$ acts by the $q \times q$ matrix whose $(j, i)^{\text{th}}$ entry is

$$\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega)_{\sigma_j \sigma_i} = \begin{cases} \pi(\sigma_j)\rho(\eta) & \text{if } \omega\sigma_i = \sigma_j\eta \text{ for some } \eta \in \hat{\mathbf{H}} \\ 0 & \text{else.} \end{cases}$$

The $(k, i)^{\text{th}}$ entry of $\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega_2) \circ \pi(\omega_2)\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega_1)$ is $\pi(\sigma_j)\rho(\omega_2') \circ \pi(\eta_2\sigma_j)\rho(\eta_1)$ if $\omega_1\sigma_i = \sigma_j\eta_1$ and $\omega_2\sigma_j = \sigma_k\eta_2$ for some $\eta_1, \eta_2 \in \hat{\mathbf{H}}$ and is zero otherwise. The component of the 2-isomorphism $\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{\omega_2, \omega_1}$ at this entry is defined to be

$$\frac{\hat{\alpha}([\omega_2|\sigma_j|\eta_1])}{\hat{\alpha}([\omega_2|\omega_1|\sigma_i])\hat{\alpha}([\sigma_k|\eta_2|\eta_1])} \cdot \pi(\sigma_k)\psi_{\eta_2, \eta_1}^{\pi(\sigma_k)}.$$

It is straightforward to verify that this defines a linear Real representation of \mathcal{G} .

7.3. Induced Real categorical and 2-characters. We compute the Real categorical and 2-characters of $\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)$. As mentioned above, the computation is a generalization of that of Ganter–Kapranov in the untwisted ungraded case. We begin with the Real categorical character.

Theorem 7.3. *There is a canonical isomorphism*

$$\text{Tr}(\text{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)) \simeq \text{Ind}_{\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{H}}}^{\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}}(\text{Tr}(\rho))$$

of $\tau_{\pi}^{\text{ref}}(\hat{\alpha})$ -twisted representations of $\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}$.

Fix $g \in \mathbf{G}$. Write $Z_{\hat{\mathbf{G}}}^{\varphi}(g)$ for the stabilizer of g under Real $\hat{\mathbf{G}}$ -conjugation. Fix an equivalence

$$\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}} \simeq \bigsqcup_{g \in \pi_0(\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}})} BZ_{\hat{\mathbf{G}}}^{\varphi}(g). \quad (24)$$

The connected components $\pi_0(\Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}})$ are identified with the set of Real conjugacy classes of \mathbf{G} . Denote by $\llbracket g \rrbracket_{\hat{\mathbf{G}}} \subset \mathbf{G}$ the Real conjugacy class of g . To prove Theorem

7.3 we first describe the action of $Z_{\hat{G}}^{\varphi}(g)$ on $\mathrm{Tr}_{\mathrm{RInd}_{\hat{H}}^{\hat{G}}(\rho)}(g)$. We require some notation. The decomposition

$$[[g]]_{\hat{G}} \cap \mathbf{H} = \bigsqcup_{i=1}^l [[h_i]]_{\hat{H}} \quad (25)$$

induces a decomposition

$$\{\sigma \in \mathcal{S} \mid \sigma^{-1}g^{\pi(\sigma)}\sigma \in \mathbf{H}\} = \bigsqcup_{i=1}^l \mathcal{S}_i$$

where $\mathcal{S}_i = \{\sigma \in \mathcal{S} \mid \sigma^{-1}g^{\pi(\sigma)}\sigma \in [[h_i]]_{\hat{H}}\}$. For each $i \in \{1, \dots, l\}$, fix an element $\sigma_i \in \mathcal{S}_i$. Relabel the representatives of the Real \hat{H} -conjugacy classes appearing in the decomposition (25) by $h_i = \sigma_i^{-1}g^{\pi(\sigma_i)}\sigma_i$.

Lemma 7.4 (cf. [15, Lemma 7.7]). *Elements of \mathcal{S}_i can be chosen so that left multiplication by σ_i^{-1} induces a bijection from \mathcal{S}_i to a complete system of representatives of $Z_{\hat{G}}^{\varphi}(h_i)/Z_{\hat{H}}^{\varphi}(h_i)$.*

Proof. The proof is nearly the same as that of [15, Lemma 7.7]; we include it for completeness. Let $\sigma \in \mathcal{S}_i$. Then $\sigma^{-1}g^{\pi(\sigma)}\sigma = \eta^{-1}h_i^{\pi(\eta)}\eta$ for some $\eta \in \hat{H}$. It follows that $\sigma\eta^{-1} = \sigma$ in \hat{G}/\hat{H} and $(\sigma\eta^{-1})^{-1}g^{\pi(\sigma\eta^{-1})}\sigma\eta^{-1} = h_i$ so that

$$(\sigma_i^{-1}\sigma\eta^{-1})^{-1}h_i^{\pi(\sigma_i^{-1}\sigma\eta^{-1})}\sigma_i^{-1}\sigma\eta^{-1} = h_i$$

and $\sigma_i^{-1}\sigma\eta^{-1} \in Z_{\hat{G}}^{\varphi}(h_i)$. Replacing σ with $\sigma\eta^{-1}$, we henceforth assume that $\sigma \in \mathcal{S}_i$ is such that $\sigma^{-1}g^{\pi(\sigma)}\sigma = h_i$ and $\sigma_i^{-1}\sigma \in Z_{\hat{G}}^{\varphi}(h_i)$.

Let $\sigma, \sigma' \in \mathcal{S}_i$ be distinct. Then $(\sigma_i^{-1}\sigma)^{-1}(\sigma_i^{-1}\sigma') = \sigma^{-1}\sigma'$ does not lie in \hat{H} . It follows that $\sigma_i^{-1}\sigma \neq \sigma_i^{-1}\sigma'$ in $Z_{\hat{G}}^{\varphi}(h_i)/Z_{\hat{H}}^{\varphi}(h_i)$, proving injectivity of the map under consideration. To prove surjectivity, let $\mu \in Z_{\hat{G}}^{\varphi}(h_i)$. Then $\sigma_i\mu = \sigma\eta$ for some $\sigma \in \mathcal{S}$ and $\eta \in \hat{H}$. We compute

$$\sigma^{-1}g^{\pi(\sigma)}\sigma = \eta h_i^{\pi(\eta)}\eta^{-1},$$

whence $\sigma \in \mathcal{S}_i$. Since $\sigma^{-1}g^{\pi(\sigma)}\sigma = h_i$, we also find that $\eta \in Z_{\hat{H}}^{\varphi}(h_i)$. So $\sigma_i^{-1}\sigma = \mu\eta^{-1}$, showing that $\sigma_i^{-1}\sigma = \mu$ in $Z_{\hat{G}}^{\varphi}(h_i)/Z_{\hat{H}}^{\varphi}(h_i)$. \square

Remark. The representatives \mathcal{S} of \hat{G}/\hat{H} can be chosen to be a subset of \mathbf{G} . Such a choice simplifies the description of induced Real 2-representations. However, it does not appear that there is a version of Lemma 7.4 which outputs a set of representatives which is again a subset of \mathbf{G} .

