Introduction of the Solution to Spatially Homogeneous Boltzmann Equation as a Probability Measure

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Abstract

In this Mini-Course, the development of the spatially homogeneous theory to the Boltzmann equation will be briefly introduced, especially for the well-posedness result of the Cauchy problem in the space of probability measure defined via the Fourier transform. Besides the original solution with finite energy, the infinite energy case is not a priori excluded from the consideration either, where the Bobylev identity and Fourier-based probability metric will be discussed as two powerful tools.

Contents

1 Personal Statement

The lecture note is based on the MATH-IMS joint Mini-Course delivered by the author in the Term 1, 2021-2022 at CUHK. The main prerequisites are a reasonable acquaintance with functional analysis, i.e., elementary topology, Fourier transform, and so forth. Preliminary knowledge about the Boltzmann equation is literally preferred, though the brief introduction will be provided at the beginning.

Due to the current limitation of the author, most likely, there are still at places inadequacies, inconsistency of notations, inadvertently omitted references... Therefore, the lecture note will be constantly updated and frequently uploaded on the website of the author, and hopefully continue to cover up the most recent results of this topic with time evolution.

Any correction and comment will be very welcomed from the readers for further improvement of the lecture note.

2 Teaching Arrangement

So far, a rough arrangement of the four lectures is provided as following, where some adjustments might happen according to the actual progress:

(I) In the lecture1 (10:00 am – 12:00 am, Oct. 7), the Fourier transformation in the kinetic equation and its induced probability metric will firstly introduced, including their basic calculus rules and contractive property, which then leads to the uniqueness of the solution to homogeneous Boltzmann equation with finite energy.

(II) In the lecture2 (3:30 pm – 5:30 pm, Oct. 11) - lecture3 (3:30 pm – 5:30 pm, Oct. 18), the well-posedness result of the homogeneous Boltzmann equation will be derived in the measure valued sense from cutoff to non-cutoff kernel, where the infinite energy solutions are also not a priori excluded from the consideration; moreover, the asymptotic behaviour towards the self-similar profile will also partially discussed.

(III) In the lecture4 (3:30 pm – 5:30 pm, Oct. 25), the contents mentioned above shall firstly be finished off, then, compared with the Maxwellian molecule, the similar methodology will be illustrated how to solve the general hard/soft potential case in the probability measure space; if time permits, some further applications to other kinetic-related model, e.g., dissipative inelastic Boltzmann equation, will be discussed as well.

3 Introduction of Boltzmann Equation

KQ: The following elementary introductions about the Boltzmann equation, especially the collision operator, are priorly assumed to be familiar with our audience, which would be no longer over-repeated during the mini-course.

3.1 The Spatially Homogeneous Boltzmann Equation

In the spatially homogeneous theory of the Boltzmann equation, one is interested in the solution $f(t,x,v)$ which does not depend on the x space variable. This view of point is pretty common in physics, especially when it comes to the problems focusing on the collision operator, as the collision integral operator only acts on the velocity dependence. On the other hand, the interests towards the spatially homogeneous study also arise from the numerical analysis, since almost all numerical schemes succeed from the splitting of the transport step and collision step.

In this case, the homogeneous Boltzmann equation in $\mathbb{R}^3$ reads:

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{3.1}$$

with the non-negative initial condition,

$$f(0,v) = F_0(v), \tag{3.2}$$

where the unknown $f = f(t,v)$ is regarded as the density function of a probability distribution, or more generally, a probability measure; and the initial datum F_0 is also assumed to be a non-negative probability measure on $\mathbb{R}^3$.

The right hand side of (3.1) is the so-called Boltzmann collision operator,

$$
\begin{aligned}
Q(f,f)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\sigma(v-v_*,\sigma)\,[f(v')f(v'_*)-f(v)f(v_*)]\,\mathrm{d}\sigma\,\mathrm{d}v_* \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\omega(v-v_*,\sigma)\,[f(v')f(v'_*)-f(v)f(v_*)]\,\mathrm{d}\omega\,\mathrm{d}v_*,
\end{aligned}
\tag{3.3}
$$

where (v', v'_*) and (v, v_*) represent the velocity pairs before and after a collision, which satisfy the conservation of momentum and energy:

$$
v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \tag{3.4}
$$

so that (v', v'_*) can be expressed in terms of (v, v_*) as

$$
\begin{aligned}
& v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \\
\text{or} \quad & v' = v - v[(v-v_*\cdot\omega)]\omega, \quad v'_* = v + v[(v-v_*\cdot\omega)]\omega,
\end{aligned}
\tag{3.5}
$$

where both of σ and ω are a vector varying over the unit sphere $\mathbb{S}^2$. And this also easily

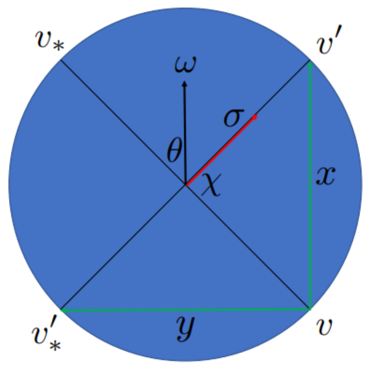

Figure 1: Velocity and unit vector during a classical elastic collision.

implies the relations

$$
v\cdot v_* = v'\cdot v'_*, \quad |v-v_*| = |v'-v'_*|, \quad (v-v_*)\cdot\omega = -(v'-v'_*)\cdot\omega. \tag{3.6}
$$

and

$$
|\langle v-v_*,\omega\rangle| = |v-v_*|\cos\alpha = |v-v_*|\cos\left(\frac{\pi-\theta}{2}\right) = |v-v_*|\sin\frac{\theta}{2}, \tag{3.7}
$$

where α denotes that angle between $v - v_*$ and ω.

Next, we have a more general relation between the σ- and ω- representation in the sense that,

Lemma 3.1. *For the change of variables:*

$$
\sigma = \frac{v-v_*}{|v-v_*|} - 2\left\langle \frac{v-v_*}{|v-v_*|},\omega\right\rangle, \tag{3.8}
$$

it has the Jacobian

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = 2^{d-1}\left|\left\langle \frac{v-v_*}{|v-v_*|},\omega\right\rangle\right|^{d-2}. \tag{3.9}
$$

Proof. Fix the unitary vector $\hat{q} = \frac{v-v_*}{|v-v_*|}$ and $\left\langle \frac{v-v_*}{|v-v_*|}, \omega \right\rangle = \hat{q} \cdot \omega$, then the change of variables can be regarded as a map $\sigma(\omega) : \mathbb{S}^{d-1} \longmapsto \mathbb{S}^{d-1}$ give by

$$\sigma(\omega) = \hat{q} - 2 \left(\hat{q} \cdot \omega \right) \omega. \tag{3.10}$$

Let $\mathcal{O}_{\hat{q}}$ be the orthogonal space to $\hat{q}$, α be the angle between $\hat{q}$ and ω, and θ be the angle between $\hat{q}$ and σ. In this way, one may write

$$\omega = \cos\alpha \hat{q} + \omega_o, \quad \sigma = \cos\theta \hat{q} + \sigma_o \tag{3.11}$$

where $\omega_o, \sigma_o \in \mathcal{O}_{\hat{q}}$. Using the spherical coordinates with north pole given by $\hat{q}$, the measures $\mathrm{d}\omega$ and $\mathrm{d}\sigma$ are given by

$$\mathrm{d}\omega = \sin^{d-2} \alpha \mathrm{d}\hat{\omega}_o \mathrm{d}\alpha, \quad \mathrm{d}\sigma = \sin^{d-2} \theta \mathrm{d}\hat{\sigma}_o \mathrm{d}\theta \tag{3.12}$$

where the measures $\mathrm{d}\hat{\omega}_o$ and $\mathrm{d}\hat{\sigma}_o$ are the Lebesgue measure in $\mathbb{S}^{d-2}(\hat{q})$ parameterized with the vectors ω_o, σ_o respectively. Directly from the expression of the map, we find,

$$\cos\theta = \hat{q} \cdot \sigma = 1 - 2 \left(\hat{q} \cdot \omega \right)^2 = 1 - 2\cos^2 \alpha. \tag{3.13}$$

Then, it follows by direct differentiation that

$$-\sin\theta \mathrm{d}\theta = 4 \cos\alpha \sin\alpha \mathrm{d}\alpha. \tag{3.14}$$

Now, choose a orthonormal base $\{\xi_j\}_{j=1}^{d-2}$ for $\mathcal{O}_{\hat{q}}$. Compute again using the explicit expression of the map

$$\begin{aligned} \sigma_o =& \sum_{j=1}^{d-2} \left(\sigma \cdot \xi_j \right) \xi_j = -2 \left(\hat{q} \cdot \omega \right) \sum_{j=1}^{d-2} \left(\omega \cdot \xi_j \right) \xi_j \\ =& -2 \left(\hat{q} \cdot \omega \right) \omega_o = -2 \cos\alpha \omega_o. \end{aligned} \tag{3.15}$$

Thus, $\hat{\omega_o} = \hat{\sigma_o}$, and as a consequence, $\mathrm{d}\hat{\omega}_o = \mathrm{d}\hat{\sigma}_o$. Gathering these relations all together and using the basic trigomometry

$$\mathrm{d}\omega = \left(\frac{\sin\alpha}{\sin\theta} \right)^{d-3} \frac{\mathrm{d}\sigma}{4|\cos\alpha|} = \frac{\mathrm{d}\sigma}{2^{d-1}|\cos\alpha|^{d-2}}. \tag{3.16}$$

This completes the proof. □

Lemma 3.2. *Fix $\sigma \in \mathbb{S}^{d-1}$ and $q = v - v_*$, the map $u : \mathbb{R}^d \longmapsto \mathbb{R}^d$ given by*

$$u(q) = \frac{q + |q|\sigma}{2} \tag{3.17}$$

has Jacobian

$$\frac{\mathrm{d}u}{\mathrm{d}q} = \frac{1 + \sigma \cdot \hat{q}}{2^d}. \tag{3.18}$$

Proof. Choose an orthonormal base $\{\sigma, \xi_j\}$ with $2 \le j \le d$. Then, the coordinates of this change of variables are

$$\begin{aligned} z_1 =& z \cdot \sigma = \frac{1}{2}\left(q \cdot \sigma + |q|\right) = \frac{1}{2}\left(q_1 + |q|\right), \\ z_j =& z \cdot \xi_j = \frac{1}{2} q_j, \quad j = 2, ..., d. \end{aligned} \tag{3.19}$$

Thus,

$$\frac{\partial z_1}{\partial q_1} = \frac{1}{2}\left(1 + \hat{q} \cdot \sigma\right), \quad \frac{\partial z_j}{\partial q_l} = \frac{1}{2}\delta_{jl}, \quad j = 2, ..., d. \tag{3.20}$$

and, therefore,

$$\frac{\mathrm{d}z}{\mathrm{d}q} = \prod_{j=1}^{d} \left|\frac{\partial z_j}{\partial q_j}\right| = \frac{1 + \hat{q} \cdot \sigma}{2^d}. \tag{3.21}$$

□

3.2 The Boltzmann collision kernel.

The collision kernel B is a non-negative function that depends only on $|v - v_*|$ and cosine of the deviation angle θ, whose specific form can be determined from the intermolecular potential using classical scattering theory. For example, in the case of **Inverse Power Law Potentials** $U(r) = r^{-(\ell-1)}, 2 < \ell < \infty$, where r is the distance between two interacting particles, B can be separated as the kinetic part and angular part:

$$B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta) = b(\cos\theta)\Phi(|v - v_*|), \quad \cos\theta = \frac{\sigma \cdot (v - v_*)}{|v - v_*|}, \tag{3.22}$$

where the kinetic part

$$\Phi(|v - v_*|) = |v - v_*|^\gamma = \begin{cases} \gamma > 0, & \text{Hard potential,} \\ \gamma = 0, & \text{Maxwellian gas,} \\ \gamma < 0, & \text{Soft potential.} \end{cases} \qquad \gamma = \frac{\ell - 5}{\ell - 1} > -3 \text{ (when } d = 3\text{)},$$

and the angular part

$$\sin^{d-2}\theta b(\cos\theta)\big|_{\theta \to 0^+} \sim K\theta^{-1-\nu}, \quad 0 < \nu = \frac{2}{\ell - 1} < 2 \text{ (when } d = 3\text{)}. \tag{3.23}$$

The kernel (3.22) encompasses a wide range of potentials, among which we mention three extreme cases [8]:

(i) $\ell = \infty$, $\gamma = 1$, $\nu = 0$ corresponds to the hard spheres, where B is only proportional to $|v - v_*|$,

$$B(|v - v_*|, \cos\theta) = K|v - v_*|, \quad K > 0; \tag{3.24}$$

(ii) $\ell = 2$, $\gamma = -3$, $\nu = 2$ corresponds to the Coulomb interaction, where B is given by the famous Rutherford formula,

$$B(|v - v_*|, \cos\theta) = \frac{1}{|v - v_*|^3 \sin^4(\theta/2)}; \tag{3.25}$$

(iii) $\ell = 5$, $\gamma = 0$, $\nu = \frac{1}{2}$ corresponds to the literally physical Maxwellian gas, where B does not depend on relative velocity $|v - v_*|$,

$$B(|v - v_*|, \cos\theta) = b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) = b(\cos\theta). \tag{3.26}$$

However, instead of this very special case above, we are interested in the more general case $B = b(\cos\theta)$, not depending on $|v - v_*|$ that,

$$\gamma = 0, \quad 0 < \nu < 2, \tag{3.27}$$

which is called Maxwellian molecules type.

The range of deviation angle θ, namely the angle between pre- and post-collisional velocities, is a full interval $[0, \pi]$, but it is customary to restrict it to $[0, \pi/2]$ mathematically, replacing $b(\cos\theta)$ by its "symmetrized" version [21]:

$$[b(\cos\theta) + b(\cos(\pi - \theta))]\, \mathbf{1}_{0 \le \theta \le \frac{\pi}{2}}, \tag{3.28}$$

which amounts more or less to forbidding the exchange of particles.

Another physically interesting example that is not explicit at all has been called **Debye-Yukawa Potential** $U(r) = \mathrm{e}^{-r}/r$, also asymptotically behaving as $\theta \to 0$:

$$\sin^{d-2}\theta B(|v - v_*|, \cos\theta)\Big|_{\theta \to 0^+} \sim K|v - v_*|\theta^{-1}|\log\theta^{-1}|. \tag{3.29}$$

3.3 Cutoff VS Non-cutoff

As it has been long known, the main difficulty in establishing the well-posedness result for Boltzmann equation is that the singularity of the collision kernel b is not locally integrable in $\sigma \in \mathbb{S}^2$. To avoid this, H. Grad gave the integrable assumption on the collision kernel b_c by a "**Cutoff** " near singularity:

$$\int_{\mathbb{S}^2} b_c\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \mathrm{d}\sigma = 2\pi \int_0^{\frac{\pi}{2}} b_c(\cos\theta)\sin\theta\, \mathrm{d}\theta < \infty. \tag{3.30}$$

However, the full singularity condition for the collision kernel with **Non-cutoff Assumption** is implicitly defined for the angular collision part $b(\cos\theta)$, which asymptotically behaves as $\theta \to 0^+$,

$$\sin\theta b(\cos\theta)\big|_{\theta \to 0^+} \sim K\theta^{-1-\nu}, \quad \nu = \frac{2}{\ell - 1}, \quad 0 < \nu < 2 \quad \text{and} \quad K > 0, \tag{3.31}$$

or in "symmetrized" manner,

$$\exists \alpha_0 \in (0, 2], \quad \text{such that} \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha_0}\left(\frac{\theta}{2}\right) b(\cos\theta)\sin\theta \mathrm{d}\theta < \infty, \tag{3.32}$$

which can handle the strongly singular kernel b in (3.31) with some $0 < \nu < 2$ and $\alpha_0 \in (\nu, 2]$. Besides, we further illustrate that the non-cutoff assumption (3.32) can be rewritten as

$$(1 - s)^{\frac{\alpha_0}{2}} b(s) \in L^1[0, 1), \quad \text{for } \alpha_0 \in (0, 2], \tag{3.33}$$

by means of the transformation of variable $s = \cos\theta$ in the symmetric version of b. As mentioned in [15, Remark 1], the full non-cutoff assumption (3.32), or equivalently (3.33), is the extension of the mild non-cutoff assumption of the collision kernel b used in [14], namely,

$$(1-s)^{\frac{\alpha_0}{4}}\,(1+s)^{\frac{\alpha_0}{4}}\,b(s) \in L^1\left(-1,1\right), \quad \text{for } \alpha_0 \in (0,2]. \tag{3.34}$$

3.4 The Weak Formulation and Conservation Law

To derive the weak formulation, a universal tool (so-called *Pre-postcollisional change of variables*) is frequently used, which is an involutive change of variables with unit Jacobian,

$$(v, v_*, \sigma) \to (v', v'_*, \hat{q}), \tag{3.35}$$

where $\hat{q}$ is the unit vector along the relative velocity $q := v - v_*$,

$$\hat{q} = \frac{v - v_*}{|v - v_*|}. \tag{3.36}$$

On the other hand, since $\sigma = (v' - v'_*)/|v' - v'_*|$, the change of variables (3.35) formally amounts to the change of (v, v_*) and (v', v'_*). Hence, under suitable integrability conditions on the measurable function F,

$$\begin{aligned}
&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v'\,\mathrm{d}v'_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v', v'_*, v, v_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma,
\end{aligned} \tag{3.37}$$

where the fact $|v' - v'_*| = |v - v_*|, \sigma\cdot\hat{q} = \hat{q}\cdot\sigma$ is used to keep the arguments of collision kernel $B(v - v_*, \sigma) = B(|v - v_*|, \hat{q}\cdot\sigma)$ unchanged. Note that the change of variables $(v, v_*) \to (v', v'_*)$ works for a fixed ω but is illegal for any given σ.

With the help of this microreversiblity of velocity from (v, v) to (v', v'_*), which leaves the collision kernel B invariant, we can obtain the following weak form for the Boltzmann collision operator.

Proposition 3.3. *For any test function ϕ that is an arbitrarily continuous function of the velocity v,*

$$\begin{aligned}
\int_{\mathbb{R}^3} Q(f,f)\phi\,\mathrm{d}v &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)(f'f'_* - ff_*)\phi\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma) ff_*(\phi' - \phi)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma) ff_*(\phi' + \phi'_* - \phi - \phi_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)(f'f'_* - ff_*)(\phi + \phi_* - \phi' - \phi'_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.
\end{aligned} \tag{3.38}$$

3.5 Boltzmann's H–Theorem and Equilibrium State

Recall the weak formulation (3.38) of the Boltzmann equation as in (3.3), there is an immediate consequence for a solution f to the Boltzmann equation that, whenever ϕ satisfies the functional equation,

$$\forall(v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \qquad \phi(v') + \phi(v'_*) = \phi(v) + \phi(v_*), \tag{3.39}$$

then, we at least formally have,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} f(t,v)\phi(v) \, \mathrm{d}v = \int_{\mathbb{R}^3} Q(f,f)\phi \, \mathrm{d}v = 0, \tag{3.40}$$

and this kind of ϕ is usually called the *collision invariant*.

Since the mass, momentum and energy are conserved during the classical elastic collisions, it is natural to find that the functions $1, v_j, 1 \le j \le 3$, and $|v|^2$ and any linear combination of them are the collision invariants, which can be actually shown as the only collision invariants. Together with the weak form, this leads to the formal conservation law of the Boltzmann equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} f(t,v) \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} \mathrm{d}v = \int_{\mathbb{R}^3} Q(f,f)(t,v) \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} \mathrm{d}v = 0, \qquad 1 \le j \le 3. \tag{3.41}$$

In particular, at a given time t, one can define the local density ρ, the local macroscopic velocity u, and the local temperature T, by

$$\rho = \int_{\mathbb{R}^3} f(t,v) \, \mathrm{d}v, \quad \rho u = \int_{\mathbb{R}^3} f(t,v) v \, \mathrm{d}v, \quad \rho |u|^2 + d\rho T = \int_{\mathbb{R}^3} f(t,v)|v|^2 \, \mathrm{d}v, \tag{3.42}$$

then the equilibrium is the *Maxwellian* distribution,

$$\mathcal{M}(v) = \mathcal{M}^f(v) = \frac{1}{(2\pi T)^{3/2}} \, \mathrm{e}^{-\frac{|v-u|^2}{2T}}. \tag{3.43}$$

If not caring about the integrability issues, we select the test function $\phi = \log f$ into the weak form (3.38), and consider the properties of the logarithm function, to find that

$$\begin{aligned} -\int_{\mathbb{R}^3} Q(f,f) \ln f \, \mathrm{d}v =& D(f) \\ =& \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(f' f'_* - f f_*) \ln \frac{f' f'_*}{f f_*} \ge 0 \end{aligned} \tag{3.44}$$

due to the fact that the function $(X, Y) \longmapsto (X - Y)(\ln X - \ln Y)$ is always non-negative. Thus, if we introduce Boltzmann's H–functional,

$$H(f) = \int_{\mathbb{R}^3} f \ln f \, \mathrm{d}v, \tag{3.45}$$

then the $H(f)$ will evolve in time because of the collisional effect that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) = -D(f(t,\cdot)) \le 0, \tag{3.46}$$

which is the well-known Boltzmann's H–Theorem: the H–functional, or entropy, is non-increasing with time evolution.

And the equality holds if and only if $\ln f$ is a collision invariant, i.e., $f = \exp(a + bv + c|v|^2)$ with a, b, c being all constants.

3.6 Fourier Transform of the Collision Operator (Bobylev Identity)

The Fourier transformation has been widely used in the analysis of various kind of partial differential equations. However, it used to be very painful to find an elegant representation of the Boltzmann equation in the Fourier space, even though the Boltzmann operator possesses a nice weak formulation. Thanks to A. V. Bobylev, this problem turned out not as intricate as one may imagine, at least for the Maxwellian molecules. Since then, the so-called "Bobylev Identity" has become an extremely powerful technique in the study of the Boltzmann equation, especially in the case of spatially homogeneous theory.

Proposition 3.4. *Consider the Boltzmann collision operator $Q(g, f)$ with its collision kernel B being the Maxwellian molecule b, i.e., B does not depend on $|v - v_*|$,*

$$Q(f, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) [f(v')f(v'_*) - f(v)f(v_*)] \, \mathrm{d}\sigma \, \mathrm{d}v_*. \tag{3.47}$$

Then, the following formulas hold,

$$\begin{aligned} \mathcal{F}\left[Q^+(g, f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi| \cdot \sigma}\right) \hat{g}(\xi^-)\hat{f}(\xi^+) \, \mathrm{d}\sigma, \\ \mathcal{F}\left[Q^-(g, f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi| \cdot \sigma}\right) \hat{g}(0)\hat{f}(\xi) \, \mathrm{d}\sigma, \end{aligned} \tag{3.48}$$

where,

$$\xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2}\sigma, \qquad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2}\sigma. \tag{3.49}$$

Proof. By performing the weak formulation, for any test function ϕ, we have,

$$\int_{\mathbb{R}^3} Q^+(g, f)(v)\phi(v)\mathrm{d}v = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) g(v_*)f(v)\phi(v') \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v. \tag{3.50}$$

Selecting $\phi(v) = \mathrm{e}^{-\mathrm{i}v\cdot\xi}$ in the identity above, we have

$$\begin{aligned} &\mathcal{F}\left[Q^+(g, f)\right](\xi) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) g(v_*)f(v) \, \mathrm{e}^{-\mathrm{i}\left(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\right)\cdot\xi} \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) g(v_*)f(v) \, \mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}\cdot\xi} \, \mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi} \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v, \end{aligned} \tag{3.51}$$

according to the general change of variable,

$$\int_{\mathbb{S}^2} F(k \cdot \sigma, l \cdot \sigma) \, \mathrm{d}\sigma = \int_{\mathbb{S}^2} F(l \cdot \sigma, k \cdot \sigma) \, \mathrm{d}\sigma, \quad |l| = |k| = 1, \tag{3.52}$$

due to the existence of an isometry on $\mathbb{S}^2$ exchanging l and k, we have, by exchanging the rule of $\frac{\xi}{|\xi|}$ and $\frac{v-v_*}{|v-v_*|}$,

$$\begin{aligned}&\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\\ &=\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\end{aligned}\tag{3.53}$$

Thus,

$$\begin{aligned}&\mathcal{F}\left[Q^+(g,f)\right](\xi)\\ &=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}v\cdot\left(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\right)}\,\mathrm{e}^{-\mathrm{i}v_*\cdot\left(\frac{\xi}{2}-\frac{|\xi|}{2}\sigma\right)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &=\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\hat{f}(\xi^+)\hat{g}(\xi^-)\,\mathrm{d}\sigma,\end{aligned}\tag{3.54}$$

where, unlike the elastic case, the ξ^+ and ξ^- are defined as

$$\xi^+=\frac{\xi}{2}+\frac{|\xi|}{2}\sigma,\qquad \xi^-=\frac{\xi}{2}-\frac{|\xi|}{2}\sigma.\tag{3.55}$$

And the formula for $\mathcal{F}\left[Q^-(g,f)\right](\xi)$ is then easily obtained by the same kind of but more simpler computations. □

For a given probability measure F or its density function f, we define the corresponding characteristic function $\varphi(\xi)$ by the Fourier transform:

$$\varphi(\xi)=\hat{f}(\xi):=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}v\cdot\xi}f(v)\,\mathrm{d}v=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}v\cdot\xi}\,\mathrm{d}F(v),\tag{3.56}$$

where the f is regarded as the distribution density function of the cumulative distribution function F in the sense of Radon-Nikodym derivative.

