STRUCTURAL STABILITY OF THE TRANSONIC SHOCK PROBLEM IN A DIVERGENT THREE DIMENSIONAL AXISYMMETRIC PERTURBED NOZZLE

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Abstract. In this paper, we prove the structural stability of the transonic shocks for three dimensional axisymmetric Euler system with swirl velocity under the perturbations for the incoming supersonic flow, the nozzle boundary, and the exit pressure. Compared with the known results on the stability of transonic shocks, one of the major difficulties for the axisymmetric flows with swirls is that corner singularities near the intersection point of the shock surface and nozzle boundary and the artificial singularity near the axis appear simultaneously. One of the key points in the analysis for this paper is the introduction of an *invertible* Lagrangian transformation which can straighten the streamlines in the whole nozzle and help to represent the solutions of transport equations explicitly.

1. Introduction and main results

The three-dimensional steady inviscid gas motion is governed by the following compressible Euler system

$$\begin{cases} \text{div } (\rho\mathbf{u}) = 0, \\ \text{div } (\rho\mathbf{u}\otimes\mathbf{u} + PI_n) = 0, \\ \text{div } (\rho(\frac{1}{2}|\mathbf{u}|^2 + \mathfrak{e})\mathbf{u} + P\mathbf{u}) = 0, \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, ρ, P, and $\mathfrak{e}$ stand for the velocity, density, pressure, and internal energy, respectively. Suppose that the gas is polytropic. Then the equation of state and the internal energy are of the form

$$P = A\rho^\gamma e^{\frac{S}{c_v}} \quad \text{and} \quad \mathfrak{e} = \frac{P}{(\gamma-1)\rho}, \tag{2}$$

respectively, where $\gamma \geq 1$, A, and c_v are positive constants, and S is called the specific entropy. The system (1) is a hyperbolic system for supersonic flows ($M_a > 1$), a hyperbolic-elliptic coupled system for subsonic flows ($M_a < 1$), and degenerate at sonic point (i.e.

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$M_a = 1$), respectively, where $M_a = \frac{|\mathbf{u}|}{c(\rho,S)}$ is called the Mach number of the flows with $c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}$ called the local sound speed.

In this paper, we are interested in the basic transonic shock problem in a De Laval nozzle described by Courant and Friedrichs ([11, Page 386]): given appropriately large receiver pressure P_e, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes P_e. The stability of transonic shocks in nozzles is a fundamental problem in gas dynamics that have been studied extensively in various situations. The early studies for transonic flows, in particular for quasi-one dimensional models, can be found in [3, 12, 24]. The structural stability of transonic shocks for multidimensional steady potential flows in nozzles was studied in [7, 27, 28]. It was showed in [27, 28] that the stability of transonic shock for potential flows is usually ill-posed under the perturbation of the exit pressure. Later on, it was proved that the transonic shock problem in the flat nozzle with small perturbations is either ill-posed under general perturbations of the exit pressure or well-posedness if the exit pressure satisfies a special constraint, see [8–10, 19, 21] and the references therein. There have been many interesting results on transonic shock problems in a nozzle for different models with various exit boundary conditions, for example, the non-isentropic potential model, the exit boundary condition for the normal velocity, the spherical flows without boundary, etc, see [1, 5, 6, 23] and references therein. The well-posedness of the transonic shock problem was first established in a special class of two dimensional divergent nozzle under the perturbations for the exit pressure in [16]. Later on, the results were generalized to the problem in general two dimensional divergent nozzles, see [17, 20]. In particular, in [20], the Courant-Friedrich's transonic shock in a two dimensional straight divergent nozzle is shown to be structurally stable under generic perturbations for both the nozzle shape and the exit pressure, and optimal regularity of solutions are also obtained. Such a structural stability also holds for perturbations of incoming supersonic flows [25]. The key idea there is to introduce a Lagrangian transformation to straighten the streamlines and reduce the Euler system with the shock to a second order elliptic equation with a nonlocal term and an unknown parameter together with an ODE for the shock front. In [18, 19], the existence and stability of transonic shock for three dimensional axisymmetric flows without swirl in a conic nozzle was proved to be structurally stable under suitable perturbations of the exit pressure.

In this paper, we study the stability of transonic shocks for 3D axisymmetric flows with swirls under the perturbations of the exit pressure, the nozzle wall, and supersonic incoming flows. First, let us introduce the standard spherical coordinates

$$\begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta\cos\varphi, \\ x_3 = r\sin\theta\sin\varphi \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{e}_r = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)^t, \\ \mathbf{e}_\theta = (-\sin\theta, \cos\theta\cos\varphi, \cos\theta\sin\varphi)^t, \\ \mathbf{e}_\varphi = (0, -\sin\varphi, \cos\varphi)^t. \end{cases}$$

Let $\mathbf{u} = U_1\mathbf{e}_r + U_2\mathbf{e}_\theta + U_3\mathbf{e}_\varphi$. The three dimensional axisymmetric Euler system can be written as

$$\begin{cases} \partial_r(r^2\rho U_1\sin\theta) + \partial_\theta(r\rho U_2\sin\theta) = 0, \\ \rho U_1\partial_r U_1 + \frac{1}{r}\rho U_2\partial_\theta U_1 + \partial_r P - \frac{\rho(U_2^2+U_3^2)}{r} = 0, \\ \rho U_1\partial_r U_2 + \frac{1}{r}\rho U_2\partial_\theta U_2 + \frac{1}{r}\partial_\theta P + \frac{\rho U_1U_2}{r} - \frac{\rho U_3^2}{r}\cot\theta = 0, \\ \rho U_1\partial_r(rU_3\sin\theta) + \frac{1}{r}\rho U_2\partial_\theta(rU_3\sin\theta) = 0, \\ \rho U_1\partial_r S + \frac{1}{r}\rho U_2\partial_\theta S = 0. \end{cases} \tag{3}$$

Suppose that $\theta_0 \in (0, \frac{\pi}{2})$, r_1, $r_2(> r_1)$ are fixed positive constants. Let $\Omega_b = \{(r,\theta) : r \in (r_1, r_2), \theta \in [0,\theta_0)\}$ be a straight divergent nozzle and $\Gamma_b = \partial\Omega_b$ be its boundary.

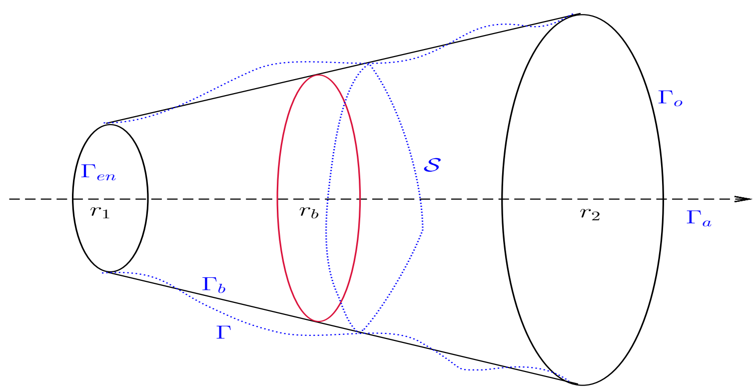

Figure 1. The straight and perturbed nozzles

Suppose that the incoming supersonic flow is prescribed at the inlet $r = r_1$, i.e.,

$$\mathbf{u}^-(\mathbf{x}) = U_b^-(r_1)\mathbf{e}_r, \quad P_b^-(\mathbf{x}) = P_b^-(r_1) > 0, \quad S_b^-(\mathbf{x}) = S_b^-, \quad \text{at } r = r_1, \tag{4}$$

where $U_b^-(r_1) > c(\rho_b^-(r_1), S_b^-) > 0$ and S_b^- is a constant. There exist two positive constants P_1 and P_2 which depend only on the incoming supersonic flows and the nozzle, such that if the pressure $P_e \in (P_1, P_2)$ is given at the exit $r = r_2$, then there exists a unique piecewise smooth spherical symmetric transonic shock solution

$$\boldsymbol{\Psi}_b(\mathbf{x}) = (\mathbf{u}_b, P_b, S_b)(\mathbf{x}) = \begin{cases} \boldsymbol{\Psi}_b^-(\mathbf{x}) := (U_b^-(r), 0, 0, P_b^-(r), S_b^-), \text{ in } \Omega_b^- \\ \boldsymbol{\Psi}_b^+(\mathbf{x}) := (U_b^+(r), 0, 0, P_b^+(r), S_b^+), \text{ in } \Omega_b^+ \end{cases} \tag{5}$$

to (1) with a shock front located at $r = r_b \in (r_1, r_2)$, where

$$\Omega_b^- = \Omega_b \cap \{r \in (r_1, r_b)\} \quad \text{and} \quad \Omega_b^+ = \Omega_b \cap \{r \in (r_b, r_2)\}. \tag{6}$$

Across the shock, the Rankine-Hugoniot conditions and the physical entropy condition are satisfied:

$$[\rho U_b]\Big|_{r=r_b} = 0, \quad [\rho_b U_b^2 + P_b]\Big|_{r=r_b} = 0, \quad [B]\Big|_{r=r_b} = 0, \quad S_b^+ > S_b^-, \tag{7}$$

where $B = \frac{|\mathbf{u}|^2}{2} + \mathfrak{e} + \frac{P}{\rho}$ is called the Bernoulli function and $[g]\Big|_{r=r_b} := g(r_b+) - g(r_b-)$ denotes the jump of g at $r = r_b$. Later on, this special solution, $\boldsymbol{\Psi}_b$, will be called the background solution. Clearly, one can extend the supersonic and subsonic parts of $\boldsymbol{\Psi}_b$ in a natural way, respectively. With an abuse of notations, we still call the extended subsonic and supersonic solutions $\boldsymbol{\Psi}_b^+$ and $\boldsymbol{\Psi}_b^-$, respectively. One can refer to [11, Section 147] or [29, Theorem 1.1] for more details of this spherical symmetric transonic shock solution. The main goal of this paper is to establish the structural stability of this spherical symmetric transonic shock solution under axisymmetric perturbations of the incoming supersonic flows, the nozzle walls, and the exit pressure.

The perturbed nozzle is $\Omega = \{(r, \theta) : r_1 < r < r_2, 0 \le \theta \le \theta_0 + \epsilon f(r)\}$, where ϵ is a small positive constant and $f \in C^{2,\alpha}([r_1, r_2])$ satisfies

$$f(r_1) = f'(r_1) = 0. \tag{8}$$

Suppose that the incoming supersonic flow at the inlet $r = r_1$ is given by

$$\boldsymbol{\Psi}\Big|_{r=r_1} := (U_1^-, U_2^-, U_3^-, P^-, S^-)\Big|_{r=r_1} = \boldsymbol{\Psi}_{en}^- = \boldsymbol{\Psi}_b^- + \epsilon \boldsymbol{\Psi}_p(\theta), \tag{9}$$

where

$$\boldsymbol{\Psi}_p(\theta) = (U_{1,p}^-, U_{2,p}^-, U_{3,p}^-, P_p^-, S_p^-)(\theta) \in (C^{2,\alpha}([0, \theta_0]))^5 \tag{10}$$

The flow satisfies the slip condition $\mathbf{u} \cdot \mathbf{n}$=0 on the nozzle wall, where $\mathbf{n}$ is the outer normal of the nozzle wall. In terms of spherical coordinates, the slip boundary condition for the axisymmetric flows can be written as

$$U_2 = \epsilon r f'(r) U_1 \quad \text{on } \Gamma := \{(r, \theta) : \theta = \theta_0 + \epsilon f(r), \quad r_1 \le r \le r_2\}. \tag{11}$$

At the exit of the nozzle, the end pressure is prescribed by

$$P(x) = P_e + \epsilon P_0(\theta) \text{ at } \Gamma_o := \{(r_2, \theta) : \theta \in (0, \theta_0)\}, \tag{12}$$

here $P_0 \in C^{1,\alpha}([0, 2\theta_0])$ (in fact, what is needed in this paper is that P_0 is a $C^{1,\alpha}$ function in a region slightly larger than $[0, \theta_0]$).