We henceforth assume that \mathcal{S} is chosen as in Lemma 7.4. We have

$$\mathrm{Tr}_{\mathrm{RInd}_{\hat{H}}^{\hat{G}}(\rho)}(g) = \bigoplus_{i=1}^l \bigoplus_{\{\sigma \in \mathcal{S}_i \mid \sigma^{-1}g\sigma \in \mathbf{H}\}} \sigma \cdot \mathrm{Tr}_{\pi(\sigma)\rho}(h_i^{\pi(\sigma)}).$$

If $\pi(\sigma) = -1$, then we have $\mathrm{Tr}_{\rho^{\mathrm{op}}}(h_i^{-1}) \simeq 2\mathrm{Hom}_{\mathrm{Cat}}(\rho(h_i^{-1}), 1_V)$. Define a map $F_i : 2\mathrm{Hom}_{\mathrm{Cat}}(\rho(h_i^{-1}), 1_V) \rightarrow 2\mathrm{Hom}_{\mathrm{Cat}}(1_V, \rho(h_i))$ by

$$\begin{array}{ccc} \begin{array}{c} h_i \\ \downarrow \\ \bullet \\ u \end{array} & \longmapsto & \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \quad \downarrow \\ \bullet \quad \bullet \\ u \quad h_i \end{array} \end{array}$$

Using equation (17), the inverse $F_i^{-1} : 2\mathrm{Hom}_{\mathrm{Cat}}(1_V, \rho(h_i)) \rightarrow 2\mathrm{Hom}_{\mathrm{Cat}}(\rho(h_i^{-1}), 1_V)$ is

$$h_i^{-1} \begin{array}{c} \bullet \\ \uparrow \\ v \end{array} \longmapsto \hat{\alpha}([h_i|h_i^{-1}|h_i]) \times \begin{array}{c} \bullet \\ \uparrow \\ v \\ \downarrow \\ h_i^{-1} \end{array}$$

Using the maps F_i , we obtain a vector space isomorphism

$$\mathrm{Tr}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\rho)}(g) \simeq \bigoplus_{i=1}^l \bigoplus_{\{\sigma \in \mathcal{S}_i | \sigma^{-1}g\sigma \in \mathbb{H}\}} \sigma \cdot \mathrm{Tr}_{\rho}(h_i).$$

Lemma 7.5 (cf. [15, Lemma 7.8]). *There is an isomorphism*

$$\mathrm{Tr}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\rho)}(g) \simeq \bigoplus_{i=1}^l \mathrm{Ind}_{Z_{\mathbb{H}}^{\hat{g}}(h_i)}^{Z_{\hat{\mathcal{G}}}^{\hat{g}}(g)}(\mathrm{Tr}_{\rho}(h_i))$$

of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ -twisted representations of $Z_{\hat{\mathcal{G}}}^{\hat{g}}(g)$, the induction being along the composition $Z_{\mathbb{H}}^{\hat{g}}(h_i) \hookrightarrow Z_{\hat{\mathcal{G}}}^{\hat{g}}(h_i) \xrightarrow{\iota \mapsto \sigma \iota \sigma^{-1}} Z_{\hat{\mathcal{G}}}^{\hat{g}}(g)$.

Proof. Let $\mu \in Z_{\hat{\mathcal{G}}}^{\hat{g}}(g)$ and $\sigma \in \mathcal{S}_i$. Then $\mu\sigma = \tilde{\sigma}\eta$ for some $\tilde{\sigma} \in \mathcal{S}$ and $\eta \in \hat{\mathbb{H}}$. It is straightforward to verify that in fact $\tilde{\sigma} \in \mathcal{S}_i$ and $\eta \in Z_{\hat{\mathbb{H}}}^{\hat{g}}(h_i)$. The equations

$$\mu^{-1}\tilde{\sigma} = \sigma\eta^{-1}, \quad g\sigma = \sigma h_i^{\pi(\sigma)}, \quad \mu\sigma = \tilde{\sigma}\eta$$

imply that μ acts on $\mathrm{Tr}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\rho)}(g)$ by a linear map

$$\xi_1(\mu) : \sigma \cdot \mathrm{Tr}_{\rho}(h_i) \rightarrow \tilde{\sigma} \cdot \mathrm{Tr}_{\rho}(h_i).$$

We claim that $\xi_1(\mu)$ is equal to $c_1 c_2 \cdot \beta_{h_i, \eta}$, where

$$c_1 = \left(\frac{\hat{\alpha}([g|\mu|\sigma])\hat{\alpha}([\tilde{\sigma}|h_i^{\pi(\tilde{\sigma})}|\eta])}{\hat{\alpha}([g|\tilde{\sigma}|\eta])} \right) \left(\frac{\hat{\alpha}([\mu|\sigma|h_i^{\pi(\mu\sigma)}])}{\hat{\alpha}([\mu|g^{\pi(\mu)}|\sigma])\hat{\alpha}([\tilde{\sigma}|\eta|h_i^{\pi(\mu\sigma)}])} \right) \times \left(\frac{\hat{\alpha}([g^{-1}|\sigma|h_i^{\pi(\sigma)}])}{\hat{\alpha}([g^{-1}|g|\sigma])\hat{\alpha}([\sigma|h_i^{-\pi(\sigma)}|h_i^{\pi(\sigma)}])} \right)^{-\frac{\pi(\mu)-1}{2}}$$

and

$$c_2 = \hat{\alpha}([h_i|h_i^{-1}|h_i])^{\delta_{\pi(\mu), \pi(\sigma), -1} - \delta_{\pi(\tilde{\sigma}), \pi(\eta), -1}} \left(\frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i^{-\pi(\eta)}|h_i^{\pi(\eta)}])}{\hat{\alpha}([h_i^{-1}|\eta|h_i^{\pi(\eta)}])} \right)^{-\frac{\pi(\tilde{\sigma})-1}{2}}.$$

