CORDIAL ELEMENTS AND DIMENSIONS OF AFFINE DELIGNE-LUSZTIG VARIETIES

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ABSTRACT. The affine Deligne-Lusztig variety $X_w(b)$ in the affine flag variety of a reductive group $\mathbf{G}$ depends on two parameters: the σ-conjugacy class $[b]$ and the element w in the Iwahori-Weyl group $\tilde{W}$ of $\mathbf{G}$. In this paper, for any given σ-conjugacy class $[b]$, we determine the nonemptiness pattern and the dimension formula of $X_w(b)$ for most $w \in \tilde{W}$.

1. INTRODUCTION

1.1. **Motivation.** The notion of affine Deligne-Lusztig variety was first introduced by Rapoport in [Ra05]. It plays an important role in arithmetic geometry and the Langlands program. One of the main motivations comes from the reduction of Shimura varieties. We focus on the affine Deligne-Lusztig varieties in the affine flag variety in this paper. In this case, the affine Deligne-Lusztig varieties are closely related to the Shimura varieties with Iwahori level structure. On the special fibers, there are two important stratifications:

- Newton stratification, indexed by specific σ-conjugacy classes $[b]$ in the associated p-adic group.

- Kottwitz-Rapoport stratification, indexed by specific elements w in the associated Iwahori-Weyl group.

A fundamental question is to determine when the intersection of a Newton stratum indexed by $[b]$ and a Kottwitz-Rapoport stratum indexed by w is nonempty and to determine its dimension. Such intersection is closely related to the affine Deligne-Lusztig variety $X_w(b)$, see for instance [HR17]. In a parallel story over function fields, affine Deligne-Lusztig varieties also arise naturally in the study of local shtukas, see for instance [HV11].

Motivated by the study of Shimura varieties and the local shtukas, one would like to understand the following fundamental questions on the affine Deligne-Lusztig varieties:

- When is the affine Deligne-Lusztig variety nonempty?

- If nonempty, what is its dimension?

It is also worth pointing out that much information on the affine Deligne-Lusztig varieties in the partial affine flag varieties (which are closely related to Shimura varieties with other parahoric level structures) can be deduced from the information on the affine Deligne-Lusztig varieties in the affine flag variety.

1.2. **The main result.** In this paper we determine, for any given σ-conjugacy class $[b]$, the nonemptiness pattern and dimension formula of $X_w(b)$ for most w in the Iwahori-Weyl group $\tilde{W}$. To state the result, we introduce some notations first. For simplicity, we will only consider the split groups **G** here. The general case will be studied in the main context.

The Iwahori-Weyl group $\tilde{W}$ is the semidirect product of the coweight lattice with the relative Weyl group W_0. We may write $\tilde{W}$ as $\tilde{W} = \sqcup_{\lambda \text{ is dominant}} W_0 t^\lambda W_0$. For any $w \in W_0 t^\lambda W_0$, we set $\lambda_w = \lambda$. The σ-conjugacy classes $[b]$ are classified by Kottwitz [Ko85] and [Ko97] via the two invariants: the image under the Kottwitz map κ and the Newton point ν_b (which is a dominant rational coweight). By the Mazur's inequality for the affine Deligne-Lusztig varieties in the affine Grassmannian [Ga10], we deduce that if $X_w(b) \neq \emptyset$, then $\kappa(w) = \kappa(b)$ and $\lambda_w \geqslant \nu_b$ with respect to the dominance order of the rational coweights.

The converse, however, is far from being true. The main result of this paper is the following.

Theorem 1.1. *Let $w \in \tilde{W}$. Suppose that w is in a Shrunken Weyl chamber. If $\kappa(w) = \kappa(b)$, $\lambda_w - \nu_b$ is a linear combination of the simple coroots with all the coefficients positive, and $\lambda_w^{\flat\flat} \geqslant \nu_b$, then we have a complete description of the nonemptiness pattern and dimension formula for $X_w(b)$.*

We refer to §2.2 for the definition of Shrunken Weyl chambers, §5.2 for the definition of $-^{\flat\flat}$ and Theorem 6.1 for the precise description of the nonemptiness pattern and dimension formula. These assumptions are satisfied for example when $\lambda_w \geqslant \nu_b + 2\rho^\vee$, where $\rho^\vee$ is the half sum of positive coroots. See Corollary 6.4.

As an application of Theorem 1.1, in the joint work [HY20+] with Q. Yu, we establish a dimension formula for the group-theoretic analogy of Newton strata for sufficiently large dominant coweights.

1.3. **Some previous results.** In [GHKR10, Conjecture 9.5.1], Görtz, Haines, Kottwitz and Reuman made several influential conjectures on the nonemptiness pattern and dimension formula of $X_w(b)$.

First, for the basic σ-conjugacy class $[b]$, they gave a conjecture in [GHKR10, Conjecture 9.5.1 (a)] on the nonemptiness pattern and dimension formula of $X_w(b)$ for w in the Shrunken Weyl chamber. This conjecture was established in [He14]. For $X_w(b)$ with $[b]$ basic, and w outside the Shrunken Weyl chamber, in [GHKR10, Conjecture 9.4.2] they gave a conjecture on the nonemptiness pattern. This conjecture is established in [GHN15]. But for $[b]$ basic and w outside the Shrunken Weyl chamber, no conjectural dimension formula of $X_w(b)$ has even been formulated so far.

For arbitrary σ-conjugacy class $[b]$, they made an interesting conjecture in [GHKR10, Conjecture 9.5.1 (b)] which predicts the difference of the dimensions of $X_w(b)$ and $X_w(b_{basic})$, where $[b_{basic}]$ is the unique basic σ-conjugacy class such that $\kappa(b) = \kappa(b_{basic})$. In this conjecture, w is not required to be shrunken, but the length of w is required to be big enough with some (unspecified) lower bound. In the later works, we studied $X_w(b)$ via a somehow different direction. First, the assumption that w is in the Shrunken Weyl chamber is added, as even for the basic b, the dimension formula of $X_w(b)$ with w outside the shrunken Weyl chamber is still very mysterious. Second, one would like to have a specific lower bound on w.

For split groups and the case where $[b]$ is represented by translation elements, under the "very shrunken" assumption, the nonemptiness pattern and the dimension formula of $X_w(b)$ were given in [He15, Theorem 2.28 & Theorem 2.34]. A similar result were obtained in [MST19] under a different condition on w.

For other nonbasic σ-conjugacy classes, little is known so far on the nonemptiness pattern and dimension formula of $X_w(b)$.

1.4. **Old Strategies.** We discuss several strategies used to study the nonemptiness pattern and dimension formula for $X_w(b)$ in the previous works.

The emptiness pattern is established via the method of P-alcove elements introduced in [GHKR10, Definition 2.1.1]. The upper bound of $\dim X_w(b)$ is given by the virtual dimension $d_w(b)$ introduced in [He14, §10.1].