Since the steady Euler system for supersonic flow is hyperbolic, if the incoming data satisfies the following compatibility conditions

$$(13)\qquad \begin{cases} U_{2,p}^-(0) = U_{3,p}^-(0) = \frac{d^2}{d\theta^2}U_{2,p}^-(0) = \frac{d}{d\theta}P_p^-(0) = \frac{d}{d\theta}U_{3,p}^-(0) = \frac{d}{d\theta}S_p^-(0) = 0, \\ U_{2,p}^-(\theta_0) = 0,\ \frac{d}{d\theta}P_p^-(\theta_0) = (U_{3,p}^-(\theta_0))^2 \cot\theta_0, \end{cases}$$

then the problem for the system (3) together with (9) and (11) can be solved by the characteristic method and Picard iteration (see [15]). Furthermore, for small $\epsilon > 0$, there exists a unique $C^{2,\alpha}(\overline{\Omega})$ solution $\mathbf{\Psi}^- = (U_1^-, U_2^-, U_3^-, P^-, S^-)(r, \theta)$ to (1), which does not depend on φ and satisfies the following properties

$$(14)\qquad \|(U_1^-, U_2^-, U_3^-, P^-, S^-) - (U_b^-, 0, 0, P_b^-, S_b^-)\|_{C^{2,\alpha}(\overline{\Omega})} \le C_0\epsilon,$$

and

$$(15)\quad U_2^- = U_3^- = \frac{\partial}{\partial\theta}(U_1^-, U_3^-, P^-, S^-) = \frac{\partial^2}{\partial\theta^2}U_2^- = 0, \text{ at } \Gamma_a := \{(r, 0) : r_1 < r < r_2\}.$$

Now we are looking for a piecewise smooth solution $\mathbf{\Psi}$ for (3) supplemented with the boundary conditions (9), (11), and (12), which jumps only at a shock front at $\mathcal{S} = \{(r, \theta) : r = \xi(\theta), 0 \le \theta \le \theta_0\}$. More precisely, $\mathbf{\Psi}$ has the form

$$(16)\qquad \mathbf{\Psi} = \begin{cases} \mathbf{\Psi}^- = (U_1^-, U_2^-, U_3^-, P^-, S^-)(r, \theta), & \text{if } r_1 < r < \xi(\theta),\ 0 \le \theta < \theta_0, \\ \mathbf{\Psi}^+ = (U_1^+, U_2^+, U_3^+, P^+, S^+)(r, \theta), & \text{if } \xi(\theta) < r < r_2,\ 0 \le \theta < \theta_0, \end{cases}$$

and the following Rankine-Hugoniot conditions on the shock surface $\mathcal{S} = \{(r, \theta) | r = \xi(\theta)\}$ are satisfied

$$(17)\qquad \begin{cases} [\rho U_1] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_2] = 0, \\ [\rho U_1^2 + P] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_1 U_2] = 0, \\ [\rho U_1 U_2] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_2^2 + P] = 0, \\ [\rho U_1 U_3] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_2 U_3] = 0, \\ [\mathfrak{e} + \frac{1}{2}|U|^2 + \frac{P}{\rho}] = 0. \end{cases}$$

To state the main results, some weighted Hölder norms are needed. For any bounded domain $\mathcal{D} \subset \mathbb{R}^n$, $\mathcal{K} \subset \partial\mathcal{D}$, and $\mathbf{x} \in \mathcal{D}$, define

$$\delta_{\mathbf{x}} := \text{dist}(\mathbf{x}, \mathcal{K}), \quad \text{and} \quad \delta_{\mathbf{x},\tilde{\mathbf{x}}} := \min(\delta_{\mathbf{x}}, \delta_{\tilde{\mathbf{x}}}).$$

For any nonnegative integer m, $\alpha\in(0,1)$ and $\sigma\in\mathbb{R}$, define weighted Hölder norms by

$$\begin{aligned}
[u]^{(\sigma;\mathcal{K})}_{k,0;\mathcal{D}} &:= \sum_{|\beta|=k}\sup_{\mathbf{x}\in\mathcal{D}}\delta_{\mathbf{x}}^{\max\{|\beta|+\sigma,0\}}|D^\beta u(\mathbf{x})|,\ k=0,1,\cdots,m,\\
[u]^{(\sigma;\mathcal{K})}_{m,\alpha;\mathcal{D}} &:= \sum_{|\beta|=m}\sup_{\mathbf{x},\tilde{\mathbf{x}}\in\mathcal{D},\mathbf{x}\neq\tilde{\mathbf{x}}}\delta_{\mathbf{x},\tilde{\mathbf{x}}}^{\max\{m+\alpha+\sigma,0\}}\frac{|D^\beta u(\mathbf{x})-D^\beta u(\tilde{\mathbf{x}})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\alpha},\\
\|u\|^{(\sigma;\mathcal{K})}_{m,\alpha;\mathcal{D}} &:= \sum_{k=0}^m[u]^{(\sigma;\mathcal{K})}_{k,0;\mathcal{D}}+[u]^{(\sigma;\mathcal{K})}_{m,\alpha;\mathcal{D}}.
\end{aligned}$$

$C^{(\sigma;\mathcal{K})}_{m,\alpha;\mathcal{D}}$ denotes the space of all smooth functions whose $\|\cdot\|^{(\sigma;\mathcal{K})}_{m,\alpha;\mathcal{D}}$ norms are finite. One can refer to [13, 14, 22] for the properties of these weighted Hölder spaces. Furthermore, $\Omega_\pm$ are defined as follows

$$\Omega_- := \{(r,\theta): r_1\le r\le \xi(\theta), 0\le\theta<\theta_0+\epsilon f(r)\}\quad\text{and}\quad \Omega_+:=\Omega\setminus\Omega_-.$$

Theorem 1. *Assume that Γ satisfies (8) and $\mathbf{\Psi}_{en}$ satisfies (13). There exists a small $\epsilon_0>0$ depending only on the background solution $\mathbf{\Psi}_b$ and boundary data $\mathbf{\Psi}_p$, f, P_0 such that if $0\le\epsilon<\epsilon_0$, the problem (3) with (9), (11), (12), and (17) has a unique solution $\mathbf{\Psi}^+=(U_1^+,U_2^+,U_3^+,P^+,S^+)(r,\theta)$ with the shock front $\mathcal{S}=\{(r,\theta): r=\xi(\theta),\theta\in[0,\theta_*]\}$ satisfying the following properties.*

(i) *The function $\xi(\theta)\in C^{(-1-\alpha;\{\theta_*\})}_{3,\alpha;(0,\theta_*)}$ satisfies*

$$\|\xi(\theta)-r_b\|^{(-1-\alpha;\{\theta_*\})}_{3,\alpha;(0,\theta_*)}\le C_0\epsilon, \tag{18}$$

where $(\xi(\theta_*),\theta_*)$ stands for the intersection circle of the shock surface with the nozzle wall and C_0 is a positive constant depending only on the supersonic incoming flow.

(ii) *The solution $\mathbf{\Psi}^+=(U_1^+,U_2^+,U_3^+,P^+,S^+)(r,\theta)\in C^{(-\alpha;\Gamma_{w,s})}_{2,\alpha;\Omega_+}$ satisfies the entropy condition*

$$P^+(\xi(\theta)+,\theta)>P^-(\xi(\theta)-,\theta)\quad \textit{for } \theta\in[0,\theta_*] \tag{19}$$

and

$$\|\mathbf{\Psi}^+-\hat{\mathbf{\Psi}}_b^+\|^{(-\alpha;\Gamma_{w,s})}_{2,\alpha;\Omega_+}\le C_0\epsilon, \tag{20}$$

where

$$\Gamma_{w,s}=\{(r,\theta):\xi(\theta)\le r\le r_2,\theta=\theta_0+\epsilon f(r)\}.$$

In fact, if the nozzle boundary is straight and the exit pressure satisfies some further compatibility conditions, we have the higher order regularity for both the flows and the shock surface. This is our second main result.

Theorem 2. *Assume that the nozzle wall is straight, i.e., $f(r) \equiv 0$. If, in addition to (13), the following compatibility conditions*

$$P_0'(0) = P_0'(\theta_0) = 0, \tag{21}$$

and

$$U_{3,p}^-(\theta_0) = 0, \quad \frac{d}{d\theta}(U_{1,p}^-, U_{3,p}^-, S_p^-)(\theta_0) = 0, \tag{22}$$

hold then the system (3) in Ω_b together with (9), (12), and the slip boundary conditions

$$U_2(r, \theta_0) = 0, \quad r \in [r_1, r_2]. \tag{23}$$

has a unique solution $\mathbf{\Psi}(r, \theta)$ with the shock surface $\mathcal{S} = \{(r, \theta) : r = \xi(\theta), \theta \in [0, \theta_0]\}$ satisfying the following properties.

(i) *The function $\xi(\theta) \in C^{3,\alpha}([0, \theta_0])$ satisfies*

$$\|\xi(\theta) - r_b\|_{C^{3,\alpha}([0,\theta_0])} \leq C_0\epsilon, \tag{24}$$

where C_0 is a positive constant depending only on the supersonic incoming flow and the background solutions.

(ii) *$\mathbf{\Psi}^+ = (U_1^+, U_2^+, U_3^+, P^+, S^+)(r, \theta) \in C^{2,\alpha}(\overline{\mathscr{R}_+})$ satisfies the entropy condition (19) with $\theta_* = \theta_0$ and*

$$\|\mathbf{\Psi}(r, \theta) - \mathbf{\Psi}_b^+(r, \theta)\|_{C^{2,\alpha}(\overline{\mathscr{R}_+})} \leq C_0\epsilon, \tag{25}$$

where $\mathscr{R}_+ = \{(r, \theta) : \xi(\theta) < r < r_2, 0 < \theta < \theta_0\}$ is the subsonic region.

We make some comments on the key ingredients of the analysis in this paper. As is well-known, the supersonic flow is fully determined in the whole nozzle when the data at the entrance is given. Therefore, the transonic shock problem is reduced to a free boundary problem in subsonic region where the unknown shock surface is a free boundary and should be determined with the subsonic flow simultaneously, see [20]. In general, the optimal boundary regularity for subsonic flow is C^α for some $\alpha \in (0, 1)$ (see [26, Remark 3.2 and Lemma 3.3]), hence the streamline may not be uniquely determined. For two dimensional problem, the strategy to overcome this difficulty is to introduce a Lagrangian transformation to straighten the streamline. However, there is a singular term $\sin\theta$ in the density equation (cf. (3)) for axisymmetric flows. This makes the Lagrangian transformation (the one used in [20]) not invertible near the axis $\theta = 0$. Our key observation is that the singular term $\sin\theta$ is of order $O(\theta)$ so that there is a simple invertible Lagrangian transformation to straighten the streamline. Although the density equation still preserves the conservation form and a potential function as in [20] can be introduced, it is not easy to represent all the quantities in terms of the potential function and the entropy because the function θ becomes a nonlocal and nonlinear term in the Lagrangian coordinates. Here we resort to the first order elliptic system for the flow angle and the pressure and look for the solution in the function space $C_{2,\alpha;\Omega_+}^{(-\alpha;\Gamma_{w,s})}$ rather than the space $C_{1,\alpha;\Omega_+}^{(-\alpha;\Gamma_{w,s})}$ used in [20]. The axisymmetric Euler system with the shock front equation can be decomposed as a boundary value problem for a first order elliptic system with a nonlocal term and a singular term together with some transport equations. Compared with the elliptic system derived in [19], the coefficients for the linearized elliptic system for the angular velocity and pressure are smooth near the axis. One may refer to Proposition 3 for more details. When the nozzle is a straight cone, even if the swirl component of the velocity is not zero, the key issue is that $U_3 = \partial_\theta U_3 = 0$ on the axis so that the singular term $\frac{U_3^2 \cot\theta}{r}$ does not cause any essential difficulty.

The rest of this paper is organized as follows. In Section 2, we introduce a new invertible Lagrangian transformation and reformulate the transonic shock problem in the new coordinates. Then the Euler system is decomposed as an elliptic system of the flow angle and the pressure together with the transport equations for the entropy, the swirl velocity, and the Bernoulli function. An iteration scheme is developed in Section 3 to prove the existence and uniqueness of the transonic shock problem. In the last section, an improved regularity of the shock front and subsonic solutions is obtained if the nozzle is kept to be straight and some further compatibility conditions are satisfied.

2. The reformulation of the transonic shock problem

In this section, we first introduce a Lagrangian transformation to rewrite the Euler system. Then we use a transformation to fix the shock front so that the problem becomes a fixed boundary problem.

