

EXISTENCE OF MULTI-DIMENSIONAL CONTACT DISCONTINUITIES FOR THE IDEAL COMPRESSIBLE MAGNETOHYDRODYNAMICS

YANJIN WANG AND ZHOUPING XIN

ABSTRACT. In this paper, we establish the local existence and uniqueness of multi-dimensional contact discontinuities for the ideal compressible magnetohydrodynamics (MHD) in Sobolev spaces, which are most typical interfacial waves for astrophysical plasmas and prototypical fundamental waves for hyperbolic systems of conservation laws. Such waves are characteristic discontinuities for which there is no flow across the discontinuity surface while the magnetic field crosses transversely, which lead to a two-phase free boundary problem where the pressure, velocity and magnetic field are continuous across the interface whereas the entropy and density may have jumps. To overcome the difficulties of possible nonlinear Rayleigh–Taylor instability and loss of derivatives, here we use crucially the Lagrangian formulation and the Cauchy formula (1882) for the magnetic field. This enables us to capture the boundary regularizing effect of the transversal magnetic field on the flow map, and on the other hand, it allows us to define a special good unknown of the magnetic field to get around the troublesome boundary integrals due to the transversality of the magnetic field. In particular, our result removes the additional assumption of the Rayleigh–Taylor sign condition required by Morando, Tarkhinin and Trebeschi (*J. Differential Equations* **258** (2015), no. 7, 2531–2571; *Arch. Ration. Mech. Anal.* **228** (2018), no. 2, 697–742) and holds for both 2D and 3D and hence gives a complete answer to the two open questions raised therein. Moreover, there is *no loss of derivatives* in our well-posedness theory. The solution is constructed as *the inviscid limit* of solutions to well-chosen nonlinear approximate problems for the two-phase compressible viscous non-resistive MHD.

1. INTRODUCTION

1.1. **Eulerian formulation.** Consider the equations for the ideal compressible magnetohydrodynamics (MHD):

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u - B \otimes B) + \nabla(P + \frac{1}{2}|B|^2) = 0 \\ \partial_t B - \operatorname{curl}(u \times B) = 0 \\ \operatorname{div} B = 0 \\ \partial_t(\varrho S) + \operatorname{div}(\varrho u S) = 0, \end{cases} \quad (1.1)$$

where ϱ denotes the density of the plasma, u the velocity, B the magnetic field, S the entropy and P the pressure given by the state equation:

$$P = A\varrho^\gamma e^S \text{ with constants } A > 0, \gamma > 1. \quad (1.2)$$

The fourth equation in (1.1) should be transformed to be the initial constraint, which follows from the third equation. Note that (1.1) is hyperbolic if $\varrho > 0$.

Let $\Sigma(t)$ be a surface of strong discontinuity for (1.1) and consider solutions that are smooth on either side of $\Sigma(t)$. According to the classification of strong discontinuities in MHD [14], for

Date: December 15, 2021.

2010 Mathematics Subject Classification. 35L65, 35R35, 76E17, 76E25, 76N10, 76W05.

Key words and phrases. Contact discontinuities; Multi-dimensional waves; Ideal compressible MHD; Plasma interface; Free boundary problems.

Y. J. Wang was supported by the National Natural Science Foundation of China (12171401, 11771360) and the Natural Science Foundation of Fujian Province of China (2019J02003). Z. P. Xin was supported by Zheng Ge Ru Foundation, HongKong RGC Earmarked Research Grants CUHK14301421, CUHK14300819, CUHK14302819, CUHK14300917, CUHK14302917, and Basic and Applied Basic Research Foundation of Guangdong Province 2020B1515310002.

contact discontinuities there is no flow across the discontinuity and the magnetic field is nowhere tangent to the discontinuity. In such case, the Rankine–Hugoniot jump conditions across $\Sigma(t)$ for (1.1) are equivalent to (see [14, 12])

$$V = u_{\pm} \cdot \nu, \quad P_+ = P_-, \quad u_+ = u_-, \quad B_+ = B_- \text{ on } \Sigma(t), \quad (1.3)$$

where V denotes the normal velocity of $\Sigma(t)$, $f_{\pm} := f|_{\Omega_{\pm}(t)}$ for $\Omega_{\pm}(t)$ the two domains separated by $\Sigma(t)$ and ν is the unit normal to $\Sigma(t)$ pointing to $\Omega_-(t)$. Note that in (1.3) the entropy S and the density ϱ may have jumps. The normal part of the last jump condition, $B_+ \cdot \nu = B_- \cdot \nu$ on $\Sigma(t)$, should be also transformed to be the initial constraint. The boundary conditions (1.3) are most typical for astrophysical plasmas where magnetic fields originate in a planet, star, or other rotating object, with a dynamo operating inside, but intersect the surface [12].