In [He14], we combined the Deligne-Lusztig reduction with some remarkable properties of minimal length elements in their conjugacy classes in $\tilde{W}$ to establish a method to compute $\dim X_w(b)$ for arbitrary w and arbitrary $[b]$. As a consequence, in [He14, Theorem 6.1] we establish the "dimension=degree" theorem which relates the dimension of affine Deligne-Lusztig varieties with the degree of the class polynomials of the affine Hecke algebras. However, the computation of the class polynomials, in general, is extremely difficult. The "dimension=degree" theorem does not lead to explicit descriptions of the nonemptiness pattern and the dimension formula of $X_w(b)$.

For basic $[b]$, assume that $X_w(b) \neq \emptyset$. It remains to show that $\dim X_w(b)$ reaches the upper bound $d_w(b)$. Note that for any Coxeter element c, $\dim X_c(b)$ is easy to compute. This will be used as the starting point. In [He14, §11], we constructed an explicit "reduction path" from an element w in the shrunken Weyl chamber to an element w' with finite part a Coxeter element. By [HY12, Theorem 1.1], the minimal length elements in the conjugacy class of w' in $\tilde{W}$ are the Coxeter elements c. This gives a "reduction path" from w to c and thus leads to a lower bound of $\dim X_w(b)$. Fortunately, the lower bound also equals to the virtual dimension $d_w(b)$. Thus we proved the nonemptiness pattern and the dimension formula of $X_w(b)$ with basic $[b]$.

For split groups and the case where $[b]$ is represented by translation elements, in [He15] we used the superset method of [GHKR10] to relate the nonemptiness pattern and dimension formula of $X_w(b)$ with $X_{w'}(1)$ for a given w'. Note that $[1]$ is a basic σ-conjugacy class. We then used the result on $X_{w'}(1)$ established in [He14] to obtain the desired result on $X_w(b)$. A very different approach is introduced in [MST19], where the authors used the alcove walk and Littelman path to study the nonemptiness pattern and dimension formula of $X_w(b)$.

I do not know how/whether the methods in [He15] or in [MST19] for the translation elements may be generalized to arbitrary σ-conjugacy class $[b]$. The reduction method introduced in [He14], in theory, works for an arbitrary σ-conjugacy class $[b]$. However, to construct an explicit "reduction path" from a given w to a minimal length element associated to a nonbasic $[b]$ is very challenging. Q. Yu has written down a computer program to construct the "reduction path" for groups with small ranks. But so far it is not clear how such a "reduction path" may be constructed in general.

1.5. **New strategy.** The new strategy in this paper is as follows. Instead of using minimal length elements as the starting point, we use the cordial elements introduced by Milićević and Viehmann in [MV20+] as the starting point instead. In section 4, we construct a new family of cordial elements. For any element w' in this family, $\dim X_{w'}(b)$ equals to the virtual dimension. We then construct in section 5 an explicit "reduction path" from an element w in the shrunken Weyl chamber to an element in this family. This is where the assumption $\lambda_w^{\flat} \geqslant \nu_b$ is used. This shows that $\dim X_w(b) \geqslant d_w(b)$. Finally we use the result that $\dim X_w(b) \leqslant d_w(b)$ established in [He14] and [He15] to prove the desired nonemptiness pattern and the dimension formula of $X_w(b)$.

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2. Preliminary

2.1. **The reductive group G and its Iwahori-Weyl group.** Let F be a non-archimedean local field and $\breve{F}$ be the completion of the maximal unramified extension of F. We write Γ for $\mathrm{Gal}(\overline{F}/F)$, and write Γ_0 for the inertia subgroup of Γ.

Let $\mathbf{G}$ be a connected reductive group over F. Let σ be the Frobenius morphism of $\breve{F}/F$. We write $\breve{G}$ for $\mathbf{G}(\breve{F})$. We use the same symbol σ for the induced Frobenius morphism on $\breve{G}$.

We fix a maximal $\breve{F}$-split torus S in $\mathbf{G}$ defined over F which contains a maximal F-split torus. Let T be the centralizer of S in $\mathbf{G}$. Then T is a maximal torus. Let $\mathcal{A}$ be the apartment of $\mathbf{G}_{\breve{F}}$ corresponding to $S_{\breve{F}}$. Thus $\mathcal{A}$ is (non-canonically) isomorphic to $V = X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$. The Frobenius σ naturally acts on $\mathcal{A}$. We fix a σ-stable alcove $\mathfrak{a}$ in $\mathcal{A}$, and let $\breve{I} \subset \breve{G}$ be the Iwahori subgroup corresponding to $\mathfrak{a}$. Thus $\breve{I}$ is σ-stable.

We denote by N the normalizer of T in $\mathbf{G}$. The *relative Weyl group* W_0 is defined to be $N(\breve{F})/T(\breve{F})$. The *Iwahori–Weyl group* (associated to S) is defined as

$$\tilde{W} = N(\breve{F})/T(\breve{F}) \cap \breve{I}.$$

For any $w \in \tilde{W}$, we choose a representative $\dot{w}$ in $N(L)$.

We have a natural short exact sequence $0 \to X_*(T)_{\Gamma_0} \to \tilde{W} \to W_0 \to 0$. We choose a special vertex of $\mathfrak{a}$ and represent $\tilde{W}$ as a semidirect product

$$\tilde{W} = X_*(T)_{\Gamma_0} \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(T)_{\Gamma_0}, w \in W_0\}.$$

The Iwahori-Weyl group $\tilde{W}$ contains the affine Weyl group W_a as a normal subgroup and we have

$$\tilde{W} = W_a \rtimes \Omega,$$

where Ω is the normalizer of $\mathfrak{a}$. The length function ℓ and Bruhat order $\leqslant$ on W_a extend in a natural way to $\tilde{W}$. The Frobenius σ naturally acts on $\tilde{W}$, in such a way that the subset $\tilde{\mathbb{S}} \subset \tilde{W}$ is stable.

For any $K \subset \tilde{\mathbb{S}}$, we denote by W_K the subgroup of $\tilde{W}$ generated by $s \in K$. Let $^K\tilde{W}$ (resp. $\tilde{W}^K$) be the set of minimal length elements in their cosets in $W_K \backslash \tilde{W}$ (resp. $\tilde{W}/W_K$).

Let $\mathbb{S} \subset \tilde{\mathbb{S}}$ be the set of simple reflections of W_0. By convention, the dominant Weyl chamber of V is opposite to the unique Weyl chamber containing $\mathfrak{a}$. Let Δ be the set of relative simple roots determined by the dominant Weyl chamber. For any $s \in \mathbb{S}$, we denote by $\alpha_s \in \Delta$ the corresponding simple root and $\alpha_s^\vee$ the corresponding simple coroot. We denote by $w_\mathbb{S}$ the longest element of W_0.

We define the σ-conjugation action on $\breve{G}$ by $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$. Let $B(\mathbf{G})$ be the set of σ-conjugacy classes on $\breve{G}$. The classification of the σ-conjugacy classes is obtained by Kottwitz in [Ko85] and [Ko97]. Any σ-conjugacy class $[b]$ is determined by two invariants:

- The element $\kappa([b]) \in \Omega_\sigma$;

- The Newton point $\nu_b \in \left((X_*(T)_{\Gamma_0,\mathbb{Q}})^+\right)^{\langle\sigma\rangle}$.