And its inversion formula by normalization writes

$$f(v)=\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}v\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}v\cdot\xi}\varphi(\xi)\,\mathrm{d}\xi.\tag{3.57}$$

4 Introduction of Probability Measure and Corresponding Characteristic Function

4.1 Probability Measures

In fact, there are two perspectives to define the probability measures

From the classical perspective: For a measure space $(\Omega,\mathcal{B}(\Omega),\mu)$, i.e., Ω is a set and

(i) $\mathcal{B}(\Omega)\subset 2^\Omega$ is a σ-algebra in Ω, i.e., $\mathcal{B}(\Omega)$ is a collection of subsets of Ω such that:

- $\emptyset \in \mathcal{B}$.
- $A \in \mathcal{B} \Longrightarrow A^c \in \mathcal{B}$.
- $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$ whenever $A_n \in \mathcal{B}, \forall n$.

(ii) μ is a measure, i.e., $\mu : \mathcal{B} \longmapsto [0, \infty]$ satisfies:

- $\mu(\emptyset) = 0$.
- $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \cup_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\}$ is a disjoint countable family of members of $\mathcal{B}$.

(iii) Ω is σ-finite, i.e., there exists a countable family $\{\Omega\}_n$ in $\mathcal{B}$ such that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty, \forall n$.

The **Borel measure**: μ is defined on every Borel set, which is the component of the Borel σ-algebra, i.e., generated by all bounded open set (smallest σ-algebra that includes all the open sets).

The **Radon measure**: is a Borel measure (every Borel set is μ-measurable) if further satisfies (as in Lawrence Evans and Ronald Gariepy's *Measure Theory and Fine Properties of Functions*),

- Locally finite: $\mu(K) < \infty$ for any compact set $K \subset \Omega$.
- Borel regular: For each $A \subset \Omega$, there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.

Remark 4.1. *(i) For $\Omega = \mathbb{R}^d$ and $\mathcal{B}(\mathbb{R}^d)$, Borel measures and Radon measures coincide.*

(ii) Another equivalent way to define the ***Radon measure*** *(as in Herbert Federer's Geometric Measure Theory) is a Borel measure, if further satisfies,*

- *Locally finite: $\mu(K) < \infty$ for any compact set $K \subset \Omega$.*
- *Inner regular on open sets: If $B \subset \Omega$ is open, then B is μ-measurable and*

$$\mu(B) = \sup \mu(K) : K \text{ is compact and } K \subset B. \tag{4.1}$$

- *Outer regular on Borel sets: For each $A \subset \Omega$ and A* ***is Borel (a plus as in Gerald Folland's Real Analysis)****, then*

$$\mu(A) = \inf \mu(B) : B \text{ is open and } A \subset B. \tag{4.2}$$

(iii) More detailed discussions can be found in https://mathoverflow.net/a/117693.

From the duality perspective:

- If $\Omega \subset \mathbb{R}^3$ is a bounded domain (bounded open set), then, the space of Radon measures $M(\overline{\Omega})$ is defined as the dual space of $C(\overline{\Omega}) = C_c$, including all the continuous linear functionals on $C(\overline{\Omega})$. Furthermore, we have $L^1(\Omega) \hookrightarrow M(\overline{\Omega})$, since for any $f \in L^1(\Omega)$,

$$\mu(\phi) := \int_\Omega f(v)\phi(v)\,\mathrm{d}v, \quad \forall \phi \in C(\overline{\Omega}) \tag{4.3}$$

hence, $\mu \in M(\overline{\Omega})$ with $\|\mu\|_{M(\overline{\Omega})} \leq \|f\|_{L^1(\Omega)}$, where $\|\cdot\|_{M(\Omega)}$ is defined as a dual norm. Thus, *the space of Radon measures represents a natural extension of the space of integrable functions* (this point will be further illustrated in the following subsection).

Or more generally, it is also sometimes to replace the space $E = C(\overline{\Omega})$ by their subspace

$$E_0 = \{\phi \in C(\overline{\Omega});\ \phi = 0 \text{ on the boundary of } \overline{\Omega}\} \tag{4.4}$$

the dual space of which is then denoted by $M(\Omega)$.

- If $\Omega = \mathbb{R}^3$ (a locally compact set which is not compact),

$$C_0(\mathbb{R}^3) := \left\{\phi(v) \in C(\mathbb{R}^3); \lim_{|v|\to\infty} \phi(v) = 0\right\} \tag{4.5}$$

then, the space of Radon measures $M(\mathbb{R}^3)$ is defined as

$$M(\mathbb{R}^3) := \big\{\mu : C_0 \longmapsto \mathbb{R};\ \mu \text{ is linear s.t. } \exists C > 0,\ |\mu(\phi)| \leq C\|\phi\|_\infty,\ \forall \phi \in C_c(\mathbb{R}^3)\big\} \tag{4.6}$$

associated with the measure norm

$$\|\mu\|_{M(\mathbb{R}^3)} := \sup_{\phi \in C_c(\mathbb{R}^3), \|\phi\|_\infty \leq 1} |\mu(\phi)| = \sup\left\{\int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}\mu(v);\ \phi \in C_c(\mathbb{R}^3),\ \|\phi\|_\infty \leq 1\right\}. \tag{4.7}$$

is a Banach space, where the definition also holds for $\phi \in C_c^\infty(\mathbb{R}^3) = \mathcal{D}(\mathbb{R}^3)$ with (?)

$$C_0(\mathbb{R}^3) = \overline{\mathcal{D}(\mathbb{R}^3)}^{\|\cdot\|_\infty}. \tag{4.8}$$

Remark 4.2. *(i) Another way to define the space M could be to replace $C_0(\mathbb{R}^3)$ by $C(\mathbb{R}^3_c)$ for non-compact space, where $\mathbb{R}^3_c$ is denoted as the compactification of $\mathbb{R}^3$ by means of a single point ∞, implying that the limit value $\phi(\infty)$ exists for any $\phi \in C(\mathbb{R}^3_c)$. This is a technical issue that we need to have convenient compactness properties for some subsets of $M(\mathbb{R}^3_c)$.*

(ii) Any $\mu \in M(\mathbb{R}^3)$ can be uniquely extended to the element of dual space to $C_0(\mathbb{R}^3)$. In the sense that,

$$\big(C_0(\mathbb{R}^3), \|\cdot\|_\infty\big)^* \equiv M(\mathbb{R}^3). \tag{4.9}$$

*(iii) (**Variant of Riesz Representation Theorem**) The functional $\mu \in M(\mathbb{R}^3)$ are called (Radon) measures, since there is a one-to-one correspondence between elements of $M(\mathbb{R}^3)$ and a class of (Borel) measures $\tilde{\mu}$ on $\mathbb{R}^3$ with finite total mass $\tilde{\mu}(\mathbb{R}^3) < \infty$, such that*

$$\mu(\phi) = \int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}\tilde{\mu}(v), \qquad \forall \phi \in C_0(\mathbb{R}^3). \tag{4.10}$$

And, as usual, we do not distinguish between μ and $\tilde{\mu}$. Hence, instead of $\mu(\phi)$, we use the standard duality notation

$$\langle \mu, \phi \rangle := \mu(\phi) = \int_{\mathbb{R}^3} \phi(v) \, \mathrm{d}\mu(v), \quad \mu \in M(\mathbb{R}^3), \ \phi \in C_0(\mathbb{R}^3). \tag{4.11}$$

Moreover, all the statements above are also valid for $\phi \in C(\overline{\Omega})$ and $\mu \in M(\Omega)$, e.g.,

$$\langle \mu, \phi \rangle := \mu(\phi) = \int_{\overline{\Omega}} \phi(v) \, \mathrm{d}\mu(v), \quad \mu \in M(\overline{\Omega}), \ \phi \in C_0(\overline{\Omega}). \tag{4.12}$$

If $\mu \in M(\mathbb{R}^3)$ with $\|\mu\|_{M(\mathbb{R}^3)} \leq \infty$ and $\mu(\phi) \geq 0$ for all $0 \leq \phi \in C_c(\mathbb{R}^3)$, we say that μ belongs to a non-negative bounded Radon measure space $M_b^+(\mathbb{R}^3)$.

Finally, the space of ***probability measure*** is defined as follows:

$$P(\mathbb{R}^3) := \left\{ \mu \in M_b^+(\mathbb{R}^3) \text{ with } \|\mu\|_{M(\mathbb{R}^3)} = 1 \right\}. \tag{4.13}$$

Remark 4.3. *In the statements above, we didn't distinguish clearly the random variable and its associated probabilistic space; however, the most formal, axiomatic definition of a random variable actually involves measure theory as following:*

(Why?) The probability cannot be defined on all of the events, (corresponding to the measure cannot be defined on all of the sets). But we hope to find a collection of events (or sets) to make them measurable and closed by some calculations such as countable intersections, unions, complements, etc. so the question is what kind of collection of sets satisfy our requirements, then the σ-algebra arises.

*Let $(E, \mathcal{E})$ be a measurable space, then there exists a smallest σ-algebra $\mathcal{B}$ in E such that every open set in E belongs to $\mathcal{B}$ or so-called generated by all open sets, which is called **Borel σ-algebra**. Usually, we take $E = \mathbb{R}$ as an example and its corresponding Borel σ-algebra is $\mathcal{B}_{\mathbb{R}}$. Then, the measurable function (corresponding to the random variable in probability language) is the mapping: $(E, \mathcal{E}) \longmapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

Precisely speaking, let $(\Omega, \mathcal{A}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space. Then an $(E, \mathcal{E})$-valued random variable is a measurable function $V : \Omega \to E$, which means that, for every subset $B \in \mathcal{E}$, its pre-image is $\mathcal{A}$-measurable, i.e., $V^{-1}(B) \in \mathcal{A}$, where $V^{-1}(B) = \{v : V(v) \in B\}$. This definition enables us to measure any subset $B \in \mathcal{E}$ in the target space by looking at its pre-image, which by assumption is measurable.

In more intuitive terms, a member of Ω is a possible outcome (event), a member of $\mathcal{A}$ is a measurable subset of possible outcomes (events), the function μ gives the probability of each such measurable subset, E represents the set of values that the random variable can take (such as the set of real numbers), and a member of $\mathcal{E}$ is a "well-behaved" (measurable) subset of E (those for which the probability may be determined). The random variable is then a function from any outcome to a quantity, such that the outcomes leading to any useful subset of quantities for the random variable have a well-defined probability.

*When $(E, \mathcal{O})$ is a topological space, then the most common choice for the σ-algebra $\mathcal{E}$ is the **Borel σ-algebra** $\mathcal{B}(E)$, which is the σ-algebra generated by the collection of all open sets in E. In such case the $(E, \mathcal{E})$-valued random variable is called an E-valued* *random variable. Moreover, when the space E is the real line $\mathbb{R}$, then such a real-valued random variable is called simply a random variable; when the space E is the surface $\mathbb{R}^2$, then that is a two-dimensional random variable. In our case, we don't distinguish the two measurable spaces by letting the random variable V be the trivial mapping $V = v : (\Omega, \mathcal{A}, P) \longmapsto (\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), F)$, where the probability measure F is sometimes also called the multivariate probability distribution on a random vector, in terms of the multivariate cumulative distribution function F_V $(F_V(v) = F((-\infty, v]) = \int_{-\infty}^{v} \mathrm{d}F(v))$:*

$$\begin{aligned} \text{1D Domain: } F_V(b) - F_V(a) = P(a < v \leq b) = \int_a^b \mathrm{d}F(v) = \int_a^b f(v)\,\mathrm{d}v \\ \text{General Domain: } F_V(A \subset \mathcal{A}) = P(v \in A \subset \mathcal{A}) = \int_A \mathrm{d}F(v) = \int_A f(v)\,\mathrm{d}v. \end{aligned} \tag{4.14}$$

that is to say, there is a one-to-one correspondence between the three concepts: Borel probability measure $\Longleftrightarrow$ Distribution function $\Longleftrightarrow$ Random variable

The following theorem will illustrate the equivalent relation between the probability measure and random variables:

Theorem 4.4 (Skorohod Representation)**.** *For probability measures $\{F_n\}_{n\geq 1}$ and F on $\big(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3)\big)$, if $F_n \rightharpoonup F$, then there exists a probability space $(\Omega, \mathcal{A}, P)$ with random variables $\{V_n\}_{n\geq 1}$ and V such that*

- *F_n is the distribution of V_n, $\forall n = 1, 2, 3, ...$;*
- *$V_n \rightharpoonup V$ in the sense that, $\forall \phi \in C_b(\mathbb{R}^3)$, $\mathbb{E}\left[\phi(V_n)\right] \to \mathbb{E}\left[\phi(V)\right]$, i.e., $\int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}F_n(v) \to \int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}F(v)$, as $n \to \infty$.*

4.2 The Characteristic Functions (Fourier Transform of a Probability Measures)

There are some heuristic reasons why we introduce the characteristic function $\varphi : \mathbb{R}^3 \longmapsto \mathbb{C}$ associated to a probability measure μ on the space $(\mathbb{R}^3, \mathcal{B})$ ($\mathcal{B}$ is a Borel σ-algebra on $\mathbb{R}^3$):

- The characteristic function uniquely determines the probability measure;
- The characteristic function can be used to prove weak convergence;
- The characteristic function can be used to obtain non-trivial information about probability measure.

4.2.1 Heuristic Glance

Generally speaking:

For $\mu \in P(\mathbb{R}^3)$, we define the Fourier transform by

$$\mathcal{F}(\mu)(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}\mu(v), \tag{4.15}$$

or without normalization,

$$\mathcal{F}(\mu)(\xi) = \hat{\mu}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i} v\cdot\xi} \,\mathrm{d}\mu(v). \tag{4.16}$$

Then, $\varphi(\xi) = \hat{\mu}(\xi)$ is called the **characteristic function** and

$$\mathcal{F} : P(\mathbb{R}^3) \longmapsto C_b(\mathbb{R}^3), \tag{4.17}$$

where $C_b(\mathbb{R}^3)$ is the space of bounded continuous functions.

Proof. Consider, for $\mu \in P(\mathbb{R}^3)$,

$$|\hat{\mu}(\xi)| \le \left| \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i} v\cdot\xi} \,\mathrm{d}\mu(v) \right| \le \int_{\mathbb{R}^3} \left|\mathrm{e}^{-\,\mathrm{i} v\cdot\xi}\right| \,\mathrm{d}\mu(v) = 1 < \infty, \tag{4.18}$$

where we observe $\left|\mathrm{e}^{-\,\mathrm{i} v\cdot\xi}\right| \le 1$ and the continuity of $\hat{\mu}(\xi)$ follows from the dominated convergence theorem. □

Proposition 4.5. *Let $\mu \in P(\mathbb{R}^3)$, then μ is uniquely determined by $\hat{\mu}$.*

Proof. We need to show that $\mu_1 = \mu_2$, if $\hat{\mu}_1 = \hat{\mu}_2$. For any $\phi \in \mathcal{S}$, we have,

$$\int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu(v) = \int_{\mathbb{R}^3} \phi(\xi)\hat{\mu}(\xi) \,\mathrm{d}\xi. \tag{4.19}$$

Indeed, by the Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\phi(\xi) \,\mathrm{d}\xi \,\mathrm{d}\mu(v) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}\xi\cdot v} \,\mathrm{d}\mu(v)\phi(\xi) \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^3} \hat{\mu}(\xi)\phi(\xi) \,\mathrm{d}\xi. \end{aligned} \tag{4.20}$$

Furthermore, if $\mu_1(v) = \mu_2(v)$, then by (4.19), we find,

$$\int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu_1(v) = \int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu_2(v) \tag{4.21}$$

for any $\phi \in \mathcal{S}$. Since the Fourier transform in invertible on $\mathcal{S}$, we can also write it as,

$$\int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu_1(v) = \int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu_2(v), \qquad \forall \phi \in \mathcal{S}. \tag{4.22}$$

By choosing the mollifier function $\phi_\epsilon \in \mathcal{S}$ such that

$$\mathbf{1}_{[a,b]} \le \phi_\epsilon \le \mathbf{1}_{[a-\epsilon,b-\epsilon]}, \tag{4.23}$$

it follows the dominated convergence theorem

$$\mu_1([a,b]) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^3} \phi_\epsilon \,\mathrm{d}\mu_1(v) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^3} \phi_\epsilon \,\mathrm{d}\mu_2(v) = \mu_2([a,b]) \tag{4.24}$$

that $\mu_1 = \mu_2$. □

Let μ_j and μ denote probability measure on $\mathbb{R}^3$, we say that μ_j convergent weakly* to μ if

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\phi\,\mathrm{d}\mu_j=\int_{\mathbb{R}^3}\phi\,\mathrm{d}\mu \tag{4.25}$$

for all $\phi\in C_0(\mathbb{R}^3)$, which is the weak* convergence of measures.

Proposition 4.6. *If μ_j and μ belong to $P(\mathbb{R}^3)$ and for each $\xi\in\mathbb{R}^3$,*

$$\lim_{j\to\infty}\hat{\mu}_j(\xi)=\hat{\mu}(\xi), \tag{4.26}$$

then,

$$\lim_{j\to\infty}\phi(v)\,\mathrm{d}\mu_j(v)=\int_{\mathbb{R}^3}\phi(v)\,\mathrm{d}\mu(v), \tag{4.27}$$

for all $\phi\in\mathcal{S}(\mathbb{R}^3)$.

Proof. Since $\mu_j\in P(\mathbb{R}^3)$ is a probability measure with $\mu_j(\mathbb{R}^3)=1$, we have,

$$\sup_j|\hat{\mu}_j(\xi)|=\sup_j\mu_j(\mathbb{R}^3)=1\leq\infty. \tag{4.28}$$

Then, for $\phi\in\mathcal{S}$, $|\phi(\xi)\hat{\mu}_j(\xi)|\leq|\phi(\xi)|$, hence, it follows the dominated convergence theorem that,

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\phi(\xi)\hat{\mu}_j(\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}^3}\phi(\xi)\hat{\mu}(\xi)\,\mathrm{d}\xi, \tag{4.29}$$

which is equivalent to, by Fubini's theorem,

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\hat{\phi}(v)\,\mathrm{d}\mu_j(v)=\int_{\mathbb{R}^3}\hat{\phi}(v)\,\mathrm{d}\mu(v). \tag{4.30}$$

Finally, the desired weak* convergence of μ_j to μ follows from the fact that the set of all $\hat{\phi}$ such that $\phi\in\mathcal{S}$ is all of $\mathcal{S}$. □

4.2.2 Precise Definition

More precisely:

Definition 4.7 (Characteristic Function). *A function $\varphi:=\mathbb{R}^3\mapsto\mathbb{C}$ is called a characteristic function if there is a probability measure F (i.e. a Borel measure with $\int_{\mathbb{R}^3}\mathrm{d}F(v)=1$) such that we have the identity $\varphi(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}v\cdot\xi}\mathrm{d}F(v)$.*

Remark 4.8. *It is worth noticing that, in probabilistic language, the characteristic function is usually defined as $\varphi(\xi)=\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi\cdot V}\right)$, which completely determines the behavior and properties of the probability distribution of the random variable V in $\left(\mathbb{R}^3,\mathcal{B}(\mathbb{R}^3),\mu\right)$.*

We will denote the set of all characteristic function $\varphi:=\mathbb{R}^3\mapsto\mathbb{C}$ by $\mathcal{K}$.

Proposition 4.9 (Basic Properties of Characteristic Function). *Let μ be a probability measure on $\left(\mathbb{R}^3,\mathcal{B}\right)$ and let φ be its corresponding characteristic function. Then,*

(i) $\varphi(0)=1$ and $|\varphi(\xi)|\leq1$, for all $\xi\in\mathbb{R}^3$.

(ii) $\overline{\varphi(\xi)}=\varphi(-\xi)$, where the bar denotes complex conjugate.

(iii) $\varphi(\xi)$ is uniformly continuous, i.e., for all $\xi \in \mathbb{R}^3$, there exists a $\psi(\eta) \to 0$ as $|\eta| \to 0$ such that

$$|\varphi(\xi+\eta) - \varphi(\xi)| \le \psi(\eta). \tag{4.31}$$

or, in probabilistic language, $|\varphi(\xi+\eta) - \varphi(\xi)| \le \mathbb{E}\left(\left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v} - 1\right|\right)$.

(iv) If V be a (real) random variable with $W = aV + b$ (a, b are constants), then

$$\varphi_W(\xi) = \mathrm{e}^{-\,\mathrm{i}b\xi}\varphi_V(a\xi). \tag{4.32}$$

Proof. For (i), for $\mu \in P(\mathbb{R}^3)$, $\varphi(0) = \hat{\mu}(0) = \int_{\mathbb{R}^3} \mathrm{d}\mu(v) = 1$, and

$$|\varphi(\xi)| \le \left|\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}\mu(v)\right| \le \int_{\mathbb{R}^3} \left|\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\right| \mathrm{d}\mu(v) = 1, \tag{4.33}$$

hence, we also have $\sup\limits_{\xi\in\mathbb{R}^3}$.

For (ii), it follows that,

$$\overline{\varphi(\xi)} = \overline{\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}\mu(v)} = \int_{\mathbb{R}^3} \overline{\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}}\,\mathrm{d}\mu(v) = \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot(-\xi)}\,\mathrm{d}\mu(v) = \varphi(-\xi). \tag{4.34}$$

For (iii), we have

$$|\varphi(\xi+\eta) - \varphi(\xi)| = \left|\int_{\mathbb{R}^3} \left(\mathrm{e}^{-\,\mathrm{i}(\xi+\eta)\cdot v} - \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\right) \mathrm{d}\mu(v)\right| \le \psi(\eta), \tag{4.35}$$

where $\psi(\eta) = \int_{\mathbb{R}^3} \left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v} - 1\right| \mathrm{d}\mu(v) = \mathbb{E}\left(\left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v} - 1\right|\right)$. Since $\left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v} - 1\right| \le 2$, the Dominated Convergence Theorem implies that

$$\lim_{\eta\to 0} \psi(\eta) = 0, \tag{4.36}$$

Hence, the $\varphi(\xi)$ is uniformly continuous on $\xi \in \mathbb{R}^3$.

