FROM THE LANDAU-DE GENNES THEORY TO THE ERICKSEN-LESLIE THEORY IN DIMENSION TWO

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Abstract. In this paper, we study the connection between the Ericksen-Leslie equations and the Beris-Edwards equations in dimension two. It is shown that the weak solutions to the Beris-Edwards equations converge to the one to the Ericksen-Leslie equations as the elastic coefficient tends to zero. Moreover, the limiting weak solutions to the Ericksen-Leslie equations may have singular points.

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1. introduction

Liquid crystals are states of matter between conventional liquid and solid crystal, they may flow like a liquid, but their molecules may be oriented in a crystal-like way. In physics, different order parameters are introduced to characterize the anisotropic behavior of liquid crystals, which lead to different theories. There are several competing mathematical theories for nematic liquid crystals in the literature, such as the Oseen-Frank theory [37, 10], the Erickse-Leslie theory [9, 23], the Landau-de Gennes theory [4], and the Doi-Onsager theory [6, 38]. The Oseen-Frank theory and the Erickse-Leslie theory are vector theories, in which the average direction of the liquid crystal molecules at a certain point is described by a unit vector. The Landau-de Gennes theory uses a 3×3 symmetric traceless tensor Q as the order parameter to describe the orientation of liquid crystal molecules. The Doi-Onsager theory is a molecular theory where the molecule has a continuous distribution of orientations. As these theories are derived from different considerations and are widely used in liquid crystal studies, it is important to explore the relationships of different theories.

The Ericksen-Leslie equations can be derived from the Doi-Onsager equations by taking small Deborah number limit in dimension three. This limit was formally derived in [21, 8], and rigorously justified before the first singular time of the Ericksen-Leslie equations in [46]. It is noted that a new dynamic Q-tensor model was derived from the Doi's kinetic theory in [13], and the Ericksen-Leslie model can also be formally derived from this dynamic Q-tensor model in dimension three, which was later rigorously justified for smooth solutions in [25]. Similarly, in dimension three, a rigorous derivation of the Ericksen-Leslie equations from the Beris-Edwards model [2] in the Landau-de Gennes framework was given in [45], and similar result was established in dimension three recently in [26] concerning the connection between the Ericksen-Leslie equations and the Qian-Sheng model in Landau-de Gennes framework. It should be noted that all these results have been established under the main assumption

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that the solutions to the Ericksen-Leslie equations are suitable smooth. If one takes no account of the velocity of the fluid, the Beris-Edwards model becomes the Q-tensor flows, and the Ericksen-Leslie system becomes the harmonic map flows. Moreover, in dimension three, [43] has shown the connection between the solutions to the Q-tensor flows and the weak solutions to the harmonic map flows which may contain singular points. However, non-trivial singular weak solutions (with non-trivial velocity) to the Ericksen-Leslie equations do exist, see [18] for three-dimensional case and [22] for two-dimensional case. Our main goal in this paper is to study the connections between the solutions to the Beris-Edwards model and the weak solutions to the Ericksen-Leslie equations in dimension two. Note that the weak solutions to the Ericksen-Leslie equations may contain singular points.

1.1. **Notations.** The following convertions will be used. $\Omega_T = (0,T)\times\mathbb{R}^2$ for $0 < T < +\infty$, and

$$||f||^q_{L^q_tL^p_x} = \int_0^T ||f(t,\cdot)||^q_{L^p(\mathbb{R}^2)}dt,\ ||f||^p_{L^p_tH^m_x} = \int_0^T ||f(t,\cdot)||^p_{H^m(\mathbb{R}^2)}dt,\ ||f||_{L^p(\Omega_T)} = ||f||_{L^p_tL^p_x},$$

for $p,q \in [1,\infty]$. For any two vectors $m = (m_1,m_2,m_3), n = (n_1,n_2,n_3) \in \mathbb{R}^3$, $m\otimes n = [m_in_j]_{1\le i,j\le 3}$. $A\cdot B$ denotes the usual matrix/vector-matrix/vector product. Einstein summation is used throughout the paper. $A : B = A_{ij}B_{ij}$ and $|A| = \sqrt{A:A}$. The divergence of a tensor is defined by $(\nabla\cdot\sigma)_i = \partial_j\sigma_{ij}$, where $\partial_j f = \partial_{x_j} f$. $(\nabla A\odot\nabla A)_{ij} = \partial_i A : \partial_j A$ for matrix A and $(\nabla d\odot\nabla d)_{ij} = \partial_i d\cdot\partial_j d$ for vector d. $\mathbb{I}$ denotes the 3×3 identity matrix. $\mathbb{S}^2 = \{d\in\mathbb{R}^2, |d| = 1\}$. For simplicity, the subsequences of $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$ and $\{Q^\epsilon\}_{\epsilon>0}$ are still denoted as $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$ and $\{Q^\epsilon\}_{\epsilon>0}$. Let $\mathcal{Q}_0 \subset M^{3\times3}$ denote the space of Q-tensors i.e.

$$\mathcal{Q}_0 = \{Q\in M^{3\times3}, \operatorname{tr} Q = 0, Q_{ij} = Q_{ji}, i,j = 1,2,3\}.$$

Set

$$\mathcal{D} = C_0^\infty(\mathbb{R}^2,\mathbb{R}^3)\cap\{\phi = (\phi_1,\phi_2,\phi_3)^T : \partial_1\phi_1 + \partial_2\phi_2 = 0\},$$

$$\mathring{H} = \text{closure of } \mathcal{D} \text{ in } L^2(\mathbb{R}^2,\mathbb{R}^3), \quad \mathring{J} = \text{closure of } \mathcal{D} \text{ in } H^1(\mathbb{R}^2,\mathbb{R}^3),$$

and

$$\overline{\nabla F} = \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 & 0 \\ \partial_1 F_2 & \partial_2 F_2 & 0 \\ \partial_1 F_3 & \partial_2 F_3 & 0 \end{pmatrix}, \quad \underline{\nabla F} = \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 \\ \partial_1 F_2 & \partial_2 F_2 \end{pmatrix} \text{ for } F : (0,T)\times\mathbb{R}^2 \mapsto \mathbb{R}^3.$$

For any $Q\in\mathcal{Q}_0$, since $Q_{ii} = 0$, it holds that

$$Q = s_1(\tilde{d}\otimes\tilde{d} - \frac{1}{3}\mathbb{I}) + s_2(\hat{d}\otimes\hat{d} - \frac{1}{3}\mathbb{I}),$$

where $s_1, s_2 \in \mathbb{R}$ and $\tilde{d}, \hat{d} \in \mathbb{S}^2$ are the eigenvectors of Q satisfying $\tilde{d}\cdot\hat{d} = 0$. When $s_1 = s_2 = 0$, the nematic liquid crystal is said to be isotropic. When s_1, s_2 are different and nonzero, it is said to be biaxial. When $s_1 = s_2 \ne 0$ or $s_1 = 0, s_2 \ne 0$ or $s_1 \ne 0, s_2 = 0$, it is said to be uniaxial and Q can be rewritten as

$$Q = s(\bar{d}\otimes\bar{d} - \frac{1}{3}\mathbb{I}), \quad \bar{d}\in\mathbb{S}^2.$$

1.2. **The Landau-de Gennes theory.** Let $Q \in \mathcal{Q}_0$. For the bulk energy density

$$F_b(Q) = -\frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}Q^3 + \frac{c}{4}|Q|^4,$$

one can verify that if $c > 0$, then $F_b(Q)$ is bounded from below([35, Proposition 1]). Note that a, b and c are material-dependent and temperature-dependent constants. Furthermore, F_b attains its minimum on the uniaxial Q-tensor with constant order parameter

$$s_+ = \frac{b + \sqrt{b^2 + 24ac}}{4c}.$$

Thus, F_b has a corresponding non-negative bulk energy density $\hat{F}_b$ defined by

$$\hat{F}_b(Q) = F_b(Q) - \min_{Q \in \mathcal{Q}_0} F_b(Q), \tag{1.1}$$

and

$$\hat{F}_b(Q) = 0 \Longleftrightarrow Q \in \mathcal{N}, \tag{1.2}$$

where

$$\mathcal{N} = \left\{ Q \in \mathcal{Q}_0, Q = s_+(d \otimes d - \frac{1}{3}\mathbb{I}), d \in \mathbb{S}^2 \right\}.$$

Then we define the following Landau-de Gennes energy functional $\mathcal{F}$ as

$$\mathcal{F}(Q, \nabla Q) = \hat{\mathcal{F}}_b(Q) + \mathcal{F}_e(Q, \nabla Q),$$

where $\hat{\mathcal{F}}_b$ and $\mathcal{F}_e$ are the bulk energy and the elastic energy defined respectively by

$$\hat{\mathcal{F}}_b(Q) = \int_{\mathbb{R}^n} \hat{F}_b(Q)dx,$$

$$\mathcal{F}_e(Q, \nabla Q) = \frac{1}{2}\int_{\mathbb{R}^n} \left(L_1|\nabla Q|^2 + L_2 Q_{ij,j}Q_{ik,k} + L_3 Q_{ij,k}Q_{ik,j} + L_4 Q_{ij}Q_{kl,i}Q_{kl,j} \right) dx,$$

with L_1, L_2, L_3 and L_4 being material dependent elastic constants and $n = 2, 3$.

The Beris-Edwards model in $\mathbb{R}^3$ takes the form:

$$v_t + v \cdot \nabla v = -\nabla P + \nabla \cdot (\sigma^s + \sigma^a + \sigma^d), \tag{1.3}$$

$$\nabla \cdot v = 0, \tag{1.4}$$

$$Q_t + v \cdot \nabla Q = \frac{1}{\Gamma}H + R(\nabla v, Q), \tag{1.5}$$

where $v : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ is the velocity of the fluid, $Q : (0,T) \times \mathbb{R}^3 \mapsto \mathcal{Q}_0$ is the macroscopic Q-tensor order parameter, $P : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{R}$ is the pressure, Γ is a collective rotational diffusion constant, $D = \frac{1}{2}(\nabla v + (\nabla v)^T)$, $\Lambda = \frac{1}{2}(\nabla v - (\nabla v)^T)$, σ^s, σ^a and σ^d are the symmetric viscous stress, antisymmetric viscous stress, and distortion stress, respectively, defined by

$$\sigma^s = \eta D - S_Q(H), \quad \sigma^a = Q \cdot H - H \cdot Q, \quad \sigma^d_{ij} = -\frac{\delta \mathcal{F}}{\delta Q_{kl,j}} Q_{kl,i},$$

$$S_Q(A) = \xi \left[A \cdot (Q + \frac{1}{3}\mathbb{I}) + (Q + \frac{1}{3}\mathbb{I}) \cdot A - 2(Q + \frac{1}{3}\mathbb{I})Q : A \right], \text{ for } Q, A \in M^{3\times 3}, \tag{1.6}$$

where η is the viscous coefficient, H is the molecular field given by $H(Q) = -\frac{\delta \mathcal{F}}{\delta Q}$, ξ is a constant depending on the molecular details of a given liquid crystal and measures the ratio between the tumbling and the aligning effect that a shear flow would exert over the liquid crystals directors. $R(\nabla v, Q)$ describes the rotating and stretching effects on the order parameter Q due to the fluid, $R(\nabla v, Q)$ is defined by

$$R(\nabla v, Q) = S_Q(D) + \Lambda \cdot Q - Q \cdot \Lambda.$$

There have been quite many works on the global existence of solutions to the system (1.3)-(1.5), see [39, 40, 3, 33, 12, 1, 44] and the references therein. In particular, the first existence of global weak solutions to the cauchy problem for (1.3)-(1.5) in the dimension two (2D) and dimension three (3D) is established by Paicu-Zarnescu in [39] under the conditions that $\xi = 0$ and

$$L_2 = L_3 = L_4 = 0, \quad c > 0, \quad \eta > 0, \quad \Gamma > 0, \tag{1.7}$$

where they also showed the existence of global regular solutions in 2D for suitably regular initial data, and the condition $\xi = 0$ can be relaxed to $|\xi|$ being small in [40]. These results in [39, 40] have been generalized to many interesting cases. In particular, for 2D periodic initial data, the global well-posedness of strong solutions to (1.3)-(1.5) was obtained in [3] under just condition (1.7), which was relaxed to allow L_2, L_3 and L_4 being non-zero with some other minor conditions recently in [33]; and for initial-boundary value problems for 2D and 3D, the global existence of weak solutions to the system (1.3)-(1.5) has been proved in [12, 1] under conditions that $\xi = 0$ and (1.7) holds. Similar results have been obtained in [44], where the bulk potential (1.2) is replaced by Ball-Majumdar type bulk potential.

In this paper, we assume that condition (1.7) holds. Since the elastic constant L_1 is typically very small compared with a, b and c, one can introduce a small parameter ϵ, and consider the following Landau-de Gennes energy functional:

$$\mathcal{F}_\epsilon(Q, \nabla Q) = \frac{1}{\epsilon}\int_{\mathbb{R}^2} \hat{F}_b(Q)dx + \frac{L_1}{2}\int_{\mathbb{R}^2} |\nabla Q|^2 dx. \tag{1.8}$$

Thus, we look for a solution (v^ϵ, Q^ϵ) to (1.3)-(1.5) which is independent of x_3. Then, $v^\epsilon = (v_1^\epsilon, v_2^\epsilon, v_3^\epsilon)^T : (0, T) \times \mathbb{R}^2 \mapsto \mathbb{R}^3$ and $Q^\epsilon : (0, T) \times \mathbb{R}^2 \mapsto \mathcal{Q}_0$ satisfy

$$\partial_3 v^\epsilon = 0, \quad \partial_3 Q^\epsilon = 0, \quad \partial_3 P^\epsilon = 0. \tag{1.9}$$

Then in this case, the system (1.3)-(1.5) is reduced to the following two-dimensional one:

$$\begin{cases} \partial_t v_i^\epsilon + v_j^\epsilon(\overline{\nabla v^\epsilon})_{ij} = -\partial_i P^\epsilon + \sum_{j=1}^2 \partial_j[\eta\overline{D^\epsilon} - S_{Q^\epsilon}(H^\epsilon) + Q^\epsilon \cdot H^\epsilon - H^\epsilon \cdot Q^\epsilon]_{ij} \\ \qquad\qquad -L_1\sum_{k=1}^2 \partial_k(\nabla Q^\epsilon \odot \nabla Q^\epsilon)_{ik}, \quad i = 1, 2, \\ \partial_t v_3^\epsilon + v_j^\epsilon(\overline{\nabla v^\epsilon})_{3,j} = \sum_{j=1}^2 \partial_j[\eta\overline{D^\epsilon} - S_{Q^\epsilon}(H^\epsilon) + Q^\epsilon \cdot H^\epsilon - H^\epsilon \cdot Q^\epsilon]_{3j}, \\ \partial_1 v_1^\epsilon + \partial_2 v_2^\epsilon = 0, \\ Q_t^\epsilon + \underline{v^\epsilon} \cdot \nabla Q^\epsilon = \frac{1}{\Gamma}H^\epsilon + S_{Q^\epsilon}(\overline{D^\epsilon}) + \overline{\Lambda^\epsilon} \cdot Q^\epsilon - Q^\epsilon \cdot \overline{\Lambda^\epsilon}, \end{cases} \tag{1.10}$$

where $P^\epsilon : (0, T) \times \mathbb{R}^2 \mapsto \mathbb{R}$, and

$$\overline{D^\epsilon} = \frac{\overline{\nabla v^\epsilon} + (\overline{\nabla v^\epsilon})^T}{2}, \quad \overline{\Lambda^\epsilon} = \frac{\overline{\nabla v^\epsilon} - (\overline{\nabla v^\epsilon})^T}{2}, \quad \underline{v^\epsilon} = (v_1^\epsilon, v_2^\epsilon)^T, \tag{1.11}$$

$$H^\epsilon = L_1\Delta Q^\epsilon - \frac{\mathcal{J}(Q^\epsilon)}{\epsilon}, \tag{1.12}$$

$$\mathcal{J}(Q) = \frac{\delta\hat{\mathcal{F}}_b(Q)}{\delta Q} = -aQ - b(Q^2 - \frac{1}{3}|Q|^2\mathbb{I}) + c|Q|^2 Q, \tag{1.13}$$

with $S_{Q^\epsilon}(H^\epsilon)$ and $S_{Q^\epsilon}(\overline{D^\epsilon})$ given in (1.6). Note that ΔQ^ϵ in (1.12) is equal to $\sum_{i=1}^2 \frac{\partial^2 Q^\epsilon}{\partial x_i^2}$ due to $Q^\epsilon : (0, T) \times \mathbb{R}^2 \mapsto \mathcal{Q}_0$.

Remark 1.1. *Let $\mathbb{T}^1$ denote the periodic interval with period $A > 0$ and $\Omega' = \mathbb{R}^2 \times \mathbb{T}^1$. It can be checked that the solution (v, Q) to the system (1.3)-(1.5) is unique when $v, \nabla Q \in L^\infty(0,T; L^2(\Omega')) \cap L^2(0,T; H^1(\Omega')) \cap L^\infty(0,T; W^{1,\infty}(\Omega'))$. Then, the condition (1.9) can be guaranteed by $\partial_3 v_0^\epsilon = 0$ and $\partial_3 Q_0^\epsilon = 0$ ($v^\epsilon|_{t=0} = v_0^\epsilon, Q^\epsilon|_{t=0} = Q_0^\epsilon$) if there exists a global smooth solution (v^ϵ, Q^ϵ) to the system (1.10) with $v^\epsilon, \nabla Q^\epsilon \in L^\infty(0,T; L^2(\mathbb{R}^2)) \cap L^2(0,T; H^1(\mathbb{R}^2)) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}^2))$, which will be given in our forthcoming paper. Meanwhile, it should be noted that this two-dimensional system (1.10) includes the two-dimensional system in [39, 40, 3, 33, 12, 1, 44], in which $v : (0,T) \times \mathbb{R}^2 \mapsto \mathbb{R}^2$, $Q : (0,T) \times \mathbb{R}^2 \mapsto \mathcal{Q}_1$ and $\mathcal{Q}_1 = \{Q \in \mathbb{M}^{2\times 2}, trQ = 0, Q_{ij} = Q_{ji}, i,j = 1,2\}$. Moreover, under conditions (1.7) and $|\xi|$ being sufficiently small, Paicu-Zarnescu [39] proved the existence of global regular solutions with sufficiently regular initial data for this two-dimensional problem.*

Corresponding to (1.10), the initial data for (v^ϵ, Q^ϵ) can be taken as

$$v^\epsilon|_{t=0} = v_0^\epsilon \in \mathring{J}, \quad Q^\epsilon|_{t=0} = Q_0^\epsilon, \quad Q_0^\epsilon - Q^\infty \in H^2(\mathbb{R}^2, \mathcal{Q}_0), \tag{1.14}$$

where $Q^\infty = s_+(d^\infty \otimes d^\infty - \frac{1}{3}\mathbb{I})$ for the constant vector $d^\infty \in \mathbb{S}^2$. Then, the energy inequality of the Beris-Edwards system (1.10) corresponding to the initial data (1.14) is read as:

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\frac{1}{2}|v^\epsilon|^2 + \frac{L_1}{2}|\nabla Q^\epsilon|^2 + \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right)(\cdot, t)dx + \int_0^t \int_{\mathbb{R}^2} \left(\eta|\overline{D^\epsilon}|^2 + \frac{1}{\Gamma}|H^\epsilon|^2 \right) dxdt \\ \le\ & \int_{\mathbb{R}^2} \left(\frac{1}{2}|v_0^\epsilon|^2 + \frac{L_1}{2}|\nabla Q_0^\epsilon|^2 + \frac{\hat{F}_b(Q_0^\epsilon)}{\epsilon} \right) dx \end{aligned} \tag{1.15}$$

for $t \in (0,T)$, see [40, Proposition 1] for the detailed derivation of (1.15). It is assumed further that

$$v_0^\epsilon \to v_0^* \text{ in } L^2(\mathbb{R}^2, \mathbb{R}^3), \quad Q_0^\epsilon - Q^\infty \to Q_0^* - Q^\infty \text{ in } H^1(\mathbb{R}^2, \mathcal{Q}_0), \quad \int_{\mathbb{R}^2} \frac{\hat{F}_b(Q_0^\epsilon)}{\epsilon} \to 0, \tag{1.16}$$

as $\epsilon \to 0$, where $Q_0^* = s_+(d_0^* \otimes d_0^* - \frac{1}{3}\mathbb{I})$. Note that (1.16) implies that there exists $E_0 > 0$ such that for suitably small and positive ϵ, it holds that

$$\int_{\mathbb{R}^2} \left(\frac{1}{2}|v_0^\epsilon|^2 + \frac{L_1}{2}|\nabla Q_0^\epsilon|^2 + \frac{\hat{F}_b(Q_0^\epsilon)}{\epsilon} \right) dx \le E_0. \tag{1.17}$$

1.3. **The Ericksen-Leslie theory.** The general Ericksen-Leslie system in $\mathbb{R}^3$ takes the form

$$v_t + v \cdot \nabla v + \nabla P = \nabla \cdot \sigma, \tag{1.18}$$

$$\nabla \cdot v = 0, \tag{1.19}$$

$$d \times (h - \gamma_1 N - \gamma_2 D \cdot d) = 0, \tag{1.20}$$

where $v : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ is the velocity of the fluid, $P : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{R}$ is the pressure, $d : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{S}^2$ is the macroscopic orientation of the nematic liquid crystal molecules, and the stress σ is modeled by the phenomenological constitutive relation

$$\sigma = \sigma^L + \sigma^E.$$

σ^L is the viscous (Leslie) stress given by

$$\sigma^L = \alpha_1(d \otimes d : D)d \otimes d + \alpha_2 N \otimes d + \alpha_3 d \otimes N + \alpha_4 D + \alpha_5 D \cdot (d \otimes d) + \alpha_6 (d \otimes d) \cdot D \tag{1.21}$$

with
$$N = d_t + v \cdot \nabla d - \Lambda \cdot d.$$
The six viscous coefficients $\alpha_1, \cdots, \alpha_6$ are called the Leslie coefficients. σ^E is the elastic (Ericksen) stress
$$\sigma_{ij}^E = -\frac{\partial E_{OF}}{\partial d_{k,j}} d_{k,i}, \tag{1.22}$$
where $E_{OF} = E_{OF}(d, \nabla d)$ is the Oseen-Frank energy density with the form
$$E_{OF} = \frac{k_1}{2}(\nabla \cdot d)^2 + \frac{k_2}{2}|d \cdot (\nabla \times d)|^2 + \frac{k_3}{2}|d \times (\nabla \times d)|^2$$
$$+\frac{k_2 + k_4}{2}\left[\mathrm{tr}(\nabla d)^2 - (\nabla \cdot d)^2\right].$$
Here k_1, k_2, k_3 and k_4 are the elastic constants. The molecular field h is given by
$$h = -\frac{\delta \mathcal{E}_{OF}}{\delta d}, \quad \mathcal{E}_{OF} = \int_{\mathbb{R}^3} E_{OF} dx.$$

In order to obtain a basic energy law to the system (1.18)-(1.20), one requires the Leslie coefficients, γ_1, and γ_2 to satisfy the following relations:
$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \tag{1.23}$$
$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \tag{1.24}$$
where (1.23) is called Parodi's relation.