Here $-_\sigma$ denotes the σ-coinvariants, $(X_*(T)_{\Gamma_0,\mathbb{Q}})^+$ denotes the intersection of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q} = X_*(T)^{\Gamma_0} \otimes \mathbb{Q}$ with the set $X_*(T)_\mathbb{Q}^+$ of dominant elements in $X_*(T)_\mathbb{Q}$; the action of σ on $(X_*(T)_{\Gamma_0,\mathbb{Q}})/W_0$ is transferred to an action on $(X_*(T)_\mathbb{Q})^+$ (*L-action*).

For any $w \in \tilde{W}$, we write $\kappa(w)$ for $\kappa(\dot{w})$. It is easy to see that $\kappa(w)$ is independent of the choice of the representative $\dot{w}$.

2.2. Affine Deligne-Lusztig varieties.
We have the following generalization of the Bruhat decomposition
$$\breve{G} = \sqcup_{w \in \tilde{W}} \breve{I}\dot{w}\breve{I},$$
due to Iwahori and Matsumoto [IM65] in the split case, and to Bruhat and Tits [BT72] in the general case. Let $Fl = \breve{G}/\breve{I}$ be the *affine flag variety*. For any $b \in \breve{G}$ and $w \in \tilde{W}$, we define the corresponding *affine Deligne-Lusztig variety* in the affine flag variety
$$X_w(b) = \{g\breve{I} \in \breve{G}/\breve{I}; g^{-1}b\sigma(g) \in \breve{I}\dot{w}\breve{I}\} \subset Fl.$$

As discussed in [GHN15, §2], the study of nonemptiness pattern and dimension formula of affine Deligne-Lusztig varieties for arbitrary reductive group may be reduced to simple and quasi-split groups over F. From now on, we assume that $\mathbf{G}$ is simple and quasi-split over F. In this case, the σ-action on $\tilde{W}$ preserves W_0 and $X_*(T)_{\Gamma_0}$ respectively. Moreover, we have $\sigma(\mathbb{S}) = \mathbb{S}$ and $\sigma(\Delta) = \Delta$.

Now we recall the definition of virtual dimension in [He14, §10.1].

Note that any element $w \in \tilde{W}$ may be written in a unique way as $w = xt^\mu y$ with μ dominant, $x, y \in W_0$ such that $t^\mu y \in {}^\mathbb{S}\tilde{W}$. In this case, we set
$$\eta_\sigma(w) = \sigma^{-1}(y)x.$$

Let $\mathbf{J}_b$ be the reductive group over F with $\mathbf{J}_b(F) = \{g \in \breve{G}; gb\sigma(g)^{-1} = b\}$. The *defect* of b is defined by $\text{def}(b) = \text{rank}_F \mathbf{G} - \text{rank}_F \mathbf{J}_b$. Here for a reductive group $\mathbb{H}$ defined over F, rank_F is the F-rank of the group $\mathbb{H}$. Let ρ be the dominant weight with $\langle\alpha^\vee, \rho\rangle = 1$ for any $\alpha \in \Delta$. The *virtual dimension* is defined to be
$$d_w(b) = \frac{1}{2}\left(\ell(w) + \ell(\eta_\sigma(w)) - \text{def}(b)\right) - \langle\nu_b, \rho\rangle.$$

The following result is proved in [He14, Corollary 10.4] for residually split groups and proved in [He15, Theorem 2.30] for the general case.

Theorem 2.1. *Let $b \in \breve{G}$ and $w \in \tilde{W}$. Then $\dim X_w(b) \leqslant d_w(b)$.*

For any $w \in W_0$, we denote by $\mathrm{supp}(w) \subset \mathbb{S}$ the set of simple reflections appears in some (or equivalently, any) reduced expression of w. We set $\mathrm{supp}_\sigma(w) = \cup_{i \in \mathbb{Z}} \sigma^i(\mathrm{supp}(w))$.

For any $w \in \tilde{W}$, let λ_w is the unique dominant coweight such that $w \in W_0 t^{\lambda_w} W_0$. For any $\lambda \in X_*(T)_{\Gamma_0}$, we denote by $\lambda^\diamond$ the average of the σ-orbit of λ. For any $\lambda, \lambda' \in X_*(T)_{\mathbb{Q}}^+$, we write $\lambda \geqslant \lambda'$ if $\lambda - \lambda' \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geqslant 0} \alpha^\vee$ and write $\lambda \geqslant_{\mathbb{Z}} \lambda'$ if $\lambda - \lambda' \in \sum_{\alpha \in \Delta} \mathbb{N} \alpha^\vee$. Here $\mathbb{N}$ is the set of natural numbers, i.e., the set of nonnegative integers.

A *critical strip* of the apartment V is the subset $\{v; -1 < \langle v, \alpha \rangle < 0\}$ for a given positive root α in the reduced root system associated to the affine Weyl group W_a. We remove all the critical strips from V and call each connected component of the remaining subset of V *a Shrunken Weyl chamber*.

3. Some combinatorial properties

3.1. **Minimal length elements.** For any σ-conjugacy class $\mathcal{O}$ in $\tilde{W}$, we denote by $\mathcal{O}_{\min}$ the set of minimal length elements in $\mathcal{O}$. For $w, w' \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \rightarrow_\sigma w'$ if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\tilde{W}$ such that for any k, $w_{k-1} \xrightarrow{s}_\sigma w_k$ for some $s \in \tilde{\mathbb{S}}$. We write $w \approx_\sigma w'$ if $w \rightarrow_\sigma w'$ and $w' \rightarrow_\sigma w$. It is easy to see that $w \approx_\sigma w'$ if $w \rightarrow_\sigma w'$ and $\ell(w) = \ell(w')$.

The following result is proved in [HN14, §2].

Theorem 3.1. *Let $\mathcal{O}$ be a σ-conjugacy class of $\tilde{W}$ and $w \in \mathcal{O}$. Then there exists $w' \in \mathcal{O}_{\min}$ such that $w \rightarrow_\sigma w'$.*

Theorem 3.2. *Let $b \in \breve{G}$ and $w \in \mathcal{O}_{\min}$ for some σ-conjugacy class $\mathcal{O}$ of $\tilde{W}$. Then $X_w(b) \neq \emptyset$ if and only if $\dot{w} \in [b]$. In this case, $\dim X_w(b) = \ell(w) - \langle \nu_b, 2\rho \rangle$.*

3.2. **Deligne-Lusztig reduction.** Now we recall the "reduction" à la Deligne and Lusztig for affine Deligne-Lusztig varieties (see [DL76, Proof of Theorem 1.6], also [GH10, Corollary 2.5.3]).

Proposition 3.3. *Let $b \in \breve{G}$. Then*

(1) Let $w, w' \in \tilde{W}$ with $w \approx_\sigma w'$, then

$$\dim X_w(b) = \dim X_{w'}(b).$$

(2) Let $w \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$ with $\ell(sw\sigma(s)) = \ell(w) - 2$. Then

$$\dim X_w(b) = \max\{\dim X_{sw}(b), \dim X_{sw\sigma(s)}(b)\} + 1.$$

Here by convention, we set $\dim \emptyset = -\infty$ and $-\infty + n = -\infty$ for any $n \in \mathbb{R}$.