The first condition in (1.3) states that contact discontinuities are characteristic discontinuities for (1.1), which are characteristic free boundaries. There is another type of characteristic discontinuities, tangential discontinuities, for which the magnetic field is tangent to the discontinuity. In such case, the Rankine–Hugoniot jump conditions are equivalent to (see [14, 12])

$$V = u_{\pm} \cdot \nu, \quad B_{\pm} \cdot \nu = 0, \quad P_+ + \frac{1}{2}|B_+|^2 = P_- + \frac{1}{2}|B_-|^2 \text{ on } \Sigma(t). \quad (1.4)$$

Note that in (1.4) the tangential velocity, tangential magnetic field and pressure may also have jumps. The boundary conditions (1.4) usually occur in laboratory plasmas aimed at thermonuclear energy production [12].

In this paper, we are concerned with the existence of MHD contact discontinuities for (1.1). For simplicity, we consider (1.1) in the horizontally periodic flat slab $\Omega = \mathbb{T}^2 \times (-1, 1)$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and assume that the interface of discontinuity $\Sigma(t)$ extends to infinity in every horizontal direction and does not intersect the upper and lower boundaries $\Sigma_{\pm} := \mathbb{T}^2 \times \{\pm 1\}$. Σ_{\pm} are assumed to be impermeable and perfectly conducting, so

$$u \cdot e_3 = 0, \quad E \times e_3 = 0 \text{ on } \Sigma_{\pm}, \quad (1.5)$$

where $E = u \times B$ is the electric field and $e_3 = (0, 0, 1)$. If $B \cdot e_3 \neq 0$ on Σ_{\pm} , then (1.5) is equivalent to (see also [30])

$$u = 0 \text{ on } \Sigma_{\pm}. \quad (1.6)$$

To complete the statement of the problem, one must specify the initial data. Suppose that the initial interface of discontinuity $\Sigma(0)$ is given, which does not intersect Σ_{\pm} . This yields the open sets $\Omega_{\pm}(0)$ on which the initial density $\varrho(0) = \varrho_0 : \Omega_{\pm}(0) \rightarrow \mathbb{R}_+$, the initial velocity $u(0) = u_0 : \Omega_{\pm}(0) \rightarrow \mathbb{R}^3$, the initial magnetic field $B(0) = B_0 : \Omega_{\pm}(0) \rightarrow \mathbb{R}^3$ and the initial entropy $S(0) = S_0 : \Omega_{\pm}(0) \rightarrow \mathbb{R}$ are specified.

1.2. Related works. Contact discontinuities (vortex sheets and entropy waves), along with shocks and rarefaction waves, are fundamental waves in the entropy solutions to multidimensional hyperbolic systems of conservation laws.

Contact discontinuities of the Euler equations are subject to both the Kelvin–Helmholtz instability and the Rayleigh–Taylor instability, see Ebin [10] for the ill-posedness of incompressible Kelvin–Helmholtz and Rayleigh–Taylor problems and Guo and Tice [13] for the ill-posedness of the compressible Rayleigh–Taylor problem. It was shown by the normal mode analysis that supersonic vortex sheets in 2D of the compressible Euler equations are neutrally linearly stable, while subsonic vortex sheets in 2D and vortex sheets in 3D are violently unstable, see Syrovatskij [26], Miles [16], Fejer and Miles [11]. Coulombel and Secchi [7, 8] proved the nonlinear stability of 2D supersonic vortex sheets for the compressible isentropic Euler equations, and the non-isentropic case was proved by Morando and Trebeschi [19] and Morando, Trebeschi and Wang [20]. Note that owing to the regularizing effect for the interface of discontinuity induced by the supersonic condition, the nonlinear Rayleigh–Taylor instability is also absent in 2D supersonic vortex sheets [7, 8, 19, 20]. It is known that the surface tension has the stabilizing effect on the Kelvin–Helmholtz and Rayleigh–Taylor instabilities, see Cheng, Coutand and Shkoller [5] and Shatah and Zeng [21, 22] for the local well-posedness of the two-phase incompressible Euler equations with surface tension and Stevens [23] for the compressible case.