For the system (1.18)-(1.20), the local well-posedness in dimension three has been proved in [47] under the physical constraints on the Leslie coefficients (1.23)-(1.24), which ensure that the energy of the system is dissipated. When $v_3 = 0$, $\partial_3 v = 0$, $d_3 = 0$ and $\partial_3 d = 0$, the system (1.18)-(1.20) becomes a two-dimensional one. For this case, the global weak solution has been shown in [17] under the conditions that (1.23) and (1.24) hold and $k_1 = k_2 = k_3 = 1, k_4 = 0$ and
$$\gamma_1 > 0, \quad \alpha_1 + \frac{\gamma_2^2}{\gamma_1} \geq 0, \quad \alpha_4 > 0, \quad \alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1} \geq 0. \tag{1.25}$$
Similar results have been obtained in [42] under weaker conditions that (1.23), (1.24), $v_3 = 0, \partial_3 v = 0, \partial_3 d = 0, \partial_1 \sigma_{31}^L + \partial_2 \sigma_{32}^L = 0$ and $\min\{k_1, k_2, k_3\} > 0$ and
$$\beta_2 \geq 0, \beta_1 + 2\beta_2 + \beta_3 \geq 0, \beta_1 < 0 \text{ or } \beta_2 \geq 0, 2\beta_2 + \beta_3 \geq 0, \beta_1 \geq 0, \tag{1.26}$$
where $\beta_1 = \alpha_1 + \frac{\gamma_2^2}{\gamma_1}, \beta_2 = \alpha_4, \beta_3 = \alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1}$.

For the simplified Ericksen-Leslie system:
$$\begin{cases} v_t - \nu \Delta v + v \cdot \nabla v + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot v = 0, \\ d_t + v \cdot \nabla d = \gamma(\Delta d + |\nabla d|^2 d). \end{cases} \tag{1.27}$$
where $v : (0, T) \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $d : (0, T) \times \mathbb{R}^n \mapsto \mathbb{S}^2$, $n = 2, 3$ and ν, λ, γ are constants. This system was proposed first by Lin [27] in 1989. It has been shown in [28, 14] that global weak solutions to (1.27) in dimension two exist, which are smooth with possible exceptions of finitely many singular times. Similar results for more general case were obtained in [16, 17, 42]. For the uniqueness of this kind of weak solutions, we refer to [31, 24]. For the three dimensional case, global weak solutions with the initial data $d_0 \in \mathbb{S}_+^2$ have been obtained in [32], which are weak limits of sequences of weak solutions to the Ginzburg-Landau approximate equations of (1.27) (see (1.28) below). Note that the weak solutions to

(1.27) with smooth initial data may not be smooth. In fact, two examples of weak solutions of finite time singularity in dimension three have been constructed in [18]. Recently, weak solutions with finite time singularities in dimension two have been constructed in [22]. For the blow-up criteria of strong solutions to the Ericksen-Leslie system, we refer to [19, 15] and the references therein.

It should be noted that there are many studies on the following Ginzburg-Landau type approximation of the simplified Ericksen-Leslie system,

$$\begin{cases} v_t^\epsilon - \nu\Delta v^\epsilon + v^\epsilon\cdot\nabla v^\epsilon + \nabla P^\epsilon = -\lambda\nabla\cdot(\nabla d^\epsilon\odot\nabla d^\epsilon), \\ \nabla\cdot v^\epsilon = 0, \\ d_t^\epsilon + v^\epsilon\cdot\nabla d^\epsilon = \gamma\left(\Delta d^\epsilon + \frac{(1-|d^\epsilon|^2)d^\epsilon}{\epsilon}\right). \end{cases} \tag{1.28}$$

For (1.28) with fixed $\epsilon > 0$ and the initial data $(v^\epsilon, d^\epsilon)|_{t=0} = (v_0, d_0)$, $v_0 \in L^2(\mathbb{R}^n, \mathbb{R}^n), d_0 \in \mathbb{S}^2, d_0 - d^\infty \in H^1(\mathbb{R}^n, \mathbb{R}^3), n = 2, 3$, the global existence of weak solutions (even strong solutions for $n = 2$) have been established by Lin-Liu [29] (see also the extension to the case with Leslie stress [30]). Such solutions satisfy the following energy inequality

$$\frac{1}{2}\int_{\mathbb{R}^n}\left(|v^\epsilon|^2 + |\nabla d^\epsilon|^2 + \frac{(1-|d^\epsilon|^2)^2}{2\epsilon}\right) + \int_0^T\int_{\mathbb{R}^n}\left(|\nabla v^\epsilon|^2 + \left|\Delta d^\epsilon + \frac{(1-|d^\epsilon|^2)d^\epsilon}{\epsilon}\right|^2\right) \le G_0$$

with $G_0 = \frac{1}{2}\int_{\mathbb{R}^n}(|v_0|^2 + |\nabla d_0|^2)dx$. This then implies that as $\epsilon \to 0^+$, $|d^\epsilon| \to 1$ as $\epsilon \to 0^+$ a.e., and (v^ϵ, d^ϵ) is expected to converge to a solution of (1.27). Indeed, this convergence has been shown in [14, 15] on the time interval where the solution to (1.27) remains regular, and the methods in [14, 15] depend crucially on the regularity of the strong solutions to (1.27). However, the extension of this approach in [14, 15] to larger times seems impossible due to the existence of singular weak solutions to the Ericksen-Leslie equations [18, 22]. In this respect, Kortum [20] proved the convergence of the weak solutions to (1.28) to the global-in-time weak solutions to (1.27) in two dimensional torus $\mathbb{T}^2$. The convergence of weak solutions to the Ginzburg-Landau approximation of the two dimensional simplified Ericksen-Leslie equations for both uniaxial and biaxial nematics has been obtained by Du-Huang-Wang [7].

For a solution (v^ϵ, Q^ϵ) to (1.10) and (1.14), due to (1.2), the energy inequality (1.15) and the condition (1.16), one may expect that $(v^\epsilon, Q^\epsilon) \to (v^*, Q^*)$ with $Q^* = s_+(d^*\otimes d^* - \frac{1}{3}\mathbb{I})$ as $\epsilon \to 0^+$, and (v^*, d^*) is a solution to the Ericksen-Leslie system with the coefficients satisfying

$$k_1 = k_2 = k_3 = 2L_1 s_+^2, \quad k_4 = 0, \quad \gamma_1 = 2\Gamma s_+^2, \quad \gamma_2 = -\frac{2\Gamma\xi s_+(s_+ + 2)}{3}, \tag{1.29}$$

$$\alpha_1 = -\frac{2\Gamma\xi^2 s_+^2(3-2s_+)(1+2s_+)}{3}, \quad \alpha_2 = -\Gamma s_+^2 - \frac{\Gamma\xi s_+(2+s_+)}{3}, \tag{1.30}$$

$$\alpha_3 = \Gamma s_+^2 - \frac{\Gamma\xi s_+(2+s_+)}{3}, \quad \alpha_4 = \eta + \frac{4\Gamma\xi^2(1-s_+)^2}{9}, \tag{1.31}$$

$$\alpha_5 = \frac{\Gamma\xi^2 s_+(4-s_+)}{3} + \frac{\Gamma\xi s_+(2+s_+)}{3}, \quad \alpha_6 = \frac{\Gamma\xi^2 s_+(4-s_+)}{3} - \frac{\Gamma\xi s_+(2+s_+)}{3}. \tag{1.32}$$

Furthermore, the solution (v^*, d^*) to the limiting Ericksen-Leslie system must satisfy

$$\partial_3 v^* = 0, \quad \partial_3 d^* = 0 \text{ and } \partial_3 P^* = 0 \tag{1.33}$$

due to (v^ϵ, Q^ϵ) satisfying (1.9). Thus, as (1.10), (v^*, d^*) solves the following two-dimensional system

$$\begin{cases} \partial_t v_i^* + v_j^*(\overline{\nabla v^*})_{ij} = -\partial_i P^* + \sum_{j=1}^2 \partial_j(\overline{\sigma_*^L})_{ij} - k_1 \sum_{l=1}^2 \partial_l(\nabla d^* \odot \nabla d^*)_{il}, \quad i = 1, 2, \\ \partial_t v_3^* + v_j^*(\overline{\nabla v^*})_{3,j} = \sum_{j=1}^2 \partial_j(\overline{\sigma_*^L})_{3j}, \\ \partial_1 v_1^* + \partial_2 v_2^* = 0, \\ k_1(\Delta d^* + |\nabla d^*|^2 d^*) - \gamma_1 \overline{N^*} - \gamma_2[\overline{D^*} \cdot d^* - (\overline{D^*} : d^* \otimes d^*) d^*] = 0, \end{cases} \tag{1.34}$$

where $v^* : (0,T) \times \mathbb{R}^2 \mapsto \mathbb{R}^3$, $d^* : (0,T) \times \mathbb{R}^2 \mapsto \mathbb{S}^2$, $P^* : (0,T) \times \mathbb{R}^2 \mapsto \mathbb{R}$,

$$\overline{D^*} = \frac{\overline{\nabla v^*} + (\overline{\nabla v^*})^T}{2},\ \overline{\Lambda^*} = \frac{\overline{\nabla v^*} - (\overline{\nabla v^*})^T}{2},\ \underline{v^*} = (v_1^*, v_2^*)^T,\ \overline{N^*} = d_t^* + \underline{v^*} \cdot \nabla d^* - \overline{\Lambda^*} \cdot d^*, \tag{1.35}$$

and

$$\overline{\sigma_*^L} = \alpha_1(d^* \otimes d^* : \overline{D^*}) d^* \otimes d^* + \alpha_2 \overline{N^*} \otimes d^* + \alpha_3 d^* \otimes \overline{N^*} + \alpha_4 \overline{D^*} + \alpha_5 \overline{D^*} \cdot (d^* \otimes d^*) + \alpha_6 (d^* \otimes d^*) \cdot \overline{D^*}.$$

Note that Δd^* in (1.34) is equal to $\sum_{i=1}^2 \frac{\partial^2 d^*}{\partial x_i^2}$ due to $d^* : (0,T) \times \mathbb{R}^2 \mapsto \mathbb{S}^2$, and the corresponding initial data for (v^*, d^*) can be taken as

$$v^*|_{t=0} = v_0^* \in \mathring{H}, \quad d^*|_{t=0} = d_0^*, \quad d_0^* - d^\infty \in H^1(\mathbb{R}^2, \mathbb{R}^3), \tag{1.36}$$

where v_0^* and d_0^* are given in (1.16). Then, the energy inequality for the limiting Ericksen-Leslie system (1.34) corresponding to the initial data (1.36) is

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\frac{1}{2}|v^*|^2 + \frac{k_1}{2}|\nabla d^*|^2 \right)(\cdot, t) dx + \int_0^t \int_{\mathbb{R}^2} \left[\alpha_4 |\overline{D^*}|^2 + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1}) |\overline{D^*} : (d^* \otimes d^*)|^2 \right] dx dt \\ &+ \int_0^t \int_{\mathbb{R}^2} \left[(\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1}) |\overline{D^*} \cdot d^*|^2 + \frac{1}{\gamma_1} |d^* \times h^*|^2 \right] dx dt \\ \leq \ &\int_{\mathbb{R}^2} \left(\frac{1}{2}|v_0^*|^2 + \frac{k_1}{2}|\nabla d_0^*|^2 \right) dx, \end{aligned} \tag{1.37}$$

where $t \in (0,T)$, see [42, Proposition 2.1] for the detailed derivation of (1.37).

Remark 1.2. *Under conditions* (1.29)-(1.32)*, the system* (1.18)-(1.20) *can be regarded as the uniaxial limit of the Beris-Edwards system* (1.3)-(1.5) *by sending* $\epsilon \to 0$*. In dimension three, this has been shown rigorously by Wang-Zhang-Zhang* [45] *before the first singular time of the Ericksen-Leslie system* (1.18)-(1.20)*. Our main goal in this paper is to show that such an asymptotic convergence holds true for weak solutions to the Beris-Edwards system* (1.3)-(1.5) *and the Ericksen-Leslie system* (1.18)-(1.20) *in the 2-dimensional case specified by* (1.9)*.*

1.4. **Main results.** We give first the definitions of weak solutions to the system (1.10) and the limiting weak solutions to the system (1.34).

Weak solutions to (1.10) subject to the initial data (1.14) can be defined as:

Definition 1.3. *For* $0 < T < \infty$*, a pair* (v^ϵ, Q^ϵ) *is a weak solution to the system* (1.10) *subject to the initial data* (1.14)*, if* $v^\epsilon \in L^\infty(0,T; \mathring{H}) \cap L^2(0,T; \mathring{J})$ *and* $Q^\epsilon \in L^\infty(0,T; H^1(\mathbb{R}^2, \mathcal{Q}_0)) \cap$

$L^2(0,T;H^2(\mathbb{R}^2,\mathcal{Q}_0))$ *satisfy the energy inequality* (1.15) *and*

$$\begin{aligned}&\int_{\Omega_T}[-v^\epsilon\cdot\psi_t-(v^\epsilon\cdot\overline{\nabla\psi})\cdot v^\epsilon+\eta\overline{D^\epsilon}:\overline{\nabla\psi}-L_1\nabla Q^\epsilon\odot\nabla Q^\epsilon:\underline{\nabla\psi}]dxdt\\&+\int_{\Omega_T}[Q^\epsilon\cdot H^\epsilon-H^\epsilon\cdot Q^\epsilon-S_{Q^\epsilon}(H^\epsilon)]:\overline{\nabla\psi}dxdt=\int_{\mathbb{R}^2}v_0^\epsilon(x)\cdot\psi(0,x)dx,\end{aligned}\tag{1.38}$$

$$\begin{aligned}&\int_{\Omega_T}[-Q^\epsilon:\varphi_t-(\underline{v^\epsilon}\cdot\nabla\varphi):Q^\epsilon-\frac{1}{\Gamma}H^\epsilon:\varphi]dxdt\\&+\quad\int_{\Omega_T}[Q^\epsilon\cdot\overline{\Lambda^\epsilon}-\overline{\Lambda^\epsilon}\cdot Q^\epsilon-S_{Q^\epsilon}(\overline{D^\epsilon})]:\varphi dxdt=\int_{\mathbb{R}^2}Q_0^\epsilon(x):\varphi(0,x)dx\end{aligned}\tag{1.39}$$

for every $\psi\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3),\partial_1\psi_1+\partial_2\psi_2=0,\varphi\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathcal{Q}_0)$, *where* H^ϵ *is given in* (1.12) *and* $\overline{D^\epsilon}$, $\overline{\Lambda^\epsilon}$, $\underline{v^\epsilon}$ *are given in* (1.11), *and* $S_{Q^\epsilon}(H^\epsilon)$, $S_{Q^\epsilon}(\overline{D^\epsilon})$ *are given in* (1.6).

While weak solutions to (1.34) subject to the initial data (1.36) are defined as:

Definition 1.4. *For* $0<T<\infty$, *a pair* (v^*,d^*) *is a weak solution to the system* (1.34) *subject to the initial data* (1.36), *if* $v^*\in L^\infty(0,T;\mathring{H})\cap L^2(0,T;\mathring{J})$ *and* $d^*-d^\infty\in L^\infty(0,T;H^1(\mathbb{R}^2,\mathbb{R}^3))$ *satisfy the energy inequality* (1.37) *and*

$$\begin{aligned}&\int_{\Omega_T}[-v^*\cdot\psi_t-(v^*\cdot\overline{\nabla\psi})\cdot v^*+\alpha_4\overline{D^*}:\overline{\nabla\psi}-2L_1s_+^2\nabla d^*\odot\nabla d^*:\underline{\nabla\psi}]dxdt\\&+\int_{\Omega_T}[\alpha_1(d^*\otimes d^*:\overline{D^*})d^*\otimes d^*+\alpha_2\overline{N^*}\otimes d^*+\alpha_3d^*\otimes\overline{N^*}+\alpha_5\overline{D^*}\cdot(d^*\otimes d^*)]:\overline{\nabla\psi}dxdt\\&+\int_{\Omega_T}\alpha_6(d^*\otimes d^*)\cdot\overline{D^*}:\overline{\nabla\psi}dxdt=\int_\Omega v_0^*(x)\cdot\psi(0,x)dx,\end{aligned}\tag{1.40}$$

$$\begin{aligned}&\int_{\Omega_T}\{\gamma_1[-d^*\cdot\zeta_t-(\underline{v^*}\cdot\nabla\zeta)\cdot d^*-(\overline{\Lambda^*}\cdot d^*)\cdot\zeta]+\gamma_2(\overline{D^*}\cdot d^*-d^*\otimes d^*:\overline{D^*}d^*)\cdot\zeta\}dxdt\\&+2L_1s_+^2\int_{\Omega_T}(\partial_kd^*\cdot\partial_k\zeta-|\nabla d^*|^2d^*\cdot\zeta)dxdt=\gamma_1\int_\Omega d_0^*(x)\cdot\zeta(0,x)dx\end{aligned}\tag{1.41}$$

for every $\psi\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3),\partial_1\psi_1+\partial_2\psi_2=0,\zeta\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3)$, *where* $\overline{D^*}$, $\overline{\Lambda^*}$, $\underline{v^*}$, $\overline{N^*}$ *are given in* (1.35), *and* $\gamma_1,\gamma_2,\alpha_1,\cdots,\alpha_6$ *are given in* (1.29)-(1.32).

Then, the main results in this paper can be stated as follows.

Theorem 1.5. *Assume that* ξ *is suitably small, the conditions* (1.7), (1.9) *and* (1.16) *hold, and the parameters* a,b, *and* c *satisfy*

$$b>0,\quad b^2+27ac>0.\tag{1.42}$$

Let (v^ϵ,Q^ϵ) *be weak solutions to the system* (1.10) *subject to the initial data* (1.14), *and* $v^\epsilon\in L_t^\infty H_x^1\cap L_t^2H_x^2,Q^\epsilon-Q^\infty\in L_t^\infty H_x^2\cap L_t^2H_x^3$. *Then, there exists a convergent subsequence of* $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$, *such that*

$$v^\epsilon\rightharpoonup v^*\ in\ L^2(0,T;\mathring{J}),\quad v^\epsilon\overset{\star}{\rightharpoonup}v^*\ in\ L^\infty(0,T;\mathring{H}),\quad\nabla Q^\epsilon\overset{\star}{\rightharpoonup}\nabla Q^*\ in\ L^\infty(0,T;L^2(\mathbb{R}^2,\mathcal{Q}_0)),$$

as $\epsilon\to0^+$. *Furthermore,* Q^* *has the form*

$$Q^*=s_+(d^*\otimes d^*-\frac{1}{3}\mathbb{I}),\quad d^*\in\mathbb{S}^2,\quad s_+=\frac{b+\sqrt{b^2+24ac}}{4c},$$

and (v^*, d^*) *is a weak solution to the Ericksen-Leslie system* (1.34) *subject to the initial data* (1.36) *with the coefficients satisfying* (1.29)-(1.32).

Remark 1.6. *In the proof of Theorem 1.5, we need only the conditions* $b > 0$ *and* $b^2+24ac > 0$. *However, to make sure that* $\min_{Q\in\mathcal{Q}_0} F_b(Q)$ *is achieved at* $Q = s_+(d\otimes d - \frac{1}{3}\mathbb{I}), d \in \mathbb{S}^2$, *one needs the second condition in (1.42). Indeed, as shown in* [36, 35], *if* $F_b(Q_m) = \min_{Q\in\mathcal{Q}_0} F_b(Q)$, *then* Q_m *must be uniaxial, i.e.* $Q_m = s(d\otimes d - \frac{1}{3}\mathbb{I}), d \in \mathbb{S}^2$. *Hence,* $F_b(Q_m)$ *can be rewritten as*

$$F_b(Q_m) = -\frac{a}{3}s^2 - \frac{2b}{27}s^3 + \frac{4c}{9}s^4 := f_b(s),$$

whose critical points are

$$s_0 = 0, \quad s_+ = \frac{b+\sqrt{b^2+24ac}}{4c}, \quad s_- = \frac{b-\sqrt{b^2+24ac}}{4c}.$$

Therefore, $\min_{Q\in\mathcal{Q}_0} F_b(Q) = \min\{f_b(s_0), f_b(s_+), f_b(s_-)\}$. *Note that* $f_b(s_\pm) = \frac{s_\pm^2}{54}(-9a - bs_\pm)$ *or* $f_b(s_\pm) = \frac{s_\pm^3}{9}(\frac{b}{3} - cs_\pm)$ *due to* $-3a - bs_\pm + 2cs_\pm^2 = 0$. *Then,*

- *if* $b > 0$ *and* $a > 0$, *one has* $f_b(s_+) < f_b(s_-) < f_b(0)$,
- *if* $b > 0$ *and* $0 \geq a > -\frac{b^2}{27c}$, *one has* $f_b(s_+) < f_b(0) \leq f_b(s_-)$,
- *if* $b > 0$ *and* $-\frac{b^2}{27c} \geq a \geq -\frac{b^2}{24c}$, *one has* $f_b(s_+) \geq f_b(0), f_b(s_-) \geq f_b(0)$.