3.3. **The relation $\Rightarrow$.** Following [GH10, Definition 3.1.4], for $w, w' \in \tilde{W}$, we write $w \Rightarrow_\sigma w'$ if for any $b \in \breve{G}$,

$$\dim X_w(b) - d_w(b) \geqslant \dim X_{w'}(b) - d_{w'}(b).$$

Here again by convention, we set $\dim \emptyset = -\infty$. If the right hand side is $-\infty$, then the inequality holds regardless of the left hand side. It is also easy to see that the relation is transitive.

Note that by definition of virtual dimension, $w \Rightarrow_\sigma w'$ if and only if for any $b \in \breve{G}$ with $X_{w'}(b) \neq \emptyset$, $X_w(b) \neq \emptyset$, and in this case,

$$\dim X_w(b) - \dim X_{w'}(b) \geqslant \frac{1}{2}\big(\ell(w) + \ell(\eta_\sigma(w)) - \ell(w') - \ell(\eta_\sigma(w'))\big).$$

We write $w \Leftrightarrow_\sigma w'$ if $w \Rightarrow_\sigma w'$ and $w' \Rightarrow_\sigma w$.

3.4. **The monoid structure on $\tilde{W}$.** By [He09, Lemma 1], for any $w, w' \in \tilde{W}$, the subset $\{uw'; u \leqslant w\}$ of $\tilde{W}$ contains a unique maximal element which we denote by $w * w'$. Moreover, $w * w' = \max\{uv; u \leqslant w, v \leqslant w'\}$. Hence $*$ is associative. This gives a monoid structure on $\tilde{W}$. If $w_1 \leqslant w$ and $w_1' \leqslant w'$, then $w_1 * w_1' \leqslant w * w'$.

4. The cordial elements

4.1. **Definition.** There is a natural partial ordering $\leqslant$ on $B(\mathbf{G})$ defined as follows. Let $[b], [b'] \in B(\mathbf{G})$. Then $[b] \leqslant [b']$ if $\kappa(b) = \kappa(b')$ and $\nu_b \leqslant \nu_{b'}$.

Now we recall the cordial elements introduced by Milićević and Viehmann in [MV20+, Definition 3.14].

For any $w \in \tilde{W}$, there is a unique maximal σ-conjugacy class $[b]$ such that $X_w(b) \neq \emptyset$. We denote this σ-conjugacy class by $[b_w]$. The element w is called *cordial* if $\dim X_w(b_w) = d_w(b_w)$. Equivalently, w is cordial if and only if $\ell(w) - \ell(\eta_\sigma(w)) = \langle \nu_{b_w}, 2\rho \rangle - \mathrm{def}(b_w)$.

By definition, if $w \Leftrightarrow_\sigma w'$, then w is a cordial element if and only if w' is a cordial element. The following result is proved in [MV20+, Theorem 1.1 & Corollary 3.17].

Theorem 4.1. *Let $w \in \tilde{W}$ be a cordial element. Then*
(1) Let $[b], [b'] \in B(\mathbf{G})$. If $[b] \leqslant [b'] \leqslant [b_w]$ and $X_w(b) \neq \emptyset$, then $X_w(b') \neq \emptyset$.
(2) If $X_w(b) \neq \emptyset$, then $\dim X_w(b) = d_w(b)$.

It is mentioned in loc. cit that to fully characterize the cordial elements is fairly difficult. In [MV20+, Theorem 1.2], some interesting families of cordial elements are provided. The main result of this section is to provide another family of cordial elements.

Theorem 4.2. *Let λ be a dominant coweight and $x \in W_0$. Then xt^λ is a cordial element and $[b_{xt^\lambda}] = [\dot{t}^\lambda]$.*

Remark 4.3. The original proof I had is a bit technical. The following proof is suggested to me by E. Viehmann.

4.2. **Mazur's inequality.** Recall that $\mathbf{G}$ is quasi-split over F. Let $\breve{K} \supset \breve{I}$ be a σ-stable special maximal parahoric subgroup of $\breve{G}$. The nonemptiness pattern of the affine Deligne-Lusztig varieties in the affine Grassmannian $\breve{G}/\breve{K}$ is determined in terms of Mazur's inequality. It was established by Gashi [Ga10, Theorem 1.1] for unramified groups and proved in general case in [He14, Theorem 7.1]. We may reformulate the result as follows.

Theorem 4.4. *Let λ be a dominant coweight and $b \in \breve{G}$. Then $[b] \cap \breve{K}\dot{t}^\lambda\breve{K} \neq \emptyset$ if and only if $\kappa(b) = \kappa(t^\lambda)$ and $\nu_b \leqslant \lambda^\diamond$.*

4.3. **Proof of Theorem 4.2.** Let $w \in \tilde{W}$. By definition, $[b_w]$ is the unique maximal σ-conjugacy class that intersects $\breve{I}\dot{w}\breve{I}$. By [Vi14, Corollary 5.6], $[b_w]$ is also the unique maximal σ-conjugacy class that intersects $\overline{\breve{I}\dot{w}\breve{I}}$. Note that

$$\breve{I}\dot{w}_{\mathbb{S}}\dot{t}^\lambda\breve{I} \subset \breve{K}\dot{t}^\lambda\breve{K} \subset \overline{\breve{I}\dot{w}_{\mathbb{S}}\dot{t}^\lambda\breve{I}}.$$

By Theorem 4.4, $[b_{w_{\mathbb{S}}t^\lambda}] = [\dot{t}^\lambda]$.

On the other hand, t^λ is a minimal length element in its σ-conjugacy class. Hence by Theorem 3.2, $[b_{t^\lambda}] = [\dot{t}^\lambda]$.

Since $t^\lambda \leqslant xt^\lambda \leqslant w_{\mathbb{S}}t^\lambda$ for any $x \in W_0$, we have that

$$\breve{I}\dot{t}^\lambda\breve{I} \subset \overline{\breve{I}\dot{x}\dot{t}^\lambda\breve{I}} \subset \overline{\breve{I}\dot{w}_{\mathbb{S}}\dot{t}^\lambda\breve{I}}.$$

Hence $[\dot{t}^\lambda] = [b_{t^\lambda}] \leqslant [b_{xt^\lambda}] \leqslant [b_{w_{\mathbb{S}}t^\lambda}] = [\dot{t}^\lambda]$. Thus $[b_{xt^\lambda}] = [\dot{t}^\lambda]$.

Now $\nu_{b_{xt^\lambda}} = \lambda^\diamond$ and $\mathrm{def}(b_{xt^\lambda}) = 0$. Hence

$$\ell(xt^\lambda) - \ell(\eta_\sigma(xt^\lambda)) = \ell(x) + \ell(t^\lambda) - \ell(x) = \ell(t^\lambda) = \langle \lambda, 2\rho \rangle = \langle \lambda^\diamond, 2\rho \rangle.$$

Thus xt^λ is a cordial element.

4.4. **Another family of cordial element.** Let $w \in \tilde{W}$ such that $w\mathfrak{a}$ is in the antidominant Weyl chamber, i.e. $w = w_{\mathbb{S}}t^\lambda y$, where λ is a dominant coweight and $y \in W_0$ with $t^\lambda y \in {}^{\mathbb{S}}\tilde{W}$. It is proved by Milićević and Viehmann in [MV20+, Theorem 1.2 (a)] that

(a) w is also a cordial element.