Chen and Wang [4] and Trakhinin [27, 28] proved the existence of MHD tangential discontinuities (current-vortex sheets) under some stability condition when the tangential magnetic fields on the two sides of the interface of discontinuity are non-colinear. Recall that one has the Syrovatskij stability criterion for the incompressible current-vortex sheets [25], and one may refer to Coulombel, Morando, Secchi and Trebeschi [6] for the a priori estimates under a stronger stability condition and Sun, Wang and Zhang [24] for the existence under the Syrovatskij stability condition. These works show the strong stabilizing effect of tangential magnetic fields.

Although MHD contact discontinuities are neutrally linearly stable, see Blokhin and Trakhinin [2], and do not admit the Kelvin–Helmholtz instability, yet they allow the possibility of the Rayleigh–Taylor instability due to the nonlinear effect and whether the magnetic field can prevent such a nonlinear instability is a subtle and difficult issue. Indeed, in contrast to the analysis of current-vortex sheets, even the a priori nonlinear tangential estimates pose essential difficulties due to the regularity issue of the interface of discontinuity. To overcome such difficulties, Morando, Trakhinin and Trebeschi [17, 18] proposed the Rayleigh–Taylor sign condition on the jump of the normal derivative of the pressure across the interface of discontinuity, and proved the existence of 2D MHD contact discontinuities under this additional Rayleigh–Taylor sign condition; however, they also pointed out that their ideas and approaches, developed in [17, 18], fail to treat some key boundary integral terms appearing in 3D. Therefore, the general 3D case and the question whether the Rayleigh–Taylor sign condition is necessary for the existence of MHD contact discontinuities in both 2D and 3D were left as open problems in [17, 18]. In this paper, by exploring the fact that the magnetic fields for MHD contact discontinuities are always transversal across the interface of discontinuity and thus may have stabilizing effects and some key observations related to the boundary integrals, we resolve these two open questions by establishing the existence of both 2D and 3D MHD contact discontinuities in Sobolev spaces without any additional condition. Thus our results show that the Rayleigh–Taylor sign condition is not necessary for the existence of MHD contact discontinuities in both 2D and 3D.

1.3. Lagrangian reformulation. Differently from [17, 18], our analysis in this paper relies crucially on the reformulation of the problem under consideration in Lagrangian coordinates. Take $\Omega_{\pm} := \{x_3 \gtrless 0\} \cap \Omega$ as the Lagrangian domains and denote the interface by $\Sigma := \{x_3 = 0\}$. Assume that there is a diffeomorphism $\eta_0 : \Omega_{\pm} \rightarrow \Omega_{\pm}(0)$ such that

$$[\eta_0] = [[\partial_3 \eta_0]] = 0 \text{ on } \Sigma, \quad \Sigma(0) = \eta_0(\Sigma) \text{ and } \Sigma_{\pm} = \eta_0(\Sigma_{\pm}). \quad (1.7)$$

Note that (1.7) is fulfilled at least when the initial interface $\Sigma(0)$ is a graph as in [17, 18]. Define the flow map $\eta(t, x) \in \Omega_{\pm}(t)$ by that for $x \in \Omega_{\pm}$,

$$\begin{cases} \partial_t \eta(t, x) = u(t, \eta(t, x)), & t > 0 \\ \eta(0, x) = \eta_0(x). \end{cases} \quad (1.8)$$

Assume that $\eta(t, \cdot)$ is invertible and define the Lagrangian unknowns in Ω_{\pm} :

$$(\rho, v, b, s, p)(t, x) := (\varrho, u, B, S, P)(t, \eta(t, x)). \quad (1.9)$$

In Lagrangian coordinates $\partial_t s = 0$, which implies $s = s_0 := S_0(\eta_0)$, and the problem under consideration can be reformulated equivalently as