These mean

$$\min_{Q\in\mathcal{Q}_0} F_b(Q) = f_b(s_+), \text{ when } b > 0 \text{ and } b^2 + 27ac > 0.$$

Remark 1.7. *It follows from direct calculations that*

$$\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1} = -\frac{8\Gamma\xi^2(1-s_+)^2}{9} \leq 0$$

due to (1.29)-(1.32). *This,* (1.25) *and* (1.26) *imply that only some special cases of Ericksen-Leslie systems can be derived from the Beris-Edwards system. The weak solutions obtained in* [28, 14, 16, 17, 42] *have at most finite number singular times and are smooth away from the singular times, furthermore, the values of the singular times can be uniquely redefined by the weak-*L^2 *limit through the energy inequalities. Therefore, the uniqueness of the weak solution between the nearest two singular times implies the uniqueness of the global weak solutions, and this kind of results are proved in* [31, 24]. *In this paper, different from* [28, 14, 16, 17, 42] *where the global weak solution is defined by extending the local strong solution to the Ericksen-Leslie system, we obtain a global-in-time solution to the Ericksen-Leslie system as a limit of the global-in-time solutions to the Beris-Edwards system, and the regularity and uniqueness of such solution are not clear.*

We now make some comments on the main ideas of the proof of Theorem 1.5. As mentioned in Remark 1.2, the asymptotic convergence of solutions to the Beris-Edwards system (1.3)-(1.5) to the regular solutions to the Ericksen-Leslie system (1.18)-(1.20) with the the coefficients satisfying (1.29)-(1.32) has been proved in [45]. However, the analysis in [45] is based on the Hilbert expansion, which depends crucially on the high order differentiability of the limiting solutions to the Ericksen-Leslie system and thus cannot be applied to the case that the solutions to the Ericksen-Leslie system have singularities whose existence had been confirmed in [18, 22]. Here we will establish the asymptotic convergence of these two systems as $\epsilon \to 0^+$ for weak solutions by analysing the a priori energy inequality (1.15) for (v^ϵ, Q^ϵ) and showing that the weak-limit (v^*, Q^*) of (v^ϵ, Q^ϵ) solves the Ericksen-Leslie system (1.34) subject to the initial data (1.36) and satisfies the energy inequalities (1.37). This approach is strongly motivated by studies in [32, 20, 7] where the weak solutions to the simplified Ericksen-Leslie system are obtained as weak limits of solutions to the Ginzburg-Landau approximation system by weak convergence methods. Here we outline some major elements of the proof of Theorem 1.5. First, the existence of the weak *-limit, (v^*, Q^*), of the (v^ϵ, Q^ϵ) is guaranteed by the basic energy inequality (1.15) and Aubin-Lious Lemma by a standard argument. The key step of the analysis is passing this weak limit into nonlinear terms in the systems. Due to the super-critical nonlinear term $\sum_{l=1}^2 \partial_l(\nabla d \odot \nabla d)_{il}, i = 1, 2$ in the Ericksen-Leslie system (see (1.34)), it turns out that the most difficult part of the proof of Theorem 1.5 is to show that there exists a subsequence of $\{Q^\epsilon\}_{\epsilon>0}$ such that

$$\int_0^T \int_{\mathbb{R}^2} \nabla Q^\epsilon(t,x) \odot \nabla Q^\epsilon(t,x) : \underline{\nabla\psi}(t,x)dxdt \to \int_0^T \int_{\mathbb{R}^2} \nabla Q^*(t,x) \odot \nabla Q^*(t,x) : \underline{\nabla\psi}(t,x)dxdt \tag{1.43}$$

as $\epsilon \to 0^+$, for each $\psi \in C_0^\infty([0,T) \times \mathbb{R}^2, \mathbb{R}^3)$, $\partial_1\psi_1 + \partial_2\psi_2 = 0$. As in [32, 20, 7], one can define a good time t such that

$$\liminf_{\epsilon\to 0^+} \int_{\mathbb{R}^2} |H^\epsilon|^2(t,x)dx < \infty. \tag{1.44}$$

By the energy inequality (1.15), (1.43) is satisfied as long as

$$\int_{\mathbb{R}^2} \nabla Q^\epsilon(t,x) \odot \nabla Q^\epsilon(t,x) : \underline{\nabla\psi}(t,x)dx \to \int_{\mathbb{R}^2} \nabla Q^*(t,x) \odot \nabla Q^*(t,x) : \underline{\nabla\psi}(t,x)dx \tag{1.45}$$

as $\epsilon \to 0^+$, for each good time and each $\psi \in C_0^\infty([0,T) \times \mathbb{R}^2, \mathbb{R}^3), \partial_1\psi_1 + \partial_2\psi_2 = 0$. To prove (1.45), we can establish the following important claim:

Claim 1.1 (the strong convergence under samll energy condition)**.** *At a good time, the local strong H^1 convergence of Q^ϵ can be obtained if the local total energy is suitably small (see Lemma* 3.1*).*

The proof of this claim is the crucial step in the proof of Theorem 1.5 and the most technical part in this paper. Once the Claim 1.1 is established, we can prove easily the convergence (1.45) by modifying the analysis in [20, 7]. Indeed, the Claim 1.1 implies that $\nabla Q^\epsilon(t,\cdot) \odot \nabla Q^\epsilon(t,\cdot)$ may concentrate on only at a finite number of points at good times. Based on this fact and

$$\nabla Q^\epsilon \odot \nabla Q^\epsilon : \underline{\nabla\psi} = \begin{pmatrix} \frac{1}{2}(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) & \partial_1 Q^\epsilon : \partial_2 Q^\epsilon \\ \partial_1 Q^\epsilon : \partial_2 Q^\epsilon & -\frac{1}{2}(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) \end{pmatrix} : \underline{\nabla\psi},$$

one can rule out the potential isolated concentrate points of $\nabla Q^\epsilon(t,\cdot) \odot \nabla Q^\epsilon(t,\cdot)$ by studying the convergence of $|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2$ and $\partial_1 Q^\epsilon : \partial_2 Q^\epsilon$ through a Pohozaev type argument as in [7] where the method was used to study the compensated compactness property of solutions to the Ginzburg-Landau approximate equations of the simplified Ericksen-Leslie equations for both uniaxial and biaxial nematics. It should be noted that the convergence (1.45) can also be proved by combining the Claim 1.1 here with the concentration-cancellation method in [20] which was developed by DiPerna and Majda [5] for the incompressible Euler equations. Note also that the method in [32] to rule out the potential concentrate points of $\nabla d^\epsilon \odot \nabla d^\epsilon$ for the Ginzburg-Landau approximate solutions cannot be used here since it depends crucially on $d^\epsilon \in \mathbb{S}^2_+$ which implies $d^* \in \mathbb{S}^2_+$ and that the Liouville theorem of harmonic maps holds.

We now make some comments on the proof of the Claim 1.1 above (for more details, see the proof of Lemma 3.1). Note first that though the corresponding results on the strong convergence under small energy conditions have been proved for the Ginzburg-Landau approximate solutions in [20, 7], yet the analysis in [20, 7] depends crucially on the geometric structure of the Ginzburg-Landau approximation, such as in the case of (1.28) for uniaxial nematics [20], it holds that

$$f(d^\epsilon) - \left(f(d^\epsilon) \cdot \frac{d^\epsilon}{|d^\epsilon|} \right) \cdot \frac{d^\epsilon}{|d^\epsilon|} = 0 \tag{1.46}$$

with $f(d^\epsilon) = (|d^\epsilon|^2 - 1)d^\epsilon$. Indeed, one of the key observations in [20] is that (1.46) implies that the phase function $\psi^\epsilon = \frac{d^\epsilon}{|d^\epsilon|}$ satisfies the following quasi-linear elliptic equation

$$\Delta \psi^\epsilon = -|\nabla \psi^\epsilon|^2 \psi^\epsilon - \frac{2}{|d^\epsilon|} \partial_k \psi^\epsilon \partial_k |d^\epsilon| + \frac{1}{|d^\epsilon|} (\tau^\epsilon - (\tau^\epsilon \cdot \psi^\epsilon) \psi^\epsilon) \tag{1.47}$$

with $\tau^\epsilon = d^\epsilon_t + v^\epsilon \cdot \nabla d^\epsilon$. Note that the right hand side of (1.47) contains no terms of order ϵ^{-1}, so by the classic theory of elliptic equations, one can obtain the ϵ-independent uniform bound of $||\nabla^2 \psi^\epsilon||_{L^{\frac{4}{3}}}$ provided that $||\nabla d^\epsilon||_{L^2}$ is suitably small. This implies the local strong H^1 convergence of d^ϵ. For the case of the Ginzburg-Landau approximate equations of the Ericksen-Leslie equations for both uniaxial and biaxial nematics, similar quasi-linear elliptic equations as (1.47) were obtained in [7] by using the geometric structure as (1.46), see the equation (2.9) in [7], which yields the local strong H^1 convergence under small energy conditions. Unfortunately, this elegant argument cannot be applied easily to the solutions (v^ϵ, Q^ϵ) to the Beris-Edwards system due to the structure of the bulk energy density. To see this, one sets

$$Q^\epsilon = e^\epsilon \phi^\epsilon, \quad e^\epsilon = |Q^\epsilon|, \quad \phi^\epsilon = \frac{Q^\epsilon}{|Q^\epsilon|}.$$

It then follows from (1.12) that

$$\Delta \phi^\epsilon = -|\nabla \phi^\epsilon|^2 \phi^\epsilon - \frac{2}{e^\epsilon} \partial_k e^\epsilon \partial_k \phi^\epsilon + \frac{1}{L_1 e^\epsilon} [H^\epsilon - (H^\epsilon : \phi^\epsilon) \phi^\epsilon] + \frac{1}{\epsilon L_1 e^\epsilon} [\mathcal{J}(Q^\epsilon) - (\mathcal{J}(Q^\epsilon) : \phi^\epsilon) \phi^\epsilon] . \tag{1.48}$$

However, it can be checked that

$$\mathcal{J}(Q^\epsilon) - (\mathcal{J}(Q^\epsilon) : \phi^\epsilon) \phi^\epsilon \neq 0, \tag{1.49}$$

and so the right hand side of (1.48) contains a term of order ϵ^{-1}, which makes it difficult to use the approach in [20, 7] to obtain the local uniform estimate of $||\nabla^2 \phi^\epsilon||_{L^{\frac{4}{3}}}$ even for small energy. Thus new ideas and techniques are needed to establish the strong convergence under small energy conditions for solutions to the Beris-Edwards system. We will prove this by making use of both the geometric structure of Q^ϵ and the bulk energy density for the Beris-Edwards system and some algebraic properties of Q^ϵ (see **Step 3** in the proof of Lemma 3.1). The main steps and ideas are sketched as follows.

- **Step 1 (L^∞ estimate)** The aim is to show that there exists a suitably small constant $\delta_0 > 0$ with the corresponding r_0 such that if

$$\int_{B_{4r_0}(x_0)} \left(|\nabla Q^\epsilon|^2 + \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right) dx < \delta_0, \tag{1.50}$$

then,

$$\text{dist}(Q^\epsilon(x), \mathcal{N}) < \delta_0^{\frac{1}{8}}, x \in B_{3r_0}(x_0). \tag{1.51}$$

Since there is no maximum principle for the system (1.5) due to $\xi \neq 0$, it is difficult to get the uniform bound for $||Q^\epsilon||_{L^\infty}$. To overcome this difficulty, we decompose $B_{3r_0}(x_0)$ as the disjoint union of $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$ defined as:

$$\mathcal{C}_1^\epsilon = \{x \in B_{3r_0}(x_0) : \mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta_0^{\frac{1}{10}}\}, \quad \mathcal{C}_2^\epsilon = B_{3r_0(x_0)} \setminus \mathcal{C}_1^\epsilon.$$

Let $\delta^* > 0$ be the geometric constant depending on the bulk energy density to be given in Lemma 2.2, and choose $\delta_0 > 0$ so that $\delta_0^{\frac{1}{10}} < \frac{1}{4}\delta^*$. It then follows from the continuity of Q^ϵ that for each $y \in \mathcal{C}_1^\epsilon$, there exists $r_y^\epsilon > 0$ such that $\mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta^*$ for $x \in B_{r_y^\epsilon}(y)$, which implies $|\mathcal{J}(Q^\epsilon)|^2 \le C\hat{F}_b(Q^\epsilon)$ on a neighbourhood of $\mathcal{C}_1^\epsilon$ by Lemma 2.2. Using this, (1.12), and (1.50), one can get by a proper scaling and elliptic estimates that

$$|Q^\epsilon(x) - Q^\epsilon(y)| \le C_1 \left[\left(\frac{r_{\bar{y}}^\epsilon}{\sqrt{\epsilon}}\right)^2 + 1\right]\left(\frac{|x-y|}{\sqrt{\epsilon}}\right)^{\frac{1}{2}} \text{ for } x, y \in B_{r_{\bar{y}}^\epsilon}(\bar{y})$$

with $\bar{y} \in \mathcal{C}_1^\epsilon$, furthermore, $r_{\bar{y}}^\epsilon \ge C_2(\delta^*)^2\sqrt{\epsilon}$, where C_2 is independent of ϵ. This implies that $\hat{F}_b(Q^\epsilon)$ cannot decay too fast on a neighbourhood of $\mathcal{C}_1^\epsilon$. Then choosing $r_{\bar{y}}^\epsilon = C_2(\delta^*)^2\sqrt{\epsilon}$, one can show by contradiction that $\mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta_0^{\frac{1}{8}}$ for $x \in \mathcal{C}_1^\epsilon$ and suitably small δ_0. This yields that $\mathcal{C}_2^\epsilon = \emptyset$ so (1.51) holds.

- **Step 2** We show that there exists $r_3^\epsilon \in (r_0, 3r_0)$ such that

$$\left|\int_{\partial B_{r_3^\epsilon}(x_0)} \frac{1}{\epsilon}\frac{\partial \hat{F}_b(Q^\epsilon)}{\partial \nu} dS\right| \le C_0, \tag{1.52}$$

where C_0 is independent of ϵ and ν is the radial direction. Note that

$$\frac{1}{\epsilon}\frac{\partial \hat{F}_b(Q^\epsilon)}{\partial \nu} = \frac{\mathcal{J}(Q^\epsilon)}{\epsilon} : (\frac{x-x_0}{|x-x_0|} \cdot \nabla Q^\epsilon), \tag{1.53}$$

$$\begin{aligned}\Delta Q^\epsilon : (\frac{x-x_0}{|x-x_0|} \cdot \nabla Q^\epsilon) &= \partial_k\left[\partial_k Q^\epsilon : (\frac{x-x_0}{|x-x_0|} \cdot \nabla Q^\epsilon) - \frac{1}{2}|\nabla Q^\epsilon|^2 \frac{(x-x_0)_k}{|x-x_0|}\right] \\ &-Q_{ij,k}^\epsilon Q_{ij,l}^\epsilon \partial_k\left(\frac{(x-x_0)_l}{|x-x_0|}\right) + \frac{1}{2}|\nabla Q^\epsilon|^2 \nabla \cdot \left(\frac{x-x_0}{|x-x_0|}\right).\end{aligned} \tag{1.54}$$

It follows from (1.50) that there exist $r_1^\epsilon \in (r_0, \frac{3}{2}r_0)$ and $r_2^\epsilon \in (2r_0, 3r_0)$ such that

$$||\nabla Q^\epsilon||^2_{L^2(\partial B_{r_1^\epsilon}(x_0))} < \frac{8\delta_0}{r_0}, \quad ||\nabla Q^\epsilon||^2_{L^2(\partial B_{r_2^\epsilon}(x_0))} < \frac{8\delta_0}{r_0}. \tag{1.55}$$

Due to (1.12), it holds that

$$\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)} (L_1 \Delta Q^\epsilon - \frac{\mathcal{J}(Q^\epsilon)}{\epsilon} - H^\epsilon) : (\frac{x-x_0}{|x-x_0|} \cdot \nabla Q^\epsilon) dx = 0. \tag{1.56}$$

Using (1.53)-(1.55), (1.44) and (1.50), one can derive from (1.56) that

$$\left|\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)} \frac{x-x_0}{|x-x_0|} \cdot \nabla \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx\right| \le C_0,$$

which yields the desired (1.52) immediately.

- **Step 3** Observe that (1.12) implies

$$\int_{B_{r_3^\epsilon}(x_0)}\left(L_1^2|\Delta Q^\epsilon|^2+\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2\right)$$
$$= -\int_{B_{r_3^\epsilon}(x_0)}\frac{2L_1}{\epsilon}\partial_k Q^\epsilon:\partial_k\mathcal{J}(Q)+\int_{B_{r_3^\epsilon}(x_0)}|H^\epsilon|^2-\int_{\partial B_{r_3^\epsilon}(x_0)}\frac{2L_1}{\epsilon}\frac{\partial\hat{F}_b(Q^\epsilon)}{\partial\nu}. \quad (1.57)$$

The last two terms on the right hand side of (1.57) have been estimated by **Step 2** and (1.44). The most difficult task is to estimate the first integral on the right hand side of (1.57). Observe that

$$\frac{1}{\epsilon}\partial_k Q^\epsilon:\partial_k\mathcal{J}(Q) = \frac{1}{\epsilon}\left[-b\lambda_1^\epsilon|\nabla Q^\epsilon|^2-b\partial_k(Q^\epsilon)^2:\partial_k Q^\epsilon+\frac{c}{2}|\nabla|Q^\epsilon|^2|^2\right]$$
$$+\frac{1}{\epsilon}\left(-a+b\lambda_1^\epsilon+c|Q^\epsilon|^2\right)|\nabla Q^\epsilon|^2 := J_1+J_2, \quad (1.58)$$

where λ_1^ϵ is the minimum eigenvalue of Q^ϵ. One of the key facts is that $J_1\geq 0$ provided that $b>0$, $b^2+24ac>0$ and $\mathrm{dist}(Q^\epsilon,\mathcal{N})$ is suitably small. This can be proved by very careful and delicate calculations based on the algebraic structure of Q^ϵ and the properties established in **Step 1** (for details, see **Step 3** in the proof of Lemma 3.1).

- **step 4** It remains to estimate the integral J_2. To this end, by using some algebraic properties of Q^ϵ, (1.51) in **Step 1**, and the geometric structure of $\mathcal{J}$ (Lemma 2.2), one can obtain easily that $|-a+b\lambda_1^\epsilon+c|Q^\epsilon|^2|\leq C|\mathcal{J}(Q^\epsilon)|$ with $\mathrm{dist}(Q^\epsilon,\mathcal{N})$ being suitably small. It then follows that

$$\int_{B_{r_3^\epsilon}}\frac{1}{\epsilon}\left(-a+b\lambda_1^\epsilon+c|Q^\epsilon|^2\right)|\nabla Q^\epsilon|^2$$
$$\leq \int_{B_{r_3^\epsilon}}\frac{1}{2}\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2+C_0\int_{B_{r_3^\epsilon}}|\nabla Q^\epsilon|^2\int_{B_{r_3^\epsilon}}(|\Delta Q^\epsilon|^2+|\nabla Q^\epsilon|^2), \quad (1.59)$$

where Hölder's and Ladyzhenskaya's inequalities have been used. Consequently, we can obtain the uniform estimate of $||\Delta Q^\epsilon||_{L^2(B_{r_3^\epsilon})}$ for small δ_0 by collecting (1.57)-(1.59), which yields the main part of the proof of the Claim 1.1.

The rest of this paper is organized as follows: In section 2, some properties of the Q-tensor and bulk energy density are discussed; In section 3, we proved the strong convergence under small energy condition at good times; Section 4 is devoted to the proof of the Theorem 1.5.

2. Properties of the Q-tensor and bulk energy density

2.1. **Properties of the Q-tensor.** For a matrix $Q\in\mathcal{N}$, $T_Q\mathcal{N}$ denotes the tangent space to $\mathcal{N}$ at Q in $\mathcal{Q}_0$, $(T_Q\mathcal{N})^\perp_{\mathcal{Q}_0}$ denotes the orthogonal complement of $T_Q\mathcal{N}$ in $\mathcal{Q}_0$, and $\mathcal{P}_\mathcal{N}$ denotes the projection operator on $\mathcal{N}$. We list some important geometric properties of $\mathcal{J}(Q^\epsilon)$(see (1.13) for the definition), which will be used later.

Lemma 2.1. [43, Lemma 2.2, Lemma 2.3] *Let $Q=s_+(d_3\otimes d_3-\frac{1}{3}\mathbb{I})\in\mathcal{N}$, and d_1,d_2 be unit perpendicular vectors in $V_{d_3}=\{d^\perp\in\mathbb{S}^2: d^\perp\cdot d_3=0\}$. Then, it holds that*

(1)
$$T_Q\mathcal{N} = Span\left\{\frac{1}{\sqrt{2}}(d_3\otimes d_2 + d_2\otimes d_3), \frac{1}{\sqrt{2}}(d_3\otimes d_1 + d_1\otimes d_3)\right\},$$
(2)
$$\begin{aligned}(T_Q\mathcal{N})^{\perp}_{\mathcal{Q}_0} \quad &= \quad Span\left\{\frac{1}{\sqrt{2}}(d_2\otimes d_1 + d_1\otimes d_2), \frac{1}{\sqrt{2}}(d_1\otimes d_1 - d_2\otimes d_2),\right.\\ &\left.\sqrt{6}(\frac{1}{2}d_1\otimes d_1 + \frac{1}{2}d_2\otimes d_2 - \frac{1}{3}\mathbb{I})\right\},\end{aligned}$$
(3) $\mathcal{Q}_0 = T_Q\mathcal{N}\oplus(T_Q\mathcal{N})^{\perp}_{\mathcal{Q}_0}$,
(4) *For* $Q\in\mathcal{Q}_0$, *there exists* $\delta^*>0$ *such that if dist* $(Q,\mathcal{N})<\delta^*$, *then*
$$\mathcal{J}(Q)\in(T_{\mathcal{P}_{\mathcal{N}}(Q)}(\mathcal{N}))^{\perp}_{\mathcal{Q}_0}.$$

2.2. **The equivalence of bulk energy density.** To estimate the term $\mathcal{J}(Q^\epsilon)$, one needs also the following equivalence of the bulk energy.

