ENTROPY-BOUNDED SOLUTIONS TO THE ONE-DIMENSIONAL HEAT CONDUCTIVE COMPRESSIBLE NAVIER–STOKES EQUATIONS WITH FAR FIELD VACUUM

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Abstract. In the presence of vacuum, the physical entropy for polytropic gases behaves singularly and it is thus a challenge to study its dynamics. It is shown in this paper that the boundedness of the entropy can be propagated up to any finite time provided that the initial vacuum presents only at far fields with sufficiently slow decay of the initial density. More precisely, for the Cauchy problem of the onedimensional heat conductive compressible Navier–Stokes equations, the global well-posedness of strong solutions and uniform boundedness of the corresponding entropy are established, as long as the initial density vanishes only at far fields with a rate no more than $O(\frac{1}{x^2})$. The main tools of proving the uniform boundedness of the entropy are some singularly weighted energy estimates carefully designed for the heat conductive compressible Navier–Stokes equations and an elaborate De Giorgi type iteration technique for some classes of degenerate parabolic equations. The De Giorgi type iterations are carried out to different equations in establishing the lower and upper bounds of the entropy.

1. Introduction

1.1. **The compressible Navier–Stokes equations.** The one dimensional heat conductive compressible Navier–Stokes equations for the polytropic gases are:

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$\rho(u_t + uu_x) - \mu u_{xx} + p_x = 0, \tag{1.2}$$

$$\rho(e_t + ue_x) + pu_x - \kappa\theta_{xx} = \mu|u_x|^2, \tag{1.3}$$

where the density $\rho \geq 0$, the velocity $u \in \mathbb{R}$, and the absolute temperature $\theta \geq 0$ are the unknowns, and the specific internal energy e and the pressure p are expressed as

$$e = c_v\theta, \quad p = R\rho\theta,$$

with R and c_v being positive constants, μ and κ are the viscous and heat conductive coefficients, respectively, which are assumed to be positive constants.

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In terms of ϑ, the energy equation becomes

$$c_v\rho(\theta_t + u\theta_x) + pu_x - \kappa\theta_{xx} = \mu|u_x|^2. \tag{1.4}$$

The entropy s is defined by the Gibb's equation $\theta Ds = De + pD(\frac{1}{\rho})$. The following equations of state hold:

$$p = Ae^{\frac{s}{c_v}}\rho^\gamma, \quad s = c_v\left(\log\frac{R}{A} + \log\theta - (\gamma-1)\log\rho\right), \tag{1.5}$$

where $\gamma - 1 = \frac{R}{c_v}$ and $\gamma > 1$. The entropy s satisfies

$$\rho(s_t + us_x) - \kappa\left(\frac{\theta_x}{\theta}\right)_x = \frac{1}{\theta}\left(\mu|u_x|^2 + \kappa\frac{|\theta_x|^2}{\theta}\right), \tag{1.6}$$

at the place where both ρ and θ are positive.

The compressible Navier–Stokes equations have been studied extensively. In the absence of vacuum, that is, the density is uniformly positive, local well-posedness of classic or strong solutions was first proved by Nash in [32] long time ago, and later by many mathematicians, see, e.g., [17, 29, 35, 36, 38]. However, the global existence of classic or strong solutions with arbitrary large initial data is not known generally. Only the one-dimensional theory is quite satisfactory: global well-posedness of strong solutions was proved by Kazhikhov–Shelukhin [21] and Kazhikhov [20]; global well-posedness in the framework of weak solutions can be also proved, see, e.g., [1, 19, 42, 43]; large time behavior of solutions with large initial data was recently proved in [25]. Compared with the one-dimensional case, the multidimensional case is much more complicated, and up to now, essentially only for the cases that the initial data is around some non-vacuum equilibrium, the global well-posedness is well understood. The results along this direction were first obtained by Matsumura–Nishida [30, 31], and later developed by many mathematicians, see, e.g., [2, 3, 7–10, 13, 22, 33, 37].

One major difference between the one-dimensional and multidimensional cases for the compressible Navier–Stokes equations is the possible formation of vacuum. As shown by Hoff-Smoller [14], for the 1D compressible Navier–Stokes equations, if there is no vacuum initially, then no vacuum will be formed later in finite time, while such a result is still open for the multidimensional case. The possible formation of vacuum is one of the main challenges.

In the presence of vacuum, the study of the compressible Navier–Stokes equations becomes much more difficult than the non-vacuum case due to the degeneracy of the system. Global existence of weak solutions to the isentropic fluids with possible vacuum was first initiated by Lions [28], and later improved by Feireisl–Novotný–Petzeltová [11] and further by Jiang–Zhang [18]. For the full case, global existence of variational weak solutions was proved by Feireisl [12] for special equations of state. Local well-posedness of strong solutions was proved in [4–6, 34]. Global existence of strong solutions, of small energy but allowing large oscillations and vacuum, was first proved by Huang–Li–Xin [16] for the isentropic case, and generalized later by the authors in [15, 24, 39] for the full case.

There are some substantial differences in the mathematical theories for the compressible Navier–Stokes equations between the vacuum and non-vacuum cases. First, in the absence of vacuum, the well-posedness holds in both the homogeneous and inhomogeneous spaces, but it is not necessarily true if the vacuum appears. In fact, if the density is compactly supported, then the well-posedness holds in the homogeneous spaces, see, e.g., [4–6, 15, 16, 39], but not in the inhomogeneous spaces, see Li–Wang–Xin [23], while if the density tends to zero sufficiently slowly at the far field, then the well-posedness holds in both the homogeneous and inhomogeneous spaces, see the recent work by the authors [26]. Second, the solution spaces guarantee the uniform boundedness of the entropy for the non-vacuum case, but may fail for the vacuum case. In fact, it follows from the blowup results of Xin [40] and Xin–Yan [41] that the corresponding entropy in [15, 39] must be unbounded, if initially there is an isolated mass group surrounded by the vacuum region.

Due to the lack of the expression of the entropy in the vacuum region and the high singularity and degeneracy of the entropy equation close to the vacuum region, in spite of its importance, the mathematical analysis of the entropy for the viscous compressible fluids in the presence of vacuum was rarely carried out before. In this paper, we continue our studies, initiated in [26], on the uniform boundedness of the entropy for the full compressible Navier–Stokes equations in the presence of vacuum. Different from the non heat conductive case in [26], for the heat conductive case, one may only need to deal with the the far field vacuum, as the heat conductivity will make the temperature strictly positive everywhere after the initial time, which implies that the entropy becomes unbounded instantaneously if the interior vacuum occurs initially. However, positive heat conductivity leads to both increase and decrease of the entropy and thus creates substantial difficulties in the analysis compared with [26].

The results of this paper are stated and proved in the Lagrangian coordinates, see Section 1.2; however, since the solutions being established are Lipschitz continuous, all results can be transformed accordingly in the Euler coordinates.

1.2. **Main results and key ideas of the analysis.** Let y be the Lagrangian coordinate and define the coordinate transform between y and the Euler coordinate x as $x = \eta(y,t)$ with $\eta(y,t)$ satisfying

$$\begin{cases} \partial_t \eta(y,t) = u(\eta(y,t),t), \\ \eta(y,0) = y. \end{cases}$$

Denote

$$\varrho(y,t) := \rho(\eta(y,t),t), \quad v(y,t) := u(\eta(y,t),t), \quad \vartheta(y,t) := \theta(\eta(y,t),t),$$

and

$$J := J(y,t) = \eta_y(y,t).$$

Then,

$$J_t = v_y, \quad J|_{t=0} \equiv 1, \quad J\varrho = \varrho_0.$$

Thus, in the Lagrangian coordinates, the system (1.1), (1.2), and (1.4) becomes

$$J_t = v_y, \tag{1.7}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y + R\left(\frac{\varrho_0}{J}\theta\right)_y = 0, \tag{1.8}$$

$$c_v\varrho_0\vartheta_t - \kappa\left(\frac{\vartheta_y}{J}\right)_y + R\frac{\varrho_0}{J}\vartheta v_y = \mu\frac{|v_y|^2}{J}. \tag{1.9}$$

The initial date will be taken as

$$(J, v, \vartheta)|_{t=0} = (J_0, v_0, \vartheta_0), \tag{1.10}$$

where J_0 has uniform positive lower and upper bounds.

It should be emphasized that here J is deliberately chosen to replace ϱ as one of the unknowns of the system, which is one of the main technical differences between the current paper and the classic works [20, 21]. Note that, by the definition of J, the initial J_0 should be identically one; however, for the aim of extending a local solution (J, v, ϑ) to be a global one, one needs the local well-posedness of solutions to the system (1.7)–(1.9) with initial J_0 not being identically one.

In the Lagrangian coordinates, the entropy can be expressed as

$$s \;=\; c_v\left(\log\frac{R}{A} + (\gamma-1)\log J + \log\vartheta - (\gamma-1)\log\varrho_0\right). \tag{1.11}$$

The effective viscous flux G, defined as

$$G := \mu\frac{v_y}{J} - R\frac{\varrho_0\vartheta}{J}, \tag{1.12}$$

is useful for proving the global existence of solutions, which satisfies

$$G_t - \frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\frac{\kappa(\gamma-1)}{J}\left(\frac{\vartheta_y}{J}\right)_y - \gamma\frac{v_y}{J}G. \tag{1.13}$$

The following conventions will be used throughout this paper. For $1 \leq q \leq \infty$ and positive integer m, $L^q = L^q(\mathbb{R})$ and $W^{1,q} = W^{m,q}(\mathbb{R})$ denote the standard Lebesgue and Sobolev spaces, respectively, and $H^m = W^{m,2}$. For simplicity, L^q and H^m denote also their N product spaces $(L^q)^N$ and $(H^m)^N$, respectively. $\|u\|_q$ is the L^q norm of u, and $\|(f_1, f_2, \cdots, f_n)\|_X$ is the sum $\sum_{i=1}^N \|f_i\|_X$ or the equivalent norm $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$.

The definition of the solutions being considered in this paper is given as follows:

Definition 1.1. *Given a positive time $\mathcal{T}$ and assume that*

$$\begin{cases} 0 < \varrho_0 \in W^{1,\infty}(\mathbb{R}), \quad \underline{J} \leq J_0 \in L^\infty(\mathbb{R}), \quad \vartheta_0 \geq 0, \\ \sqrt{\varrho_0}v_0, \sqrt{\varrho_0}v_0^2, \sqrt{\varrho_0}\vartheta_0, \sqrt{\varrho_0}J_0', v_0', \varrho_0^{\frac{3}{2}}\vartheta_0' \in L^2(\mathbb{R}), \end{cases} \tag{H0}$$

where $\underline{J}$ is a positive constant. A triple (J, v, ϑ) is called a solution to the system (1.7)–(1.9), subject to (1.10), in $\mathbb{R}\times(0,T)$, if it has the regularities

$$0 < J \in L^\infty(\mathbb{R}\times(0,\mathcal{T})), \quad 0 \le \vartheta \in L^\infty(\mathbb{R}\times(0,\mathcal{T})),$$
$$J_t, \sqrt{\varrho_0}J_y, \sqrt{\varrho_0}v, \sqrt{\varrho_0}v^2, v_y, \sqrt{\varrho_0}\vartheta, \varrho_0^{\frac{3}{2}}\vartheta_y \in L^\infty(0,\mathcal{T};L^2(\mathbb{R})),$$
$$\sqrt{\varrho_0}J_{yt}, vv_y, \sqrt{\varrho_0}v_t, \sqrt{\varrho_0}v_{yy}, \vartheta_y, \varrho_0(\tfrac{\vartheta_y}{J})_y, \varrho_0^2\vartheta_t \in L^2(0,\mathcal{T};L^2(\mathbb{R})),$$

satisfies (1.7)–(1.9) a.e. in $\mathbb{R}\times(0,\mathcal{T})$, and fulfills the initial condition (1.10).

Remark 1.1. *It can be checked easily that (J, v, ϑ) in Definition 1.1 has the regularities*

$$J \in C([0,\mathcal{T}];H^1((-R,R))), \quad J_t \in L^2(0,\mathcal{T};H^1((-R,R))),$$
$$v,\vartheta \in C([0,\mathcal{T}];H^1((-R,R))) \cap L^2(0,\mathcal{T};H^2((-R,R))),$$
$$v_t,\vartheta_t \in L^2(0,\mathcal{T};L^2((-R,R))),$$

for any $R>0$ and, in particular, $(J,v,\vartheta)|_{t=0}$ is well-defined.

The main results of this paper are summarized in the following theorems, whose precise statements will be given in the subsequent sections, and the major ideas of the proofs are sketched here.

First, the following well-posedness results hold.

Theorem 1.1. *(i) Assume that (H0) holds. Then there is a local solution (J, v, ϑ) to the system (1.7)–(1.9) with initial data (1.10).*

(ii) Under the additional assumption that

$$\left(\frac{1}{\sqrt{\varrho_0}}\right)' \in L^\infty(\mathbb{R}), \quad \varrho_0 \in L^1(\mathbb{R}), \quad \sqrt{\varrho_0}\vartheta_0' \in L^2(\mathbb{R}) \tag{H1}$$

the solution (J, v, ϑ) established in (i) is unique and exists globally in time.

The local existence part of Theorem 1.1 can be proven in the standard way. For the global existence, one may try to follow the arguments for the non-vacuum case in [21]. Unfortunately, it does not work directly here. Indeed, one of the key observations used in [21] is the following inequality (see (3.11) there)

$$m_\varrho(t) \ge C\left[1+\int_0^t M_\vartheta(\tau)d\tau\right]^{-1}, \tag{1.14}$$

where m_ϱ and M_ϑ are the lower bound of ϱ and upper bound of ϑ, respectively, which is employed to obtain the $L^\infty(0,T;L^2)$ type a priori estimates (see (4.7) in [21]) and consequently the high order estimates. However, (1.14) fails in the presence of vacuum where $m_\varrho \equiv 0$ and M_ϑ is finite.

The key step of proving the global existence here is to get the a priori $L^\infty(0,T;L^2)$ estimate of $(\sqrt{\varrho_0}v^2, \sqrt{\varrho_0}\vartheta)$ and upper bound of J. These are achieved by the L^2 type energy estimate for $E := \frac{v^2}{2} + c_v\vartheta$ and the observation that $J = B(J_0 + \frac{R}{\mu}\int_0^t \frac{\varrho_0\vartheta}{B}d\tau)$

for some function B having positive lower and upper bounds (see Proposition 4.2, below), which, in particular, implies

$$\|J\|_\infty \le C\left(1+\int_0^t \|\varrho_0\vartheta\|_\infty d\tau\right).$$

It is noted that this inequality holds for both the vacuum and non-vacuum cases, and it reduces to (1.14) for the non-vacuum case.

Now, we turn to the major issue of this paper: the uniform boundedness of the entropy. For the lower bound, we need the following key assumption:

$$\left(\frac{1}{\varrho_0}\right)'' \in L^\infty(\mathbb{R}). \tag{H2}$$

Theorem 1.2. *Under the assumptions (H0)–(H2), the entropy of the solution in Theorem 1.1 is uniformly bounded from below, up to any finite time, as long as it holds initially.*

Note that the entropy s satisfies

$$c_v\varrho_0 s_t - \kappa\left(\frac{s_y}{J}\right)_y = \kappa R\left(\frac{\varrho_0'}{J\varrho_0} - \frac{J_y}{J^2}\right)_y + \frac{c_v}{J\vartheta}\left(\mu|v_y|^2 + \frac{\kappa}{\vartheta}|\vartheta_y|^2\right). \tag{1.15}$$

So in the non-heat conductive case, $\kappa = 0$, the entropy can only increase in time and thus is bounded from below trivially, while the upper bound of the entropy is achieved by carrying a certain class of singular type energy estimates in [26]. However, in the general case $\kappa > 0$, the term $\kappa R\big(\frac{\varrho_0'}{J\varrho_0} - \frac{J_y}{J^2}\big)_y$ may cause both the increasing and decreasing of s and gives some major technical difficulties to get the uniform bounds on s. In particular, though the idea of estimating the entropy by singularly weighted energy estimates may still be useful here, yet it is not enough to yield the uniform bounds for the entropy. Some additional ideas are needed for the heat conductive case. Indeed, here are some new key observations:

For the uniform lower bound of s, it suffices to estimate a new quantity $S := \log\vartheta - (\gamma-1)\log\varrho_0$, which can be shown to satisfy

$$c_v\varrho_0 S_t - \kappa\left(\frac{S_y}{J}\right)_y = F_{gd} + F_{ok} + F_{bd}, \tag{1.16}$$

where $F_{gd} = \varrho_0 f_{gd}$ and $F_{ok} = \varrho_0 f_{ok}$ for some $f_{gd} \in L^\infty(0,T;L^2)$ and $f_{ok} \in L^\infty(0,T;L^\infty)$, while F_{bd} is given by

$$F_{bd} = \frac{\mu}{J\vartheta}\left(v_y - \frac{R}{2\mu}\varrho_0\vartheta\right)^2 + \kappa\frac{|\vartheta_y|^2}{J\vartheta^2}. \tag{1.17}$$

The uniform lower bound of S is achieved by applying some modified De Giorgi type iterations to (1.16). Note that F_{bd} is nonnegative and thus causes no difficulty in proving the uniform lower bound of S. The contributions due to the source term $F_{ok} = \varrho_0 f_{ok}$ are dealt with by introducing an auxiliary function $\tilde{S} := S + Mt$, with a sufficient large M, which satisfies a similar equation as S, but with the term corresponding to F_{ok} having desired sign. To deal with the source term F_{gd}, one notes that $\frac{F}{\varrho_0} \in L^\infty(0,T;L^2)$ is sufficient to get the lower bound of the solution to the model equation $\varrho_0 V_t - V_{yy} = F$, by applying a modified De Giorgi type iteration. Thus, since $\frac{F_{gd}}{\varrho_0} \in L^\infty(0,T;L^2)$, the contributions due to the term F_{gd} can also be handled.

Technically, due to the degeneracy of equation (1.16), different from the classic De Giorgi iteration for uniform parabolic equations, the testing function used in our iteration is $\frac{(S-\ell)_-}{\varrho_0}$ instead of $(S-\ell)_-$. In other words, our energy estimates needed in the De Giorgi iteration should be of singular type, to which our idea of singular energy estimates in [26] will be useful here. Moreover, due to the unboundedness of the domain and the lack of integrability of S, some suitable cut-off and delicate approximations will be used to justify rigorously the arguments, see Proposition 5.1 in Section 5.

For the upper bound of the entropy, we need also the following compatibility condition:

$$\varrho_0^{\frac{1-\gamma}{2}} v_0, \varrho_0^{1-\frac{\gamma}{2}} \vartheta_0, \varrho_0^{-\frac{\gamma}{2}} G_0 \in L^2(\mathbb{R}). \tag{HS}$$

Theorem 1.3. *Under the conditions (H0)–(H2) and (HS), the entropy of the unique solution in Theorem 1.1 is uniformly bounded from above, up to any finite time, as long as it holds initially.*

As J is uniformly positive, a necessary and sufficient condition for the uniform boundedness of the entropy is that ϑ tends to zero at the same rate as $\varrho_0^{\gamma-1}$ at the far field, which unfortunately is not guaranteed by the solution spaces used in [4–6, 11, 12, 16, 18, 23, 28, 34, 39]. Indeed, the solutions established in these papers have the L^2 integrability of $\sqrt{\varrho_0}\vartheta$, but not of ϑ itself, which allows ϑ not to decay to zero or even to grow to infinity at the far field.

Due to the singular term $\frac{c_v}{J\vartheta}\left(\mu|v_y|^2 + \frac{\kappa}{\vartheta}|\vartheta_y|^2\right)$ in (1.15), performing the same type of De Giorgi iteration to (1.15) as before will not lead to the desired upper bound for the entropy. In fact, for this case, instead of working on the entropy equation ((1.15)or (1.16)) directly, we will apply a modified De Giorgi iteration to the temperature equation, with some elaborate singular type energy estimates. The main steps can be sketched as follows. Note that the entropy has uniform upper bound iff

$$\vartheta_\ell := \vartheta - \ell\varrho_0^{\gamma-1}e^{Mt} \le 0, \quad \text{or equivalently} \quad (\vartheta_\ell)_+ = 0,$$

for some positive numbers ℓ and M. ϑ_ℓ satisfies

$$c_v\varrho_0\partial_t\vartheta_\ell - \kappa\partial_y\left(\frac{\partial_y\vartheta_\ell}{J}\right) = v_yG + \text{“other terms”}.$$

Testing the above equation with $\varrho_0^{1-2\gamma}(\vartheta_\ell)_+$ yields

$$\frac{c_v}{2}\frac{d}{dt}\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\|_2^2 + \kappa\|\varrho_0^{\frac{1}{2}-\gamma}\partial_y(\vartheta_\ell)_+\|_2^2$$

$$\leq \ C\int_{\mathbb{R}}(|\varrho_0^{-\frac{\gamma}{2}}G|^2+|\varrho_0^{1-\frac{\gamma}{2}}\vartheta|^2)\varrho_0^{1-\gamma}\vartheta_\ell dy+\text{“other terms”},$$

see Proposition 6.3, below. It should be noted here that the choice of the singularly weighted test function $\varrho_0^{1-2\gamma}(\vartheta_\ell)_+$ is crucial. The above inequality indicates the necessity of carrying out the energy estimates for $\varrho_0^{1-\frac{\gamma}{2}}\vartheta$ and $\varrho_0^{-\frac{\gamma}{2}}G$; these estimates, thanks to the assumption (H1), can be achieved by testing (1.8), (1.9), and (1.13) with $\varrho_0^{-\gamma}v$, $\varrho_0^{1-\gamma}\vartheta$, and $J\varrho_0^{-\gamma}G$, respectively, see Propositions 6.1 and 6.2, below. With these estimates in hand, one can proceed the iteration to get finally $(\vartheta_\ell)_+\equiv 0$ for some positive ℓ, which yields the desired upper bound of the entropy.

Some remarks are in order.