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega \\ \frac{1}{\gamma p} \partial_t p + \operatorname{div}_{\mathcal{A}} v = 0 & \text{in } \Omega \\ \rho \partial_t v + \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2) = b \cdot \nabla_{\mathcal{A}} b & \text{in } \Omega \\ \partial_t b + b \operatorname{div}_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} v & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} b = 0 & \text{in } \Omega \\ [[p]] = 0, \quad [[v]] = 0, \quad [[b]] = 0 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_{\pm} \\ (\eta, p, v, b) |_{t=0} = (\eta_0, p_0, v_0, b_0), \end{cases} \quad (1.10)$$

where, by (1.2),

$$\rho = \rho_0 p_0^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}} \text{ with } \rho_0 := A^{-\frac{1}{\gamma}} e^{-\frac{s_0}{\gamma}} p_0^{\frac{1}{\gamma}}. \quad (1.11)$$

Here $\mathcal{A} := (\nabla\eta)^{-T}$, $(\nabla_{\mathcal{A}})_i = \partial_i^{\mathcal{A}} := \mathcal{A}_{ij}\partial_j$, $\operatorname{div}_{\mathcal{A}} := \nabla_{\mathcal{A}} \cdot$ and $\llbracket f \rrbracket := f_+ - f_-$ for $f_{\pm} := f|_{\Omega_{\pm}}$. Denote also $J := \det(\nabla\eta)$, the Jacobian of the coordinate transformation, and

$$\mathcal{N} := J\mathcal{A}e_3 = \partial_1\eta \times \partial_2\eta. \quad (1.12)$$

Recall that (1.7) and the following initial conditions have been required:

$$\rho_0, p_0 > 0, J_0 \neq 0 \text{ and } \operatorname{div}_{\mathcal{A}_0} b_0 = 0 \text{ in } \Omega, \llbracket b_0 \rrbracket \cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, b_0 \cdot \mathcal{N}_0 \neq 0 \text{ on } \Sigma \cup \Sigma_{\pm}, \quad (1.13)$$

where $J_0 = J(\eta_0)$, $\mathcal{A}_0 = \mathcal{A}(\eta_0)$ and $\mathcal{N}_0 = \mathcal{N}(\eta_0)$. Some additional conditions for the initial data will be specified later.

1.4. Expressions of ρ, p and b . It is crucial to deduce from (1.10) some important facts. First, by the first equation in (1.10), one has

$$\partial_t J = J \operatorname{div}_{\mathcal{A}} \partial_t \eta = J \operatorname{div}_{\mathcal{A}} v. \quad (1.14)$$

Thus (1.14), the second equation in (1.10) and (1.11) yield $\partial_t(\rho J) = 0$ and hence

$$\rho = \rho_0 J_0 J^{-1} \text{ and } p = p_0 J_0^{\gamma} J^{-\gamma}, \quad (1.15)$$

while (1.14) and the fourth and first equations in (1.10) imply

$$\partial_t(Jb) = Jb \cdot \nabla_{\mathcal{A}} v \equiv J \nabla v \mathcal{A}^T b = J \nabla \partial_t \eta \mathcal{A}^T b = -J \nabla \eta \partial_t \mathcal{A}^T b, \quad (1.16)$$

which shows $\partial_t(J\mathcal{A}^T b) = 0$ and hence

$$b = J^{-1} J_0 \mathcal{A}_0^T b_0 \cdot \nabla \eta. \quad (1.17)$$

We may refer to (1.17) as the Cauchy formula for the magnetic field in Lagrangian coordinates as its analogue to Cauchy's vorticity formula for the compressible Euler equations [3].

As a consequence, one has the following facts for (1.10).

Proposition 1.1. (i) $\partial_t(J \operatorname{div}_{\mathcal{A}} b) = 0$; (ii) $\partial_t(b \cdot \mathcal{N}) = 0$.

Proof. By (1.17), one has that, using the Piola identity $\partial_j(J\mathcal{A}_{ij}) = 0$,

$$\partial_t(J \operatorname{div}_{\mathcal{A}} b) = \partial_t \operatorname{div}(J\mathcal{A}^T b) = 0 \quad (1.18)$$

and that, recalling (1.12),

$$\partial_t(b \cdot \mathcal{N}) = \partial_t(J\mathcal{A}^T b)_3 = 0. \quad (1.19)$$