Another proof for (iii), note that

$$\left|\mathrm{e}^{-\,\mathrm{i}(\xi+\eta)\cdot v} - \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\right| = \left|\mathrm{i}\int_{v\cdot\xi}^{v\cdot(\xi+\eta)} \mathrm{e}^{-\,\mathrm{i}s}\,\mathrm{d}s\right| \le |v||\eta|. \tag{4.37}$$

Since $F \in P_\alpha(\mathbb{R}^3) \subset M_b^+(\mathbb{R}^3)$ is bound, (i.e., $\|F\|_{M_b^+(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \mathrm{d}F(v) = 1$), then for any $\epsilon > 0$, we find a ball $B_R(0)$ of radius $R = R(\epsilon)$ such that $F(B_R^c(0)) < \epsilon$ (i.e., $\int_{\mathbb{R}^3 - B_R(0)} \mathrm{d}F(v) < \epsilon$),

$$\begin{aligned}
|\varphi(\xi+\eta) - \varphi(\xi)| =& \left|\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot(\xi+\eta)}\mathrm{d}F(v) - \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\mathrm{d}F(v)\right| \\
\le& \int_{\mathbb{R}^3} \left|\mathrm{e}^{-\,\mathrm{i}(\xi+\eta)\cdot v} - \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\right| \mathrm{d}F(v) \\
=& \int_{B_R(0)} \left|\mathrm{e}^{-\,\mathrm{i}(\xi+\eta)\cdot v} - \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\right| \mathrm{d}F(v) + \int_{\mathbb{R}^3 - B_R(0)} \left|\mathrm{e}^{-\,\mathrm{i}(\xi+\eta)\cdot v} - \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\right| \mathrm{d}F(v) \\
\le& R|\eta| \int_{B_R(0)} \mathrm{d}F(v) + 2\int_{\mathbb{R}^3 - B_R(0)} \mathrm{d}F(v) \\
\le& R|\eta| + 2\epsilon,
\end{aligned} \tag{4.38}$$

which implies the uniform continuity of φ.

For (iv), it formally follows that,

$$\varphi_W(\xi) = \mathbb{E}\left(\mathrm{e}^{-\mathrm{i}\xi\cdot(a+bV)}\right) = \mathrm{e}^{-\mathrm{i}a\xi}\mathbb{E}\left(\mathrm{e}^{-\mathrm{i}\xi\cdot(bV)}\right) = \mathrm{e}^{-\mathrm{i}a\xi}\varphi_V(b\xi) \tag{4.39}$$

□

4.2.3 Positive Definite Function

In order to study the further properties and estimates of the characteristic functions (or characterise the Fourier transform of bounded measures), we need to rely on the following Positive Definite Functions:

Definition 4.10 (Positive Definite Functions)**.** *A function $\varphi := \mathbb{R}^3 \mapsto \mathbb{C}$ is called positive definite if, for every $k \in \mathbb{N}$ and every vector $\xi^1, ..., \xi^k \in \mathbb{R}^3$, the matrix $\left[\varphi(\xi^j - \xi^l)\right]_{j,l=1,...,k}$ is positive Hermitian, i.e., for all $\zeta_1, ..., \zeta_k \in \mathbb{C}$, we have,*

$$\sum_{j,l=1}^{k} \zeta_j \varphi(\xi^j - \xi^l)\overline{\zeta_l} \geq 0. \tag{4.40}$$

Or, in matrix language, by letting $A = (a_{jl})_{1\leq j,l\leq k}$ given by

$$a_{jl} = \varphi(\xi^j - \xi^l) \tag{4.41}$$

is Hermitian and positive semi-definite, i.e., $A^* = A$ and $\zeta^T A\bar{\zeta} \geq 0$, for all $\zeta \in \mathbb{C}^3$.

Remark 4.11. *Let's give a simplest example for $k = 2$: for $\varphi : \mathbb{R}^d \longmapsto \mathbb{C}$ that is a continuous Positive Definite Function, we select $\xi^1 = 0 \in \mathbb{R}^d$ and $\xi^2 = \xi \in \mathbb{R}^d$, then the positive Hermitianness of the matrix*

$$\begin{pmatrix} \varphi(0) & \varphi(\xi) \\ \varphi(-\xi) & \varphi(0) \end{pmatrix} \tag{4.42}$$

implies that

$$\overline{\varphi(\xi)} = \varphi(-\xi) \quad \text{and} \quad \varphi(0) \geq 0, \tag{4.43}$$

and therefore

$$|\varphi(\xi)| \leq u(0). \tag{4.44}$$

Thus, any continuous Positive Definite Function is bounded, and therefore it belongs to $\mathcal{S}'(\mathbb{R}^d)$.

Why we prefer to introduce the larger set of **Positive Definite Functions**, instead of just working on the simple characteristic functions? Because it is easily for us to derive estimates on a certain product of positive definite functions that will be useful for the study of the following collision operator.

Lemma 4.12 (**Properties of Positive Definite Function**)**.** *Every positive definite function φ satisfies:*

- $\overline{\varphi(\xi)} = \varphi(-\xi)$ *and* $\varphi(0) \geq 0$.
- $|\varphi(\xi)| \leq \varphi(0) \implies \sup_{\xi \in \mathbb{R}^d} = |\varphi(0)|$.
- *Any linear combination with positive coefficients of positive definite functions is a positive definite function.*
- *The production of two positive definite functions is a positive definite function. (Since from the Definition 4.10, the product two positive Hermitian matrices is still positive Hermitian.)*
- *The set of positive definite functions is a positive definite function.*
- *If φ is a positive definite function, so are $\overline{\varphi}$ and* $\mathrm{Re}\varphi$

Lemma 4.13 (Estimates of Positive Definite Function). *For any positive definite function $\varphi = \varphi(\xi)$ such that $\varphi(0) = 1$, we have*

$$|\varphi(\xi) - \varphi(\eta)|^2 \leq 2\left(1 - \mathrm{Re}\left[\varphi(\xi - \eta)\right]\right), \tag{4.45}$$

and

$$|\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)|^2 \leq \left(1 - |\varphi(\xi)|^2\right)\left(1 - |\varphi(\eta)|^2\right). \tag{4.46}$$

for all $\xi, \eta \in \mathbb{R}^3$.

Proof. As for the estimate (4.45), the proof is based on the inequality (4.40) in the definition of the Positive Definite Function with suitably chosen vectors ξ^j and λ_j: indeed, for $\xi, \eta \in \mathbb{R}^3$, we consider the Hermitian matrix

$$\begin{pmatrix} \varphi(0) & \overline{\varphi(\xi)} & \overline{\varphi(\eta)} \\ \varphi(\xi) & \varphi(0) & \varphi(\xi - \eta) \\ \varphi(\eta) & \overline{\varphi(\xi - \eta)} & \varphi(0) \end{pmatrix} \tag{4.47}$$

where $\varphi(0) = 1$. Next, with arbitrarily given $s \in \mathbb{R}$, we define,

$$\zeta_1 = r, \quad \zeta_2 = \frac{r|\varphi(\xi) - \varphi(\eta)|}{\varphi(\xi) - \varphi(\eta)}, \quad \zeta_3 = -\zeta_2. \tag{4.48}$$

Hence, by applying the inequality (4.40), we find by a straightforward calculation

$$2r^2 - 2r^2\mathrm{Re}[\varphi(\xi - \eta)] + 2\mathrm{r}\,|\varphi(\xi) - \varphi(\eta)| + 1 \geq 0. \tag{4.49}$$

which implies that the discriminant of the quadratic form on the left hand side (as a function of r) has to be non-positive; thus,

$$2\,|\varphi(\xi) - \varphi(\eta)|^2 \leq 4\left(2 - 2\mathrm{Re}[\varphi(\xi - \eta)]\right), \tag{4.50}$$

which completes the proof of (4.45).

As for the estimate (4.46), it follows from the fact that the determinant of the Hermitian matrix (4.47) with $\varphi(0)=1$ is non-negative; indeed, suppose that for $a,b,c\in\mathbb{C}$, the matrix

$$\begin{pmatrix} 1 & a & b \\ \bar{a} & 1 & c \\ \bar{b} & \bar{c} & 1 \end{pmatrix} \tag{4.51}$$

is positive Hermitian, then its determinant if non-negative, which gives

$$1 + a\bar{b}c + \bar{a}b\bar{c} \geq |a|^2 + |b|^2 + |c|^2, \tag{4.52}$$

which is equivalent to

$$|c - \bar{a}b|^2 \leq (1 - |a|^2)(1 - |b|^2). \tag{4.53}$$

Thus, when φ is a Positive Definite Function with $\varphi(0)=1$ and $\overline{\varphi(\xi)} = \varphi(-\xi)$, it gives the estimate (4.46). □

The following celebrated theorem by Bochner plays a fundamental role in the theory of positive definite functions, since it states that *the set of continuous positive definite functions coincides with the set of characteristic functions.*

Theorem 4.14 (**Bochner's Theorem**). *A function $\varphi : \mathbb{R}^3 \to \mathbb{C}$ is a characteristic function (i.e., the Fourier transform of a probability measure) if and only if the following conditions hold:*

(i) φ is a continuous function on $\mathbb{R}^3$.
(ii) $\varphi(0) = 1$.
(iii) φ is positive definite.

Proof. "$\Longrightarrow$" This implication is directly obtained, since the characteristic function $\varphi : \mathbb{R}^3 \to \mathbb{C}$ is the Fourier transform of a probability measure in $P(\mathbb{R}^3)$.

"$\Longleftarrow$" For the converse implication, we know already that φ is an element of $\mathcal{S}'(\mathbb{R}^3)$ and therefore there exists a tempered distribution μ such that $\varphi = \hat{\mu}$. We will prove that $\langle \mu, \phi\rangle = \int_{\mathbb{R}^d} \phi(v)\mathrm{d}\mu(v) \geq 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^3)$ such that $\phi \geq 0$. Then, by our assumption on φ, we can find for this measure that $\hat{\mu}(0) = \varphi(0) = 1$, which implies that $\mu \in P(\mathbb{R}^3)$ is a probability measure.

To prove $\langle \mu, \phi\rangle = \int_{\mathbb{R}^d} \phi(v)\mathrm{d}\mu(v) \geq 0$ for all $0 \leq \phi \in \mathcal{S}(\mathbb{R}^3)$. Denote by $\tilde{\phi}$ the function $\phi(z) = \overline{\phi(-z)}$. Since φ is continuous and positive definite, we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi(x-y)\overline{\phi(y)}\phi(x)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^3} \varphi(z)\left(\tilde{\phi} * \phi\right)(z)\,\mathrm{d}z \geq 0. \tag{4.54}$$

For $\phi \in \mathcal{S}(\mathbb{R}^3)$ the function $\tilde{\phi} * \phi$ belongs to $\mathcal{S}(\mathbb{R}^d)$, and since $\varphi \in \mathcal{S}'(\mathbb{R}^3)$, we have,

$$\left\langle \varphi, \tilde{\phi} * \phi \right\rangle \geq 0. \tag{4.55}$$

However, this yields

$$0 \leq \left\langle \varphi, \tilde{\phi} * \phi \right\rangle = \left\langle \hat{\mu}, \tilde{\phi} * \phi \right\rangle = \left\langle \mu, \widehat{\tilde{\phi} * \phi} \right\rangle = \left\langle \mu, \left|\hat{\phi}\right|^2 \right\rangle \tag{4.56}$$

thus, $\left\langle \mu, |\phi|^2 \right\rangle \geq 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^3)$, which is equivalent to

$$\left\langle \mu, \left|\hat{\phi}\right|^2 \right\rangle \geq 0, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^3). \tag{4.57}$$

It then sufficient to follows that there exists a probability measure $\mu \in P(\mathbb{R}^3)$ such that $\langle \mu, \phi \rangle = \int_{\mathbb{R}^3} \phi(v) \mathrm{d}\mu(v) \geq 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^3)$, which finally proves the theorem. □

Therefore, we can denote the set of **all Characteristic Functions (Continuous Positive Definite Functions)** $\varphi : \mathbb{R}^3 \mapsto \mathbb{C}$ by $\mathcal{K}$.

4.3 Different Kinds of Convergence (Relation between Probability Measures and their Corresponding Characteristic Functions)

KQ: Some of the motivations and heuristic statement might stem from the personal view of the author, which is just for reference and further discussion. Some concepts might be slightly different in various places, such as the different kinds of "weak" convergence, etc.

Some motivations: Since originally what we want to find is the L^1-solution to the homogeneous Boltzmann equation, however, the L^1-space does NOT have some good compactness property, e.g., bounded sets of L^1 do Not play an important role w.r.t the weak topology $\sigma\left(L^1, L^\infty\right)$, since L^1 is not reflexive, indeed,

$$\left(L^1\right)^* = L^\infty, \quad \text{but } \left(L^\infty\right)^* \supset L^1. \tag{4.58}$$

So what does the weakly compact sets of L^1 look like? The following **Dunford-Pettis** criterion provides a useful characterization of weakly compact sets of L^1:

Theorem 4.15 (**Dunford-Pettis Criterion**). *Let $\{f_n\}$ be a bounded set in $L^1(\Omega)$. Then $\{f_n\}$ has compact closure in the weak topology $\sigma\left(L^1, L^\infty\right)$ if and only if $\{f_n\}$ is equi-integrable, that is,*

$$(a) \begin{cases} \forall \epsilon > 0, \exists \delta > 0 \text{ such that} \\ \displaystyle\int_A |f_n| \, \mathrm{d}v \leq \epsilon, \ \forall A \subset \Omega \text{ that is measureable with } |A| \leq \delta, \quad \forall f \in \{f_n\}, \end{cases}$$

and

$$(b) \begin{cases} \forall \epsilon > 0, \exists \omega \subset \Omega \text{ that is measureable with } |\omega| \leq \infty \text{ such that} \\ \displaystyle\int_{\Omega - \omega} |f_n| \, \mathrm{d}v \leq \epsilon, \quad \forall f \in \{f_n\}. \end{cases}$$

Then, since the bounded sets of L^1 enjoy no compactness properties, to overcome this drawback it is sometimes useful to *embed L^1 into a large space: the space of Radon measures.*

So let's recall the natural extension mentioned before, $L^1(\Omega) \hookrightarrow M(\overline{\Omega})$:

Precisely speaking, let $\Omega \subset \mathbb{R}^d$ be a bounded open set with usual Lebesgue measure and space $C(\overline{\Omega})$ be equipped with norm $\|\phi\|_\infty = \sup_{v\in\overline{\Omega}} |\phi(v)|$, the dual space of which is then denoted by $M(\overline{\Omega})$ called *Radon measures* on $\overline{\Omega}$.

So how can we identity/embed $L^1(\Omega)$ with a subspace of $M(\overline{\Omega})$? For this purpose, we need to introduce the mapping $T : L^1(\Omega) \longmapsto M(\overline{\Omega})$ in the sense that: given $f \in L^1(\Omega)$, the mapping $\phi \in C(\overline{\Omega}) \longmapsto \int_\Omega f\phi \,\mathrm{d}v$ is a continuous linear functional on $C(\overline{\Omega})$, i.e.,

$$\langle Tf, \phi\rangle_{(C(\overline{\Omega}))^*, C(\overline{\Omega})} = \int_\Omega f\phi \,\mathrm{d}v, \quad \forall \phi \in C(\overline{\Omega}). \tag{4.59}$$

Clearly T is not only linear, but also an isometry, since

$$\|Tf\|_{M(\overline{\Omega})} = \sup_{\phi\in C(\overline{\Omega}), \|\phi\|_\infty \le 1} \int_\Omega f(v)\phi(v)\,\mathrm{d}v = \|f\|_{L^1(\Omega)}. \tag{4.60}$$

Using T we can identify $L^1(\Omega)$ with a subspace of $M(\overline{\Omega})$. Since $M(\overline{\Omega})$ is the dual space of the separable space $C(\overline{\Omega})$, it has some compactness properties in the weak* topology, thanks to the following **Banach-Alaoglu-Bourbaki** Theorem:

Theorem 4.16 (**Banach-Alaoglu-Bourbaki Theorem**). *The closed unit ball*

$$B_{E*} = \{f \in E^*; \|f\|_{E*} \le 1\} \tag{4.61}$$

is compact in the weak* topology $\sigma(E^*, E)$.

In particular, if $\{f_n\}$ is a bounded sequence in $L^1(\Omega)$, there exits a subsequence $\{f_{n_k}\}$ and a Radon measure μ such that $f_{n_k} \overset{*}{\rightharpoonup} \mu$ in the weak* topology $\sigma(E^*, E)$, that is,

$$\int_\Omega f_{n_k}(v)\phi(v) \to \langle \mu, \phi\rangle, \quad \forall \phi \in C(\overline{\Omega}). \tag{4.62}$$

Let's introduce some topologies on $M_b^+(\Omega)$ (which is definitely also applicable on $P(\Omega)$) with Ω being a locally compact space (i.e., it works for $\mathbb{R}^3$): let $\{\mu_n\}_{n\in\mathbb{N}}$ be sequence in $M_b^+(\Omega)$ and $\mu \in M_b^+(\Omega)$, we say,

- $\{\mu_n\}_{n\in\mathbb{N}}$ converges in norm (**norm(strong) convergence**) to $\mu \Longleftrightarrow \lim\limits_{n\to\infty} \|\mu_n - \mu\| = 0$.
- $\{\mu_n\}_{n\in\mathbb{N}}$ converges narrowly (**narrow convergence**) to $\mu \Longleftrightarrow \lim\limits_{n\to\infty} \int_\Omega \phi(v)\,\mathrm{d}\mu_n(v) = \int_\Omega \phi(v)\,\mathrm{d}\mu(v)$, for all $\phi \in C_b(\Omega)$.
- $\{\mu_n\}_{n\in\mathbb{N}}$ converges weakly* (**weak* convergence**) to $\mu \Longleftrightarrow \lim\limits_{n\to\infty} \int_\Omega \phi(v)\,\mathrm{d}\mu_n(v) = \int_\Omega \phi(v)\,\mathrm{d}\mu(v)$, for all $\phi \in C_0(\Omega)$.
- $\{\mu_n\}_{n\in\mathbb{N}}$ converges vaguely (**vague convergence**) to $\mu \Longleftrightarrow \lim\limits_{n\to\infty} \int_\Omega \phi(v)\,\mathrm{d}\mu_n(v) = \int_\Omega \phi(v)\,\mathrm{d}\mu(v)$, for all $\phi \in C_c(\Omega)$ or $C(\overline{\Omega})$.

  where $C_b(\Omega)$ denotes bounded continuous functions, $C_0(\Omega)$ denotes continuous functions vanishing at infinity, and $C_c(\Omega)$ (or $C(\overline{\Omega})$) denotes continuous functions with compact support.

Remark 4.17. *Obviously, we have the implications: since $C_c(\Omega) \subset C_0(\Omega) \subset C_b(\Omega)$,*

$$\textit{Strong Convergence} \Longrightarrow \textit{Narrow Convergence} \Longrightarrow \textit{Weak}^* \textit{ Convergence} \Longrightarrow \textit{Vague Convergence} \tag{4.63}$$

However, on the other hand, we actually possess the inverse embedding relation,

Theorem 4.18. *Suppose that $\{\mu_n\}_{n\in\mathbb{N}} \in M_b^+(\Omega)$ converges vaguely to $\mu \in M_b^+(\Omega)$ and that $\lim_{n\to\infty} \mu_n(\Omega) = \mu(\Omega)$. Then $\{\mu_n\}_{n\in\mathbb{N}}$ converges narrowly to μ.*

Remark 4.19. *It is supposed to require more on Ω, e.g., locally compact metric space. (from Jose. Canizo).*

which then implies the following corollary:

Corollary 4.20. *A sequence of probability measures on Ω converges narrowly if and only if it converges vaguely.*

Notice that $M_b^+(\mathbb{R}^d)$ is a convex cone and $P(\mathbb{R}^d) \subset M_b^+(\mathbb{R}^d)$ is a convex subset.

And Bochner's Theorem states that the Fourier transform is a bijective mapping from $M_b^+(\mathbb{R}^3)$ onto the Set of Continuous Positive Definite functions $CP(\mathbb{R}^3)$, i.e.,

$$M_b^+(\mathbb{R}^3) \overset{\mathcal{F}}{\Longleftrightarrow} CP(\mathbb{R}^3) \tag{4.64}$$

as well as $P(\mathbb{R}^d)$ is mapped on to the Convex Set of Continuous Positive Definite functions $CP(\mathbb{R}^3)$ with $\varphi(0) = \hat{F}(0) = 1$, i.e.,

$$P(\mathbb{R}^3) \overset{\mathcal{F}}{\Longleftrightarrow} \left\{F \in CP(\mathbb{R}^3) \mid \varphi(0) = 1\right\}. \tag{4.65}$$

In fact, we have a stronger result:

Theorem 4.21 (**Bi-continuous Theorem**)**.** *The Fourier transform $\mathcal{F}$ is a bi-continuous mapping from $M_b^+(\mathbb{R}^d)$ equipped with the narrow topology onto $CP(\mathbb{R}^d)$ equipped with the topology of uniform convergence on compact sets.*

Proof. See N. Jacob's book Page112-114. □

In what follows, an even stronger theorem **Levy's Continuity Theorem** is presented:

Theorem 4.22 (Complete Version)**.** *Suppose we have a sequence of the probability measure $\{F_n\}_{n\in\mathbb{N}}$ (as the distribution of corresponding random variables $\{V_n\}_{n\in\mathbb{N}}$ in probability language), if their corresponding characteristic functions φ_n satisfy*

$$\varphi_n(\xi) \to \varphi(\xi),\ \forall \xi \in \mathbb{R}^d. \tag{4.66}$$

Then the following statements are equivalent:

(i) F_n is tight, i.e., $\lim_{|v|\to\infty} \sup_{n\in\mathbb{N}} \int_v^\infty \mathrm{d}F_n(v) = 0$, or, $\forall \epsilon > 0, \exists K \subset \mathbb{R}^d$ is compact, then $\sup_{n\in\mathbb{N}} = \int_{K^c} \mathrm{d}F_n(v) < \epsilon$;

(ii) $F_n \rightharpoonup F$ in narrow sense for F;

(iii) φ is a characteristic function of some F, i.e., $\varphi(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}\, \mathrm{d}F(v)$;

(iv) φ is a continuous function of ξ;

(v) φ is continuous at $\xi = 0$.

Remark 4.23. *If all the conditions (i)-(v) hold, then $F_n \rightharpoonup F$ for F as in (iii).*

Proof. (i)$\Longrightarrow$(ii): Since $\mathrm{e}^{-\,\mathrm{i}\xi\cdot v}$ is uniformly bounded and continuous, and the uniqueness theorem of characteristic functions implies that they are the determining class. Hence, by Helly's selection theorem, the tightness implies the existence of the probability measure F such that $F_n \rightharpoonup F$ in narrow sense. (This is actually one side of Prokhorov's Theorem).

(ii)$\Longrightarrow$(iii): Assume (ii) holds, then $\int_{\mathbb{R}^d} \phi(v)\,\mathrm{d}F_n(v) \to \int_{\mathbb{R}^d} \phi(v)\,\mathrm{d}F(v)$ for all $\phi \in C_b(\mathbb{R}^d)$. By selecting $\phi = \mathrm{e}^{-\,\mathrm{i}\xi\cdot v}$, then we have $\varphi_n \to \varphi$, where we have assumed the uniqueness of a limit.

(iii)$\Longrightarrow$(iv): the continuity of characteristic function has been proved in the Proposition 4.9.

(iv)$\Longrightarrow$(v): If φ is continuous everywhere, it is then continuous at $\xi = 0$.