Remark 1.2. *(i) Conditions $(\frac{1}{\sqrt{\varrho_0}})', (\frac{1}{\varrho_0})''\in L^\infty(\mathbb{R})$ in (H1)–(H2) are essentially slow decay assumptions on ϱ_0 at the far field. In fact, for $\varrho_0(y)=\frac{K_\varrho}{\langle y\rangle^{\ell_\varrho}}$, with $\langle y\rangle=(1+y^2)^{\frac12}$ and positive constants K_ϱ and ℓ_ϱ, it holds that*

$$\left(\frac{1}{\sqrt{\varrho_0}}\right)'\in L^\infty\Leftrightarrow 0\leq \ell_\varrho\leq 2 \quad \textit{and} \quad \left(\frac{1}{\varrho_0}\right)''\in L^\infty\Leftrightarrow 0\leq \ell_\varrho\leq 2.$$

(ii) All results in the above theorems still hold true if replacing the assumptions $(\frac{1}{\sqrt{\varrho_0}})', (\frac{1}{\varrho_0})''\in L^\infty(\mathbb{R})$ in (H1) and (H2) by the following weaker one:

$$\frac{\underline{K}_\varrho}{\langle y\rangle^{\bar{\ell}_\varrho}}\leq\varrho_0(y)\leq\frac{\bar{K}_\varrho}{\langle y\rangle^{\underline{\ell}_\varrho}},\quad \forall y\in\mathbb{R},$$

for some constants $0<\underline{K}_\varrho\leq\bar{K}_\varrho$ and $0\leq\underline{\ell}_\varrho\leq\bar{\ell}_\varrho\leq 2$.

Remark 1.3. *Let K_ϱ and $\frac{1}{\gamma}<\ell_\varrho\leq 2$ be positive constants. Choose*

$$\varrho_0(y)=\frac{K_\varrho}{\langle y\rangle^{\ell_\varrho}},\quad J_0\equiv 1,\quad v_0\in C_c^\infty(\mathbb{R}),\quad s_0\in W^{1,\infty}(\mathbb{R}),\quad \vartheta_0=\frac{A}{R}e^{\frac{s_0}{c_v}}\varrho_0^{\gamma-1}.$$

Then, one can verify easily that (H0)–(H2) and (HS) hold. Therefore, the set of the initial data that fulfills the conditions in the above theorems is not empty.

Remark 1.4. *Both the assumptions that there is no interior vacuum and that the initial density decays slowly at the far field are necessary conditions for guaranteeing the uniform boundedness of the entropy. In fact, if either there is an interior point vacuum or the density decays to vacuum sufficient fast at the far field, then the entropy will become unbounded immediately after the initial time, see Li-Xin [27].*

Remark 1.5. *It should be emphasized that though we deal with only the one dimensional case here, the main ideas of combining singularly weighted energy estimates with some deliberately modified De Giorgi iterations can be used to derive the uniform boundedness of the entropy for the multi-dimensional case at least locally in time. Indeed, by adapting these ideas with some more involved and complicated calculations,* *one can obtain that the boundedness of the entropy can be propagated by the multi-dimensional compressible Navier-Stokes system up to the maximal existing time of the strong solution under similar conditions on the initial density. However, the global in time existence of strong solutions for general initial data is still unknown.*

The rest of this paper is arranged as follows: in Section 2, we consider the system with the initial density uniformly away from zero, prove the local existence of solutions, and carry out some a priori estimates independent of the positive lower bound of the initial density; Section 3 is devoted to the proof of the local existence of solutions in the presence of far field vacuum; while the global existence and uniqueness of solutions are shown in Section 4; and finally in Section 5 and Section 6, we establish the uniform lower and upper bounds of the entropy, respectively, by performing the singular type energy estimates and using some suitably modified De Giorgi type iterations.

Throughout this paper, C will denote a genetic positive constant, which may vary from line to line. For simplicity of presentations, the quantities, on which the constant C depends, will be emphasized only in the statements, but not in the proofs, of the theorems, propositions, and corollaries.

2. Local existence and a priori estimates in the absence of vacuum

Let $\underline{\varrho}, \bar{\varrho}, \underline{J}$, and $\bar{J}$ be positive constants. Assume that

$$\begin{cases} 0<\underline{\varrho}\leq\varrho_0(y)\leq\bar{\varrho}<\infty, \quad 0<\underline{J}\leq J_0(y)\leq\bar{J}<\infty, \quad \forall y\in\mathbb{R}, \\ \varrho_0'\in L^\infty(\mathbb{R}), \quad J_0'\in L^2(\mathbb{R}), \quad v_0\in H^1(\mathbb{R}), \quad 0\leq\vartheta_0\in H^1(\mathbb{R}). \end{cases} \tag{2.1}$$

The following local existence result holds.

Proposition 2.1. *Under the assumption (2.1), there is a positive time T_0 depending only on $\underline{\varrho}, \bar{\varrho}, \underline{J}, \bar{J}, \|\varrho_0'\|_\infty, \|J_0'\|_2, \|v_0\|_{H^1}$, and $\|\theta_0\|_{H^1}$, such that the problem (1.7)–(1.10) with the following far field condition*

$$(v,\vartheta)\to 0, \quad \text{as } y\to\infty, \tag{2.2}$$

has a unique solution (J,v,θ), on $\mathbb{R}\times(0,T_0)$, satisfying

$$\begin{aligned} &\frac{\underline{J}}{2}\leq J\leq 2\bar{J}, \quad \text{on } \mathbb{R}\times[0,T_0], \quad J-J_0\in C([0,T_0];H^1), \\ &v\in C([0,T_0];H^1)\cap L^2(0,T_0;H^2), \quad 0\leq\vartheta\in C([0,T_0];H^1)\cap L^2(0,T_0;H^2), \\ &J_t\in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;H^1), \quad v_t\in L^2(0,T_0;L^2), \quad \vartheta_t\in L^2(0,T_0;L^2). \end{aligned}$$

Proof. This can be proved in the standard way by using the fixed point argument based on the following linearized system

$$J_t = V_y, \tag{2.3}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y = -R\left(\frac{\varrho_0}{J}\Theta\right)_y, \tag{2.4}$$

$$c_v\varrho_0\vartheta_t - \kappa\left(\frac{\vartheta_y}{J}\right)_y = \mu\frac{V_y^2}{J} - R\frac{\varrho_0}{J}\Theta V_y, \tag{2.5}$$

subject to (1.10) and (2.2), for given (V,Θ). Indeed, the classic theory for uniformly parabolic equations yields a unique global solution (v,ϑ) to the system (2.4)–(2.5), subject to (1.10) and (2.2). Thus, one can define a solution mapping $(V,\Theta)\to(v,\vartheta)$. Then, by carrying out the energy estimates, similar to (actually easier than) those we will derive in the rest of this section, one can see that this solution mapping fulfills all the conditions of the Banach's contracting fixed point theorem, and thus has a unique fixed point in the corresponding Banach space, which yields the unique solution to the system (1.7)–(1.9), subject to (1.10) and (2.2). □

By applying Proposition 2.1 iteratively, one can extend the local solution (J,v,ϑ) uniquely to the maximal time $T_{\max}$ of existence, which is characterized as

$$\limsup_{T\to T_{\max}^-}\left((\inf_{y\in\mathbb{R}} J)^{-1} + \sup_{y\in\mathbb{R}} J + \|J_y\|_2 + \|v\|_{H^1} + \|\vartheta\|_{H^1}\right) = \infty. \tag{2.6}$$

In the rest of this section, it is always assumed that the unique solution (ρ,v,θ) has already been extended uniquely to the maximal time of existence $T_{\max}$.

One aim of this section is to show $T_{\max}$ is independent of $\underline{\varrho}$. To this end, we set

$$T_* := \max\left\{T\in(0,T_{\max})\;\Big|\;\frac{\underline{J}}{3}\le J\le 3\bar{J}\text{ on }\mathbb{R}\times[0,T]\right\}. \tag{2.7}$$

In the rest of this section, we will focus on the solutions in the time interval $(0,T_*)$, so that J has the positive lower and upper bounds stated in (2.7).

2.1. A priori L^2 estimates.

Proposition 2.2. *There is a positive time T_{ode} depending only on $c_v, R, \mu, \kappa, \|\varrho_0\|_{W^{1,\infty}}$, $\underline{J}$, and $\bar{J}$, such that*

$$\sup_{0\le t\le T_{ode}^*}\|(\sqrt{\varrho_0}v,\sqrt{\varrho_0}E)\|_2^2 + \int_0^{T_{ode}^*}(\|\varrho_0\vartheta\|_\infty^2 + \|(v_y, vv_y, \vartheta_y)\|_2^2)dt \le \mathcal{E}_0,$$

where $E=\frac{v^2}{2}+c_v\vartheta$, $T_{ode}^* := \min\{T_*, T_{ode}, 1\}$, and $\mathcal{E}_0$ is a positive constant depending only on $c_v, R, \mu, \kappa, \underline{J}, \bar{J}, \|\varrho_0\|_{W^{1,\infty}}, \|\sqrt{\varrho_0}v_0\|_2$, and $\|\sqrt{\varrho_0}E_0\|_2$.

Proof. It follows from (1.8) and the Cauchy inequality that

$$\frac{d}{dt}\|\sqrt{\varrho_0}v\|_2^2 + \mu\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2 \le \frac{R^2}{\mu}\left\|\frac{\varrho_0}{\sqrt{J}}\vartheta\right\|_2^2. \tag{2.8}$$

Set $E=\frac{v^2}{2}+c_v\vartheta$. Then,

$$\varrho_0 E_t - \kappa\left(\frac{\vartheta_y}{J}\right)_y = \left(\left(\mu\frac{v_y}{J} - R\frac{\varrho_0}{J}\vartheta\right)v\right)_y. \tag{2.9}$$

Testing (2.9) with E yields

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\varrho_0}E\|_2^2+\kappa\int_{\mathbb{R}}\frac{\vartheta_y}{J}E_y dy=-\int_{\mathbb{R}}\left(\mu\frac{v_y}{J}-R\frac{\varrho_0}{J}\vartheta\right)vE_y dy.$$

Direct estimates show that

$$\begin{aligned}\int_{\mathbb{R}}\frac{\vartheta_y}{J}E_y dy &\geq \frac{3c_v}{4}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2-\frac{1}{c_v}\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2,\\ -\int_{\mathbb{R}}\left(\mu\frac{v_y}{J}-R\frac{\varrho_0}{J}\vartheta\right)vE_y dy &\leq \frac{c_v\kappa}{4}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2+C\left(\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2+\int_{\mathbb{R}}\frac{\varrho_0^2}{J}\vartheta^2v^2dy\right),\end{aligned}$$

and, consequently,

$$\frac{d}{dt}\|\sqrt{\varrho_0}E\|_2^2+\kappa c_v\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2\leq C\left(\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2+\int_{\mathbb{R}}\frac{\varrho_0^2}{J}\vartheta^2v^2dy\right). \tag{2.10}$$

Test (1.8) with v^3 and apply the Cauchy-Schwaz inequality to get

$$\frac{d}{dt}\|\sqrt{\varrho_0}v^2\|_2^2+8\mu\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2\leq\frac{9R^2}{\mu}\int_{\mathbb{R}}\frac{\varrho_0^2\vartheta^2v^2}{J}dy. \tag{2.11}$$

By (2.8), (2.10), and (2.11), one can choose A_1 sufficiently large such that

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\varrho_0}v\|_2^2+\|\sqrt{\varrho_0}E\|_2^2+A_1\|\sqrt{\varrho_0}v^2\|_2^2)+\mu\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2\\ &\qquad+\kappa c_v\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2+A_1\mu\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2\leq C\int_{\mathbb{R}}\left(\frac{\varrho_0^2\vartheta^2}{J}+\frac{\varrho_0^2\vartheta^2v^2}{J}\right)dy.\end{aligned} \tag{2.12}$$

Due to the definition of T_*, one has

$$\int_{\mathbb{R}}\left(\frac{\varrho_0^2\vartheta^2}{J}+\frac{\varrho_0^2\vartheta^2v^2}{J}\right)dy \leq C(1+\|\varrho_0\vartheta\|_\infty)\|\sqrt{\varrho_0}E\|_2^2.$$

Note that

$$\begin{aligned}\|\varrho_0\vartheta\|_\infty^2 &\leq 2\int_{\mathbb{R}}(\varrho_0|\varrho_0'|\vartheta^2+\varrho_0^2\vartheta|\vartheta_y|)dy\\ &\leq C\left(\|\sqrt{\varrho_0}E\|_2^2+\|\sqrt{\varrho_0}E\|_2\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2\right).\end{aligned} \tag{2.13}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}}\left(\frac{\varrho_0^2\vartheta^2}{J}+\frac{\varrho_0^2\vartheta^2v^2}{J}\right)dy &\leq C\left(1+\|\sqrt{\varrho_0}E\|_2+\|\sqrt{\varrho_0}E\|_2^{\frac{1}{2}}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^{\frac{1}{2}}\right)\|\sqrt{\varrho_0}E\|_2^2\\ &\leq \varepsilon\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2+C_\varepsilon(1+\|\sqrt{\varrho_0}E\|_2^2)^2,\end{aligned}$$

for any $t \in [0, T_*]$, and for any $\varepsilon > 0$.

Choosing ε sufficient small, one obtains from this and (2.12) that

$$\frac{d}{dt}(\|\sqrt{\varrho_0}v\|_2^2 + \|\sqrt{\varrho_0}E\|_2^2 + A_1\|\sqrt{\varrho_0}v^2\|_2^2) + \mu\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2 + \frac{\kappa c_v}{2}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2 + A_1\mu\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2 \leq C(1 + \|\sqrt{\varrho_0}E\|_2^2)^2,$$

for any $t \in [0, T_*]$. This and solving an ordinary differential inequality of the type $f' \leq Cf^2$ yield that there is a positive time T_{ode} such that

$$\sup_{0\leq t\leq T^*_{\text{ode}}} \|(\sqrt{\varrho_0}v, \sqrt{\varrho_0}E)\|_2^2 + \int_0^{T^*_{\text{ode}}} \|(v_y, vv_y, \vartheta_y)\|_2^2\, dt \leq \mathcal{E}_0',$$

where $T^*_{\text{ode}} := \min\{T_*, T_{\text{ode}}, 1\}$. Then, it follows from (2.13) that $\int_0^{T^*_{\text{ode}}} \|\varrho_0\vartheta\|_\infty^2 dt \leq \mathcal{E}_0''$. This proves the conclusion. □

2.2. A priori H^1 estimates.

Proposition 2.3. *Let T^*_{ode} be as in Proposition 2.2 and G be given by (1.12). Set $G_0 = \frac{1}{J_0}(\mu v_0' - R\varrho_0\vartheta_0)$. Then, there is a positive constant $\mathcal{E}_1$, depending only on $\mu, \kappa, c_v, R, \bar\varrho, \underline{J}, \bar J, \|\varrho_0\|_{W^{1,\infty}}, \|\sqrt{\varrho_0}v_0\|_2, \|\sqrt{\varrho_0}E_0\|_2$, $\|G_0\|_2$, and $\left\|\varrho_0^{\frac{3}{2}}\vartheta_0'\right\|_2^2$, such that*

$$\sup_{0\leq t\leq T^*_{ode}} \|G\|_2^2 + \int_0^{T^*_{ode}} \left(\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + \|G\|_\infty^2\right)dt \leq \mathcal{E}_1,$$

$$\sup_{0\leq t\leq T^*_{ode}} \left\|\varrho_0^{\frac{3}{2}}\vartheta_y\right\|_2^2 + \int_0^{T^*_{ode}} \left(\|\varrho_0^2\vartheta_t\|_2^2 + \left\|\varrho_0\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2 + \|v_y\|_\infty^2\right)dt \leq \mathcal{E}_1.$$

Proof. We start with the estimate on G. Testing (1.13) with JG yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 &= (2\gamma - 1)\int_{\mathbb{R}} vGG_y dy - \kappa(\gamma - 1)\int_{\mathbb{R}} \frac{\vartheta_y G_y}{J} dy \\ &\leq \frac{\mu}{4}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + C\left(\|\sqrt{\varrho_0}v\|_2^2\|G\|_\infty^2 + \|\vartheta_y\|_2^2\right).\end{aligned}$$

It follows from Proposition 2.2 and the inequality above that

$$\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \frac{3\mu}{2}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \leq C\left(\|G\|_\infty^2 + \|\vartheta_y\|_2^2\right). \tag{2.14}$$

Note that

$$\|G\|_\infty^2 \leq \int_{\mathbb{R}} |\partial_y G^2| dy \leq C\|G\|_2\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 \leq \varepsilon\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + C_\varepsilon\|\sqrt{J}G\|_2^2, \tag{2.15}$$

for any positive ε. Choosing ε sufficient small, one gets from (2.14) and (2.15) that

$$\frac{d}{dt}\|\sqrt{J}G\|_2^2+\mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2\le C\left(\|\sqrt{J}G\|_2^2+\|\vartheta_y\|_2^2\right).$$

Consequently, Proposition 2.2 and the Gronwall inequality show that

$$\sup_{0\le t\le T^*_{\mathrm{ode}}}\|G\|_2^2+\int_0^{T^*_{\mathrm{ode}}}\left(\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+\|G\|_\infty^2\right)dt\le\mathcal{E}_1'. \tag{2.16}$$

Next we estimate ϑ. It follows from (1.9) that

$$\int_{\mathbb{R}}\varrho_0^2\left(c_v\varrho_0\vartheta_t-\kappa\left(\frac{\vartheta_y}{J}\right)_y\right)^2=\int_{\mathbb{R}}\varrho_0^2v_y^2G^2dy. \tag{2.17}$$

Using (1.7), one deduces

$$\begin{aligned}-2\int_{\mathbb{R}}\varrho_0^3\vartheta_t\left(\frac{\vartheta_y}{J}\right)_y dy&=\frac{d}{dt}\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2+\int_{\mathbb{R}}\left(v_y\frac{\varrho_0^3}{J^2}\vartheta_y^2+6\varrho_0^2\varrho_0'\frac{\vartheta_y\vartheta_t}{J}\right)dy\\
&\ge\ \frac{d}{dt}\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2-\frac{c_v}{2\kappa}\|\varrho_0^2\vartheta_t\|_2^2-\left\|\frac{v_y}{J}\right\|_\infty\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2-\frac{18\kappa}{c_v}\left\|\frac{\varrho_0'\vartheta_y}{J}\right\|_2^2.\end{aligned}$$

This, together with (2.16) and (2.17), yields

$$\begin{aligned}&c_v\kappa\frac{d}{dt}\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2+\frac{c_v^2}{2}\|\varrho_0^2\vartheta_t\|_2^2+\kappa^2\left\|\varrho_0\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2\\
\le\ &c_v\kappa\left\|\frac{v_y}{J}\right\|_\infty\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2+18\kappa^2\left\|\frac{\varrho_0'\vartheta_y}{J}\right\|_2^2+\int_{\mathbb{R}}\varrho_0^2v_y^2G^2dy\\
\le\ &C\left[(1+\|v_y\|_\infty^2)\left(\left\|\sqrt{\frac{\varrho_0^3}{J}}\vartheta_y\right\|_2^2+\mathcal{E}_1'\right)+\|\vartheta_y\|_2^2\right].\end{aligned} \tag{2.18}$$

Since $v_y=\frac{1}{\mu}(JG+R\varrho_0\vartheta)$, by Proposition 2.2 and (2.16), one has

$$\int_0^{T^*_{\mathrm{ode}}}\|v_y\|_\infty^2dt\ \le\ C\int_0^{T^*_{\mathrm{ode}}}(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2)dt\le C(\mathcal{E}_0+\mathcal{E}_1').$$

This, together with (2.18) and Proposition 2.2, yields

$$\sup_{0\le t\le T^*_{\mathrm{ode}}}\left\|\sqrt{\varrho_0^3}\vartheta_y\right\|_2^2+\int_0^{T^*_{\mathrm{ode}}}\left(\|\varrho_0^2\vartheta_t\|_2^2+\left\|\varrho_0\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2\right)dt$$

$$\leq \quad e^{C(1+\mathcal{E}_0+\mathcal{E}_1')}\left(\left\|\sqrt{\frac{\varrho_0^3}{J_0}}\vartheta_0'\right\|_2^2+\mathcal{E}_1'+\mathcal{E}_0\right)=:\mathcal{E}_1''.$$

The conclusion follows by setting $\mathcal{E}_1=\max\{\mathcal{E}_1',\mathcal{E}_1''\}$. □

Proposition 2.4. *Let T^*_{ode} be as in Proposition 2.2. Then, there is a positive constant $\mathcal{E}_2$ depending only on $\mu,\kappa,c_v,R,\underline{\varrho},\underline{J},\bar{J},\|\varrho_0\|_{W^{1,\infty}},\|\sqrt{\varrho_0}v_0\|_2,\|\sqrt{\varrho_0}E_0\|_2$, $\|G_0\|_2$, $\left\|\varrho_0^{\frac32}\vartheta_0'\right\|_2^2$, and $\|\sqrt{\varrho_0}J_0'\|_2^2$, such that*

$$\sup_{0\leq t\leq T^*_{ode}}\|v_y\|_2^2+\int_0^{T^*_{ode}}(\|\sqrt{\varrho_0}v_t\|_2^2+\|\sqrt{\varrho_0}v_{yy}\|_2^2)dt\leq\mathcal{E}_2,$$

$$\sup_{0\leq t\leq T^*_{ode}}(\|\sqrt{\varrho_0}J_y\|_2^2+\|J_t\|_2^2)+\int_0^{T^*_{ode}}\|\sqrt{\varrho_0}J_{yt}\|_2^2dt\leq\mathcal{E}_2.$$

Proof. Since $J_t=v_y$ and $J_{yt}=v_{yy}$, it suffices to prove the first conclusion and the estimate $\sup_{0\leq t\leq T^*_{\text{ode}}}\|\sqrt{\varrho_0}J_y\|_2^2$. Besides, since $\sqrt{\varrho_0}v_t=\frac{G_y}{\sqrt{\varrho_0}}$, the estimate for $\int_0^{T^*_{\text{ode}}}\|\sqrt{\varrho_0}v_t\|_2^2dt$ follows from Proposition 2.3 directly.