Indeed, the factor c_1 is due to the scalars relating $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\psi)_{g^{-1}, g}^{-1}$, $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\psi)_{\mu, g^{\pi(\mu)}}$ and ${}^{\mathrm{op}}\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\psi)_{g^{-1}, g}$ to $\psi_{h_i^{\pi(\tilde{\sigma})}, \eta}^{-1}$, $\psi_{\eta, h_i^{\pi(\mu\sigma)}}$ and $\psi_{h_i^{-\pi(\sigma)}, h_i^{\pi(\sigma)}}$, respectively. Note that ${}^{\mathrm{op}}\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{g}}(\psi)_{g^{-1}, g}$ appears only when $\pi(\mu) = -1$. These maps contribute to $\xi_1(\mu)$ regardless of the degrees of σ and $\tilde{\sigma}$. The factor c_2 is related to the appearance of the maps $F_i^{\pm 1}$ and is best understood using string diagrams. For example, when μ

and η are of degree +1 and σ and $\tilde{\sigma}$ are of degree -1, we have

$$\xi_1(\mu)(\phi) = c_1 \cdot \text{Diagram}$$

This string diagram can be simplified as follows:

$$\begin{aligned} & \text{Diagram} \xrightarrow[\text{Eq. (16)}]{\hat{\alpha}([h_i^{-1}|\eta|\eta^{-1}])} \text{Diagram} \xrightarrow[\text{Eq. (13)}]{\hat{\alpha}([\eta|h_i^{-1}|\eta^{-1}]^{-1})} \\ & \text{Diagram} \xrightarrow[\text{Eq. (13)}]{\hat{\alpha}([h_i|\eta|\eta^{-1}h_i^{-1}])} \text{Diagram} \xrightarrow[\text{Eq. (13)}]{\hat{\alpha}([\eta h_i|h_i^{-1}|\eta^{-1}])} \\ & \text{Diagram} \xrightarrow[\text{Eq. (16)}]{\hat{\alpha}([\eta|h_i|h_i^{-1}]^{-1})} \text{Diagram} \xrightarrow[\text{Eq. (17)}]{\hat{\alpha}([h_i|h_i^{-1}|h_i])^{-1}} \text{Diagram} \end{aligned}$$

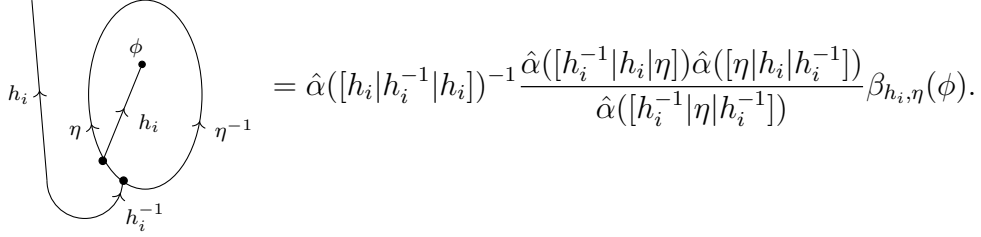
so that, after a short calculation, we arrive at

$$\text{Diagram} = \frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i^{-1}|h_i])}{\hat{\alpha}([h_i^{-1}|\eta|h_i])} \beta_{h_i,\eta}(\phi) = c_2 \beta_{h_i,\eta}(\phi).$$

Similarly, when μ, σ are of degree +1 and $\tilde{\sigma}, \eta$ are of degree -1, we have

$$\xi_1(\mu)(\phi) = c_1 \cdot \text{Diagram}$$

This string diagram can be evaluated in the same way as the previous diagram, the main difference being that the final step is not required, leading to an additional factor of $\hat{\alpha}([h_i|h_i^{-1}|h_i])^{-1}$:



$$= \hat{\alpha}([h_i|h_i^{-1}|h_i])^{-1} \frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i|h_i^{-1}])}{\hat{\alpha}([h_i^{-1}|\eta|h_i^{-1}])} \beta_{h_i,\eta}(\phi).$$

The coefficient of $\beta_{h_i,\eta}(\phi)$ is again c_2 . The remaining cases are dealt with similarly.

On the other hand, as

$$(\sigma_i^{-1}\mu\sigma_i)(\sigma_i^{-1}\sigma) = (\sigma_i^{-1}\tilde{\sigma})\eta,$$

the results of Section 6.1 imply that $\sigma_i^{-1}\mu\sigma_i \in Z_{\hat{\mathcal{G}}}^{\varphi}(h_i)$ acts on

$$\text{Ind}_{Z_{\hat{\mathcal{H}}}^{\varphi}(h_i)}^{Z_{\hat{\mathcal{G}}}^{\varphi}(h_i)}(\text{Tr}_{\rho}(h_i)) \simeq \bigoplus_{\sigma \in \mathcal{S}_i} \sigma_i^{-1}\sigma \cdot \text{Tr}_{\rho}(h_i)$$

by the linear map $\xi_2(\sigma_i^{-1}\mu\sigma_i) : \sigma_i^{-1}\sigma \cdot \text{Tr}_{\rho}(h_i) \rightarrow \sigma_i^{-1}\tilde{\sigma} \cdot \text{Tr}_{\rho}(h_i)$ given by

$$\xi_2(\sigma_i^{-1}\mu\sigma_i) = \frac{\theta_{h_i}([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}\sigma])}{\theta_{h_i}([\sigma_i^{-1}\tilde{\sigma}|\eta])} \beta_{h_i,\eta}.$$

Here $\theta_{\gamma}([\omega_2|\omega_1]) = \tau_{\pi}^{\text{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma)$. It follows from [33, §2.4.1] that

$$\xi_2(\mu) = \frac{\theta_g([\sigma_i^{-1}|\mu])}{\theta_g([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}])} \xi_2(\sigma_i^{-1}\mu\sigma_i).$$

Closedness of $\tau_{\pi}^{\text{ref}}(\hat{\alpha})$ then gives

$$\xi_2(\mu) = \frac{\theta_{h_i}([\mu|\sigma])}{\theta_{h_i}([\tilde{\sigma}|\eta])} \frac{\theta_{h_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{h_i}([\sigma_i^{-1}|\sigma])} \beta_{h_i,\eta}.$$