(v)$\Longrightarrow$(i): The idea is to get a bound by using the continuity of φ at $\xi = 0$ and show the sequence in (i) is tight. ☐

Theorem 4.24 (Classical Version)**.** *Let $\{F_n\}_{n\in\mathbb{N}}$ and F be probability measures on $\mathbb{R}^d$. Then, $\{F_n\}_{n\in\mathbb{N}}$ converges to F in the narrow topology if and only if $\{\varphi_n\}_{n\in\mathbb{N}}$ converges pointwise to φ, i.e.,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(v)\,\mathrm{d}F_n(v) = \int_{\mathbb{R}^d} \phi(v)\,\mathrm{d}F(v),\ \forall \phi \in C_b(\mathbb{R}^d) \Longleftrightarrow \lim_{n\to\infty} \varphi_n(\xi) \to \varphi(\xi),\ \forall \xi \in \mathbb{R}^d \tag{4.67}$$

Proof. " $\Longrightarrow$ ": This implication is direct, if $\{F_n\}_{n\in\mathbb{N}}$ converges to F in the narrow topology, it is clear that its Fourier transform $\{\varphi_n\}_{n\in\mathbb{N}}$ converges pointwise to φ, since both the real and imaginary parts of $v \longmapsto \mathrm{e}^{-\,\mathrm{i}v\cdot\xi}$ are in $C_b(\mathbb{R}^d)$.

" $\Longleftarrow$ ": Another implication is a bit subtle. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ converge pointwise to φ, the Dominated Convergence Theorem shows that, for any $\delta > 0$,

$$\frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} [1 - \varphi_n(\xi)]\,\mathrm{d}\xi \to \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} [1 - \varphi(\xi)]\,\mathrm{d}\xi, \quad \text{as } n \to \infty, \tag{4.68}$$

where $[-\delta, \delta]^d$ is the cube of side 2δ on $\mathbb{R}^d$. Since φ is continuous at 0, the right-hand side can be made small enough by choosing δ appropriately, and then one can see that after choosing the suitable δ, the left-hand side can be made as small as we wish, uniformly for all n; that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} [1 - \varphi_n(\xi)]\,\mathrm{d}\xi \le \epsilon, \quad \forall n \ge 1. \tag{4.69}$$

which then implies the tightness of the sequence $\{F_n\}_{n\in\mathbb{N}}$ by noticing the following Lemma 4.25. Furthermore, it must have a subsequence which converges weakly to a probability measure by the other side of Prokhorov's Theorem. This probability measure must be F, due to the implication we proved first. In fact, this reasoning applies to any subsequence of $\{F_n\}_{n\in\mathbb{N}}$, so the whole sequence must converge weakly to F. ☐

Lemmas 4.25 is provided in the sense that:

Lemma 4.25. *For a probability measure F on $\mathbb{R}$, we have,*

$$\int_{|v|\geq 2/\delta} \mathrm{d}F(v) \leq \frac{1}{\delta}\int_{-\delta}^{\delta} [1-\varphi(\xi)]\,\mathrm{d}\xi, \quad \forall \delta > 0. \tag{4.70}$$

For a probability measure F on $\mathbb{R}^d$, we have,

$$\frac{1}{2}\int_{\mathbb{R}^d - [-2/\delta, 2/\delta]^d} \mathrm{d}F(v) \leq \frac{1}{(2\delta)^d}\int_{[-\delta,\delta]^d} [1-\varphi(\xi)]\,\mathrm{d}\xi, \quad \forall \delta > 0. \tag{4.71}$$

Proof. *(i) For the case of $d=1$:* by observation and Fubini's Theorem,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi(\xi)\,\mathrm{d}\xi = \frac{1}{2\delta}\int_{-\infty}^{\infty} \hat{\phi}_\delta(v)\,\mathrm{d}F(v), \tag{4.72}$$

where $\phi_\delta := \mathbf{1}_{[-\delta,\delta]}$. The left-hand side is the average of the Fourier transform on $[-\delta,\delta]$; the right-hand side is not far from the integral of F on a large set. We can calculate explicitly $\hat{\phi}_\delta$:

$$\hat{\phi}_\delta(v) = \int_{-\delta}^{\delta} \mathrm{e}^{-\,\mathrm{i}v\xi}\,\mathrm{d}\xi = \frac{2}{v}\sin(\delta v), \quad v \in \mathbb{R}. \tag{4.73}$$

Since $\sin y/y \leq 1$ for all $y \in \mathbb{R}$, we have,

$$\begin{aligned}\int_{-\infty}^{\infty} \left[1 - \frac{1}{2\delta}\hat{\phi}_\delta(v)\right] \mathrm{d}F(v) &\geq \int_{|v|\geq 2/\delta} \left[1 - \frac{|\sin(\delta v)|}{\delta|v|}\right] \mathrm{d}F(v)\\ &\geq \int_{|v|\geq 2/\delta} \left[1 - \frac{1}{\delta|v|}\right] \mathrm{d}F(v)\\ &\geq \frac{1}{2}\int_{|v|\geq 2/\delta} \mathrm{d}F(v),\end{aligned} \tag{4.74}$$

that it to say,

$$\begin{aligned}\frac{1}{2}\int_{|v|\geq 2/\delta} \mathrm{d}F(v) \leq 1 - \frac{1}{2\delta}\int_{-\infty}^{\infty} \hat{\phi}_\delta(v)\,\mathrm{d}F(v) &= 1 - \frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi(\xi)\,\mathrm{d}\xi\\ &= \frac{1}{2\delta}\int_{-\delta}^{\delta} [1-\varphi(\xi)]\,\mathrm{d}\xi.\end{aligned} \tag{4.75}$$

(ii) For the case of general dimension $d \geq 1$: The result can be proved in $\mathbb{R}^d$ for any $d \geq 1$ with essentially the same calculation, and the parallel proof is provided as following: let $\Omega_\delta = [-\delta,\delta]^d$, we have, again by Fubini's Theorem,

$$\frac{1}{(2\delta)^d}\int_{\Omega_\delta} \varphi(\xi)\,\mathrm{d}\xi = \frac{1}{(2\delta)^d}\int_{\mathbb{R}^d} \hat{\Phi}_\delta(v)\,\mathrm{d}F(v), \tag{4.76}$$

where now $\Phi_\delta := \mathbf{1}_{\Omega_\delta} = \prod_{j=1}^d \phi_\delta$ and $\hat{\Phi}_\delta$ becomes,

$$\hat{\Phi}_\delta(v) = \prod_{j=1}^{d} \hat{\phi}_\delta(v_j) = \prod_{j=1}^{d} \frac{2}{v_j}\sin(\delta v_j), \quad v \in \mathbb{R}^d. \tag{4.77}$$

Notice that $(2\delta)^{-d}\hat{\Phi}_\delta(v) \leq 1$ for all $v \in \mathbb{R}^d$, and

$$(2\delta)^{-d}\left|\hat{\Phi}_\delta(v)\right| \leq \frac{1}{2}, \quad \forall v \in \mathbb{R}^d \setminus \Omega_{2/\delta}, \tag{4.78}$$

since outside the cube $\Omega_{2/\delta}$ at least one of the coordinate must be larger than $2/\delta$. Then,

$$\begin{aligned}\int_{\mathbb{R}^d}\left[1-\frac{1}{2\delta}\hat{\Phi}_\delta(v)\right]\mathrm{d}F(v) \geq& \int_{\mathbb{R}^d\setminus\Omega_{2/\delta}}\left[1-\frac{1}{(2\delta)^d}\left|\hat{\Phi}_\delta(v)\right|\right]\mathrm{d}F(v)\\ \geq&\frac{1}{2}\int_{\mathbb{R}^d\setminus\Omega_{2/\delta}}\mathrm{d}F(v),\end{aligned} \tag{4.79}$$

that is to say,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d\setminus\Omega_{2/\delta}}\mathrm{d}F(v) \leq 1-\frac{1}{(2\delta)^d}\int_{\mathbb{R}^d}\hat{\Phi}_\delta(v)\,\mathrm{d}F(v) =& 1-\frac{1}{(2\delta)^d}\int_{\Omega_{2/\delta}}\varphi(\xi)\,\mathrm{d}\xi\\ =&\frac{1}{(2\delta)^d}\int_{\Omega_{2/\delta}}\left[1-\varphi(\xi)\right]\mathrm{d}\xi.\end{aligned} \tag{4.80}$$

□

Corollary 4.26 (Useful Version)**.** *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^d$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ be their corresponding characteristic counterparts. The following are equivalent:*

- *$\{\varphi_n\}_{n\in\mathbb{N}}$ converges pointwise to a function φ which is continuous at 0.*
- *$\{F_n\}_{n\in\mathbb{N}}$ converges in the narrow sense to a probability measure F.*

If these equivalent statements hold, then $\hat{F} = \varphi$.

Proof. The proof is essentially a repetition of the same idea which led to Theorem 4.24:

" $\Longleftarrow$ " If the second statement holds then from Theorem 4.24, we already know that $\varphi_n \to \varphi$, so the first statement hods since φ must be continuous.

" $\Longrightarrow$ " Conversely, if the first statement holds, then same proof of Theorem 4.24 shows that the sequence $\{F_n\}_{n\in\mathbb{N}}$ is tight, since in that part of the proof, we only used the continuity of the pointwise limit of $\{\varphi_n\}_{n\in\mathbb{N}}$. Again by Prokhorov's Theorem, $\{\varphi_n\}_{n\in\mathbb{N}}$ has a subsequence which converges in the narrow sense to some measure F. This F must then be non-negative, and actually be a probability measure since

$$\int_{\mathbb{R}^d}\mathrm{d}F(v) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\mathrm{d}F_n(v) = 1 \tag{4.81}$$

by the definition of narrow convergence.

And $\hat{F} = \varphi$ holds due to the first implication " $\Longleftarrow$ ". In fact, this reasoning holds for all subsequences of $\{F_n\}_{n\in\mathbb{N}}$, so we conclude the whole sequence must converge in the narrow sense to the only measure F such that $\hat{F} = \varphi$, which is in addition a probability measure. □

Remark 4.27. *Why continuity of the limit at 0 is needed in (v)? The following examples could give us an insight into the connection between the continuity at $\xi = 0$ and the tightness of random variable (or narrow convergence of its corresponding probability measure).*

- *First Example: Let F_n be the uniform distribution over $(-n, n)$, its characteristic function $\varphi_n = \frac{\sin n\xi}{n\xi}$, which converges as,*

$$\varphi_n(\xi) = \frac{\sin n\xi}{n\xi} \xrightarrow{n\to\infty} \varphi(\xi) = \begin{cases} 1, & \textit{for } \xi = 0, \\ 0, & \textit{for } \xi \neq 0, \end{cases}$$

  In this case, $\{F_n\}$ is NOT tight, the limit of their characteristic functions is NOT continuous at $\xi = 0$, and hence, $\{F_n\}$ does NOT converge narrowly.

- *Second Example: Let Z be a standard normal random variable, i.e., having standard normal distribution $\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{v^2}{2}}$ with expected value 0 and standard deviation 1; then if $V_n = nZ$, V_n will have a normal distribution F_n with the same expected value 0 but standard deviation (standard variance) n, i.e.,*

$$f_n(v) = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-\frac{v^2}{2n^2}} \overset{\mathcal{F}}{\Longrightarrow} \varphi_n(\xi) = \mathrm{e}^{-\frac{n^2\xi^2}{2}} \xrightarrow{n\to\infty} \varphi(\xi) = \begin{cases} 1, & \textit{for } \xi = 0, \\ 0, & \textit{for } \xi \neq 0, \end{cases}$$

  So the sequence of probability measures F_n does NOT narrowly converge to any probability measure F, or the sequence of random variable V_n does NOT converge to any random variable in distribution, since $\lim\limits_{V\to\infty} \int_{-\infty}^{V} \mathrm{d}F_n(v) \to \frac{1}{2}$ for all $v \in \mathbb{R}$, which implies that V_n is not tight, or that is to say, $V_n \to V$ in distribution such that $P(V = \infty) = P(V = -\infty) = \frac{1}{2}$.

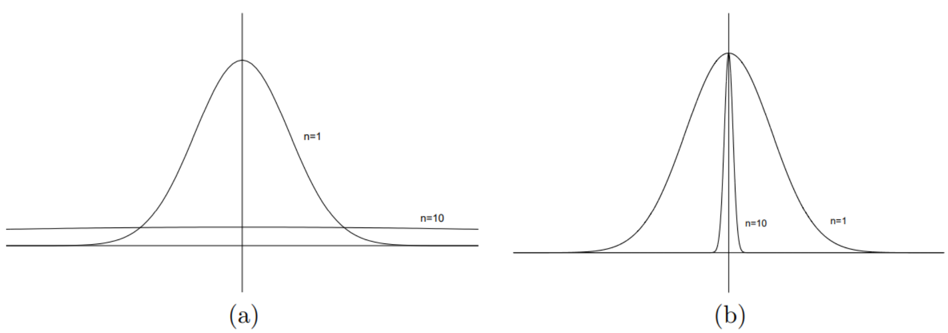

Figure 2: (a) Density function of V_n for $n = 1$ and $n = 10$; (b) Characteristic function of V_n for $n = 1$ and $n = 10$.

- *Third Example: Let $f : \mathbb{R}^d \longmapsto \mathbb{R}$ be a non-negative, continuous, compactly supported fubushanction and the measures F_n be equipped with density function,*

$$f_n(v) := \frac{1}{n} f\left(\frac{v}{n}\right), \ \forall v \in \mathbb{R}^d \tag{4.82}$$

Their Fourier transforms converge pointwise to 0 *everywhere except at* $\xi = 0$*, where they are constantly equal to* 1*, which implies the pointwise limit of* φ_n *is NOT continuous at* 0*. However, one can easily see that the sequence* $\{F_n\}_{n\in\mathbb{N}}$ *does NOT converge in the narrow sense (though it does converge to* 0 *in the weak* sense of measures).*

4.4 Probability Measures with α-Finite Moments and their Fourier Transform

To study the specific problem, let us denote the subspace of $P(\mathbb{R}^3)$ by $P_\alpha(\mathbb{R}^3)$ with $\alpha \in [0, \infty)$, which is the probability measure F on $\mathbb{R}^3$ having finite moments up to the order α:

$$\begin{aligned} P_\alpha(\mathbb{R}^3) = \Big\{ f \in P(\mathbb{R}^3) \Big| \int_{\mathbb{R}^3} f \, \mathrm{d}v = 1, \ \int_{\mathbb{R}^3} |v|^\alpha f \, \mathrm{d}v < \infty \\ \text{and if } \alpha > 1, \int_{\mathbb{R}^3} v_j f \, \mathrm{d}v = 0, \ j = 1, 2, 3 \Big\} \end{aligned} \tag{4.83}$$

and $\mathcal{K}^\alpha$ the subspace of $\mathcal{K}$, defined as following:

$$\mathcal{K}^\alpha = \left\{ \varphi \in \mathcal{K}; \ \|\varphi - 1\|_\alpha < \infty \right\}, \tag{4.84}$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \tag{4.85}$$

Lemma 4.28 (**Basic Properties of** $\mathcal{K}^\alpha$)**.** *Some basic properties of* $\mathcal{K}^\alpha$ *are provided:*

- *The space* $\mathcal{K}^\alpha$ *is NOT a vector space. (since, for example,* $\varphi(\xi) = 0$ *does NOT belong to* $\mathcal{K}^\alpha$*, no zero element.)*
- $\varphi(\xi) = 1 \in \mathcal{K}^\alpha$ *for every* $\alpha \geq 0$*.*
- $|\varphi(\xi)| \leq \varphi(0) = 1$*, for every* $\varphi \in \mathcal{K}^\alpha$*.*
- $\varphi\tilde{\varphi} \in \mathcal{K}^\alpha$*, for the product of all* $\varphi, \tilde{\varphi} \in \mathcal{K}^\alpha$*.*
- *Any linear and convex combination of functions from* $\mathcal{K}^\alpha$ *belongs to* $\mathcal{K}^\alpha$*.*

The set $\mathcal{K}^\alpha$ endowed with the Fourier-based distance $\|\cdot\|_\alpha$, for any $\varphi, \tilde{\varphi} \in \mathcal{K}^\alpha$,

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}, \tag{4.86}$$

is a complete metric space, with the following embedding relation:

Proposition 4.29 (**Completeness of** $\mathcal{K}^\alpha$)**.** *For every* $\alpha \in [0, 2]$*, the set* $\mathcal{K}^\alpha$ *endowed with the distance* (4.86) *is a complete metric space.*

Proof. The proof is immediate because the set of characteristic functions is closed with respect to the pointwise convergence:

More specifically, an inspection reveals that if $\{\varphi_n\}$ is a Cauchy sequence w.r.t the distance $\|\cdot\|_\alpha$ as in (4.86), then it satisfies the pointwise Cauchy condition (even the uniform Cauchy condition) on every compact subset of $\mathbb{R}^d$ w.r.t to the standard Euclidean metric, and hence $\{\varphi_n\}$ converges pointwise to a continuous function φ, which belongs to $\mathcal{K}$ (since the set of characteristic functions is closed with respect to the pointwise convergence, or by the Levy's Continuous Theorem as in the subsection above).

To verify the limit function φ actually lies in $\mathcal{K}^\alpha$: on one side, the Cauchy condition implies the uniform boundedness,

$$\sup_{\xi\in\mathbb{R}^d} \frac{|\varphi_n(\xi)-1|}{|\xi|^\alpha} \le C, \quad \forall n \in \mathbb{N}. \tag{4.87}$$

on the other side, we have the pointwise convergence $\varphi_n(\xi)-1 \to \varphi(\xi)-1$. Thus, $\varphi \in \mathcal{K}^\alpha$ follows (or from the $\|\varphi_n - \varphi\|_\alpha \to 0$) that, for any $\xi \in \mathbb{R}^d$,

$$\sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi)-1|}{|\xi|^\alpha} \le \overline{\lim_{n\to\infty}} \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi_n(\xi)-1|}{|\xi|^\alpha} \le \infty. \tag{4.88}$$

□

Proposition 4.30 (**Embedding Relation of $\mathcal{K}^\alpha$**).

$$\{1\} \subseteq \mathcal{K}^\alpha \subseteq \mathcal{K}^{\alpha_0} \subseteq \mathcal{K}^0, \quad \textit{for all} \quad 2 \ge \alpha \ge \alpha_0 \ge 0. \tag{4.89}$$

Proof. To prove the embedding relation, we have to show the following points:

- For $\mathcal{K}^0 = \mathcal{K}$, it suffices to show that, for any $\varphi \in \mathcal{K}$, we have $\|\varphi - 1\|_0 \le \infty$; indeed, from the boundedness of characteristic function,

$$\|\varphi - 1\|_0 = \sup_{\xi\in\mathbb{R}^3} |\varphi(\xi)-1| \le \varphi(0)+1 = 2. \tag{4.90}$$

- For $\mathcal{K}^\alpha \subseteq \mathcal{K}^{\alpha_0}$ if $\alpha \ge \alpha_0$, we proceed as follows: for any $\varphi \in \mathcal{K}^\alpha$,

$$\begin{aligned} \|\varphi - 1\|_{\alpha_0} &\le \sup_{|\xi|\le 1} \frac{|\varphi(\xi)-1|}{|\xi|^{\alpha_0}} + \sup_{|\xi|>1} \frac{|\varphi(\xi)-1|}{|\xi|^{\alpha_0}} \\ &\le \sup_{|\xi|\le 1} \frac{|\varphi(\xi)-1|}{|\xi|^{\alpha}} + \sup_{|\xi|>1} |\varphi(\xi)-1| \\ &\le \|\varphi - 1\|_\alpha + \varphi(0) + 1. \end{aligned} \tag{4.91}$$

  since $\alpha \ge \alpha_0$. Hence, $\varphi \in \mathcal{K}^{\alpha_0}$.

- For $\mathcal{K}^\alpha = \{1\}$ if $\alpha > 2$, it follows immediately from fact that,

$$\begin{aligned} \|\varphi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\varphi(\xi)-1|}{|\xi|^\alpha} < \infty \Longrightarrow & \frac{|\varphi(\xi)-1|}{|\xi|^\alpha} \le \|\varphi - 1\|_\alpha \\ \Longrightarrow & |\varphi(\xi)-1| \le \|\varphi - 1\|_\alpha |\xi|^\alpha \end{aligned} \tag{4.92}$$

then, for any $\varphi \in \mathcal{K}^\alpha$ with $\alpha > 2$ satisfies,

$$\left|\frac{1-\varphi(\xi)}{|\xi|^2}\right| \leq |\xi|^{\alpha-2}\|\varphi - 1\|_\alpha \to 0, \quad \text{as } |\xi| \to 0. \tag{4.93}$$

Next, by using the inequality (4.45), we find, for any unit vector $\zeta \in \mathbb{R}^3$ and all $\xi \in \mathbb{R}^3$,

$$\left|\frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h}\right|^2 \leq 2\frac{[1 - \mathrm{Re}[\varphi(h\zeta)]]}{h^2} \leq 2\left|\frac{1-\varphi(h\zeta)}{h^2}\right|, \tag{4.94}$$

thus, combining the (4.93) and (4.94), we have,

$$\lim_{h\to 0} \frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h} = 0. \tag{4.95}$$

Hence, for all $\zeta \in \mathbb{R}^3$ the directional derivative $\zeta \cdot \nabla\varphi(\xi)$ exists and is always equal to 0, which implies that φ is a constant and finally concludes by noting that $\varphi(0) = 1$.