Note that

$$\sqrt{\varrho_0}J_{yt}=\sqrt{\varrho_0}v_{yy}=\frac{\sqrt{\varrho_0}}{\mu}(JG_y+J_yG+R\varrho_0'\vartheta+R\varrho_0\vartheta_y).$$

Multiplying the equation before by $\sqrt{\varrho_0}J_y$ yields

$$\begin{aligned}\frac12\frac{d}{dt}\|\sqrt{\varrho_0}J_y\|_2^2 &\leq \frac{\|G\|_\infty}{\mu}\|\sqrt{\varrho_0}J_y\|_2^2+\frac{\|\sqrt{\varrho_0}J_y\|_2}{\mu}\|J\sqrt{\varrho_0}G_y+R\varrho_0'\sqrt{\varrho_0}\vartheta+R\varrho_0^{\frac32}\vartheta_y\|_2\\ &\leq C(\|G\|_\infty^2+1)\|\sqrt{\varrho_0}J_y\|_2^2+C(\|G_y\|_2^2+\|\sqrt{\varrho_0}\vartheta\|_2^2+\|\vartheta_y\|_2^2),\end{aligned}$$

which, together with Propositions 2.2–2.3, yields

$$\begin{aligned}\sup_{0\leq t\leq T^*_{\text{ode}}}\|\sqrt{\varrho_0}J_y\|_2^2 &\leq e^{C\int_0^{T^*_{\text{ode}}}(1+\|G\|_\infty^2)dt}\left[\|\sqrt{\varrho_0}J_0'\|_2^2+C\int_0^{T^*_{\text{ode}}}\|(G_y,\sqrt{\varrho_0}\vartheta,\vartheta_y)\|_2^2dt\right]\\ &\leq Ce^{C\mathcal{E}_1}(\|\sqrt{\varrho_0}J_0'\|_2^2+C\mathcal{E}_0+C\mathcal{E}_1)=:\mathcal{E}_2'.\end{aligned}\tag{2.19}$$

It follows from direct calculations, (2.19), and Propositions 2.2–2.3 that

$$\begin{aligned}&\sup_{0\leq t\leq T^*_{\text{ode}}}\|v_y\|_2^2+\int_0^{T^*_{\text{ode}}}\|\sqrt{\varrho_0}v_{yy}\|_2^2dt\\ =&\sup_{0\leq t\leq T^*_{\text{ode}}}\left\|\frac1\mu(JG+R\varrho_0\vartheta)\right\|_2^2+\int_0^{T^*_{\text{ode}}}\left\|\frac{\sqrt{\varrho_0}}{\mu}(JG_y+J_yG+R\varrho_0'\vartheta+R\varrho_0\vartheta_y)\right\|_2^2dt\\ \leq& C\sup_{0\leq t\leq T^*_{\text{ode}}}\|(G,\sqrt{\varrho_0}\vartheta)\|_2^2+C\int_0^{T^*_{\text{ode}}}(\|(G_y,\sqrt{\varrho_0}\vartheta,\vartheta_y)\|_2^2+\|\sqrt{\varrho_0}J_y\|_2^2\|G\|_\infty^2)dt\end{aligned}$$

$$\leq \quad C(\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_1\mathcal{E}_2') =: \mathcal{E}_2''.$$

Setting $\mathcal{E}_2 = \mathcal{E}_2' + \mathcal{E}_2''$ gives the desired conclusion. □

2.3. **Estimate on the life span and a summary of a priori estimates.**

Proposition 2.5. *Let T_{ode} and T^*_{ode} be as in Proposition 2.2, and $\mathcal{E}_1$ in Proposition 2.3. Then, $T^*_{ode} = \min\left\{T_{ode}, 1, \frac{\underline{J}^2}{4\mathcal{E}_1}\right\}$.*

Proof. Note that $\varrho_0 \geq \underline{\varrho} > 0$. Propositions 2.2–2.4 imply

$$\sup_{0\leq t\leq T^*_{\mathrm{ode}}} (\|J_y\|_2 + \|v\|_{H^1} + \|\vartheta\|_{H^1}) < \infty.$$

It follows from the definition of T_* and $T^*_{\mathrm{ode}} = \min\{T_*, T_{\mathrm{ode}}, 1\}$ that

$$\sup_{0\leq t\leq T^*_{\mathrm{ode}}} \left((\inf_{y\in\mathbb{R}} J)^{-1} + \sup_{y\in\mathbb{R}} J\right) < \infty.$$

Thus, $T^*_{\mathrm{ode}} < T_{\max}$.

Then, (1.7) and Proposition 2.3 imply

$$J = J_0 + \int_0^t v_y d\tau \geq \underline{J} - \left(\int_0^t \|v_y\|_\infty^2 d\tau\right)^{\frac{1}{2}} t^{\frac{1}{2}} \geq \underline{J} - \mathcal{E}_1^{\frac{1}{2}} t^{\frac{1}{2}} \geq \frac{\underline{J}}{2},$$

$$J \leq \bar{J} + t^{\frac{1}{2}} \left(\int_0^t \|v_y\|_\infty^2 d\tau\right)^{\frac{1}{2}} \leq \bar{J} + \mathcal{E}_1^{\frac{1}{2}} t^{\frac{1}{2}} \leq \frac{3\bar{J}}{2},$$

for all $t \leq T_{**}$, where

$$T_{**} := \min\left\{T^*_{\mathrm{ode}}, \frac{\underline{J}^2}{4\mathcal{E}_1}, \frac{\bar{J}^2}{4\mathcal{E}_1}\right\} = \min\left\{T_*, T_{\mathrm{ode}}, 1, \frac{\underline{J}^2}{4\mathcal{E}_1}\right\},$$

with T_{ode} given in Proposition 2.2.

Note that $\frac{\underline{J}}{3} < \frac{\underline{J}}{2} \leq J \leq \frac{3\bar{J}}{2} < 3\bar{J}$ on $\mathbb{R} \times [0, T_{**}]$, $H^1(\mathbb{R}) \hookrightarrow C(\overline{\mathbb{R}})$, $J - J_0 \in C([0, T_{\max}); H^1(\mathbb{R}))$, and $T_{**} \leq T^*_{\mathrm{ode}} < T_{\max}$. There is a positive constant $T^+_{**} \in (T_{**}, T_{\max})$, such that $\frac{\underline{J}}{3} \leq J \leq 3\bar{J}$ on $\mathbb{R} \times [0, T^+_{**}]$. Thanks to this and the definition of T_* in (2.7), one has

$$\min\left\{T_*, T_{\mathrm{ode}}, 1, \frac{\underline{J}^2}{4\mathcal{E}_1}\right\} = T_{**} < T^+_{**} \leq T_*,$$

which implies $T_* > \min\left\{T_{\mathrm{ode}}, 1, \frac{\underline{J}^2}{4\mathcal{E}_1}\right\}$. Thus,

$$T^*_{\mathrm{ode}} = \min\{T_{\mathrm{ode}}, 1, T_*\} = \min\left\{T_{\mathrm{ode}}, 1, \frac{\underline{J}^2}{4\mathcal{E}_1}\right\},$$

which yields the desired conclusion. □

As a consequence of Propositions 2.1–2.5, we have the following:

Theorem 2.1. *Under the assumption (2.1), there are two positive constants $\mathcal{T}$ and $\mathcal{E}$ depending only on $c_v, R, \mu, \kappa, \bar\varrho, \underline{J}, \bar J, \|\varrho_0'\|_\infty, \|\sqrt{\varrho_0}v_0\|_2, \|\sqrt{\varrho_0}v_0^2\|_2, \|\sqrt{\varrho_0}\vartheta_0\|_2, \|v_0'\|_2$, $\|\sqrt{\varrho_0}J_0'\|_2^2$, and $\left\|\varrho_0^{\frac32}\vartheta_0'\right\|_2^2$, but independent of $\underline{\varrho}$, such that the problem (1.7)–(1.10) has a unique solution (J, v, ϑ) on $\mathbb{R}\times[0,\mathcal{T}]$, satisfying*

$$
\begin{aligned}
\frac{\underline{J}}{2}\leq J\leq 2\bar J, \quad \vartheta\geq 0, \qquad on\ \mathbb{R}\times[0,\mathcal{T}],&\\
\sup_{0\leq t\leq\mathcal{T}}\|(J_t,\sqrt{\varrho_0}J_y)\|_2^2+\int_0^{\mathcal{T}}\|\sqrt{\varrho_0}J_{yt}\|_2^2dt &\leq \mathcal{E},\\
\sup_{0\leq t\leq\mathcal{T}}\|(\sqrt{\varrho_0}v,\sqrt{\varrho_0}v^2,v_y)\|_2^2+\int_0^{\mathcal{T}}\|(vv_y,\sqrt{\varrho_0}v_t,\sqrt{\varrho_0}v_{yy})\|_2^2)dt &\leq \mathcal{E},\\
\sup_{0\leq t\leq\mathcal{T}}\|(\sqrt{\varrho_0}\vartheta,\varrho_0^{\frac32}\vartheta_y)\|_2^2+\int_0^{\mathcal{T}}\left\|\left(\vartheta_y,\varrho_0^2\vartheta_t,\varrho_0\left(\frac{\vartheta_y}{J}\right)_y\right)\right\|_2^2dt &\leq \mathcal{E}.
\end{aligned}
$$

3. Local existence in the presence of far field vacuum

The aim of this section is to establish the local existence of solutions to the problem (1.7)–(1.10), with vacuum at the far field only.

Theorem 3.1. *Let $\bar\varrho, \underline{J}$, and $\bar J$ be positive constants. Assume that*

$$
\begin{cases}
0<\varrho_0(y)\leq\bar\varrho, \quad \underline{J}\leq J_0(y)\leq\bar J, \quad \vartheta_0(y)\geq0, \quad \forall y\in\mathbb{R},\\
\varrho_0'\in L^\infty(\mathbb{R}), \quad \left(\sqrt{\varrho_0}J_0',\sqrt{\varrho_0}v_0,\sqrt{\varrho_0}v_0^2,v_0',\sqrt{\varrho_0}\vartheta_0,\varrho_0^{\frac32}\vartheta_0'\right)\in L^2(\mathbb{R}).
\end{cases}
\tag{3.1}
$$

Then, there is a positive time $\mathcal{T}$ depending only on c_v, R, μ, κ, and

$$
\begin{cases}
\bar\varrho, \underline{J}, \bar J, \|\varrho_0'\|_\infty, \|\sqrt{\varrho_0}v_0\|_2, \|\sqrt{\varrho_0}v_0^2\|_2,\\
\|\sqrt{\varrho_0}\vartheta_0\|_2, \|v_0'\|_2, \|\sqrt{\varrho_0}J_0'\|_2, \|\varrho_0^{\frac32}\vartheta_0'\|_2,
\end{cases}
$$

such that the problem (1.7)–(1.10), on $\mathbb{R}\times[0,\mathcal{T}]$, has a solution (J, v, ϑ), satisfying $\frac{\underline{J}}{2}\leq J\leq 2\bar J$ and $\vartheta\geq0$ on $\mathbb{R}\times[0,\mathcal{T}]$, and

$$
\begin{gathered}
J_t,\sqrt{\varrho_0}J_y,\sqrt{\varrho_0}v,\sqrt{\varrho_0}v^2,v_y,\sqrt{\varrho_0}\vartheta,\varrho_0^{\frac32}\vartheta_y\in L^\infty(0,\mathcal{T};L^2(\mathbb{R})),\\
\sqrt{\varrho_0}J_{yt},vv_y,\sqrt{\varrho_0}v_t,\sqrt{\varrho_0}v_{yy},\vartheta_y,\varrho_0(\tfrac{\vartheta_y}{J})_y,\varrho_0^2\vartheta_t\in L^2(0,\mathcal{T};L^2(\mathbb{R})).
\end{gathered}
$$

Proof. We first construct a sequence $\{(\varrho_{0n},J_{0n},v_{0n},\vartheta_{0n})\}_{n=1}^\infty$ approximating $(\varrho_0,J_0,v_0,\vartheta_0)$, so that Theorem 2.1 applies.

For each integer $n\geq1$, choose $0<\delta_n\leq\frac1n$ sufficiently small such that

$$
\delta_n\max\left\{\|v_0\|_{L^2(I_{2n})}^2,\|v_0\|_{L^4(I_{2n})}^4,\|\vartheta_0\|_{L^2(I_{2n})}^2,\|\vartheta_0'\|_{L^2(I_{2n})}^2\right\}\leq1,
\tag{3.2}
$$

where $I_{2n}=(-2n,2n)$. Choose $\varphi\in C_0^\infty((-2,2))$, with $\varphi\equiv 1$ on $(-1,1)$, $0\leq\varphi\leq 1$ on $(-2,2)$, and $|\varphi'|\leq 2$. Since $v_0',\vartheta_0'\in L^2(\mathbb{R})$, it is clear $v_0,\vartheta_0\in C(\mathbb{R})$, and

$$|v_0(y)| \leq |v_0(0)|+\|v_0'\|_2\sqrt{|y|}\leq A_0\sqrt{1+|y|}, \tag{3.3}$$

where $A_0=\max\{|v_0(0)|,\|v_0'\|_2\}$.

Define $\varrho_{0n}, J_{0n}, v_{0n}$, and ϑ_{0n} as

$$\varrho_{0n}=\delta_n+\varrho_0,\quad J_{0n}=J_0,\quad v_{0n}=\varphi\left(\frac{\cdot}{n}\right)v_0,\quad \vartheta_{0n}=\varphi\left(\frac{\cdot}{n}\right)\vartheta_0.$$

Then

$$0<\delta_n\leq\varrho_{0n}\leq\bar\varrho+1,\quad \|\varrho_{0n}'\|_\infty=\|\varrho_0'\|_\infty. \tag{3.4}$$

(3.2) and (3.3) imply that

$$\begin{aligned}
\|\sqrt{\varrho_{0n}}v_{0n}\|_2^2 &\leq \|\sqrt{\varrho_0}v_0\|_2^2+\delta_n\|v_0\|_{L^2((-2n,2n))}^2\leq\|\sqrt{\varrho_0}v_0\|_2^2+1, &(3.5)\\
\|\sqrt{\varrho_{0n}}v_{0n}^2\|_2^2 &\leq \|\sqrt{\varrho_0}v_0^2\|_2^2+\delta_n\|v_0\|_{L^4((-2n,2n))}^4\leq\|\sqrt{\varrho_0}v_0^2\|_2^2+1,\\
\|v_{0n}'\|_2^2 &\leq 2\|v_0'\|_2^2+64A_0^2, &(3.6)\\
\|\sqrt{\varrho_{0n}}\vartheta_{0n}\|_2^2 &\leq \|\sqrt{\varrho_0}\vartheta_0\|_2^2+1. &(3.7)
\end{aligned}$$

Due to (3.2) and $0\leq\delta_n\leq\frac{1}{n}$, one can get

$$\begin{aligned}
\|\varrho_{0n}^{\frac32}\vartheta_{0n}'\|_2^2 &\leq 8\int_{-2n}^{2n}(\varrho_0^3+\delta_n^3)\left(|\vartheta_0'|^2+\frac{4}{n^2}\vartheta_0^2\right)dy\\
&\leq 8\left(\|\varrho_0^{\frac32}\vartheta_0'\|_2^2+\delta_n^3\|\vartheta_0'\|_2^2+4\bar\varrho^2\|\sqrt{\varrho_0}\vartheta_0\|_2^2+4\delta_n^3\|\vartheta_0\|_{L^2((-2n,2n))}^2\right)\\
&\leq 8\left(\|\varrho_0^{\frac32}\vartheta_0'\|_2^2+5+4\bar\varrho^2\|\sqrt{\varrho_0}\vartheta_0\|_2^2\right). &(3.8)
\end{aligned}$$

Since $(\varrho_{0n},J_{0n},v_{0n},\vartheta_{0n})$ fulfills the assumption (2.1), with $\underline{\varrho}$ and $\bar\varrho$ replaced by δ_n and $\bar\varrho+1$, respectively, by (3.4)–(3.8) and Theorem 2.1, there is a positive time $\mathcal{T}$ depending only on the quantities stated in Theorem 2.1, which in particular is independent of n, such that the problem (1.7)–(1.10), has a unique solution (J_n,v_n,ϑ_n), satisfying

$$\frac{\underline{J}}{2}\leq J_n\leq 2\bar J,\quad \vartheta_n\geq 0,\quad \text{on } \mathbb{R}\times[0,\mathcal{T}], \tag{3.9}$$

$$\sup_{0\leq t\leq\mathcal{T}}\|(\partial_tJ_n,\sqrt{\varrho_{0n}}\partial_yJ_n)\|_2^2+\int_0^{\mathcal{T}}\|\sqrt{\varrho_{0n}}\partial_{yt}J_n\|_2^2dt\leq\mathcal{E}, \tag{3.10}$$

$$\sup_{0\leq t\leq\mathcal{T}}\|(\varrho_{0n}^{\frac12}v_n,\varrho_{0n}^{\frac12}v_n^2,\partial_yv_n)\|_2^2+\int_0^{\mathcal{T}}\|(v_n\partial_yv_n,\varrho_{0n}^{\frac12}\partial_tv_n,\varrho_{0n}^{\frac12}\partial_{yy}v_n)\|_2^2dt\leq\mathcal{E}, \tag{3.11}$$

$$\sup_{0\leq t\leq\mathcal{T}}\left\|\left(\sqrt{\varrho_{0n}}\vartheta_n,\varrho_{0n}^{\frac32}\partial_y\vartheta_n\right)\right\|_2^2+\int_0^{\mathcal{T}}\left\|\left(\partial_y\vartheta_n,\varrho_{0n}\left(\frac{\partial_y\vartheta_n}{J_n}\right)_y,\varrho_{0n}^2\partial_t\vartheta_n\right)\right\|_2^2dt\leq\mathcal{E}, \tag{3.12}$$

for a positive constant $\mathcal{E}$ independent of n.

Since $\varrho_0' \in L^\infty(\mathbb{R})$ and $\varrho_0(y) > 0$ for all $y \in \mathbb{R}$, so $\min_{|y|\leq R} \varrho_0 > 0$ for any $R \in \mathbb{R}$. Thus, it follows from (3.9)–(3.12) that

$$\begin{aligned} \|(J_n, v_n, \vartheta_n)\|_{L^\infty(0,\mathcal{T};H^1(I_k))}, \|v_n\|_{L^2(0,\mathcal{T};H^2(I_k))} &\leq \mathcal{E}_k, \\ \|\partial_t J_n\|_{L^2(0,\mathcal{T};H^1(I_k))}, \|(\partial_t v_n, \partial_t \vartheta_n)\|_{L^2(0,\mathcal{T};L^2(I_k))} &\leq \mathcal{E}_k, \end{aligned}$$

for any positive integer k, where $I_k = (-k, k)$ and $\mathcal{E}_k$ is a positive constant independent of n. Then, by the Cantor's diagonal argument in both n and k, there is a subsequence, denoted still by $\{(J_n, v_n, \vartheta_n)\}_{n=1}^\infty$, and (J, v, ϑ), such that

$$(J_n, v_n, \vartheta_n) \rightharpoonup^* (J, v, \vartheta) \quad \text{in } L^\infty(0, \mathcal{T}; H^1(I_R)), \tag{3.13}$$

$$v_n \rightharpoonup v \quad \text{in } L^2(0, \mathcal{T}; H^2(I_R)), \tag{3.14}$$

$$\partial_t J_n \rightharpoonup^* \partial_t J \quad \text{in } L^\infty(0, \mathcal{T}; L^2(I_R)), \tag{3.15}$$

$$\partial_t J_n \rightharpoonup \partial_t J_n \quad \text{in } L^2(0, \mathcal{T}; H^1(I_R)), \tag{3.16}$$

$$(\partial_t v_n, \partial_t \vartheta_n) \rightharpoonup (\partial_t v, \partial_t \vartheta) \quad \text{in } L^2(0, \mathcal{T}; L^2(I_R)), \tag{3.17}$$

for any $R \in (0, \infty)$, where $\rightharpoonup$ and $\rightharpoonup^*$ denote the weak and weak-* convergence, respectively, in the corresponding spaces, and $I_R = (-R, R)$. Moreover, noticing that $H^1(I_R) \hookrightarrow\hookrightarrow C(\overline{I_R})$, by the Aubin-Lions lemma, and using the Cantor's diagonal argument again (in both n and k), one can get a subsequence of the previous subsequence, denoted still by $\{(J_n, v_n, \vartheta_n)\}_{n=1}^\infty$, such that

$$(J_n, v_n, \vartheta_n) \to (J, v, \vartheta) \quad \text{in } C([0, \mathcal{T}]; C(\overline{I_R})), \tag{3.18}$$

$$v_n \to v \quad \text{in } L^2(0, \mathcal{T}; H^1(I_R)), \tag{3.19}$$

for any $R \in (0, \infty)$. These and (3.9) imply that

$$\frac{\underline{J}}{2} \leq J \leq 2\bar{J}, \quad \vartheta \geq 0, \quad \text{on } \mathbb{R} \times [0, \mathcal{T}]. \tag{3.20}$$

It follows from (3.9), (3.12), (3.13), (3.18), and (3.20) that for any $R \in (0, \infty)$

$$\frac{\partial_y \vartheta_n}{J_n} \rightharpoonup \frac{\vartheta_y}{J} \quad \text{in } L^2(0, \mathcal{T}; H^1(I_R)). \tag{3.21}$$

Thanks to the convergences (3.13)–(3.19), and (3.21), as well as the a priori estimates (3.10)–(3.12), one can obtain by the weakly lower semi-continuity of norms that (J, v, ϑ) has the regularities stated in the proposition. Besides, by (3.13)–(3.19) and (3.21), one can take the limit, as $n \to \infty$, to conclude that (J, v, ϑ) satisfies equations (1.7)–(1.9), in the sense of distribution. However, due to the regularities of (J, v, ϑ) and the positivity of ϱ_0 on $\mathbb{R}$, one can show that the equations are satisfied a.e. in $\mathbb{R} \times (0, \mathcal{T})$. While the initial condition (1.10) is guaranteed by (3.18) and (3.19). Therefore, (J, v, ϑ) is the desired solution to the problem (1.7)–(1.10). This completes the proof. $\square$

4. Global well-posedness in the presence of far field vacuum

This section is devoted to proving the global existence and uniqueness of solutions in the presence of far field vacuum via establishing a series of a priori estimates, which are finite up to any finite time. Throughout this section, we will suppose, in addition to the assumption (3.1), that

$$\varrho_0 \in L^1(\mathbb{R}), \quad \frac{J_0'}{\sqrt{\varrho_0}}, \quad \sqrt{\varrho_0}\vartheta_0' \in L^2(\mathbb{R}), \tag{4.1}$$

and, for some given positive constant K_1,

$$|\varrho_0'(y)| \leq K_1 \varrho_0^{\frac{3}{2}}(y), \quad y \in \mathbb{R}. \tag{4.2}$$

Remark 4.1. *It should be noticed that though (4.2) is assumed throughout this section, yet it is not needed for some results (say, Propositions 4.1–4.2 and Corollary 4.1), while for some others (Proposition 4.3 and Proposition 4.4), one needs only the following weaker assumption*

$$|\varrho_0'| \leq \tilde{K}_1 \varrho_0, \ \text{on } \mathbb{R}, \ \text{for some positive constant } \tilde{K}_1.$$

Note that the above weaker assumption can be satisfied even if the initial density decays very fast. It is an interesting problem to see if all the results in this section (and thus the well-posedness) still hold without (4.2) or under the weaker assumption.