The proposition is thus concluded. \square

Proposition 1.2. Assume that $\llbracket \eta_0 \rrbracket = \llbracket \partial_3 \eta_0 \rrbracket = \llbracket p_0 \rrbracket = \llbracket b_0 \rrbracket = 0$ and $b_0 \cdot \mathcal{N}_0 \neq 0$ on Σ . Then

$$\llbracket \partial_3 v \rrbracket = \llbracket \eta \rrbracket = \llbracket \partial_3 \eta \rrbracket = 0 \text{ on } \Sigma. \quad (1.20)$$

Proof. Since $\partial_t \eta = v$, by (1.17) and (1.15), one has

$$\begin{aligned} b_0 \cdot \mathcal{N}_0 \partial_3 v &\equiv J_0 \mathcal{A}_0^T b_0 \cdot \nabla v - (J_0 \mathcal{A}_0^T b_0)_{\beta} \partial_{\beta} v \\ &= \partial_t(Jb) - (J_0 \mathcal{A}_0^T b_0)_{\beta} \partial_{\beta} v = J_0 p_0^{\frac{1}{\gamma}} \partial_t(p^{-\frac{1}{\gamma}} b) - (J_0 \mathcal{A}_0^T b_0)_{\beta} \partial_{\beta} v. \end{aligned} \quad (1.21)$$

By $\llbracket \eta_0 \rrbracket = \llbracket \partial_3 \eta_0 \rrbracket = \llbracket p_0 \rrbracket = \llbracket b_0 \rrbracket = 0$ on Σ and the jump conditions in (1.10), one then obtains

$$b_0 \cdot \mathcal{N}_0 \llbracket \partial_3 v \rrbracket = J_0 p_0^{\frac{1}{\gamma}} \llbracket \partial_t(p^{-\frac{1}{\gamma}} b) \rrbracket - (J_0 \mathcal{A}_0^T b_0)_{\beta} \llbracket \partial_{\beta} v \rrbracket = 0 \text{ on } \Sigma, \quad (1.22)$$

which implies $\llbracket \partial_3 v \rrbracket = 0$ on Σ since $b_0 \cdot \mathcal{N}_0 \neq 0$ on Σ . Since $\partial_t \llbracket \eta \rrbracket = \llbracket v \rrbracket = 0$ and $\partial_t \llbracket \partial_3 \eta \rrbracket = \llbracket \partial_3 v \rrbracket = 0$ on Σ , $\llbracket \eta \rrbracket = \llbracket \partial_3 \eta \rrbracket = 0$ on Σ thus follows again from $\llbracket \eta_0 \rrbracket = \llbracket \partial_3 \eta_0 \rrbracket = 0$ on Σ . \square

2. MAIN RESULTS

2.1. Statement of the results. We will work in a high-regularity context with regularity up to m temporal derivatives, which requires one to use the data $(\eta_0, p_0, v_0, b_0, \rho_0)$ of (1.10) to construct the initial data $(\partial_t^j p(0), \partial_t^j v(0), \partial_t^j b(0))$ for $j = 1, \dots, m$ (and $\partial_t^j \eta(0) = \partial_t^{j-1} v(0)$ for $j = 1, \dots, m-1$) recursively by using (3.1). These data need to satisfy various conditions (3.4), which require $(\eta_0, p_0, v_0, b_0, \rho_0)$ to satisfy the necessary $(m-1)$ -th order compatibility conditions that are natural for the local well-posedness of (1.10) in the functional framework below.

Let $H^k(\Omega_\pm)$, $k \geq 0$ and $H^s(\Sigma)$, $s \in \mathbb{R}$ be the usual Sobolev spaces with norms denoted by $\|\cdot\|_k$ and $|\cdot|_s$, respectively. For $f = f_\pm$ in Ω_\pm , denote $\|f\|_k^2 := \|f_+\|_{H^k(\Omega_+)}^2 + \|f_-\|_{H^k(\Omega_-)}^2$. For an integer $m \geq 0$, define the high-order energy as

$$\mathcal{E}_m := \sum_{j=0}^m \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b) \right\|_{m-j}^2 + \|\eta\|_m^2 + |\eta|_m^2. \quad (2.1)$$

Denote

$$\mathcal{M}_0^m := P \left(\|(\eta_0, p_0, v_0, b_0, \rho_0)\|_m^2 + |\eta_0|_m^2 \right), \quad (2.2)$$

where P is a generic polynomial.

Our main result of this paper is stated as follows.