2.1. **Lagrangian formulation.** As we mentioned before, in general, one can only expect the C^α boundary regularity for the solution in subsonic region ([26, Remark 3.2]). To avoid the difficulty to determine the streamline uniquely, we introduce a Lagrangian transformation to straighten the streamline. Note that there is a singular factor $\sin\theta$ in the density equation of (3), the standard Lagragian coordinates used in [20] is not invertible near the axis $\theta = 0$. Observing that $\sin\theta$ is of order $O(\theta)$ near $\theta = 0$, there indeed exists a simple invertible Lagrangian coordinates so that the streamlines can be straightened. Define $(\tilde{y}_1, \tilde{y}_2) = (r, \tilde{y}_2(r,\theta))$ such that

$$\begin{cases} \dfrac{\partial \tilde{y}_2}{\partial r} = -r\rho^- U_2^- \sin\theta, \\ \dfrac{\partial \tilde{y}_2}{\partial \theta} = r^2 \rho^- U_1^- \sin\theta, \\ \tilde{y}_2(r_1, 0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial \tilde{y}_2}{\partial r} = -r\rho^+ U_2^+ \sin\theta, \\ \dfrac{\partial \tilde{y}_2}{\partial \theta} = r^2 \rho^+ U_1^+ \sin\theta, \\ \tilde{y}_2(r_1, 0) = 0 \end{cases} \tag{26}$$

for $(r,\theta)\in\overline{\Omega_-}$ and $\overline{\Omega_+}$, respectively. It is clear that $\tilde{y}_2\geq 0$ in $\overline{\Omega}$ as long as $U_1^\pm>0$ in $\overline{\Omega^\pm}$. On the axis $\theta=0$ and the nozzle wall Γ, one has

$$\frac{d}{dr}\tilde{y}_2(r,0)=0\quad\text{and}\quad\frac{d}{dr}\tilde{y}_2(r,\theta_0+\epsilon f(r))=0.$$

Without loss of generality, assume that

$$\tilde{y}_2(r,0)=0\quad\text{for all } r\in[r_1,r_2].$$

Then there exist two positive constants M and M_1 satisfying

$$\tilde{y}_2(r,\theta_0+\epsilon f(r))=M^2 \text{ for } r\in[r_1,r_*] \text{ and } \tilde{y}_2(r,\theta_0+\epsilon f(r))=M_1^2 \text{ for } r\in[r_*,r_2]$$

respectively, where $(r_*,\theta_0+\epsilon f(r_*))$ is the intersection point of the shock front $\mathcal{S}$ with the nozzle wall Γ. We claim that $\tilde{y}_2(r,\theta)$ is well-defined in $\bar{\Omega}$ and belongs to $\mathrm{Lip}(\bar{\Omega})$. Using the first equation in (17) yields

$$\frac{d}{d\theta}\tilde{y}_2(\xi(\theta)+0,\theta)=\frac{d}{d\theta}\tilde{y}_2(\xi(\theta)-0,\theta).$$

This implies $M_1=M$ which can be computed as follows

$$M^2=r_1^2\int_0^{\theta_0}(\rho^-U_1^-)(r_1,\theta)\sin\theta d\theta>0.$$

Set

$$y_1=r,\quad y_2=\tilde{y}_2^{\frac{1}{2}}(r,\theta). \tag{27}$$

Under the transformation (27), the domains Ω, Ω_-, and Ω_+ are changed into $D=(r_1,r_2)\times(0,M)$,

$$D_-=\{(y_1,y_2):r_1<y_1<\psi(y_2),y_2\in(0,M)\},\quad\text{and}\quad D_+=D\setminus\overline{D_-}, \tag{28}$$

respectively. Note that if $(\rho^\pm,U_1^\pm,U_2^\pm)$ are close to the background solution $(\rho_b^\pm,U_b^\pm,0)$, then there exist two positive constants C_1 and C_2 depending only on the background solution such that

$$C_1\theta^2\leq\tilde{y}_2(r,\theta)=r^2\int_0^\theta(\rho^\pm U_1^\pm)(r,\tau)\sin\tau d\tau\leq C_2\theta^2.$$

Hence $\sqrt{C_1}\theta\leq y_2(r,\theta)\leq\sqrt{C_2}\theta$ and the Jacobian of the transformation $\mathcal{L}:(r,\theta)\in\bar{\Omega}\mapsto(y_1,y_2)=(r,y_2(r,\theta))\in\bar{D}$ satisfies

$$\det\begin{pmatrix}\frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial\theta}\\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial\theta}\end{pmatrix}=\det\begin{pmatrix}1 & 0\\ -\frac{r\rho U_2\sin\theta}{2y_2} & \frac{r^2\rho U_1\sin\theta}{2y_2}\end{pmatrix}=\frac{r^2\rho U_1\sin\theta}{2y_2}\geq C_3>0, \tag{29}$$

where C_3 is a constant depending only on the background solution. Hence the inverse transformation $\mathcal{L}^{-1}:(y_1,y_2)\mapsto(r,\theta)$ exists. To simplify the notations, we neglect the superscript "+" for the solutions in the subsonic region. Under the transformation (26), the Euler system (3) can be written as

$$
\begin{cases}
\partial_{y_1}\left(\frac{2y_2}{y_1^2\rho U_1\sin\theta}\right)-\partial_{y_2}\left(\frac{U_2}{y_1U_1}\right)=0,\\
\partial_{y_1}(U_1+\frac{P}{\rho U_1})-\frac{y_1\sin\theta}{2y_2}\partial_{y_2}(\frac{PU_2}{U_1})-\frac{2P}{y_1\rho U_1}-\frac{PU_2\cos\theta}{y_1\rho U_1^2\sin\theta}-\frac{(U_2^2+U_3^2)}{y_1U_1}=0,\\
\partial_{y_1}(y_1U_2)+\frac{y_1^2\sin\theta}{2y_2}\partial_{y_2}P-\frac{U_3^2}{U_1}\cot\theta=0,\\
\partial_{y_1}(y_1U_3\sin\theta)=0,\\
\partial_{y_1}B=0.
\end{cases}
\tag{30}
$$

The nozzle wall $\Gamma_{w,s}$ is straightened to be $\Gamma_{w,y}=(\psi(M),r_2)\times\{M\}$. Suppose that the shock front $\mathcal{S}$ and the flows ahead and behind $\mathcal{S}$ are denoted by $y_1=\psi(y_2)$ and $(U_1^\pm,U_2^\pm,U_3^\pm,P^\pm,S^\pm)(y)$, respectively. Then the Rankine-Hugoniot conditions on $\mathcal{S}$, (17), become

$$
\begin{cases}
\frac{2y_2}{\psi(y_2)\sin\theta}[\frac{1}{\rho U_1}]+\psi'(y_2)[\frac{U_2}{U_1}]=0,\\
[U_1+\frac{P}{\rho U_1}]+\psi'(y_2)\frac{\psi(y_2)\sin\theta}{2y_2}[\frac{PU_2}{U_1}]=0,\\
[U_2]-\psi'(y_2)\frac{\psi(y_2)\sin\theta}{2y_2}[P]=0,\\
[U_3]=0,\\
[B]=0,
\end{cases}
\tag{31}
$$

where $[g]=g(\psi(y_2)+,y_2)-g(\psi(y_2)-,y_2)$.

It should be emphasized that in terms of the new coordinates (y_1,y_2), θ becomes nonlinear and nonlocal. Indeed, one has

$$
\frac{\partial\theta}{\partial y_1}=\frac{U_2}{y_1U_1},\quad \frac{\partial\theta}{\partial y_2}=\frac{2y_2}{y_1^2\rho U_1\sin\theta},\quad \theta(y_1,0)=0.
\tag{32}
$$

Thus it holds that

$$
\theta(y_1,y_2)=\arccos\left(1-\int_0^{y_2}\frac{2s}{y_1^2(\rho U_1)(y_1,s)}ds\right).
\tag{33}
$$

For the background solution $(\rho_b^\pm,U_b^\pm)$, the similar Lagrangian transformation yields

$$
\frac{\partial\theta_b}{\partial y_2}=\frac{2y_2}{y_1^2(\rho_bU_b)(y_1)\sin\theta}=\frac{2\kappa_by_2}{\sin\theta_b},
$$

where

$$
\kappa_b=\frac{1}{y_1^2(\rho_bU_b)(y_1)}
\tag{34}
$$

is a positive constant for any $y_1\in[r_b,r_2]$. Hence

$$
\theta_b(y_2)=\arccos(1-\kappa_by_2^2).
\tag{35}
$$

2.2. **The elliptic modes.** Note that there is a singular factor $\cot\theta$ in (30), which is also a nonlinear and nonlocal term because of (33). In order to study the system (30), we need to focus on the governing equations for the pressure and the flow angle. Denote $\varpi = \frac{U_2}{U_1}$. Due to the first equation in (30), the second and third equations in (30) can be written as

$$\begin{cases} \partial_{y_1}\varpi - \dfrac{y_1\rho U_1\varpi\sin\theta}{2y_2}\partial_{y_2}\varpi - \dfrac{\varpi}{y_1} - \dfrac{\varpi^2}{y_1}\cot\theta + \dfrac{y_1\sin\theta}{2y_2U_1}\partial_{y_2}P \\ \qquad - \dfrac{\varpi}{\rho c^2(\rho,S)}\partial_{y_1}P - \dfrac{U_3^2}{y_1U_1^2}\cot\theta = 0, \\ \partial_{y_1}P - \dfrac{\rho c^2(\rho,S)U_1^2}{y_1(c^2(\rho,S)-U_1^2)}\dfrac{y_1^2\rho U_1\sin\theta}{2y_2}\partial_{y_2}\varpi - \dfrac{y_1\rho c^2(\rho,S)U_1\varpi\sin\theta}{2y_2(c^2(\rho,S)-U_1^2)}\partial_{y_2}P \\ \qquad - \dfrac{\rho c^2(\rho,S)U_1^2}{y_1(c^2(\rho,S)-U_1^2)}(\varpi^2+\varpi\cot\theta+2) - \dfrac{\rho c^2(\rho,S)U_3^2}{y_1(c^2(\rho,S)-U_1^2)} = 0, \end{cases} \tag{36}$$

where one used the following equation for the entropy,

$$\partial_{y_1}S = 0. \tag{37}$$

In fact, the equation (37) can be obtained from (30) together with the definition of the equation of the state (2). It follows from (11) and (12) that the corresponding boundary conditions for ϖ and P read

$$\begin{cases} \varpi(y_1,0)=0, \quad \varpi(y_1,M)=\epsilon y_1 f'(y_1), \quad \text{for any } y_1\in[r_1,r_2], \\ P(r_2,y_2) = P_e + \epsilon P_0(\theta(r_2,y_2)), \quad \text{for any } y_2\in[0,M]. \end{cases} \tag{38}$$

By the third equation in (31), one has

$$\psi'(y_2) = \frac{2y_2}{\sin\theta(\psi(y_2),y_2)}\frac{U_2(\psi(y_2),y_2)-U_2^-(\psi(y_2),y_2)}{\psi(y_2)(P(\psi(y_2),y_2)-P^-(\psi(y_2),y_2))}. \tag{39}$$

Substituting (39) into the first two equations in (31) yields that

$$\begin{cases} [\rho U_1] = \rho U_1\rho^-U_1^-\frac{[U_2]}{[P]}\left[\frac{U_2}{U_1}\right], \\ [\rho U_1^2+P] = -\rho^-U_1^-\frac{[U_2]}{[P]}\left[\frac{PU_2}{U_1}\right] + (\rho(U_1)^2+P)\rho^-U_1^-\frac{[U_2]}{[P]}\left[\frac{U_2}{U_1}\right]. \end{cases} \tag{40}$$

Furthermore, the last two equations in (31) are equivalent to

$$U_3(\psi(y_2),y_2) = U_3^-(\psi(y_2),y_2) \quad \text{and} \quad B(\psi(y_2),y_2) = B^-(\psi(y_2),y_2). \tag{41}$$

It follows from the Bernoulli's law, the last equation in (30), that one can represent U_1 as

$$U_1 = \sqrt{\frac{2B - U_3^2 - \frac{2A^{\frac{1}{\gamma}}\gamma}{\gamma-1}P^{\frac{\gamma-1}{\gamma}}e^{\frac{S}{\gamma c_v}}}{1+\varpi^2}}.$$

Hence we can write ρU_1 and $\rho U_1^2 + P$ as smooth functions of P, S, B, U_3, and ϖ. Note that

$$(\rho_b^+ U_b^+)(r_b) = (\rho_b^- U_b^-)(r_b) \quad \text{and} \quad (\rho_b^+ (U_b^+)^2 + P_b^+)(r_b) = (\rho_b^- (U_b^-)^2 + P_b^-)(r_b)$$

Applying the Taylor's expansion for (40) yields

$$\begin{cases} a_{11}(P(\psi(y_2), y_2) - P_b^+(r_b)) + a_{12}(S(\psi(y_2), y_2) - S_b^+) \\ \quad = -\dfrac{\rho_b^+(r_b)}{U_b^+(r_b)}(B(\psi(y_2), y_2) - B_b^+) - \dfrac{2(\rho_b^- U_b^-)(r_b)}{r_b}(\psi(y_2) - r_b) + R_1, \\ a_{21}(P(\psi(y_2), y_2) - P_b^+(r_b)) + a_{22}(S(\psi(y_2), y_2) - S_b^+) \\ \quad = -2\rho_b^+(r_b)(B(\psi(y_2), y_2) - B_b^+) - \dfrac{2(\rho_b^- (U_b^-)^2)(r_b)}{r_b}(\psi(y_2) - r_b) + R_2, \end{cases} \tag{42}$$

where

$$a_{11} = \frac{(U_b^+(r_b))^2 - c^2(\rho_b^+(r_b), S_b^+)}{U_b^+(r_b) c^2(\rho_b^+(r_b), S_b^+)}, \; a_{12} = -\frac{(U_b^+(r_b))^2 + \frac{1}{\gamma-1} c^2(\rho_b^+(r_b), S_b^+)}{c_v U_b^+(r_b) c^2(\rho_b^+(r_b), S_b^+)} P_b^+(r_b),$$
$$a_{21} = \frac{(U_b^+(r_b))^2 - c^2(\rho_b^+(r_b), S_b^+)}{c^2(\rho_b^+(r_b), S_b^+)}, \; a_{22} = -\frac{(U_b^+(r_b))^2 + \frac{2}{\gamma-1} c^2(\rho_b^+(r_b), S_b^+)}{c_v c^2(\rho_b^+(r_b), S_b^+)} P_b^+(r_b)$$

and $R_i = R_i(\mathbf{\Phi}^+(\psi(y_2), y_2) - \mathbf{\Phi}_b^+(r_b), \psi(y_2) - r_b, \mathbf{\Phi}^-(\psi(y_2), y_2) - \mathbf{\Phi}_b^-(\psi(y_2)))$ $(i = 1, 2)$ denotes the error term with

$$\mathbf{\Phi}^\pm := (U_1^\pm, \varpi^\pm, U_3^\pm, P^\pm, S^\pm) \quad \text{and} \quad \mathbf{\Phi}_b^\pm := (U_b^\pm, 0, 0, P_b^\pm, S_b^\pm) \tag{43}$$