The explicit expression for $\tau_{\pi}^{\text{ref}}(\hat{\alpha})$ shows that $\frac{\theta_{h_i}([\mu|\sigma])}{\theta_{h_i}([\tilde{\sigma}|\eta])}$ is equal to c_1c_2 above. We therefore arrive at the equality

$$\xi_2(\mu) = \frac{\theta_{h_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{h_i}([\sigma_i^{-1}|\sigma])} \xi_1(\mu).$$

In other words, the diagram

$$\begin{array}{ccc} \sigma \cdot \text{Tr}_{\rho}(h_i) & \xrightarrow{\xi_1(\mu)} & \tilde{\sigma} \cdot \text{Tr}_{\rho}(h_i) \\ \tau_{\pi}^{\text{ref}}(\hat{\alpha})([\sigma_i^{-1}|\sigma]h_i) \downarrow & & \downarrow \tau_{\pi}^{\text{ref}}(\hat{\alpha})([\sigma_i^{-1}|\tilde{\sigma}]h_i) \\ \sigma_i^{-1}\sigma \cdot \text{Tr}_{\rho}(h_i) & \xrightarrow{\xi_2(\mu)} & \sigma_i^{-1}\tilde{\sigma} \cdot \text{Tr}_{\rho}(h_i) \end{array}$$

commutes. The vertical (scalar multiplication) maps of this diagram assemble to define the desired isomorphism $\text{Tr}_{\text{RInd}_{\hat{\mathcal{H}}}^{\varphi}(\rho)}(g) \xrightarrow{\sim} \bigoplus_{i=1}^l \text{Ind}_{Z_{\hat{\mathcal{H}}}^{\varphi}(h_i)}^{Z_{\hat{\mathcal{G}}}^{\varphi}(g)}(\text{Tr}_{\rho}(h_i))$. \square

Theorem 7.3 follows at once from Lemma 7.5.

Theorem 7.6. *The Real 2-character of $\mathrm{RInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)$ is given by*

$$\chi_{\mathrm{RInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)}(g, \omega) = \frac{1}{2|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega)\sigma^{-1} \in \hat{\mathbf{H}}^2}} \tau\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\sigma]g \xrightarrow{\omega} g)^{-1} \cdot \chi_{\rho}(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}).$$

Proof. By Theorem 7.3, it suffices to compute the character of $\mathrm{Ind}_{\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathbf{H}}}^{\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathbf{G}}}(\mathrm{Tr}(\rho))$. As the canonical functor $\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathbf{H}} \rightarrow \Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathbf{G}}$ is faithful, this character can be computed using Proposition 6.2. Doing so gives the desired result. \square

7.4. Hyperbolic 2-induction. We now turn to the categorified hyperbolic construction. Let $\hat{\mathbf{G}}$ be a finite \mathbb{Z}_2 -graded group. Fix $\hat{\alpha} \in Z^3(B\hat{\mathbf{G}}, k_{\pi}^{\times})$ and $\varsigma \in \hat{\mathbf{G}} \setminus \mathbf{G}$. Let ρ be a representation of $\mathcal{G} = \mathcal{G}(\mathbf{G}, \alpha)$ on V . The underlying category of $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)$ is $V \times V^{\mathrm{op}}$. The required 1-morphisms are

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(\varsigma^{-1}g\varsigma)^{\mathrm{op}} \end{pmatrix}, \quad g \in \mathbf{G}$$

and

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)(\omega) = \begin{pmatrix} 0 & \rho(\omega\varsigma) \\ \rho(\varsigma^{-1}\omega)^{\mathrm{op}} & 0 \end{pmatrix}, \quad \omega \in \hat{\mathbf{G}} \setminus \mathbf{G}.$$

The associativity 2-isomorphisms are

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{g_2, g_1} = \begin{pmatrix} \psi_{g_2, g_1} & 0 \\ 0 & \frac{\hat{\alpha}([g_2|\varsigma|\varsigma^{-1}g_1\varsigma])}{\hat{\alpha}([g_2|g_1|\varsigma])\hat{\alpha}([\varsigma|\varsigma^{-1}g_2\varsigma|\varsigma^{-1}g_1\varsigma])} \psi_{\varsigma^{-1}g_2\varsigma, \varsigma^{-1}g_1\varsigma}^{-\mathrm{op}} \end{pmatrix}$$

and

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{\omega_2, g_1} = \begin{pmatrix} \frac{\hat{\alpha}([\omega_2|\varsigma|\varsigma^{-1}g_1\varsigma])}{\hat{\alpha}([\omega_2|g_1|\varsigma])} \psi_{\omega_2\varsigma, \varsigma^{-1}g_1\varsigma} & 0 \\ 0 & \hat{\alpha}([\varsigma|\varsigma^{-1}\omega_2|g_1])^{-1} \psi_{\varsigma^{-1}\omega_2, g_1}^{-\mathrm{op}} \end{pmatrix}$$

and

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{g_2, \omega_1} = \begin{pmatrix} \hat{\alpha}([g_2|\omega_1|\varsigma])^{-1} \psi_{g_2, \omega_1\varsigma} & 0 \\ 0 & \frac{\hat{\alpha}([g_2|\varsigma|\varsigma^{-1}\omega_1])}{\hat{\alpha}([\varsigma|\varsigma^{-1}g_2\varsigma|\varsigma^{-1}\omega_1])} \psi_{\varsigma^{-1}g_2\varsigma, \varsigma^{-1}\omega_1}^{-\mathrm{op}} \end{pmatrix}$$

and

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{\omega_2, \omega_1} = \begin{pmatrix} \hat{\alpha}([\omega_2|\varsigma|\varsigma^{-1}\omega_1]) \psi_{\omega_2\varsigma, \varsigma^{-1}\omega_1} & 0 \\ 0 & \frac{1}{\hat{\alpha}([\omega_2|\omega_1|\varsigma])\hat{\alpha}([\varsigma|\varsigma^{-1}\omega_2|\omega_2\varsigma])} \psi_{\varsigma^{-1}\omega_2, \omega_1\varsigma}^{-\mathrm{op}} \end{pmatrix}$$

where $g_1, g_2 \in \mathbf{G}$ and $\omega_1, \omega_2 \in \hat{\mathbf{G}} \setminus \mathbf{G}$. These expressions are obtained by interpreting Real representations of \mathcal{G} as homotopy fixed points of $\mathrm{Rep}_{\mathrm{Cat}_k, k}(\mathcal{G})$ (see Section 3.4) and applying a categorified hyperbolic construction. In any case, it is straightforward to verify that this defines a linear Real representation of \mathcal{G} .