Lemma 2.2. [34, 43] *There exists* $\delta^*>0$ *such that if dist*$(Q,\mathcal{N})<\delta^*$, *then*
$$\frac{1}{C}dist(Q,\mathcal{N})^2 \le \hat{F}_b(Q)\le C\,dist(Q,\mathcal{N})^2, \tag{2.1}$$
$$\frac{1}{C}\hat{F}_b(Q)\le|\mathcal{J}(Q)|^2\le C\hat{F}_b(Q), \tag{2.2}$$
where C *depends on* a,b *and* c, *but independent of* Q, *and*
$$dist(Q,\mathcal{N}) = |Q-\mathcal{P}_{\mathcal{N}}(Q)| = \min_{A\in\mathcal{N}}\{|Q-A|\}.$$

3. The strong convergence under small energy condition

As discussed in the introduction, in this section, we establish the strong convergence in H^1 under the small energy condition for solutions Q^ϵ to (1.5), which is crucial to estimate the set of potential concentration points of $\nabla Q^\epsilon\odot\nabla Q^\epsilon$.

Lemma 3.1. *For* $x_0\in\mathbb{R}^2, r_0>0$ *and* $B_{4r_0}(x_0)\subset\mathbb{R}^2$, *let*
$$L_1\Delta Q^\epsilon - \frac{\mathcal{J}(Q^\epsilon)}{\epsilon} = H^\epsilon \quad in\ B_{4r_0}(x_0), \tag{3.1}$$
and $Q^\epsilon\in H^3(B_{4r_0}(x_0))$. *Assume that*

(I): *there exists small* $\delta_0>0$ *such that*
$$\int_{B_{4r_0}(x_0)}\left(|\nabla Q^\epsilon|^2 + \frac{\hat{F}_b(Q^\epsilon)}{\epsilon}\right)dx<\delta_0,$$
(II): $||H^\epsilon||_{L^2(B_{4r_0}(x_0))}<C_0$,
(III): $b>0$, $b^2+24ac>0$,

where δ_0 *and* C_0 *are independent of* ϵ. *Then there exists a subsequence of* $\{Q^\epsilon\}_{\epsilon>0}$ *such that*
$$Q^\epsilon\to Q^*,\ in\ H^1(B_{r_0}(x_0)),\quad \frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\rightharpoonup\mathcal{J}^*,\ in\ L^2(B_{r_0}(x_0)),\quad \int_{B_{r_0}(x_0)}\frac{\hat{F}_b(Q^\epsilon)}{\epsilon}dx\to 0,$$

as $\epsilon \to 0$, where $Q^* = s_+(d^* \otimes d^* - \frac{1}{3}\mathbb{I})$, $d^* \in \mathbb{S}^2$ satisfies

$$L_1 \Delta Q^* - \mathcal{J}^* = H^*$$

in the weak sense, with $H^\epsilon \rightharpoonup H^*$ in $L^2(B_{r_0}(x_0))$ and

$$\mathcal{J}^* \in (T_{Q^*}\mathcal{N})^{\perp}_{\mathcal{Q}_0} \text{ a.e. in } B_{r_0}(x_0).$$

Proof. This will be proved by the following four steps.

Step 1. Claim: $\mathrm{dist}(Q^\epsilon, \mathcal{N}) < \delta_0^{\frac{1}{8}}$, for all $x \in B_{3r_0}(x_0)$. To prove this claim, we decompose $B_{3r_0}(x_0)$ as a disjoint union of $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$ defined as

$$\mathcal{C}_1^\epsilon = \{x \in B_{3r_0}(x_0) : \mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta_0^{\frac{1}{10}}\}, \quad \mathcal{C}_2^\epsilon = B_{3r_0}(x_0) \setminus \mathcal{C}_1^\epsilon.$$

Since $\hat{F}_b(Q) = 0$ if and only if $Q \in \mathcal{N}$, one has

$$\hat{F}_b(Q^\epsilon(x)) \geq C_* \delta_0^{\frac{1}{5}} \text{ for all } x \in \mathcal{C}_2^\epsilon$$

with some $C_* > 0$. This and the condition **(I)** imply that

$$|\mathcal{C}_2^\epsilon| \leq \frac{\epsilon}{C_*} \delta_0^{\frac{4}{5}}.$$

Therefore, $\mathcal{C}_1^\epsilon$ is not empty when ϵ is sufficiently small. Let δ^* be the geometric quantity given in Lemma 2.2 and choose δ_0 suitably small so that $\delta_0^{\frac{1}{10}} < \frac{1}{4}\delta^*$. Then it follows from the continuity of Q^ϵ that for each $y \in \mathcal{C}_1^\epsilon$, there exists $r_y^\epsilon > 0$ such that

$$\mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta^* \text{ for } x \in B_{r_y^\epsilon}(y). \tag{3.2}$$

For fixed $\bar{y} \in \mathcal{C}_1^\epsilon$, $0 < \sqrt{\epsilon} < r_0$, the rescaled quantity $\hat{Q}^\epsilon(x) = Q^\epsilon(\bar{y} + \sqrt{\epsilon}x)$ satisfies

$$L_1 \Delta \hat{Q}^\epsilon - \mathcal{J}(\hat{Q}^\epsilon) = \hat{H}^\epsilon \text{ in } B_{r_{\bar{y}}^\epsilon/\sqrt{\epsilon}}(\mathbf{0}),$$

where $\hat{H}^\epsilon(x) = \epsilon H(\bar{y} + \sqrt{\epsilon}x)$ and $r_{\bar{y}}^\epsilon$ is given in (3.2). It follows from (II), (I), (3.2), and the structure of $\mathcal{N}$ that

$$\int_{B_{r_{\bar{y}}^\epsilon/\sqrt{\epsilon}}(\mathbf{0})} |\hat{H}^\epsilon|^2 dy = \epsilon \int_{B_{r_{\bar{y}}^\epsilon}(\bar{y})} |H^\epsilon|^2 dx \leq \epsilon C_0$$

and

$$\int_{B_{r_{\bar{y}}^\epsilon/\sqrt{\epsilon}}(\mathbf{0})} |\hat{Q}^\epsilon|^2 dy \leq \pi \left(\frac{r_{\bar{y}}^\epsilon}{\sqrt{\epsilon}}\right)^2 \left(\sqrt{\frac{2}{3}} s_+ + \delta^*\right)^2.$$

Meanwhile, Lemma 2.2 and condition **(I)** imply that

$$\int_{B_{r_{\bar{y}}^\epsilon/\sqrt{\epsilon}}(\mathbf{0})} |\mathcal{J}(\hat{Q}^\epsilon)|^2 dy = \int_{B_{r_{\bar{y}}^\epsilon}(\bar{y})} \frac{|\mathcal{J}(Q^\epsilon)|^2}{\epsilon} dx \leq C \int_{B_{r_{\bar{y}}^\epsilon}(\bar{y})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx \leq C\delta_0.$$

Then, these and the classic elliptic theory [11, Theorem 9.9] yield

$$||\hat{Q}^\epsilon||_{H^2(B_{r_{\bar{y}}^\epsilon/\sqrt{\epsilon}}(\mathbf{0}))} \leq C\left[\left(\frac{r_{\bar{y}}^\epsilon}{\sqrt{\epsilon}}\right)^2 + 1\right],$$

where C is independent of ϵ. Therefore, one obtains that

$$|Q^\epsilon(x) - Q^\epsilon(y)| \leq C_1 \left[\left(\frac{r_{\bar{y}}^\epsilon}{\sqrt{\epsilon}}\right)^2 + 1\right] \left(\frac{|x-y|}{\sqrt{\epsilon}}\right)^{\frac{1}{2}} \text{ in } B_{r_{\bar{y}}^\epsilon}(\bar{y}) \tag{3.3}$$

by the embedding theorem.

Next we show that $r^\epsilon_{\bar y} \ge C_2(\delta^*)^2\sqrt{\epsilon}$ for some $C_2 > 0$. If $r^\epsilon_{\bar y} \le (\delta^*)^2\sqrt{\epsilon}$ and $r^\epsilon_{\bar y} \le \frac{1}{2}r_0$, one can find a $x^* \in B_{4r_0}(x_0) \cap B_{r^\epsilon_{\bar y}}(\bar y)$ such that $\mathrm{dist}(Q^\epsilon(x^*), \mathcal{N}) \ge \frac{3}{4}\delta^*$. Then, taking $y = \bar y$ and $x = x^*$ in the estimate (3.3) leads to

$$\frac{\delta^*}{2} \le |Q^\epsilon(x^*) - Q^\epsilon(\bar y)| \le 2C_1\left(\frac{|x^* - \bar y|}{\sqrt{\epsilon}}\right)^{\frac{1}{2}} \le 2C_1\left(\frac{r^\epsilon_{\bar y}}{\sqrt{\epsilon}}\right)^{\frac{1}{2}},$$

which yields

$$r^\epsilon_{\bar y} \ge \left(\frac{\delta^*}{4C_1}\right)^2\sqrt{\epsilon}.$$

Therefore, setting

$$C_2 = \min\left\{1, \frac{1}{16C_1^2}, \frac{r_0}{2(\delta^*)^2}\right\},$$

one gets that $r^\epsilon_{\bar y} \ge C_2(\delta^*)^2\sqrt{\epsilon}$.

We are now ready to prove that $\mathrm{dist}(Q^\epsilon(x), \mathcal{N}) < \delta_0^{\frac{1}{8}}$ for $x \in \mathcal{C}^\epsilon_1$. If not, there exists $x_1 \in \mathcal{C}^\epsilon_1$ with $\mathrm{dist}(Q^\epsilon(x_1), \mathcal{N}) \ge \delta_0^{\frac{1}{8}}$. Then, using Lemma 2.2 and the estimate (3.3) with $r^\epsilon_{x_1} = C_2(\delta^*)^2\sqrt{\epsilon}$($r^\epsilon_{x_1}$ is given in (3.2)), one can get that

$$\hat F_b(Q^\epsilon) \ge C\mathrm{dist}(Q, \mathcal{N})^2 \ge C\frac{1}{4}\delta_0^{\frac{1}{4}} = C_3\delta_0^{\frac{1}{4}} \text{ in } B_{\delta_\epsilon}(x_1), \text{ and } \delta_\epsilon = \frac{\delta_0^{\frac{1}{4}}\sqrt{\epsilon}}{4C_1^2[C_2^2(\delta^*)^4 + 1]^2}.$$

Note that $\delta_\epsilon < r^\epsilon_{x_1}$ for suitably small δ_0. Then,

$$\int_{B_{\delta_\epsilon}(x_1)} \frac{\hat F_b(Q^\epsilon)}{\epsilon}dx \ge \pi\frac{\delta_0^{\frac{1}{2}}}{16C_1^4[C_2^2(\delta^*)^4 + 1]^4}C_3\delta_0^{\frac{1}{4}} = C_+\delta_0^{\frac{3}{4}},$$

which contradicts the assumption that

$$\int_{B_{4r_0}(x_0)} \frac{\hat F_b(Q^\epsilon)}{\epsilon} \le \delta_0$$

for a sufficiently small $\delta_0 > 0$(for example $\delta_0 < C_+^4$). Thus one has shown that

$$\mathrm{dist}(Q^\epsilon(x), \mathcal{N}) \le \delta_0^{\frac{1}{8}} \text{ for } x \in C^\epsilon_1. \tag{3.4}$$

If C^ϵ_2 is not empty, by the definition of C^ϵ_1, then $\mathrm{dist}(Q^\epsilon(z), \mathcal{N}) = \delta_0^{\frac{1}{10}}$ for $z \in \partial\mathcal{C}^\epsilon_1$. This contradicts the estimate (3.4) when $\delta_0 < 1$. Consequently, $\mathcal{C}^\epsilon_2 = \emptyset$. Thus the desired claim holds.

Step 2. The goal is to show that there exists a $r^\epsilon_3 \in (r_0, 3r_0)$ such that

$$\left|\int_{\partial B_{r^\epsilon_3}(x_0)} \frac{1}{\epsilon}\frac{\partial \hat F_b(Q^\epsilon)}{\partial\nu}dS\right| \le C_0. \tag{3.5}$$

It follows from $||\nabla Q^\epsilon||^2_{L^2(B_{4r_0}(x_0))} \le \delta_0$ that for every $\epsilon > 0$, there exist $r^\epsilon_1 \in (r_0, \frac{3}{2}r_0)$ and $r^\epsilon_2 \in (2r_0, 3r_0)$ such that

$$||\nabla Q^\epsilon||^2_{L^2(\partial B_{r^\epsilon_1}(x_0))} < \frac{8\delta_0}{r_0}, \quad ||\nabla Q^\epsilon||^2_{L^2(\partial B_{r^\epsilon_2}(x_0))} < \frac{8\delta_0}{r_0}. \tag{3.6}$$

Multiplying (3.1) by $\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon$ and integrating over $B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)$, one gets

$$0=\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}(L_1\Delta Q^\epsilon-\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}-H^\epsilon):(\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon)dx=I_1+I_2+I_3. \quad (3.7)$$

Now we estimate I_1, I_2 and I_3 respectively as follows.

$$\begin{aligned} I_1 &= L_1\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}\Delta Q^\epsilon:(\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon)dx \\ &= -L_1\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}\left\{Q^\epsilon_{ij,k}Q^\epsilon_{ij,l}\partial_k\left(\frac{x_l-x_{0l}}{|x-x_0|}\right)-\frac{1}{2}|\nabla Q^\epsilon|^2\nabla\cdot\left(\frac{x-x_0}{|x-x_0|}\right)\right\}dx \\ &\quad +L_1\int_{\partial(B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0))}\left\{\frac{\partial Q^\epsilon}{\partial\nu}:(\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon)-\frac{1}{2}|\nabla Q^\epsilon|^2\frac{(x-x_0)\cdot\nu}{|x-x_0|}\right\}dS, \end{aligned}$$

where ν is the external normal vector of $B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)$. Therefore, the estimate (3.6) and condition **(I)** imply that

$$|I_1|\le C_0.$$

$$-I_2=\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}:(\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon)dx=\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}\frac{x-x_0}{|x-x_0|}\cdot\nabla\frac{\hat{F}_b(Q^\epsilon)}{\epsilon}dx.$$

$$|I_3|=\left|\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}H^\epsilon:(\frac{x-x_0}{|x-x_0|}\cdot\nabla Q^\epsilon)dx\right|\le||\nabla Q^\epsilon||_{L^2(B_{4r_0}(x_0))}||H^\epsilon||_{L^2(B_{4r_0}(x_0))}\le C_0.$$

Substituting the above three estimates into (3.7) yields

$$\left|\int_{B_{r_2^\epsilon}(x_0)\setminus B_{r_1^\epsilon}(x_0)}\frac{x-x_0}{|x-x_0|}\cdot\nabla\frac{\hat{F}_b(Q^\epsilon)}{\epsilon}dx\right|\le C_0.$$

Note that $Q^\epsilon\in H^3(B_{4r_0}(x_0))$. Then ∇Q^ϵ is continuous. Therefore, there exists $r_3^\epsilon\in(r_1^\epsilon,r_2^\epsilon)$ such that

$$\left|\int_{\partial B_{r_3^\epsilon}(x_0)}\frac{1}{\epsilon}\frac{\partial\hat{F}_b(Q^\epsilon)}{\partial\nu}dS\right|=\left|\int_{\partial B_{r_3^\epsilon}(x_0)}\frac{x-x_0}{|x-x_0|}\cdot\nabla\frac{\hat{F}_b(Q^\epsilon)}{\epsilon}dS\right|\le\frac{2C_0}{r_0}\le C_0,$$

which gives the desired estimate (3.5).

To obtain the strong convergence, we are going to derive the uniform estimate on $||\Delta Q^\epsilon||_{L^2(B_{r_3^\epsilon}(x_0))}$ as follows. Multiplying (3.1) by $L_1\Delta Q^\epsilon-\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}$ and integrating over $B_{r_3^\epsilon}(x_0)$ lead to

$$\begin{aligned} &\int_{B_{r_3^\epsilon}(x_0)}\left(L_1^2|\Delta Q^\epsilon|^2+\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2\right)dx \\ =&\int_{B_{r_3^\epsilon}(x_0)}|H^\epsilon|^2dx+\int_{\partial B_{r_3^\epsilon}(x_0)}\frac{2L_1\partial\hat{F}_b(Q^\epsilon)}{\epsilon\partial\nu}dS-\int_{B_{r_3^\epsilon}(x_0)}\frac{2L_1}{\epsilon}\partial_kQ^\epsilon:\partial_k\mathcal{J}(Q)dx \end{aligned} \quad (3.8)$$

Due to condition (**II**) and (3.5), the desired uniform estimate on the left hand side of (3.8) can be achieved once the last integral on the right hand side of (3.8) can be handled.

To this end, one calculates that

$$\begin{aligned}\frac{1}{\epsilon}\partial_k Q^\epsilon : \partial_k \mathcal{J}(Q) &= \frac{1}{\epsilon}\left[-b\lambda_1^\epsilon|\nabla Q^\epsilon|^2 - b\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon + \frac{c}{2}|\nabla|Q^\epsilon|^2|^2\right]\\ &+\frac{1}{\epsilon}\left(-a + b\lambda_1^\epsilon + c|Q^\epsilon|^2\right)|\nabla Q^\epsilon|^2 := J_1 + J_2, \end{aligned} \tag{3.9}$$

where λ_1^ϵ is the smallest eigenvalue of Q^ϵ. Then our key observation here is that J_1 has the favorable sign and the integral involving J_2 can be bounded uniformly, which will be derived in **Step 3** and **Step 4** respectively.

Step 3. we will show that J_1 is nonnegative, i.e.

$$\boldsymbol{J_1(x) \geq 0 \text{ for all } x \in B_{3r_0}(x_0)}. \tag{3.10}$$

To prove this, we will examine the structure of Q^ϵ in details. Let $\lambda_1^\epsilon, \lambda_2^\epsilon$ and λ_3^ϵ be the eigenvalues of Q^ϵ with the corresponding eigenvectors $d_1^\epsilon, d_2^\epsilon$ and $d_3^\epsilon \in \mathbb{S}^2$ respectively. Then, Since Q^ϵ is symmetric and with zero trace, Q^ϵ can be represented as

$$Q^\epsilon = \lambda_1^\epsilon d_1^\epsilon \otimes d_1^\epsilon + \lambda_2^\epsilon d_2^\epsilon \otimes d_2^\epsilon + \lambda_3^\epsilon d_3^\epsilon \otimes d_3^\epsilon, \quad \lambda_1^\epsilon + \lambda_2^\epsilon + \lambda_3^\epsilon = 0. \tag{3.11}$$

Without loss of generality, it can be assumed that

$$\lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \lambda_3^\epsilon.$$

Set

$$D_{12}^\epsilon = \sum_{k=1}^3 |d_1^\epsilon \cdot \partial_k d_2^\epsilon|^2, \quad D_{13}^\epsilon = \sum_{k=1}^3 |d_1^\epsilon \cdot \partial_k d_3^\epsilon|^2, \quad D_{23}^\epsilon = \sum_{k=1}^3 |d_2^\epsilon \cdot \partial_k d_3^\epsilon|^2. \tag{3.12}$$

Since $d_1^\epsilon, d_2^\epsilon$ and d_3^ϵ form an orthonormal basis to $\mathbb{R}^3$, it holds that

$$|\nabla d_1^\epsilon|^2 = D_{12}^\epsilon + D_{13}^\epsilon, \quad |\nabla d_2^\epsilon|^2 = D_{12}^\epsilon + D_{23}^\epsilon, \quad |\nabla d_3^\epsilon|^2 = D_{13}^\epsilon + D_{23}^\epsilon. \tag{3.13}$$

It follows from the structure of Q^ϵ ((3.11)), (3.12), (3.13) and detailed calculations that

$$\begin{aligned}|\nabla Q^\epsilon|^2 &= |\nabla\lambda_1^\epsilon|^2 + |\nabla\lambda_2^\epsilon|^2 + |\nabla\lambda_3^\epsilon|^2 + 2(\lambda_1^\epsilon)^2|\nabla d_1^\epsilon|^2 + 2(\lambda_2^\epsilon)^2|\nabla d_2^\epsilon|^2 + 2(\lambda_3^\epsilon)^2|\nabla d_3^\epsilon|^2\\ &\quad -4\lambda_1^\epsilon\lambda_2^\epsilon D_{12}^\epsilon - 4\lambda_1^\epsilon\lambda_3^\epsilon D_{13}^\epsilon - 4\lambda_2^\epsilon\lambda_3^\epsilon D_{23}^\epsilon\\ &= |\nabla\lambda_1^\epsilon|^2 + |\nabla\lambda_2^\epsilon|^2 + |\nabla\lambda_3^\epsilon|^2 + 2(\lambda_3^\epsilon - \lambda_2^\epsilon)^2|\nabla d_3^\epsilon|^2 + E_1^\epsilon, \end{aligned} \tag{3.14}$$

where

$$E_1^\epsilon = 6\lambda_3^\epsilon(\lambda_2^\epsilon - \lambda_1^\epsilon)D_{13}^\epsilon + 2(\lambda_2^\epsilon - \lambda_1^\epsilon)^2 D_{12}^\epsilon \geq 0.$$

Meanwhile, it holds that

$$(Q^\epsilon)^2 = (\lambda_1^\epsilon)^2 d_1^\epsilon \otimes d_1^\epsilon + (\lambda_2^\epsilon)^2 d_2^\epsilon \otimes d_2^\epsilon + (\lambda_3^\epsilon)^2 d_3^\epsilon \otimes d_3^\epsilon.$$