Later on, we denote $\mathbf{\Phi}^+$ by $\mathbf{\Phi}$ for simplicity. Furthermore, for $i = 1$ and 2, straightforward computations give

$$|R_i| \le C(|\mathbf{\Phi}(\psi(y_2), y_2) - \mathbf{\Phi}_b^+(r_b)|^2 + |\psi(y_2) - r_b|^2 + |\mathbf{\Phi}^-(\psi(y_2), y_2) - \mathbf{\Phi}_b^-(\psi(y_2))|). \tag{44}$$

It follows from (1) and (7) that $B_b^+ = B_b^-$. This, together with (41), yields

$$B(\psi(y_2), y_2) - B_b^+ = B^-(\psi(y_2), y_2) - B_b^-.$$

Hence one has

$$\begin{cases} P(\psi(y_2), y_2) - P_b^+(r_b) = e_1(\psi(y_2) - r_b) + R_3, \\ S(\psi(y_2), y_2) - S_b^+ = e_2(\psi(y_2) - r_b) + R_4, \end{cases} \tag{45}$$

where R_i $(i = 3, 4)$ satisfies the similar estimate as (44),

$$\begin{aligned} e_1 =& 2\frac{c_v(\rho_b^- U_b^-)(r_b) c^2(\rho_b^+(r_b), S_b^+)}{r_b((U_b^+(r_b))^2 - c^2(\rho_b^+(r_b), S_b^+))}\Bigg(U_b^-(r_b)\Bigg((U_b^+(r_b))^2 + \frac{1}{\gamma-1}c^2(\rho_b^+(r_b), S_b^+)\Bigg) \\ & - U_b^+(r_b)\Bigg((U_b^+(r_b))^2 + \frac{2}{\gamma-1}c^2(\rho_b^+(r_b), S_b^+)\Bigg)\Bigg), \end{aligned}$$

and

$$e_2 = \frac{2(\gamma-1)c_v}{r_b}\frac{(\rho_b^- U_b^-)(r_b)}{P_b^+(r_b)}(U_b^-(r_b) - U_b^+(r_b)). \tag{46}$$

Clearly, $e_2 > 0$.

2.3. **Fix the domain and the reformulation of the problem.** To fix the shock front, we introduce the following coordinate transformation

$$z_1 = \frac{y_1 - \psi(y_2)}{r_2 - \psi(y_2)}N \quad \text{and} \quad z_2 = y_2 \quad \text{with} \quad N = r_2 - r_b.$$

Clearly, the domain D_+ and the wall $\Gamma_{w,y}$ are changed into

$$E_+ = (0,N)\times(0,M) \quad \text{and} \quad \Gamma_{w,z} = (0,N)\times\{M\},$$

respectively. Define

$$\begin{aligned}
&(\tilde\rho_b^+, \tilde U_b^+, \tilde P_b^+)(z_1) = (\rho_b^+, U_b^+, P_b^+)(r_b + z_1),\\
&(\tilde\rho, \tilde U_1, \tilde\varpi, \tilde U_3, \tilde P, \tilde S, \tilde B, \tilde\theta)(z) = (\rho, U_1, \varpi, U_3, P, S, B, \theta)\left(\psi(z_2) + \frac{r_2 - \psi(z_2)}{N}z_1, z_2\right).
\end{aligned}$$

Set $\mathbf{W} := (W_1, W_2, W_3, W_4, W_5, W_6)$ with

$$\begin{aligned}
&W_1(z) = \tilde U_1(z) - \tilde U_b^+(z_1),\ W_2(z) = \tilde\varpi(z), \quad W_3(z) = \tilde U_3(z),\\
&W_4(z) = \tilde P(z) - \tilde P_b^+(z_1),\ W_5(z) = \tilde S(z) - S_b^+,\ W_6(z_2) = \psi(z_2) - r_b,
\end{aligned}$$

and

$$W_6^\diamondsuit(z_2) = r_b + W_6(z_2), \quad W_6^\#(z_1,z_2) = r_b + z_1 + \frac{N - z_1}{N}W_6(z_2). \tag{47}$$

In terms of the coordinates (z_1, z_2), the equation (39) becomes

$$W_6'(z_2) = \frac{2z_2}{\sin\theta(0,z_2)}\frac{(\tilde U_b^+(0) + W_1(0,z_2))W_2(0,z_2) - U_2^-(W_6^\diamondsuit(z_2), z_2)}{W_6^\diamondsuit(z_2)((\tilde P_b^+(0) + W_4(0,z_2)) - P^-(W_6^\diamondsuit(z_2), z_2))}. \tag{48}$$

It follows from the last equation in (30) and (37) that one has

$$\partial_{z_1} W_5 = 0 \quad \text{and} \quad \partial_{z_1}\tilde B = 0, \quad \text{in } E_+. \tag{49}$$

This, together with (41) and the second equation in (45), gives

$$\begin{aligned}
W_5(z) =& W_5(0,z_2) = e_2 W_6(z_2)\\
&+ R_4(\mathbf{\Phi}(W_6^\diamondsuit(z_2), z_2) - \mathbf{\Phi}_b^+(r_b), W_6(z_2), \mathbf{\Phi}^-(W_6^\diamondsuit(z_2), z_2) - \mathbf{\Phi}_b^-(W_6^\diamondsuit(z_2))),
\end{aligned} \tag{50}$$

and

$$B(z) - B_b^+ = B(0,z_2) - B_b^+ = B^-(W_6^\diamondsuit(z_2), z_2) - B_b^-. \tag{51}$$

It follows from the fourth equations in (30) and (31) that

$$\begin{cases}\partial_{z_1}[W_6^{\#}(z_1,z_2)W_3\sin\theta(z_1,z_2)]=0,\\ W_3(0,z_2)=U_3^-(W_6^{\diamondsuit}(z_2),z_2).\end{cases}\tag{52}$$

This yields

$$W_3(z) \quad = \quad \frac{W_6^{\diamondsuit}(z_2)}{W_6^{\#}(z_1,z_2)}\frac{\sin\theta(0,z_2)}{\sin\theta(z_1,z_2)}U_3^-(W_6^{\diamondsuit}(z_2),z_2).\tag{53}$$

Note that

$$U_1(y_1,y_2)=(\tilde{U}_b^++W_1)\left(\frac{y_1-W_6^{\diamondsuit}(y_2)}{N-W_6(y_2)}N,y_2\right).$$

Then it follows from (33) that

$$\theta(z_1,z_2)=\arccos(1-\vartheta(z_1,z_2)),\tag{54}$$

where

$$\vartheta(z_1,z_2)=\int_0^{z_2}\frac{2s}{(W_6^{\#}(z_1,z_2))^2\{\varrho(W_4,W_5)(\tilde{U}_b^++W_1)\}\left(\frac{W_6^{\#}(z_1,z_2)-W_6^{\diamondsuit}(s)}{N-W_6(s)}N,s\right)}ds\tag{55}$$

with

$$\varrho(W_4,W_5)=A^{-\frac{1}{\gamma}}(\tilde{P}_b^++W_4)^{\frac{1}{\gamma}}e^{-\frac{S_b^++W_5}{\gamma c_v}}.\tag{56}$$

The Bernoulli's law (51) together with the Rankine-Hugoniot conditions (41) yields

$$\begin{aligned}&\left\{\frac{1}{2}(\tilde{U}_b^++W_1)^2(1+W_2^2)+\frac{1}{2}W_3^2+h(\tilde{P}_b^++W_4,S_b^++W_5)\right\}(W_6^{\diamondsuit}(z_2),z_2)\\ =&B^-(W_6^{\diamondsuit}(z_2),z_2).\end{aligned}\tag{57}$$

Since $B_b^-=B_b^+=\frac{1}{2}(\tilde{U}_b^+)^2+h(\tilde{P}_b^+,S_b^+)$, one has

$$\begin{aligned}W_1=&\frac{1}{\tilde{U}_b^+}\left\{B^-(W_6^{\diamondsuit}(z_2),z_2)-B_b^--[h(\tilde{P}_b^++W_4,S_b^++W_5)-h(\tilde{P}_b^+,S_b^+)]\right\}\\ &-\frac{1}{2\tilde{U}_b^+}[W_1^2+(\tilde{U}_b^++W_1)^2W_2^2+W_3^2].\end{aligned}\tag{58}$$

Finally, we rewrite the system (36) in terms of W_2 and W_4. Note that

$$\frac{d}{dz_1}\tilde{P}_b^+-\frac{2\gamma\tilde{P}_b^+(\tilde{U}_b^+)^2}{(r_b+z_1)(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}=0.\tag{59}$$

Then straightforward calculations yield that

$$
\begin{aligned}
&-\frac{2\gamma\tilde{P}\tilde{U}_1^2}{\left(\psi(z_2)+\frac{r_2-\psi(z_2)}{N}z_1\right)(c^2(\tilde{\rho},\tilde{S})-\tilde{U}_1^2)}+\frac{2\gamma}{r_b+z_1}\frac{\tilde{P}_b^+(\tilde{U}_b^+)^2}{c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2}\\
=\;& e_3(z_1)(\tilde{B}(z)-B_b^+)+e_4(z_1)W_4+e_5(z_1)W_5+\tilde{e}_6(z_1)W_6(z_2)+R_5(\mathbf{W}),
\end{aligned}
$$

where

$$
\begin{aligned}
e_3(z_1) &= \frac{4\gamma\tilde{P}_b^+c^2(\tilde{\rho}_b^+,S_b^+)}{c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2},\\
e_4(z_1) &= \frac{2\gamma}{(r_b+z_1)\tilde{\rho}_b^+(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}(\tilde{\rho}_b^+(\tilde{U}_b^+)^4-P_b^+(\tilde{U}_b^+)^2+2\tilde{P}_b^+c^2(\tilde{\rho}_b^+,S_b^+)),\\
e_5(z_1) &= \frac{2\gamma(\tilde{P}_b^+)^2((\tilde{U}_b^+)^2+\frac{2}{\gamma-1}c^2(\tilde{\rho}_b^+,S_b^+))}{c_v(r_b+z_1)\tilde{\rho}_b^+(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)^2},\\
\tilde{e}_6(z_1) &= \frac{2\gamma(N-z_1)\tilde{P}_b^+(\tilde{U}_b^+)^2}{N(r_b+z_1)^2(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)},
\end{aligned}
$$

and R_5 is quadratic with respect to $\mathbf{W}$. Clearly, one has

$$e_3,\ e_4,\ e_5>0.$$

Therefore, it follows from (36) that

$$
\begin{cases}
\partial_{z_1}W_2-\dfrac{c^2(\tilde{\rho}_b^+,S_b^+)+(\tilde{U}_b^+)^2}{(r_b+z_1)(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}W_2+\dfrac{r_b+z_1}{\tilde{U}_b^+}\dfrac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}W_4\\
\qquad+\dfrac{r_b+z_1}{\tilde{U}_b^+}\dfrac{z_1-N}{N}\dfrac{d}{dz_1}\tilde{P}_b^+\dfrac{\sin\theta_b(z_2)}{2z_2}W_6'(z_2)=F_1(\mathbf{W},\nabla\mathbf{W},\mathbf{\Phi}^--\mathbf{\Phi}_b^-),\\
\partial_{z_1}W_4-\dfrac{\gamma\tilde{P}_b^+(\tilde{U}_b^+)^2}{c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2}\dfrac{1}{\kappa_b(r_b+z_1)}\dfrac{\sin\theta_b(z_2)}{2z_2}\left(\partial_{z_2}W_2+\dfrac{2\kappa_bz_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}W_2\right)\\
\qquad+e_4(z_1)W_4(z)+e_5(z_1)W_5(z)+e_6(z_1)W_6(z_2)=F_2(\mathbf{W},\nabla\mathbf{W},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)
\end{cases}
\tag{60}
$$

where $F_1(\mathbf{W},\nabla\mathbf{W},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)$ and $F_2(\mathbf{W},\nabla\mathbf{W},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)$ are quadratic with respect to $\mathbf{W}$ and $\nabla\mathbf{W}$ and

$$e_6(z_1)=\tilde{e}_6(z_1)+\frac{1}{N}\frac{d}{dz_1}\tilde{P}_b^+(z_1)=\frac{2\gamma r_2\tilde{P}_b^+(\tilde{U}_b^+)^2}{N(r_b+z_1)^2(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}.$$

Clearly, the system (60) should be supplemented with the following boundary conditions

$$
\begin{aligned}
&W_4(0,z_2) = e_1 W_6(z_2) + R_3(\mathbf{W}(0,z_2), \mathbf{\Phi}^- - \mathbf{\Phi}_b^-),\\
&W_2(z_1,0) = 0, \quad \text{for } z_1 \in [0,N],\\
&W_2(z_1,M) = \epsilon W_6^{\#}(r_1,M)) f'(W_6^{\#}(r_1,M)), \quad \text{for } z_1 \in [0,N],\\
&W_4(N,z_2) = \epsilon P_0(\theta(N,z_2)), \quad \text{for } z_2 \in [0,M].
\end{aligned}
\tag{61}
$$

Therefore, the original problem is equivalent to (48), (50), (53), (58), and (60)-(61).