□

Lemma 4.31 (Estimate of Real and Imaginary Parts). *Let* $\alpha \in [0, 2]$ *and* $\varphi \in \mathcal{K}^\alpha$, *then* $\mathrm{Re}[\varphi(\cdot)] \in \mathcal{K}^\alpha$,

$$\|\mathrm{Re}[\varphi(\cdot)] - 1\|_\alpha \leq \|\varphi - 1\|_\alpha, \tag{4.96}$$

and

$$\sup_{\xi\in\mathbb{R}^3/\{0\}} \frac{|\mathrm{Im}[\varphi(\xi)]|}{|\xi|^\alpha} \leq \|\varphi - 1\|_\alpha. \tag{4.97}$$

Proof. In fact, for any characteristic function $\varphi \in \mathcal{K}^\alpha$, its real part $\mathrm{Re}[\varphi(\cdot)]$ is the characteristic function as well, thanks to the identity $\mathrm{Re}[\varphi] = (\varphi + \bar{\varphi})/2$. Then, by the Pythagorean Theorem, we have,

$$|\varphi(\xi) - 1|^2 = |\mathrm{Im}\,[\varphi(\xi)]|^2 + |\mathrm{Re}\,[\varphi(\xi) - 1]|^2 \geq |\mathrm{Re}\,[\varphi(\xi)] - 1|^2. \tag{4.98}$$

After dividing the equation above by $|\xi|^\alpha$ and calculating the supremum with respect to $\xi \in \mathbb{R}^3/\{0\}$, we obtain,

$$\|\varphi(\cdot) - 1\|_\alpha \geq \|\mathrm{Re}[\varphi(\cdot)] - 1\|_\alpha. \tag{4.99}$$

Besides, considering the inequality $|\varphi(\xi) - 1| \geq |\mathrm{Im}[\varphi(\xi)]|$, we find that,

$$\sup_{\xi\in\mathbb{R}^3/\{0\}} \frac{|\mathrm{Im}[\varphi(\xi)]|}{|\xi|^\alpha} \leq \|\varphi - 1\|_\alpha. \tag{4.100}$$

□

Lemma 4.32 (Continuity Type Estimate). *For all* $\xi, \eta \in \mathbb{R}^3$ *and* $\varphi \in \mathcal{K}^\alpha$, *then*

$$|\varphi(\xi) - \varphi(\xi + \eta)| \leq \|\varphi - 1\|_\alpha \left(4|\xi|^{\frac{\alpha}{2}}|\eta|^{\frac{\alpha}{2}} + |\eta|^\alpha\right). \tag{4.101}$$

Proof. Since $|\varphi| \leq 1$, it follows from (4.46) that

$$\begin{aligned}
|\varphi(\xi)-\varphi(\xi+\eta)| \leq & |\varphi(\xi)\varphi(\eta)-\varphi(\xi+\eta)| + |\varphi(\xi)|\,|\varphi(\eta)-1| \\
\leq & \left(1-|\varphi(\xi)|^2\right)^{\frac{1}{2}}\left(1-|\varphi(\eta)|^2\right)^{\frac{1}{2}} + |\varphi(\eta)-1| \\
\leq & (1+|\varphi(\xi)|)^{\frac{1}{2}}(1-|\varphi(\xi)|)^{\frac{1}{2}}(1+|\varphi(\eta)|)^{\frac{1}{2}}(1-|\varphi(\eta)|)^{\frac{1}{2}} + |\varphi(\eta)-1| \\
\leq & 4(1-|\varphi(\xi)|)^{\frac{1}{2}}(1-|\varphi(\eta)|)^{\frac{1}{2}} + |\varphi(\eta)-1| \\
\leq & 4\|1-\varphi\|_{\alpha}^{\frac{1}{2}}|\xi|^{\frac{\alpha}{2}}\|1-\varphi\|_{\alpha}^{\frac{1}{2}}|\eta|^{\frac{\alpha}{2}} + \|\varphi-1\|_{\alpha}|\eta|^{\alpha} \\
= & \|\varphi-1\|_{\alpha}\left(4|\xi|^{\frac{\alpha}{2}}|\eta|^{\frac{\alpha}{2}} + |\eta|^{\alpha}\right),
\end{aligned} \tag{4.102}$$

which gives the desired estimate (4.101). □

The following technical Lemma illustrates that the **Fourier transform of** $P_{\alpha}(\mathbb{R}^3)$, any probability measure with α-finite moments, **belongs to** $\mathcal{K}^{\alpha}$:

Lemma 4.33. *Let $\alpha \in [0,2]$. Assume that μ is a probability measure on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3}|v|^{\alpha}\mathrm{d}\mu(v)$ is finite; if, moreover, $\alpha \in (1,2]$, assume that $\int_{\mathbb{R}^3} v_j\,\mathrm{d}\mu(v) = 0$ for $j = 1,2,3$. (i.e., $\mu \in P_{\alpha}(\mathbb{R}^3)$). Then $\hat{\mu} \in \mathcal{K}^{\alpha}$.*

Proof. First of all, for $\alpha \in (0,1]$. Using the definition of the Fourier transform of a probability measure μ, we obtain,

$$\frac{|\hat{\mu}(\xi)-1|}{|\xi|^{\alpha}} \leq \int_{\mathbb{R}^3}\frac{\left|\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}-1\right|}{|\xi|^{\alpha}}\,\mathrm{d}\mu(v). \tag{4.103}$$

Then, by substituting $\xi = \eta/|v|$, we have,

$$\sup_{|\xi|\leq R}\frac{\left|\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}-1\right|}{|\xi|^{\alpha}} = |v|^{\alpha}\sup_{|\eta|\leq R'}\frac{\left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v/|v|}-1\right|}{|\eta|^{\alpha}} \leq C|v|^{\alpha}, \tag{4.104}$$

where, in view of the elementary inequality $|\,\mathrm{e}^{\mathrm{i}s}-1| \leq \min\{|s|,2\}$ for all $s \in \mathbb{R}$, the constant

$$C = \sup_{v,\eta\in\mathbb{R}^3}\frac{\left|\mathrm{e}^{-\,\mathrm{i}\eta\cdot v/|v|}-1\right|}{|\eta|^{\alpha}} \Longrightarrow \begin{cases} = |\eta|^{1-\alpha}, & \text{for } \eta < R, \\ \leq \dfrac{2}{|\eta|^{\alpha}}, & \text{for } \eta \geq R, \end{cases}$$

is finite for $\alpha \in (0,1]$. Hence, we deduce from (4.103) by combining the discussions (4.104)-(4.105) above,

$$\|\hat{\mu}-1\|_{\alpha} \leq C\int_{\mathbb{R}^3}|v|^{\alpha}\,\mathrm{d}\mu(v) < \infty. \tag{4.105}$$

Secondly, for $\alpha \in (1,2]$, one should proceed analogously using the following counterpart of inequality (4.103):

$$\frac{|\hat{\mu}(\xi)-1|}{|\xi|^{\alpha}} \leq \int_{\mathbb{R}^3}\left|\frac{\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}+\mathrm{i}v\cdot\xi-1}{|\xi|^{\alpha}}\right|\mathrm{d}\mu(v) \tag{4.106}$$

which then becomes the simple consequence of the additional assumption $\int_{\mathbb{R}^3} v_i\mathrm{d}\mu(v)$ for every i $= \{1,2,3\}$. □

Note that the Fourier transform of every probability measure in $P_\alpha(\mathbb{R}^3)$ belongs to $\mathcal{K}^\alpha$, however, the set $\mathcal{K}^\alpha$ is bigger than the $\mathcal{F}\left[P_\alpha(\mathbb{R}^3)\right]$, see [14, Remark 3.16].

Remark 4.34. *Let us provide a counterexample that the reverse implication above for $\alpha \in (0,2)$ is NOT true: it is well-known that the function $\varphi_\alpha(\xi) = \mathrm{e}^{-|\xi|^\alpha}$ with $\alpha \in (0,2)$ is the Fourier transform of the probability density $g_\alpha(t,v)$ of an α-stable symmetric Levy process.*

Obviously, $\varphi_\alpha(\xi) = \mathrm{e}^{-|\xi|^\alpha} \in \mathcal{K}^\alpha$, on the other hand, we now claim that $g_\alpha(t,v) \notin P_\alpha(\mathbb{R}^3)$: indeed, for every $\alpha \in (0,2)$, though implicitly defined, the density function is smooth, non-negative, and satisfies the estimate $0 < g_\alpha(t,v) \leq C(1+|v|)^{-(\alpha+d)}$ for all $v \in \mathbb{R}^d$. Moreover, we have,

$$\frac{g_\alpha(t,v)}{|v|^{\alpha+d}} \to c_0, \quad \text{as } |v| \to \infty, \tag{4.107}$$

where $c_0 = \alpha 2^{\alpha-1}\pi^{-(d+2)/2}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$, which then implies that

$$\int_{\mathbb{R}^3} g_\alpha(t,v)|v|^\alpha \,\mathrm{d}v = \infty. \tag{4.108}$$

So the following question is whether this gap could be filled? To achieve this, we need to introduce *a new classification on the Characteristic Functions*, by setting

$$\mathcal{M}^\alpha = \left\{\varphi \in \mathcal{K};\ \|\varphi - 1\|_{\mathcal{M}^\alpha} < \infty\right\}, \quad \alpha \in (0,2), \tag{4.109}$$

where

$$\|\varphi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^{d+\alpha}} \,\mathrm{d}\xi. \tag{4.110}$$

For $\varphi, \tilde{\varphi} \in \mathcal{M}^\alpha$, put

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{d+\alpha}} \,\mathrm{d}\xi, \tag{4.111}$$

and, for any $\beta \in (0,\alpha]$, we introduce the distance

$$\mathrm{dis}_{\alpha,\beta}(\varphi,\tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta \tag{4.112}$$

in this case, we have $\mathcal{F}\left[P_\alpha(\mathbb{R}^d)\right] = M^\alpha(\mathbb{R}^d)$, see [20] for detailed discussion of the new space.

4.5 First Application: Uniqueness of the Solution to the Boltzmann Equation with Finite Energy

Theorem 4.35. *Let the non-cutoff collision kernel b satisfy the condition* (3.32) *with $0 < \mu < 1$. For all (non-negative) energy-conserving solution f, g (the corresponding probability density function of probability measure F, G) of homogeneous Boltzmann equation with respect to the initial datum f_0, g_0 (the corresponding probability density function of probability measure F_0, G_0), satisfying*

$$\int_{\mathbb{R}^3} \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} f_0(v)\,\mathrm{d}v = \int_{\mathbb{R}^3} \begin{pmatrix} 0 \\ v_j \\ |v|^2 \end{pmatrix} g_0(v)\,\mathrm{d}v = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \qquad 1 \leq j \leq 3. \tag{4.113}$$

Then, we have,

$$\|f(t,\cdot)-g(t,\cdot)\|_2 \le \|f_0(\cdot)-g_0(\cdot)\|_2, \tag{4.114}$$

where,

$$\|f(t,\cdot)-g(t,\cdot)\|_2 = \sup_{\xi\in\mathbb{R}^3} \frac{\left|\hat{f}(t,\xi)-\hat{g}(t,\xi)\right|}{|\xi|^2}. \tag{4.115}$$

Proof. Let f, g be the solution to the homogeneous Boltzmann equation with respect to the initial datum f_0, g_0, and $\hat{f}, \hat{g}$ be their Fourier transform, which then satisfies,

$$\begin{aligned} \partial_t \hat{f}(t,\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{f}(t,0)\hat{f}(t,\xi)\right] \mathrm{d}\sigma, \\ \partial_t \hat{g}(t,\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\hat{g}(t,\xi^+)\hat{g}(t,\xi^-)-\hat{g}(t,0)\hat{g}(t,\xi)\right] \mathrm{d}\sigma, \end{aligned} \tag{4.116}$$

after the subtraction, we obtain,

$$\begin{aligned} &\partial_t\left[\frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi|^2}\right] \\ &= \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{g}(t,\xi^+)\hat{g}(t,\xi^-)}{|\xi|^2} - \frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi|^2}\right] \mathrm{d}\sigma. \end{aligned} \tag{4.117}$$

Then, we make the usual splitting,

$$\begin{aligned} &\left|\frac{\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{g}(t,\xi^+)\hat{g}(t,\xi^-)}{|\xi|^2}\right| \\ &\le \left|\hat{f}(t,\xi^+)\right|\left|\frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi^-|^2}\right|\frac{|\xi^-|^2}{|\xi|^2} + \left|\hat{g}(t,\xi^-)\right|\left|\frac{\hat{f}(t,\xi^+)-\hat{g}(t,\xi^+)}{|\xi^+|^2}\right|\frac{|\xi^+|^2}{|\xi|^2} \\ &\le \sup_{\xi\in\mathbb{R}^3}\left|\frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi|^2}\right|\left(\frac{|\xi^+|^2+|\xi^-|^2}{|\xi|^2}\right) = \sup_{\xi\in\mathbb{R}^3}\left|\frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi|^2}\right|. \end{aligned} \tag{4.118}$$

By setting the function $d\,(t,\xi)$ as following:

$$d\,(t,\xi) := \frac{\hat{f}(t,\xi)-\hat{g}(t,\xi)}{|\xi|^2}, \tag{4.119}$$

(i) For the cutoff collision kernel b_c, by the rotational invariance, for all $\xi \neq 0$,

$$\gamma_2 = \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \mathrm{d}\sigma, \tag{4.120}$$

as a result, the equation (4.117) leads to the inequality satisfied by $d(t,\xi)$,

$$|\partial_t d\,(t,\xi)+\gamma_2 d(t,\xi)| \le \gamma_2 \sup_{\xi\in\mathbb{R}^3}|d(t,\xi)|\,. \tag{4.121}$$

The desired estimate (4.114) will be obtained from (4.121) by the generalized Grönwall's inequality.

(ii) For the non-cutoff collision kernel b, considering the fact that,

$$\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)=\infty, \tag{4.122}$$

we make the following split toward the integration domain $\sigma\in\mathbb{S}^2$,

$$\begin{cases}\left|1-\dfrac{\xi\cdot\sigma}{|\xi|}\right|\geq\epsilon : \text{Cutoff part}\\ \left|1-\dfrac{\xi\cdot\sigma}{|\xi|}\right|<\epsilon : \text{Non-cutoff part}\end{cases}$$

Notice that

$$\begin{aligned}&\left|\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{f}(t,\xi)\hat{f}(t,0)\right|\\ \leq&\left|\hat{f}(t,\xi^-)\right|\left|\hat{f}(t,\xi^+)-\hat{f}(t,\xi)\right|+\left|\hat{f}(t,\xi)\right|\left|\hat{f}(t,\xi^-)-\hat{f}(t,0)\right|\\ \leq&\sup_{|\eta|\leq\sup(|\xi|,|\xi^+|)}\left|\nabla\hat{f}(\eta)\right||\xi^+-\xi|+\sup_{|\eta|\leq|\xi^-|}\left|\nabla\hat{f}(\eta)\right||\xi^-|\end{aligned} \tag{4.123}$$

since $|\xi^+|,|\xi^-|\leq|\xi|$ and $\nabla\hat{f}(0)=0$, we conclude that

$$\left|\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{f}(t,\xi)\hat{f}(t,0)\right|\leq C|\xi||\xi^-|\leq C|\xi|^2\left(1-\frac{\xi\cdot\sigma}{|\xi|}\right)^{\frac{1}{2}} \tag{4.124}$$

where the constant C depends only on the dimension. This implies that the right-hand side of the equations (4.116) is well-defined when the non-cutoff collision kernel has the mild singularity, i.e., $0\leq\nu\leq 1$.

As a conclusion, by setting,

$$\gamma_2^\epsilon=\int_{\mathbb{S}^2}\mathbf{1}_{|1-\frac{\xi\cdot\sigma}{|\xi|}|\geq\epsilon}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{d}\sigma \tag{4.125}$$

and

$$r_\epsilon=\sup_{\xi\in\mathbb{R}^3,t\in[0,\infty)}\left|\int_{\mathbb{S}^2}\mathbf{1}_{|1-\frac{\xi\cdot\sigma}{|\xi|}|<\epsilon}b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\hat{f}(t,\xi^+)\hat{f}(t,\xi^-)-\hat{f}(t,\xi)\hat{f}(t,0)\right]\mathrm{d}\sigma\right| \tag{4.126}$$

we obtain that $r_\epsilon\to 0$ as $\epsilon\to 0$, and

$$|\partial_t d\,(t,\xi)+\gamma_2^\epsilon d(t,\xi)|\leq\gamma_2^\epsilon\sup_{\xi\in\mathbb{R}^3}|d(t,\xi)|+r_\epsilon, \tag{4.127}$$

which is equivalent to

$$\left|\partial_t\left[\mathrm{e}^{\gamma_2^\epsilon t}d(t,\xi)\right]\right|\leq\gamma_2^\epsilon\sup_{\xi\in\mathbb{R}^3}\left|\mathrm{e}^{\gamma_2^\epsilon t}d(t,\xi)\right|+\mathrm{e}^{\gamma_2^\epsilon t}r_\epsilon. \tag{4.128}$$

Integrating the time variable from 0 to t, we have,

$$\left|\mathrm{e}^{\gamma_2^\epsilon t}d(t,\xi)\right|\leq|h(0,\xi)|+\int_0^t\left[\gamma_2^\epsilon\sup_{\xi\in\mathbb{R}^3}\left|\mathrm{e}^{\gamma_2^\epsilon t}d(t,\xi)\right|+\mathrm{e}^{\gamma_2^\epsilon t}r_\epsilon\right]\mathrm{d}\tau. \tag{4.129}$$

Hence, if we further denote $H_\epsilon(t) = \sup_{\xi\in\mathbb{R}^3} \left| \mathrm{e}^{\gamma_2^\epsilon t} d(t,\xi) \right|$,

$$H_\epsilon(t) \le H_\epsilon(0) + \int_0^t \left[\gamma_2^\epsilon H_\epsilon(\tau) + \mathrm{e}^{\gamma_2^\epsilon t} r_\epsilon \right] \,\mathrm{d}\tau, \tag{4.130}$$

then, by the generalized Grönwall's inequality, we find,

$$H_\epsilon(t) \le \mathrm{e}^{\gamma_2^\epsilon t} H_\epsilon(0) + t\,\mathrm{e}^{\gamma_2^\epsilon t} r_\epsilon, \tag{4.131}$$

namely,

$$\sup_{\xi\in\mathbb{R}^3} |d(t,\xi)| \le \sup_{\xi\in\mathbb{R}^3} |d(0,\xi)| + r_\epsilon t. \tag{4.132}$$

By letting $\epsilon \to 0$, we finally obtain the desired estimate (4.114). □

5 Well-posedness Theory of Homogeneous Boltzmann Equation in Probability Measure Space

5.1 Definition of the Solution

In this section, the solution that we try to look for is a probability measure F_t for any $t \ge 0$, *i.e.*, $F_t \in C\left([0,\infty), P_\alpha(\mathbb{R}^3)\right)$, where $P_\alpha(\mathbb{R}^3)$ is the set of probability measures on $\mathbb{R}^3$ with finite moments up to the order $\alpha \in [0,2]$ as in (4.83), which implies the possible existence of infinite energy solution.

The main purpose is to show that, the Cauchy problem of the spatially homogeneous Boltzmann equation admits a measure-valued solution in the case of Maxwellian molecule.

Hence, for the completeness, we first introduce the precise definition of measure-valued solution to the Boltzmann equation following the definition in [21].

Definition 5.1. *[21, Definition 1.1] ("Weak" measure-valued solution to spatially homogeneous Boltzmann equation) Let the collision kernel b satisfy* (3.32)*. For any $F_0 \in P_\alpha(\mathbb{R}^3)$ with $0 < \alpha \le 2$. We define $F_t \in C\left([0,\infty), P_\alpha(\mathbb{R}^3)\right)$ as a measure-valued solution to the Cauchy problem* (3.1)-(3.2) *if it satisfies:*
(1) For every $\phi(v) \in C_0^2(\mathbb{R}^3)$ and $t > 0$,

$$\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) |\phi(v_*') + \phi(v') - \phi(v_*) - \phi(v)| \,\mathrm{d}\sigma\,\mathrm{d}F_\tau(v)\,\mathrm{d}F_\tau(v_*)\,\mathrm{d}\tau \tag{5.1}$$

is finite.
(2) For every $\phi(v) \in C_0^2(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} \phi(v)\,\mathrm{d}F_0(v) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) [\phi(v_*') + \phi(v') \\ - \phi(v_*) - \phi(v)]\,\mathrm{d}\sigma\,\mathrm{d}F_\tau(v)\,\mathrm{d}F_\tau(v_*)\,\mathrm{d}\tau.\end{aligned} \tag{5.2}$$

(3) If $\alpha \geq 1$, then the momentum conservation law holds:

$$\forall t \geq 0, \quad \int_{\mathbb{R}^3} v_j \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} v_j \,\mathrm{d}F_0(v), \quad j = 1, 2, 3. \tag{5.3}$$

(4) If $\alpha = 2$, then $F_t \in C\left([0, \infty), P_2(\mathbb{R}^3)\right)$ and then energy conservation law holds:

$$\forall t \geq 0, \quad \int_{\mathbb{R}^3} |v|^2 \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} |v|^2 \,\mathrm{d}F_0(v). \tag{5.4}$$

Remark 5.2. *The definition of the measure-valued solution to the Boltzmann equation is absolutely not unique, and here is mainly following the definition as in [21, Definition 1.1]. Hence, the following points should be noted for consistent purpose:*

(i) The Definition 5.1 above does NOT require finite entropy condition.

(ii) The continuity of the map $t \in [0, \infty) \to F_t \in P_\alpha(\mathbb{R}^3)$ is in the weak* sense that,

$$\lim_{t \to t_0} \int_{\mathbb{R}^3} \phi \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} \phi \,\mathrm{d}F_{t_0}(v), \quad \forall \phi \in C_0(\mathbb{R}^3), \tag{5.5}$$

note that the space $C_0(\mathbb{R}^3)$ includes all the continuous with certain type of decay condition at the infinity of $v \in \mathbb{R}^3$, which is defined as following,

$$C_0(\mathbb{R}^3) := \left\{ \phi \in C(\mathbb{R}^3); \quad \sup_{v \in \mathbb{R}^3} \frac{|\phi(v)|}{\langle v \rangle^\alpha} < \infty, \quad \langle v \rangle = \sqrt{1 + |v|^2} \right\}. \tag{5.6}$$

As usual, we also define the similar operator $L_b[\phi](v, v_*)$: for any $\phi \in C_0^2(\mathbb{R}^3)$,

$$L_b[\phi](v, v_*) := \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) [\phi(v_*') + \phi(v') - \phi(v_*) - \phi(v)] \,\mathrm{d}\sigma. \tag{5.7}$$

such that the equation (5.2) is rewritten as

$$\int_{\mathbb{R}^3} \phi(v) \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} \phi(v) \,\mathrm{d}F_0(v) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_b[\phi](v, v_*) \,\mathrm{d}F_\tau(v) \,\mathrm{d}F_\tau(v_*) \,\mathrm{d}\tau. \tag{5.8}$$

and moreover the map

$$t \longmapsto \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_b[\phi](v, v_*) \,\mathrm{d}F_t(v) \,\mathrm{d}F_t(v_*) \tag{5.9}$$

belongs to $C((0, \infty))$.

As mentioned in the Remark (5.2), we present another definition of (weak) measure-valued solution to the Boltzmann equation frequently used by N. Fournier and his collaborators such as in [13, 19] for reference to the readers.

Definition 5.3. *[19, Definition 1.1] Assume that for $\mu \in (0, 1)$ and $\gamma \in (-1, 1)$ as in* (3.22)-(3.23)*. A family $\{F_t\}_{t \geq 0} \subset P_2(\mathbb{R}^3)$ is a (weak) measure-valued solution to* (3.1)-(3.2)*, if for all $t \geq 0$,*

$$\begin{gathered} \int_{\mathbb{R}^3} \mathrm{d}F_t(v) = 1, \quad \int_{\mathbb{R}^3} v_j \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} v_j \,\mathrm{d}F_0(v), \; j = 1, 2, 3. \\ \int_{\mathbb{R}^3} |v|^2 \,\mathrm{d}F_t(v) = \int_{\mathbb{R}^3} |v|^2 \,\mathrm{d}F_0(v) < \infty \end{gathered} \tag{5.10}$$

and if for any $\phi \in Lip_b(\mathbb{R}^3)$ *and any* $t \geq 0$,

$$\int_{\mathbb{R}^3} \phi(v)\, \mathrm{d}F_t(v) = \int_{\mathbb{R}^3} \phi(v)\, \mathrm{d}F_0(v) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) [\phi(v') - \phi(v)]\, \mathrm{d}\sigma\, \mathrm{d}F_\tau(v)\, \mathrm{d}F_\tau(v_*)\, \mathrm{d}\tau \tag{5.11}$$

where $Lip_b(\mathbb{R}^3)$ *is the set of bounded globally Lipschitz-continuous functions such that, for* $\phi \in Lip(\mathbb{R}^3)$ *associated with,*

$$\|\phi\|_{Lip_b(\mathbb{R}^3)} = \sup_{v_1 \neq v_2} \frac{|\phi(v_1) - \phi(v_2)|}{|v_1 - v_2|} \leq \infty. \tag{5.12}$$

Remark 5.4. *(i) In this case, the operator* $L_b[\phi](v, v_*)$ *in* (5.7) *is replaced by the following* $L'_b[\phi](v, v_*)$*: for any* $\phi \in Lip_b(\mathbb{R}^3)$,

$$L'_b[\phi](v, v_*) := \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) [\phi(v') - \phi(v)]\ \mathrm{d}\sigma, \tag{5.13}$$

such that the equation (5.11) *is rewritten as*

$$\int_{\mathbb{R}^3} \phi(v)\, \mathrm{d}F_t(v) = \int_{\mathbb{R}^3} \phi(v)\, \mathrm{d}F_0(v) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L'_b[\phi](v, v_*)\, \mathrm{d}F_\tau(v)\, \mathrm{d}F_\tau(v_*)\, \mathrm{d}\tau. \tag{5.14}$$

and moreover the map

$$t \longmapsto \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L'_b[\phi](v, v_*)\, \mathrm{d}F_t(v)\, \mathrm{d}F_t(v_*) \tag{5.15}$$

belongs to $C((0, \infty))$.