Then, a detailed calculation using (3.11)-(3.13) yields

$$\begin{aligned}\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon &= 2\lambda_1^\epsilon|\nabla\lambda_1^\epsilon|^2 + 2\lambda_2^\epsilon|\nabla\lambda_2^\epsilon|^2 + 2\lambda_3^\epsilon|\nabla\lambda_3^\epsilon|^2 + 2(\lambda_1^\epsilon)^3|\nabla d_1^\epsilon|^2\\ &\quad +2(\lambda_2^\epsilon)^3|\nabla d_2^\epsilon|^2 + 2(\lambda_3^\epsilon)^3|\nabla d_3^\epsilon|^2 - 2[(\lambda_1^\epsilon)^2\lambda_2 + (\lambda_2^\epsilon)^2\lambda_1^\epsilon]D_{12}^\epsilon\\ &\quad -2[(\lambda_1^\epsilon)^2\lambda_3 + (\lambda_3^\epsilon)^2\lambda_1^\epsilon]D_{13}^\epsilon - 2[(\lambda_2^\epsilon)^2\lambda_3 + (\lambda_3^\epsilon)^2\lambda_2^\epsilon]D_{23}^\epsilon. \end{aligned}$$

Due to (3.11) and (3.13) again, $\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon$ can be rewritten as

$$\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon = 2\lambda_1^\epsilon|\nabla\lambda_1^\epsilon|^2 + 2\lambda_2^\epsilon|\nabla\lambda_2^\epsilon|^2 + 2\lambda_3^\epsilon|\nabla\lambda_3^\epsilon|^2 - 2\lambda_1^\epsilon(\lambda_3^\epsilon - \lambda_2^\epsilon)^2|\nabla d_3^\epsilon|^2 + E_2^\epsilon, \tag{3.15}$$

where

$$E_2^\epsilon = 2(\lambda_1^\epsilon + \lambda_2^\epsilon)(\lambda_1^\epsilon - \lambda_2^\epsilon)^2 D_{12}^\epsilon + 2(\lambda_1^\epsilon - \lambda_2^\epsilon)[(\lambda_1^\epsilon)^2 + (\lambda_2^\epsilon)^2 + \lambda_1^\epsilon\lambda_2^\epsilon]D_{13}^\epsilon \leq 0.$$

It follows from **Step 1** that $\lambda_1^\epsilon < 0, \lambda_2^\epsilon < 0$ and $\lambda_3^\epsilon > 0$ in $B_{3r_0}(x_0)$ for suitable small δ_0. This and $\lambda_1^\epsilon \le \lambda_2^\epsilon \le \lambda_3^\epsilon$ imply that

$$-b\lambda_1^\epsilon|\nabla Q^\epsilon|^2 - b\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon + \frac{1}{2}c|\nabla|Q^\epsilon|^2|^2 \ge E_3^\epsilon, \tag{3.16}$$

where

$$E_3^\epsilon = -b\lambda_1^\epsilon(|\nabla\lambda_1^\epsilon|^2+|\nabla\lambda_2^\epsilon|^2+|\nabla\lambda_3^\epsilon|^2) - b(2\lambda_1^\epsilon|\nabla\lambda_1^\epsilon|^2+2\lambda_2^\epsilon|\nabla\lambda_2^\epsilon|^2+2\lambda_3^\epsilon|\nabla\lambda_3^\epsilon|^2)+\frac{c}{2}|\nabla|Q^\epsilon|^2|^2.$$

Since $\lambda_1^\epsilon + \lambda_2^\epsilon + \lambda_3^\epsilon = 0$, one has

$$\begin{aligned}|\nabla|Q^\epsilon|^2|^2 &= 4|2\lambda_1^\epsilon\nabla\lambda_1^\epsilon + 2\lambda_2^\epsilon\nabla\lambda_2^\epsilon + \lambda_2^\epsilon\nabla\lambda_1^\epsilon + \lambda_1^\epsilon\nabla\lambda_2^\epsilon|^2 \\ &= 4(2\lambda_1^\epsilon+\lambda_2^\epsilon)^2|\nabla\lambda_1^\epsilon|^2 + 8(2\lambda_1^\epsilon+\lambda_2^\epsilon)(2\lambda_2^\epsilon+\lambda_1^\epsilon)\nabla\lambda_1^\epsilon\cdot\nabla\lambda_2^\epsilon + 4(2\lambda_2^\epsilon+\lambda_1^\epsilon)^2|\nabla\lambda_2^\epsilon|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&-b[\lambda_1^\epsilon(|\nabla\lambda_1^\epsilon|^2+|\nabla\lambda_2^\epsilon|^2+|\nabla\lambda_3^\epsilon|^2)+(2\lambda_1^\epsilon|\nabla\lambda_1^\epsilon|^2+2\lambda_2^\epsilon|\nabla\lambda_2^\epsilon|^2+2\lambda_3^\epsilon|\nabla\lambda_3^\epsilon|^2)] \\ = \quad &-b[(2\lambda_1^\epsilon-2\lambda_2^\epsilon)|\nabla\lambda_1^\epsilon|^2 - 2(\lambda_1^\epsilon+2\lambda_2^\epsilon)\nabla\lambda_1^\epsilon\cdot\nabla\lambda_2^\epsilon] \ge 2b(\lambda_1^\epsilon+2\lambda_2^\epsilon)\nabla\lambda_1^\epsilon\cdot\nabla\lambda_2^\epsilon.\end{aligned}$$

Therefore, these together with condition **(III)** imply that

$$\begin{aligned}E_3^\epsilon \quad \ge \quad & 2c(2\lambda_1^\epsilon+\lambda_2^\epsilon)^2|\nabla\lambda_1^\epsilon|^2 + 2c(2\lambda_2^\epsilon+\lambda_1^\epsilon)^2|\nabla\lambda_2^\epsilon|^2 \\ & +2\left[2c(2\lambda_1^\epsilon+\lambda_2^\epsilon)+b\right](2\lambda_2^\epsilon+\lambda_1^\epsilon)\nabla\lambda_1^\epsilon\cdot\nabla\lambda_2^\epsilon \ge 0\end{aligned} \tag{3.17}$$

for sufficiently small δ_0, where one has used the expression of s_+ and the fact that

$$\mathrm{dist}(Q^\epsilon(x),\mathcal{N})^2 = (\lambda_1^\epsilon(x)+\frac{s_+}{3})^2 + (\lambda_2^\epsilon(x)+\frac{s_+}{3})^2 + (\lambda_3^\epsilon(x)-\frac{2s_+}{3})^2 \le \delta_0^{\frac{1}{4}},\ x\in B_{3r_0}(x_0).$$

Substituting the estimate (3.17) into (3.16) yeilds

$$-b\lambda_1^\epsilon|\nabla Q^\epsilon|^2 - b\partial_k(Q^\epsilon)^2 : \partial_k Q^\epsilon + \frac{c}{2}|\nabla|Q^\epsilon|^2|^2 \ge 0 \text{ in } B_{3r_0}(x_0) \tag{3.18}$$

for sufficiently small δ_0, which proves the desired estimate (3.10).

Step 4. We now estimate the integral involving J_2 in (3.9) and establish finally the uniform estimate on $||\Delta Q^\epsilon||_{L^2(B_{r_3^\epsilon}(x_0))}$. Set

$$g(\lambda_1^\epsilon,\lambda_3^\epsilon) = -a + b\lambda_1^\epsilon + c|Q^\epsilon|^2 = -a + b\lambda_1^\epsilon + 2c[(\lambda_1^\epsilon)^2+(\lambda_3^\epsilon)^2+\lambda_1^\epsilon\lambda_3^\epsilon].$$

It follows from the definition of s_+ that $g(-\frac{s_+}{3},\frac{2s_+}{3}) = 0$. Then, it holds that

$$g(\lambda_1^\epsilon,\lambda_3^\epsilon) = b\left(\lambda_1^\epsilon+\frac{s_+}{3}\right) + 2cs_+\left(\lambda_3^\epsilon-\frac{2s_+}{3}\right) + o\left(\sqrt{|\lambda_1^\epsilon+\frac{s_+}{3}|^2+|\lambda_3^\epsilon-\frac{2s_+}{3}|^2}\right)$$

near the point $(-\frac{s_+}{3},\frac{2s_+}{3})$. Note that

$$\mathrm{dist}(Q^\epsilon,\mathcal{N})^2 = (\lambda_1^\epsilon+\frac{s_+}{3})^2 + (\lambda_2^\epsilon+\frac{s_+}{3})^2 + (\lambda_3^\epsilon-\frac{2s_+}{3})^2.$$

Thus, for suitably small δ_0, one has that

$$|g(\lambda_1^\epsilon,\lambda_3^\epsilon)| \le C\sqrt{|\lambda_1^\epsilon+\frac{s_+}{3}|^2+|\lambda_3^\epsilon-\frac{2s_+}{3}|^2} \le C\mathrm{dist}(Q^\epsilon,\mathcal{N}) \le C|\mathcal{J}(Q^\epsilon)|,$$

where Lemma 2.2 and the estimate (3.4) have been used. Hence

$$|J_2| \le \frac{C}{\epsilon}|\mathcal{J}(Q^\epsilon)||\nabla Q^\epsilon|^2 \text{ in } B_{3r_0}(x_0). \tag{3.19}$$

It then follows from (3.5), (3.8)-(3.10), and (3.18)-(3.19) that

$$\begin{aligned}&\int_{B_{r_3^\epsilon}(x_0)}\left(L_1^2|\Delta Q^\epsilon|^2+\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2\right)dx\\ \leq\ &\int_{B_{r_3^\epsilon}(x_0)}\left(|H^\epsilon|^2+\frac{C}{\epsilon}|\mathcal{J}(Q^\epsilon)||\nabla Q^\epsilon|^2\right)dx+C_0\\ \leq\ &\int_{B_{r_3^\epsilon}(x_0)}\left(|H^\epsilon|^2+\frac{1}{2}\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2+C_0|\nabla Q^\epsilon|^4\right)dx+C_0.\end{aligned}$$

Meanwhile, by Ladyzhenskaya's inequality, one has

$$\int_{B_{r_3^\epsilon}(x_0)}|\nabla Q^\epsilon|^4dx\leq C_0\int_{B_{r_3^\epsilon}(x_0)}|\nabla Q^\epsilon|^2dx\int_{B_{r_3^\epsilon}(x_0)}(|\Delta Q^\epsilon|^2+|\nabla Q^\epsilon|^2)dx.$$

Since $||\nabla Q^\epsilon||^2_{L^2(B_{4r_0}(x_0))}<\delta_0$ and δ_0 can be chosen very small, one can get

$$\int_{B_{r_3^\epsilon}(x_0)}\left(|\Delta Q^\epsilon|^2+\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|^2\right)dx\leq C_4, \tag{3.20}$$

where C_4 is independent of ϵ. Therefore, there exists a subsequence of $\{Q^\epsilon\}_{\epsilon>0}$ such that

$$Q^\epsilon\to Q^*=s_+(d^*\otimes d^*-\frac{1}{3}\mathbb{I})\text{ in }H^1(B_{r_0}(x_0)),d^*\in\mathbb{S}^2, \tag{3.21}$$

$$\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\rightharpoonup\mathcal{J}^*\text{ in }L^2(B_{r_0}(x_0)), \tag{3.22}$$

as $\epsilon\to0$. Furthermore, by Step 1 and lemma 2.2, one can get

$$\begin{aligned}\int_{B_{r_0}(x_0)}\frac{\hat{F}_b(Q^\epsilon)}{\epsilon}dx\ &\leq\ C\int_{B_{r_0}(x_0)}|Q^\epsilon-Q^*|\left|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right|dx\\ &\leq\ C||Q^\epsilon-Q^*||_{L^2(B_{r_0}(x_0))}\left\|\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}\right\|_{L^2(B_{r_0}(x_0))}\to0\end{aligned} \tag{3.23}$$

as $\epsilon\to0$.

Step 5. Finally, we show that

$$\boldsymbol{\mathcal{J}^*\in(T_{Q^*}\mathcal{N})^{\perp}_{\mathfrak{Q}_0}\text{ a.e. in }B_{r_0}(x_0).} \tag{3.24}$$

Recalling (3.14) in **Step 3**, one has that

$$\begin{aligned}|\nabla Q^\epsilon|^2\ =\ &|\nabla\lambda_1^\epsilon|^2+|\nabla\lambda_2^\epsilon|^2+|\nabla\lambda_3^\epsilon|^2+2(\lambda_3^\epsilon-\lambda_2^\epsilon)^2|\nabla d_3^\epsilon|^2\\ &+6\lambda_3^\epsilon(\lambda_2^\epsilon-\lambda_1^\epsilon)D_{13}^\epsilon+2(\lambda_2^\epsilon-\lambda_1^\epsilon)^2D_{12}^\epsilon\geq2(\lambda_3^\epsilon-\lambda_2^\epsilon)^2|\nabla d_3^\epsilon|^2.\end{aligned}$$

On the other hand, the claim in **Step 1** implies that $|\lambda_3^\epsilon-\lambda_2^\epsilon|\geq\frac{1}{2}s_+$ for all $x\in B_{3r_0}(x_0)$. Therefore $|\nabla d_3^\epsilon|^2\leq\frac{2}{s_+^2}|\nabla Q^\epsilon|^2$ holds, which yields that

$$||\nabla d_3^\epsilon||_{L^p(B_{r_0}(x_0))}\leq C||\nabla Q^\epsilon||_{L^p(B_{r_0}(x_0))},\quad 1\leq p<\infty. \tag{3.25}$$

Note also that (3.20) in **Step 4** implies $||Q^\epsilon||_{H^2(B_{r_0}(x_0))}\leq C$. Thus

$$||\nabla Q^\epsilon||_{L^p(B_{r_0}(x_0))}\leq C,\quad 1\leq p<\infty$$

by the embedding theorem, which together with (3.25), yields that

$$d_3^\epsilon\to d^*\text{ in }C^\alpha(B_{r_0}(x_0)),0<\alpha<1,\text{ as }\epsilon\to0. \tag{3.26}$$

Set

$$e_1^\epsilon=\frac{1}{\sqrt{2}}(d_3^\epsilon\otimes d_2^\epsilon+d_2^\epsilon\otimes d_3^\epsilon),e_2^\epsilon=\frac{1}{\sqrt{2}}(d_3^\epsilon\otimes d_1^\epsilon+d_1^\epsilon\otimes d_3^\epsilon),e_3^\epsilon=\frac{1}{\sqrt{2}}(d_2^\epsilon\otimes d_1^\epsilon+d_1^\epsilon\otimes d_2^\epsilon),$$

$$e_4^\epsilon=\frac{1}{\sqrt{2}}(d_1^\epsilon\otimes d_1^\epsilon-d_2^\epsilon\otimes d_2^\epsilon),e_5^\epsilon=\sqrt{6}(\frac{1}{2}d_1^\epsilon\otimes d_1^\epsilon+\frac{1}{2}d_2^\epsilon\otimes d_2^\epsilon-\frac{1}{3}\mathbb{I}),$$

where $d_1^\epsilon, d_2^\epsilon\in V_{d_3^\epsilon}$(see Lemma 2.1 for the definition of $V_{d_3^\epsilon}$) and $d_1^\epsilon\cdot d_2^\epsilon=0$. Similarly, e_1^* and e_2^* can also be defined in a same way as e_1^ϵ and e_2^ϵ with d^*, d_1^*, d_2^*, where d^* is given in (3.21) and $d_1^*, d_2^*\in V_{d^*}, d_1^*\cdot d_2^*=0$. Obviously, $(T_{\mathcal{P}_\mathcal{N}(Q^\epsilon)}\mathcal{N})^\perp_{\mathcal{Q}_0}=Span\{e_3^\epsilon,e_4^\epsilon,e_5^\epsilon\}$. Lemma 2.1 yields that

$$\mathcal{J}(Q^\epsilon)=a_1^\epsilon e_3^\epsilon+a_2^\epsilon e_4^\epsilon+a_3^\epsilon e_5^\epsilon,$$

where $a_1^\epsilon,a_2^\epsilon,a_3^\epsilon\in L^2(B_{r_0}(x_0))$. Then, by taking the limit $\epsilon\to0^+$ and (3.22) and (3.26), one can get

$$\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}:(d_3^\epsilon\otimes d_2^*+d_2^*\otimes d_3^\epsilon)=-\frac{2\sqrt{6}}{3}a_3^\epsilon d_3^\epsilon\cdot d_2^*\rightharpoonup\mathcal{J}^*:e_1^*=0\text{ in }L^2(B_{r_0}(x_0)),$$

$$\frac{\mathcal{J}(Q^\epsilon)}{\epsilon}:(d_3^\epsilon\otimes d_1^*+d_1^*\otimes d_3^\epsilon)=-\frac{2\sqrt{6}}{3}a_3^\epsilon d_3^\epsilon\cdot d_1^*\rightharpoonup\mathcal{J}^*:e_2^*=0\text{ in }L^2(B_{r_0}(x_0)).$$

This implies (3.24) by Lemma 2.1.

□

4. Proof of main results

Proof of Theorem 1.5 **Step 1. Convergence of (v^ϵ,Q^ϵ).** It follows from the energy inequality (1.15) that there exists a subsequence of $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$ such that

$$v^\epsilon\overset{\star}{\rightharpoonup}v^*\text{ in }L^\infty(0,T;L^2(\mathbb{R}^2)),\quad v^\epsilon\rightharpoonup v^*\text{ in }L^2(0,T;H^1(\mathbb{R}^2)),\tag{4.1}$$

$$\nabla Q^\epsilon\overset{\star}{\rightharpoonup}\nabla Q^*\text{ in }L^\infty(0,T;L^2(\mathbb{R}^2)),\quad H^\epsilon\rightharpoonup H^*\text{ in }L^2(0,T;L^2(\mathbb{R}^2)),\tag{4.2}$$

where $Q^*=s_+(d^*\otimes d^*-\frac{1}{3}\mathbb{I})$ a.e. in $\mathbb{R}^2$ and $d^*:(0,T)\times\mathbb{R}^2\mapsto\mathbb{S}^2$.

For any smooth bounded domain $\tilde\Omega\subset\mathbb{R}^2$ and any $\psi\in L^2H_0^3((0,T)\times\tilde\Omega,\mathbb{R}^3),\partial_1\psi_1+\partial_2\psi_2=0$, one can derive from (1.38), Hölder inequalities, and the Sobolev embedding theorem that

$$\begin{aligned}
&|<v_t^\epsilon,\psi>|\\
\le&\left|\int_0^T\int_{\mathbb{R}^2}(\eta\overline{D^\epsilon}:\overline{\nabla\psi}-v^\epsilon\otimes v^\epsilon:\overline{\nabla\psi}-L_1\nabla Q^\epsilon\odot\nabla Q^\epsilon:\underline{\nabla\psi})dxdt\right|\\
&+\left|\int_0^T\int_{\mathbb{R}^2}[Q^\epsilon\cdot H^\epsilon-H^\epsilon\cdot Q^\epsilon-S_{Q^\epsilon}(H^\epsilon)]:\overline{\nabla\psi}dxdt\right|\\
\le&C\left(||D^\epsilon||_{L^2}||\nabla\psi||_{L^2}+||v^\epsilon||_{L_t^\infty L_x^2}||\nabla v^\epsilon||_{L^2}||\psi||_{L_t^2L_x^\infty}+||\nabla Q^\epsilon||^2_{L_t^\infty L_x^2}||\nabla\psi||_{L_t^2L_x^\infty}\right)\\
&+C||H^\epsilon||_{L^2}(||Q^\epsilon||^2_{L_t^\infty L_x^4(\tilde\Omega)}+||Q^\epsilon||_{L_t^\infty L_x^2(\tilde\Omega)})||\nabla\psi||_{L_t^2L_x^\infty}+C||H^\epsilon||_{L^2}||\nabla\phi||_{L^2}\\
\le&C||\psi||_{L_t^2H_x^3}
\end{aligned}\tag{4.3}$$

with $\overline{D^\epsilon}$ given in (1.11). Therefore, v_t^ϵ is uniformly bounded in $L^2(0,T;H^{-3}(\tilde{\Omega}))$. Similarly, for any $\varphi \in L^4((0,T)\times\tilde{\Omega},\mathcal{Q}_0)$, one can get from (1.39) that

$$\begin{aligned}
&|<Q_t^\epsilon,\varphi>| \\
= &\left|\int_0^T\int_{\mathbb{R}^2}[-\underline{v^\epsilon}\cdot\nabla Q^\epsilon+\frac{1}{\Gamma}H^\epsilon+S_{Q^\epsilon}(\overline{D^\epsilon})+\overline{\Lambda^\epsilon}\cdot Q^\epsilon-Q^\epsilon\cdot\overline{\Lambda^\epsilon}]:\varphi dxdt\right| \\
\leq\ &C(||v^\epsilon||_{L^4(\tilde{\Omega})}||\nabla Q^\epsilon||_{L^2}||\varphi||_{L^4}+||H^\epsilon||_{L^2}||\varphi||_{L^2}+||D^\epsilon||_{L^2}||\varphi||_{L^2}) \\
&+C||D^\epsilon||_{L^2}(||Q^\epsilon||^2_{L^8(\tilde{\Omega})}+||Q^\epsilon||_{L^4(\tilde{\Omega})})||\varphi||_{L^4}\leq C||\varphi||_{L^4}, \qquad (4.4)
\end{aligned}$$

where $\underline{v^\epsilon}$, $\overline{D^\epsilon}$ and $\overline{\Lambda^\epsilon}$ are given in (1.11). This implies that Q_t^ϵ is uniformly bounded in $L^{\frac{4}{3}}((0,T)\times\tilde{\Omega})$ for any smooth bounded domain $\tilde{\Omega}\subset\mathbb{R}^2$. Then, it follows from the energy inequality (1.15) and Aubin-Lious Lemma that there exists a subsequence of $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$ such that

$$v^\epsilon\to v^* \text{ in } L^p(0,T;L^p(\tilde{\Omega})),\quad 2\leq p<4, \qquad (4.5)$$

$$Q^\epsilon\to Q^* \text{ in } L^q(0,T;L^q(\tilde{\Omega})),\quad 1\leq q<\infty, \qquad (4.6)$$

where $\tilde{\Omega}$ is any smooth bounded domain in $\mathbb{R}^2$.