In the rest of this section, we always assume that (J, v, ϑ) is a solution to the problem (1.7)–(1.10), in $\mathbb{R} \times (0, T)$, for some positive time T, satisfying

$$\begin{aligned}
&0 < J, J^{-1} \in L^\infty(0, T; L^\infty(\mathbb{R})), \quad \vartheta \geq 0,\\
&J_t, \sqrt{\varrho_0} J_y, \sqrt{\varrho_0} v, \sqrt{\varrho_0} v^2, v_y, \sqrt{\varrho_0}\vartheta, \varrho_0^{\frac{3}{2}}\vartheta_y \in L^\infty(0, T; L^2(\mathbb{R})),\\
&\sqrt{\varrho_0} J_{yt}, v v_y, \sqrt{\varrho_0} v_t, \sqrt{\varrho_0} v_{yy}, \vartheta_y, \varrho_0^2 \vartheta_t \in L^2(0, T; L^2(\mathbb{R})).
\end{aligned}$$

4.1. Basic estimates and the control of J.

The basic energy estimates, uniform positive lower bound of J, and a control on the upper bound of J are derived in this subsection. We start with the conservation of the energy.

Proposition 4.1. *Set* $\mathscr{E}_0 := \int_{\mathbb{R}} \varrho_0 \left(\frac{v_0^2}{2} + c_v \vartheta_0 \right) dy$. *Then*

$$\left[\int_{\mathbb{R}} \varrho_0 \left(\frac{v^2}{2} + c_v \vartheta \right) dy \right](t) = \mathscr{E}_0.$$

Proof. Let φ be the cut-off function given in the proof of Theorem 3.1, and set $\varphi_r(\cdot) = \varphi\left(\frac{\cdot}{r}\right)$. Testing (2.9) by φ_r yields

$$\int_{\mathbb{R}} \varrho_0 E \varphi_r dy = \int_{\mathbb{R}} \varrho_0 E_0 \varphi_r dy - \int_0^t \int_{\mathbb{R}} \frac{\varphi_r'}{J} \left(\kappa \vartheta_y + \mu v v_y - R \varrho_0 \vartheta v \right) dy d\tau. \tag{4.3}$$

Direct calculations show that

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}\frac{\varphi_r'}{J}\left(\kappa\vartheta_y+\mu v v_y-R\varrho_0\vartheta v\right)dyd\tau\right|\\
\le\ & \frac{C}{r}\int_0^t\int_{r\le|y|\le 2r}\left(|\vartheta_y|+|v||v_y|+\varrho_0\vartheta|v|\right)dyd\tau\\
\le\ & \frac{C}{r}\int_0^t\left[(\|\vartheta_y\|_2+\|vv_y\|_2)\sqrt r+\|\sqrt{\varrho_0}\vartheta\|_2\|\sqrt{\varrho_0}v\|_2\right]d\tau\\
\le\ & \frac{C}{\sqrt r}\left(1+\|\left(\vartheta_y,vv_y,\sqrt{\varrho_0}v,\sqrt{\varrho_0}\vartheta\right)\|_{L^2(\mathbb{R}\times(0,t))}^2\right),
\end{aligned}$$

for any $r\ge 1$, where C is a positive constant independent of r but may depend on t. Then, taking $r\to\infty$ in (4.3) gives the desired identity. □

The equality for J in the next proposition is in the spirit of Kazhikov-Shelukin [21], where the mass Lagrangian coordinate, rather than the flow map, was considered.

Proposition 4.2. *It holds that for any* $(y,t)\in\mathbb{R}\times(0,T)$

$$J(y,t)=B(y,t)\left(J_0(y)+\frac{R}{\mu}\int_0^t\frac{\varrho_0(y)\vartheta(y,\tau)}{B(y,\tau)}d\tau\right),$$

where $B(y,t)=\exp\left\{\frac1\mu\int_{-\infty}^y\varrho_0(v-v_0)dy'\right\}$.

Proof. It follows from (1.7) and (1.8) that

$$\varrho_0 v-\varrho_0 v_0-\mu\left[(\log J)_y-(\log J_0)'\right]+R\int_0^t\left(\frac{\varrho_0\vartheta}{J}\right)_y d\tau=0.$$

Integrating the above equation in the spatial variable over (z,y) yields

$$\begin{aligned}
&\int_{-\infty}^y(\varrho_0v-\varrho_0v_0)dy'-\mu\left(\log J(y,t)-\log J_0(y)\right)+R\int_0^t\frac{\varrho_0(y)\vartheta(y,\tau)}{J(y,\tau)}d\tau\\
=&\int_{-\infty}^z(\varrho_0v-\varrho_0v_0)dy'-\mu\left(\log J(z,t)-\log J_0(z)\right)+R\int_0^t\frac{\varrho_0(z)\vartheta(z,\tau)}{J(z,\tau)}d\tau.
\end{aligned}$$

Therefore, there is a function $f(t)$ independent of y such that

$$\int_{-\infty}^y(\varrho_0v-\varrho_0v_0)dy'-\mu\left(\log J-\log J_0\right)+R\int_0^t\frac{\varrho_0\vartheta}{J}d\tau=f(t). \tag{4.4}$$

We claim that $f\equiv 0$. Set $\delta_T:=\inf_{(y,t)\in\mathbb{R}\times(0,T)}J(y,t)>0$. It follows from (1.7) and $v_y\in L^2(\mathbb{R}\times(0,T))$ that

$$\begin{aligned}
\left|\int_{-(k+1)}^{-k}(\log J-\log J_0)dy\right|&=\left|\int_{-(k+1)}^{-k}\int_0^t\frac{v_y}{J}d\tau dy\right|\\
&\le\delta_T^{-1}\sqrt t\|v_y\|_{L^2((-(k+1),-k)\times(0,t))}\ \to\ 0,\quad \text{as } k\to\infty.
\end{aligned}$$

While $\varrho_0\vartheta \in L^1(0,T;L^1(\mathbb{R}))$ yields

$$\int_{-(k+1)}^{-k}\int_0^t \frac{\varrho_0\vartheta}{J}d\tau dy \leq \delta_T^{-1}\|\varrho_0\vartheta\|_{L^1((-(k+1),-k)\times(0,t))} \to 0, \quad \text{as } k\to\infty.$$

Since $\varrho_0 \in L^1(\mathbb{R})$ and $\sqrt{\varrho_0}v \in L^\infty(0,T;L^2(\mathbb{R}))$, one has

$$\left|\int_{-(k+1)}^{-k}\int_{-\infty}^{y}\varrho_0 v dy'dy\right| \leq \left(\int_{-\infty}^{-k}\varrho_0 dy\right)^{\frac12}\left(\int_{-\infty}^{-k}\varrho_0 v^2\right)^{\frac12} \to 0, \quad \text{as } k\to\infty.$$

Hence, $f(t)\equiv 0$, and, consequently, (4.4) gives

$$\int_{-\infty}^{y}\varrho_0 v dy' - \mu\log J + R\int_0^t\frac{\varrho_0\vartheta}{J}d\tau = \int_{-\infty}^{y}\varrho_0 v_0 dy' - \mu\log J_0.$$

Dividing both sides by μ and taking the exponential yield

$$\frac{1}{J}\exp\left\{\frac{R}{\mu}\int_0^t\frac{\varrho_0\vartheta}{J}d\tau\right\} = \exp\left\{\frac{1}{\mu}\int_{-\infty}^{y}\varrho_0(v_0-v)dy'\right\}\frac{1}{J_0}. \tag{4.5}$$

Multiplying (4.5) by $\frac{R}{\mu}\varrho_0\vartheta$ and integrating in t yield

$$\exp\left\{\frac{R}{\mu}\int_0^t\frac{\varrho_0\vartheta}{J}d\tau\right\} = 1+\frac{R\varrho_0}{\mu J_0}\int_0^t\exp\left\{\frac{1}{\mu}\int_{-\infty}^{y}\varrho_0(v_0-v)dy'\right\}\vartheta d\tau.$$

Substituting the above into (4.5) gives

$$\frac{1}{J}\left(1+\frac{R\varrho_0}{\mu J_0}\int_0^t e^{\frac{1}{\mu}\int_{-\infty}^{y}\varrho_0(v_0-v)dy'}\vartheta d\tau\right) = e^{\frac{1}{\mu}\int_{-\infty}^{y}\varrho_0(v_0-v)dy'}\frac{1}{J_0},$$

which yields the desired expression for J. □

As a corollary of Propositions 4.1 and 4.2, one can obtain the uniform positive lower bound of J and the upper control of J stated as follows.

Corollary 4.1. *It holds that*

$$\begin{aligned} J &\geq \underline{J}e^{-\frac{2\sqrt2}{\mu}\sqrt{\|\varrho_0\|_1\mathcal{E}_0}}, \qquad \textit{and} \\ \|J\|_\infty(t) &\leq e^{\frac{4\sqrt2}{\mu}\sqrt{\|\varrho_0\|_1\mathcal{E}_0}}\left(\bar{J}+\frac{R}{\mu}\int_0^t\|\varrho_0\vartheta\|_\infty d\tau\right). \end{aligned}$$

Proof. Proposition 4.1 implies

$$\begin{aligned} \left|\int_{-\infty}^{y}\varrho_0(v-v_0)dy'\right| &\leq \left(\int_{\mathbb{R}}\varrho_0 dz\right)^{\frac12}\left[\left(\int_{\mathbb{R}}\varrho_0 v^2 dy\right)^{\frac12}+\left(\int_{\mathbb{R}}\varrho_0 v_0^2 dy\right)^{\frac12}\right] \\ &\leq 2\sqrt{2\|\varrho_0\|_1\mathscr{E}_0}. \end{aligned}$$

Therefore, it follows from the definition of B in Proposition 4.2 that

$$\exp\left\{-\frac{2\sqrt{2}}{\mu}\sqrt{\|\varrho_0\|_1\mathscr{E}_0}\right\}\leq B(y,t)\leq\exp\left\{\frac{2\sqrt{2}}{\mu}\sqrt{\|\varrho_0\|_1\mathscr{E}_0}\right\}.$$

Due to this and $\vartheta\geq 0$, the conclusions follow easily from Proposition 4.2. □

4.2. L^2 **estimates.** We now turn to derive the $L^\infty(0,T;L^2(\mathbb{R}))$ a priori estimates on (J,v,ϑ). We need an elementary lemma.

Lemma 4.1. *Let ω and η be nonnegative and bounded on $\mathbb{R}$, satisfying $|\omega'|\leq K|\omega|$ and $\eta>0$ on $\mathbb{R}$, for some positive constant K. Assume that f is a nonnegative measurable function on $\mathbb{R}$ such that $\sqrt{\omega}f, \frac{f'}{\sqrt{\eta}}\in L^2(\mathbb{R})$, and $\omega f\in L^1(\mathbb{R})$. Then,*

$$\|\sqrt{\omega}f\|_\infty^2\leq 2K\|\sqrt{\omega}f\|_2^2+8\|\omega\|_\infty^{\frac13}\|\omega f\|_1^{\frac23}\left\|\frac{f'}{\sqrt{\eta}}\right\|_2^{\frac43}\|\eta\|_\infty^{\frac23}.$$

Proof. By assumptions and elementary calculations, one deduces

$$\begin{aligned}\|\sqrt{\omega}f\|_\infty^2 &\leq \int_{\mathbb{R}}(|\omega'|f^2+2\omega f|f'|)dz\\ &\leq K\|\sqrt{\omega}f\|_2^2+2\|\sqrt{\omega}f\|_\infty^{\frac12}\|\omega\|_\infty^{\frac14}\|\omega f\|_1^{\frac12}\left\|\frac{f'}{\sqrt{\eta}}\right\|_2\|\eta\|_\infty^{\frac12}\\ &\leq 2^{\frac53}\|\omega\|_\infty^{\frac13}\|\omega f\|_1^{\frac23}\left\|\frac{f'}{\sqrt{\eta}}\right\|_2^{\frac43}\|\eta\|_\infty^{\frac23}+\frac12\|\sqrt{\omega}f\|_\infty^2+K\|\sqrt{\omega}f\|_2^2,\end{aligned}$$

which yields the desired conclusion. □

Now we are ready to derive the $L^\infty(0,T;L^2)$ estimates.

Proposition 4.3. *It holds that*

$$\begin{aligned}\sup_{0\leq t\leq T}(\|\sqrt{\varrho_0}E\|_2^2+\|J\|_\infty^2)+\int_0^T(\|v_y\|_2^2+\||v|v_y\|_2^2+\|\vartheta_y\|_2^2+\|\sqrt{\varrho_0}\vartheta\|_\infty^2)dt\\ \leq C(1+\|\sqrt{\varrho_0}E_0\|_2^2),\end{aligned}$$

for a positive constant C depending only on $\mu,\kappa,c_v,R,K_1,\bar{\varrho},\underline{J},\bar{J},\|\varrho_0\|_1,T,$ and $\mathcal{E}_0$.

Proof. Let φ_r be given as in the proof of Proposition 4.1. Testing (1.8) with $v\varphi_r^2$ yields

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{R}}\varrho_0v^2\varphi_r^2dy+\mu\int_{\mathbb{R}}\frac{v_y^2}{J}\varphi_r^2dy\\ =&-2\mu\int_{\mathbb{R}}\frac{v_y}{J}v\varphi_r\varphi_r'dy+R\int_{\mathbb{R}}\frac{\varrho_0\vartheta}{J}(v_y\varphi_r^2+2v\varphi_r\varphi_r')dy\end{aligned}$$

$$
\begin{aligned}
\le\ & \frac{C}{r}\left(\int_{r\le|y|\le 2r}|vv_y|dy+\|\sqrt{\varrho_0}v\|_2\|\sqrt{\varrho_0}\vartheta\|_2\right)\\
&+\frac{\mu}{2}\int_{\mathbb{R}}\frac{v_y^2}{J}\varphi_r^2dy+C\int_{\mathbb{R}}\varrho_0E^2\varphi_r^2dy,
\end{aligned}
$$

where Corollary 4.1 has been used. Therefore, recalling Proposition 4.1, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\varrho_0v^2\varphi_r^2dy+\mu\int_{\mathbb{R}}\frac{v_y^2}{J}\varphi_r^2dy\\
\le\ & \frac{C}{r}\Big(\int_{r\le|y|\le 2r}|vv_y|dy+\|\sqrt{\varrho_0}\vartheta\|_2\Big)+C\int_{\mathbb{R}}\varrho_0E^2\varphi_r^2dy. \qquad (4.6)
\end{aligned}
$$

Rewrite equation (2.9) as $\varrho_0E_t-\frac{\kappa}{c_v}\left(\frac{E_y}{J}\right)_y=\left(\mu-\frac{\kappa}{c_v}\right)\left(\frac{vv_y}{J}\right)_y-R\left(\frac{\varrho_0\vartheta v}{J}\right)_y$ and test it with $E\varphi_r^2$ to get

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb{R}}\varrho_0E^2\varphi_r^2dy+\frac{\kappa}{c_v}\int_{\mathbb{R}}\frac{E_y^2}{J}\varphi_r^2dy\\
=\ & \int_{\mathbb{R}}\left[R\varrho_0\vartheta v+\left(\frac{\kappa}{c_v}-\mu\right)vv_y\right]\frac{E_y}{J}\varphi_r^2dy\\
&+2\int_{\mathbb{R}}\left[-\frac{\kappa}{c_v}E_y+R\varrho_0\vartheta v+\left(\frac{\kappa}{c_v}-\mu\right)vv_y\right]\frac{E}{J}\varphi_r\varphi_r'dy\\
\le\ & \int_{\mathbb{R}}\left[\frac{\kappa}{2c_v}E_y^2+C\left(v^2v_y^2+\varrho_0^2\vartheta^2v^2\right)\right]\frac{\varphi_r^2}{J}dy\\
&+\frac{C}{r}\int_{r\le|y|\le 2r}E(|E_y|+\varrho_0\vartheta|v|+|v||v_y|)dy\\
\le\ & \frac{\kappa}{2c_v}\int_{\mathbb{R}}\frac{E_y^2}{J}\varphi_r^2dy+C\int_{\mathbb{R}}\left(\frac{v^2v_y^2}{J}+\varrho_0^2\vartheta^2v^2\right)\varphi_r^2dy\\
&+\frac{C}{r}\int_{r\le|y|\le 2r}\left[E(|\vartheta_y|+|v||v_y|)+\varrho_0E^{\frac52}\right]dy,
\end{aligned}
$$

where Corollary 4.1 has been used. Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\varrho_0E^2\varphi_r^2dy+\frac{\kappa}{c_v}\int_{\mathbb{R}}\frac{E_y^2}{J}\varphi_r^2dy\\
\le\ & C\int_{\mathbb{R}}\left(\frac{v^2v_y^2}{J}+\varrho_0^2\vartheta^2v^2\right)\varphi_r^2dy+\frac{C}{r}\int_{r\le|y|\le 2r}\left[E(|\vartheta_y|+|v||v_y|)+\varrho_0E^{\frac52}\right]dy. \qquad (4.7)
\end{aligned}
$$

Similarly, taking the inner product of (1.8) with $v^3\varphi_r^2$ leads to

$$
\frac{d}{dt}\int_{\mathbb{R}}\varrho_0v^4\varphi_r^2dy+8\mu\int_{\mathbb{R}}\frac{v_y^2}{J}v^2\varphi_r^2dy
$$

$$\leq \frac{C}{r}\int_{r\leq|y|\leq 2r}(\varrho_0E^{\frac52}+E|v||v_y|)dy+C\int_{\mathbb{R}}\varrho_0^2v^2\vartheta^2\varphi_r^2dy. \tag{4.8}$$

Multiplying (4.8) with a sufficiently large positive number M and adding the resultant with (4.6) and (4.7), one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\varrho_0(v^2+E^2+Mv^4)\varphi_r^2dy+\int_{\mathbb{R}}\frac1J\left(\mu v_y^2+\frac{\kappa}{c_v}E_y^2+\mu Mv^2v_y^2\right)\varphi_r^2dy\\
&\quad\leq C\int_{\mathbb{R}}(\varrho_0E^2+\varrho_0^2\vartheta^2v^2)\varphi_r^2dy+\frac{C}{r}\left(\int_{r\leq|y|\leq 2r}|v||v_y|dy+\|\sqrt{\varrho_0}\vartheta\|_2\right)\\
&\qquad+\frac{C}{r}\int_{r\leq|y|\leq 2r}\left[\varrho_0E^{\frac52}+(|v||v_y|+|\vartheta_y|)E\right]dy.
\end{aligned}$$

Integrating the above inequality in t yields

$$\begin{aligned}
&\left(\int_{\mathbb{R}}\varrho_0E^2\varphi_r^2dy\right)(t)+\int_0^t\int_{\mathbb{R}}\frac{\varphi_r^2}{J}\left(v_y^2+E_y^2+v^2v_y^2+\vartheta_y^2\right)dyd\tau\\
&\quad\leq C\left(1+\|\sqrt{\varrho_0}E_0\|_2^2+\int_0^t\int_{\mathbb{R}}(\varrho_0E^2+\varrho_0^2\vartheta^2v^2)\varphi_r^2dyd\tau\right)\\
&\qquad+\frac{C}{r}\int_0^t\int_{r\leq|y|\leq 2r}\left[\varrho_0E^{\frac52}+(|v||v_y|+|\vartheta_y|)E\right]dyd\tau\\
&\qquad+\frac{C}{r}\int_0^t\left(\int_{r\leq|y|\leq 2r}|v||v_y|dy+\|\sqrt{\varrho_0}\vartheta\|_2\right)d\tau.
\end{aligned}\tag{4.9}$$

We claim that the last two terms on the right-hand side of (4.9) tend to zero, as $r\to\infty$. Since $vv_y\in L^2(0,T;L^2)$ and $\sqrt{\varrho_0}\vartheta\in L^\infty(0,T;L^2)$, one deduces

$$\begin{aligned}
I_1 &:= \frac{C}{r}\int_0^t\left(\int_{r\leq|y|\leq 2r}|v||v_y|dy+\|\sqrt{\varrho_0}\vartheta\|_2\right)d\tau\\
&\leq \frac{Ct^{\frac12}}{r^{\frac12}}\left(\int_0^t\int_{r\leq|y|\leq 2r}v^2v_y^2dyd\tau\right)^{\frac12}+\frac{Ct}{r}\sup_{0\leq\tau\leq t}\|\sqrt{\varrho_0}\vartheta\|_2\to0,\quad \text{as } r\to\infty.
\end{aligned}$$

For $t\in(0,T)$, choose $\xi(t)\in(-1,1)$ such that $E^2(\xi(t),t)\leq\frac{2\int_{-1}^1\varrho_0E^2dz}{\int_{-1}^1\varrho_0dz}$. Then,

$$|E(y,t)|=\left|E(\xi(t),t)+\int_{\xi(t)}^yE_y(z,t)dz\right|\leq C\|\sqrt{\varrho_0}E\|_2+\|E_y\|_2^{\frac12}(|y|+1)^{\frac12},\quad\forall y\in\mathbb{R}.$$

Hence, for any $r\geq1$, it holds that

$$|E(y,t)|\leq C\left(\|\sqrt{\varrho_0}E\|_2+\|E_y\|_2^{\frac12}r^{\frac12}\right),\quad\forall r\leq|y|\leq 2r. \tag{4.10}$$

It follows from (4.10), $\sqrt{\varrho_0}E\in L^\infty(0,T;L^2(\mathbb{R}))$, $E_y\in L^2(0,T;L^2(\mathbb{R}))$, and

$$
\begin{aligned}
I_2 &:= \frac{1}{r}\int_0^t\int_{r\leq|y|\leq 2r}\varrho_0 E^{\frac52}dyd\tau\\
&\leq \frac{C}{r}\int_0^t\int_{r\leq|y|\leq 2r}\left(\|\sqrt{\varrho_0}E\|_2^{\frac12}+\|E_y\|_2^{\frac14}r^{\frac14}\right)\varrho_0E^2dyd\tau\\
&\leq \frac{Ct}{r}\sup_{0\leq s\leq t}\|\sqrt{\varrho_0}E\|_2^{\frac52}+\frac{Ct^{\frac78}}{r^{\frac34}}\sup_{0\leq s\leq t}\|\sqrt{\varrho_0}E\|_2^2\left(\int_0^t\|E_y\|_2^2d\tau\right)^{\frac18},
\end{aligned}
$$

that $I_2\to 0$, as $r\to\infty$. Similarly, it follows from (4.10), $(\sqrt{\varrho_0}E, vv_y,\vartheta_y,E_y)\in L^2(0,T;L^2(\mathbb{R}))$, and

$$
\begin{aligned}
I_3 &:= \frac{1}{r}\int_0^t\int_{r\leq|y|\leq 2r}(|v||v_y|+|\vartheta_y|)Edyd\tau\\
&\leq \frac{C}{r}\int_0^t\int_{r\leq|y|\leq 2r}(|v||v_y|+|\vartheta_y|)\left(\|\sqrt{\varrho_0}E\|_2+\|E_y\|_2^{\frac12}r^{\frac12}\right)dyd\tau\\
&\leq \frac{C}{r}\int_0^t\left[\int_{r\leq|y|\leq 2r}(v^2v_y^2+\vartheta_y^2)dy\right]^{\frac12}r^{\frac12}\left(\|\sqrt{\varrho_0}E\|_2+\|E_y\|_2^{\frac12}r^{\frac12}\right)d\tau\\
&\leq \frac{C}{r^{\frac12}}\left(\int_0^t\|\sqrt{\varrho_0}E\|_2^2d\tau\right)^{\frac12}\left[\int_0^t\int_{r\leq|y|\leq 2r}(v^2v_y^2+\vartheta_y^2)dyd\tau\right]^{\frac12}\\
&\quad +Ct^{\frac14}\left[\int_0^t\int_{r\leq|y|\leq 2r}(v^2v_y^2+\vartheta_y^2)dyd\tau\right]^{\frac12}\left(\int_0^t\|E_y\|_2^2d\tau\right)^{\frac14},
\end{aligned}
$$

that $I_3\to 0$, as $r\to\infty$.