Theorem 2.1. *Let $m \geq 4$ be an integer. Assume that $\eta_0 \in H^m(\Omega_\pm) \cap H^m(\Sigma)$ and $p_0, v_0, b_0, \rho_0 \in H^m(\Omega_\pm)$ are given such that $\operatorname{div}_{\mathcal{A}_0} b_0 = 0$ in Ω ,*

$$[\eta_0] = [\partial_3 \eta_0] = 0 \text{ and } [b_0] \cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, \quad \eta_{0,3} = \pm 1 \text{ on } \Sigma_\pm, \quad (2.3)$$

$$\rho_0, p_0, |J_0| \geq c_0 > 0 \text{ in } \Omega \text{ and } |b_0 \cdot \mathcal{N}_0| \geq c_0 > 0 \text{ on } \Sigma \cup \Sigma_\pm \text{ for some constant } c_0 > 0 \quad (2.4)$$

and the $(m-1)$ -th order compatibility conditions (3.4) are satisfied. Then there exist a $T_0 > 0$ and a unique solution (η, p, v, b) to (1.10) on the time interval $[0, T_0]$ which satisfies

$$\sup_{t \in [0, T_0]} \mathcal{E}_m(t) \leq \mathcal{M}_0^m. \quad (2.5)$$

Remark 2.2. *Our result in particular removes the assumption of the Rayleigh–Taylor sign condition required in [17, 18] and holds for both 2D and 3D and hence gives a complete answer to the two open questions raised therein. This shows also the strong stabilizing effect of the transversal magnetic field on the Rayleigh–Taylor instability. The key ingredient here is the new boundary regularity $|\eta|_m^2$, which is captured from the regularizing effect of the transversal magnetic field.*

Remark 2.3. *Note that there is no loss of derivatives in our well-posedness theory in Sobolev spaces, which is in contrast to [17, 18] where the solution is constructed by employing the Nash–Moser-type linearized iteration scheme and thus has a loss of derivatives.*

Our solution to (1.10) is constructed as the inviscid limit of solutions to “well-chosen” nonlinear viscous approximate problems. For this, we need to first smooth the data $(\eta_0, p_0, v_0, b_0, \rho_0)$ given in Theorem 2.1 to produce the regular enough data $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ for (1.10), with the smoothing parameters $\delta > 0$, which satisfies all the initial conditions (with c_0 replaced by $c_0/2$) assumed in Theorem 2.1 except that $\operatorname{div}_{\mathcal{A}_0^\delta} b_0^\delta = 0$ in Ω may not hold, where $\mathcal{A}_0^\delta = \mathcal{A}(\eta_0^\delta)$; such construction will be elaborated in Appendix A. Now we consider the following viscous (and non-resistive) approximate problem: for the artificial viscosity $\varepsilon > 0$,

$$\begin{cases} \partial_t \eta^{\varepsilon, \delta} = v^{\varepsilon, \delta} & \text{in } \Omega \\ \frac{1}{\gamma p^{\varepsilon, \delta}} \partial_t p^{\varepsilon, \delta} + \operatorname{div}_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = 0 & \text{in } \Omega \\ \rho^{\varepsilon, \delta} \partial_t v^{\varepsilon, \delta} + \nabla_{\mathcal{A}^{\varepsilon, \delta}} (p^{\varepsilon, \delta} + \frac{1}{2} |b^{\varepsilon, \delta}|^2) - \varepsilon \Delta_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = b^{\varepsilon, \delta} \cdot \nabla_{\mathcal{A}^{\varepsilon, \delta}} b^{\varepsilon, \delta} + \Psi^{\varepsilon, \delta} & \text{in } \Omega \\ \partial_t b^{\varepsilon, \delta} + b^{\varepsilon, \delta} \operatorname{div}_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = b^{\varepsilon, \delta} \cdot \nabla_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} & \text{in } \Omega \\ [p^{\varepsilon, \delta}] = 0, \quad [v^{\varepsilon, \delta}] = 0, \quad [b^{\varepsilon, \delta}] = 0, \quad [\partial_3 v^{\varepsilon, \delta}] = 0 & \text{on } \Sigma \\ v^{\varepsilon, \delta} = 0 & \text{on } \Sigma_\pm \\ (\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta})|_{t=0} = (\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta) & \end{cases} \quad (2.6)$$