Now we show that it can also be deduced from Theorem 4.2.

Set $w' = \sigma^{-1}(y)w_{\mathbb{S}}t^\lambda$. By Theorem 4.2, w' is a cordial element. Note that $\eta_\sigma(w') = \eta_\sigma(w) = \sigma^{-1}(y)w_{\mathbb{S}}$. Moreover, it is easy to see that $w \approx_\sigma w'$. Hence $w \Leftrightarrow_\sigma w'$ and w is also a cordial element.

It is also worth mentioning that not all the elements of the form xt^λ is $\approx_\sigma$-equivalent to an element in the antidominant Weyl chamber.

5. FROM w TO A CORDIAL ELEMENT

We first show that

Proposition 5.1. *Let λ, λ' be dominant coweights. Then the set*

$$\{\mu'; \mu' \text{ is dominant}, \mu' + \lambda' \geqslant_\mathbb{Z} \lambda\}$$

contains a unique minimal element with respect the dominance order $\geqslant_\mathbb{Z}$.

Remark 5.2. The proof is due to S. Nie.

Proof. Let μ_1', μ_2' be dominant coweights with $\mu_1' + \lambda' \geqslant_{\mathbb{Z}} \lambda$ and $\mu_2' + \lambda' \geqslant_{\mathbb{Z}} \lambda$. We may write $\mu_1' - \mu_2'$ as $\mu_1' - \mu_2' = \gamma_1 - \gamma_2$, where $\gamma_1 \in \sum_{\alpha \in J_1} \mathbb{Z}_{>0}\alpha$, $\gamma_2 \in \sum_{\alpha \in J_2} \mathbb{Z}_{>0}\alpha$ for some $J_1, J_2 \subset \Delta$ with $J_1 \cap J_2 = \emptyset$.

Set $\mu = \mu_1' - \gamma_1 = \mu_2' - \gamma_2$. Let $\alpha \in \Delta$. Since $J_1 \cap J_2 = \emptyset$, we have $\alpha \notin J_1$ or $\alpha \notin J_2$. If $\alpha \notin J_1$, then $\langle \mu, \alpha \rangle \geqslant \langle \mu_1', \alpha \rangle \geqslant 0$. If $\alpha \notin J_2$, then $\langle \mu, \alpha \rangle \geqslant \langle \mu_2', \alpha \rangle \geqslant 0$. Thus μ is dominant. By definition, $\mu_1' \geqslant_{\mathbb{Z}} \mu$ and $\mu_2' \geqslant_{\mathbb{Z}} \mu$. Moreover,

$$\lambda' - \lambda + \mu_1' = \lambda' - \lambda + \mu_2' + \gamma_1 - \gamma_2 \in \left(\sum_{\alpha \in \Delta} \mathbb{Z}_{>0}\alpha + \gamma_1 - \gamma_2 \right) \cap \sum_{\alpha \in \Delta} \mathbb{Z}_{>0}\alpha.$$

Since $J_1 \cap J_2 = \emptyset$, we have $\lambda' - \lambda + \mu_1' - \gamma_1 \in \sum_{\alpha \in \Delta} \mathbb{Z}_{>0}\alpha$. In other words, $\lambda' + \mu \geqslant_{\mathbb{Z}} \lambda$.

The statement is proved. $\qquad\qquad\square$

5.1. The normalized subtraction.

For any dominant coweights λ, λ', we denote by $\lambda -_{\mathrm{dom}} \lambda'$ the unique minimal element in the set

$$\{\mu'; \mu' \text{ is dominant}, \mu' + \lambda' \geqslant_{\mathbb{Z}} \lambda\}.$$

It is easy to see that if $\lambda - \lambda'$ is dominant, then $\lambda -_{\mathrm{dom}} \lambda' = \lambda - \lambda'$. We call $-_{\mathrm{dom}}$ the *normalized subtraction*. Now we prove some properties of the normalized subtraction $-_{\mathrm{dom}}$.

Corollary 5.3. *Let λ, λ' be dominant coweights. Let λ'' be a dominant coweight with $\lambda' \geqslant_{\mathbb{Z}} \lambda''$. Let $x \in W_0$ and let μ be the unique dominant coweight in the W_0-orbit of $\lambda - x(\lambda'')$. Then $\mu \geqslant_{\mathbb{Z}} \lambda -_{dom} \lambda'$.*

Proof. Note that $\mu - (\lambda - x(\lambda'')) \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$, $\lambda'' - x(\lambda'') \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$ and $\lambda' - \lambda'' \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$. Thus

$$\begin{aligned}
\mu + \lambda' &= (\mu - \lambda + x(\lambda'')) + \lambda - x(\lambda'') + \lambda' \\
&= (\mu - \lambda + x(\lambda'')) + (\lambda'' - x(\lambda'')) + (\lambda' - \lambda'') + \lambda \\
&\geqslant_{\mathbb{Z}} \lambda. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square
\end{aligned}$$

Corollary 5.4. *Let $\lambda, \lambda_1, \lambda_2$ be dominant coweights. Then*

$$(\lambda -_{dom} \lambda_1) -_{dom} \lambda_2 = \lambda -_{dom} (\lambda_1 + \lambda_2).$$

Proof. Set $\mu_1 = (\lambda -_{\mathrm{dom}} \lambda_1) -_{\mathrm{dom}} \lambda_2$ and $\mu_2 = \lambda -_{\mathrm{dom}} (\lambda_1 + \lambda_2)$. By definition,

$$(\lambda_1 + \lambda_2) + \mu_1 = \lambda_1 + (\lambda_2 + \mu_1) \geqslant_{\mathbb{Z}} \lambda_1 + (\lambda -_{\mathrm{dom}} \lambda_1) \geqslant_{\mathbb{Z}} \lambda.$$

So $\mu_1 \geqslant_{\mathbb{Z}} \mu_2$.

On the other hand,

$$\lambda_1 + (\lambda_2 + \mu_2) = (\lambda_1 + \lambda_2) + \mu_2 \geqslant_{\mathbb{Z}} \lambda.$$

So by definition, $\lambda_2 + \mu_2 \geqslant_{\mathbb{Z}} \lambda -_{\mathrm{dom}} \lambda_1$ and $\mu_2 \geqslant_{\mathbb{Z}} \mu_1$. $\qquad\square$

5.2. The double flat operator.

For any subset J of $\mathbb{S}$, we denote by $\rho_J^\vee$ the dominant coweight with

$$\langle \rho_J^\vee, \alpha_s \rangle = \begin{cases} 1, & \text{if } s \in J; \\ 0, & \text{if } s \notin J. \end{cases}$$

Let $\eta_J^\vee$ be the unique dominant coweight in the W_0-orbit of $-\sigma^{-1}(\rho_J^\vee)$.