Thus, taking the limit as $r\uparrow\infty$ in (4.9) gives

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\varrho_0E^2dy\right)(t)+\int_0^t\int_{\mathbb{R}}\frac{1}{J}\left(v_y^2+E_y^2+v^2v_y^2+\vartheta_y^2\right)dyd\tau\\
&\quad\leq C\left(1+\|\sqrt{\varrho_0}E_0\|_2^2+\int_0^t\int_{\mathbb{R}}(\varrho_0E^2+\varrho_0^2\vartheta^2v^2)dyd\tau\right). \qquad (4.11)
\end{aligned}
$$

By Proposition 4.1, one has

$$
\int_0^t\int_{\mathbb{R}}\varrho_0^2\vartheta^2v^2dyd\tau\leq\int_0^t\|\sqrt{\varrho_0}v\|_2^2\|\sqrt{\varrho_0}\vartheta\|_\infty^2d\tau\leq C\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2d\tau.
$$

Therefore, it follows from (4.11) that

$$
\begin{aligned}
&\|\sqrt{\varrho_0}E\|_2^2(t)+\int_0^t\int_{\mathbb{R}}\frac{1}{J}\left(v_y^2+E_y^2+v^2v_y^2+\vartheta_y^2\right)dyd\tau\\
\leq\ & C\left(1+\|\sqrt{\varrho_0}E_0\|_2^2+\int_0^t\|\sqrt{\varrho_0}E\|_2^2d\tau\right)+A_2\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2d\tau, \qquad (4.12)
\end{aligned}
$$

with a positive constant A_2.

It remains to estimate $\|\sqrt{\varrho_0}\vartheta\|_\infty^2$ in (4.12). Note that (4.2) implies $|\varrho_0'| \leq \sqrt{\bar\varrho}K_1\varrho_0$. One can apply Lemma 4.1, with $\omega = \varrho_0$, $f = \vartheta$, and $\eta = J$, to obtain

$$\|\sqrt{\varrho_0}\vartheta\|_\infty^2 \leq 2\sqrt{\bar\varrho}K_1\|\sqrt{\varrho_0}\vartheta\|_2^2 + 8\|\varrho_0\|_\infty^{\frac13}\|\varrho_0\vartheta\|_1^{\frac23}\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^{\frac43}\|J\|_\infty^{\frac23}. \tag{4.13}$$

It follows from (4.13), Proposition 4.1, and Corollary 4.1 that

$$\begin{aligned}
\|\sqrt{\varrho_0}\vartheta\|_\infty^2(t) &\leq C\|\sqrt{\varrho_0}E\|_2^2 + C\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^{\frac43}\|J\|_\infty^{\frac23}\\
&\leq \frac{1}{4A_2}\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2 + C(\|J\|_\infty^2 + \|\sqrt{\varrho_0}E\|_2^2)\\
&\leq \frac{1}{4A_2}\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2 + C\left(1 + \|\sqrt{\varrho_0}E\|_2^2 + \int_0^t \|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right),
\end{aligned}$$

and, thus,

$$2A_2\int_0^t \|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau \leq \frac12\int_0^t\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2 d\tau + C + C\int_0^t\left(\|\sqrt{\varrho_0}E\|_2^2 + \int_0^\tau \|\sqrt{\varrho_0}\vartheta\|_\infty^2 ds\right)d\tau. \tag{4.14}$$

Summing (4.14) with (4.12) yields

$$\begin{aligned}
&\|\sqrt{\varrho_0}E\|_2^2(t) + \int_0^t\left[\int_{\mathbb R}\frac1J\left(v_y^2 + E_y^2 + v^2v_y^2 + \vartheta_y^2\right)dy + \|\sqrt{\varrho_0}\vartheta\|_\infty^2\right]d\tau\\
&\quad \leq C(1 + \|\sqrt{\varrho_0}E_0\|_2^2) + C\int_0^t\left(\|\sqrt{\varrho_0}E\|_2^2 + \int_0^\tau \|\sqrt{\varrho_0}\vartheta\|_\infty^2 ds\right)d\tau.
\end{aligned}$$

Thus the Gronwall inequality yields

$$\begin{aligned}
\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}E\|_2^2 + \int_0^T\left[\int_{\mathbb R}\frac1J\left(v_y^2 + E_y^2 + v^2v_y^2 + \vartheta_y^2\right)dy + \|\sqrt{\varrho_0}\vartheta\|_\infty^2\right]dt&\\
\leq C(1 + \|\sqrt{\varrho_0}E_0\|_2^2).&
\end{aligned} \tag{4.15}$$

This and Corollary 4.1 show that

$$\sup_{0\leq t\leq T}\|J\|_\infty^2 \leq C(1 + \|\sqrt{\varrho_0}E_0\|_2^2). \tag{4.16}$$

Then the conclusion follows from (4.15) and (4.16). □

4.3. H^1 **estimates.** In this subsection, we establish the $L^\infty(0,T;H^1)$ type a priori estimates for (J, v, ϑ).

We start with the H^1 estimate of J.

Proposition 4.4. *It holds that*

$$\sup_{0\leq t\leq T}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2\leq C\left(\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2+\|\sqrt{\varrho_0}E_0\|_2\right),$$

for a positive constant C depending only on $\mu,\kappa,c_v,R,T,K,\bar{\varrho},\underline{J},\bar{J},\|\varrho_0\|_1$, and $\mathscr{E}_0$.

Proof. Recalling Proposition 4.2 and the following estimate on B obtained in the proof of Corollary 4.1

$$\exp\left\{-\frac{2\sqrt{2}}{\mu}\sqrt{\|\varrho_0\|_1\mathcal{E}_0}\right\}\leq B\leq\exp\left\{\frac{2\sqrt{2}}{\mu}\sqrt{\|\varrho_0\|_1\mathcal{E}_0}\right\}, \tag{4.17}$$

one can get from

$$J_y=B\left[J_0'+\frac{R}{\mu}\int_0^t\left(\frac{\varrho_0'\vartheta+\varrho_0\vartheta_y}{B}-\frac{B_y}{B^2}\varrho_0\vartheta\right)d\tau\right]+B_y\left(J_0+\frac{R}{\mu}\int_0^t\frac{\varrho_0\vartheta}{B}d\tau\right),$$

(4.17), and (4.2) that

$$\begin{aligned}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2 \leq\ & C\left[\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2+\int_0^t(\|\sqrt{\varrho_0}\vartheta\|_2+\|\vartheta_y\|_2+\|B_y\|_2\|\sqrt{\varrho_0}\vartheta\|_\infty)d\tau\right]\\ &+\left\|\frac{B_y}{\sqrt{\varrho_0}}\right\|_2\left(\|J_0\|_\infty+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty d\tau\right)\\ \leq\ & C\left[\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2+\int_0^t(\|\sqrt{\varrho_0}\vartheta\|_2+\|\vartheta_y\|_2+\|\sqrt{\varrho_0}(v-v_0)\|_2\|\sqrt{\varrho_0}\vartheta\|_\infty)d\tau\right]\\ &+\|\sqrt{\varrho_0}(v-v_0)\|_2\left(\|J_0\|_\infty+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty d\tau\right).\end{aligned}$$

Then the desired conclusion follows by Proposition 4.1 and Proposition 4.3. □

Next, we carry out the estimate on the effective viscous flux G, which is the key to get the corresponding $L^\infty(0,T;H^1)$ estimates of v and ϑ.

For simplicity of presentations, the proofs of Proposition 4.5 and Proposition 4.7 in this subsection, as well as the uniqueness part of Theorem 4.1 in the next one, will be given "formally". However, similar to the proof of Proposition 4.3, one can easily adopt the cut-off procedure there to justify the arguments rigorously.

Proposition 4.5. *It holds that*

$$\sup_{0\leq t\leq T}\|G\|_2^2+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2d\tau\leq C,$$

for a positive constant C depending only on $\mu,\kappa,c_v,R,T,K_1,\bar{\varrho},\underline{J},\bar{J},\|\varrho_0\|_1,\mathcal{E}_0$, $\|\sqrt{\varrho_0}E_0\|_2$, and $\|G_0\|_2$, where G is defined by (1.12), and $G_0=\frac{1}{J_0}(v_0'-R\varrho_0\vartheta_0)$.

Proof. Taking the inner product of (1.13) with JG yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} JG^2dy+\mu\int_{\mathbb{R}}\frac{G_y^2}{\varrho_0}dy\leq\frac{\mu}{2}\int_{\mathbb{R}}\frac{G_y^2}{\varrho_0}dy+C\int_{\mathbb{R}}\left(|v_y|G^2+\varrho_0\vartheta_y^2\right)dy, \tag{4.18}$$

where Corollary 4.1 has been used. By the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\int_{\mathbb{R}}|v_y|G^2dy &\leq \|v_y\|_2\|G\|_4^2\leq C\|v_y\|_2\|G\|_2^{\frac{3}{2}}(\|G\|_2+\|G_y\|_2)^{\frac{1}{2}}\\ &\leq \varepsilon\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+C_\varepsilon(1+\|v_y\|_2^2)\|G\|_2^2,\end{aligned}$$

for any $\varepsilon>0$. Thanks to this and choosing ε suitably small, one obtains from Corollary 4.1, Proposition 4.3, and (4.18) that

$$\sup_{0\leq t\leq T}\|G\|_2^2+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2d\tau\leq Ce^{C\int_0^T(1+\|v_y\|_2^2)d\tau}(1+\|G_0\|_2^2)\leq C.$$

This proves the conclusion. □

Then, we derive the $L^\infty(0,T;H^1)$ estimate on v.

Proposition 4.6. *It holds that*

$$\sup_{0\leq t\leq T}\|v_y\|_2^2+\int_0^T\left(\|\sqrt{\varrho_0}v_t\|_2^2+\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2\right)dt\leq C,$$

for a positive constant C depending only on c_v,R,μ,κ,T,$\bar\varrho$,K_1,$\underline{J}$,$\bar J$,$\|\varrho_0\|_1$,$\mathcal{E}_0$, $\|\sqrt{\varrho_0}E_0\|_2$, $\|G_0\|_2$, and $\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2$.

Proof. Since $v_y=\frac{1}{\mu}(JG+R\varrho_0\vartheta)$ and $\varrho_0v_t=G_y$, it follows from Corollary 4.1, Proposition 4.3, and Proposition 4.5 that

$$\begin{aligned}\sup_{0\leq t\leq T}\|v_y\|_2^2+\int_0^T\|\sqrt{\varrho_0}v_t\|_2^2dt &= \frac{1}{\mu}\sup_{0\leq t\leq T}\|JG+R\varrho_0\vartheta\|_2^2+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2dt\\ &\leq C\sup_{0\leq t\leq T}(\|G\|_2^2+\|\varrho_0\vartheta\|_2^2)+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2dt\leq C.\end{aligned}$$

Since

$$v_{yy}=\frac{1}{\mu}(JG_y+J_yG+R\varrho_0'\vartheta+R\varrho_0\vartheta_y),$$

it follows from (4.2), the Sobolev inequality, and Propositions 4.3–4.5 that

$$\int_0^T\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2dt\leq C\int_0^T\left(\left\|\left(\frac{G_y}{\sqrt{\varrho_0}},\varrho_0\vartheta,\sqrt{\varrho_0}\vartheta_y\right)\right\|_2^2+\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\|G\|_\infty^2\right)dt$$

$$\leq \ C\int_0^T \left(\left\|\left(\frac{G_y}{\sqrt{\varrho_0}},\varrho_0\vartheta,\sqrt{\varrho_0}\vartheta_y\right)\right\|_2^2+\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\|G\|_{H^1}^2\right)dt\leq C,$$

which yields the conclusion. □

Finally, we give the corresponding weighted $L^\infty(0,T;H^1)$ estimates on ϑ.

Proposition 4.7. *The following estimate holds*

$$\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\int_0^T\left(\|\varrho_0\vartheta_t\|_2^2+\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2+\|\vartheta_{yy}\|_2^2\right)dt\leq C,$$

for a positive constant C depending only on $\mu,\kappa,c_v,R,T,K_1,\bar\varrho,\underline{J},\bar J,\|\varrho_0\|_1,\mathcal{E}_0,\|\sqrt{\varrho_0}E_0\|_2$, $\|G_0\|_2,\|\sqrt{\varrho_0}\vartheta_0'\|_2$, and $\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2$.

Proof. Rewrite (1.9) as $c_v\varrho_0\vartheta_t-\kappa\left(\frac{\vartheta_y}{J}\right)_y=v_yG$. Then,

$$-2c_v\kappa\int_{\mathbb{R}}\varrho_0\vartheta_t\left(\frac{\vartheta_y}{J}\right)_y dy+\int_{\mathbb{R}}\left(c_v^2\varrho_0^2\vartheta_t^2+\kappa\left|\left(\frac{\vartheta_y}{J}\right)_y\right|^2\right)dy=\int_{\mathbb{R}}v_y^2G^2dy. \tag{4.19}$$

By direct calculations, one can get that

$$\begin{aligned}
&c_v\kappa\frac{d}{dt}\int_{\mathbb{R}}\frac{\varrho_0}{J}\vartheta_y^2dy+\int_{\mathbb{R}}\left(c_v^2\varrho_0^2\vartheta_t^2+\kappa^2\left|\left(\frac{\vartheta_y}{J}\right)_y\right|^2\right)dy\\
=&\ -c_v\kappa\int_{\mathbb{R}}\left(\frac{\varrho_0}{J^2}v_y\vartheta_y^2+2\varrho_0'\vartheta_t\frac{\vartheta_y}{J}\right)dy+\int_{\mathbb{R}}v_y^2G^2dy\\
\leq&\ \frac{c_v^2}{2}\int_{\mathbb{R}}\varrho_0^2\vartheta_t^2dy+C\int_{\mathbb{R}}(\varrho_0|v_y|\vartheta_y^2+\varrho_0\vartheta_y^2+v_y^2G^2)dy,
\end{aligned}\tag{4.20}$$

where (4.2) and Corollary 4.1 have been used. Then, Propositions 4.3 and 4.5 imply

$$\int_{\mathbb{R}}\varrho_0|v_y|\vartheta_y^2dy = \frac{1}{\mu}\int_{\mathbb{R}}\varrho_0|JG+R\varrho_0\vartheta|\vartheta_y^2dy\leq C(\|G\|_{H^1}+\|\varrho_0\vartheta\|_\infty)\|\sqrt{\varrho_0}\vartheta_y\|_2^2,$$

and

$$\begin{aligned}
\int_{\mathbb{R}}v_y^2G^2dy &= \frac{1}{\mu^2}\int_{\mathbb{R}}(JG+R\varrho_0\vartheta)^2G^2dy\\
&\leq C\left[\|G\|_2^3(\|G\|_2+\|G_y\|_2)+\|\varrho_0\vartheta\|_\infty^2\|G\|_2^2\right]\\
&\leq C(1+\|G_y\|_2+\|\varrho_0\vartheta\|_\infty^2).
\end{aligned}$$

Substituting the above two estimates into (4.20) gives

$$
\begin{aligned}
&c_v\kappa\frac{d}{dt}\int_{\mathbb{R}}\frac{\varrho_0}{J}\vartheta_y^2dy+\frac{1}{2}\int_{\mathbb{R}}\left(c_v^2\varrho_0^2\vartheta_t^2+\kappa^2\left|\left(\frac{\vartheta_y}{J}\right)_y\right|^2\right)dy\\
&\qquad\leq C(1+\|\varrho_0\vartheta\|_\infty^2+\|G\|_{H^1})(1+\|\sqrt{\varrho_0}\vartheta_y\|_2^2),
\end{aligned}
$$

which, together with Corollary 4.1 and Propositions 4.3 and 4.5, implies

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\int_0^T\left(\|\varrho_0\vartheta_t\|_2^2+\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2\right)dt\\
&\qquad\leq e^{C\int_0^T(1+\|\varrho_0\vartheta\|_\infty^2+\|G\|_{H^1})dt}(1+\|\sqrt{\varrho_0}\vartheta_0'\|_2^2)\leq C. \qquad (4.21)
\end{aligned}
$$

It remains to estimate ϑ_{yy}. Direct calculations show that

$$
\vartheta_{yy}=J\left(\frac{\vartheta_y}{J}\right)_y+\frac{J_y}{\sqrt{\varrho_0}}\frac{\sqrt{\varrho_0}}{J}\vartheta_y,
$$

and

$$
\begin{aligned}
\left\|\varrho_0\left(\frac{\vartheta_y}{J}\right)^2\right\|_\infty &\leq \int_{\mathbb{R}}\left|\partial_y\left(\varrho_0\left(\frac{\vartheta_y}{J}\right)^2\right)\right|dy=\int_{\mathbb{R}}\left|\varrho_0'\frac{\vartheta_y^2}{J^2}+2\varrho_0\frac{\vartheta_y}{J}\left(\frac{\vartheta_y}{J}\right)_y\right|dy\\
&\leq C\left(\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\|\sqrt{\varrho_0}\vartheta_y\|_2\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2\right),
\end{aligned}
$$

which gives

$$
\left\|\sqrt{\varrho_0}\frac{\vartheta_y}{J}\right\|_\infty^2\leq C\left(1+\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2\right), \qquad (4.22)
$$

where one has used Corollary 4.1, (4.2), and (4.21). It follows from (4.21), (4.22), and Proposition 4.4 that

$$
\begin{aligned}
\int_0^T\|\vartheta_{yy}\|_2^2dt &\leq C\int_0^T\left(\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2+\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\left\|\frac{\sqrt{\varrho_0}}{J}\vartheta_y\right\|_\infty^2\right)dt\\
&\leq C\int_0^T\left(1+\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2+\left\|\left(\frac{\vartheta_y}{J}\right)_y\right\|_2^2\right)dt\leq C.
\end{aligned}
$$

Combining this with (4.21) yields the desired conclusion. □

4.4. **Global existence and uniqueness.** Based on the a priori estimates in the previous subsections, we are now ready to prove the following global well-posedness.