More generally, for a subgroup $\mathbf{H} \subset \mathbf{G}$, define $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}} = \mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}} \circ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$.

7.5. Hyperbolically induced Real categorical and 2-characters. We compute the Real categorical and 2-characters of $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)$. Since the method is similar to that of Section 7.3, we will at points be brief.

Theorem 7.7. *There is a canonical isomorphism*

$$\mathrm{Tr}(\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)) \simeq \mathrm{Ind}_{\Lambda_{BH}^{\mathrm{ref}} B\hat{\mathcal{G}}}^{\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathcal{G}}}(\mathrm{Tr}(\rho))$$

of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ -twisted representations of $\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathcal{G}}$.

It suffices to prove Theorem 7.7 under the assumption that $\mathbf{H} = \mathbf{G}$. Indeed, standard properties of induction for groupoids yield an isomorphism

$$\mathrm{Ind}_{\Lambda_{BH}^{\mathrm{ref}} B\mathbf{G}}^{\Lambda_{\pi}^{\mathrm{ref}} B\mathbf{G}}(\mathrm{Tr}_{\rho}(g)) \simeq \mathrm{Ind}_{\Lambda_{BG}^{\mathrm{ref}} B\mathbf{G}}^{\Lambda_{\pi}^{\mathrm{ref}} B\mathbf{G}}(\mathrm{Ind}_{\Lambda_{BH}^{\mathrm{ref}} B\mathbf{G}}^{\Lambda_{BG}^{\mathrm{ref}} B\mathbf{G}}(\mathrm{Tr}_{\rho}(g)))$$

of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ -twisted representations. On the other hand, $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}$ is by definition $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}} \circ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$ and Theorem 7.3 gives an isomorphism

$$\mathrm{Tr}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)) \simeq \mathrm{Ind}_{\Lambda_{BH}^{\mathrm{ref}} B\mathbf{G}}^{\Lambda_{BG}^{\mathrm{ref}} B\mathbf{G}}(\mathrm{Tr}(\rho)).$$

We therefore assume that $\mathbf{H} = \mathbf{G}$ for the remainder of this section.

We proceed as in the proof of Theorem 7.3. Fix again an equivalence of the form (24). Instead of (25) we consider a decomposition

$$[[g]]_{\mathbf{G}} = \bigsqcup_{i=1}^l [g_i]_{\mathbf{G}} \quad (26)$$

with $[g_i]_{\mathbf{G}} \subset \mathbf{G}$ the conjugacy class of g_i . We have $l \in \{1, 2\}$ according to whether or not the conjugacy class $[g_1]_{\mathbf{G}}$ is Real ($l = 1$) or non-Real ($l = 2$). The Real and non-Real cases have $Z_{\mathbf{G}}(g) \subsetneq Z_{\hat{\mathcal{G}}}^{\varphi}(g)$ and $Z_{\mathbf{G}}(g) = Z_{\hat{\mathcal{G}}}^{\varphi}(g)$, respectively. There is an induced decomposition

$$\mathcal{S} = \bigsqcup_{i=1}^l \mathcal{S}_i$$

with $\mathcal{S}_i = \{\sigma \in \mathcal{S} \mid \sigma^{-1}g^{\pi(\sigma)}\sigma \in [g_i]_{\mathbf{G}}\}$. We set $\sigma_1 = e$ and, in the non-Real case, $\sigma_2 = \varsigma$. Relabel the representatives of the conjugacy classes appearing in the decomposition (26) by $g_i = \sigma_i^{-1}g^{\pi(\sigma_i)}\sigma_i$.

The obvious analogue of Lemma 7.4 holds by inspection. Explicitly, in the Real case we take ς to be any element of $Z_{\hat{\mathcal{G}}}^{\varphi}(g) \setminus Z_{\mathbf{G}}(g)$. The maps F_i^{\pm} then yield the identification

$$\mathrm{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \mathrm{Tr}_{\rho}(g) \oplus \varsigma \cdot \mathrm{Tr}_{\rho}(g_2)$$

where, by convention, $g_2 = g$ in the Real case.

Lemma 7.8. *There is an isomorphism*

$$\mathrm{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \bigoplus_{i=1}^l \mathrm{Ind}_{Z_{\mathbf{G}}(g_i)}^{Z_{\hat{\mathcal{G}}}^{\varphi}(g)}(\mathrm{Tr}_{\rho}(g_i))$$

of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ -twisted representations of $Z_{\hat{\mathcal{G}}}^{\varphi}(g)$, the induction being along the composition

$$Z_{\mathbf{G}}(g_i) \hookrightarrow Z_{\hat{\mathcal{G}}}^{\varphi}(g_i) \xrightarrow{l \mapsto \sigma_i l \sigma_i^{-1}} Z_{\hat{\mathcal{G}}}^{\varphi}(g).$$

Proof. Consider first the non-Real case. Let $\mu \in Z_G^\varphi(g) = Z_G(g)$ and $\sigma \in \mathcal{S}_i$. Then $\mu\sigma = \tilde{\sigma}p$ for some $\tilde{\sigma} \in \mathcal{S}_i$ and $p \in Z_G(g_i)$. For $i = 1$ we have $\sigma = e = \tilde{\sigma}$ and μ corresponds to the map

$$\xi_1(\mu) : \mathrm{Tr}_\rho(g) \rightarrow \mathrm{Tr}_\rho(g)$$

given by $\beta_{g,p}$. For $i = 2$ we have $\sigma = \varsigma = \tilde{\sigma}$ and μ induces a map

$$\xi_1(\mu) : \varsigma \cdot \mathrm{Tr}_\rho(g_2) \rightarrow \varsigma \cdot \mathrm{Tr}_\rho(g_2).$$

which is equal to $c_1 c_2 \cdot \beta_{g_2,p}$, where

$$c_1 = \left(\frac{\hat{\alpha}([g_2^{-1}|p|\varsigma^{-1}])\hat{\alpha}([\varsigma^{-1}|g|\mu])}{\hat{\alpha}([g_2^{-1}|\varsigma^{-1}|\mu])} \right) \left(\frac{\hat{\alpha}([p|\varsigma^{-1}|g])}{\hat{\alpha}([p|g_2^{-1}|\varsigma^{-1}])\hat{\alpha}([\varsigma^{-1}|\mu|g])} \right)$$

and

$$c_2 = \frac{\hat{\alpha}([g_2^{-1}|g_2|p])\hat{\alpha}([p|g_2^{-1}|g_2])}{\hat{\alpha}([g_2^{-1}|p|g_2])}.$$