3. Iteration scheme and Proof of Theorem 1

We are now in position to design an iteration scheme to prove Theorem 1. The approach is motivated by [20]. Define

$$
\Xi_\delta = \left\{ \mathbf{W} \;\middle|\; \begin{aligned} &|||\mathbf{W}||| \le \delta;\ \partial_{z_2} W_j(z_1,0) = 0, j = 1,3,4,5;\\ &W_2(z_1,0) = \partial_{z_2}^2 W_2(z_1,0) = W_5(z_1,0) = 0;\ W_6'(0) = W_6^{(3)}(0) = 0 \end{aligned} \right\},
\tag{62}
$$

where

$$
|||\mathbf{W}||| = \sum_{i=1}^5 \|W_i\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|W_6\|_{3,\alpha;(0,M)}^{(-1-\alpha;\{M\})}.
$$

Clearly, Ξ_δ is a complete metric space under the metric $d(\mathbf{W}, \hat{\mathbf{W}}) = |||\mathbf{W} - \hat{\mathbf{W}}|||$. Given any $\hat{\mathbf{W}} \in \Xi_\delta$, we use an iteration to define a mapping with $\mathcal{T}\hat{\mathbf{W}} = \mathbf{W}$ from Ξ_δ to itself by choosing suitable small δ.

3.1. The iteration scheme for W_6, W_5, and W_3.

It follows from (48) that W_6 is required to satisfy the following equation

$$
W_6'(z_2) = a\frac{2z_2}{\sin\theta_b(z_2)} W_2(0,z_2) + R_{11}(\hat{\mathbf{W}}(0,z_2), \mathbf{\Phi}^-(\hat{W}_6^{\diamondsuit}(z_2), z_2) - \mathbf{\Phi}_b^-(\hat{W}_6^{\diamondsuit}(z_2))),
$$

where $\hat{W}_6^{\diamondsuit}(z_2) = \hat{W}_6(z_2) + r_b$, R_{11} is quadratic with respect to $\hat{\mathbf{W}}(0,z_2)$, and

$$
a = \frac{\tilde{U}_b^+(0)}{r_b(\tilde{P}_b^+(0) - P_b^-(r_b))}.
\tag{63}
$$

Hence W_6 can be solved as follows

$$
W_6(z_2) = W_6(M) - a\int_{z_2}^M \frac{2s}{\sin\theta_b(s)} W_2(0,s)ds + R_{12},
\tag{64}
$$

where θ_b is defined in (35) and

$$
R_{12}(\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-) = -\int_{z_2}^M R_{11}(\hat{\mathbf{W}}(0,s), \mathbf{\Phi}^-(\hat{W}_6^{\diamondsuit}(s), s) - \mathbf{\Phi}_b^-(r_b))ds.
$$

We also note that for $\hat{\mathbf{W}} \in \Xi_\delta$, $R_{11}(z_1,0) = \partial_{z_2}^2 R_{11}(z_1,0) = 0$ for any $z_1 \in [0,N]$.

Since $\partial_{z_1} W_5 = 0$, one has

$$W_5(z) = W_5(0, z_2) = e_2 W_6(z_2) + R_4(\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-), \tag{65}$$

where e_2 is defined in (46). It is easy to verify that $\partial_{z_2} R_4(z_1, 0) = 0$ for $\hat{\mathbf{W}} \in \Xi_\delta$.

It follows from (53) that one defines

$$W_3(z_1, z_2) \;=\; \frac{\hat{W}_6^{\diamondsuit}(z_2)}{\hat{W}_6^{\#}(z_1, z_2)} \frac{\sin \hat{\theta}(0, z_2)}{\sin \hat{\theta}(z_1, z_2)} U_3^-(\hat{W}_6^{\diamondsuit}(z_2), z_2), \tag{66}$$

where $\hat{W}_6^{\#}(z_1, z_2) = r_b + z_1 + \frac{N - z_1}{N} \hat{W}_6(z_2)$ and $\hat{\theta}(z_1, z_2) = \arccos(1 - \hat{\vartheta}(z_1, z_2))$ with

$$\hat{\vartheta}(z_1, z_2) = \int_0^{z_2} \frac{2s}{(\hat{W}_6^{\#}(z_1, z_2))^2 \Big\{ \varrho(\hat{W}_4, \hat{W}_5)(\tilde{U}_b^+ + \hat{W}_1) \Big\} \Big(\frac{\hat{W}_6^{\#}(z_1, z_2) - \hat{W}_6^{\diamondsuit}(s)}{N - \hat{W}_6(s)} N, s \Big)} ds, \tag{67}$$

where ϱ is the function defined in (56). Note that $\frac{\hat{W}_6^{\#}(z_1, z_2) - \hat{W}_6^{\diamondsuit}(s)}{N - \hat{W}_6(s)} N$ may exceed the interval $[0, N]$, hence we extend the functions $\hat{\mathbf{W}}$ to a larger domain $[-N, 2N] \times [0, M]$ as follows

$$\hat{\mathbf{W}}^e(z_1, z_2) = \begin{cases} \sum_{k=1}^3 c_k \hat{\mathbf{W}}(-\frac{z_1}{k}, z_2), & -N \le z_1 < 0, \\ \sum_{k=1}^3 c_k \hat{\mathbf{W}}(\frac{2N - z_1}{k}, z_2), & N < z_1 \le 2N, \end{cases} \tag{68}$$

where the constants c_k $(k = 1, 2, 3)$ satisfy the following algebraic relations

$$\sum_{k=1}^3 c_k = 1, \quad -\sum_{k=1}^3 \frac{c_k}{k} = 1, \quad \sum_{k=1}^3 \frac{c_k}{k^2} = 1. \tag{69}$$

It is easy to see that the extended functions $\hat{\mathbf{W}}^e$ belong to C^2 as long as $\hat{\mathbf{W}} \in C^2$. For ease of notations, we still denote these extended functions by $\hat{\mathbf{W}}$.

3.2. **The iteration scheme for W_2 and W_4.** Substituting (64) and (65) into (60) yields that W_2 and W_4 satisfy the following first order elliptic system with a nonlocal term and a parameter,

(70)

$$\begin{cases}\partial_{z_1}W_2-\frac{c^2(\tilde{\rho}_b^+,S_b^+)+(\tilde{U}_b^+)^2}{(r_b+z_1)(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}W_2+\frac{r_b+z_1}{\tilde{U}_b^+}\frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}W_4+a\frac{r_b+z_1}{\tilde{U}_b^+}\frac{z_1-N}{N}\frac{d}{dz_1}\tilde{P}_b^+W_2(0,z_2)\\ \qquad\qquad =F_3(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-),\\ \partial_{z_1}W_4-\frac{\gamma\tilde{P}_b^+(\tilde{U}_b^+)^2}{\kappa_b(r_b+z_1)(c^2(\tilde{\rho}_b^+,S_b^+)-(\tilde{U}_b^+)^2)}\frac{\sin\theta_b(z_2)}{2z_2}\left(\partial_{z_2}W_2+\frac{2\kappa_bz_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}W_2\right)+r_4(z_1)W_4\\ \quad+\Big(e_6(z_1)+e_2e_5(z_1)\Big)\left(W_6(M)-a\int_{z_2}^M\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\right)=F_4(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-),\\ W_4(0,z_2)=e_1\left(W_6(M)-a\int_{z_2}^M\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\right)+e_1R_{12}+R_5(\hat{\mathbf{W}}(0,z_2),\mathbf{\Phi}^--\mathbf{\Phi}_b^-),\\ W_2(z_1,0)=0,\quad z_1\in[0,N],\\ W_2(z_1,M)=\epsilon\hat{W}_6^{\#}(M)f'(\hat{W}_6^{\#}(M)),\quad z_1\in[0,N],\\ W_4(N,z_2)=\epsilon P_0(\hat{\theta}(N,z_2)),\quad z_2\in[0,M],\end{cases}$$

where $F_3(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)$ and $F_4(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)$ are quadratic with respect to $\mathbf{W}$ and $\nabla\mathbf{W}$. Since the values $\hat{W}_6^{\#}(z_1,M)$ and $\hat{\theta}(N,z_2)$ may exceed the interval $[r_b,r_2]$ and $[0,\theta_0+\epsilon f(r_2)]$, respectively, one can also extend the functions f and P_0 smoothly to a larger interval as in (68) and (69). The straightforward computations show

$$F_3(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)(z_1,0)=0\quad\text{and}\quad\partial_{z_2}F_4(\hat{\mathbf{W}},\nabla\hat{\mathbf{W}},\mathbf{\Phi}^--\mathbf{\Phi}_b^-)(z_1,0)=0.$$

To obtain the estimate for F_3 and F_4, we should be careful about the singular terms involving sine and cotangent functions of $\hat{\theta}(z)$ and $\theta_b(z_2)$. Note that there exists $\kappa_i(i=1,2)$ depending only on the background solutions such that

$$\kappa_1z_2\le\hat{\theta}(z)\le\kappa_2z_2\quad\text{for any }z\in\overline{E_+}.S$$

Since $\hat{W}_2(z_1,0)=\hat{W}_3(z_1,0)=0$, it is easy to see that

$$\sum_{j=2}^3\|\hat{W}_j^2\cot\hat{\theta}(z)\|_{1,\alpha;E_+}^{(1-\alpha;\Gamma_{w,z})}\le C\||\hat{\mathbf{W}}\||^2. \tag{71}$$

Also by (67) and (35), one has

$$\cos\hat{\theta}(z)-\cos\theta_b(z_2)=\frac{1}{(r_b+z_1)^2\tilde{\rho}_b^+(z_1)\tilde{U}_b^+(z_1)}z_2^2-\hat{\vartheta}(z_1,z_2)$$

and

$$\begin{aligned}(\cot\hat{\theta}(z)-\cot\theta_b(z_2))\hat{W}_2(z)&=\frac{\hat{W}_2(z)}{\sin\theta_b(z_2)}(\cos\hat{\theta}(z)-\cos\theta_b(z_2))\\&\quad+\frac{\cos\hat{\theta}(\cos\hat{\theta}(z)+\cos\theta_b(z_2))}{\sin\hat{\theta}(z)+\sin\theta_b(z_2)}\frac{\cos\hat{\theta}(z)-\cos\theta_b(z_2)}{\sin\hat{\theta}(z)\sin\theta_b(z_2)}\hat{W}_2(z).\end{aligned} \tag{72}$$

With the aid of (71), one has

$$\sum_{j=3}^{4} \|F_i(\hat{\mathbf{W}}, \nabla\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-)\|_{1,\alpha;E_+}^{(1-\alpha;\Gamma_{w,z})} \le C(\epsilon + \||\hat{\mathbf{W}}\||^2). \tag{73}$$

The crucial part for the analysis is to get the existence of solutions for the problem (70). Set

$$\begin{aligned}
\lambda_1(z_1) &= \exp\bigg(-\int_0^{z_1} \frac{c^2(\tilde{\rho}_b^+, S_b^+) + (\tilde{U}_b^+)^2}{(r_b+z_1)(c^2(\tilde{\rho}_b^+, S_b^+) - (\tilde{U}_b^+)^2)} ds\bigg),\\
\lambda_2(z_1) &= \frac{r_b+z_1}{\tilde{U}_b^+(z_1)}\lambda_1(z_1),\ \ \lambda_3(z_1) = a\frac{r_b+z_1}{\tilde{U}_b^+(z_1)}\frac{(z_1-N)\partial_{z_1}\tilde{P}_b^+}{N}\lambda_1(z_1),\\
\lambda_4(z_1) &= \exp\bigg(\int_0^{z_1} e_3(s)ds\bigg),\ \ \lambda_5(z_1) = \frac{\gamma\tilde{P}_b^+(\tilde{U}_b^+)^2}{\kappa_b(r_b+z_1)(c^2(\tilde{\rho}_b^+, S_b^+) - (\tilde{U}_b^+)^2)}\lambda_4(z_1),\\
\lambda_6(z_1) &= \bigg(e_6(z_1) + e_2e_4(z_1)\bigg)\lambda_4(z_1).
\end{aligned}$$

It is clear that

$$\lambda_1, \lambda_2, \lambda_4 > 0 \quad \text{and} \quad \lambda_3 \le 0. \tag{74}$$