Theorem 4.1. *Assume that (3.1), (4.1), and (4.2) hold. Then, there is a unique global solution* (J, v, ϑ) *to the problem (1.7)–(1.10), such that, for any finite* T,

$$0 < J, J^{-1} \in L^\infty(0,T;L^\infty), \quad \vartheta \geq 0,$$

$$J_{yt}, vv_y, \frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}v_t, \vartheta_y, \left(\frac{\vartheta_y}{J}\right)_y, \vartheta_{yy}, \varrho_0\vartheta_t \in L^2(0,T;L^2(\mathbb{R})), \tag{4.23}$$

$$J_t, \frac{J_y}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}v, \sqrt{\varrho_0}v^2, v_y, \sqrt{\varrho_0}\vartheta, \sqrt{\varrho_0}\vartheta_y \in L^\infty(0,T;L^2(\mathbb{R})), \tag{4.24}$$

$$J - J_0, \sqrt{\varrho_0}v, \varrho_0\vartheta \in C([0,T];L^2). \tag{4.25}$$

Proof. We start with the uniqueness. Let (J_1, v_1, ϑ_1) and (J_2, v_2, ϑ_2) be two solutions to problem (1.7)–(1.10), satisfying the regularities in the theorem. Set

$$(J, v, \theta) = (J_1 - J_2, v_1 - v_2, \vartheta_1 - \vartheta_2).$$

Then, straightforward calculations yield

$$J_t = v_y, \tag{4.26}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J_1}\right)_y = (\omega_1 J + \omega_2\varrho_0\vartheta)_y, \tag{4.27}$$

$$c_v\varrho_0\vartheta_t - \kappa\left(\frac{\vartheta_y}{J_1}\right)_y = (\varpi_1 J)_y + \varpi_2 v_y + \varpi_3 J + \varpi_4\varrho_0\vartheta, \tag{4.28}$$

where

$$\omega_1 := \tfrac{R\varrho_0\vartheta_2 - \mu\partial_y v_2}{J_1 J_2}, \qquad \omega_2 := -\tfrac{R}{J_1},$$
$$\varpi_1 := -\kappa\tfrac{\partial_y\vartheta_2}{J_1 J_2}, \qquad \varpi_2 := \tfrac{1}{J_1}(\mu\partial_y(v_1+v_2) - R\varrho_0\vartheta_1),$$
$$\varpi_3 := \tfrac{\partial_y v_2}{J_1 J_2}(R\varrho_0\vartheta_2 - \mu\partial_y v_2), \qquad \varpi_4 := -R\tfrac{\partial_y v_2}{J_1}.$$

Taking the inner product of (4.26) with J yields

$$\frac{d}{dt}\int_{\mathbb{R}} J^2 dy = \int_{\mathbb{R}} v_y J dy \leq \varepsilon\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy + C_\varepsilon\int_{\mathbb{R}} J_1 J^2 dy, \tag{4.29}$$

for any positive $\varepsilon > 0$. Taking the inner product of (4.27) with v leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\varrho_0 v^2 dy + \mu\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy &\leq C\int_{\mathbb{R}}(|\omega_1||J| + |\omega_2|\varrho_0|\vartheta|)|v_y|dy \\ &\leq \frac{\mu}{2}\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy + C\int_{\mathbb{R}}(\omega_1^2 J^2 + \omega_2^2\varrho_0^2\vartheta^2).\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_{\mathbb{R}}\varrho_0 v^2 dy+\mu\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy\leq C\int_{\mathbb{R}}(\omega_1^2+\omega_2^2)(J^2+\varrho_0^2\vartheta^2)dy. \tag{4.30}$$

Taking the inner product of (4.28) with $\varrho_0\vartheta$ and using (4.2), one can get

$$\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\int_{\mathbb{R}}\varrho_0^2\vartheta^2dy+\kappa\int_{\mathbb{R}}\varrho_0\frac{\vartheta_y^2}{J_1}dy\\
\leq\ & C\int_{\mathbb{R}}\frac{|\vartheta_y|}{J_1}\varrho_0^{\frac32}|\vartheta|dy+C\int_{\mathbb{R}}|\varpi_1||J|(\varrho_0^{\frac32}|\vartheta|+\varrho_0|\vartheta_y|)dy\\
&+C\int_{\mathbb{R}}(|\varpi_2||v_y|+|\varpi_3||J|+|\varpi_4|\varrho_0|\vartheta|)\varrho_0|\vartheta|dy\\
\leq\ & \frac{\kappa}{2}\int_{\mathbb{R}}\varrho_0\frac{\vartheta_y^2}{J_1}dy+C\int_{\mathbb{R}}(\varrho_0^2\vartheta^2+|\varpi_1|^2\varrho_0J^2)dy+\varepsilon\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy\\
&+C_\varepsilon\int_{\mathbb{R}}\varpi_2^2\varrho_0^2\vartheta^2dy+C\int_{\mathbb{R}}(|\varpi_3|+|\varpi_4|)(J^2+\varrho_0^2\vartheta^2)dy,
\end{aligned}$$

which yields

$$\begin{aligned}
c_v\frac{d}{dt}\int_{\mathbb{R}}\varrho_0^2\vartheta^2dy+\kappa\int_{\mathbb{R}}\varrho_0\frac{\vartheta_y^2}{J_1}dy\ \leq\ & 2\varepsilon\int_{\mathbb{R}}\frac{v_y^2}{J_1}dy+C_\varepsilon\int_{\mathbb{R}}(1+\varrho_0\varpi_1^2+\varpi_2^2\\
&+|\varpi_3|+|\varpi_4|)(J^2+\varrho_0^2\vartheta^2)dy.
\end{aligned}\tag{4.31}$$

It follows from (4.29)–(4.31) and choosing ε sufficiently small that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(J^2+\varrho_0v^2+c_v\varrho_0^2\vartheta^2\right)dy+\int_{\mathbb{R}}\frac{1}{J_1}\left(\frac{\mu}{2}v_y^2+\kappa\varrho_0\vartheta_y^2\right)dy\\
\leq\ & C(1+\|(\omega_1,\omega_2,\sqrt{\varrho_0}\varpi_1,\varpi_2)\|_\infty^2+\|(\varpi_3,\varpi_4)\|_\infty)\|(J,\sqrt{\varrho_0}v,\varrho_0\vartheta)\|_2^2.
\end{aligned}$$

Thanks to this and that

$$\omega_1,\omega_2,\sqrt{\varrho_0}\varpi_1,\varpi_2\in L^2(0,T;L^\infty(\mathbb{R}))\quad\text{and}\quad\varpi_3,\varpi_4\in L^1(0,T;L^\infty(\mathbb{R})),$$

which can be easily verified by the regularities of (J_i,v_i,ϑ_i), $i=1,2$, the uniqueness follows by the Gronwall inequality.

Next we prove the global existence. The local existence of solutions in the class stated in the theorem follows from Theorem 3.1 and Propositions 4.4, 4.6, and 4.7. Note that the regularities $J_t\in L^\infty(0,T;L^2)$ and $J_{yt}\in L^2(0,T;L^2)$ follow directly from Proposition 4.6 and equation (1.7), while the regularities in (4.25) follow from those in (4.23)–(4.24). The global existence is then the corollary of the local existence and uniqueness and the a priori estimates obtained in Propositions 4.1–4.7. This completes the proof of Theorem 4.1. □

5. Uniform lower bound of the entropy

In this section, we establish the uniform lower bound for the entropy. This is proved by a De Giorgi type iteration which will be carried out for a suitably modified entropy equation. To this end, we assume that (3.1), (4.1), and (4.2) hold, and the initial entropy is bounded from below. Furthermore, we require that

$$|\varrho_0''| \le K_2 \varrho_0^2, \quad \text{on } \mathbb{R}, \tag{5.1}$$

with any given positive constant K_2. Let (ϱ, v, ϑ) be the unique global solution guaranteed by Theorem 4.1 (for this section and the next one).

Set

$$\underline{\ell}_0 := \log\left(\frac{\min\left\{1, \frac{A}{R} e^{\frac{\underline{s}_0}{c_v}}\right\}}{\max\{1, 2^{\gamma-2}\}}\right), \tag{5.2}$$

$$\underline{J}_T := \inf_{(y,t)\in\mathbb{R}\times(0,T)} J(y,t), \quad \bar{J}_T := \sup_{(y,t)\in\mathbb{R}\times(0,T)} J(y,t), \tag{5.3}$$

$$\mathscr{Z}_J(T) := \sup_{0\le t\le T}\left(\left\|\varrho_0^{-\frac{1}{2}} J_y\right\|_2^2 + \|\sqrt{\varrho_0}\vartheta\|_2^2\right), \tag{5.4}$$

where $\underline{s}_0 := \inf_{y\in\mathbb{R}} s_0(y)$.

Due to (1.11) and that J is uniformly positive, to get a uniform lower bound for s, it suffices to obtain that for $\log\vartheta - (\gamma-1)\log\varrho_0$. For $\varepsilon \in (0,1)$, set

$$S_\varepsilon := \log\vartheta_\varepsilon - (\gamma-1)\log\tilde{\varrho}_\varepsilon, \quad \text{with } \vartheta_\varepsilon = \vartheta + \varepsilon \text{ and } \tilde{\varrho}_\varepsilon = \varrho_0 + \varepsilon^{\frac{1}{\gamma-1}}.$$

Then, by direct calculations,

$$c_v\varrho_0\partial_t S_\varepsilon - \kappa\partial_y\left(\frac{\partial_y S_\varepsilon}{J}\right) = \kappa(\gamma-1)\left(\frac{1}{J}\left(\frac{\varrho_0'}{\tilde{\varrho}_\varepsilon}\right)' - \frac{\varrho_0' J_y}{\tilde{\varrho}_\varepsilon J^2}\right) - \frac{R^2}{4\mu}\frac{\varrho_0^2\vartheta^2}{J\vartheta_\varepsilon} + H_\varepsilon. \tag{5.5}$$

where $H_\varepsilon = \frac{\mu}{J\vartheta_\varepsilon}\left(v_y - \frac{R}{2\mu}\varrho_0\vartheta\right)^2 + \kappa\frac{|\partial_y\vartheta_\varepsilon|^2}{J\vartheta_\varepsilon^2}$. Define

$$s_\varepsilon := S_\varepsilon + \underline{M}_T t \tag{5.6}$$

with

$$\underline{M}_T := \frac{\kappa(\gamma-1)}{c_v\underline{J}_T}\left(K_1^2 + K_2\right). \tag{5.7}$$

Then, it follows from (5.5) that

$$c_v\varrho_0\partial_t s_\varepsilon - \kappa\partial_y\left(\frac{\partial_y s_\varepsilon}{J}\right) = -\kappa(\gamma-1)\frac{\varrho_0' J_y}{\tilde{\varrho}_\varepsilon J^2} - \frac{R^2}{4\mu}\frac{\varrho_0^2\vartheta^2}{J\vartheta_\varepsilon} + \widetilde{H}_\varepsilon, \tag{5.8}$$

where $\widetilde{H}_\varepsilon = H_\varepsilon + c_v \underline{M}_T \varrho_0 + \kappa(\gamma-1)\frac{1}{J}\left(\frac{\varrho_0'}{\tilde{\varrho}_\varepsilon}\right)' \geq 0$. The nonnegativity of $\widetilde{H}_\varepsilon$ can be verified easily. Indeed, since $\tilde{\varrho}_\varepsilon > \varrho_0$, it follows from (4.2) and (5.1) that

$$\left|\frac{1}{J}\left(\frac{\varrho_0'}{\tilde{\varrho}_\varepsilon}\right)'\right| \leq \frac{\varrho_0}{\underline{J}_T}\left(\left|\frac{\varrho_0''}{\varrho_0^2}\right| + \left|\frac{\varrho_0'}{\varrho_0^{\frac{3}{2}}}\right|^2\right) \leq \frac{\varrho_0}{\underline{J}_T}(K_1^2 + K_2) = \frac{c_v \underline{M}_T}{\kappa(\gamma-1)}\varrho_0.$$

Thus, $\kappa(\gamma-1)\frac{1}{J}\left(\frac{\varrho_0'}{\tilde{\varrho}_\varepsilon}\right)' + c_v\underline{M}_T\varrho_0 \geq 0$. This and $H_\varepsilon \geq 0$ imply that $\widetilde{H}_\varepsilon \geq 0$.

Now, we are going to derive an uniform lower bound for s_ε, independent of ε, which will be achieved by using a De Giorgi type iteration. To this end, as a preparation, we state the following iterative lemma whose proof is given in the Appendix.

Lemma 5.1. *Let $m_0 \in [0,\infty)$ be given and f be a nonnegative non-increasing function on $[m_0,\infty)$ satisfying*

$$f(\ell) \leq \frac{M_0(\ell+1)^\alpha}{(\ell-m)^\beta} f^\sigma(m), \quad \forall \ell > m \geq m_0,$$

for some nonnegative constants M_0, α, β, and σ, with $0 \leq \alpha < \beta$ and $\sigma > 1$. Then,

$$f(m_0 + d) = 0,$$

where

$$d = \left[2f^\sigma(m_0)(m_0 + M_0 + 2)^{\frac{2\alpha+2\beta+1}{\sigma-1} + \frac{\beta}{(\sigma-1)^2} + 2\alpha+\beta+1}\right]^{\frac{1}{\beta-\alpha}} + 2.$$

5.1. L^2 **estimate on** s_ε**.** The following L^2 energy inequality holds for s_ε.

Proposition 5.1. *Let s_ε be defined as (5.6). Then, it holds that*

$$\begin{aligned}\sup_{0\leq t\leq T}\|(s_\varepsilon-\ell)_-\|_2^2 &+ \int_0^T \left\|\frac{\partial_y(s_\varepsilon-\ell)_-}{\sqrt{\varrho_0}}\right\|_2^2 dt\\ &\leq C\int_0^T\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2 + \varrho_0^2\vartheta^2\right)\Big|_{\{s_\varepsilon<\ell\}} dydt,\end{aligned}$$

for any $\ell \leq \underline{\ell}_0$, where $\underline{\ell}_0$ is given by (5.2), and C is a positive constant depending only on $R, \gamma, \kappa, \mu, \underline{J}_T, \bar{J}_T, T$, and K_1.

Proof. For $\delta > 0$, set $\varrho_\delta = \varrho_0 + \delta$. Testing (5.8) with $-\frac{\varrho_0}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2$ and recalling $\widetilde{H}_\varepsilon \geq 0$, one obtains

$$\begin{aligned}\frac{c_v}{2}\frac{d}{dt}\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2\varphi_r^2 dy &+ \kappa\int_{\mathbb{R}}\partial_y\left(\frac{\partial_y s_\varepsilon}{J}\right)\frac{\varrho_0}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2 dy\\ &\leq \int_{\mathbb{R}}\left(\kappa(\gamma-1)\frac{\varrho_0' J_y}{\tilde{\varrho}_\varepsilon J^2} + \frac{R^2}{4\mu}\frac{\varrho_0^2\vartheta^2}{J\vartheta_\varepsilon}\right)\frac{\varrho_0}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2 dy. \qquad (5.9)\end{aligned}$$

Integration by parts and using the Cauchy inequality yield

$$
\begin{aligned}
&\int_{\mathbb{R}}\partial_y\left(\frac{\partial_y s_\varepsilon}{J}\right)\frac{\varrho_0}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2dy\\
=\ &\int_{\mathbb{R}}\frac{\varrho_0}{J\varrho_\delta^2}|\partial_y(s_\varepsilon-\ell)_-|^2\varphi_r^2dy+2\int_{\mathbb{R}}\frac{\varrho_0}{J\varrho_\delta^2}\partial_y(s_\varepsilon-\ell)_-(s_\varepsilon-\ell)_-\varphi_r\varphi_r'dy\\
&+\int_{\mathbb{R}}\frac{\partial_y(s_\varepsilon-\ell)_-}{J}\frac{\varrho_0\varrho_0'}{\varrho_\delta^2}\left(\frac{1}{\varrho_0}-\frac{2}{\varrho_\delta}\right)(s_\varepsilon-\ell)_-\varphi_r^2dy\\
\geq\ &\frac{3}{4}\int_{\mathbb{R}}\frac{\varrho_0}{J\varrho_\delta^2}|\partial_y(s_\varepsilon-\ell)_-|^2\varphi_r^2dy-C\int_{\mathbb{R}}\frac{\varrho_0}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2|\varphi_r'|^2dy\\
&-C\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2\varphi_r^2dy, \qquad (5.10)
\end{aligned}
$$

where (4.2) has been used. Note that $\tilde{\varrho}_\varepsilon>\varrho_0$ and $\frac{\vartheta}{\vartheta_\varepsilon}\leq 1$. It follows from (4.2) that

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\kappa(\gamma-1)\frac{\varrho_0'J_y}{\tilde{\varrho}_\varepsilon J^2}+\frac{R^2}{4\mu}\frac{\varrho_0^2\vartheta^2}{J\vartheta_\varepsilon}\right)\frac{\varrho_0}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2dy\\
\leq\ &C\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|+\varrho_0\vartheta\right)\frac{\varrho_0^2}{\varrho_\delta^2}(s_\varepsilon-\ell)_-\varphi_r^2dy\\
\leq\ &C\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}\left[\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)\Bigg|_{\{s_\varepsilon<\ell\}}+|(s_\varepsilon-\ell)_-|^2\right]\varphi_r^2dy. \qquad (5.11)
\end{aligned}
$$

Substituting (5.10) and (5.11) into (5.9) and applying the Gronwall inequality yield

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2\varphi_r^2dy\right)(t)+\int_0^t\int_{\mathbb{R}}\frac{\varrho_0}{\varrho_\delta^2}|\partial_y(s_\varepsilon-\ell)_-|^2\varphi_r^2dy\\
\leq\ &Ce^{Ct}\left(\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2\varphi_r^2dy\Bigg|_{t=0}+\int_0^t\int_{\mathbb{R}}\frac{\varrho_0}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2|\varphi_r'|^2dyd\tau\right)\\
&+Ce^{Ct}\int_0^t\int_{\mathbb{R}}\frac{\varrho_0^2}{\varrho_\delta^2}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)\Bigg|_{\{s_\varepsilon<\ell\}}\varphi_r^2dyd\tau. \qquad (5.12)
\end{aligned}
$$

Due to the definition of s_ε, it holds that

$$
s_\varepsilon \ \geq\ \log\varepsilon-(\gamma-1)\log\left(\|\varrho_0\|_\infty+\varepsilon^{\frac{1}{\gamma-1}}\right),
$$

and, thus,

$$
0\leq(s_\varepsilon-\ell)_-\leq\max\left\{0,\ell-\log\varepsilon+(\gamma-1)\log\left(\|\varrho_0\|_\infty+\varepsilon^{\frac{1}{\gamma-1}}\right)\right\}:=A_{\ell,\varepsilon}.
$$

Therefore,

$$\int_0^t\int_{\mathbb{R}}\frac{\varrho_0}{\varrho_\delta^2}|(s_\varepsilon-\ell)_-|^2|\varphi_r'|^2dyd\tau\leq CA_{\ell,\varepsilon}^2\delta^{-2}t\int_{r\leq|y|\leq2r}\frac{\varrho_0}{r^2}dy$$
$$\leq CA_{\ell,\varepsilon}^2\delta^{-2}\|\varrho_0\|_\infty tr^{-1}\to0,\quad\text{as }r\to\infty. \tag{5.13}$$

Thanks to (5.13), one can take the limits $r\uparrow\infty$ first and then $\delta\downarrow0$ in (5.12) to get

$$\left(\int_{\mathbb{R}}|(s_\varepsilon-\ell)_-|^2dy\right)(t)+\int_0^t\int_{\mathbb{R}}\frac{|\partial_y(s_\varepsilon-\ell)_-|^2}{\varrho_0}dy$$
$$\leq\ e^{Ct}\left[\int_0^t\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)\Bigg|_{\{s_\varepsilon<\ell\}}dyd\tau+\int_{\mathbb{R}}|(s_\varepsilon-\ell)_-|^2dy\Bigg|_{t=0}\right], \tag{5.14}$$

where the monotone convergence theorem has been used.

Using the elementary inequalities that for any $a,b>0$, $(a+b)^\sigma\leq2^{\sigma-1}(a^\sigma+b^\sigma)$, if $\sigma\geq1$, and $(a+b)^\sigma\leq(a^\sigma+b^\sigma)$, if $0<\sigma<1$, one can deduce easily

$$\left(\varrho_0+\varepsilon^{\frac{1}{\gamma-1}}\right)^{\gamma-1}\leq\max\left\{1,2^{\gamma-2}\right\}(\varrho_0^{\gamma-1}+\varepsilon).$$

On the other hand,

$$\vartheta_0+\varepsilon=\frac{A}{R}e^{\frac{s_0}{c_v}}\varrho_0^{\gamma-1}+\varepsilon\geq\frac{A}{R}e^{\frac{s_0}{c_v}}\varrho_0^{\gamma-1}+\varepsilon\geq\min\left\{1,\frac{A}{R}e^{\frac{s_0}{c_v}}\right\}\left(\varrho_0^{\gamma-1}+\varepsilon\right).$$

Therefore, recalling (5.2), one has

$$s_\varepsilon\Big|_{t=0}\ =\ \log\left(\frac{\vartheta_0+\varepsilon}{\left(\varrho_0+\varepsilon^{\frac{1}{\gamma-1}}\right)^{\gamma-1}}\right)\geq\log\left(\frac{\min\left\{1,\frac{A}{R}e^{\frac{s_0}{c_v}}\right\}}{\max\left\{1,2^{\gamma-2}\right\}}\right)=\underline{\ell}_0,$$

and, consequently,

$$(s_\varepsilon-\ell)_-\Big|_{t=0}\equiv0,\quad\forall\ell\leq\underline{\ell}_0. \tag{5.15}$$

Combining (5.14) with (5.15) yields the conclusion. □

As a straightforward corollary of Proposition 5.1, we have the following:

Corollary 5.1. *Let $\underline{\ell}_0,\mathcal{Z}_J$ and s_ε be defined by (5.2), (5.4), and (5.6), respectively. Then, for any $\ell\leq\underline{\ell}_0$, it holds that*

$$\sup_{0\leq t\leq T}\|(s_\varepsilon-\ell)_-\|_2^2+\int_0^T\left\|\frac{\partial_y(s_\varepsilon-\ell)_-}{\sqrt{\varrho_0}}\right\|_2^2dt\leq C\mathcal{Z}_J(T),$$

where C is a positive constant depending only on $R,\gamma,\kappa,\mu,\underline{J}_T,\bar{J}_T,T,$ and K_1.

5.2. **The De Giorgi iteration for s_ε.** The De Giorgi iteration for s_ε is stated in the following proposition.

Proposition 5.2. *Let $\underline{\ell}_0, \mathcal{Z}_J$ and s_ε be defined by (5.2), (5.4), and (5.6), respectively, and denote*

$$q_\ell = \sup_{0\leq t\leq T}\|(s_\varepsilon-\ell)_-\|_2^2+\int_0^T\left\|\frac{\partial_y(s_\varepsilon-\ell)_-}{\sqrt{\varrho_0}}\right\|_2^2dt.$$

Then, it holds that

$$q_\ell\leq\frac{C\mathcal{Z}_J(T)}{(m-\ell)^4}q_m^2,\qquad \text{for any } -\infty<\ell<m\leq\underline{\ell}_0,$$

with a positive constant C depending only on $R,\gamma,\kappa,\mu,\bar{\varrho},\underline{J}_T,\bar{J}_T,T$, and K_1.