(ii) The right-hand side of the equation (5.11) *is well-defined. Indeed, there holds*

$$|v' - v| = |v - v_*| \sqrt{\frac{1 - \cos\theta}{2}} \leq |v - v_*||\theta|, \tag{5.16}$$

such that

$$\begin{aligned} |L'_b[\phi](v, v_*)| \leq & C_\phi \int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) |v - v_*||\theta|\, \mathrm{d}\sigma \\ \leq & C_\phi |v - v_*|^{1+\gamma} \int_0^{\frac{\pi}{2}} |\theta|^{-\nu}\, \mathrm{d}\theta \\ \leq & C_\phi \left(1 + |v|^2 + |v_*|^2\right). \end{aligned} \tag{5.17}$$

As noticed in Remark 5.4(ii), generally speaking, the different selection of the test function ϕ in the measure-valued definition is actually based on the consideration of "how large" about the θ and $|v - v_*|$ that is provided in the weak formulation. More precisely speaking, let

$$\begin{cases} \hat{q} = \dfrac{v - v_*}{|v - v_*|}, \text{ if } v \neq v_*; \\ \hat{q} = (1, 0, 0), \text{ if } v = v_*. \end{cases}$$

Under the spherical coordinate transform of $\sigma \in \mathbb{S}^{d-1}$,

$$\sigma = \cos\theta \hat{q} + \sin\theta \hat{k}, \tag{5.18}$$

for $\theta \in [0,\pi]$ and $\hat{k} \in \mathbb{S}^{d-2}(\hat{q})$, we have

$$\begin{cases} v' = \cos^2\left(\frac{\theta}{2}\right) v + \sin^2\left(\frac{\theta}{2}\right) v_* + \frac{1}{2}|v-v_*|\sin\theta\hat{k}, \\ v'_* = \sin^2\left(\frac{\theta}{2}\right) v + \cos^2\left(\frac{\theta}{2}\right) v_* - \frac{1}{2}|v-v_*|\sin\theta\hat{k}, \end{cases}$$

$$\implies \begin{cases} |v'-v| = |v'_*-v_*| = |v-v_*|\sin\frac{\theta}{2}, \\ |v'-v_*| = |v'_*-v| = |v-v_*|\cos\frac{\theta}{2}, \end{cases}$$

since

$$\begin{aligned} \mathbb{S}^{d-2}(\hat{q}) =& \{\hat{k} \in \mathbb{S}^{d-1} |\ \hat{k}\cdot\hat{q} = 0\} \quad (d \geq 3), \\ \mathbb{S}^{0}(\hat{q}) =& \{-\hat{q}^{\perp}, \hat{q}^{\perp}\} \qquad\qquad\quad (d = 2), \end{aligned} \tag{5.19}$$

where $\hat{q}^{\perp} \in \mathbb{S}^1$ satisfies $\hat{q}^{\perp}\cdot\hat{q} = 0$. Then, for any $g \in L^1(\mathbb{S}^{d-1})$ or any measurable functions $g \geq 0$ on $\mathbb{S}^{d-1}$, we have,

$$\int_{\mathbb{S}^{d-1}} g(\sigma)\,\mathrm{d}\sigma = \int_0^{\pi} \sin^{d-2}\theta \left[\int_{\mathbb{S}^{d-2}(\hat{q})} g(\cos\theta\hat{q} + \sin\theta\hat{k})\,\mathrm{d}\hat{k}\right]\mathrm{d}\theta, \tag{5.20}$$

and in the case of $d = 2$,

$$\int_{\mathbb{S}^0(\hat{q})} f(\hat{k})\,\mathrm{d}\hat{k} = f(-\hat{q}^{\perp}) + f(\hat{q}^{\perp}). \tag{5.21}$$

There is another more precise spherical coordinate transform for $\sigma \in \mathbb{S}^{d-1}(\mathbb{R}^d)$, e.g., for $d = 3$ we can decompose the $\sigma \in \mathbb{S}^2(\mathbb{R}^3)$ as: for $\theta \in [0,\pi]$ and $\vartheta \in [0, 2\pi]$,

$$\sigma = \cos\theta\hat{q} + \sin\theta\left(\cos\vartheta\hat{h} + \sin\vartheta\hat{j}\right), \tag{5.22}$$

by the following orthogonal basis constructed by $\hat{q}$ and

$$\hat{j} := \frac{v \times v_*}{|v \times v_*|}, \qquad \hat{h} := \hat{j} \times \hat{q} = \frac{((v-v_*)\cdot v)v_* - ((v-v_*)\cdot v_*)v}{|v-v_*||v \times v_*|}. \tag{5.23}$$

Given the shorthand notations $\phi'_* = \phi(v'_*), \phi' = \phi(v'), \phi_* = \phi(v_*), \phi = \phi(v)$ for simplicity, and the pre-post collision velocities relation that,

$$\begin{aligned} v' - v =& \frac{v_* - v}{2} + \frac{1}{2}|v-v_*|\sigma \\ =& \frac{|v-v_*|}{2}\left[\sigma - \left(\sigma\cdot\frac{v-v_*}{|v-v_*|}\right)\frac{v-v_*}{|v-v_*|}\right] + \frac{v-v_*}{2}\left[\left(\sigma\cdot\frac{v-v_*}{|v-v_*|}\right) - 1\right], \end{aligned} \tag{5.24}$$

the Taylor expansion up to the second order gives that,

$$\begin{aligned}\phi' - \phi =& \nabla\phi(v)(v'-v) + \int_0^1 (1-\tau)\nabla^2\phi\left[v+\tau(v'-v)\right]\,\mathrm{d}\tau(v'-v)^2\\ =& \frac{|v-v_*|}{2}\nabla\phi(v)\left[\sigma-(\sigma\cdot\hat{q})\hat{q}\right] + \nabla\phi(v)\frac{v-v_*}{2}\left[(\sigma\cdot\hat{q})-1\right] + O\left(|v-v_*|^2\theta^2\right),\end{aligned} \tag{5.25}$$

and the similar expansion can be obtained for $\phi'_* - \phi_*$ as well, which further implies that,

$$\phi'_* + \phi' - \phi_* - \phi + \frac{|v-v_*|}{2}\left[\nabla\phi(v)-\nabla\phi(v_*)\right]\left[(\sigma\cdot\hat{q})\hat{q}-\sigma\right] = O\left(|v-v_*|^2\theta^2\right). \tag{5.26}$$

On the other hand, noticing that the symmetry on the sphere integral that,

$$\begin{aligned}&\int_{\mathbb{S}^2} b(\sigma\cdot\hat{q})\left[(\sigma\cdot\hat{q})\hat{q}-\sigma\right]\,\mathrm{d}\sigma\\ =&\int_0^{2\pi}\int_0^{\pi} b(\cos\theta)\left(\hat{h}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\vartheta = 0.\end{aligned} \tag{5.27}$$

Lemma 5.5. *Let $\phi \in C_b^2(\mathbb{R}^d)$ and $\triangle\phi = \phi(v'_*) + \phi(v') - \phi(v_*) - \phi(v)$. Then, for all $\sigma \in \mathbb{S}^{d-1}$ and $v, v_* \in \mathbb{R}^d$,*

$$|\triangle\phi| \le 2^{\frac{4-3m}{2}}\left(\sup_{|u|\le\sqrt{|v|^2+|v_*|^2}}|\partial^m\phi(u)|\right)|v-v_*|^m\sin\theta, \quad m=1,2. \tag{5.28}$$

$$\frac{1}{|\mathbb{S}^{d-2}|}\left|\int_{\mathbb{S}^{d-1}(\hat{q})}\triangle\phi\,\mathrm{d}\hat{k}\right| \le \left(\sup_{|u|\le\sqrt{|v|^2+|v_*|^2}}\left|\partial^2\phi(u)\right|\right)|v-v_*|^2\sin^2\theta. \tag{5.29}$$

Proof. By noticing that

$$-\sigma = \cos(\pi-\theta)\hat{q} + \sin(\pi-\theta)(-\hat{k}) \tag{5.30}$$

and $\triangle\phi$ is invariant under the reflective transformation $\sigma \longmapsto -\sigma$, we can assume, without loss of generality, that $\theta \in [0, \frac{\pi}{2}]$. In this case, we have $\sin\frac{\theta}{2} \le \frac{\sin\theta}{\sqrt{2}}$.

By writing $\triangle\phi = (\phi'-\phi) + (\phi'_* - \phi_*)$, we can find that (5.28) for $m=1$ follows from the first equality in (5.19).

Then, by writing $\triangle\phi = (\phi'-\phi) + (\phi'_* - \phi_*)$ and noticing $v_* - v'_* = v' - v$, we have,

$$\begin{aligned}\triangle\phi =& \int_0^1 \left\langle \partial\phi\left[v+t(v'-v)\right] - \partial\phi\left[v'_*+t(v'-v)\right], v'-v\right\rangle\,\mathrm{d}t\\ =& \int_0^1\int_0^1 (v-v'_*)\partial^2\phi(\xi_{t,\tau})(v-v'_*)^T\,\mathrm{d}\tau\,\mathrm{d}t\end{aligned} \tag{5.31}$$

with $|\xi_{t,\tau}| \le \max\{|v|, |v'|, |v_*|, |v'_*|\} \le \sqrt{|v|^2+|v_*|^2}$. Since $|v'_*-v||v'-v| = \frac{1}{2}|v-v_*|^2\sin\theta$, this gives (5.28) for $m=2$.

To prove (5.29), we write $\triangle\phi = (\phi' - \phi) + (\phi'_* - \phi_*)$ and notice $v'_* - v_* = -(v' - v)$. Then,

$$\begin{aligned}\triangle\phi =& \langle \partial\phi(v) - \partial\phi(v_*) \rangle \\ &+ \int_0^1 (1-t)(v'-v)\partial^2\phi\left[v + t(v'-v)\right](v'-v)^T \,\mathrm{d}t \\ &+ \int_0^1 (1-t)(v'_*-v_*)\partial^2\phi\left[v_* + t(v'_*-v_*)\right](v'_*-v_*)^T \,\mathrm{d}t.\end{aligned} \tag{5.32}$$

Since by (5.19),

$$\frac{1}{|\mathbb{S}^{d-2}(\hat{q})|}\int_{\mathbb{S}^{d-1}(\hat{q})} \langle \partial\phi(v) - \partial\phi(v_*), v'-v\rangle \,\mathrm{d}\hat{k} = \langle \partial\phi(v) - \partial\phi(v_*), v_*-v\rangle \sin^2\left(\frac{\theta}{2}\right), \tag{5.33}$$

where the fact that $\int_{\mathbb{S}^{d-1}(\hat{q})} \left\langle \partial\phi(v) - \partial\phi(v_*), \hat{k}\right\rangle \mathrm{d}\hat{k} = 0$ is utilized, and it finally follows that,

$$\frac{1}{|\mathbb{S}^{d-2}(\hat{q})|}\left|\int_{\mathbb{S}^{d-1}(\hat{q})} \triangle\phi \,\mathrm{d}\hat{k}\right| \leq 2\left(\sup_{|u|\leq\sqrt{|v|^2+|v_*|^2}} \left|\partial^2\phi(u)\right|\right)|v-v_*|^2 \sin^2\left(\frac{\theta}{2}\right). \tag{5.34}$$

□

For further discussion, we are now to state a definition of "strong" measure-valued solution to the spatially homogeneous Boltzmann equation in the case of finite energy, for which some time-differentiability is assumed in total variation topology. More details can be found in [16].

Definition 5.6. *[16, Definition 1.2] ("Strong" measure-valued solution to spatially homogeneous Boltzmann equation) Let the collision kernel b satisfy* (3.32). *For any* $F_0 \in P_2(\mathbb{R}^3)$. *We define* $F_t \in C\left([0,\infty), P_2(\mathbb{R}^3)\right)$ *as a (strong) measure-valued solution to the Cauchy problem* (3.1)-(3.2) *if it satisfies:*
(1) For $t \in [0,\infty)$,

$$\sup_{t\geq 0}\int_{\mathbb{R}^3} \langle v\rangle^2 \,\mathrm{d}F_t(v) < \infty. \tag{5.35}$$

(2) For $t \in [0,\infty)$, $t \longmapsto F_t \in C\left([0,\infty), P_2(\mathbb{R}^3)\right) \bigcap C^1\left([0,\infty), P(\mathbb{R}^3)\right)$, *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_t = Q(F_t, F_t). \tag{5.36}$$

Remark 5.7. *(i) The strong continuity of*

$$t \longmapsto F_t \in C\left([0,\infty), P_2(\mathbb{R}^3)\right) \tag{5.37}$$

implies the strong continuity

$$t \longmapsto Q(F_t, F_t) \in C\left([0,\infty), P(\mathbb{R}^3)\right) \tag{5.38}$$

so that the differential equation (5.36) *is equivalent to the integral equation,*

$$F_t = F_0 + \int_0^t Q(F_\tau, F_\tau)\,\mathrm{d}\tau, \quad t \geq 0, \tag{5.39}$$

where the integral is taken in the Riemann sense or generally in the Bochner sense. Recall also that here the derivative $\frac{\mathrm{d}}{\mathrm{d}t}F_t$ *and integral* $\int_a^b F_t \,\mathrm{d}t$ *as measures are defined by,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}F_t\right)(\Omega) = \frac{\mathrm{d}}{\mathrm{d}t}F_t(\Omega), \quad \left(\int_a^b F_t \,\mathrm{d}t\right)(\Omega) = \int_a^b F_t(\Omega)\,\mathrm{d}t \tag{5.40}$$

for all Borel sets $\Omega \subset \mathbb{R}^3$.

(ii) Note also that if a strong measure-valued solution F_t *is absolutely continuous with respect to the Lebesgue measure for all* $t \geq 0$, *i.e.,* $\mathrm{d}F_t(v) = f_t(v)$ *where* $f_t(v)$ *is the corresponding probability density function, then it is easily seen that* f_t *(after modification on a* v*-null set) is a mild solution to the the Cauchy problem* (3.1)-(3.2).

That ism $(t, v) \longmapsto f_t(v)$ *is non-negative and Lebesgue measure on* $[0, \infty) \times \mathbb{R}^3$ *and for every* $t \geq 0$, $v \longmapsto f_t(v)$ *belongs to* $L^1_2(\mathbb{R}^3)$ *with* $\sup_{t\geq 0} \|f_t\|_{L^1_2} < \infty$, *and there is a Lebesgue null set* $\Omega_0 \subset \mathbb{R}^3$ *(which is independent of* t*) such that*

$$\begin{cases} \displaystyle\int_0^t Q^{\pm}(f_\tau, f_\tau)\,\mathrm{d}\tau < \infty, \quad t \in [0,\infty), \quad \forall v \in \mathbb{R}^3 \setminus \Omega_0, \\ \displaystyle f_t(v) = f_0(v) + \int_0^t Q(f_\tau, f_\tau)(v)\,\mathrm{d}\tau, \quad t \in [0,\infty), \quad \forall v \in \mathbb{R}^3 \setminus \Omega_0, \end{cases}$$

where

$$L^1_2(\mathbb{R}^3) = \left\{ f \in L^1(\mathbb{R}^3) \middle| \|f\|_{L^1_2} := \int_{\mathbb{R}^3} |f(v)| \langle v \rangle^2 \,\mathrm{d}v < \infty \right\}. \tag{5.41}$$

5.2 Big Picture and Preliminary Results

Thanks to the discussions above, especially the Bobylev Identity, in the following subsections, we will follow the strategy of proof as in the Figure 3 as blow:

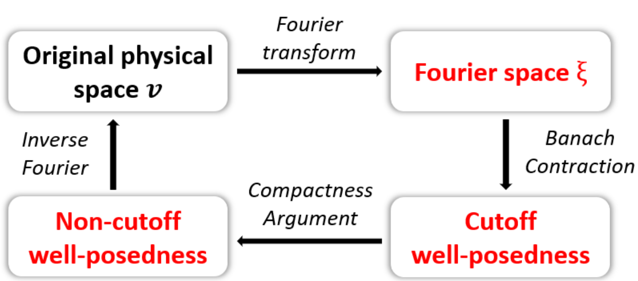

Figure 3: Flow of proof in the following subsections

First of all, the spatially homogeneous Boltzmann equation can be converted into the following equation for the new unknown function $\varphi = \varphi(t, \xi)$ by applying the Fourier transformation to both hand sides of (3.1):

$$\partial_t \varphi(t,\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left[\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,0)\varphi(t,\xi)\right] \mathrm{d}\sigma, \tag{5.42}$$

where the ξ^+ and ξ^- have the same definition as in (3.55),

$$\xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2}\sigma, \qquad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2}\sigma. \tag{5.43}$$

which also satisfy the following relations:

$$\xi^+ + \xi^- = \xi \quad \text{and} \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2, \tag{5.44}$$

as well as

$$|\xi^+|^2 = |\xi|^2 \frac{1 + \frac{\xi}{|\xi|} \cdot \sigma}{2} \quad \text{and} \quad |\xi^-|^2 = |\xi|^2 \frac{1 - \frac{\xi}{|\xi|} \cdot \sigma}{2}. \tag{5.45}$$

Therefore, our main object will be the equation (5.42) associated with the following initial condition:

$$\varphi(0, \xi) = \varphi_0(\xi) = \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}v\cdot\xi} \, \mathrm{d}F_0(v), \tag{5.46}$$

where if $\varphi_0 \in \mathcal{K}^\alpha$ defined as (4.84) is the Fourier transform of a probability measure F_0 satisfying (4.83), then the corresponding solution $\varphi = \varphi(t, \xi)$ to (5.42)-(5.46) is the Fourier transform of the solution $f = f(t, v)$ to the original initial value problem (3.1)-(3.2), see more explanations in [14].

Therefore, except those estimates for all characteristic function φ such as (4.13) and (4.32), we will introduce another technical estimate of the function φ with respect to the variable ξ^+ and ξ^- based on the observation and some elementary inequalities, which will then play a key role in proving the well-definedness of the right hand side of (5.42).

Lemma 5.8. *Let $\alpha \in [0, 2]$. For each $\xi \in \mathbb{R}^3$, the variables ξ^+ and ξ^- are defined as* (3.55) *with some fixed $\sigma \in \mathbb{S}^2$ respectively. Then, for $\varphi \in \mathcal{K}^\alpha$,*

$$\left|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)\varphi(0)\right| \leq 4 \left|\xi^+\right|^{\frac{\alpha}{2}} \left|\xi^-\right|^{\frac{\alpha}{2}} \|\varphi - 1\|_\alpha . \tag{5.47}$$

Proof. Start from the following identity

$$1 - |\varphi(\xi)|^2 = [1 - \varphi(\xi)] \left[1 + \overline{\varphi(\xi)}\right] + 2\mathrm{Im}\,[\varphi(\xi)]\,\mathrm{i}, \tag{5.48}$$

together with the estimate (4.97) in Lemma 4.31 and the following inequality,

$$\left|1 + \overline{\varphi(\xi)}\right| \leq 1 + |\varphi(\xi)| \leq 2, \tag{5.49}$$

we can deduce from the inequality (5.48) that

$$0 \leq 1 - |\varphi(\xi)|^2 \leq 4 \, |\xi|^\alpha \, \|\varphi - 1\|_\alpha . \tag{5.50}$$

In fact, the (5.50) holds if we substitute ξ^+ and ξ^- into it. Recalling that $\varphi(0) = 1$ and the relation $\xi^+ + \xi^- = \xi$, consequently, we are able to apply the inequality (4.46),

$$\begin{aligned} \left|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)\right| &\leq \sqrt{\left(1 - |\varphi(\xi^+)|^2\right)\left(1 - |\varphi(\xi^-)|^2\right)} \\ &\leq 4 \left|\xi^+\right|^{\frac{\alpha}{2}} \left|\xi^-\right|^{\frac{\alpha}{2}} \|\varphi - 1\|_\alpha , \end{aligned} \tag{5.51}$$

for all $\xi \in \mathbb{R}^3$. □

With the help of the preliminary estimates (5.8) and (4.32) above, we are able to prove the following technical Lemma 5.9 to show that the nonlinear term in the right hand side of (5.42) is well-defined for any function $\varphi \in \mathcal{K}^\alpha$, even the strong singularity condition (3.32) of the collision kernel b holds.

Lemma 5.9. *Assume that the collision kernel b satisfies the non-cutoff assumption* (3.32) *for $\alpha_0 \in (0,2]$. If $\varphi \in \mathcal{K}^\alpha$ for $\alpha \in [\alpha_0, 2]$, then*

$$\begin{aligned} &\left| \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)\varphi(\xi^-) - \varphi(0)\varphi(\xi)\right] \mathrm{d}\sigma \right| \\ &\qquad \lesssim \left[\int_0^{\frac{\pi}{2}} \sin^\alpha\left(\frac{\theta}{2}\right) b(\cos\theta)\sin\theta \mathrm{d}\theta\right] \|1-\varphi\|_\alpha |\xi|^\alpha < \infty. \end{aligned} \tag{5.52}$$

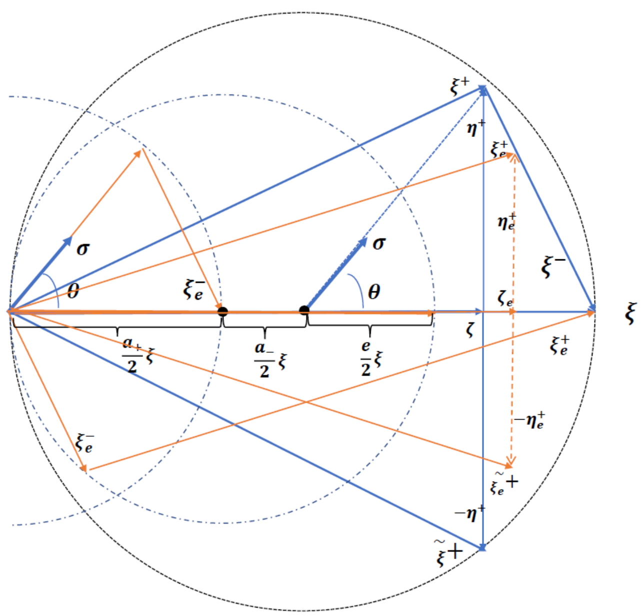

Figure 4: Illustration of the inelastic collision mechanism with $\cos\theta = \frac{\xi\cdot\sigma}{|\xi|}$ and $\eta^+ = \xi^+ - \zeta$.

Proof. By introducing $\zeta = \left(\xi^+ \cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|}$ as the middle variable as well as considering the fact that $\varphi(0)=1$,

$$
\begin{aligned}
&\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)\varphi(\xi^-)-\varphi(0)\varphi(\xi)\right]\mathrm{d}\sigma\\
=&\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)\varphi(\xi^-)-\varphi(\xi^+)+\varphi(\xi^+)-\varphi(\xi)\right]\mathrm{d}\sigma\\
=&\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)-\varphi(\xi)\right]\mathrm{d}\sigma+\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\left[\varphi(\xi^-)-1\right]\mathrm{d}\sigma\\
=&\frac{1}{2}\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)+\varphi(\tilde{\xi}^+)-2\varphi(\xi)\right]\mathrm{d}\sigma+\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\left[\varphi(\xi^-)-1\right]\mathrm{d}\sigma\\
=&\frac{1}{2}\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\xi^+)+\varphi(\tilde{\xi}^+)-2\varphi(\zeta)\right]\mathrm{d}\sigma+\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(\zeta)-\varphi(\xi)\right]\mathrm{d}\sigma\\
&+\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\left[\varphi(\xi^-)-1\right]\mathrm{d}\sigma\\
:=&I_1+I_2+I_3
\end{aligned}
\tag{5.53}
$$

(i) For the first part I_1, by considering the symmetric geometry relation $\xi^+=\zeta+\eta^+$ and $\tilde{\xi}^+=\zeta+(-\eta^+)$ as in Figure 1, we obtain,

$$
\begin{aligned}
&\left|\varphi(\xi^+)+\varphi(\tilde{\xi}^+)-2\varphi(\zeta)\right|=\left|\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\zeta\cdot v}\left(\mathrm{e}^{-\mathrm{i}\eta^+\cdot v}+\mathrm{e}^{\mathrm{i}\eta^+\cdot v}-2\right)\mathrm{d}F(v)\right|\\
\le&\int_{\mathbb{R}^3}\left|\mathrm{e}^{-\mathrm{i}\zeta\cdot v}\right|\left(2-\mathrm{e}^{-\mathrm{i}\eta^+\cdot v}-\mathrm{e}^{\mathrm{i}\eta^+\cdot v}\right)\mathrm{d}F(v)\\
=&2-\varphi(\eta^+)-\varphi(-\eta^+)\\
=&\left[1-\varphi(\eta^+)\right]+\left[1-\varphi(-\eta^+)\right]\\
\le&2\left\|1-\varphi\right\|_\alpha\left|\eta^+\right|^\alpha\le 2\left\|1-\varphi\right\|_\alpha|\xi|^\alpha\sin^\alpha\left(\frac{\theta}{2}\right),
\end{aligned}
\tag{5.54}
$$

where, in the first inequality, we utilize the fact that $\left(2-\mathrm{e}^{-\mathrm{i}\eta^+\cdot v}-\mathrm{e}^{\mathrm{i}\eta^+\cdot v}\right)$ is positive, since the imaginary parts are cancelled thanks to the symmetric relation; whereas the relationship $|\eta^+|=|\xi^+|\sin\left(\frac{\theta}{2}\right)$ and $|\xi^+|\le|\xi|$ are noticed in the last inequality.