In terms of λ_i $(i = 1, \cdots, 6)$, the problem (70) can be rewritten as

$$\begin{cases}
\partial_{z_1}(\lambda_1(z_1)W_2) + \frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}(\lambda_2(z_1)W_4) + \lambda_3(z_1)W_2(0,z_2) = G_1(z),\\
\partial_{z_1}(\lambda_4(z_1)W_4) - \lambda_5(z_1)\frac{\sin\theta_b(z_2)}{2z_2}(\partial_{z_2}W_2 + \frac{2\kappa_b z_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}W_2)\\
\qquad +\lambda_6(z_1)\bigg(W_6(M) - a\int_{z_2}^M \frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\bigg) = G_2(z),\\
W_4(0,z_2) = e_1a\bigg(\frac{W_6(M)}{a} - \int_{z_2}^M \frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\bigg) + G_3(z_2),\\
W_4(N,z_2) = \epsilon G_4(z_2),\\
W_2(z_1,0) = 0, \quad W_2(z_1,M) = \epsilon G_5(z_1),
\end{cases} \tag{75}$$

where a is given in (63) and

$$\begin{aligned}
&G_1(z) = \lambda_1(z_1)F_3(\hat{\mathbf{W}}, \nabla\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-),\ \ G_2(z) = \lambda_4(z_1)F_4(\hat{\mathbf{W}}, \nabla\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-),\\
&G_3(z_2) = e_1R_{12}(\hat{\mathbf{W}}(0,z_2), \mathbf{\Phi}^- - \mathbf{\Phi}_b^-) + R_5(\hat{\mathbf{W}}(0,z_2)),\ \ G_4(z_2) = P_0(\hat{\theta}(N,z_2)),\\
&G_5(z_1) = \hat{W}_6^{\#}(z_1,M)f'(\hat{W}_6^{\#}(z_1,M)).
\end{aligned}$$

Note that the first equation in (75) can be written as follows

$$\partial_{z_1}\left(\frac{2z_2}{\sin\theta_b(z_2)}\lambda_1(z_1)W_2\right)+\partial_{z_2}\left\{\lambda_2(z_1)W_4+\lambda_3(z_1)\left(\frac{W_6(M)}{a}-\int_{z_2}^M\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\right)\right.$$
$$\left.-\int_{z_2}^M G_1(z_1,s)ds\right\}=0.$$

Hence there exists a potential function ϕ satisfying

$$(76)\quad \begin{cases}\partial_{z_1}\phi=\lambda_2(z_1)W_4+\lambda_3(z_1)\left(\frac{W_6(M)}{a}-\int_{z_2}^M\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\right)-\int_{z_2}^M G_1(z_1,s)ds,\\ \partial_{z_2}\phi=-\lambda_1(z_1)\frac{2z_2}{\sin\theta_b(z_2)}W_2(z),\quad \phi(0,M)=0.\end{cases}$$

Therefore, W_2 and W_4 can represented in terms of ϕ as follows

$$(77)\quad \begin{cases}W_2(z)=-\frac{1}{\lambda_1(z_1)}\frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}\phi,\\ W_4(z)=\frac{\partial_{z_1}\phi}{\lambda_2(z_1)}-\frac{\lambda_3(z_1)}{\lambda_2(z_1)}\left(\frac{W_6(M)}{a}-\phi(0,z_2)\right)+\frac{1}{\lambda_2(z_1)}\int_{z_2}^M G_1(z_1,s)ds.\end{cases}$$

Now, substituting (77) into the second equation and the boundary conditions in (75) gives

$$(78)\quad \begin{cases}\partial_{z_1}\left(\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\partial_{z_1}\phi\right)-\left\{a\lambda_6(z_1)+\frac{d}{dz_1}\left(\frac{\lambda_4(z_1)\lambda_3(z_1)}{\lambda_2(z_1)}\right)\right\}(\phi(0,z_2)-\frac{W_6(M)}{a})\Big)\\ \quad+\frac{\lambda_5(z_1)}{\lambda_1(z_1)}\left(\frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}\left(\frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}\phi\right)+\frac{\kappa_b\cos\theta_b(z_2)}{2z_2}\partial_{z_2}\phi\right)\\ \qquad=\partial_{z_2}\left(\int_0^{z_2}G_2(z_1,s)ds\right)-\partial_{z_1}\left(\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\int_{z_2}^M G_1(z_1,s)ds\right),\\ \partial_{z_1}\phi(0,z_2)+(a\lambda_2(0)e_1+\lambda_3(0))\left(\phi(0,z_2)-\frac{W_6(M)}{a}\right)=\lambda_2(0)G_3(z_2)-\int_{z_2}^M G_1(0,s)ds,\\ \partial_{z_1}\phi(N,z_2)=\epsilon\lambda_2(N)P_0(\hat{\theta}(N,z_2))-\int_{z_2}^M G_1(N,s)ds,\\ \partial_{z_2}\phi(z_1,0)=0,\\ \partial_{z_2}\phi(z_1,M)=-\frac{2M}{\sin\theta_b(M)}\lambda_1(z_1)\epsilon(\hat{W}_6^{\#}(z_1,M))f'(\hat{W}_6^{\#}(z_1,M)).\end{cases}$$

To simplify the notations, we define

$$a_1(z_1) = \frac{\lambda_4(z_1)}{\lambda_2(z_1)},\ a_2(z_1) = \frac{\lambda_5(z_1)}{\lambda_1(z_1)},\ a_3(z_1) = \left\{a\lambda_6(z_1) + \frac{d}{dz_1}\left(\frac{\lambda_4(z_1)\lambda_3(z_1)}{\lambda_2(z_1)}\right)\right\},$$

$$a_4 = ae_1\lambda_2(0) + \lambda_3(0),\ \mu = -\frac{W_6(M)}{a},\ \mathcal{G}_1(z_2) = \lambda_2(0)G_3(z_2) - \int_{z_2}^{M} G_1(0,s)ds$$

$$\mathcal{F}_1(z) = -\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\int_{z_2}^{M} G_1(z_1,s)ds, \quad \mathcal{F}_2(z) = \int_0^{z_2} G_2(z_1,s)ds,$$

$$\mathcal{G}_2(z_2) = \epsilon\lambda_2(N)P_0(\hat{\theta}(N,z_2)) - \int_{z_2}^{M} G_1(N,s)ds, \quad \mathcal{G}_3(z_1) = -\frac{2M}{\sin\theta_b(M)}\lambda_1(z_1)G_5(z_1),$$

$$\mathfrak{d}_1(z_2) = \frac{\sin\theta_b(z_2)}{2z_2}, \quad \mathfrak{d}_2(z_2) = \frac{\kappa_b\cos\theta_b(z_2)}{2z_2}.$$

It follows from (73) that

$$\sum_{i=1}^{2}\|\mathcal{F}_i\|_{1,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \sum_{i=1}^{2}\|\mathcal{G}_i\|_{1,\alpha;(0,M)}^{(-\alpha;\{M\})} \le C(\epsilon + \||\hat{\mathbf{W}}\||^2). \tag{79}$$

To deal with the singularity near $z_2 = 0$, we define

$$\zeta_1 = z_1,\ \zeta_2 = z_2\cos\tau,\ \zeta_3 = z_2\sin\tau, \quad \text{for } z_1\in[0,N],\ z_2\in[0,M],\ \tau\in[0,2\pi].$$

and denote

$$E_1 = \{(\zeta_1,\zeta_2,\zeta_3) : 0<\zeta_1<N,\ \zeta_2^2+\zeta_3^2\le M^2\},\ \Gamma_{w,\zeta} = [0,N]\times\{(\zeta_2,\zeta_3):\zeta_2^2+\zeta_3^2 = M^2\},$$

$$E_2 = \{(\zeta_2,\zeta_3) : \zeta_2^2+\zeta_3^2\le M^2\}, \quad \Gamma'_\zeta = \{(\zeta_2,\zeta_3):\zeta_2^2+\zeta_3^2 = M^2\},$$

$$\Upsilon(\boldsymbol{\zeta}) = \phi(\zeta_1, \sqrt{\zeta_2^2+\zeta_3^2}),$$

Denote $\Upsilon^*(\boldsymbol{\zeta}) = \Upsilon(\boldsymbol{\zeta}) + \mu$ where $\boldsymbol{\zeta} = (\zeta_1,\zeta_2,\zeta_3)$. Then Υ^* satisfies the following problem

$$\begin{cases} \partial_{\zeta_1}(a_1(\zeta_1)\partial_{\zeta_1}\Upsilon^*) - \frac{\kappa_b^2}{4}a_2(\zeta_1)(\zeta_2\partial_{\zeta_2}\Upsilon^* + \zeta_3\partial_{\zeta_3}\Upsilon^*) + a_3(\zeta_1)\Upsilon^*(0,\zeta_2,\zeta_3) \\ \quad + a_2(\zeta_1)\mathfrak{d}_1(\sqrt{\zeta_2^2+\zeta_3^2})\left[\partial_{\zeta_2}(\mathfrak{d}_1(\sqrt{\zeta_2^2+\zeta_3^2})\partial_{\zeta_2}\Upsilon^*) + \partial_{\zeta_3}(\mathfrak{d}_1(\sqrt{\zeta_2^2+\zeta_3^2})\partial_{\zeta_3}\Upsilon^*)\right] \\ = \partial_{\zeta_1}\mathcal{F}_1(\zeta_1,\sqrt{\zeta_2^2+\zeta_3^2}) + \sum_{i=2}^{3}\partial_{\zeta_i}\left(\frac{\zeta_i\mathcal{F}_2(\zeta_1,\sqrt{\zeta_2^2+\zeta_3^2})}{\sqrt{\zeta_2^2+\zeta_3^2}}\right) - \frac{\mathcal{F}_2(\zeta_1,\sqrt{\zeta_2^2+\zeta_3^2})}{\sqrt{\zeta_2^2+\zeta_3^2}}, \\ \partial_{\zeta_1}\Upsilon^*(0,\zeta_2,\zeta_3) + a_4\Upsilon^*(0,\zeta_2,\zeta_3) = \mathcal{G}_1(\sqrt{\zeta_2^2+\zeta_3^2}), \\ \partial_{\zeta_1}\Upsilon^*(N,\zeta_2,\zeta_3) = \mathcal{G}_2(\sqrt{\zeta_2^2+\zeta_3^2}), \\ (\zeta_2\partial_{\zeta_2}+\zeta_3\partial_{\zeta_3})\Upsilon^*(\zeta_1,\zeta_2,\zeta_3) = M\mathcal{G}_3(\zeta_1), \quad \text{on } \zeta_2^2+\zeta_3^2 = M^2. \end{cases} \tag{80}$$

Proposition 3. *For any* $(\mathcal{F}_1,\mathcal{F}_2)\in C^{(-\alpha;\Gamma_{w,\zeta})}_{1,\alpha;E_1}$ *and* $\mathcal{F}_2(x_1,0)=0$, $\mathcal{G}_1$, $\mathcal{G}_2\in C^{(-\alpha;\Gamma'_{\zeta})}_{1,\alpha;E_2}$, *then the problem* (80) *has a unique solution* $\Upsilon^*(\boldsymbol{\zeta})=\tilde{\Upsilon}^*(\zeta_1,\sqrt{\zeta_2^2+\zeta_3^2})\in C^{(-1-\alpha;\Gamma_{w,\zeta})}_{2,\alpha;E_1}$, *which satisfies the following estimate*

$$\|\Upsilon^*\|^{(-1-\alpha;\Gamma_{w,\zeta})}_{2,\alpha;E_1}\le C\left(\sum_{i=1}^2\|\mathcal{F}_i\|^{(-\alpha;\Gamma_{w,\zeta})}_{1,\alpha;E_1}+\sum_{j=1}^2\|\mathcal{G}_j\|^{(-\alpha;\Gamma'_{\zeta})}_{1,\alpha;E_2}+\|\mathcal{G}_3\|_{1,\alpha;[0,N]}\right).\tag{81}$$

Proof. Note that the coefficients in the first equation of (80) are infinitely smooth near the axis $\zeta_2^2+\zeta_3^2=0$, which is quite different from the elliptic system in [19, Lemma 4.3]. So we do not need to take much care of the regularity near the axis. This advantage comes essentially from our new Lagrangian transformation. The system (80) has a variational structure similar to the one in the proof of [19, Lemma 4.3], one can obtain the existence and uniqueness of $H^1(E_1)$ weak solution by Lax-Milgram theorem and Fredholm alternative theorem as in [19].

To get the estimate (81), one can put the term $a_3(\zeta_1)\Upsilon^*(0,\zeta_2,\zeta_3)$ on the right hand side, so by the trace theorem, the right hand side belongs to $L^2(E_1)$ and the interior estimates can be obtained by a standard way. Furthermore, one can use [22, Theorems 5.36 and 5.45] to obtain global L^∞ bound and C^α norm estimates for Υ^* with some Hölder exponent $\alpha\in(0,1)$. Hence the nonlocal term $a_3(\zeta_1)\Upsilon^*(0,\zeta_2,\zeta_3)$ becomes C^α and (81) follows by employing [22, Theorem 4.6]. ☐

Proposition 3 actually implies the following estimates for W_2 and W_4.