Then taking the limiting $\epsilon\to 0^+$ in the equality (1.39) for (v^ϵ,Q^ϵ) yields:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}[-Q^*:\varphi_t-(\underline{v^*}\cdot\nabla\varphi):Q^*-\frac{1}{\Gamma}H^*:\varphi]dxdt \\
+&\int_0^T\int_{\mathbb{R}^2}[Q^*\cdot\overline{\Lambda^*}-\overline{\Lambda^*}\cdot Q^*-S_{Q^*}(\overline{D^*})]:\varphi dxdt=\int_{\mathbb{R}^2}Q_0^*(x):\varphi(0,x)dx, \qquad (4.7)
\end{aligned}$$

where $\overline{D^*}$, $\overline{\Lambda^*}$, $\underline{v^*}$ are given in (1.35), $\varphi\in C_0^\infty((0,T)\times\mathbb{R}^2,\mathcal{Q}_0)$, and $S_{Q^*}(D^*)$ is given in (1.6). Note that condition (1.16) implies

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}Q_0^\epsilon:\varphi(0,x)dx=\int_{\mathbb{R}^2}Q_0^*:\varphi(0,x)dx,\quad \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}v_0^\epsilon\cdot\psi(0,x)dx=\int_{\mathbb{R}^2}v_0^*\cdot\psi(0,x)dx.$$

Next, we show that one can pass limit in (1.38) to obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}[-v^*\cdot\psi_t-(v^*\cdot\overline{\nabla\psi})\cdot v^*+\eta\overline{D^*}:\overline{\nabla\psi}+L_1\nabla Q^*\odot\nabla Q^*:\underline{\nabla\psi}]dxdt \\
+&\int_0^T\int_{\mathbb{R}^2}[Q^*\cdot H^*-H^*\cdot Q^*-S_{Q^*}(H^*)]:\overline{\nabla\psi}dxdt=\int_{\mathbb{R}^2}v_0^*(x)\cdot\psi(0,x)dx, \qquad (4.8)
\end{aligned}$$

where $S_{Q^*}(H^*)$ is defined by (1.6). To this end, due to (1.38) for (v^ϵ,Q^ϵ) and (4.1)-(4.6), it suffices to show only that there exists a subsequence of $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$ such that

$$\int_0^T\int_{\mathbb{R}^2}\nabla Q^\epsilon\odot\nabla Q^\epsilon:\underline{\nabla\psi}dxdt\to\int_0^T\int_{\mathbb{R}^2}\nabla Q^*\odot\nabla Q^*:\underline{\nabla\psi}dxdt,\ \text{as } \epsilon\to 0. \qquad (4.9)$$

Note that the energy inequality (1.15) for (v^ϵ,Q^ϵ) implies that there exists a subsequence of $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$ such that

$$\nabla Q^\epsilon\odot\nabla Q^\epsilon:\underline{\nabla\psi}dxdt\rightharpoonup\nabla Q^*\odot\nabla Q^*:\underline{\nabla\psi}dxdt+\kappa,\ \text{as } \epsilon\to 0$$

for a possibly non-vanishing measure κ. Our aim is to show that there exists a subsequence of $\{(v^\epsilon,Q^\epsilon)\}_{\epsilon>0}$ such that $\kappa=0$, which implies (4.9). To this end, we observe that

$$\int_0^T\liminf_{\epsilon\to 0^+}\int_{\mathbb{R}^2}|H^\epsilon|^2dxdt\leq E_0$$

by the energy inequality (1.15) and Fatou's lemma. Set

$$\mathcal{L}_\infty = \left\{ t : \liminf_{\epsilon\to 0^+} \int_{\mathbb{R}^2} |H^\epsilon(t,\cdot)|^2 dx < \infty \right\}.$$

Then, it holds that

$$|\mathcal{L}_\infty| = T.$$

Meanwhile, we claim that there exists a set $\mathcal{A}_\infty \subset (0,T)$ such that $|\mathcal{A}_\infty| = T$ and

$$\nabla Q^\epsilon(t) \rightharpoonup \nabla Q^*(t) \text{ in } L^2(\mathbb{R}^2) \tag{4.10}$$

for any $t \in \mathcal{A}_\infty$. To prove this claim, one notes that (4.2) implies

$$\lim_{\epsilon\to 0^+} \int_0^T \int_{\mathbb{R}^2} \partial_i Q^\epsilon : \hat{\varphi}(x)\hat{\phi}(t) dx dt = \int_0^T \int_{\mathbb{R}^2} \partial_i Q^* : \hat{\varphi}(x)\hat{\phi}(t) dx dt, \quad i = 1,2, \tag{4.11}$$

for any $\hat{\varphi} \in C_0^\infty(\mathbb{R}^2, \mathbb{M}^{3\times 3})$ and any $\hat{\phi} \in C_0^\infty((0,T))$. Since $||\nabla Q^\epsilon||_{L_t^\infty L_x^2}$ is uniformly bounded, (4.11) holds true for $\hat{\varphi} \in L^2(\mathbb{R}^2, \mathbb{M}^{3\times 3})$ and $\hat{\phi} \in L^1((0,T))$. For fixed $\hat{\varphi} \in L^2(\mathbb{R}^2, \mathbb{M}^{3\times 3})$, define

$$g_i^\epsilon(t) = \int_{\mathbb{R}^2} \partial_i Q^\epsilon(t,x) : \hat{\varphi}(x) dx \text{ and } g_i^*(t) = \lim_{\epsilon\to 0^+} g_i^\epsilon(t),\ i = 1,2.$$

Note that g_1^* and g_2^* can also be regarded as the weak limits of g_1^ϵ and g_2^ϵ in $L^p((0,T)), 1 < p < \infty$ ($||g_1^\epsilon||_{L^p((0,T))}$ and $||g_2^\epsilon||_{L^p((0,T))}$ are uniformly bounded due to the uniformly bound for $||\nabla Q^\epsilon||_{L_t^\infty L_x^2}$), hence g_1^* and g_2^* are measurable. Then, taking $\hat{\phi} \equiv 1$ into (4.11) yields that

$$\lim_{\epsilon\to 0^+} \int_0^T g_i^\epsilon(t) dt = \int_0^T \lim_{\epsilon\to 0^+} g_i^\epsilon(t) dt = \int_0^T g_i^*(t) dt < \infty,\ i = 1,2,$$

due to the Lebesgue dominated convergence theorem and the energy inequality (1.15). Next, we prove that there exists a set $\mathcal{A}_0 \subset (0,T)$ such that $|\mathcal{A}_0| = T$ and

$$g_i^*(t) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^2} \partial_i Q^\epsilon(t,x) : \hat{\varphi}(x) dx = \int_{\mathbb{R}^2} \partial_i Q^*(t,x) : \hat{\varphi}(x) dx \tag{4.12}$$

for any $t \in \mathcal{A}_0$. Since $\nabla Q^* \in L_t^\infty L_x^2$, one can define the following measurable functions:

$$\tilde{\delta}_i(t) = g_i^*(t) - \int_{\mathbb{R}^2} \partial_i Q^*(t,x) : \hat{\varphi}(x) dx$$

and

$$\hat{\phi}_i(t) = \begin{cases} 1, & \tilde{\delta}_i(t) > 0; \\ -1, & \tilde{\delta}_i(t) < 0; \\ 0, & \text{others}, \end{cases}$$

$i = 1,2$. Note that $\hat{\phi}_i \in L^1((0,T)), i = 1,2$. Then, these and (4.11) yield that

$$\int_0^T \left| g_i^*(t) - \int_{\mathbb{R}^2} \partial_i Q^*(t,x) : \hat{\varphi}(x) dx \right| dt = 0,\ i = 1,2,$$

which implies (4.12). Since $L^2(\mathbb{R}^2)$ is separable, one can find a countable set $\{\hat{\varphi}_j\}_{j=1}^\infty$ such that $\overline{\{\hat{\varphi}_j\}_{j=1}^\infty} = L^2(\mathbb{R}^2)$. For every fixed $\hat{\varphi}_j \in \{\hat{\varphi}_j\}_{j=1}^\infty$, as in (4.12), one can find a set $\mathcal{A}_j \subset (0,T)$ such that $|\mathcal{A}_j| = T$ and

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^2} \partial_i Q^\epsilon(t,x) : \hat{\varphi}_j(x) dx = \int_{\mathbb{R}^2} \partial_i Q^*(t,x) : \hat{\varphi}_j(x) dx,\ i = 1,2,$$

for any $t \in \mathcal{A}_j$. Then, let $\mathcal{A}_\infty = \cap_{j=1}^\infty \mathcal{A}_j$, one has that

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^2} \partial_i Q^\epsilon(t,x) : \hat{\varphi}(x)dx = \int_{\mathbb{R}^2} \partial_i Q^*(t,x) : \hat{\varphi}(x)dx,\ i = 1, 2,$$

for any $t \in \mathcal{A}_\infty$ and any $\hat{\varphi} \in L^2(\mathbb{R}^2, \mathbb{M}^{3\times 3})$. Hence (4.10) is proved.

Therefore, (4.9) holds true provided that there exists a subsequence of $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$ such that for $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$,

$$\int_{\mathbb{R}^2} \nabla Q^\epsilon(t,\cdot) \odot \nabla Q^\epsilon(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx \to \int_{\mathbb{R}^2} \nabla Q^*(t,\cdot) \odot \nabla Q^*(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx \tag{4.13}$$

as $\epsilon \to 0^+$. It should be noted that, for fixed $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$, there exists a subsequence of $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$ such that

$$\nabla Q^\epsilon(t,\cdot) \odot \nabla Q^\epsilon(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx \rightharpoonup \nabla Q^*(t,\cdot) \odot \nabla Q^*(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx + \kappa_1 \text{ as } \epsilon \to 0^+$$

for a possibly non-vanishing measure κ_1. For this subsequence $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$, $\kappa_1 = 0$ if there exists a subsequence $\{(v^{\epsilon_j}, Q^{\epsilon_j})\}_{j=1}^\infty$ of $\{(v^\epsilon, Q^\epsilon)\}_{\epsilon>0}$ such that

$$\int_{\mathbb{R}^2} \nabla Q^{\epsilon_j}(t,\cdot) \odot \nabla Q^{\epsilon_j}(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx \to \int_{\mathbb{R}^2} \nabla Q^*(t,\cdot) \odot \nabla Q^*(t,\cdot) : \underline{\nabla\psi}(t,\cdot)dx \text{ as } \epsilon_j \to 0^+.$$

For each time $t_0 \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$, there exists a subsequence of $\{H^\epsilon\}_{\epsilon>0}$ such that the L^2-norm of this subsequence is uniformly bounded, i.e. $||H^\epsilon||_{L^2(\mathbb{R}^2)} < \infty$. This and Lemma 3.1 imply that the concentration point y_0 of $\nabla Q^\epsilon \odot \nabla Q^\epsilon$ at time t_0 must satisfy

$$\liminf_{r\to 0} \left\{ \liminf_{\epsilon\to 0} \int_{B_r(y_0)} \left(|\nabla Q^\epsilon|^2 + \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right)(t_0, x)dx \right\} \geq \delta_0. \tag{4.14}$$

Then, this and the energy inequality (1.15) imply that there exist at most finite points where $\nabla Q^\epsilon \odot \nabla Q^\epsilon$ may concentrate on and the set of these points is denoted as

$$\begin{aligned} \mathcal{B}(t_0) &= \bigcap_{r>0} \left\{ x : \liminf_{\epsilon\to 0} \int_{B_r(x)} \left(|\nabla Q^\epsilon|^2 + \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right)(t_0, y)dy \geq \delta_0, t_0 \in \mathcal{L}_\infty \cap \mathcal{A}_\infty \right\} \\ &= \left\{ x_1(t_0), x_2(t_0), \cdots, x_{L(t_0)}(t_0) : t_0 \in \mathcal{L}_\infty \cap \mathcal{A}_\infty \right\}, \end{aligned}$$

where $0 \leq L(t_0) \leq [\frac{E_0}{\delta_0}] + 1$ is an integer depending on t_0, and E_0 is given in (1.17).

Based on this fact, without loss of generality, one can assume that $\mathcal{B}(t) = \{\mathbf{0}\}, t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$, consists of a single point at the origin. Since $\partial_1\psi_1 + \partial_2\psi_2 = 0$, it is easy to see

$$\nabla Q^\epsilon \odot \nabla Q^\epsilon : \underline{\nabla\psi} = (\nabla Q^\epsilon \odot \nabla Q^\epsilon - \frac{1}{2}|\nabla Q^\epsilon|^2 I) : \underline{\nabla\psi},$$

where I is the identity matrix in $\mathbb{M}^{2\times 2}$. Then, direct calculations give

$$\nabla Q^\epsilon \odot \nabla Q^\epsilon - \frac{1}{2}|\nabla Q^\epsilon|^2 I = \begin{pmatrix} \frac{1}{2}(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) & \partial_1 Q^\epsilon : \partial_2 Q^\epsilon \\ \partial_1 Q^\epsilon : \partial_2 Q^\epsilon & -\frac{1}{2}(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) \end{pmatrix}.$$

Therefore, it suffices to show that

$$\int_{\mathbb{R}^2} (|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2)\phi dx \to \int_{\mathbb{R}^2} (|\partial_1 Q^*|^2 - |\partial_2 Q^*|^2)\phi dx \tag{4.15}$$

and

$$\int_{\mathbb{R}^2} \partial_1 Q^\epsilon : \partial_2 Q^\epsilon \phi dx \to \int_{\mathbb{R}^2} \partial_1 Q^* : \partial_2 Q^* \phi dx \tag{4.16}$$

as $\epsilon \to 0$ for every $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$. We prove (4.15) and (4.16) by a Pohozaev type argument. First, we claim that there exists $R > 0$ such that

$$\int_{B_R(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon(t,x))}{\epsilon} dx \to 0, \quad t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty, \text{ as } \epsilon \to 0. \tag{4.17}$$

Note that this claim implies that

$$\int_{\mathbb{R}^2} \frac{\hat{F}_b(Q^\epsilon(t,x))}{\epsilon} dx \to 0, \quad t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty, \text{ as } \epsilon \to 0$$

due to the assumption that $\mathcal{B}(t) = \{\mathbf{0}\}$ for $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$. By Lemma 3.1, the small energy condition implies the strong convergence of $\frac{1}{\epsilon}\hat{F}_b(Q^\epsilon)$ in L^1 and Q^ϵ in H^1. Then, this and the energy inequality (1.15) imply that $\frac{1}{\epsilon}\hat{F}_b(Q^\epsilon)$ converges strongly to 0 in $L^1_{loc}(\mathbb{R}^2 \setminus \mathcal{B}(t))$ for $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$. Therefore, if (4.17) fails, one can assume that

$$\frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx \rightharpoonup \beta_1 \delta(\mathbf{0}), \quad \beta_1 > 0, \text{ as } \epsilon \to 0 \tag{4.18}$$

in $B_r(\mathbf{0})$ for any $r > 0$, where $\delta(\mathbf{0})$ is the dirac measure centered at the origin. Recall the definition of H^ϵ:

$$L_1 \Delta Q^\epsilon - \frac{\mathcal{J}(Q^\epsilon)}{\epsilon} = H^\epsilon. \tag{4.19}$$

Multiplying (4.19) by $\frac{x}{r} \cdot \nabla Q^\epsilon$ and integrating over $B_r(\mathbf{0})$, one gets after integration by parts that

$$\begin{aligned} & \int_{\partial B_r(\mathbf{0})} \left(L_1 \left| \frac{\partial Q^\epsilon}{\partial \nu} \right|^2 - \frac{L_1}{2} |\nabla Q^\epsilon|^2 - \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right) dS \\ = & -\frac{1}{r} \int_{B_r(\mathbf{0})} \left(2\frac{\hat{F}_b(Q^\epsilon)}{\epsilon} - x \cdot \nabla Q^\epsilon : H^\epsilon \right) dx. \end{aligned}$$

Then, integrating above identity from τ to R yields

$$\begin{aligned} & 2\int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx dr \\ = & -\int_{B_R(\mathbf{0}) \setminus B_\tau(\mathbf{0})} \left(L_1 \left| \frac{\partial Q^\epsilon}{\partial \nu} \right|^2 - \frac{L_1}{2} |\nabla Q^\epsilon|^2 - \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} \right) dx + \int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} x \cdot \nabla Q^\epsilon : H^\epsilon dx dr. \end{aligned}$$

For $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$, it holds that

$$\left| \frac{1}{r} \int_{B_r(\mathbf{0})} x \cdot \nabla Q^\epsilon : H^\epsilon dx \right| \le C ||\nabla Q^\epsilon||_{L^2} ||H^\epsilon||_{L^2} \le C E_0$$

thanks to the energy inequality (1.15) and the definition of $\mathcal{L}_\infty \cap \mathcal{A}_\infty$. Therefore,

$$\int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx dr \le 2E_0 + C E_0 (R - \tau) \tag{4.20}$$

for all $0 < \tau < R < \infty$.

On the other hand, it follows from (1.15) and (4.18) that

$$\lim_{\epsilon \to 0} \int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx dr = \int_\tau^R \frac{1}{r} \lim_{\epsilon \to 0} \int_{B_r(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx dr = \int_\tau^R \frac{\beta_1}{r} dr = \beta_1 \ln\left(\frac{R}{\tau}\right)$$

by the Lebesgue dominated convergence theorem, which contradicts (4.20) when τ is very small. This proves the claim.

We now verify (4.15). Otherwise, one can assume that there exists a real nonzero number β_2 such that

$$(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2)dx \rightharpoonup (|\partial_1 Q^*|^2 - |\partial_2 Q^*|^2)dx + \beta_2\delta(\mathbf{0}) \text{ as } \epsilon \to 0 \tag{4.21}$$

in $B_r(\mathbf{0})$ for any $r > 0$. Multiplying (4.19) by $\frac{x_1}{r}\partial_1 Q^\epsilon$ and integrating over $B_r(\mathbf{0})$, one can get through integration by parts that

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{0})} \left(\frac{L_1 x_1}{r}\partial_1 Q^\epsilon : \frac{\partial Q^\epsilon}{\partial \nu} - \frac{L_1}{2}\frac{x_1^2}{r^2}|\nabla Q^\epsilon|^2 - \frac{x_1^2 \hat{F}_b(Q^\epsilon)}{r^2\epsilon} \right) dS \\
= \; & \frac{1}{r}\int_{B_r(\mathbf{0})} \left[\frac{L_1}{2}(|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) - \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} + x_1\partial_1 Q^\epsilon : H^\epsilon \right].
\end{aligned}$$

It follows from this and a similar way as for (4.20) that

$$\begin{aligned}
&\left| \lim_{\epsilon\to 0} \int_\tau^R \frac{L_1}{r} \int_{B_r(\mathbf{0})} (|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) dx dr \right| \\
\le \; & \int_\tau^R \frac{1}{r} \lim_{\epsilon\to 0} \int_{B_r(\mathbf{0})} \frac{\hat{F}_b(Q^\epsilon)}{\epsilon} dx dr + 2E_0 + CE_0(R-\tau) \le E_0[2 + C(R-\tau)]
\end{aligned} \tag{4.22}$$

by the Lebesgue dominated convergence theorem, where C_0 is independent of ϵ and τ.

On the other hand, it follows from the energy inequality (1.15) and (4.21) that

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} (|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) dx dr &= \int_\tau^R \frac{1}{r} \lim_{\epsilon\to 0} \int_{B_r(\mathbf{0})} (|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) dx dr \\
&= \int_\tau^R \frac{1}{r} \left[\int_{B_r(\mathbf{0})} (|\partial_1 Q^*|^2 - |\partial_2 Q^*|^2) dx + \beta_2 \right] dr
\end{aligned}$$

by the Lebesgue dominated convergence theorem. Then, let r be small enough such that

$$\left| \int_{B_r(\mathbf{0})} (|\partial_1 Q^*|^2 - |\partial_2 Q^*|^2) dx \right| \le \frac{|\beta_2|}{2}.$$

Therefore, one has

$$\left| \lim_{\epsilon\to 0} \int_\tau^R \frac{1}{r} \int_{B_r(\mathbf{0})} (|\partial_1 Q^\epsilon|^2 - |\partial_2 Q^\epsilon|^2) dx dr \right| \ge \frac{1}{2} \int_\tau^R \frac{|\beta_2|}{r} dr = \frac{1}{2}|\beta_2|(\ln R - \ln \tau)$$

for small R, which contradicts (4.22) when τ is very small and $0 < \tau << R$. This proves (4.15).