The factor c_1 is due to the scalars relating $\mathrm{HInd}_G^{\hat{G}}(\psi)_{g,\mu}^{-1}$ and $\mathrm{HInd}_G^{\hat{G}}(\psi)_{\mu,g}$ to $\psi_{g_2^{-1},p}^{\mathrm{op}}$ and $\psi_{p,g_2^{-1}}^{\mathrm{op}}$, respectively, while c_2 is due to the maps $F_i^{\pm 1}$, as in Lemma 7.5.

On the other hand, as

$$(\sigma_i^{-1}\mu\sigma_i)(\sigma_i^{-1}\sigma) = (\sigma_i^{-1}\tilde{\sigma})p$$

the element $\sigma_i^{-1}\mu\sigma_i \in Z_G(g_i)$ acts on $\mathrm{Ind}_{Z_G(g_i)}^{Z_G^\varphi(g_i)}(\mathrm{Tr}_\rho(g_i)) = \mathrm{Tr}_\rho(g_i)$ by the map

$$\xi_2(\sigma_i^{-1}\mu\sigma_i) = \frac{\theta_{g_i}([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}\sigma])}{\theta_{g_i}([\sigma_i^{-1}\tilde{\sigma}|p])} \beta_{g_i,p}.$$

Again, $\theta_\gamma([\omega_2|\omega_1]) = \tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma)$. As in Lemma 7.5, this leads to the expression

$$\xi_2(\mu) = \frac{\theta_{g_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{g_i}([\sigma_i^{-1}|\sigma])} \frac{\theta_{g_i}([\mu|\sigma])}{\theta_{g_i}([\tilde{\sigma}|p])} \beta_{g_i,p}.$$

Since $\sigma = \tilde{\sigma}$, we have $\frac{\theta_{g_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{g_i}([\sigma_i^{-1}|\sigma])} = 1$. When $i = 1$ this gives $\xi_2(\mu) = \beta_{g,p}$. When $i = 2$ the explicit expression for $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$ instead gives $\xi_2(\mu) = c_1 c_2 \cdot \beta_{g_2,p}$.

Consider now the Real case. Let $\mu \in Z_G^\varphi(g)$ and $\sigma \in \mathcal{S} = \mathcal{S}_1$. Then $\mu\sigma = \tilde{\sigma}p$ for some $\tilde{\sigma} \in \mathcal{S}$ and $p \in Z_G(g)$. It follows that the action of μ on $\mathrm{Tr}_{\mathrm{HInd}_G^{\hat{G}}(\rho)}(g)$ induces a linear map

$$\xi_1(\mu) : \sigma \cdot \mathrm{Tr}_\rho(g) \rightarrow \tilde{\sigma} \cdot \mathrm{Tr}_\rho(g)$$

of the form $c_1 c_2 \cdot \beta_{g,p}$. The factors c_1 and c_2 , whose explicit forms we omit, arise in the same way as above; note that c_1 may now receive contributions from ${}^{\mathrm{op}}\mathrm{HInd}_G^{\hat{G}}(\psi)_{g^{-1},g}$.

On the other hand, since $\mu\sigma = \tilde{\sigma}p$, the action of $\mu \in Z_G(g)$ on $\mathrm{Tr}_{\mathrm{HInd}_G^{\hat{G}}(\rho)}(g)$ induces a linear map $\xi_2(\mu) : \sigma \cdot \mathrm{Tr}_\rho(g) \rightarrow \tilde{\sigma} \cdot \mathrm{Tr}_\rho(g)$ given by

$$\xi_2(\mu) = \frac{\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\mu|\sigma]g)}{\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\tilde{\sigma}|p]g)} \beta_{g,p}.$$

For example, when $\pi(\mu) = -1$, $\sigma = \varsigma$ and $\tilde{\sigma} = e$, the coefficient of $\beta_{g,p}$ reads

$$\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\mu|\varsigma]g) = \hat{\alpha}([g|g^{-1}|g]) \frac{\hat{\alpha}([g^{-1}|\varsigma|g^{-1}])}{\hat{\alpha}([g^{-1}|g|\varsigma])\hat{\alpha}([\varsigma|g|g^{-1}])} \frac{\hat{\alpha}([\mu|\varsigma]g)\hat{\alpha}([g|\mu|\varsigma])}{\hat{\alpha}([\mu|g^{-1}|\varsigma])}.$$

The factor $\hat{\alpha}([g|g^{-1}|g])$ is c_2 while the remaining terms multiply to c_1 . Note that ${}^{\text{op}}\text{HInd}_{\hat{\mathbf{G}}}^{\hat{\mathbf{G}}}(\psi)_{g^{-1},g}$ contributes a factor of 1 to c_1 in this case. The other cases are treated in the same way. \square

Theorem 7.9. *The Real 2-character of $\text{HInd}_{\mathcal{H}}^{\hat{\mathbf{G}}}(\rho)$ is given by*

$$\chi_{\text{HInd}_{\mathcal{H}}^{\hat{\mathbf{G}}}(\rho)}(g, \omega) = \frac{1}{|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega)\sigma^{-1} \in \mathbf{H}^2}} \tau \tau_{\pi}^{\text{ref}}(\hat{\alpha})([\sigma]g \xrightarrow{\omega} g)^{-1} \cdot \chi_{\rho}(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}).$$

In particular, $\chi_{\text{HInd}_{\mathcal{H}}^{\hat{\mathbf{G}}}(\rho)}$ is supported on the subset $\pi_0(\Lambda^2 B\mathbf{G}) \subset \pi_0(\Lambda \Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}})$.