with

$$\rho^{\varepsilon,\delta} = \rho_0^\delta (p_0^\delta)^{-\frac{1}{\gamma}} (p^{\varepsilon,\delta})^{\frac{1}{\gamma}}, \quad (2.7)$$

where $\mathcal{A}^{\varepsilon,\delta} = \mathcal{A}(\eta^{\varepsilon,\delta})$, $\Delta_{\mathcal{A}^{\varepsilon,\delta}} = \operatorname{div}_{\mathcal{A}^{\varepsilon,\delta}} \nabla_{\mathcal{A}^{\varepsilon,\delta}}$. Note that all the arguments in Section 1.4 hold also for (2.6); in particular, according to (i) in Proposition 1.1, one has

$$J^{\varepsilon,\delta} \operatorname{div}_{\mathcal{A}^{\varepsilon,\delta}} b^{\varepsilon,\delta} = J_0^\delta \operatorname{div}_{\mathcal{A}_0^\delta} b_0^\delta \text{ in } \Omega, \quad (2.8)$$

where $J^{\varepsilon,\delta} = J(\eta^{\varepsilon,\delta})$ and $J_0^\delta = J(\eta_0^\delta)$. It should be pointed out that it is important to introduce the correctors $\Psi^{\varepsilon,\delta}$ in (2.6), defined according to (3.16) and (3.17), that vanish as $\varepsilon \rightarrow 0$, so that the smoothed data $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ satisfies the $(m-1)$ -th order compatibility conditions (3.25) and thus can be taken as the data for (2.6). It is crucial that the boundary conditions of (2.6) are essentially same as those of (1.10) (cf. Proposition 1.2), but the jump conditions on Σ are not standard for solving the viscous MHD and so it is not direct to get the local well-posedness of (2.6), even with $\varepsilon > 0$. Our way of getting around this difficulty is to follow first those of the compressible Navier–Stokes equations (see for instance [9] for the references) to get the local well-posedness of the modified problem (3.32), *i.e.*, the corresponding problem with the jump conditions in (2.6) replaced by the following “standard” jump conditions

$$\llbracket v^{\varepsilon,\delta} \rrbracket = 0, \quad \llbracket \nabla_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \rrbracket \mathcal{N}^{\varepsilon,\delta} = 0 \text{ on } \Sigma, \quad (2.9)$$

where $\mathcal{N}^{\varepsilon,\delta} = \mathcal{N}(\eta^{\varepsilon,\delta})$. The crucial point is then that under the initial conditions these two sets of jump conditions are indeed equivalent. The full details will be provided in Theorem 3.2.

In order to pass to the limit as $\varepsilon, \delta \rightarrow 0$ in (2.6), one needs to show that the solution $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$ constructed in Theorem 3.2 actually exists on an (ε, δ) -independent time interval and satisfies certain uniform estimates. Set

$$Z_3 = x_3(x_3^2 - 1)\partial_3 \text{ and } Z^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} Z_3^{\alpha_3} \text{ for } \alpha \in \mathbb{N}^3. \quad (2.10)$$

Define the anisotropic Sobolev norm

$$\|f\|_{k,\ell} := \sum_{\alpha \in \mathbb{N}^3, |\alpha| \leq \ell} \|Z^\alpha f\|_k. \quad (2.11)$$

Note that $\|f\|_k = \|f\|_{k,0}$. For an integer $m \geq 4$ and each $\varepsilon > 0$, define the energy functional

$$\mathcal{G}_m^\varepsilon(t) := \sup_{[0,t]} \mathfrak{E}_m^\varepsilon + \int_0^t (\bar{\mathfrak{D}}_m^\varepsilon + \mathfrak{D}_m^\varepsilon), \quad (2.12)$$

where

$$\begin{aligned} \mathfrak{E}_m^\varepsilon := & \sum_{j=0}^m \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b) \right\|_{0,m-j}^2 + \sum_{j=0}^{m-1} \left\| \partial_t^j \partial_3 v \right\|_{0,m-j-1}^2 + \sum_{j=0}^{m-1} \left\| \partial_t^j \partial_3 b \cdot \mathcal{N} \right\|_{0,m-j-1}^2 \\ & + \|\eta\|_m^2 + \|\eta\|_m^2 + \varepsilon \|\eta\|_{1,m}^2 + \varepsilon^2 \|\eta\|_{m+1}^2, \end{aligned} \quad (2.13)$$