Similarly, we can prove (4.16). Indeed, if (4.16) fails, one can assume that there exists a real nonzero number β_3 such that

$$\partial_1 Q^\epsilon : \partial_2 Q^\epsilon dx \rightharpoonup \partial_1 Q^* : \partial_2 Q^* dx + \beta_3\delta(\mathbf{0}) \text{ as } \epsilon \to 0 \tag{4.23}$$

in $B_r(\mathbf{0})$ for any $r>0$. Multiplying (4.19) by $\frac{x_2}{r}\partial_1 Q^\epsilon$ and integrating over $B_r(\mathbf{0})$ yield

$$\begin{aligned}&\int_{\partial B_r(\mathbf{0})}\left(\frac{L_1x_2}{r}\partial_1Q^\epsilon:\frac{\partial Q^\epsilon}{\partial\nu}-\frac{L_1}{2}\frac{x_1x_2}{r^2}|\nabla Q^\epsilon|^2-\frac{x_1x_2\hat{F}_b(Q^\epsilon)}{r^2\epsilon}\right)dS\\ =\ &\frac{1}{r}\int_{B_r(\mathbf{0})}(L_1\partial_1Q^\epsilon:\partial_2Q^\epsilon+x_2\partial_1Q^\epsilon:H^\epsilon)\,dx.\end{aligned}$$

As for the derivation of (4.20), one can obtain that as $\epsilon\to0$,

$$\left|\int_\tau^R\frac{L_1}{r}\int_{B_r(\mathbf{0})}\partial_1Q^\epsilon:\partial_2Q^\epsilon dxdr\right|\le 2E_0+CE_0(R-\tau). \tag{4.24}$$

On the other hand, (1.15) and (4.23) imply that

$$\begin{aligned}\lim_{\epsilon\to0}\int_\tau^R\frac{1}{r}\int_{B_r(\mathbf{0})}\partial_1Q^\epsilon:\partial_2Q^\epsilon dxdr &= \int_\tau^R\frac{1}{r}\lim_{\epsilon\to0}\int_{B_r(\mathbf{0})}\partial_1Q^\epsilon:\partial_2Q^\epsilon dxdr\\ &=\int_\tau^R\frac{1}{r}\left[\int_{B_r(\mathbf{0})}\partial_1Q^*:\partial_2Q^*dx+\beta_3\right]dr\end{aligned}$$

by the Lebesgue dominated convergence theorem. Then, let r be small enough such that

$$\left|\int_{B_r(\mathbf{0})}\partial_1Q^*:\partial_2Q^*\right|\le\frac{|\beta_3|}{2}.$$

Therefore, one has

$$\left|\lim_{\epsilon\to0}\int_\tau^R\frac{1}{r}\int_{B_r(\mathbf{0})}\partial_1Q^\epsilon:\partial_2Q^\epsilon dxdr\right|\ge\frac{1}{2}\int_\tau^R\frac{|\beta_3|}{r}dr=\frac{1}{2}|\beta_3|(\ln R-\ln\tau)$$

for small R, which contradicts (4.24) when τ is very small and $0<\tau<<R$. This proves (4.16) and hence completes the proof of (4.13).

Step 2. We prove that $(v^*(\cdot,t),\nabla d^*(\cdot,t))\to(v_0^*,\nabla d_0^*)$ as $t\to0$ in $L^2(\mathbb{R}^2)$. To this end, we prove first that

$$\frac{1}{2}||v^*(t,\cdot)||^2_{L^2(\mathbb{R}^2)}+L_1s_+^2||\nabla d^*(t,\cdot)||^2_{L^2(\mathbb{R}^2)}\le\frac{1}{2}||v_0^*||^2_{L^2(\mathbb{R}^2)}+L_1s_+^2||\nabla d_0^*||^2_{L^2(\mathbb{R}^2)}\le E_0 \tag{4.25}$$

for all $t\in[0,T]$, where E_0 is given in (1.17). The uniform estimates (4.3) and (4.4) imply that $v_t^*\in L^2(0,T;H^{-3}(\tilde\Omega))$ and $Q_t^*\in L^{\frac{4}{3}}((0,T)\times\tilde\Omega)$. Therefore, v^* is weakly continuous on $H^3(\tilde\Omega)$, thus

$$f_1(t)=\int_{\tilde\Omega}v^*(t,x)\cdot\tilde\phi(x)dx$$

is continuous in $[0,T]$ for any $\tilde\phi\in H^3(\tilde\Omega)$. Similarly, ∇Q^* and ∇d^* are weakly continuous on $W^{1,4}(\tilde\Omega)$. Note that $|Q_t^*|^2=2s_+^2|d_t^*|^2$ and $|\nabla Q^*|^2=2s_+^2|\nabla d^*|^2$. Then, as shown in [41, Chapter III, Lemma 1.4], $v^*,\nabla Q^*$ and $\nabla d^*\in L^\infty(0,T;L^2(\mathbb{R}^2)$ yield that

$$v^*,\nabla Q^*\text{ and }\nabla d^*\text{ are weakly continuous on }L^2(\tilde\Omega) \tag{4.26}$$

for any smooth bounded domain $\tilde\Omega\subset\mathbb{R}^2$. Based on this fact, one has

$$||v^*(t)||_{L^2(\tilde\Omega)}\le||v^*||_{L_t^\infty L_x^2},\quad ||\nabla Q^*(t)||_{L^2(\tilde\Omega)}\le||\nabla Q^*||_{L_t^\infty L_x^2}$$

for all $t\in[0,T]$. Then, letting $\tilde\Omega=B_R$ and $R\to\infty$ yields

$$||v^*(t)||_{L^2(\mathbb{R}^2)}\le||v^*||_{L_t^\infty L_x^2},\quad ||\nabla Q^*(t)||_{L^2(\mathbb{R}^2)}\le||\nabla Q^*||_{L_t^\infty L_x^2}$$

for all $t \in [0,T]$. Meanwhile, similar arguments as for (4.10) imply that

$$v^\epsilon(t) \rightharpoonup v^*(t) \text{ and } \nabla Q^\epsilon(t) \rightharpoonup \nabla Q^*(t) \text{ in } L^2(\mathbb{R}^2) \text{ as } \epsilon \to 0^+ \tag{4.27}$$

for any fixed $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$. Then, it follows from the energy inequality (1.15) and the lower semicontinuity that

$$\frac{1}{2}||v^*(t)||^2_{L^2(\mathbb{R}^2)} + \frac{L_1}{2}||\nabla Q^*(t)||^2_{L^2(\mathbb{R}^2)} \le \frac{1}{2}||v_0^*||^2_{L^2(\mathbb{R}^2)} + \frac{L_1}{2}||\nabla Q_0^*||^2_{L^2(\mathbb{R}^2)}$$

for any fixed $t \in \mathcal{L}_\infty \cap \mathcal{A}_\infty$. Since $|\mathcal{L}_\infty \cap \mathcal{A}_\infty| = T$, one has that

$$\frac{1}{2}||v^*||^2_{L^\infty_t L^2_x} + \frac{L_1}{2}||\nabla Q^*||^2_{L^\infty_t L^2_x} \le \frac{1}{2}||v_0^*||^2_{L^2(\mathbb{R}^2)} + \frac{L_1}{2}||\nabla Q_0^*||^2_{L^2(\mathbb{R}^2)}.$$

Therefore, one gets the desired (4.25).

Next, We show that v^* and ∇d^* are weakly continuous to the initial data v_0^* and ∇d_0^* as $t \to 0$ on $L^2(\tilde{\Omega})$. Let

$$\tilde{\varphi}(t) = \begin{cases} e^{\frac{1}{|t|^2-1}}, & |t| < 1; \\ 0, & |t| \ge 1, \end{cases}$$

and $\overline{\varphi} = \tilde{\varphi}/\int_{\mathbb{R}} \tilde{\varphi}(t)dt$. Then, it is easy to see that $\overline{\varphi}$ is smooth and $\int_{\mathbb{R}} \overline{\varphi}(t)dt = 1$. For $t_2 \in (0,T)$ and $\tilde{\phi} \in \mathcal{D}$, set

$$\psi_{\tilde{\epsilon}}(t,x) = \tilde{\phi}(x)\left(\int_0^t \frac{1}{\tilde{\epsilon}}\overline{\varphi}(\frac{t_2-\tau}{\tilde{\epsilon}})d\tau - 1\right).$$

Note that $|\int_0^t \frac{1}{\tilde{\epsilon}}\overline{\varphi}(\frac{t_2-\tau}{\tilde{\epsilon}})d\tau - 1| \le 1$ for all $t \ge 0$ and

$$\lim_{\tilde{\epsilon}\to 0^+}\int_0^t \frac{1}{\tilde{\epsilon}}\overline{\varphi}(\frac{t_2-\tau}{\tilde{\epsilon}})d\tau - 1 = \begin{cases} 0, & t > t_2; \\ -1, & t < t_2. \end{cases}$$

Then, it follows from (4.26) that

$$\lim_{\tilde{\epsilon}\to 0^+}\int_0^T\int_{\mathbb{R}^2} v^*(t,x)\cdot\partial_t\psi_{\tilde{\epsilon}}(t,x) = \lim_{\tilde{\epsilon}\to 0^+}\int_0^T \frac{1}{\tilde{\epsilon}}\overline{\varphi}(\frac{t_2-t}{\tilde{\epsilon}})\int_{\mathbb{R}^2} v^*(t,x)\cdot\tilde{\phi}(x) = \int_{\mathbb{R}^2} v^*(t_2,x)\cdot\tilde{\phi}(x).$$

Using this, taking $\psi_{\tilde{\epsilon}}$ into (4.8) and sending $\tilde{\epsilon} \to 0^+$ yield that

$$\begin{aligned} &-\int_0^{t_2}\int_{\mathbb{R}^2}[-(v^*\cdot\overline{\nabla\tilde{\phi}})\cdot v^* + \eta\overline{D^*} : \overline{\nabla\tilde{\phi}} + L_1\nabla Q^*\odot\nabla Q^* : \underline{\nabla\tilde{\phi}}]dxdt \\ &-\int_0^{t_2}\int_{\mathbb{R}^2}[Q^*\cdot H^* - H^*\cdot Q^* - S_{Q^*}(H^*)] : \overline{\nabla\tilde{\phi}}dxdt \\ = &\int_{\mathbb{R}^2} v^*(t_2)\cdot\tilde{\phi}(x)dx - \int_{\mathbb{R}^2} v_0^*\cdot\tilde{\phi}(x)dx \end{aligned}$$

by (4.1)-(4.2) and the Lebesgue dominated convergence theorem. Therefore, one has that

$$\lim_{t_2\to 0}\int_{\mathbb{R}^2} v^*(t_2,x)\cdot\tilde{\phi}(x)dx = \int_{\mathbb{R}^2} v_0^*(x)\cdot\tilde{\phi}(x)dx,$$

which implies that

$$v^* \text{ is weakly continuous on } L^2(\tilde{\Omega}) \text{ to the initial data } v_0^* \tag{4.28}$$

by (4.26). Similarly, for $\tilde{\zeta} \in C_0^\infty(\mathbb{R}^2, \mathcal{Q}_0)$, taking $\varphi_{\tilde{\epsilon}}(t,x) = \tilde{\zeta}(x)(\int_0^t \frac{1}{\tilde{\epsilon}}\overline{\varphi}(\frac{t_2-\tau}{\tilde{\epsilon}})d\tau - 1)$ into (4.7) and sending $\tilde{\epsilon} \to 0^+$, one can get

$$\begin{aligned} & \int_0^{t_2}\int_{\mathbb{R}^2}[(\underline{v^*}\cdot\nabla\tilde{\zeta}) : Q^* + \frac{1}{\Gamma}H^* : \tilde{\zeta}] - \int_0^{t_2}\int_{\mathbb{R}^2}[Q^*\cdot\overline{\Lambda^*} - \overline{\Lambda^*}\cdot Q^* - S_{Q^*}(\overline{D^*})] : \tilde{\zeta} \\ = & \int_{\mathbb{R}^2} Q^*(t_2,x) : \tilde{\zeta}(x)dx - \int_{\mathbb{R}^2} Q_0^* : \tilde{\zeta}(x)dx \end{aligned}$$

by (4.26), (4.1)-(4.2) and the Lebesgue dominated convergence theorem. Therefore,

$$\lim_{t_2\to 0}\int_{\mathbb{R}^2} Q^*(t_2,x) : \tilde{\zeta}(x)dx = \int_{\mathbb{R}^2} Q_0^* : \tilde{\zeta}(x)dx. \tag{4.29}$$

Note that (4.26) implies that d^* is weakly continuous on $L^2(\tilde{\Omega})$. Due to $\nabla Q^* \in L_t^\infty L_x^2$, one can take $\tilde{\zeta} = (d_0^* \otimes d_0^* - \mathbb{I})\hat{\zeta}$ with $\hat{\zeta} \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ into (4.29), then

$$\lim_{t_2\to 0}\int_{\mathbb{R}^2} d^*(t_2,x)\cdot d_0^*(x)\hat{\zeta}dx = \int_{\mathbb{R}^2} d^*(0,x)\cdot d_0^*(x)\hat{\zeta}dx = \int_{\mathbb{R}^2}\hat{\zeta}dx,$$

which implies that d^* is weakly continuous on $L^2(\tilde{\Omega})$ to the initial data d_0^*. Then, this together with (4.26), implies that

$$\nabla d^* \text{ is weakly continuous on } L^2(\tilde{\Omega}) \text{ to the initial data } \nabla d_0^*. \tag{4.30}$$

Finally we show that $(v^*(\cdot,t), \nabla d^*(t,\cdot)) \to (v_0^*, \nabla d_0^*)$ as $t \to 0$ in $L^2(\mathbb{R}^2)$. If not, there exist a real number $\delta_4 > 0$ and a subsequence of $\{(v^*(t,\cdot), d^*(t,\cdot))\}_{t>0}$ such that

$$\lim_{t\to 0^+}\left(\frac{1}{2}||v^*(t,\cdot) - v_0^*||^2_{L^2(\mathbb{R}^2)} + L_1 s_+^2||\nabla d^*(t,\cdot) - \nabla d_0^*||^2_{L^2(\mathbb{R}^2)}\right) \geq \delta_4 > 0,$$

which yields

$$\begin{aligned} & \lim_{t\to 0^+}\left[\int_{\mathbb{R}^2}(\frac{1}{2}|v^*|^2 + L_1 s_+^2|\nabla d^*|)dx - \int_{\mathbb{R}^2}(v^*\cdot v_0^* + 2L_1 s_+^2\nabla d^* : \nabla d_0^*)dx\right] \\ \geq \;& \delta_4 - \int_{\mathbb{R}^2}(\frac{1}{2}|v_0^*|^2 + L_1 s_+^2|\nabla d_0^*|)dx. \end{aligned} \tag{4.31}$$

It follows from (4.28) and (4.30) that

$$\lim_{t\to 0^+}\int_{\tilde{\Omega}}(v^*\cdot v_0^* + 2L_1 s_+^2\nabla d^* : \nabla d_0^*)dx = \int_{\tilde{\Omega}}(|v_0^*|^2 + 2L_1 s_+^2|\nabla d_0^*|^2)dx \tag{4.32}$$

for any smooth bounded domain $\tilde{\Omega} \subset \mathbb{R}^2$. Choose suitable big $\tilde{\Omega}$ such that

$$\int_{\mathbb{R}^2\setminus\tilde{\Omega}}(|v_0^*|^2 + 2L_1 s_+^2|\nabla d_0^*|^2)dx \leq \min\left\{\frac{\delta_4^2}{32E_0}, \frac{\delta_4}{4}\right\},$$

where E_0 is given in (1.17). Based on this fact, it follows from (4.25) that

$$\left|\lim_{t\to 0^+}\int_{\mathbb{R}^2\setminus\tilde{\Omega}}(v^*\cdot v_0^* + 2L_1 s_+^2\nabla d^* : \nabla d_0^*)dx\right| \leq \frac{\delta_4}{2}.$$

Then, substituting the above estimate and (4.32) into (4.31), one has

$$\lim_{t\to0^+}\int_{\mathbb{R}^2}(\frac{1}{2}|v^*|^2+L_1s_+^2|\nabla d^*|)dx\geq\int_{\mathbb{R}^2}(\frac{1}{2}|v_0^*|^2+L_1s_+^2|\nabla d_0^*|)+\delta_4$$
$$-\int_{\mathbb{R}^2\backslash\tilde{\Omega}}(|v_0^*|^2+2L_1s_+^2|\nabla d_0^*|^2)dx+\lim_{t\to0^+}\int_{\mathbb{R}^2\backslash\tilde{\Omega}}(v^*\cdot v_0^*+2L_1s_+^2\nabla d^*:\nabla d_0^*)dx$$
$$\geq\int_{\mathbb{R}^2}(\frac{1}{2}|v_0^*|^2+L_1s_+^2|\nabla d_0^*|)+\frac{\delta_4}{4},$$

which contradicts (4.25). Hence the desired conclusion follows.

Step 3. We prove that the limit (v^*,d^*) satisfies the equalities (1.40) and (1.41). We show first that (v^*,d^*) satisfies the equalities (1.41). To this end, one can show first that

$$\int_0^T\int_{\mathbb{R}^2}d_t^*\cdot\zeta dxdt=-\int_{\mathbb{R}^2}d_0^*\cdot\zeta(0,\cdot)dx-\int_0^T\int_{\mathbb{R}^2}d^*\cdot\zeta_t dxdt. \tag{4.33}$$

for any $\zeta\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3)$. Note that $|\nabla Q^*|^2=2s_+^2|\nabla d^*|^2$ and $|Q_t^*|^2=2s_+^2|d_t^*|^2$. Then, it follows from $\nabla Q^*\in L_t^\infty L_x^2$ and $Q_t^*\in L^{\frac{4}{3}}((0,T)\times\tilde{\Omega})$ that $\nabla d^*\in L_t^\infty L_x^2$ and $d_t^*\in L^{\frac{4}{3}}((0,T)\times\tilde{\Omega})$ for any smooth bounded domain $\tilde{\Omega}\subset\mathbb{R}^2$. By **Step 2** in this section, d^* is weakly continuous on $L^2(\tilde{\Omega})$ to the initial data d_0^*. Therefore, (4.33) holds.

It follows from (4.1)-(4.2) and (4.4) that (4.7) holds for $\varphi\in L^4((0,T)\times\tilde{\Omega},\mathcal{Q}_0)$ with any smooth bounded domain $\tilde{\Omega}\subset\mathbb{R}^2$. Therefore, one can take $\varphi=d^*\otimes[\zeta-(d^*\cdot\zeta)d^*]+[\zeta-(d^*\cdot\zeta)d^*]\otimes d^*$ in (4.7) due to $d^*\in\mathbb{S}^2$ and $\zeta\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3)$. Then, one has

$$\int_0^T\int_{\mathbb{R}^2}Q^*:\varphi_t dxdt=-2s_+\int_0^T\int_{\mathbb{R}^2}d_t^*\cdot\zeta dxdt=2s_+\int_0^T\int_{\mathbb{R}^2}d^*\cdot\zeta_t dxdt+2s_+\int_{\mathbb{R}^2}d_0^*\cdot\zeta(0,\cdot)dx \tag{4.34}$$

by using (4.33). Note that

$$\int_{\mathbb{R}^2}Q_0^*:\varphi(0,\cdot)dx=2\int_{\mathbb{R}^2}Q_0^*:\{d_0^*\otimes[\zeta(0,\cdot)-(d_0^*\cdot\zeta(0,\cdot))d_0^*]\}dx=0. \tag{4.35}$$

Since $\partial_1v_1^\epsilon+\partial_2v_2^\epsilon=0$ for all $\epsilon>0$, therefore the limit v^* of v^ϵ satisfies $\partial_1v_1^*+\partial_2v_2^*=0$. Then, direct calculations yield that

$$\int_0^T\int_{\mathbb{R}^2}(\underline{v^*}\cdot\nabla\varphi):Q^*dxdt=-2s_+\int_0^T\int_{\mathbb{R}^2}(\underline{v^*}\cdot\nabla d^*)\cdot\zeta dxdt=2s_+\int_0^T\int_{\mathbb{R}^2}(\underline{v^*}\cdot\nabla\zeta)\cdot d^*dxdt, \tag{4.36}$$

$$\int_0^T\int_{\mathbb{R}^2}(Q^*\cdot\overline{\Lambda^*}-\overline{\Lambda^*}\cdot Q^*):\varphi dxdt=-2s_+\int_0^T\int_{\mathbb{R}^2}(\overline{\Lambda^*}\cdot d^*)\cdot\zeta dxdt, \tag{4.37}$$

$$\int_0^T\int_{\mathbb{R}^2}S_{Q^*}(\overline{D^*}):\varphi dxdt=\frac{2\xi(s_++2)}{3}[\overline{D^*}\cdot d^*-(\overline{D^*}:d^*\otimes d^*)d^*]\cdot\zeta dxdt. \tag{4.38}$$

It remains to calculate $\int_0^T\int_{\mathbb{R}^2}H^*:\varphi dxdt$. For any fixed $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$, recall the definition of $\mathcal{B}(t)$ in **Step 1** in this section. For each $x_i(t)\in\mathcal{B}(t),1\leq i\leq L(t)$, define a smooth function

$$\chi_{r(t)}^{x_i(t)}(y)=\begin{cases}1, & y\in B_{r(t)}(x_i(t))\ ;\\ 0, & y\notin B_{2r(t)}(x_i(t)),\end{cases}$$

with $0\leq\chi_{r(t)}^{x_i(t)}\leq1,|\nabla\chi_{r(t)}^{x_i(t)}|\leq\frac{C}{r(t)}$. Then, one can choose $r(t)$ small enough such that $B_{2r(t)}(x_1(t)),\cdots,B_{2r(t)}(x_{L(t)}(t))$ are disjoint to each other. Then, it follows from $\varphi\in T_{Q^*}\mathcal{N}$

and Lemma 3.1 that

$$\int_{\mathbb{R}^2}(H^*:\varphi)(t,x)dx=\lim_{r(t)\to 0}\int_{\mathbb{R}^2}H^*(t,x):\varphi(t,x)\left(1-\sum_{i=1}^{L(t)}\chi_{r(t)}^{x_i(t)}\right)dx$$
$$=\lim_{r(t)\to 0}\int_{\mathbb{R}^2}L_1\Delta Q^*(t,x):\varphi(t,x)\left(1-\sum_{i=1}^{L(t)}\chi_{r(t)}^{x_i(t)}\right)dx$$
$$=-2L_1s_+\int_{\mathbb{R}^2}(\partial_k d^*\cdot\partial_k\zeta-|\nabla d^*|^2d^*\cdot\zeta)(t,x)dx, \tag{4.39}$$

where one has used the geometric structure of $\mathcal{J}^*(t,x)$ for $x\in\mathbb{R}^2\setminus\mathcal{B}(t), t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$. Therefore, it follows from (4.7) and (4.34)-(4.39) that (1.41) holds true.