Proof. For any $\ell\leq\underline{\ell}_0$, Corollary 5.1 implies that

$$(s_\varepsilon-\ell)_-\in L^\infty(0,T;L^2(\mathbb{R}))\cap L^2(0,T;H^1(\mathbb{R})),$$

and Proposition 5.1 shows that

$$q_\ell\leq C\int_0^T\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)\Bigg|_{\{s_\varepsilon<\ell\}}dydt. \tag{5.16}$$

Let $-\infty<\ell<m\leq\underline{\ell}_0$. Then, it is clear that

$$1<\frac{(s_\varepsilon(y,t)-m)_-}{m-\ell},\qquad \text{for any } (y,t) \text{ such that } s_\varepsilon(y,t)<\ell.$$

It follows from this, (4.2), (5.4), and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)\Bigg|_{\{s_\varepsilon<\ell\}}dydt\\
\leq\ &\frac{1}{(m-\ell)^4}\int_0^T\int_{\mathbb{R}}\left(\left|\frac{J_y}{\sqrt{\varrho_0}}\right|^2+\varrho_0^2\vartheta^2\right)|(s_\varepsilon-m)_-|^4\,dydt\\
\leq\ &\frac{C}{(m-\ell)^4}\int_0^T\left(\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2+\|\sqrt{\varrho_0}\vartheta\|_2^2\right)\|(s_\varepsilon-m)_-\|_\infty^4dt\\
\leq\ &\frac{C\mathcal{Z}_J(T)}{(m-\ell)^4}\int_0^T\|(s_\varepsilon-m)_-\|_2^2\|\partial_y(s_\varepsilon-m)_-\|_2^2dt\\
\leq\ &\frac{C\mathcal{Z}_J(T)}{(m-\ell)^4}\sup_{0\leq t\leq T}\|(s_\varepsilon-m)_-\|_2^2\int_0^T\left\|\frac{\partial_y(s_\varepsilon-m)_-}{\sqrt{\varrho_0}}\right\|_2^2dt.
\end{aligned}\tag{5.17}$$

Combining (5.16) and (5.17) yields the conclusion. ∎

5.3. **Lower bound of the entropy.** As a corollary of Proposition 5.2 and Lemma 5.1, we have the following uniform lower bound of the entropy.

Theorem 5.1. *Assume that (3.1), (4.1), (4.2), and (5.1) hold, and that the initial entropy is bonded from below. Let* $\underline{\ell}_0$, $\underline{J}_T, \mathcal{Z}_J(T)$, *and* $\underline{M}_T$ *be given by (5.2), (5.3), (5.4), and (5.7), respectively. Then, the unique global solution obtained in Theorem 4.1 satisfies*

$$\inf_{(y,t)\in\mathbb{R}\times(0,T)} s \;\geq\; c_v\left[\log\frac{R}{A}+\underline{\ell}_0+(\gamma-1)\log\underline{J}_T-\underline{M}_TT-C\Big(\mathcal{Z}_J(T)+1-\underline{\ell}_0\Big)^5\right],$$

for any positive time T, *with a positive constant* C *depending only on* R, γ, κ, μ, $\underline{J}_T$, $\bar{J}_T$, T, *and* K_1.

Proof. Set $m_0=-\underline{\ell}_0\geq 0$, and define $f(\ell):=q_{-\ell}$, for $\ell\geq m_0$, with q_ℓ given in Proposition 5.2. Then, f is nonnegative and non-increasing on $[m_0,\infty)$. It follows from Proposition 5.2 that

$$f(\ell)=q_{-\ell}\leq\frac{C\mathcal{Z}_J(T)}{(\ell-m)^4}f^2(m),\quad \forall\ell>m\geq m_0.$$

Applying Lemma 5.1, with $M_0=C\mathcal{Z}_J(T),\alpha=0,\beta=4$, and $\sigma=2$, one can get

$$f(m_0+d_0)=q_{-(m_0+d_0)}=q_{\underline{\ell}_0-d_0}=0, \tag{5.18}$$

where $d_0=\left[2q^2_{\underline{\ell}_0}\big(-\ell_0+C\mathcal{Z}_J(T)+2\big)^{18}\right]^{\frac14}+2$. Thus,

$$(s_\varepsilon-(\underline{\ell}_0-d_0))_-=0,\qquad \text{on }\mathbb{R}\times(0,T),$$

which, due to the definition of s_ε, implies that

$$\vartheta+\varepsilon\geq e^{\ell_0-d_0-\underline{M}_TT}\left(\varrho_0+\varepsilon^{\frac{1}{\gamma-1}}\right)^{\gamma-1}.$$

This, passing limit $\varepsilon\to0$, shows that $\vartheta\geq e^{\ell_0-d_0-\underline{M}_TT}\varrho_0^{\gamma-1}$. Therefore,

$$\begin{aligned} s &= c_v\left(\log\frac{R}{A}+\log\vartheta-(\gamma-1)\log\varrho_0+(\gamma-1)\log J\right)\\ &\geq c_v\left(\log\frac{R}{A}+\underline{\ell}_0-d_0-\underline{M}_TT+(\gamma-1)\log\underline{J}_T\right), \end{aligned}\tag{5.19}$$

for any $(y,t)\in\mathbb{R}\times(0,T)$. Corollary 5.1 and the expression of d_0 imply that $d_0\leq C(\mathcal{Z}_J(T)+1-\ell_0)^5$, which, together with (5.19), leads to the conclusion. □

6. Uniform upper bound of the entropy

This section is devoted to deriving the uniform upper bound for the entropy. Due to the degeneracy of equations (1.8)–(1.9) at the far fields, some singular type estimates on (v,ϑ,G) will be needed, which require some additional compatibility conditions on the initial data. Indeed, in addition to (3.1), (4.1), (4.2), and (5.1), used in Theorem 5.1, we assume further that the initial entropy is bounded from above, and

$$\varrho_0^{\frac{1-\gamma}{2}}v_0,\varrho_0^{1-\frac{\gamma}{2}}\vartheta_0,\varrho_0^{-\frac{\gamma}{2}}G_0\in L^2(\mathbb{R}), \tag{6.1}$$

where $G_0=\mu\frac{v_0'}{J_0}-R\frac{\varrho_0}{J_0}\vartheta_0$.

All the notations in Section 5 will be adopted in this section. Furthermore, set

$$\bar\ell_0:=\frac{A}{R}e^{\frac{\bar s_0}{c_v}},$$

where $\bar s_0:=\sup_{y\in\mathbb{R}}s_0(y)$, and, for any positive time T,

$$\mathscr{Z}_\vartheta(T) := \sup_{0\le t\le T}\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2+\int_0^T\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta\|_2^2dt, \tag{6.2}$$

$$\mathscr{Z}_G(T) := \sup_{0\le t\le T}\|\varrho_0^{-\frac{\gamma}{2}}G\|_2^2+\int_0^T\|\varrho_0^{-\frac{\gamma+1}{2}}G\|_2^2dt. \tag{6.3}$$

The following lemma holds.

Lemma 6.1. *Let $\sigma\neq0$ and (4.2) hold. Then, it holds that*

$$\|\varrho_0^\sigma f\|_q\le C\|\varrho_0\|_\infty^{\frac14-\frac{1}{2q}}\left(\|\varrho_0^\sigma f\|_2+\|\varrho_0^\sigma f\|_2^{\frac12+\frac1q}\|\varrho_0^{\sigma-\frac12}\partial_yf\|_2^{\frac12-\frac1q}\right),\quad 2\le q\le\infty,$$

for any f with $\varrho_0^\sigma f\in H^1(\mathbb{R})$, where positive constant C depends only on σ,q, and K_1.

Proof. It follows from the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\varrho_0^\sigma f\|_q &\le C\|\varrho_0^\sigma f\|_2^{\frac12+\frac1q}\left(\|\varrho_0^\sigma\partial_yf\|_2+\|\varrho_0^{\sigma-1}\varrho_0'f\|_2\right)^{\frac12-\frac1q}\\ &\le C\|\varrho_0^\sigma f\|_2^{\frac12+\frac1q}\left(\|\varrho_0^\sigma\partial_yf\|_2+\|\varrho_0^{\sigma+\frac12}f\|_2\right)^{\frac12-\frac1q}\\ &\le C\|\varrho_0\|_\infty^{\frac14-\frac{1}{2q}}\left(\|\varrho_0^\sigma f\|_2+\|\varrho_0^\sigma f\|_2^{\frac12+\frac1q}\|\varrho_0^{\sigma-\frac12}\partial_yf\|_2^{\frac12-\frac1q}\right),\end{aligned}$$

which yields the conclusion. □

As mentioned already in the Introduction, the uniform upper bound for s is achieved by applying a modified De Giorgi iteration to the temperature equation rather than to the entropy equation itself. As preparations, a series of singular energy estimates will be carried out in the following three subsections. These estimates will be proven in a brief way to make the ideas clear. However, as indicated in the proof of Proposition 5.1, one can adopt similar cut-off and approximations there to justify the arguments rigorously. In particular, one can choose $\frac{\varrho_0}{\varrho_\delta^{\gamma+1}}v\varphi_r^2$ and $\frac{\varrho_0^2}{\varrho_\delta^{\gamma+1}}\vartheta\varphi_r^2$, $\frac{\varrho_0}{\varrho_\delta^{\gamma+1}}JG\varphi_r^2$, and $\frac{\varrho_0}{\varrho_\delta^{2\gamma}}(\vartheta_\ell)_+\varphi_r^2$, respectively, as testing functions in Propositions 6.1, 6.2, and 6.3, and pass the limits $r\uparrow\infty$ and $\delta\downarrow 0$ to give the rigorous proofs.

6.1. **Singular weighted estimates on** (v,ϑ).

Proposition 6.1. *It holds that*

$$\sup_{0\le t\le T}\left(\left\|\varrho_0^{\frac{1-\gamma}{2}}v\right\|_2^2+\left\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\right\|_2^2\right)+\int_0^T\left(\left\|\varrho_0^{-\frac{\gamma}{2}}v_y\right\|_2^2+\left\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\right\|_2^2\right)dt$$
$$\le C\left(\left\|\varrho_0^{\frac{1-\gamma}{2}}v_0\right\|_2^2+\left\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta_0\right\|_2^2\right)e^{C\int_0^T\|v_y\|_2^4dt},$$

for a positive constant C depending only on $\mu,\kappa,\gamma,R,\bar\varrho,K_1,T,\underline{J}_T$, and $\bar J_T$.

Proof. Taking the inner product of (1.8) with $\frac{v}{\varrho_0^\gamma}$ leads to

$$\frac12\frac{d}{dt}\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+\mu\int_{\mathbb R}\frac{v_y}{J}\partial_y\left(\frac{v}{\varrho_0^\gamma}\right)dy=R\int_{\mathbb R}\frac{\varrho_0\vartheta}{J}\partial_y\left(\frac{v}{\varrho_0^\gamma}\right)dy. \tag{6.4}$$

Direct estimates give

$$\int_{\mathbb R}\frac{v_y}{J}\partial_y\left(\frac{v}{\varrho_0^\gamma}\right)dy\ge\frac34\int_{\mathbb R}\frac{v_y^2}{J\varrho_0^\gamma}dy-C\int_{\mathbb R}\frac{v^2}{J\varrho_0^\gamma}\left|\frac{\varrho_0'}{\varrho_0}\right|^2dy, \tag{6.5}$$

and

$$\int_{\mathbb R}\frac{\varrho_0\vartheta}{J}\partial_y\left(\frac{v}{\varrho_0^\gamma}\right)dy\ \le\ \frac{\mu}{4R}\int_{\mathbb R}\frac{v_y^2}{J\varrho_0^\gamma}dy+C\int_{\mathbb R}\frac{1}{J\varrho_0^\gamma}\left(\varrho_0^2\vartheta^2+v^2\left|\frac{\varrho_0'}{\varrho_0}\right|^2\right)dy. \tag{6.6}$$

It follows from (4.2) and (6.4)–(6.5) that

$$\frac{d}{dt}\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+\frac{\mu}{\bar J_T}\|\varrho_0^{-\frac{\gamma}{2}}v_y\|_2^2\le C(\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2). \tag{6.7}$$

Next, taking the inner product of (1.9) with $\frac{\vartheta}{\varrho_0^{\gamma-1}}$ and estimating as for (6.5), one can get from (4.2) that

$$c_v\frac{d}{dt}\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2+\frac{\kappa}{\bar J_T}\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2\le C\int_{\mathbb R}\left(\frac{\vartheta^2}{J\varrho_0^{\gamma-2}}+\frac{|v_y|\vartheta^2}{J\varrho_0^{\gamma-2}}+\frac{v_y^2\vartheta}{J\varrho_0^{\gamma-1}}\right)dy. \tag{6.8}$$

Summing (6.7) with (6.8) leads to

$$\frac{d}{dt}(\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+c_v\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2)+\frac{1}{\bar J_T}(\mu\|\varrho_0^{-\frac{\gamma}{2}}v_y\|_2^2+\kappa\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2)$$
$$\le\ C(\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2)+C\int_{\mathbb R}\left(\varrho_0^{2-\gamma}|v_y|\vartheta^2+\varrho_0^{1-\gamma}v_y^2\vartheta\right)dy. \tag{6.9}$$

It follows from Lemma 6.1 that

$$\begin{aligned}\int_{\mathbb{R}} \varrho_0^{2-\gamma}|v_y|\vartheta^2 dy &\le \|v_y\|_2\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_4^2 \\ &\le C\|v_y\|_2\left(\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2+\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^{\frac{3}{2}}\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^{\frac{1}{2}}\right) \\ &\le \eta\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2+C_\eta\left(\|v_y\|_2+\|v_y\|_2^{\frac{4}{3}}\right)\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2, \qquad (6.10)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} \varrho_0^{1-\gamma}v_y^2\vartheta dy &\le \|v_y\|_2\|\varrho_0^{-\frac{\gamma}{2}}v_y\|_2\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_\infty \\ &\le C\|v_y\|_2\|\varrho_0^{-\frac{\gamma}{2}}v_y\|_2\left(\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2+\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^{\frac{1}{2}}\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^{\frac{1}{2}}\right) \\ &\le \eta\left\|(\varrho_0^{-\frac{\gamma}{2}}v_y,\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y)\right\|_2^2+C_\eta(\|v_y\|_2^2+\|v_y\|_2^4)\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2. \quad (6.11)\end{aligned}$$

Substituting (6.10) and (6.11) into (6.9) and choosing η sufficiently small yield

$$\begin{aligned}&2\frac{d}{dt}(\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+c_v\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2)+\frac{1}{\bar{J}_T}(\mu\|\varrho_0^{-\frac{\gamma}{2}}v_y\|_2^2+\kappa\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2) \\ &\qquad\le C(1+\|v_y\|_2^4)(\|\varrho_0^{\frac{1-\gamma}{2}}v\|_2^2+\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2),\end{aligned}$$

from which, by the Gronwall inequality, the conclusion follows. □

6.2. **A singular weighted estimate on G.** Based on Proposition 6.1, one can derive the corresponding weighted a priori estimates on G.

Proposition 6.2. *It holds that*

$$\sup_{0\le t\le T}\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_2^2(t)+\int_0^T\left\|\varrho_0^{-\frac{1+\gamma}{2}}G_y\right\|_2^2dt\le Ce^{\int_0^T\|v_y\|_2^2dt}\left\|(\varrho_0^{\frac{1-\gamma}{2}}v_0,\varrho_0^{1-\frac{\gamma}{2}}\vartheta_0,\varrho_0^{-\frac{\gamma}{2}}G_0)\right\|_2^2,$$

for a positive constant C depending only on $\mu,\kappa,\gamma,K_1,T,\underline{J}_T$, and $\bar{J}_T$.

Proof. Taking the inner product of (1.13) with $\frac{JG}{\varrho_0^\gamma}$ and using (4.2), one deduces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\frac{JG^2}{\varrho_0^\gamma}dy+\mu\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2 \\ \le\ & C\int_{\mathbb{R}}\left[|\vartheta_y|\left(\varrho_0^{-\gamma}|G_y|+\varrho_0^{\frac{1}{2}-\gamma}|G|\right)+|v_y|\varrho_0^{-\gamma}G^2+\varrho_0^{-\frac{1}{2}-\gamma}|G||G_y|\right]dy \\ \le\ & \frac{\mu}{8}\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2+C\left[\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2+\int_{\mathbb{R}}(1+|v_y|)\varrho_0^{-\gamma}G^2dy\right].\end{aligned}$$

Similar to (6.10), one can get

$$\int_{\mathbb{R}}|v_y|\varrho_0^{-\gamma}G^2dyd\tau \le \eta\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2+C_\eta\left(\|v_y\|_2+\|v_y\|_2^{\frac{4}{3}}\right)\|\varrho_0^{-\frac{\gamma}{2}}G\|_2^2$$

$$\leq \quad \eta\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2+C_\eta\left(1+\|v_y\|_2^2\right)\|\varrho_0^{-\frac{\gamma}{2}}G\|_2^2.$$

Combining the two inequalities above and choosing η sufficiently small yield

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{JG^2}{\varrho_0^\gamma}dy+\mu\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2\leq C\left[\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2+\left(1+\|v_y\|_2^2\right)\|\varrho_0^{-\frac{\gamma}{2}}G\|_2^2\right],$$

which leads to the conclusion by the Gronwall inequality and Proposition 6.1. □

6.3. **Higher singular weighted estimates on ϑ.** In this subsection, we derive some estimates of ϑ with weights which are more singular than those in Section 6.1.

Denote

$$\vartheta_\ell:=\vartheta-\ell\varrho_0^{\gamma-1}e^{\overline{M}_Tt},\quad \ell\geq\bar{\ell}_0, \tag{6.12}$$

where

$$\overline{M}_T:=\frac{\kappa(\gamma-1)}{c_v\underline{J}_T}(|\gamma-2|K_1^2+K_2). \tag{6.13}$$

Then,

$$c_v\varrho_0\partial_t\vartheta_\ell-\kappa\partial_y\left(\frac{\partial_y\vartheta_\ell}{J}\right)=v_yG-\ell\kappa(\gamma-1)e^{\overline{M}_Tt}\varrho_0^{\gamma-2}\varrho_0'J^{-2}J_y+N_\ell, \tag{6.14}$$

where $N_\ell:=\ell e^{\overline{M}_Tt}(\frac{\kappa}{J}(\varrho_0^{\gamma-1})''-c_v\overline{M}_T\varrho_0^\gamma)$. Note that $N_\ell\leq0$. Indeed, since $\frac{\varrho_0''}{\varrho_0^2}\leq K_2$ and $\left|\frac{\varrho_0'}{\varrho_0^{3/2}}\right|^2\leq K_1^2$, it follows from (6.13) and direct calculations that

$$\begin{aligned}
N_\ell &= \ell\kappa(\gamma-1)e^{\overline{M}_Tt}\left[\frac{1}{J}\left(\frac{\varrho_0''}{\varrho_0^2}+(\gamma-2)\left|\frac{\varrho_0'}{\varrho_0^{\frac{3}{2}}}\right|^2\right)\varrho_0^\gamma-\frac{1}{\underline{J}_T}(|\gamma-2|K_1^2+K_2)\varrho_0^\gamma\right]\\
&\leq \ell\kappa(\gamma-1)e^{\overline{M}_Tt}\left[\frac{1}{J}\left(K_2+|\gamma-2|K_1^2\right)\varrho_0^\gamma-\frac{1}{\underline{J}_T}(|\gamma-2|K_1^2+K_2)\varrho_0^\gamma\right]\\
&= \ell e^{\overline{M}_Tt}\kappa(\gamma-1)\left(K_2+|\gamma-2|K_1^2\right)\varrho_0^\gamma\frac{\underline{J}_T-J}{J\underline{J}_T}\leq0.
\end{aligned}$$

The main singularly weighted estimates on ϑ_ℓ are stated as follows:

Proposition 6.3. *There exists a positive constant C depending only on $c_v,\kappa,\gamma,\bar{\varrho},K_1,K_2,T$, $\underline{J}_T$, and $\bar{J}_T$, such that, for any $\ell\geq\bar{\ell}_0$,*

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\right\|_2^2+\int_0^T\left\|\varrho_0^{\frac{1}{2}-\gamma}\partial_y(\vartheta_\ell)_+\right\|_2^2dt\\
\leq\ & C\int_0^T\int_{\mathbb{R}}\left(|\varrho_0^{-\frac{\gamma}{2}}G|^2+|\varrho_0^{1-\frac{\gamma}{2}}\vartheta|^2+\ell|\varrho_0^{-\frac{1}{2}}J_y|\right)\varrho_0^{1-\gamma}(\vartheta_\ell)_+dydt
\end{aligned}$$

and

$$\sup_{0\leq t\leq T}\left\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\right\|_2^2+\int_0^T\left\|\varrho_0^{\frac{1}{2}-\gamma}\partial_y(\vartheta_\ell)_+\right\|_2^2dt\leq C(\ell^2\mathcal{Z}_J+\mathcal{Z}_\vartheta^2+\mathcal{Z}_G^2).$$

Proof. Testing (6.14) with $\varrho_0^{1-2\gamma}(\vartheta_\ell)_+$ and recalling $N_\ell \le 0$, one obtains

$$\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\|_2^2 + \kappa\int_{\mathbb{R}}\frac{\partial_y\vartheta_\ell}{J}\partial_y\left(\varrho_0^{1-2\gamma}(\vartheta_\ell)_+\right)dy \\
\le\ & \int_{\mathbb{R}}\left(v_yG - \ell\kappa(\gamma-1)e^{\overline{M}_Tt}\varrho_0^{\gamma-2}\varrho_0'J^{-2}J_y\right)\varrho_0^{1-2\gamma}(\vartheta_\ell)_+dy =: I. \qquad (6.15)
\end{aligned}$$

Similar to (6.5), one can get by using (4.2) that

$$\begin{aligned}
\int_{\mathbb{R}}\frac{\partial_y\vartheta_\ell}{J}\partial_y\left(\frac{(\vartheta_\ell)_+}{\varrho_0^{2\gamma-1}}\right)dy &\ge \frac{3}{4}\int_{\mathbb{R}}\frac{|\partial_y(\vartheta_\ell)_+|^2}{J\varrho_0^{2\gamma-1}}dy - C\int_{\mathbb{R}}\frac{|(\vartheta_\ell)_+|^2}{J\varrho_0^{2\gamma-1}}\left|\frac{\varrho_0'}{\varrho_0}\right|^2dy \\
&\ge \frac{3}{4\bar{J}_T}\|\varrho_0^{\frac12-\gamma}\partial_y(\vartheta_\ell)_+\|_2^2 - C\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\|_2^2. \qquad (6.16)
\end{aligned}$$

Due to (4.2) and $|v_y| \le C(|G| + \varrho_0\vartheta)$,

$$\begin{aligned}
I &\le C\int_{\mathbb{R}}\left[(|G| + \varrho_0\vartheta)|G| + \ell\varrho_0^{\gamma-\frac12}|J_y|\right]\varrho_0^{1-2\gamma}(\vartheta_\ell)_+dy \\
&\le C\int_{\mathbb{R}}(G^2 + \varrho_0^2\vartheta^2 + \ell\varrho_0^{\gamma-\frac12}|J_y|)\varrho_0^{1-2\gamma}(\vartheta_\ell)_+dy. \qquad (6.17)
\end{aligned}$$

Substituting (6.16) and (6.17) into (6.15) and noticing that $(\vartheta_0 - \ell\varrho_0^{\gamma-1})_+ \equiv 0$ for any $\ell \ge \bar{\ell}_0$, one obtains the first conclusion by the Gronwall inequality.