As a result, we have, according to the assumption (3.32),

$$
|I_1|\le C_1\left\|1-\varphi\right\|_\alpha|\xi|^\alpha\int_0^{\frac{\pi}{2}}\sin^\alpha\left(\frac{\theta}{2}\right)b(\cos\theta)\sin\theta\,\mathrm{d}\theta<\infty.
\tag{5.55}
$$

(ii) For the second part I_2, with the help of the inequality (4.101) in Lemma 4.32 and $\zeta-\xi=\eta$ in Figure 1, we have

$$
|\varphi(\zeta)-\varphi(\xi)|\le\left\|\varphi-1\right\|_\alpha\left(4|\xi|^{\frac{\alpha}{2}}|\eta|^{\frac{\alpha}{2}}+|\eta|^\alpha\right),
\tag{5.56}
$$

together with the geometric relation $|\eta|=|\zeta-\xi|=|\xi|\sin^2\left(\frac{\theta}{2}\right)$, we can further obtain that

$$
\begin{aligned}
|I_2|\le&\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left\|\varphi-1\right\|_\alpha\left(4|\xi|^{\frac{\alpha}{2}}|\eta|^{\frac{\alpha}{2}}+|\eta|^\alpha\right)\mathrm{d}\sigma\\
\le&C_2\left\|1-\varphi\right\|_\alpha|\xi|^\alpha\int_0^{\frac{\pi}{2}}\sin^\alpha\left(\frac{\theta}{2}\right)b(\cos\theta)\sin\theta\,\mathrm{d}\theta<\infty.
\end{aligned}
\tag{5.57}
$$

(iii) For the last part I_3, following the similar estimates above and considering the fact that $|\varphi(\xi^+)| \leq 1$, we have,

$$
\begin{aligned}
|I_3| =& \left| \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \varphi(\xi^+) \left[\varphi(\xi^-) - 1\right] \,\mathrm{d}\sigma \right| \\
\leq & \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left|\varphi(\xi^-) - 1\right| \,\mathrm{d}\sigma \\
\leq & \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \|1-\varphi\|_\alpha \left|\xi^-\right|^\alpha \,\mathrm{d}\sigma \\
\leq & \|1-\varphi\|_\alpha \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left(\frac{1-\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{2}} |\xi|^\alpha \,\mathrm{d}\sigma \\
\leq & C_3 \|1-\varphi\|_\alpha |\xi|^\alpha \int_0^{\frac{\pi}{2}} \sin^\alpha\left(\frac{\theta}{2}\right) b(\cos\theta)\sin\theta \,\mathrm{d}\theta,
\end{aligned}
\tag{5.58}
$$

where we use the fact that $\frac{\xi\cdot\sigma}{|\xi|} = \cos\theta$. Summing up the estimates in (i), (ii) and (iii), we obtain the desired estimate (5.52). □

Remark 5.10. *(i) In fact, without considering the geometric relation in Figure 1, we can still find that the initial value problem* (5.42)-(5.46) *is well-defined if there is only mild singularity assumption* (3.34)*, which is the main bottleneck in the original paper [14],*

$$
\begin{aligned}
\partial_t\varphi(t,\xi) =& \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left[\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,\xi)\varphi(t,0)\right] \,\mathrm{d}\sigma \\
\leq & 4\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left|\xi^+\right|^{\frac{\alpha}{2}} \left|\xi^-\right|^{\frac{\alpha}{2}} \|\varphi-1\|_\alpha \,\mathrm{d}\sigma \\
\leq & 4|\xi|^\alpha \|\varphi-1\|_\alpha \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left(\frac{1-\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{4}} \left(\frac{1+\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{4}} \,\mathrm{d}\sigma \\
= & 8\pi|\xi|^\alpha \|\varphi-1\|_\alpha \int_0^\pi b(\cos\theta) \left(\frac{1-\cos\theta}{2}\right)^{\frac{\alpha}{4}} \left(\frac{1+\cos\theta}{2}\right)^{\frac{\alpha}{4}} \sin\theta \,\mathrm{d}\theta \\
= & 8\pi|\xi|^\alpha \|\varphi-1\|_\alpha \int_0^\pi b(\cos\theta) \sin^{\frac{\alpha}{2}}\left(\frac{\theta}{2}\right) \cos^{\frac{\alpha}{2}}\left(\frac{\theta}{2}\right) \sin\theta \,\mathrm{d}\theta \leq \infty
\end{aligned}
\tag{5.59}
$$

where we utilize the estimate (5.47) *of Lemma 5.8 in the first inequality.*

(ii) The variables with subscript e in Figure 4 represents the geometric relation of the inelastic collision, provided that the restitution coefficient $0 < e \leq 1$ *is the constant.*

Moreover, we introduce some corresponded parameters that will appear systematically in our following proof.

Lemma 5.11. *[14, Lemma 4.1 and Corrollary 4.2]*

(i) Assume that the collision kernel b_c *satisfy the cutoff assumption* (3.30)*, for all*

$\alpha \in [0,2]$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, we define the parameter γ_α,

$$\begin{aligned}\gamma_\alpha \equiv \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha}\,\mathrm{d}\sigma =& 2\pi\int_0^\pi b_c(\cos\theta)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}\right)\sin\theta\,\mathrm{d}\theta\\ =& 2\pi\int_{-1}^1 b_c(s)\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}}+\left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}}\right]\mathrm{d}s,\end{aligned} \tag{5.60}$$

is finite and independent of ξ. Moreover,

$$\gamma_\alpha > \gamma_2 = \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{d}\sigma = 2\pi\int_{-1}^1 b_c(s)\,\mathrm{d}s, \tag{5.61}$$

if $0<\alpha<2$.

(ii) Furthermore, if the collision kernel b satisfies the non-cutoff assumption (3.32)*, we have the parameter λ_α defined as following, for every $\xi \in \mathbb{R}^3\setminus\{0\}$,*

$$\lambda_\alpha \equiv \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha+|\xi^-|^\alpha}{|\xi|^\alpha}-1\right)\mathrm{d}\sigma. \tag{5.62}$$

Then λ_α is finite, independent of ξ, and positive provided that $0<\alpha<2$.

Proof. Let $\sigma = (\sigma_1,\sigma_2,\sigma_3)\in\mathbb{S}^2$. Rotating $\mathbb{R}^3$ (if necessary) and using spherical coordinates, we obtain the equality,

$$\int_{\mathbb{S}^2} B\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{d}\sigma = \int_{\mathbb{S}^2} B(\sigma_3)\,\mathrm{d}\sigma = 2\pi\int_{-1}^1 B(s)\,\mathrm{d}s, \tag{5.63}$$

which is valid for every $B\in L^1(-1,1)$ and $\xi\in\mathbb{R}^3\setminus\{0\}$. Hence, if set $B=b_c$, we have,

$$\gamma_2 = \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{d}\sigma = 2\pi\int_{-1}^1 b_c(s)\,\mathrm{d}s. \tag{5.64}$$

For $0<\alpha<2$, we recall the equalities of $|\xi^+|^\alpha$ and $|\xi^-|^\alpha$ as,

$$\begin{aligned}|\xi^+|^\alpha &= |\xi|^\alpha\left(\frac{1+\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{2}},\\ |\xi^-|^\alpha &= |\xi|^\alpha\left(\frac{1-\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{2}},\end{aligned} \tag{5.65}$$

we then obtain that

$$\begin{aligned}\gamma_\alpha &= \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\left(\frac{1+\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{2}}+\left(\frac{1-\frac{\xi\cdot\sigma}{|\xi|}}{2}\right)^{\frac{\alpha}{2}}\right]\mathrm{d}\sigma\\ &\overset{\cos\theta:=\frac{\xi\cdot\sigma}{|\xi|}}{=\!=\!=} 2\pi\int_0^\pi b_c(\cos\theta)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}\right)\sin\theta\mathrm{d}\theta\\ &\overset{s:=\cos\theta}{=\!=\!=} 2\pi\int_{-1}^1 b_c(s)\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}}+\left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}}\right]\mathrm{d}s,\end{aligned} \tag{5.66}$$

which is finite because the function in brackets is bounded for $s \in [-1, 1]$.

In order to show that $\gamma_\alpha > \gamma_2$, whenever $0 < \alpha < 2$, it suffices to use the elementary inequality

$$\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} > 1, \tag{5.67}$$

which is valid for all $s \in (-1, 1)$.

For λ_α under cutoff assumption, the finiteness can be immediately found with the help of γ_α in (3.30); then, to handle the non-cutoff collision kernel b satisfying (3.32), we have the following estimate,

$$\begin{aligned} \lambda_\alpha =& \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1\right) \mathrm{d}\sigma \\ =& 2\pi \int_{-1}^{1} b(s)\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} - 1\right] \mathrm{d}s, \end{aligned} \tag{5.68}$$

it then suffices to consider the following estimate,

$$0 \le b(s)\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} - 1\right] \le C_\alpha b(s)(1-s^2)^{\frac{\alpha}{2}}, \tag{5.69}$$

for every $\alpha \in (0, 2)$, a constant $C_\alpha > 0$, and all $s \in [-1, 1]$, which can be checked by the following limit,

$$\lim_{s\to\pm1} \frac{\left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} - 1\right]}{(1-s^2)^{\frac{\alpha}{2}}} = 1, \tag{5.70}$$

provided that $\alpha \in (0, 2)$. Hence, both functions in the numerator and denominator are comparable in the sense that there are two positive constants c_α and C_α such that

$$c(1-s^2)^{\frac{\alpha}{2}} \le \left[\left(\frac{1+s}{2}\right)^{\frac{\alpha}{2}} + \left(\frac{1-s}{2}\right)^{\frac{\alpha}{2}} - 1\right] \le C(1-s^2)^{\frac{\alpha}{2}}. \tag{5.71}$$

This completes the proof of the Lemma 5.11. □

Remark 5.12. *Note that $\lambda_\alpha = \gamma_\alpha - \gamma_2$ is valid for any collision kernel $b_c \in L^1(-1, 1)$.*

5.3 Well-posedness under Cutoff Assumption

In this section, we first construct the solution of the initial value problem (5.42)-(5.46), and study its stability in the space $\mathcal{K}^\alpha$ under the *cutoff assumption* on the collision kernel b_c in the sense that, for all $\xi \in \mathbb{R}^3 \setminus \{0\}$,

$$\int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \mathrm{d}\sigma < \infty, \tag{5.72}$$

in fact, we will dispense with the assumption and prove the existence of solutions to the initial value problem (5.42)-(5.46) without cutoff assumption by compactness argument in next subsection.

5.3.1 Preparation under Cutoff Assumption

Now we are ready to give the construction of solution to the initial value equation (5.42)-(5.46) in space $\mathcal{K}^\alpha$. Firstly, based on the cutoff assumption (3.30), we denote the consistent notation as in [14],

$$\gamma_2 = \int_{\mathbb{S}^2} b_c \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \mathrm{d}\sigma = 2\pi \int_0^\pi b_c(\cos\theta) \sin\theta \, \mathrm{d}\theta < \infty, \tag{5.73}$$

meanwhile, considering the fact that $\varphi(0,\xi) = 1$ for all $t \geq 0$, we are able to rewrite the equation (5.42) into the following form:

$$\partial_t \varphi(t,\xi) + \gamma_2 \varphi(t,\xi) = \mathcal{G}[\varphi](t,\xi). \tag{5.74}$$

In order to construct the solution by Banach fixed point theorem, we also present another technical Lemma 5.13 about the nonlinear operator $\mathcal{G}[\varphi]$, defined as following:

$$\mathcal{G}[\varphi](\xi) := \int_{\mathbb{S}^2} b_c \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \varphi(\xi^-) \, \mathrm{d}\sigma, \tag{5.75}$$

where ξ^+ and ξ^- are defined in (3.55).

Meanwhile, the following technical Lemma 5.13 about nonlinear operator $\mathcal{G}[\varphi]$ is presented as following:

Lemma 5.13. *Let $\alpha \in [0,2]$ and the collision kernel b_c satisfy the cutoff assumption* (3.30). *For any $\varphi \in \mathcal{K}^\alpha$, the function $\mathcal{G}[\varphi]$ is continuous and positive definite. Moreover, we have,*

$$\left| \mathcal{G}[\varphi](\xi) - \mathcal{G}\left[\tilde{\varphi}\right](\xi) \right| \leq \gamma_\alpha \left\| \varphi - \tilde{\varphi} \right\|_\alpha |\xi|^\alpha, \tag{5.76}$$

for all $\varphi, \tilde{\varphi} \in \mathcal{K}^\alpha$ and all $\xi \in \mathbb{R}^3 \setminus \{0\}$.

Proof. For all $\varphi \in \mathcal{K}^\alpha$, to show that $\mathcal{G}[\varphi]$ is continuous and positive definite, it suffices to following the reasoning from [4, Lemma 2.1], for completeness, the proof will be included in the following Lemma 5.14.

To show the estimate (5.76) holds, since the properties for $\varphi \in \mathcal{K}^\alpha$, we have $|\varphi(\xi^-)| \leq 1$, $|\tilde{\varphi}(\xi^+)| \leq 1$, we obtain

$$\begin{aligned}
&\left| \mathcal{G}[\varphi](\xi) - \mathcal{G}\left[\tilde{\varphi}\right](\xi) \right| \\
=& \left| \int_{\mathbb{S}^2} b_c \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\left(\varphi(\xi^+) - \tilde{\varphi}(\xi^+) \right) \varphi(\xi^-) + \tilde{\varphi}(\xi^+) \left(\varphi(\xi^-) - \tilde{\varphi}(\xi^-) \right) \right] \, \mathrm{d}\sigma \right| \\
\leq& \int_{\mathbb{S}^2} b_c \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\left\| \varphi - \tilde{\varphi} \right\|_\alpha \left| \xi^+ \right|^\alpha + \left\| \varphi - \tilde{\varphi} \right\|_\alpha \left| \xi^- \right|^\alpha \right) \, \mathrm{d}\sigma \\
=& \left\| \varphi - \tilde{\varphi} \right\|_\alpha \int_{\mathbb{S}^2} b_c \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\left| \xi^+ \right|^\alpha + \left| \xi^- \right|^\alpha \right) \, \mathrm{d}\sigma \\
\leq& \gamma_\alpha \left\| \varphi - \tilde{\varphi} \right\|_\alpha |\xi|^\alpha
\end{aligned} \tag{5.77}$$

for all $\xi \in \mathbb{R}^3$. □

The complete proof the Lemma of Pulvirenti-Toscani in [4] is presented as blow:

Lemma 5.14. *[4, Lemma 2.1] Let b_c satisfy the cutoff assumption* (5.72). *The bilinear operator $\mathcal{G}[\varphi,\psi](\xi)$, defined by*

$$\mathcal{G}[\varphi,\psi](\xi) = \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\psi(\xi^-)\,\mathrm{d}\sigma \tag{5.78}$$

maps characteristic functions into characteristic functions.

Proof. Since $\mathcal{G}[\varphi,\psi]$ is continuous at $\xi=0$, it is sufficient to prove that it can be defined as the pointwise limit of a sequence of characteristic functions. Define:

$$\mathcal{G}_m(\xi) = \mathcal{G}[\varphi,\psi](\xi)\,\mathrm{e}^{-\frac{1}{2m}\xi^2} = \int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(\xi^+)\,\mathrm{e}^{-\frac{1}{2m}|\xi^+|^2}\psi(\xi^-)\,\mathrm{e}^{-\frac{1}{2m}|\xi^-|^2}\,\mathrm{d}\sigma. \tag{5.79}$$

The sequence $\{\mathcal{G}_m\}$ converges pointwise to $\{\mathcal{G}\}$. We are going to prove that is a sequence of characteristic functions. To this end, let F and G be the probability measures, related to φ and ψ respectively, i.e.,

$$\begin{aligned}\varphi(\xi) &= \int_{\mathbb{R}^3}\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}F(v),\\ \psi(\xi) &= \int_{\mathbb{R}^3}\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}G(v).\end{aligned} \tag{5.80}$$

Then $\varphi(\xi^+)\,\mathrm{e}^{-\frac{1}{2m}|\xi^+|^2}$ and $\psi(\xi^-)\,\mathrm{e}^{-\frac{1}{2m}|\xi^-|^2}$ are the characteristic functions of the following probability density functions:

$$\begin{aligned}f_m(v) &= \int_{\mathbb{R}^3}\omega_m(v-w)\,\mathrm{d}F(w),\\ g_m(v) &= \int_{\mathbb{R}^3}\omega_m(v-w)\,\mathrm{d}G(w),\end{aligned} \tag{5.81}$$

where,

$$\omega_m(v) = \frac{1}{\left(2\pi\frac{1}{m}\right)^{\frac{3}{2}}}\,\mathrm{e}^{-\frac{mv^2}{2}}. \tag{5.82}$$

Let us observe that, since $\varphi(\xi^+)\,\mathrm{e}^{-\frac{1}{2m}|\xi^+|^2}$ and $\psi(\xi^-)\,\mathrm{e}^{-\frac{1}{2m}|\xi^-|^2}$ belong to $L^2(\mathbb{R}^3)$, f_m and g_m are univocally determined. On the other hand, given the probability density functions f_m and g_m,

$$\left(\mathcal{G}[\varphi,\psi]\right)^{\vee}(t,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b_c\left(\frac{\xi\cdot\sigma}{|\xi|}\right) f_m(t,v')g_m(t,w')\,\mathrm{d}\sigma\,\mathrm{d}w \tag{5.83}$$

is a probability density function as well. More precisely, it is the probability density function related to $\mathcal{G}_m$, namely,

$$\mathcal{G}_m[\varphi,\psi] = \int_{\mathbb{R}^3}\mathcal{G}[f_m,g_m]\,\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}v. \tag{5.84}$$

Therefore, $\mathcal{G}_m$ is a characteristic function. In such a way, we have constructed a sequence of characteristic functions, which converges pointwise to $\mathcal{G}$. This concludes the proof. □

5.3.2 Existence and Uniqueness of Cutoff Solution

Now we are ready to give the construction of solution to the initial value equation (5.42)-(5.46) in space $\mathcal{K}^\alpha$ by applying the Banach Contraction Theorem.

After multiplying (5.74) by a factor $\mathrm{e}^{\gamma_2 t}$ and integrating with respect to t, we obtain the following equivalent formulation of equation (5.42)-(5.46):

$$\varphi(t,\xi) = \varphi_0(\xi)\,\mathrm{e}^{-\gamma_2 t} + \int_0^t \mathrm{e}^{-\gamma_2(t-\tau)}\mathcal{G}[\varphi](\tau,\xi)\,\mathrm{d}\tau. \tag{5.85}$$

Theorem 5.15. (Well-posedness under cutoff assumption) *Assume that $\alpha \in [0,2]$ and the collision kernel b_c satisfy the cutoff assumption* (5.72)*.*

Then, for each initial datum $\varphi_0 \in \mathcal{K}^\alpha$, there exists a unique solution $\varphi(t,\xi)$ to problem (5.42)-(5.46) *such that $\varphi \in \chi^\alpha := C([0,\infty), \mathcal{K}^\alpha)$.*

Proof. For fixed $\varphi_0 \in \mathcal{K}^\alpha$ and any $\varphi \in \mathcal{K}^\alpha$, we are ready to apply the Banach fixed point theorem to the non-linear operator,

$$\mathcal{P}[\varphi](t,\xi) \equiv \varphi_0(\xi)\,\mathrm{e}^{-\gamma_2 t} + \int_0^t \mathrm{e}^{-\gamma_2(t-\tau)}\mathcal{G}[\varphi](\tau,\xi)\,\mathrm{d}\tau. \tag{5.86}$$

We first prove the local existence and uniqueness by showing that operator $\mathcal{P} : \chi_T^\alpha \mapsto C([0,T], \mathcal{K}^\alpha)$ has a unique fixed point in the space $\chi_T^\alpha \subset C([0,T], \mathcal{K}^\alpha)$ defined as

$$\chi_T^\alpha := \left\{ \varphi \in C([0,T], \mathcal{K}^\alpha) : \sup_{t\in[0,T]} \|\varphi(t,\cdot)\|_\alpha < \infty \right\}, \tag{5.87}$$

which is a complete metric space with respect to the induced norm

$$\|\cdot\|_{\chi_T^\alpha} := \sup_{t\in[0,T]} \|\cdot\|_\alpha\,. \tag{5.88}$$

Step (I): To show that, for any $\varphi \in \chi_T^\alpha$ and every $t \in [0,T]$, the function $\mathcal{P}[\varphi](t,\xi) \in \chi_T^\alpha$, which means that $\mathcal{P}[\varphi](\cdot,\xi)$ is still continuous and positive definite: in fact, considering the Lemma 5.13, we find that $\mathcal{G}[\varphi](\tau,\cdot)$ is continuous and positive definite for every $\tau \in [0,t]$, then $\mathcal{P}[\varphi](t,\xi) \in \mathcal{K}^\alpha$ can directly follow the [14, Lemma 3.5] (which implies that the linear combination with positive coefficients of positive definite functions is still a positive definite function), if one approximates the integral on the right hand side of (5.86) by finite sums with positive coefficients.

Hence, for every $\varphi \in \chi_T^\alpha$, by noticing the integration that $\gamma_2 \int_0^t \mathrm{e}^{-\gamma_2(t-\tau)}\mathrm{d}\tau = 1 - \mathrm{e}^{-\gamma_2 t}$, we rewrite the equation (5.86) as following,

$$\mathcal{P}[\varphi](t,\xi) - 1 = [\varphi_0(\xi) - 1]\,\mathrm{e}^{-\gamma_2 t} + \int_0^t \mathrm{e}^{-\gamma_2(t-\tau)}\left[\mathcal{G}[\varphi](\tau,\xi) - \gamma_2\right]\,\mathrm{d}\tau. \tag{5.89}$$

Furthermore, by the observation that $\gamma_2 = \mathcal{G}[1]$ as well as $\mathrm{e}^{-\gamma_2(t-\tau)} \leq 1$ for every $\tau \in [0,t]$, we obtain,

$$|\mathcal{P}[\varphi](t,\xi) - 1| \leq \|\varphi_0 - 1\|_\alpha\,|\xi|^\alpha + \gamma_\alpha \int_0^t \|\varphi(\tau,\xi) - 1\|_\alpha\,\mathrm{d}\tau\,|\xi|^\alpha\,. \tag{5.90}$$

After dividing the inequality above by $|\xi|^\alpha$ and computing the supremum with respect to the variable $\xi \in \mathbb{R}^3$ and $t \in [0,T]$, we can finally prove that $\mathcal{P} : \chi_T^\alpha \mapsto \chi_T^\alpha$ satisfying the following estimate:

$$\|\mathcal{P}[\varphi] - 1\|_{\chi_T^\alpha} \leq \|\varphi_0 - 1\|_\alpha + \gamma_\alpha T \|\varphi - 1\|_{\chi_T^\alpha} < \infty, \tag{5.91}$$

which show that the function $\mathcal{P}[\varphi](t,\xi) \in \chi_T^\alpha$ for any $\varphi \in \chi_T^\alpha$.