Next, we prove that (v^*,d^*) satisfies the equality (1.40). By direct calculations, one has

$$\int_0^T\int_{\mathbb{R}^2}\nabla Q^*\odot\nabla Q^*:\underline{\nabla\psi}dxdt=2s_+^2\int_0^T\int_{\mathbb{R}^2}(\nabla d^*\odot\nabla d^*):\underline{\nabla\psi}dxdt \tag{4.40}$$

and

$$[Q^*\cdot H^*-H^*\cdot Q^*-S_{Q^*}(H^*)]:\overline{\nabla\psi}$$
$$=[(1-\xi)Q^*\cdot H^*-(1+\xi)H^*\cdot Q^*-\frac{2}{3}\xi H^*+2\xi(Q^*:H^*)(Q^*+\frac{1}{3}\mathbb{I})]:\overline{\nabla\psi}. \tag{4.41}$$

It follows from (4.1)-(4.2), (4.4), (4.26) and (4.29) that the equality (4.7) can be rewritten as

$$\int_0^T\int_{\mathbb{R}^2}H^*:\varphi dxdt=\Gamma\int_0^T\int_{\mathbb{R}^2}[Q_t^*+\underline{v^*}\cdot\nabla Q^*+Q^*\cdot\overline{\Lambda^*}-\overline{\Lambda^*}\cdot Q^*-S_{Q^*}(\overline{D^*})]:\varphi dxdt, \tag{4.42}$$

where $\varphi\in L^4((0,T)\times\tilde{\Omega},\mathcal{Q}_0)$ for any smooth bounded domain $\tilde{\Omega}\subset\mathbb{R}^2$. Since $\nabla Q^*\in L_t^\infty L_x^2$ and $\psi\in C_0^\infty([0,T)\times\mathbb{R}^2,\mathbb{R}^3),\partial_1\psi_1+\partial_2\psi_2=0$, it can be checked that the test function φ can be taken as $\varphi=\frac{1}{2}[Q^*\cdot\overline{\nabla\psi}+(\overline{\nabla\psi})^T\cdot Q^*-2(Q^*:\overline{\nabla\psi})\mathbb{I}]$. Then, one can get that

$$\int_0^T\int_{\mathbb{R}^2}(Q^*\cdot H^*):\overline{\nabla\psi}dxdt=\frac{1}{2}\int_0^T\int_{\mathbb{R}^2}H^*:[Q^*\cdot\overline{\nabla\psi}+(\overline{\nabla\psi})^T\cdot Q^*-2(Q^*:\overline{\nabla\psi})\mathbb{I}]dxdt$$
$$=\Gamma s_+^2\int_0^T\int_{\mathbb{R}^2}(\frac{2}{3}d^*\otimes\overline{N^*}-\frac{1}{3}\overline{N^*}\otimes d^*):\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi\int_0^T\int_{\mathbb{R}^2}\left[\frac{s_+^2(1-2s_+)}{3}(\overline{D^*}:d^*\otimes d^*)d^*\otimes d^*-\frac{2s_+(1-s_+)}{9}\overline{D^*}\right]:\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi\int_0^T\int_{\mathbb{R}^2}\left(\frac{2s_+}{3}d^*\otimes d^*\cdot\overline{D^*}-\frac{s_+^2}{3}\overline{D^*}\cdot d^*\otimes d^*\right):\overline{\nabla\psi}dxdt. \tag{4.43}$$

where $\overline{N^*}=d_t^*+\underline{v^*}\cdot\nabla d^*-\overline{\Lambda^*}\cdot d^*$, and one has used the fact that Q^*,H^* and $\overline{D^*}$ are symmetric with zero traces. Similarly, choosing $\varphi=\frac{1}{2}[\overline{\nabla\psi}\cdot Q^*+Q^*\cdot(\overline{\nabla\psi})^T-2(Q^*:\overline{\nabla\psi})\mathbb{I}]$

leads to

$$\int_0^T\int_{\mathbb{R}^2}(H^*\cdot Q^*):\overline{\nabla\psi}dxdt=\Gamma s_+^2\int_0^T\int_{\mathbb{R}^2}(-\frac{1}{3}d^*\otimes\overline{N^*}+\frac{2}{3}\overline{N^*}\otimes d^*):\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi\int_0^T\int_{\mathbb{R}^2}\left[\frac{s_+^2(1-2s_+)}{3}(\overline{D^*}:d^*\otimes d^*)d^*\otimes d^*-\frac{2s_+(1-s_+)}{9}\overline{D^*}\right]:\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi\int_0^T\int_{\mathbb{R}^2}\left(-\frac{s_+^2}{3}d^*\otimes d^*\cdot\overline{D^*}+\frac{2s_+}{3}\overline{D^*}\cdot d^*\otimes d^*\right):\overline{\nabla\psi}dxdt. \tag{4.44}$$

Next, choosing $\varphi=\frac{1}{2}[\overline{\nabla\psi}+(\overline{\nabla\psi})^T]$, one can get

$$\int_0^T\int_{\mathbb{R}^2}H^*:\overline{\nabla\psi}dxdt=\Gamma s_+\int_0^T\int_{\mathbb{R}^2}(d^*\otimes\overline{N^*}+\overline{N^*}\otimes d^*):\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi\int_0^T\int_{\mathbb{R}^2}\left[-2s_+^2(\overline{D^*}:d^*\otimes d^*)d^*\otimes d^*+\frac{2(1-s_+)}{3}\overline{D^*}\right]:\overline{\nabla\psi}dxdt$$
$$-\Gamma\xi s_+\int_0^T\int_{\mathbb{R}^2}\left(d^*\otimes d^*\cdot\overline{D^*}+\overline{D^*}\cdot d^*\otimes d^*\right):\overline{\nabla\psi}dxdt. \tag{4.45}$$

Finally, choosing $\varphi=Q^*(Q^*+\frac{1}{3}\mathbb{I}):\overline{\nabla\psi}$, one can get

$$\int_0^T\int_{\mathbb{R}^2}(H^*:Q^*)(Q^*+\frac{1}{3}\mathbb{I}):\overline{\nabla\psi}dxdt$$
$$=\quad -2\Gamma\xi\frac{s_+^2+s_+^3-2s_+^4}{3}\int_0^T\int_{\mathbb{R}^2}(\overline{D^*}:d^*\otimes d^*)d^*\otimes d^*:\overline{\nabla\psi}dxdt \tag{4.46}$$

Then, substituting (4.40)-(4.46) into the equality (4.8), we obtain the equality (1.40).

Step 4. This step is aim to prove that (v^*,d^*) satisfies the energy inequality (1.37). By the definition of $\mathcal{L}_\infty\cap\mathcal{A}_\infty$ in the **Step 1** in this section, it follows from the energy inequality (1.15), (4.27) and the lower semicontinuity that

$$\int_{\mathbb{R}^2}\left(\frac{1}{2}|v^*|^2+\frac{L_1}{2}|\nabla Q^*|^2\right)(\cdot,t)dx+\int_0^t\int_{\mathbb{R}^2}\left(\eta|\overline{D^*}|^2+\frac{1}{\Gamma}|H^*|^2\right)dxdt$$
$$\leq\quad\int_{\mathbb{R}^2}\left(\frac{1}{2}|v_0^*|^2+\frac{L_1}{2}|\nabla Q_0^*|^2\right)dx \tag{4.47}$$

for any $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$. The remaining task is to show that (1.37) and (4.47) are equivalent for any $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$. To this end, due to $Q^*=s_+(d^*\otimes d^*-\frac{1}{3}\mathbb{I})$, it suffices to show that

$$\int_{\mathbb{R}^2}|H^*(t,x)|^2dx=\int_{\mathbb{R}^2}\left[\Gamma(\alpha_1+\frac{\gamma_2^2}{\gamma_1})|\overline{D^*}:d^*\otimes d^*|^2+\Gamma(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1})|\overline{D^*}\cdot d^*|^2\right](t,x)dx$$
$$+\int_{\mathbb{R}^2}\left[\frac{4\Gamma^2\xi^2(1-s_+)^2}{9}|\overline{D^*}|^2+2L_1^2s_+^2|\Delta d^*+|\nabla d^*|^2d^*|^2\right](t,x)dx \tag{4.48}$$

for any $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$.

For fixed $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$ and $x\in\mathbb{R}^2\setminus\mathcal{B}(t)$, Lemma 3.1 implies that

$$\mathcal{J}^*(x,t)=a_3(x,t)e_3+a_4(x,t)e_4+a_5(x,t)e_5,$$

where

$$e_3=\frac{1}{\sqrt{2}}(d_2\otimes d_1+d_1\otimes d_2),e_4=\frac{1}{\sqrt{2}}(d_1\otimes d_1-d_2\otimes d_2),$$

$$e_5 = \sqrt{6}(\frac{1}{2}d_1 \otimes d_1 + \frac{1}{2}d_2 \otimes d_2 - \frac{1}{3}\mathbb{I}),$$

and $d_1, d_2 \in \mathbb{S}^2, d_1 \cdot d_2 = 0, d_1 \cdot d^* = 0, d_2 \cdot d^* = 0$. Since d_1, d_2 and d^* form an orthonormal basis to $\mathbb{R}^3$, one has that

$$|\partial_k d^*|^2 = |\partial_k d^* \cdot d_1|^2 + |\partial_k d^* \cdot d_2|^2, \quad k = 1, 2, \tag{4.49}$$

$$|\overline{D^*} \cdot d_1|^2 = (\overline{D^*} : d_1 \otimes d_1)^2 + (\overline{D^*} : d_2 \otimes d_1)^2 + (\overline{D^*} : d^* \otimes d_1)^2, \tag{4.50}$$

$$|\overline{D^*} \cdot d_2|^2 = (\overline{D^*} : d_1 \otimes d_2)^2 + (\overline{D^*} : d_2 \otimes d_2)^2 + (\overline{D^*} : d^* \otimes d_2)^2, \tag{4.51}$$

$$|\overline{D^*} \cdot d^*|^2 = (\overline{D^*} : d_1 \otimes d^*)^2 + (\overline{D^*} : d_2 \otimes d^*)^2 + (\overline{D^*} : d^* \otimes d^*)^2, \tag{4.52}$$

$$|\overline{D^*}|^2 = |\overline{D^*} \cdot d_1|^2 + |\overline{D^*} \cdot d_2|^2 + |\overline{D^*} \cdot d^*|^2. \tag{4.53}$$

For any $\phi \in C_0^\infty([0,T) \times \mathbb{R}^2, \mathbb{R})$, one can take $\varphi = e_3\phi$ into the equality (4.42) due to $|d_1| = |d_2| = 1$. Then direct calculations yield that

$$\begin{aligned} 0 &= \int_{\mathcal{L}_\infty \cap \mathcal{A}_\infty} \int_{\mathbb{R}^2 \setminus \mathcal{B}(t)} \left[\frac{1}{\Gamma}(L_1 \Delta Q^* - \mathcal{J}^*) + S_{Q^*}(\overline{D^*}) \right] : e_3 \phi dx dt \\ &= \int_{\mathcal{L}_\infty \cap \mathcal{A}_\infty} \int_{\mathbb{R}^2 \setminus \mathcal{B}(t)} \left[\sqrt{8}\frac{L_1}{\Gamma} s_+ (\partial_k d^* \cdot d_1)(\partial_k d^* \cdot d_2) - \frac{a_3}{\Gamma} + \sqrt{8}\xi \frac{1 - s_+}{3} \overline{D^*} : d_2 \otimes d_1 \right] \phi. \end{aligned}$$

This means

$$a_3 = 2\sqrt{2} L_1 s_+ (\partial_k d^* \cdot d_1)(\partial_k d^* \cdot d_2) - 2\sqrt{2}\Gamma\xi \frac{s_+ - 1}{3} \overline{D^*} : d_2 \otimes d_1$$

in $(\mathcal{L}_\infty \cap \mathcal{A}_\infty) \times (\mathbb{R}^2 \setminus \mathcal{B}(t))$. Similarly, it holds that

$$a_4 = \sqrt{2} L_1 s_+ \sum_{k=1}^{2} (|\partial_k d^* \cdot d_1|^2 - |\partial_k d^* \cdot d_2|^2) - \sqrt{2}\Gamma\xi \frac{s_+ - 1}{3} \overline{D^*} : (d_1 \otimes d_1 - d_2 \otimes d_2)$$

and

$$a_5 = \sqrt{6} L_1 s_+ |\nabla d^*|^2 + \frac{\sqrt{6}\Gamma\xi(2s_+^2 - s_+ - 1)}{3} \overline{D^*} : d^* \otimes d^* \tag{4.54}$$

in $(\mathcal{L}_\infty \cap \mathcal{A}_\infty) \times (\mathbb{R}^2 \setminus \mathcal{B}(t))$, where one has used the equality (4.49), $\mathbb{I} = d_1 \otimes d_1 + d_2 \otimes d_2 + d^* \otimes d^*$ and $\overline{D^*}_{ii} = 0$ in (4.54). Set

$$b_1 = -2\sqrt{2}\Gamma\xi \frac{s_+ - 1}{3} \overline{D^*} : d_2 \otimes d_1, \quad b_2 = -\sqrt{2}\Gamma\xi \frac{s_+ - 1}{3} \overline{D^*} : (d_1 \otimes d_1 - d_2 \otimes d_2),$$

$$b_3 = \frac{\sqrt{6}\Gamma\xi(2s_+^2 - s_+ - 1)}{3} \overline{D^*} : d^* \otimes d^*, \quad c_1 = a_3 - b_1 = 2\sqrt{2} L_1 s_+ (\partial_k d^* \cdot d_1)(\partial_k d^* \cdot d_2),$$

and

$$c_2 = a_4 - b_2 = \sqrt{2} L_1 s_+ \sum_{k=1}^{2} (|\partial_k d^* \cdot d_1|^2 - |\partial_k d^* \cdot d_2|^2), \quad c_3 = a_5 - b_3 = \sqrt{6} L_1 s_+ |\nabla d^*|^2.$$

Then, it follows from the definition of $Q^*, \mathcal{J}^*, a_i$ $(i = 3,4,5)$, b_j and c_j $(j = 1,2,3)$ and direct calculations that for $(t,x) \in (\mathcal{L}_\infty \cap \mathcal{A}_\infty) \times (\mathbb{R}^2 \setminus \mathcal{B}(t))$, it holds that

$$\begin{aligned} |L_1\Delta Q^* - \mathcal{J}^*|^2 &= L_1^2(\Delta Q^* : \Delta Q^*) - 2L_1(\Delta Q^* : \mathcal{J}^*) + (\mathcal{J}^* : \mathcal{J}^*) \\ &= 2L_1^2 s_+^2 \left(|\Delta d^*|^2 + |\nabla d^*|^4 + 2\sum_{k,l=1}^2 |\partial_k d^* \cdot \partial_l d^*|^2 \right) \\ &\quad -2(a_3c_1 + a_4c_2 + a_5c_3) + a_1^2 + a_2^2 + a_3^2 \\ &= 2L_1 s_+^2 \left(|\Delta d^*|^2 + |\nabla d^*|^4 + 2\sum_{k,l=1}^2 |\partial_k d^* \cdot \partial_l d^*|^2 \right) \\ &\quad -(c_1^2 + c_2^2 + c_3^2) + b_1^2 + b_2^2 + b_3^2 \\ &= 2L_1 s_+^2(|\Delta d^*|^2 - |\nabla d^*|^4) + b_1^2 + b_2^2 + b_3^2 \\ &\quad +4L_1^2 s_+^2 \sum_{k,l=1}^2 |\partial_k d^* \cdot \partial_l d^*|^2 - (c_1^2 + c_2^2 + 2L_1^2 s_+^2 |\nabla d^*|^4). \end{aligned}$$

Using (4.49) and the fact that

$$\partial_k d^* \cdot \partial_l d^* = (d_1 \cdot \partial_k d^*)(d_1 \cdot \partial_l d^*) + (d_2 \cdot \partial_k d^*)(d_2 \cdot \partial_l d^*),$$

one can calculate that

$$4L_1^2 s_+^2 \sum_{k,l=1}^2 |\partial_k d^* \cdot \partial_l d^*|^2 - (c_1^2 + c_2^2 + 2L_1^2 s_+^2 |\nabla d^*|^4) = 0.$$

Thus, one obtains that on $(\mathcal{L}_\infty \cap \mathcal{A}_\infty) \times (\mathbb{R}^2 \setminus \mathcal{B}(t))$,

$$\begin{aligned} |L_1\Delta Q^* - \mathcal{J}^*|^2 &= 2L_1 s_+^2(|\Delta d^*|^2 - |\nabla d^*|^4) + b_1^2 + b_2^2 + b_3^2 \\ &= 2L_1 s_+^2 \left|\Delta d^* + |\nabla d^*|^2 d^*\right|^2 + b_1^2 + b_2^2 + b_3^2. \end{aligned} \tag{4.55}$$

It remains to calculate $b_1^2 + b_2^2 + b_3^2$. It follows from the definitions of b_i $(i = 1,2)$ that

$$b_1^2 + b_2^2 = \frac{2\Gamma^2\xi^2(1-s_+)^2}{9}\left[4(\overline{D^*} : d_2 \otimes d_1)^2 + (\overline{D^*} : d_1 \otimes d_1 - \overline{D^*} : d_2 \otimes d_2)^2\right]. \tag{4.56}$$

Note that direct computations yield that

$$\begin{aligned} &(\overline{D^*} : d_1 \otimes d_1 - \overline{D^*} : d_2 \otimes d_2)^2 \\ = &-(\overline{D^*} : d_1 \otimes d_1 + \overline{D^*} : d_2 \otimes d_2)^2 + 2(\overline{D^*} : d_1 \otimes d_1)^2 + 2(\overline{D^*} : d_2 \otimes d_2)^2. \end{aligned} \tag{4.57}$$

Since $d_1 \otimes d_1 + d_2 \otimes d_2 + d^* \otimes d^* = \mathbb{I}$ and $\overline{D^*}_{ii} = 0$, so

$$\overline{D^*} : (d_1 \otimes d_1 + d_2 \otimes d_2) = -\overline{D^*} : d^* \otimes d^*. \tag{4.58}$$

(4.57) and (4.58) yield

$$(\overline{D^*} : d_1 \otimes d_1 - \overline{D^*} : d_2 \otimes d_2)^2 = 2(\overline{D^*} : d_1 \otimes d_1)^2 + 2(\overline{D^*} : d_2 \otimes d_2)^2 - (\overline{D^*} : d^* \otimes d^*)^2. \tag{4.59}$$

On the other hand, (4.50) and (4.51) imply that

$$(\overline{D^*} : d_1 \otimes d_1)^2 + (\overline{D^*} : d_2 \otimes d_1)^2 = |\overline{D^*} \cdot d_1|^2 - (\overline{D^*} : d^* \otimes d_1)^2,$$

$$(\overline{D^*} : d_1 \otimes d_2)^2 + (\overline{D^*} : d_2 \otimes d_2)^2 = |\overline{D^*} \cdot d_2|^2 - (\overline{D^*} : d^* \otimes d_2)^2.$$

These, together (4.59), (4.52) and (4.53), imply that

$$\begin{aligned}
& 4(\overline{D^*}:d_2\otimes d_1)^2+(\overline{D^*}:d_1\otimes d_1-\overline{D^*}:d_2\otimes d_2)^2 \\
= & \; 2(\overline{D^*}:d_1\otimes d_1)^2+2(\overline{D^*}:d_2\otimes d_2)^2+4(\overline{D^*}:d_2\otimes d_1)^2-(\overline{D^*}:d^*\otimes d^*)^2 \\
= & \; 2(|\overline{D^*}|^2-2|\overline{D^*}\cdot d^*|^2)+(\overline{D^*}:d^*\otimes d^*)^2.
\end{aligned}$$

This and (4.56) yield that

$$b_1^2+b_2^2=\frac{2\Gamma^2\xi^2(1-s_+)^2}{9}(2|\overline{D^*}|^2-4|\overline{D^*}\cdot d^*|^2+|\overline{D^*}:d^*\otimes d^*|^2). \tag{4.60}$$

Consequently,

$$\begin{aligned}
b_1^2+b_2^2+b_3^2 = & \; \frac{4\Gamma^2\xi^2(1-s_+)^2}{9}|\overline{D^*}|^2-\frac{8\Gamma^2\xi^2(1-s_+)^2}{9}|\overline{D^*}\cdot d^*|^2 \\
& +\frac{(1-s_+)^2+3(2s_+-s_+-1)^2}{9}2\Gamma^2\xi^2(\overline{D^*}:d^*\otimes d^*)^2 \\
= & \; \frac{4\Gamma^2\xi^2(1-s_+)^2}{9}|\overline{D^*}|^2+\Gamma(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1})|\overline{D^*}\cdot d^*|^2 \\
& +\Gamma(\alpha_1+\frac{\gamma_2^2}{\gamma_1})(\overline{D^*}:d^*\otimes d^*)^2.
\end{aligned} \tag{4.61}$$

Therefore, for any $t\in\mathcal{L}_\infty\cap\mathcal{A}_\infty$, we have from (4.55) and (4.61) that

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus\mathcal{B}(t)}|H^*|^2dx = & \int_{\mathbb{R}^2\setminus\mathcal{B}(t)}\left[2L_1^2s_+^2|\Delta d^*+|\nabla d^*|^2d^*|^2+b_1^2+b_2^2+b_3^2\right]dx \\
= & \int_{\mathbb{R}^2\setminus\mathcal{B}(t)}\left[2L_1^2s_+^2|\Delta d^*+|\nabla d^*|^2d^*|^2+\Gamma(\alpha_1+\frac{\gamma_2^2}{\gamma_1})|\overline{D^*}:d^*\otimes d^*|^2\right]dx \\
& +\int_{\mathbb{R}^2\setminus\mathcal{B}(t)}\left[\frac{4\Gamma^2\xi^2(1-s_+)^2}{9}|\overline{D^*}|^2+\Gamma(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1})|\overline{D^*}\cdot d^*|^2\right]dx.
\end{aligned}$$

Note $\int_{\mathbb{R}^2\setminus\mathcal{B}(t)}|H^*|^2dx=\int_{\mathbb{R}^2}|H^*|^2dx$. Thus, the desired (4.48) is proved.

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