Proof. This is proved in the same way as Theorem 7.6, using Theorem 7.7 instead of Theorem 7.3. In the present case we apply Proposition 6.2 to the functor $\Lambda B\mathbf{H} \rightarrow \Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}$. This explains the coefficient $\frac{1}{|\mathbf{H}|}$, as opposed to $\frac{1}{2|\mathbf{H}|}$, in $\chi_{\text{HInd}_{\mathcal{H}}^{\hat{\mathbf{G}}}(\rho)}$. \square

7.6. Conjectural applications to Real Hopkins–Kuhn–Ravenel characters.

Let \mathbf{G} be a finite group. For each $n \geq 1$, denote by $\mathbf{G}^{(n)} \subset \mathbf{G}^n$ the subset of commuting n -tuples. The group \mathbf{G} acts on $\mathbf{G}^{(n)}$ by simultaneous conjugation. The groupoid $\mathbf{G}^{(n)} // \mathbf{G}$ is equivalent to the iterated loop groupoid $\Lambda^n B\mathbf{G}$. The space of locally constant functions on $\Lambda^n B\mathbf{G}$ valued in a ring S is denoted by $\text{Cl}_n(\mathbf{G}, S)$.

Fix a prime p and denote by E_n^{\bullet} the corresponding Morava E -theory. Let \mathbf{BG} be a classifying space of \mathbf{G} . Hopkins, Kuhn and Ravenel proved in [17, Theorem C] that there is a generalized character map²

$$E_n^0(\mathbf{BG}) \rightarrow \text{Cl}_{n,p}(\mathbf{G}, C_0)$$

which, after tensoring the source with C_0 , is an isomorphism. Here C_0 is a $p^{-1}E_n^0$ -algebra, constructed in [17, §6.2], which can be viewed as an E_n^0 -theoretic analogue of the field obtained from \mathbb{Q} by adjoining all roots of unity, the latter field being that which one would consider in classical character theory. The subscript p in $\text{Cl}_{n,p}$ indicates that we restrict attention to functions defined on commuting n -tuples of \mathbf{G} which have p^{th} power order. Given a subgroup $i : \mathbf{H} \hookrightarrow \mathbf{G}$, it was proved in [17, Theorem D] that the induced transfer map τi is

$$E_n^0(\tau i)(\chi)(g_1, g_2, \dots, g_n) = \frac{1}{|\mathbf{H}|} \sum_{\substack{g \in \mathbf{G} \\ g(g_1, g_2, \dots, g_n)g^{-1} \in \mathbf{H}^n}} \chi(gg_1g^{-1}, \dots, gg_ng^{-1}).$$

For $n \leq 3$ this is a p -completed version of the formula for induced n -characters; for $n = 1$ this is classical, while for $n = 2$ and $n = 3$ this is proved in [15, Corollary 7.6] and [32, Theorem 6.7], respectively.

Let now $\hat{\mathbf{G}}$ be a finite \mathbb{Z}_2 -graded group. Define

$$\hat{\mathbf{G}}^{(n)} = \{(g, \omega_2, \dots, \omega_n) \in \mathbf{G} \times \hat{\mathbf{G}}^{n-1} \mid \omega_i g = g^{\pi(\omega_i)} \omega_i, \omega_i \omega_j = \omega_j \omega_i, 2 \leq i, j \leq n\}.$$

The group $\hat{\mathbf{G}}$ acts on $\hat{\mathbf{G}}^{(n)}$ by Real conjugation on \mathbf{G} and conjugation on the remaining factors. The resulting groupoid $\hat{\mathbf{G}}^{(n)} // \hat{\mathbf{G}}$ is equivalent to the iterated loop groupoid $\Lambda^{n-1} \Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}$. Denote by $\text{Cl}_{\mathbb{R},n}(\mathbf{G}, S)$ the vector space of locally constant S -valued functions on $\Lambda^{n-1} \Lambda_{\pi}^{\text{ref}} B\hat{\mathbf{G}}$.

²More precisely, [17] treats a generalized character map $E_n^{\bullet}(\mathbf{BG}) \rightarrow \text{Cl}_{n,p}(\mathbf{G}, C_{\bullet})$.

Suppose now that $p = 2$. In this case, Hu and Kriz [20] constructed a Real version ER of Morava E -theory. The following conjecture concerns a generalized character theoretic description of $ER_n^0(\mathbf{BG})$.

Conjecture 7.10. *There exists a $p^{-1}ER_n^0$ -algebra \tilde{C}_0 and a Real n -character map*

$$\chi : ER_n^0(\mathbf{BG}) \rightarrow \mathrm{ClR}_{n,p}(\mathbf{G}, \tilde{C}_0)$$

which induces an isomorphism $ER_n^0(\mathbf{BG}) \otimes_{ER_n^0} \tilde{C}_0 \xrightarrow{\sim} \mathrm{ClR}_{n,p}(\mathbf{G}, \tilde{C}_0)$.

We outline an approach to the construction of χ . Associated to each (possibly empty) subset $J \subset \{2, \dots, n\}$ is a group homomorphism

$$f_J : \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}_2, \quad (\alpha_2, \dots, \alpha_n) \mapsto (-1)^{\sum_{j \in J} \alpha_j}.$$

Identify \mathbb{Z}_2 with $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{Z})$ and write $\mathbb{Z} \rtimes_J \mathbb{Z}^{n-1}$ for $\mathbb{Z} \rtimes_{f_J} \mathbb{Z}^{n-1}$. The group $\mathbb{Z} \rtimes_J \mathbb{Z}^{n-1}$ is canonically \mathbb{Z}_2 -graded; this grading is non-trivial when $J \neq \emptyset$, in which case the ungraded subgroup is isomorphic to \mathbb{Z}^n . Observe that there is a bijection between the set of \mathbb{Z}_2 -graded group homomorphisms $\mathbb{Z} \rtimes_J \mathbb{Z}^{n-1} \rightarrow \hat{\mathbf{G}}$ and the set of graded commuting tuples in $\mathbf{G} \times \prod_{j \notin J} \mathbf{G} \times \prod_{j \in J} \hat{\mathbf{G}} \setminus \mathbf{G}$.