Let $w \in \tilde{W}$. We write w as $w = xt^\lambda y$ with λ dominant, $x, y \in W_0$ and $t^\lambda y \in {}^{\mathbb{S}}\tilde{W}$. Let $J = \{s \in \mathbb{S}; sy < y\}$. Since $t^\lambda y \in {}^{\mathbb{S}}\tilde{W}$, we have $\langle \lambda, \alpha_s \rangle > 0$ for any $s \in J$. In particular, $\lambda - \rho_J^\vee$ is dominant. We set

$$\lambda_w^\flat = (\lambda - \rho_J^\vee) -_{\text{dom}} \eta_J^\vee = \lambda -_{\text{dom}} (\rho_J^\vee + \eta_J^\vee).$$

The main result of this section is as follows.

Theorem 5.5. *Assume that* **G** *is simple and quasi-split over* F. *Let* $w \in \tilde{W}$. *Suppose that* $\text{supp}_\sigma(\eta_\sigma(w)) = \mathbb{S}$. *Then there exists a dominant coweight* γ *with* $\gamma \geqslant_{\mathbb{Z}} \lambda_w^\flat$ *and* $a \in W_0$ *with* $\text{supp}_\sigma(a) = \mathbb{S}$ *such that*

$$w \Rightarrow_\sigma at^\gamma.$$

5.3. **A convenient notation.** Following [GH10, §2.4], we give a convenient notation for varieties of tuples of elements in Fl. We explain the notation by examples. Let $\mathcal{O}_w = \{(g\breve{I}, g\dot{w}\breve{I}); g \in \breve{G}\} \subset Fl \times Fl$. Then we set

$$\{ g \xrightarrow{w} g'' \xrightarrow{w'} g' \} = \{(g, g', g'') \in (Fl)^3; (g, g'') \in \mathcal{O}_w, (g'', g') \in \mathcal{O}_{w'}\}.$$

Similarly,

$$\{ g \underset{w''}{\overset{w}{\longrightarrow}} g'' \xrightarrow{w'} g' \} = \{(g, g', g'') \in (Fl)^3; (g, g'') \in \mathcal{O}_w, (g'', g') \in \mathcal{O}_{w'}, (g, g') \in \mathcal{O}_{w''}\}.$$

The affine Deligne-Lusztig varieties can be written as

$$X_w(b) = \{ g \xrightarrow{w} b\sigma(g) \}.$$

In all these cases, we do not distinguish between the sets given by the conditions on the relative position, and the corresponding locally closed sub-ind-schemes of the product of affine flag varieties. The following result is proved in [GH10, Proposition 2.5.2].

Proposition 5.6. *Let* $w, w' \in \tilde{W}$, *and let* $w'' \in \{ww', w * w'\}$. *Then the map*

$$\pi: \{ g \underset{w''}{\overset{w}{\longrightarrow}} g'' \xrightarrow{w'} g' \} \longrightarrow \{ g \xrightarrow{w''} g' \}, \qquad (g, g', g'') \longmapsto (g, g')$$

is surjective. Moreover all the fibers have dimension

$$\dim \pi^{-1}((g, g')) \geqslant \begin{cases} \ell(w) + \ell(w') - \ell(w * w'), & \text{if } w'' = w * w'; \\ \frac{1}{2}(\ell(w) + \ell(w') - \ell(ww')), & \text{if } w'' = ww'. \end{cases}$$

5.4. **Proof of Theorem 5.5.** We write w as $w = xt^{\lambda_w} y$ with $x, y \in W_0$ and $t^{\lambda_w} y \in {}^{\mathbb{S}}\tilde{W}$. Let $J = \{s \in \mathbb{S}; sy < y\}$.

Let $J' = \{s \in \mathbb{S}; s(\lambda_w - \rho_J^\vee) = \lambda_w - \rho_J^\vee\}$. We write $\sigma^{-1}(y)x$ as $\sigma^{-1}(y)x = x'z$ for some $w \in W_0^{J'}$ and $z \in W_{J'}$. Let γ be the unique dominant coweight in the W_0-orbit of $\lambda_w - \rho_J^\vee + (x')^{-1}\sigma^{-1}(\rho_J^\vee)$. By Corollary 5.3, $\gamma \geqslant_{\mathbb{Z}} \lambda_w^\flat$.

Let $K = \{s \in \mathbb{S}; s(\gamma) = \gamma\}$ and $y' \in W_0^K$ with $\lambda_w - \rho_J^\vee + (x')^{-1}\sigma^{-1}(\rho_J^\vee) = y'(\gamma)$.

Set $w_1 = xz^{-1}t^{\lambda_w - \rho_J^\vee}y'$ and $w_2 = (y')^{-1}zt^{\rho_J^\vee}y$. Then $w = w_1w_2$. Set $a = ((y')^{-1}z) * \sigma(x'y')$. Then

$$\mathrm{supp}_\sigma(a) = \mathrm{supp}_\sigma((y')^{-1}z) \cup \mathrm{supp}_\sigma(\sigma(x'y')) = \mathrm{supp}_\sigma((y')^{-1}z) \cup \mathrm{supp}_\sigma(x'y')$$
$$\supset \mathrm{supp}_\sigma(x'y'(y')^{-1}z) = \mathrm{supp}_\sigma(x'z) = \mathrm{supp}_\sigma(\eta_\sigma(w)) = \mathbb{S}.$$

By the proof of [He14, Proposition 11.5], $\ell(w) = \ell(w_1) + \ell(w_2)$, $\ell((y')^{-1}z) + \ell(x'y') = \ell(\sigma^{-1}(y)x)$ and $\mathrm{supp}_\sigma(a) = \mathbb{S}$.

Now

$$X_w(b) = \{\, g \xrightarrow{w} b\sigma(g) \,\} = \{\, g \xrightarrow{w_1} g_1 \xrightarrow{w_2} b\sigma(g) \,\}.$$

Set

$$X_1 = \{\, g_1 \xrightarrow{w_2} g_2 \xrightarrow{\sigma(w_1)} b\sigma(g_1) \,\}$$

$$\cong \{\, g_1 \xrightarrow{(y')^{-1}z} g_3 \xrightarrow{t^{\rho_J^\vee}y} g_2 \xrightarrow{\sigma(w_1)} b\sigma(g_1) \,\}.$$

The map $(g, g_1) \mapsto (g_1, b\sigma(g))$ is a universal homeomorphism from $X_x(b)$ to X_1. We have that $y\sigma(xz^{-1}) = \sigma(\sigma^{-1}(y)xz^{-1}) = \sigma(x')$ and

$$t^{\rho_J^\vee}y\sigma(w_1) = t^{\rho_J^\vee}\sigma(x't^{\lambda_w - \rho_J^\vee}y') = \sigma(x')\sigma(t^{y'(\gamma)})\sigma(y') = \sigma(x'y't^\gamma).$$

Let

$$X_2 = \{\, g_1 \xrightarrow{(y')^{-1}z} g_3 \xrightarrow{t^{\rho_J^\vee}y} g_2 \xrightarrow{\sigma(w_1)} b\sigma(g_1) \,\} \subset X_1;$$

with $\sigma(wy't^\gamma)$ from g_3 to $b\sigma(g_1)$.