Proposition 4. *The probelm* (75) *has a unique solution* $(W_2,W_4,W_6(M))\in(C^{(-\alpha;\Gamma_{w,z})}_{2,\alpha;E_+})^2\times\mathbb{R}$ *satisfying*

$$\|W_2\|^{(-\alpha;\Gamma_{w,z})}_{2,\alpha,E_+}+\|W_4\|^{(-\alpha;\Gamma_{w,z})}_{2,\alpha,E_+}+|W_6(M)|\le C(\delta^2+\epsilon)\tag{82}$$

and

$$W_2(z_1,0)=\partial^2_{z_2}W_2(z_1,0)=0,\ \ \partial_{z_2}W_4(z_1,0)=0.\tag{83}$$

Proof. It follows from Proposition 3 and the equivalence between $\|\cdot\|^{(-\alpha;\Gamma_{w,z})}_{1,\alpha;E_+}$ and $\|\cdot\|^{(-\alpha;\Gamma_{w,\zeta})}_{1,\alpha;E_1}$ that the system (75) has a unique solution $(W_2,W_4,W_6(M))\in(C^{(-\alpha;\Gamma_{z,w})}_{1,\alpha;E_+})^2\times\mathbb{R}$ satisfying

$$\begin{aligned}&\|W_2\|^{(-\alpha;\Gamma_{w,z})}_{1,\alpha,E_+}+\|W_4\|^{(-\alpha;\Gamma_{w,z})}_{1,\alpha,E_+}+|W_6(M)|\\ \le&C(\sum_{i=1}^2\|G_i\|^{(1-\alpha;\Gamma_{w,z})}_{1,\alpha;E_+}+\|G_3\|^{(-\alpha;\Gamma_{w,z})}_{1,\alpha;E_+}+\epsilon)\\ \le&C(\||\hat{\mathbf{W}}\||^2+\epsilon)\le C(\delta^2+\epsilon).\end{aligned}$$

In addition, $W_2(z_1,0)=\partial_{z_2}W_4(z_1,0)=0$.

Rewrite the problem (75) as

$$
\begin{cases}
\partial_{z_1}(\lambda_1(z_1)W_2) + \partial_{z_2}(\lambda_2(z_1)W_4) = G_5(z), \\
\partial_{z_1}(\lambda_4(z_1)W_4) - \lambda_5(z_1)\frac{\sin\theta_b(z_2)}{2z_2}(\partial_{z_2}W_2 + \frac{2\kappa_b z_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}W_2) = G_6(z), \\
W_4(0,z_2) = G_8(z_2),\ W_4(N,z_2) = \epsilon G_4(z_2), \\
W_2(z_1,0) = 0,\ W_2(z_1,M) = \epsilon G_5(z_1),
\end{cases}
\tag{84}
$$

where

$$
\begin{aligned}
&G_5(z) = G_1(z) - \lambda_3(z_1)W_2(0,z_2), \\
&G_6(z) = G_2(z) + \lambda_6(z_1)\Big(W_6(M) - a\int_{z_2}^{M}\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\Big), \\
&G_7(z) = e_1 a\Big(\frac{W_6(M)}{a} - \int_{z_2}^{M}\frac{2s}{\sin\theta_b(s)}W_2(0,s)ds\Big) + G_3(z_2).
\end{aligned}
$$

Hence W_4 satisfies

$$
\begin{cases}
\partial_{z_1}\Big(\frac{2z_2}{\sin\theta_b(z_2)}\frac{\lambda_1(z_1)}{\lambda_5(z_1)}\partial_{z_1}(\lambda_4(z_1)W_4)\Big) + \lambda_2(z_1)\Big(\partial_{z_2}^2 W_4 + \frac{2\kappa_b z_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}\partial_{z_2}W_4\Big) \\
\qquad = \partial_{z_1}\Big(\frac{2z_2}{\sin\theta_b(z_2)}\frac{\lambda_1(z_1)}{\lambda_5(z_1)}G_6(z)\Big) + \partial_{z_2}G_5(z) + \frac{2\kappa_b z_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}G_5(z), \\
W_4(0,z_2) = G_7(z_2), \quad W_4(N,z_2) = \epsilon G_4(z_2), \quad \partial_{z_2}W_4(z_1,0) = 0.
\end{cases}
\tag{85}
$$

Similar to the proof of Proposition 3, one has

$$
\begin{aligned}
\|W_4\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \le & C\Big(\sum_{i=5}^{6}\|G_i\|_{1,\alpha;E_+}^{(1-\alpha;\Gamma_{z,w})} + \|G_7\|_{1,\alpha;(0,M)}^{(-\alpha;\{M\})} + \epsilon\Big) \\
\le & C(\||\hat{\mathbf{W}}\||^2 + \epsilon) \le C(\delta^2 + \epsilon).
\end{aligned}
\tag{86}
$$

This, together with the first equation in (84), gives

$$
\|(\partial_{z_1}^2 W_2, \partial_{z_1 z_2}^2 W_2)\|_{\alpha;E_+}^{(2-\alpha;\Gamma_{w,z})} \le C(\|W_4\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|W_2\|_{1,\alpha,E_+}^{(-\alpha;\Gamma_{w,z})}) \le C(\delta^2 + \epsilon).
$$

Finally, note that

$$
W_2(z) = \frac{2}{\lambda_5(z_1)\sin\theta_b(z_2)}\int_0^{z_2} s(\partial_{z_1}(\lambda_4(z_1)W_4)(z_1,s) - G_6(z_1,s))ds.
$$

Similar to [19, Lemma B.3], we conclude that W_2 satisfies (82) and $\partial_{z_2}^2 W_2(z_1,0) = 0$. $\square$

3.3. **The iteration scheme for W_1 and the estimate for W_1, W_3, W_5, and W_6.** It follows from (58) that W_1 can be solved as follows

$$\begin{aligned} W_1 =& \frac{1}{\tilde{U}_b^+}\{B^-(\hat{W}_6^{\diamondsuit}(z_2), z_2) - B_b^- - [h(\tilde{P}_b^+ + W_4, S_b^+ + W_5) - h(\tilde{P}_b^+, S_b^+)]\} \\ & - \frac{1}{2\tilde{U}_b^+}[\hat{W}_1^2 + (\tilde{U}_b^+ + \hat{W}_1)^2\hat{W}_2^2 + \hat{W}_3^2]. \end{aligned} \tag{87}$$

Now we are ready to estimate W_1, W_3, W_5, and W_6.

Proposition 5. *With $(W_2, W_4) \in \left(H_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})}\right)^2$ obtained in Proposition 4, W_6, W_5, W_3, and W_1 are uniquely determined by (64), (65), (66) and (87) and satisfy*

$$\sum_{j=1,3,5} \|W_j|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|W_6\|_{3,\alpha,[0,M)}^{(-1-\alpha;\{M\})} \le C(\delta^2 + \epsilon). \tag{88}$$

Proof. It follows from (64) that

$$\begin{aligned} W_6(z_2) =& W_6(M) - a\int_{z_2}^M \frac{2s}{\sin\theta_b(s)} W_2(0,s)ds \\ & - \int_{z_2}^M R_{11}(\hat{\mathbf{W}}(0,s), \mathbf{\Phi}^-(r_b + \hat{W}_6(s), s) - \mathbf{\Phi}_b^-(r_b + \hat{W}_6(s)))ds. \end{aligned} \tag{89}$$

Thus $W_6'(0) = 0$ and the following estimate holds

$$\begin{aligned} \|W_6\|_{3,\alpha,[0,M)}^{(-1-\alpha;\{M\})} \le& C(|W_6(M)| + \|W_2\|_{2,\alpha,E_+}^{(-\alpha;\Gamma_{w,z})} + \|R_{11}(\hat{W}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-)\|_{2,\alpha,E_+}^{(-\alpha;\Gamma_{w,z})}) \\ \le& C(\delta^2 + \epsilon). \end{aligned} \tag{90}$$

It follows from (65) that

$$W_5(z) = W_5(0, z_2) = e_2 W_6(z_2) + R_4(\hat{\mathbf{W}}, \mathbf{\Phi}^- - \mathbf{\Phi}_b^-). \tag{91}$$

Hence $\partial_{z_2} W_5(z_1, 0) = 0$ and

$$\|W_5\|_{2,\alpha,E_+}^{(-\alpha;\Gamma_{w,z})} \le e_2\|W_6\|_{3,\alpha,[0,M)}^{(-1-\alpha;\{M\})} + \|R_4\|_{2,\alpha,E_+}^{(-\alpha;\Gamma_{w,z})} \le C(\delta^2 + \epsilon). \tag{92}$$

Using (66) gives

$$\|W_3\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \le C\|\hat{\mathbf{W}}\|_{\Xi_\delta}\|U_3^-\|_{C^{2,\alpha}(\Omega)} \le C\epsilon\delta. \tag{93}$$

It follows from (87) that

$$\|W_1\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \le C\left(\epsilon + \sum_{j=3}^4 \|W_i\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \||\hat{\mathbf{W}}|\|^2\right) \le C(\epsilon + \delta^2). \tag{94}$$

Combining (90) with (92)-(94) together finishes the proof of the proposition. ∎

3.4. **Proof of Theorem 1.** Now we are in position to prove Theorem 1.

Proof of Theorem 1. The proof is divided into three steps.

Step 1. Boundedness. Given any $\hat{\mathbf{W}} \in \Xi_\delta$, let $\mathbf{W} = \mathcal{T}(\hat{\mathbf{W}})$ be the solutions obtained in Propositions 4 and 5. Thus one has

$$|||\mathbf{W}||| \leq C_*(\epsilon + \delta^2). \tag{95}$$

Let $\delta = 2C_*\epsilon$ and choose ϵ_0 small enough satisfying $2C_*^2\epsilon_0 \leq \frac{1}{2}$. Therefore, for any $0 < \epsilon \leq \epsilon_0$, one has

$$C_*(\epsilon + \delta^2) = \frac{\delta}{2} + 2C_*^2\epsilon\delta \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

This implies that $\mathcal{T}$ maps Ξ_δ into itself.

Step 2. Contraction. Given any $\hat{\mathbf{W}}^{(i)} \in \Xi_\delta$ ($i = 1,\, 2$), let $\mathbf{W}^{(i)} = \mathcal{T}\hat{\mathbf{W}}^{(i)}$ ($i = 1,\, 2$) be obtained in Step 1. Denote

$$\hat{\mathbf{Y}} = \hat{\mathbf{W}}^{(1)} - \hat{\mathbf{W}}^{(2)} \quad \text{and} \quad \mathbf{Y} = \mathbf{W}^{(1)} - \mathbf{W}^{(2)}.$$

It follows from (70) that Y_2 and Y_4 satisfies

$$\begin{cases} \partial_{z_1}(\lambda_1(z_1)Y_2) + \frac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}(\lambda_2(z_2)Y_4) + \lambda_3 Y_2(0,z_2) = G_1^{(1)}(z) - G_1^{(2)}(z), \\ \partial_{z_1}(\lambda_4(z_1)Y_4) - \lambda_5(z_1)\frac{\sin\theta_b(z_2)}{2z_2}(\partial_{z_2}Y_2 + \frac{2\kappa_b z_2\cos\theta_b(z_2)}{\sin^2\theta_b(z_2)}Y_2) \\ \qquad -\lambda_6(z_1)\left(Y_6(M) - a\int_{z_2}^M \frac{2s}{\sin\theta_b(s)}Y_2(0,s)ds\right) = G_2^{(1)}(z) - G_2^{(2)}(z), \\ Y_4(0,z_2) = e_1 a\left(\frac{Y_6(M)}{a} - \int_{z_2}^M \frac{2s}{\sin\theta_b(s)}Y_2(0,s)ds\right) + G_3^{(1)}(z_2) - G_3^{(2)}(z_2), \\ Y_4(N,z_2) = G_4^{(1)}(z_2) - G_4^{(2)}(z_2), \\ Y_2(z_1,0) = 0, \quad Y_2(z_1,M) = G_5^{(1)}(z_1) - G_5^{(2)}(z_1). \end{cases} \tag{96}$$

Using Proposition 4 gives

$$\begin{aligned} \sum_{i=2,4}\|Y_i\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + |Y_6(M)| \leq & C\sum_{i=1}^{2}\|G_i^{(1)} - G_i^{(2)}\|_{1,\alpha;E_+}^{(1-\alpha;\Gamma_{w,z})} + \|G_3^{(1)} - G_3^{(2)}\|_{1,\alpha;[0,M)}^{(-\alpha;\{M\})} \\ & + \epsilon\|P_0(\hat{\theta}^{(1)}) - P_0(\hat{\theta}^{(2)})\|_{1,\alpha;E_+}^{(-\alpha;\{M\})} + C\epsilon|\hat{Y}(M)| \\ \leq & C\epsilon\left(\sum_{i=1}^{5}\|\hat{Y}_i\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|\hat{Y}_6\|_{3,\alpha;[0,M)}^{(-1-\alpha;\{M\})}\right). \end{aligned} \tag{97}$$

It follows from (89) that Y_6 satisfies

$$Y_6(z_2) = Y_6(M) - \int_{z_2}^M \frac{2s}{\sin\theta_b(s)}Y_2(0,s)ds + R_{12}^{(1)} - R_{12}^{(2)}. \tag{98}$$

Therefore, one has

$$\begin{aligned}\|Y_6\|_{3,\alpha;[0,M)}^{(-1-\alpha;\{M\})} &\leq |Y_6(M)| + C\|Y_2\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|R_{11}^{(1)} - R_{11}^{(2)}\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \\ &\leq C\epsilon|||\hat{\mathbf{Y}}|||.\end{aligned} \tag{99}$$

It follows from (91) that

$$Y_5(z) = e_2 Y_6(z_2) + R_4^{(1)} - R_4^{(2)}. \tag{100}$$

Thus it holds that

$$\|Y_5\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \leq C\|Y_6\|_{3,\alpha;[0,M)}^{(-1-\alpha;\{M\})} + \|R_4^{(1)} - R_4^{(2)}\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \leq C\epsilon|||\hat{\mathbf{Y}}|||. \tag{101}$$

The equation (66) implies

$$Y_3(z_1,z_2) = \frac{\hat{W}_6^{\diamondsuit}(z_2)}{\hat{W}_6^{\#}(z_1,z_2)} \frac{\sin\hat{\theta}(0,z_2)}{\sin\hat{\theta}(z_1,z_2)} U_3^-(\hat{W}_6^{\diamondsuit}(z_2), z_2). \tag{102}$$

Thus one has

$$\|Y_3\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} \leq C\epsilon|||\hat{\mathbf{Y}}|||. \tag{103}$$

Finally, (87) implies that

$$\begin{aligned}\|Y_1\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} &\leq C(\epsilon\|\hat{Y}_6\|_{3,\alpha;(0,M)}^{(-1-\alpha;\{M\})} + \sum_{j=3}^{4}\|Y_j\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + C\epsilon|||\hat{\mathbf{Y}}||| \\ &\leq C\epsilon|||\hat{\mathbf{Y}}|||.\end{aligned} \tag{104}$$

Collecting all the estimates (97), (99), (101), (103), and (104) together gives

$$|||\mathbf{Y}||| \leq C_\sharp\epsilon|||\hat{\mathbf{Y}}|||. \tag{105}$$

Obviously, if one chooses $\epsilon_0 \leq \min\{\frac{1}{4C_*^2}, \frac{1}{2C_\sharp}\}$, then $\mathcal{T}$ is a contraction mapping for Ξ_δ to Ξ_δ. Hence $\mathcal{T}$ must have a fixed point in Ξ_δ. It is easy to see that this fixed point is a solution for the problem (48), (50), (53), (58), and (60). Furthermore, since the Lagrangian transformation is invertible, the associated solution $(U_1^+, U_2^+, U_3^+, P^+, S^+)$ and ξ satisfy the properties listed in (18) and (20).