Note that, as $p = 2$, we can replace \mathbb{Z} with the p -adic integers $\mathbb{Z}_{(p)}$ in the above paragraph. Let $(g_1, \omega_2, \dots, \omega_n) \in \hat{\mathbf{G}}^{(n)}$ with p th power order, interpreted as a morphism $f : \mathbb{Z}_{(p)} \rtimes_J \mathbb{Z}_{(p)}^{n-1} \rightarrow \hat{\mathbf{G}}$ of \mathbb{Z}_2 -graded groups. We obtain a map

$$ER_n^\bullet(\mathbf{BG}) \xrightarrow{Bf^*} ER_n^\bullet(B\mathbb{Z}_{(p)}^n; J)$$

if $J \neq \emptyset$ and

$$ER_n^\bullet(\mathbf{BG}) \xrightarrow{Bf^*} E_n^\bullet(B\mathbb{Z}_{(p)}^n)$$

if $J = \emptyset$. These morphisms factor through $ER_n^\bullet(B\mathbb{Z}_{p^i}^n; J)$ or $E_n^\bullet(B\mathbb{Z}_{p^i}^n)$ for sufficiently large $i \geq 1$. Suppose that, for each non-empty subset J , there exists a compatible system of morphisms $\varphi_i^J : ER_n^0(B\mathbb{Z}_{p^i}^n; J) \rightarrow \tilde{C}_0$. In the setting of [17], the existence of the analogous maps follows from the Lubin–Tate construction. We can then define a map

$$\chi_{(g_1, \omega_2, \dots, \omega_n)} : ER_n^0(\mathbf{BG}) \xrightarrow{Bf^*} ER_n^0(B\mathbb{Z}_{(p)}^n, J) \xrightarrow{\varphi_i^J} \tilde{C}_0.$$

By adjunction, we would then obtain the desired Real n -character map

$$\chi : ER_n^0(\mathbf{BG}) \rightarrow \mathrm{Map}_{\hat{\mathbf{G}}} \left(\bigsqcup_{J \subset \{2, \dots, n\}} \mathrm{Hom}_{\mathrm{Grp}/\mathbb{Z}_2}(\mathbb{Z}_{(p)} \rtimes_J \mathbb{Z}_{(p)}^{n-1}, \hat{\mathbf{G}}), \tilde{C}_0 \right) = \mathrm{ClR}_{n,p}(\mathbf{G}, \tilde{C}_0).$$

We can also incorporate into our conjecture various transfer-type maps. For each \mathbf{G} there is a forgetful map $ER_n^\bullet(\mathbf{BG}) \rightarrow E_n^\bullet(\mathbf{BG})$ which, in terms of n -characters, we expect to be restriction along the inclusion $\Lambda^n \mathbf{BG} \rightarrow \Lambda^{n-1} \Lambda_n^{\mathrm{ref}} \mathbf{B}\hat{\mathbf{G}}$. More interesting is the transfer map

$$ER_n^\bullet(\tau \hat{i}) : ER_n^\bullet(\mathbf{BH}) \rightarrow ER_n^\bullet(\mathbf{BG})$$

associated to a \mathbb{Z}_2 -graded subgroup $\hat{i} : \hat{\mathbf{H}} \hookrightarrow \hat{\mathbf{G}}$ which reduces to Real induction for p -completed KR -theory when $n = 1$. Similarly, given a subgroup $i : \mathbf{H} \hookrightarrow \mathbf{G}$, there is a map

$$ER_n^\bullet(hi) : E_n^\bullet(\mathbf{BH}) \rightarrow ER_n^\bullet(\mathbf{BG})$$

which reduces to the hyperbolic map from p -completed K - to KR -theory when $n = 1$.

Motivated by the results of Sections 6.2, 6.3 and 7.3, 7.5, which should be viewed as being of height $n = 1$ and $n = 2$, respectively, we make the following conjecture.

Conjecture 7.11. *Assume that Conjecture 7.10 holds.*

(1) *The Real transfer map $E\mathbb{R}_n^0(\tau\hat{i}) : E\mathbb{R}_n^0(\mathbf{BH}) \rightarrow E\mathbb{R}_n^0(\mathbf{BG})$ is given by*

$$E\mathbb{R}_n^0(\tau\hat{i})(\chi)(g, \omega_2, \dots, \omega_n) = \frac{1}{2|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega_2, \dots, \omega_n)\sigma^{-1} \in \hat{\mathbf{H}}^n}} \chi(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega_2 \sigma^{-1}, \dots, \sigma \omega_n \sigma^{-1}).$$

(2) *The hyperbolic transfer map $E_n^0(hi) : E_n^0(\mathbf{BH}) \rightarrow E\mathbb{R}_n^0(\mathbf{BG})$ is given by*

$$E_n^0(hi)(\chi)(g, \omega_2, \dots, \omega_n) = \frac{1}{|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega_2, \dots, \omega_n)\sigma^{-1} \in \mathbf{H}^n}} \chi(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega_2 \sigma^{-1}, \dots, \sigma \omega_n \sigma^{-1}).$$

Both conjectures admit natural twisted generalizations. In view of the twists of KR -theory [8] and E -theory [27], it is natural to expect that a cocycle $\hat{\beta} \in Z^n(B\hat{\mathbf{G}}, k_\pi^\times)$ defines a twisted Real theory $E\mathbb{R}_n^{\bullet+\hat{\beta}}(B\mathbf{G})$. In this setting, $\text{Cl}\mathbb{R}_{n,p}(\mathbf{G}, \tilde{C}_0)$ is replaced by $\text{Cl}\mathbb{R}_{n,p}^{\hat{\beta}}(\mathbf{G}, \tilde{C}_0)$, the space of flat sections of the line bundle $\tau^{n-1}\tau_\pi^{\text{ref}}(\hat{\beta}) \rightarrow \Lambda^{n-1}\Lambda_\pi^{\text{ref}}B\hat{\mathbf{G}}$ over \tilde{C}_0 . Such sections admit an explicit description as \tilde{C}_0 -valued functions on $\hat{\mathbf{G}}^{(n)}$ with prescribed $\hat{\beta}$ -dependent $\hat{\mathbf{G}}$ -equivariance. The first part of Conjecture 7.11, for example, becomes the conjecture that $E\mathbb{R}_n^{0+\hat{\beta}}(\tau\hat{i})(\chi)(g, \omega_2, \dots, \omega_n)$ is equal to

$$\frac{1}{2|\mathbf{H}|} \sum_{\substack{\sigma \in \hat{\mathbf{G}} \\ \sigma(g, \omega_2, \dots, \omega_n)\sigma^{-1} \in \hat{\mathbf{H}}^n}} \tau^{n-1}\tau_\pi^{\text{ref}}(\hat{\beta})([\sigma](g, \omega_2, \dots, \omega_n))^{-1} \times \chi(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega_2 \sigma^{-1}, \dots, \sigma \omega_n \sigma^{-1}).$$

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