$$X_3 = \{\, g_1 \xrightarrow{(y')^{-1}z} g_3 \xrightarrow{\sigma(x'y't^\gamma)} b\sigma(g_1) \,\} = \{\, g_1 \xrightarrow{(y')^{-1}z} g_3 \xrightarrow{\sigma(x'y')} g_4 \xrightarrow{\sigma(t^\gamma)} b\sigma(g_1) \,\},$$

By Proposition 5.6,

$$\dim(X_x(b)) \geqslant \dim(X_2) \geqslant \dim(X_3) + \frac{\ell(t^{\rho_J^\vee}y) + \ell(w_1) - \ell(x'y't^\gamma)}{2}.$$

Recall that $a = (y^{-1}z) * (x'y')$. We set

$$X_4 = \{\, g_1 \xrightarrow{(y')^{-1}z} g_3 \xrightarrow{\sigma(x'y')} g_4 \xrightarrow{\sigma(t^\gamma)} b\sigma(g_1) \,\} \subset X_3,$$

with a from g_1 to g_4.

$$X_5 = \{\, g_1 \xrightarrow{a} g_4 \xrightarrow{\sigma(t^\gamma)} b\sigma(g_1) \,\}.$$

By Proposition 5.6,

$$\dim(X_3) \geqslant \dim(X_5) + \ell((y')^{-1}z) + \ell(x'y') - \ell(a) = \ell(\eta_\sigma(w)) - \ell(a).$$

Notice that $\ell(at^\gamma) = \ell(a) + \ell(t^\gamma)$. Thus the map $(g_1, g_4) \mapsto g_1$ gives an isomorphism $X_5 \cong X_{at^\gamma}(b)$. If $X_{at^\gamma}(b) \neq \emptyset$, then $X_x(b) \neq \emptyset$. Note that $\ell((y')^{-1}z) +

$\ell(x'y') = \ell(\sigma^{-1}(y)x) = \ell(\eta_\sigma(w))$. Therefore

$$\dim X_w(b) - \dim X_{at^\gamma}(b)$$

$$\geqslant \frac{\ell(t^{\rho_J^\vee}y) + \ell(w_1) - \ell(x'y't^\gamma)}{2} + \ell(\eta_\sigma(w)) - \ell(a)$$

$$= \frac{\ell(t^{\rho_J^\vee}y) + \ell((y')^{-1}z) + \ell(w_1) - \ell(x'y't^\gamma) + \ell(x'y') - \ell(a)}{2} + \frac{\ell(\eta_\sigma(w))}{2} - \frac{\ell(a)}{2}$$

$$= \frac{\ell(w_2) + \ell(w_1) - \ell(at^\gamma)}{2} + \frac{\ell(\eta_\sigma(w))}{2} - \frac{\ell(a)}{2}$$

$$= \frac{\ell(w) - \ell(at^\gamma)}{2} + \frac{\ell(\eta_\sigma(w))}{2} - \frac{\ell(a)}{2} = d_w(b) - d_{at^\gamma}(b).$$

So $w \Rightarrow_\sigma at^\gamma$. The theorem is proved.

6. Proof of main theorem

Now we state our main result.

Theorem 6.1. *Suppose that* $\mathbf{G}$ *is simple and quasi-split over* F. *Let* $b \in \breve{G}$ *and* $w \in \tilde{W}$ *such that* $w\mathfrak{a}$ *is in a Shrunken Weyl chamber,* $\lambda_w^\diamond - \nu_b \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^\vee$ *and* $(\lambda_w^\flat)^\diamond \geqslant \nu_b$. *Then* $X_w(b) \neq \emptyset$ *if and only if* $\kappa(b) = \kappa(w)$ *and* $\mathrm{supp}_\sigma(\eta_\sigma(w)) = \mathbb{S}$. *In this case,* $\dim X_w(b) = d_w(b)$.

Remark 6.2. It is worth mentioning that in most cases, $\lambda_w - \lambda_w^\flat$ is dominant and nonzero. In this case, $\lambda_w^\diamond - (\lambda_w^\flat)^\diamond \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^\vee$. However, if $\mathbf{G}$ is split over F and λ_w is a minuscule coweight, then $\lambda_w^\flat = \lambda_w$. Thus the assumption $\lambda_w^\diamond - \nu_b \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^\vee$ is needed in our statement.

We first prove the theorem and then discuss the assumptions in the statement. In particular, we will give a simple condition where the assumptions are satisfied in Corollary 6.4.

6.1. **The** (J, w, δ)**-alcove elements.** We recall the alcove elements introduced in [GHKR10] for split groups and then generalized to quasi-split groups in [GHN15].

For any $J \subset \mathbb{S}$ with $\sigma(J) = J$, we denote by $\mathbb{M}_J \subset \mathbf{G}$ the standard Levi subgroup corresponding to J and $\mathbb{P}_J \supset \mathbb{M}_J$ be the standard parabolic subgroup. Let $\mathbb{U}_{\mathbb{P}_J}$ be the unipotent radical of $\mathbb{P}_J$.

Let $J \subset \mathbb{S}$ with $\sigma(J) = J$ and $x \in W_0$. Let $w \in \tilde{W}$. We say that w is a (J, x, σ)-*alcove element* if $x^{-1}w\sigma(x) \in \tilde{W}_J$ and ${}^{\dot{x}}\mathbb{U}_{\mathbb{P}_J}(\breve{F}) \cap {}^{\dot{w}}\breve{I} \subseteq {}^{\dot{x}}\mathbb{U}_{\mathbb{P}_J}(\breve{F}) \cap \breve{I}$. The following result is proved in [GHN15, Corollary 3.6.1][1].

Theorem 6.3. *Let* $[b] \in B(\mathbf{G})$ *and* $w \in \tilde{W}$. *Suppose that* w *is a* (J, x, σ)-*alcove element. Let* $\kappa_{\mathbb{M}_J}$ *be the Kottwitz map for the group* $\mathbb{M}_J$. *If* $\kappa_{\mathbb{M}_J}(x^{-1}w\sigma(x)) \neq \kappa_{\mathbb{M}_J}(b')$ *for any* $b' \in [b] \cap \mathbb{M}_J(\breve{F})$, *then* $X_w(b) = \emptyset$.

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[1] In loc.cit., we put the assumption that $[b]$ is basic. In fact, the assumption is required in [GHN15, Propositon 3.5.1 & Remark 3.6.2] but is not need in [GHN15, Corollary 3.6.1].

6.2. **The emptiness pattern.** Suppose that $w\mathfrak{a}$ is in a Shrunken Weyl chamber and $(\lambda_w^{\flat})^{\diamond} \geqslant \nu_b$. We write w as $w = xt^{\lambda}y$ with $x, y \in W_0$ and $t^{\lambda}y \in {}^{\mathbb{S}}\tilde{W}$. If $\kappa(b) \neq \kappa(w)$, then $X_w(b) = \emptyset$.