Step 3. Uniqueness. Suppose that there are two solutions $(U_1^{+,(j)}, U_2^{+,(j)}, U_3^{+,(j)}, P^{+,(j)}, S^{+,(j)})$ and ξ_j $(j = 1,\, 2)$ satisfying the properties (18) and (20). We can perform the corresponding Lagrangian transformation and decompose the Euler system as above, in this case we do not need to use the extension (68) any more because the existence of solutions has been assumed. It is the same as the proof for that the operator $\mathcal{T}$ is a contraction mapping. Therefore, these two solutions are indeed the same. □

4. High order regularity of the transonic shock solution

In this section, we show that the regularity of the shock front and subsonic solutions can be improved if the nozzle wall is not perturbed and the supersonic incoming flow satisfies some additional compatibility conditions.

In the following lemma, we show that the compatibility conditions (13) and (21) for the supersonic solutions are preserved along the straight wall.

Lemma 6. *If (13) and (21) hold, the system (3) supplemented with (9) and (23) has a unique smooth solution* $\mathbf{\Psi}^- = (U_1^-, U_2^-, U_3^-, P^-, S^-)(r,\theta) \in C^{2,\alpha}(\bar\Omega)$. *Moreover, this solution* $\mathbf{\Psi}^-$ *satisfies*

$$\|(U_1^-, U_2^-, U_3^-, P^-, S^-) - (U_0^-, 0, 0, \hat P_0^-, \hat S_0^-)\|_{C^{2,\alpha}(\overline{\Omega})} \le C_0\epsilon, \tag{106}$$

where the positive constant C_0 *depends only on* α *and the supersonic incoming flow.*

If, in addition, Ψ_{en}^- *satisfies (22), then the solutions* $\mathbf{\Psi}^-$ *satisfies*

$$\frac{\partial}{\partial\theta}(U_1^-, U_3^-, P^-, S^-)(r,\theta_0) = 0. \tag{107}$$

Proof. Since $U_2(r,\theta_0) \equiv 0$, it follows from the third, fourth and fifth equation of (3) that one has

$$\partial_\theta P - (\rho U_3^2)\cot\theta = 0, \quad (r\partial_r U_3 + U_3) = 0, \quad \partial_r S = 0 \quad \text{for } \theta = \theta_0. \tag{108}$$

Furthermore, differentiating the fifth equation of (3) with respect to θ yields

$$\rho U_1 \partial_r(\partial_\theta S)(r,\theta_0) + \frac{\rho}{r}\partial_\theta U_2 \partial_\theta S(r,\theta_0) = 0.$$

Therefore, $\partial_\theta S(r,\theta_0) \equiv 0$ as long as $\partial_\theta S(r_1,\theta_0) = 0$.

If $U_3(r_1,\theta_0) \equiv 0$, then one can conclude $U_3(r,\theta_0) \equiv 0$ from (108). Using (108) again yields $\partial_\theta P(r,\theta_0) = 0$ and $\partial_r U_3(r,\theta_0) \equiv 0$. Differentiating the second equation of (3) with respect to θ gives

$$\rho U_1 \partial_r(\partial_\theta U_1)(r,\theta_0) + \rho\partial_r U_1 \partial_\theta U_1(r,\theta_0) + \frac{\rho}{r}\partial_\theta U_2 \partial_\theta U_1(r,\theta_0) = 0.$$

Hence, $\partial_\theta U_1(r,\theta_0) \equiv 0$ provided $\partial_\theta U_1(r_0,\theta_0) = 0$. The compatibility conditions at $\theta = 0$ can be obtained similarly except for the second derivative $\partial_\theta^2 U_2(r,0) = 0$, which can be obtained by differentiating the first equation of (3) with respect to θ. □

In the next lemma, we give the compatibility conditions of the subsonic flows at the intersection circles of the shock front and the nozzle wall as long as the assumptions of Lemma 6 hold.

Lemma 7. *If the system (3) with (9), (12), (23) and (22), has a solution*

$$(U_1^{\pm}(r,\theta), U_2^{\pm}(r,\theta), U_3^{\pm}(r,\theta), P^{\pm}(r,\theta), S^{\pm}(r,\theta)) \in C^{2,\alpha}(\overline{\Omega^{\pm}})$$

and $\xi(\theta) \in C^{3,\alpha}([0,\theta_0])$*, then the following compatibility conditions on the nozzle wall and the symmetry axis hold*

$$\begin{cases} \partial_\theta(U_1^+, U_3^+, P^+, S^+)(r,\theta_0) \equiv 0, \quad \partial_\theta(U_1^+, U_3^+, P^+, S^+)(r,0) \equiv 0, \\ U_2(r,0)^+ = U_3^+(r,0) = U_2^+(r,\theta_0) = U_3^+(r,\theta_0) = 0, \ \partial_\theta^2 U_2^+(r,0) = \partial_\theta^2 U_2^+(r,\theta_0) = 0, \\ \xi'(0) = \xi'(\theta_0) = 0, \quad \xi^{(3)}(0) = 0. \end{cases} \tag{109}$$

Proof. It follows from the boundary condition (23) and the jump conditions (17) that

$$U_2^+(r,0) = U_2^+(r,\theta_0) = 0, \quad \xi'(0) = \xi'(\theta_0) = 0.$$

Furthermore, the fourth equation in (17) implies that $U_3^+(\xi(\theta_0),\theta_0) = U_3^-(\xi(\theta_0),\theta_0) = 0$. Thus it follows from the fourth equation in (3) that $U_3^+(r,\theta_0) = 0$ for any $r \in [\xi(\theta_0), r_2]$. Therefore, $\frac{\partial}{\partial\theta}P^+(r,\theta_0) \equiv 0$.

Differentiating the first, the second, the fourth, and the fifth equations in (17) along the shock front gives

$$\begin{cases} \partial_\theta(\rho^+ U_1^+)(\xi(\theta_0)+,\theta_0) = \partial_\theta(\rho^- U_1^-)(\xi(\theta_0)-,\theta_0), \\ \partial_\theta(\rho^+(U_1^+)^2 + P^+)(\xi(\theta_0)+,\theta_0) = \partial_\theta(\rho^-(U_1^-)^2 + P^-)(\xi(\theta_0)-,\theta_0), \\ \partial_\theta U_3^+(\xi(\theta_0)+,\theta_0) = \partial_\theta U_3^-(\xi(\theta_0)-,\theta_0), \\ \partial_\theta\left(e^+ + \frac{|U^+|^2}{2} + \frac{P^+}{\rho^+}\right)(\xi(\theta_0)+,\theta_0) = \partial_\theta\left(e^- + \frac{|U^-|^2}{2} + \frac{P^-}{\rho^-}\right)(\xi(\theta_0)-,\theta_0). \end{cases}$$

It follows from Lemma 6 that $\partial_\theta(U_1^-, U_3^-, P^-, S^-)(r,\theta_0) = 0$. These then imply that $\partial_\theta U_3^+(\xi(\theta_0),\theta_0) = 0$ and

$$\begin{cases} \partial_\theta(\rho^+ U_1^+)(\xi(\theta_0)+,\theta_0) = 0, \\ (\rho U_1^+ \partial_\theta U_1^+ + \partial_\theta P^+)(\xi(\theta_0)+,\theta_0) = 0, \\ \partial_\theta(e^+ + \frac{|U^+|^2}{2} + \frac{P^+}{\rho^+})(\xi(\theta_0)+,\theta_0) = 0, \end{cases}$$

which yields

$$\partial_\theta U_1^+(\xi(\theta_0),\theta_0) = \partial_\theta S^+(\xi(\theta_0),\theta_0) = \partial_\theta \rho^+(\xi(\theta_0),\theta_0) = 0. \tag{110}$$

Differentiating the second and the fifth equation in (17) with respect to θ yields

$$\begin{cases} \left\{U_1\partial_r(\partial_\theta U_1^+) + (\partial_r U_1^+ + \frac{1}{r}\partial_\theta U_2^+)\partial_\theta U_1^+ + \frac{U_1^+\partial_r U_1^+ \partial_S \rho}{\rho}\partial_\theta S^+\right\}(r,\theta_0) = 0, \\ \left\{U_1\partial_r(\partial_\theta S^+) + \frac{1}{r}\partial_\theta U_2^+ \partial_\theta S^+ + \partial_r S^+ \partial_\theta U_1^+\right\}(r,\theta_0) = 0. \end{cases}$$

This, together with (110), implies

$$\partial_\theta U_1^+(r,\theta_0) = \partial_\theta S^+(r,\theta_0) = \partial_\theta \rho^+(r,\theta_0) \equiv 0 \quad \text{for } r \in (\xi(\theta_0), r_2].$$

It follows from the equation for U_3^+ (the fourth equation in (3)) that one has

$$\begin{cases} \left\{ U_1^+ \partial_r(\partial_\theta U_3^+) + \frac{U_1^+}{r}\partial_\theta U_3^+ + \frac{\partial_\theta U_2^+}{r}\partial_\theta U_3^+ \right\}(r,\theta_0) = 0, \\ \partial_\theta U_3^+(\xi(\theta_0),\theta_0) = 0. \end{cases}$$

Hence $\partial_\theta U_3^+(r,\theta_0) \equiv 0$.

In addition, differentiating the first equation of (3) with respect to θ leads to

$$\partial_\theta^2 U_2^+(r,0) = 0.$$

Furthermore, differentiating the third equation of (17) along the shock front twice yields

$$\xi^{(3)}(0) = 0.$$

Hence The proof of Lemma 7 is completed. □

With the help of Lemmas 6 and 7, one can prove Theorem 2.

Proof of Theorem 2. First, if the nozzle boundary is straight, then ϖ and P satisfy the following system

$$(111)\begin{cases} \partial_\theta \varpi + \varpi\cot\theta - r\left(\frac{1}{\rho U_1^2} - \frac{1}{\rho c^2(\rho,S)}\right)\partial_r P + \frac{\varpi}{\rho c^2(\rho,S)}\partial_\theta P + (\varpi^2+2) + \frac{U_3^2}{U_1^2} = 0, \\ \partial_r\varpi - \frac{\varpi}{r} - \frac{\varpi^2}{r}\cot\theta + \left(\frac{1}{\rho U_1^2} - \frac{\varpi^2}{\rho c^2(\rho,S)}\right)\frac{1}{r}\partial_\theta P - \frac{\varpi}{\rho c^2(\rho,S)}\partial_r P - \frac{U_3^2}{rU_1^2}\cot\theta = 0. \end{cases}$$

Comparing with [19, equation (2.20)], both of the additional terms $\frac{U_3^2}{U_1^2}$ and $\frac{U_3^2}{rU_1^2}\cot\theta$ in (111) can be regarded as error terms and do not cause any trouble. Moreover, U_3 satisfies

$$(112)\qquad \begin{cases} U_1\partial_r(rU_3\sin\theta) + \frac{U_2}{r}\partial_\theta(rU_3\sin\theta) = 0, \\ U_3(\xi(\theta),\theta) = U_3^-(\xi(\theta),\theta). \end{cases}$$

The transport equation (112) can be uniquely solved by characteristic method. Furthermore, we can use the standard even extension (a simple modification for [26, Lemma A]) to get $C^{2,\alpha}(\Omega^+)$ regularity near the corner. The detailed proof of Theorem 2 is very similar to the proof for [19, Theorem 1.1], so we omit it here. □

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