By the Cauchy inequality, one can derive easily from the first conclusion that

$$\begin{aligned}
&\sup_{0\le t\le T}\left\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\right\|_2^2 + \int_0^T\left\|\varrho_0^{\frac12-\gamma}\partial_y(\vartheta_\ell)_+\right\|_2^2dt \\
\le\ & C\ell^2\mathcal{Z}_J(T) + C\int_0^T\left(\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_4^4 + \left\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\right\|_4^4\right)dt,
\end{aligned}$$

for any $\ell \ge \bar{\ell}_0$. Next, it follows from Lemma 6.1 that

$$\begin{aligned}
\int_0^T\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_4^4dt &\le C\int_0^T\left(\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_2^4 + \left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_2^3\left\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\right\|_2\right)dt \\
&\le C\left(\sup_{0\le t\le T}\|\varrho_0^{-\frac{\gamma}{2}}G\|_2^2 + \int_0^T\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\|_2^2dt\right)^2 = C\mathcal{Z}_G^2(T).
\end{aligned}$$

Similarly,

$$\int_0^T\left\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\right\|_4^4dt \le C\left(\sup_{0\le t\le T}\|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\|_2^2 + \int_0^T\|\varrho_0^{\frac{1-\gamma}{2}}\vartheta_y\|_2^2dt\right)^2 = C\mathcal{Z}_\vartheta^2(T).$$

Therefore, the second conclusion holds. ∎

6.4. **The De Giorgi iteration.** In this subsection, we derive the estimates for ϑ_ℓ by the De Giorgi iteration.

Proposition 6.4. *Set*

$$Q_\ell := \sup_{0\leq t\leq T}\|\varrho_0^{1-\gamma}(\vartheta_\ell)_+\|_2^2 + \int_0^T \|\varrho_0^{\frac12-\gamma}\partial_y(\vartheta_\ell)_+\|_2^2 dt.$$

Then, it holds that

$$Q_\ell \leq \frac{C(1+\ell)^2}{(\ell-m)^3}(\mathcal{Z}_J^{\frac12}(T)+\mathcal{Z}_\vartheta(T)+\mathcal{Z}_G(T))Q_m^2 \qquad \forall \ell > m \geq \bar{\ell}_0,$$

where C is a positive constant depending only on $\kappa, \gamma, c_v, \bar{\varrho}, K_1, K_2, T, \underline{J}_T$, and $\bar{J}_T$.

Proof. By Proposition 6.3, one has that, for any $\ell \geq \bar{\ell}_0$,

$$Q_\ell \;\leq\; C\int_0^T\int_{\mathbb{R}}\left(\left|\varrho_0^{-\frac{\gamma}{2}}G\right|^2 + \left|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\right|^2 + \ell\left|\varrho_0^{-\frac12}J_y\right|\right)\varrho_0^{1-\gamma}(\vartheta_\ell)_+ dydt. \tag{6.18}$$

For any $(y,t)\in\left\{(y,t)\middle|\vartheta_\ell>0\right\}$ and $m<\ell$, it is clear that

$$(\vartheta_m)_+(y,t) \;\geq\; (\ell-m)\varrho_0^{\gamma-1}(y)e^{\overline{M}_T t} \geq (\ell-m)\varrho_0^{\gamma-1}(y),$$

and, thus,

$$1 \leq \frac{\varrho_0^{1-\gamma}}{\ell-m}(\vartheta_m)_+, \qquad \text{on } \left\{(y,t)\middle|\vartheta_\ell>0\right\} \quad \forall m<\ell. \tag{6.19}$$

Using (6.19) and noticing that $(\vartheta_\ell)_+ \leq (\vartheta_m)_+$, for $m<\ell$, one can get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\left|\varrho_0^{-\frac{\gamma}{2}}G\right|^2\varrho_0^{1-\gamma}(\vartheta_\ell)_+ dydt\\
\leq\;& \int_0^T\int_{\mathbb{R}}\left|\varrho_0^{-\frac{\gamma}{2}}G\right|^2\varrho_0^{1-\gamma}(\vartheta_m)_+\left|\frac{\varrho_0^{1-\gamma}(\vartheta_m)_+}{\ell-m}\right|^3 dydt\\
\leq\;& \frac{1}{(\ell-m)^3}\left(\int_0^T\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_6^6 dt\right)^{\frac13}\left(\int_0^T\left\|\varrho_0^{1-\gamma}(\vartheta_m)_+\right\|_6^6 dt\right)^{\frac23}.
\end{aligned} \tag{6.20}$$

Lemma 6.1 implies that

$$\int_0^T\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_6^6 dt \leq C\int_0^T\left(\left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_2^6 + \left\|\varrho_0^{-\frac{\gamma}{2}}G\right\|_2^4\left\|\varrho_0^{-\frac{\gamma+1}{2}}G_y\right\|_2^2\right)dt \leq C\mathcal{Z}_G^3, \tag{6.21}$$

and, similarly,

$$\int_0^T\left\|\varrho_0^{1-\gamma}(\vartheta_m)_+\right\|_6^6 dt \leq CQ_m^3. \tag{6.22}$$

Substituting (6.21)–(6.22) into (6.20) yields

$$\int_0^T\int_{\mathbb{R}}\left|\varrho_0^{-\frac{\gamma}{2}}G\right|^2\varrho_0^{1-\gamma}(\vartheta_\ell)_+ dydt \leq \frac{C\mathcal{Z}_G Q_m^2}{(\ell-m)^3}, \quad \ell>m\geq\bar{\ell}_0. \tag{6.23}$$

Similarly, one can show that

$$\int_0^T\int_{\mathbb{R}}\left|\varrho_0^{1-\frac{\gamma}{2}}\vartheta\right|^2\varrho_0^{1-\gamma}(\vartheta_\ell)_+dydt\leq\frac{C\mathcal{Z}_\vartheta Q_m^2}{(\ell-m)^3},\quad \ell>m\geq\bar{\ell}_0. \tag{6.24}$$

Next, it follows from (6.19) and the fact that $(\vartheta_\ell)_+\leq(\vartheta_m)_+$, for $\ell>m$, that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\left|\frac{J_y}{\sqrt{\varrho_0}}\right|\varrho_0^{1-\gamma}(\vartheta_\ell)_+dydt &\leq \int_0^T\int_{\mathbb{R}}\left|\frac{J_y}{\sqrt{\varrho_0}}\right|\varrho_0^{1-\gamma}(\vartheta_m)_+\left|\frac{\varrho_0^{1-\gamma}(\vartheta_m)_+}{\ell-m}\right|^3dydt\\ &\leq \frac{1}{(\ell-m)^3}\sup_{0\leq t\leq T}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2\int_0^T\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_8^4dt\\ &\leq \frac{\mathcal{Z}_J^{\frac12}}{(\ell-m)^3}\int_0^T\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_8^4dt. \end{aligned}\tag{6.25}$$

By Lemma 6.1, it holds that

$$\begin{aligned}&\int_0^T\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_8^4dt\\ \leq\ & C\int_0^T\left(\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_2^4+\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_2^{\frac52}\|\varrho_0^{\frac12-\gamma}\partial_y(\vartheta_m)_+\|_2^{\frac32}\right)dt\\ \leq\ & C\left(\sup_{0\leq t\leq T}\|\varrho_0^{1-\gamma}(\vartheta_m)_+\|_2^2+\int_0^T\|\varrho_0^{\frac12-\gamma}\partial_y(\vartheta_m)_+\|_2^2dt\right)^2=CQ_m^2. \end{aligned}\tag{6.26}$$

Combining (6.25) with (6.26) leads to

$$\int_0^T\int_{\mathbb{R}}\left|\frac{J_y}{\sqrt{\varrho_0}}\right|\varrho_0^{1-\gamma}(\vartheta_\ell)_+dydt\ \leq\ \frac{C\mathcal{Z}_J^{\frac12}Q_m^2}{(\ell-m)^3}. \tag{6.27}$$

Substituting (6.23), (6.24), and (6.27) into (6.18) yields the conclusion. □

6.5. **Upper bound of the entropy.** We are now ready to establish the uniform upper bound for the entropy.

Theorem 6.1. *Assume that (3.1), (4.1), (4.2), (5.1), and (6.1) hold, and the initial entropy is bounded from above. Then, the unique global solution obtained in Theorem 4.1 satisfies*

$$\sup_{(y,t)\in\mathbb{R}\times(0,T)}s\leq C\log(2+\bar{\ell}_0+\mathcal{Z}(T)),$$

for any positive time T, where $\mathcal{Z}(T)=\mathcal{Z}_J^{\frac12}(T)+\mathcal{Z}_\vartheta(T)+\mathcal{Z}_G(T)$ and C is a positive constant depending only on c_v, κ, γ, $\bar{\varrho}$, K_1, K_2, T, $\underline{J}_T$, and $\bar{J}_T$.

Proof. It follows from Proposition 6.4 that

$$Q_\ell\leq\frac{C(1+\ell)^2}{(\ell-m)^3}\mathcal{Z}(T)Q_m^2,\qquad\forall\ell>m\geq\bar{\ell}_0.$$

One can check easily that Q_ℓ is non-increasing in ℓ. Therefore, Lemma 5.1 implies $Q_{\bar\ell_0+d}=0$, with $d=2+2\big(2+\bar\ell_0+C\mathcal{Z}(T)\big)^{22}Q_{\bar\ell_0}^2$. Hence, $(\vartheta_{\bar\ell_0+d})_+\equiv 0$, which gives

$$\vartheta\le\left(\bar\ell_0+d\right)\varrho_0^{\gamma-1}e^{\overline{M}_Tt}\le\left(\bar\ell_0+d\right)\varrho_0^{\gamma-1}e^{\overline{M}_TT},$$

and, consequently,

$$\begin{aligned} s &= c_v\left(\log\frac{R}{A}+(\gamma-1)\log J+\log\vartheta-(\gamma-1)\log\varrho_0\right)\\ &\le c_v\left(\log\frac{R}{A}+(\gamma-1)\log\bar J_T+\log(\bar\ell_0+d)+\overline{M}_TT\right). \end{aligned}\tag{6.28}$$

Proposition 6.3 shows that $Q_{\bar\ell_0}\le C(1+\bar\ell_0^2)\mathcal{Z}^2(T)$, and, thus, $d\le C(2+\bar\ell_0+\mathcal{Z}(T))^{30}$. This and (6.28) give the desired conclusion. □

7. Appendix

In this appendix, we prove Lemma 5.1.

Proof of Lemma 5.1. It follows from the assumption that

$$f(\ell)\le\frac{2^\alpha M_0\ell^\alpha}{(\ell-m)^\beta}f^\sigma(m),\quad\forall\ell>m\ge m_0+1.\tag{7.1}$$

Let $d_0\ge1$ be a positive number to be determined later, and set

$$\ell_k=m_0+1+\left(1-\frac{1}{2^k}\right)d_0,\quad k=0,1,2,\cdots.$$

Then, choosing $\ell=\ell_{k+1}$ and $m=\ell_k$ in (7.1), and noticing that $\ell_{k+1}\le m_0+1+d_0$, one deduces that

$$\begin{aligned} f(\ell_{k+1}) &\le M_02^\alpha\ell_{k+1}^\alpha(\ell_{k+1}-\ell_k)^{-\beta}f^\sigma(\ell_k)\\ &\le M_02^\alpha(m_0+1+d_0)^\alpha(2^{-(k+1)}d_0)^{-\beta}f^\sigma(\ell_k)\\ &= M_02^{k\beta+\alpha+\beta}\left(\frac{m_0+1}{d_0^{\beta/\alpha}}+\frac{1}{d_0^{\beta/\alpha-1}}\right)^\alpha f^\sigma(\ell_k), \end{aligned}$$

from which, recalling that $d_0\ge1$ and noticing that $\frac{\beta}{\alpha}>1$, one obtains

$$f(\ell_{k+1})\le M^{k\beta+2\alpha+\beta+1}f^\sigma(\ell_k),\quad k=0,1,2,\cdots,$$

with $M=M_0+m_0+2$, which can be written equivalently as

$$M^{a(k+1)+b}f(\ell_{k+1})\le[M^{ak+b}f(\ell_k)]^\sigma,\quad k=0,1,2,\cdots,\tag{7.2}$$

where $a=\frac{\beta}{\sigma-1}$ and $b=\frac{2\alpha+\beta+1}{\sigma-1}+\frac{\beta}{(\sigma-1)^2}$. It follows from (7.2) that

$$M^{a(k+1)+b}f(\ell_{k+1})\le(M^{a+b}f(\ell_1))^{\sigma^k},$$

which implies, due to $M \geq 2, a > 0$, and $b > 0$, that

$$f(\ell_{k+1}) \leq (M^{a+b} f(\ell_1))^{\sigma^k}, \quad k = 1, 2, \cdots . \tag{7.3}$$

Choosing $\ell = \ell_1$ and $m = \ell_0$ in (7.1) leads to

$$f(\ell_1) \;\; \leq \;\; 2^\alpha M_0 \ell_1^\alpha (\ell_1 - \ell_0)^{-\beta} f^\sigma(\ell_0) \leq \frac{2^{\alpha+\beta} M_0}{d_0^{\beta-\alpha}} (m_0 + 2)^\alpha f^\sigma(m_0 + 1).$$

It follows from this and the monotonicity of f that

$$f(\ell_1) \;\; \leq \;\; \frac{M^{2\alpha+\beta+1}}{d_0^{\beta-\alpha}} f^\sigma(m_0).$$

Therefore,

$$M^{a+b} f(\ell_1) = M^{\frac{2\alpha+2\beta+1}{\sigma-1} + \frac{\beta}{(\sigma-1)^2}} f(\ell_1) \leq M^{\frac{2\alpha+2\beta+1}{\sigma-1} + \frac{\beta}{(\sigma-1)^2} + 2\alpha+\beta+1} \frac{f^\sigma(m_0)}{d_0^{\beta-\alpha}} \leq \frac{1}{2},$$

provided that $d_0 = \left(2 f_0^\sigma M^{\frac{2\alpha+2\beta+1}{\sigma-1} + \frac{\beta}{(\sigma-1)^2} + 2\alpha+\beta+1}\right)^{\frac{1}{\beta-\alpha}} + 1$.

It follows from (7.3) that

$$f(m_0 + 1 + d_0) \leq f(\ell_{k+1}) \leq (M^{a+b} f(\ell_1))^{\sigma^k} \leq \left(\frac{1}{2}\right)^{\sigma^k}, \quad k = 0, 1, 2, \cdots .$$

Passing $k \to \infty$ in the above yields $f(m_0+1+d_0) = 0$, so the conclusion follows. $\square$

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References

[1] Chen, G.-Q.; Hoff, D.; Trivisa, K.: *Global solutions of the compressible Navier–Stokes equations with large discontinuous initial data*, Comm. Partial Differential Equations, **25** (2000), 2233–2257.

[2] Chen, Q.; Miao, C.; Zhang, Z.: *Global well-posedness for compressible Navier–Stokes equations with highly oscillating initial velocity*, Communications on Pure and Applied Mathematics, **63** (2010), 1173–1224.

[3] Chikami, N.; Danchin, R.: *On the well-posedness of the full compressible Naiver-Stokes system in critical Besov spaces*, J. Differential Equations, **258** (2015), 3435–3467.

[4] Cho, Y.; Choe, H. J.; Kim, H.: *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl., **83** (2004), 243–275.

[5] Cho, Y.; Kim, H.: *On classical solutions of the compressible Navier–Stokes equations with nonnegative initial densities*, Manuscripta Math., **120** (2006), 91–129.

[6] Cho, Y.; Kim, H.: *Existence results for viscous polytropic fluids with vacuum*, J. Differential Equations, **228** (2006), 377–411.

[7] Danchin, R.: *Global existence in critical spaces for flows of compressible viscous and heat-conductive gases*, Arch. Ration. Mech. Anal., **160** (2001), 1–39.

[8] Danchin, R.; Xu, J.: *Optimal time-decay estimates for the compressible Navier–Stokes equations in the critical Lp framework*, Arch. Ration. Mech. Anal., **224** (2017), no. 1, 53–90.

[9] Deckelnick, K.: *Decay estimates for the compressible Navier–Stokes equations in unbounded domains*, Math. Z., **209** (1992), 115–130.

[10] Fang, D.; Zhang, T.; Zi, R.: *Global solutions to the isentropic compressible Navier–Stokes equations with a class of large initial data*, SIAM J. Math. Anal., **50** (2018), no. 5, 4983–5026.

[11] Feireisl, E.; Novotný, A.; Petzeltová, H.: *On the existence of globally defined weak solutions to the Navier–Stokes equations*, J. Math. Fluid Mech., **3** (2001), 358–392.

[12] Feireisl, E.: *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its Applications, 26. Oxford University Press, Oxford, 2004. xii+212 pp.

[13] Hoff, D.: *Discontinuous solutions of the Navier–Stokes equations for multidimensional flows of heat-conducting fluids*, Arch. Rational Mech. Anal., **139** (1997), 303–354.

[14] Hoff, D.; Smoller, J.: *Non-formation of vacuum states for compressible Navier–Stokes equations*, Comm. Math. Phys., **216** (2001), 255–276.

[15] Huang, X.; Li, J.: *Global classical and weak solutions to the three-dimensional full compressible Navier–Stokes system with vacuum and large oscillations*, Arch. Rational Mech. Anal., **227** (2018), 995–1059.

[16] Huang, X.; Li, J.; Xin, Z.: *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations*, Comm. Pure Appl. Math., 65 (2012), 549–585.

[17] Itaya, N.: *On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluids*, Kodai Math. Sem. Rep., **23** (1971), 60–120.

[18] Jiang, S.; Zhang, P.: *Axisymmetric solutions of the 3D Navier–Stokes equations for compressible isentropic fluids*, J. Math. Pures Appl., **82** (2003), 949–973.

[19] Jiang, S.; Zlotnik, A.: *Global well-posedness of the Cauchy problem for the equations of a one-dimensional viscous heat-conducting gas with Lebesgue initial data*, Proc. Roy. Soc. Edinburgh Sect. A, **134** (2004), 939–960.

[20] Kazhikhov, A. V.: *Cauchy problem for viscous gas equations*, Siberian Math. J., **23** (1982), 44–49.

[21] Kazhikhov, A. V.; Shelukhin, V. V.: *Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., **41** (1977), 273–282.

[22] Kobayashi, T.; Shibata, Y.: *Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in* $\mathbb{R}^3$, Commun. Math. Phys., **200** (1999), 621–659.

[23] Li, H.; Wang, Y.; Xin, Z.: *Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier–Stokes equations*, Arch. Ration. Mech. Anal., **232** (2019), no. 2, 557–590.

[24] Li, J.: *Global small solutions of heat conductive compressible Navier–Stokes equations with vacuum: smallness on scaling invariant quantity*, arXiv:1906.08712 [math.AP]

[25] Li, J.; Liang, Z.: *Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data*, Arch. Rational Mech. Anal., **220** (2016), 1195–1208.

[26] Li, J.; Xin, Z.: *Entropy bounded solutions to the one-dimensional compressible Navier-Stokes equations with zero heat conduction and far field vacuum*, Adv. Math., **361** (2020), 106923, 50 pp.

[27] Li, J.; Xin, Z.: *Instantaneously blow up of entropy for heat conductive ideal gases with interior vacuum or far field strong vacuum*, in preparation.

[28] Lions, P. L.: *Mathematical Topics in Fluid Mechanics*, Vol. 2, Clarendon, Oxford, 1998.

[29] Lukaszewicz, G.: *An existence theorem for compressible viscous and heat conducting fluids*, Math. Methods Appl. Sci., **6** (1984), 234–247.

[30] Matsumura, A.; Nishida, T.: *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ., **20** (1980), 67–104.

[31] Matsumura, A.; Nishida, T.: *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Commun. Math. Phys., **89** (1983), 445–464.

[32] Nash, J.: *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. Fr., **90** (1962), 487–497.

[33] Ponce, G.: *Global existence of small solutions to a class of nonlinear evolution equations*, Nonlinear Anal., **9** (1985), 399–418.

[34] Salvi, R.; Straškraba, I.: *Global existence for viscous compressible fluids and their behavior as* $t \to \infty$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **40** (1993), 17–51.

[35] Tani, A.: *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. Res. Inst. Math. Sci., **13** (1977), 193–253.

[36] Valli, A.: *An existence theorem for compressible viscous fluids*, Ann. Mat. Pura Appl., **130** (1982), 197–213; **132** (1982), 399–400.

[37] Valli, A.; Zajaczkowski, W. M.: *Navier–Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case*, Commun. Math. Phys., **103** (1986), 259–296.

[38] Vol'pert, A. I., Hudjaev, S. I.: *On the Cauchy problem for composite systems of nonlinear differential equations*, Math. USSR-Sb, **16** (1972), 517–544 [previously in Mat. Sb. (N.S.), **87** (1972), 504–528(in Russian)].

[39] Wen, H.; Zhu, C.: *Global solutions to the three-dimensional full compressible Navier–Stokes equations with vacuum at infinity in some classes of large data*, SIAM J. Math. Anal., **49** (2017), 162–221.

[40] Xin, Z.: *Blowup of smooth solutions to the compressible Navier–Stokes equation with compact density*, Comm. Pure Appl. Math., **51** (1998), 229–240.

[41] Xin, Z.; Yan, W.: *On blowup of classical solutions to the compressible Navier–Stokes equations*, Comm. Math. Phys., **321** (2013), 529–541.

[42] Zlotnik, A. A.; Amosov, A. A.: *On stability of generalized solutions to the equations of one-dimensional motion of a viscous heat-conducting gas*, Siberian Math. J., **38** (1997), 663–684.

[43] Zlotnik, A. A.; Amosov, A. A.: *Stability of generalized solutions to equations of one-dimensional motion of viscous heat conducting gases*, Math. Notes, **63** (1998), 736–746.

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