Step (II): To prove that the operator $\mathcal{P}[\varphi]$ is a contraction in χ_T^α, we introduce another $\mathcal{P}[\tilde{\varphi}](t,\xi)$, and make the subtraction between them. Then for the same initial datum φ_0, we have,

$$\begin{aligned}|\mathcal{P}[\varphi](t,\xi) - \mathcal{P}[\tilde{\varphi}](t,\xi)| \leq & \int_0^t \mathrm{e}^{-\gamma_2(t-\tau)} \left[\mathcal{G}[\varphi](\tau,\xi) - \mathcal{G}[\tilde{\varphi}](\tau,\xi)\right] \mathrm{d}\tau \\ \leq & \gamma_\alpha T \|\varphi - \tilde{\varphi}\|_{\chi_T^\alpha} |\xi|^\alpha\end{aligned} \tag{5.92}$$

where we utilize the Lemma 5.13 in the last inequality. Consequently, after dividing the inequality above by $|\xi|^\alpha$ with respect to the variable $\xi \in \mathbb{R}^3$, we can obtain,

$$\|\mathcal{P}[\varphi] - \mathcal{P}[\tilde{\varphi}]\|_{\chi_T^\alpha} \leq \gamma_\alpha T \|\varphi - \tilde{\varphi}\|_{\chi_T^\alpha}. \tag{5.93}$$

Combining the step (I) and (II), the Banach Contraction Theorem provides the unique solution to the equation (5.85) in the space χ_T^α provided that $T < 1/\gamma_\alpha$.

Step (III): Note that we have constructed the unique solution on the time interval $[0,T]$, where T is independent of the initial datum, therefore, by the continuation argument, we can extend the unique solution to $[T, 2T]$ by choosing $\varphi(T,\xi)$ as the initial datum. Consequently, repeating the same procedure, we manage to construct the unique solution on any finite time interval. □

5.3.3 Stability of Cutoff Solution

Proposition 5.16. *Assume that $\alpha \in [0,2]$ and the collision kernel b_c satisfy the cutoff assumption (5.72). If $\varphi, \tilde{\varphi} \in C([0,\infty), \mathcal{K}^\alpha)$ are two solutions obtained in Theorem 5.15, corresponding to the initial datum $\varphi_0, \tilde{\varphi}_0$ respectively.*

Then, for every $t \geq 0$ and $R \in (0,\infty]$,

$$\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_{\alpha,R} \leq \mathrm{e}^{\lambda_\alpha t} \|\varphi_0 - \tilde{\varphi}_0\|_{\alpha,R}, \tag{5.94}$$

in the sense of the quasi-metric as following: for any $R \in (0,\infty]$ and $\varphi, \tilde{\varphi} \in \mathcal{K}^\alpha$,

$$\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_{\alpha,R} \equiv \sup_{|\xi|\leq R} \frac{|\varphi(t,\xi) - \tilde{\varphi}(t,\xi)|}{|\xi|^\alpha}, \tag{5.95}$$

where the constant $\lambda_\alpha = \gamma_\alpha - \gamma_2$.

Proof. Starting from the function $d\,(t,\xi)$ defined as following:

$$d\,(t,\xi) := \frac{\varphi(t,\xi) - \tilde{\varphi}(t,\xi)}{|\xi|^\alpha}, \tag{5.96}$$

and recalling equation (5.74) and the fact $\varphi(t,0) = 1$, we can obtain the equation satisfied by function $d\,(t,\xi)$ after making subtraction between the equation (5.42) with respect to φ and $\tilde{\varphi}$ separately:

$$\partial_t d\,(t,\xi) + \gamma_2 d(t,\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\varphi(t,\xi^+)\varphi(t,\xi^-) - \tilde{\varphi}(t,\xi^+)\tilde{\varphi}(t,\xi^-)}{|\xi|^\alpha}\right]\mathrm{d}\sigma. \tag{5.97}$$

Then, note that for $|\xi^+| \le |\xi| \le R$ and $|\xi^-| \le |\xi| \le R$, we have the following inequality,

$$\begin{aligned}
&\left|\varphi(t,\xi^+)\varphi(t,\xi^-) - \tilde{\varphi}(t,\xi^+)\tilde{\varphi}(t,\xi^-)\right| \\
\le& \left|\varphi(t,\xi^+)\varphi(t,\xi^-) - \tilde{\varphi}(t,\xi^+)\varphi(t,\xi^-) + \tilde{\varphi}(t,\xi^+)\varphi(t,\xi^-) - \tilde{\varphi}(t,\xi^+)\tilde{\varphi}(t,\xi^-)\right| \\
\le& \left|\varphi(t,\xi^+) - \tilde{\varphi}(t,\xi^+)\right|\left|\varphi(t,\xi^-)\right| + \left|\varphi(t,\xi^-) - \tilde{\varphi}(t,\xi^-)\right|\left|\tilde{\varphi}(t,\xi^+)\right| \\
\le& \left\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\right\|_{\alpha,R}\left(\left|\xi^+\right|^\alpha + \left|\xi^-\right|^\alpha\right),
\end{aligned} \tag{5.98}$$

as a result, we further deduce the inequality satisfied by $d(t,\xi)$,

$$\left|\partial_t d\,(t,\xi) + \gamma_2 d(t,\xi)\right| \le \gamma_\alpha \left\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\right\|_{\alpha,R} \tag{5.99}$$

with the constants γ_2 and γ_α. Moreover, we are able to solve the inequality (5.99) by multiplying $\mathrm{e}^{\gamma_2 t}$ to both sides of it,

$$\left|\partial_t\left(\mathrm{e}^{\gamma_2 t} d(t,\xi)\right)\right| \le \gamma_\alpha\,\mathrm{e}^{\gamma_2 t}\left\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\right\|_{\alpha,R}, \tag{5.100}$$

and integrating the time variable from 0 to t, hence,

$$\left|\mathrm{e}^{\gamma_2 t} d(t,\xi)\right| \le |d(0,\xi)| + \gamma_\alpha \int_0^t \mathrm{e}^{\gamma_2 \tau}\left\|\varphi(\tau,\cdot) - \tilde{\varphi}(\tau,\cdot)\right\|_{\alpha,R}\mathrm{d}\tau. \tag{5.101}$$

Finally, we compute the supremum with respect to $|\xi| \le R$,

$$\mathrm{e}^{\gamma_2 t}\left\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\right\|_{\alpha,R} \le \left\|\varphi_0 - \tilde{\varphi}_0\right\|_{\alpha,R} + \gamma_\alpha \int_0^t \mathrm{e}^{\gamma_2 \tau}\left\|\varphi(\tau,\cdot) - \tilde{\varphi}(\tau,\cdot)\right\|_{\alpha,R}\mathrm{d}\tau, \tag{5.102}$$

and apply the integral form of Grönwall's inequality to obtain

$$\left\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\right\|_{\alpha,R} \le \mathrm{e}^{(\gamma_\alpha - \gamma_2)t}\left\|\varphi_0 - \tilde{\varphi}_0\right\|_{\alpha,R}, \tag{5.103}$$

where note that$\gamma_\alpha - \gamma_2 = \lambda_\alpha$ under cutoff assumption. □

Remark 5.17. *In fact, though here the stability result* (5.94) *is proved in the case of cutoff collision kernel b_c, but it can be generalized for the solutions to initial value problem* (5.42)-(5.46) *with any non-cutoff collision kernel satisfying* (3.32) *in the next subsection.*

5.4 Well-posedness under Non-cutoff Assumption

In this section, we complete the proof of the well-posedness of solutions to the initial value problem (5.42)-(5.46) with non-cutoff assumption on the collision kernel, which implies that

$$\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \mathrm{d}\sigma = \infty, \tag{5.104}$$

more precisely, b satisfies the singularity condition (3.32).

Our main theorem will be first presented as following:

Theorem 5.18. (Well-posedness under non-cutoff assumption) *Assume that the collision kernel b satisfies the non-cutoff assumption* (3.32) *for some $\alpha_0 \in [0,2]$.*

Then for each $\alpha \in [\alpha_0, 2]$ and initial condition $\varphi_0 \in \mathcal{K}^\alpha$, there exists a solution $\varphi \in C([0,\infty), \mathcal{K}^\alpha)$ to the initial value problem (5.42)-(5.46) *and the solution φ is unique in the space $C([0,\infty), \mathcal{K}^{\alpha_0})$.*

Furthermore, for two solutions $\varphi, \tilde{\varphi} \in C([0,\infty), \mathcal{K}^\alpha)$ corresponding to the initial datum $\varphi_0, \tilde{\varphi}_0$ respectively, we have the stability result, for every $t \geq 0$,

$$\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_\alpha \leq \mathrm{e}^{\lambda_\alpha t} \|\varphi_0 - \tilde{\varphi}_0\|_\alpha, \tag{5.105}$$

where the finite parameter λ_α is defined as in (5.62)*,*

$$\lambda_\alpha = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1\right) \mathrm{d}\sigma. \tag{5.106}$$

5.4.1 Preparation under Non-Cutoff Assumption

In fact, our strategy is to construct the solutions to (5.42)-(5.46) with non-cutoff collision kernel b based on compactness argument, hence, we first consider the increasing sequence of bounded collision kernels,

$$b_n(s) \equiv \min\{b(s), n\} \leq b(s), \quad n \in \mathbb{N}, \tag{5.107}$$

and, for every $\alpha \in [\alpha_0, 2]$, the sequence of $\varphi_n \in C([0,\infty), \mathcal{K}^\alpha)$ of corresponding solutions to (5.42)-(5.46) with cutoff collision kernels b_n and with the same initial datum $\varphi_0 \in \mathcal{K}^\alpha$. Furthermore, under the non-cutoff assumption (3.32), we have,

$$\lambda_{\alpha,n} \equiv \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1\right) \mathrm{d}\sigma \leq \lambda_\alpha, \tag{5.108}$$

therefore, by the stability result (5.94) with $R = \infty$, it follows that,

$$\|\varphi_n(t,\cdot) - 1\|_\alpha \leq \mathrm{e}^{\lambda_{\alpha,n} t} \|\varphi_0 - 1\|_\alpha \leq \mathrm{e}^{\lambda_\alpha t} \|\varphi_0 - 1\|_\alpha, \tag{5.109}$$

for all $t \geq 0$.

Before the specific proof of Well-posedness Theorem 5.18, we give the following Lemma 5.19 about the properties satisfied by the sequence of solution $\varphi_n \in C([0,\infty), \mathcal{K}^\alpha)$,

Lemma 5.19. *Assume that the collision kernel b satisfies the non-cutoff assumption (3.32) with some $\alpha_0 \in [0,2]$. Let $\alpha \in [\alpha_0, 2]$, then the sequence of solutions $\{\varphi_n\}_{n=1}^{\infty} \subset C([0,\infty), \mathcal{K}^\alpha)$ is bounded in $C(\mathbb{R}^3 \times [0,\infty))$ and equi-continuous.*

Proof. Step (I): Uniform Bound: According to Theorem 5.15, the sequence of solution $\varphi_n(t,\cdot) \in \mathcal{K}^\alpha$ under cutoff assumption are all chacteristic function for every $t \geq 0$, hence, we have

$$|\varphi_n(t,\xi)| \leq \varphi_n(t,0) = 1, \tag{5.110}$$

for all $\xi \in \mathbb{R}^3$ and $t \geq 0$, which illustrates the uniform bound of $\varphi_n(t,\xi)$.

Step (II): Equi-continuity with respect to time variable t. We utilize the equation satisfied by φ_n as well as Lemma 5.8 to obtain that

$$\begin{aligned}|\partial_t\varphi_n(t,\xi)| \leq& \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left|\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,\xi)\varphi(t,0)\right| \mathrm{d}\sigma \\ \leq& C\,\|\varphi_n(t,\cdot)-1\|_\alpha\,|\xi|^\alpha\left[\int_0^{\frac{\pi}{2}}\sin^\alpha\left(\frac{\theta}{2}\right)b_n(\cos\theta)\sin\theta\,\mathrm{d}\theta\right] \\ \leq& C\,\mathrm{e}^{\lambda_\alpha t}\,\|\varphi_0-1\|_\alpha\,|\xi|^\alpha\left[\int_0^{\frac{\pi}{2}}\sin^\alpha\left(\frac{\theta}{2}\right)b_n(\cos\theta)\sin\theta\mathrm{d}\theta\right],\end{aligned} \tag{5.111}$$

for all $\xi \in \mathbb{R}^3$ and $t \geq 0$, where we apply the stability result (5.109) in the last inequality.

Step (III): Equi-continuity with respect to Fourier variable ξ. To prove this, it suffices to apply Lemma 4.32, combined with Lemma 4.31 to obtain the following estimate:

$$\begin{aligned}|\varphi_n(t,\xi)-\varphi_n(t,\eta)| \leq& \sqrt{2\left[1-\mathrm{Re}[\varphi_n(t,\xi-\eta)]\right]} \\ \leq& \sqrt{2}\,|\xi-\eta|^{\frac{\alpha}{2}}\,\|\varphi_n(t,\cdot)-1\|_\alpha^{\frac{1}{2}} \\ \leq& \sqrt{2}\,|\xi-\eta|^{\frac{\alpha}{2}}\,\mathrm{e}^{\frac{\lambda_\alpha}{2}}\,\|\varphi_0-1\|_\alpha^{\frac{1}{2}},\end{aligned} \tag{5.112}$$

for all $t \geq 0$, where the stability result (5.109) is used in the last inequality and the right-hand side is independent of n. □

5.4.2 Existence of Non-cutoff Solution

Now we are in the position to complete the proof of the existence part of Theorem 5.18, which is guaranteed by the standard compactness argument.

Proof. According to the Ascoli-Arzelà Theorem and the Cantor Diagnal Argument, we can deduce that there exists a subsequence of solutions $\{\varphi_{n_k}\}_{n_k\in\mathbb{N}}$ converging uniformly in any compact set of $\mathbb{R}^3 \times [0,\infty)$ based on the Lemma 5.19.

Then, we need to prove the limit of functions $\{\varphi_{n_k}\}_{n_k\in\mathbb{N}}$,

$$\varphi(t,\xi) = \lim_{n_k\to\infty}\varphi_{n_k}(t,\xi) \tag{5.113}$$

is the solution to the initial value problem (5.42)-(5.46) under non-cutoff assumption (3.32). Note that $\varphi(t, \cdot)$ is a characteristic function for every $t \geq 0$, as the pointwise limit of characteristic functions.

Here we can apply the Lebesgue Dominated Convergence Theorem to take the limit $n_k \to \infty$ in the Boltzmann collision operator,

$$\int_{\mathbb{S}^2} b_{n_k}\left(\frac{\xi \cdot \sigma}{|\xi|}\right)\left[\varphi_{n_k}(t, \xi^+)\varphi_{n_k}(t, \xi^-) - \varphi_{n_k}(t, \xi)\varphi_{n_k}(t, 0)\right] \mathrm{d}\sigma \tag{5.114}$$

which, according to the calculation (5.111) in the proof of Lemma 5.19, can be controlled by the integrable function as following:

$$4\, \mathrm{e}^{\lambda_\alpha t} \left\|\varphi_0 - 1\right\|_\alpha b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left|\xi^+\right|^{\frac{\alpha}{2}} \left|\xi^-\right|^{\frac{\alpha}{2}}. \tag{5.115}$$

On the other hand, since the Boltzmann collision operator in (5.114) converges uniformly on every compact subset of $\mathbb{R}^3 \times [0, \infty)$, there exists a continuous function $\varsigma = \varsigma(t, \xi)$ such that $\partial_t \varphi_{n_k} \to \varsigma$ as $n_k \to \infty$. Meanwhile, considering the limit relation (5.113), we immediately conclude that $\varsigma = \partial_t \varphi$. Hence, the limit function $\varphi(t, \xi)$ is a solution to the initial value problem (5.42)-(5.46).

Finally, to show the limit function $\varphi(\cdot, \xi) \in \mathcal{K}^\alpha$, it suffices to pass to the limit $n_k \to \infty$ in the stability result (5.109) in the following equivalent way,

$$\|\varphi - 1\|_\alpha = \lim_{n_k \to \infty} \frac{|\varphi_{n_k}(t, \xi) - 1|}{|\xi|^\alpha} \leq \mathrm{e}^{\lambda_{\alpha, n} t} \left\|\varphi_0 - 1\right\|_\alpha \leq \mathrm{e}^{\lambda_\alpha t} \left\|\varphi_0 - 1\right\|_\alpha, \tag{5.116}$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$ and $t \geq 0$. □

5.4.3 Stability and Uniqueness of Non-cutoff Solution

As for the uniqueness of the solution that we construct above, we have to extend the stability results to the non-cutoff case, which requires us to split the integrated domain into the integrable parts and remainder part. The precise proof is given as following:

Proof. If we consider two sequences of solution $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\varphi}_n\}_{n \in \mathbb{N}}$ to the equation (5.42) with the cutoff kernel b_n as well as corresponding to the initial condition φ_0 and $\tilde{\varphi}_0$, respectively.

By the compactness argument from Lemma 5.19, there exists a subsequence $n_k \to \infty$ and the solution to (5.42) by taking limit in the sense that

$$\varphi(t, \xi) = \lim_{n_k \to \infty} \varphi_{n_k}(t, \xi) \quad \text{and} \quad \tilde{\varphi}(t, \xi) = \lim_{n_k \to \infty} \tilde{\varphi}_{n_k}(t, \xi). \tag{5.117}$$

Thus, in order to prove the uniqueness, we need to check the stability results under non-cutoff assumption: similar to the procedures under cutoff assumption, we have the following estimate by introducing the same $d(t, \xi)$ as in (5.96) and dividing the integral domain of σ into four parts,

$$\begin{aligned}\partial_t d(t,\xi) =& \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\varphi(t,\xi^+)\varphi(t,\xi^-)-\tilde\varphi(t,\xi^+)\tilde\varphi(t,\xi^-)}{|\xi|^\alpha} - d(t,\xi)\right] \mathrm{d}\sigma\\ =& \int_{\mathbb{S}^2\cap\Omega_\epsilon^c} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\varphi(t,\xi^+)\varphi(t,\xi^-)\tilde\varphi(t,\xi^+)\tilde\varphi(t,\xi^-)}{|\xi|^\alpha}\right] \mathrm{d}\sigma\\ &-\left[\int_{\mathbb{S}^2\cap\Omega_\epsilon^c} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{d}\sigma\right] d(t,\xi)\\ &+\int_{\mathbb{S}^2\cap\Omega_\epsilon} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\varphi(t,\xi^+)\varphi(t,\xi^-)-\varphi(t,0)\varphi(t,\xi)}{|\xi|^\alpha}\right] \mathrm{d}\sigma\\ &-\int_{\mathbb{S}^2\cap\Omega_\epsilon} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\frac{\tilde\varphi(t,\xi^+)\tilde\varphi(t,\xi^-)-\tilde\varphi(t,0)\tilde\varphi(t,\xi)}{|\xi|^\alpha}\right] \mathrm{d}\sigma\\ :=& I_\epsilon(t,\xi)-\gamma_\epsilon d(t,\xi)+R_{\varphi,\epsilon}(t,\xi)-R_{\tilde\varphi,\epsilon}(t,\xi),\end{aligned} \tag{5.118}$$

where Ω_ϵ (Ω_ϵ^c denotes its complement) is defined as

$$\Omega_\epsilon := \Omega_\epsilon(\xi) = \left\{\sigma\in\mathbb{S}^2; 1-\frac{\xi}{|\xi|}\cdot\sigma \le 2\left(\frac{\epsilon}{\pi}\right)^2\right\}, \tag{5.119}$$

for any $\epsilon>0$ and then γ_ϵ can represented as

$$\gamma_\epsilon = 2\pi\int_{[0,\frac{\pi}{2}]\cap\{\sin\frac{\theta}{2}>\frac{\epsilon}{\pi}\}} b(\cos\theta)\sin\theta\,\mathrm{d}\theta\to\infty, \tag{5.120}$$

as $\epsilon\to 0^+$.

Let $R>0$ and then with the help of (5.98), we have, for any $|\xi|\le R$,

$$\left|\frac{\varphi(t,\xi^+)\varphi(t,\xi^-)-\tilde\varphi(t,\xi^+)\tilde\varphi(t,\xi^-)}{|\xi|^\alpha}\right| \le \|\varphi(t,\cdot)-\tilde\varphi(t,\cdot)\|_{\alpha,R}\frac{|\xi^+|^\alpha+|\xi^-|^\alpha}{|\xi|^\alpha}, \tag{5.121}$$

combined the fact that $|\xi^\pm|\le|\xi|$, we further obtain,

$$|I_\epsilon(t,\xi)| \le \gamma_{\alpha,\epsilon}\|\varphi(t,\cdot)-\tilde\varphi(t,\cdot)\|_{\alpha,R} \le 2\gamma_{\alpha,\epsilon}\|\varphi(t,\cdot)-\tilde\varphi(t,\cdot)\|_{\alpha,R}, \tag{5.122}$$

where

$$\gamma_{\alpha,\epsilon} = 2\pi\int_{[0,\frac{\pi}{2}]\cap\{\sin\frac{\theta}{2}>\frac{\epsilon}{\pi}\}} b(\cos\theta)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}\right)\sin\theta\,\mathrm{d}\theta<\infty. \tag{5.123}$$

Since the solutions $\varphi(t,\xi),\tilde\varphi(t,\xi)\in C([0,\infty),\mathcal{K}^\alpha)$, it follows that for any fixed $T>0$,

$$\sup_{t\in(0,T],|\xi|\le R}\left(|R_{\varphi,\epsilon}(t,\xi)|+|R_{\tilde\varphi,\epsilon}(t,\xi)|\right) = r_\epsilon\to 0, \tag{5.124}$$

as $\epsilon\to 0^+$, which can be obtained by the following estimate with the help of Lemma 5.9,

$$\begin{aligned}|R_{\epsilon,\varphi}(t,\xi)| =& \left|\int_{\mathbb{S}^2\cap\Omega_\epsilon} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\frac{[\varphi(t,\xi^+)\varphi(t,\xi^-)-\varphi(t,\xi)]}{|\xi|^\alpha}\mathrm{d}\sigma\right|\\ \le& C\|1-\varphi(t,\cdot)\|_\alpha\int_0^\epsilon \sin^\alpha\left(\frac{\theta}{2}\right) b(\cos\theta)\sin\theta\,\mathrm{d}\theta\to 0\end{aligned} \tag{5.125}$$

as $\epsilon \to 0^+$.

Hence, we obtain the differential inequality of $d(t,\xi)$, for any $|\xi| \le R$,

$$|\partial_t d(t,\xi) + \gamma_\epsilon d(t,\xi)| \le \gamma_{\alpha,\epsilon} \|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_{\alpha,R} + r_\epsilon, \tag{5.126}$$

and furthermore, by computing the supremum with respect to $|\xi| \le R$, we have

$$\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_{\alpha,R} \le \mathrm{e}^{(\gamma_{\alpha,\epsilon}-\gamma_\epsilon)t} \|\varphi_0 - \tilde{\varphi}_0\|_{\alpha,R} + \frac{r_\epsilon}{\gamma_{\alpha,\epsilon} - \gamma_\epsilon}\left[\mathrm{e}^{(\gamma_{\alpha,\epsilon}-\gamma_\epsilon)t} - 1\right]. \tag{5.127}$$

By taking the limit $\epsilon \to 0$ and letting $R \to \infty$, we finally prove the stability result under non-cutoff assumption,

$$\|\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)\|_{\alpha} \le \mathrm{e}^{\lambda_\alpha t} \|\varphi_0 - \tilde{\varphi}_0\|_{\alpha}, \tag{5.128}$$

which then, implies the uniqueness of solution to (5.42)-(5.46) in the space $C([0,\infty), \mathcal{K}^\alpha)$.

□

6 Corresponding and Relevant Materials

The following materials are, in chronological order, referred to the development of the study about the solution to spatially homogeneous Boltzmann equation as a probability measure, where the Fourier Transformation plays a critical role.

KQ: This list is not intended to be completely covered in the mini-course, which is definitely impossible, but to partly reflect the history and hopefully present a big picture about how the research of the homogeneous Boltzmann equation in probability measure sense developed: from cutoff to non-cutoff, from the Maxwellian molecule to hard/soft potential, from higher-order moments requirement to lower-order... The selection is biased in favor of personal taste.

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