Now suppose that $\kappa(b) = \kappa(w)$ and $\mathrm{supp}_{\sigma}(\sigma^{-1}(y)x) \neq \mathbb{S}$. Set $J = \mathrm{supp}_{\sigma}(\sigma^{-1}(y)x)$. By [GHN15, Lemma 3.6.3], w is a $(J, \sigma^{-1}(y), \sigma)$-alcove element. Let $b' \in [b] \cap \mathbb{M}_J(\breve{F})$. We denote by $\nu_{b'}^{\mathbb{M}_J}$ the image of b' under the Newton map for $\mathbb{M}_J$. Then $\nu_{b'}^{\mathbb{M}_J} \in W_0(\nu_b)$. Hence $\nu_b - \nu_{b'}^{\mathbb{M}_J} \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geqslant 0}\alpha^{\vee}$.

By assumption, $\lambda^{\diamond} - \nu_b \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^{\vee}$. Thus $\lambda^{\diamond} - \nu_{b'}^{\mathbb{M}_J} \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^{\vee}$ and can not be written as a linear combination of the coroots in $\mathbb{M}_J$. Therefore $\kappa_{\mathbb{M}_J}(\sigma^{-1}(y)wy^{-1}) \neq \kappa_{\mathbb{M}_J}(b')$. By Theorem 6.3, $X_w(b) = \emptyset$.

6.3. **Dimension formula.** Suppose that $\kappa(w) = \kappa(b)$ and $\mathrm{supp}_{\sigma}(\eta_{\sigma}(w)) = \mathbb{S}$. By Theorem 5.5 (1), there exists a dominant coweight $\gamma \geqslant_{\mathbb{Z}} \lambda_w^{\flat}$ and $a \in W_0$ with $\mathrm{supp}_{\sigma}(a) = \mathbb{S}$ such that

$$w \Rightarrow_{\sigma} at^{\gamma}.$$

By our assumption, $\gamma^{\diamond} \geqslant (\lambda_w^{\flat})^{\diamond} \geqslant \nu_b$. By Theorem 4.2, $[\dot{t}^{\gamma}] = [b_{at^{\gamma}}]$. Since $\kappa(w) = \kappa(t^{\gamma}) = \kappa(b)$, we have $[b] \leqslant [\dot{t}^{\gamma}]$.

By [He15, Theorem 2.27], $X_{at^{\gamma}}(\dot{\tau}) \neq \emptyset$, where $\tau \in \Omega$ with $\kappa(w) = \kappa(t^{\gamma}) = \kappa(\tau)$. Since $\kappa(w) = \kappa(b)$, we have $[\dot{\tau}] \leqslant [b]$.

By Theorem 4.2, at^{γ} is a cordial element. Hence by Theorem 4.1 (1), $X_{at^{\gamma}}(b) \neq \emptyset$ and by Theorem 4.1 (2), $\dim X_{at^{\gamma}}(b) = d_{at^{\gamma}}(b)$.

So by definition of $\Rightarrow_{\sigma}$, we have $X_w(b) \neq \emptyset$ and

$$\dim X_w(b) - d_w(b) \geqslant \dim X_{at^{\gamma}}(b) - d_{at^{\gamma}}(b) = 0.$$

Hence $\dim X_w(b) \geqslant d_w(b)$. On the other hand, by Theorem 2.1, $\dim X_w(b) \leqslant d_w(b)$. So $\dim X_w(b) = d_w(b)$.

6.4. **Some remarks on the condition $(\lambda_w^{\flat})^{\diamond} \geqslant \nu_b$.** We first consider the case where $[b]$ is basic. In this case, the condition $(\lambda_w^{\flat})^{\diamond} \geqslant \nu_b$ follows directly from the condition $\kappa(b) = \kappa(w)$.

Now we consider the nonbasic $[b]$. Suppose that $\lambda_w^{\diamond} \geqslant \nu_b + 2\rho^{\vee}$. In this case, although $\lambda_w - 2\rho^{\vee}$ may not be dominant, its σ-average is dominant and is larger than or equal to ν_b. By definition, $\lambda_w^{\flat} - (\lambda_w - \rho_J^{\vee} - \eta_J^{\vee}) \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geqslant 0}\alpha^{\vee}$ for some J. Note that $2\rho_J^{\vee} - \rho_J^{\vee} - \eta_J^{\vee} \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geqslant 0}\alpha^{\vee}$. We have $\lambda_w^{\flat} - (\lambda_w - 2\rho^{\vee}) \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geqslant 0}\alpha^{\vee}$. Hence $(\lambda_w^{\flat})^{\diamond} \geqslant \lambda_w^{\diamond} - 2\rho^{\vee} \geqslant \nu_b$. It is also easy to see that $\lambda_w^{\diamond} - \nu_b \in \sum_{\alpha \in \Delta} \mathbb{Q}_{>0}\alpha^{\vee}$.

In particular, if $\lambda_w = n\omega^{\vee}$, where $\omega^{\vee}$ is a fundamental coweight and $n \gg 0$ with respect to $[b]$. Then $\lambda_w^{\diamond} \geqslant \nu_b + 2\rho^{\vee}$ and hence the condition $(\lambda_w^{\flat})^{\diamond} \geqslant \nu_b$ is satisfied in this case.

Corollary 6.4. *Suppose that* $\mathbf{G}$ *is simple and quasi-split over* F. *Let* $b \in \breve{G}$ *and* $w \in \tilde{W}$ *such that* $w\mathfrak{a}$ *is in a Shrunken Weyl chamber. Suppose that* $\lambda_w^{\diamond} \geqslant \nu_b + 2\rho^{\vee}$. *Then* $X_w(b) \neq \emptyset$ *if and only if* $\kappa(b) = \kappa(w)$ *and* $\mathrm{supp}_{\sigma}(\eta_{\sigma}(w)) = \mathbb{S}$. *In this case,* $\dim X_w(b) = d_w(b)$.

6.5. **A side remark.** By Theorem 4.4, if $X_w(b) \neq \emptyset$, then $\kappa(b) = \kappa(w)$ and $\nu_b \leqslant \lambda_w^{\diamond}$.

Let $w \in \tilde{W}$ such that $w\mathfrak{a}$ is in a Shrunken Weyl chamber. If $\mathrm{supp}_{\sigma}(\eta_{\sigma}(w)) = \mathbb{S}$, then Theorem 6.1 describes the nonemptiness pattern and the dimension formula of $X_w(b)$ for most of the σ-conjugacy classes $[b]$ with $\kappa(b) = \kappa(w)$ and $\nu_b \leqslant \lambda_w^{\diamond}$.

If $\mathrm{supp}_\sigma(\eta_\sigma(w)) = J \subsetneq \mathbb{S}$, then by [GHN15, Lemma 3.6.3], w is a $(J, \sigma^{-1}(y), \sigma)$-alcove element for some $y \in W_0$. Then the Hodge-Newton decomposition (see [GHKR10, Theorem 2.1.4] for the split group and [GHN15, Propositon 2.5.1 & Theorem 3.3.1] in general) deduces the study of $X_w(b)$ to the study of a suitable affine Deligne-Lusztig variety associated to the Levi subgroup $\mathbb{M}_J$. One may apply Theorem 6.1 to the latter one. In this way, one also obtains an explicit description of the nonemptiness pattern and the dimension formula of $X_w(b)$ for most of the σ-conjugacy classes $[b]$ with $\kappa(b) = \kappa(w)$ and $\nu_b \leqslant \lambda_w^\diamond$.

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