$$\bar{\mathfrak{D}}_m^\varepsilon := \varepsilon \sum_{j=0}^m \left\| \partial_t^j v \right\|_{1,m-j}^2, \quad (2.14)$$

$$\mathfrak{D}_m^\varepsilon := \sum_{j=0}^{m-1} \left\| (\partial_t^j \partial_3 p, \partial_t^j \partial_3 b) \right\|_{m-j-1}^2 + \sum_{j=0}^{m-2} \left\| \partial_t^j \partial_3^2 v \right\|_{m-j-2}^2 + \varepsilon^2 \sum_{j=0}^{m-1} \left\| \partial_t^j \partial_3^2 v \right\|_{m-j-1}^2. \quad (2.15)$$

Theorem 2.4. *Let $m \geq 4$ be an integer. Let $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ be the smoothed data constructed in Appendix A and $\Phi^{\varepsilon,\delta}$ be the corrector defined according to (3.16) and (3.17). Then there exist a $T_0 > 0$ and a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ so that for $0 < \varepsilon \leq \varepsilon_0$ the unique solution $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$ to (2.6), constructed in Theorem 3.2, exists on $[0, T_0]$ and satisfies*

$$\mathcal{G}_m^\varepsilon(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})(T_0) \leq \mathcal{M}_0^m. \quad (2.16)$$

By Theorem 2.4, one can easily pass to the limit as first $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$ in (2.6) to find that the limit (η, p, v, b) of $(\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta})$ solves (1.10) on $[0, T_0]$, where the constraint that $\operatorname{div}_{\mathcal{A}} b = 0$ in Ω is recovered by using (i) in Proposition 1.1 as $\operatorname{div}_{\mathcal{A}_0} b_0 = 0$ in Ω . Moreover, the solution satisfies the estimate $\mathcal{G}_m^0(T_0) \leq \mathcal{M}_0^m$. By the a posteriori estimates, one can improve those L^2 -in-time estimates in $\mathcal{G}_m^0(T_0)$ to be L^∞ -in-time so that the estimate (2.5) holds. After proving the uniqueness of the solutions, Theorem 2.1 is thus concluded.

Remark 2.5. *Our analysis can be applied to justify the inviscid limit of the compressible viscous non-resistive MHD in bounded domains with the no-slip boundary condition in Sobolev spaces when the magnetic field is nowhere tangent to the boundary. It should be pointed out that unlike the Navier–Stokes equations, there are no boundary layers in the inviscid limit here. This is due to that here the viscous and ideal problems have the exactly same boundary conditions.*

2.2. Strategy of the uniform estimates. The main part of the paper will be devoted to prove Theorem 2.4, where the key step is to derive the uniform estimate (2.16) on a time interval small but independent of ε, δ . Suppress the dependence of solutions on ε, δ . Note that the estimates of the lower order derivatives of (p, v, b) and the last three terms of η in (2.13) can be easily controlled by using the transported estimates as recorded in Section 4. It then suffices to estimate the highest order derivatives of (p, v, b) and the boundary regularity of η .

To derive the highest order tangential energy estimates, we shall use the equations (5.1) (derived from (2.6)) for (q, v, b) with $q = p + \frac{1}{2}|b|^2$ the total pressure. One starts with applying the tangential *spatial* derivatives Z^m , any Z^α for $\alpha \in \mathbb{N}^3$ with $|\alpha| = m$, to (5.1). Typically, the estimate of the commutator between Z^m and ∂_i^A needs a control of $\|Z^m \nabla \eta\|_0$, which yields a loss of one derivative. Motivated by Alinhac [1], a natural way to get around this difficulty is to introduce the good unknowns,

$$\mathcal{V}^m = Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v, \quad \mathcal{Q}^m = Z^m q - Z^m \eta \cdot \nabla_{\mathcal{A}} q \text{ and } \mathcal{B}^m = Z^m b - Z^m \eta \cdot \nabla_{\mathcal{A}} b. \quad (2.17)$$

Then the highest order term of η will be canceled when considering the equations satisfied by good unknowns. This leads to that, by using $\partial_t \eta = v$ and (ii) in Proposition 1.1,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} J \left(\frac{1}{\gamma p} |\mathcal{Q}^m - b \cdot \mathcal{B}^m|^2 + \rho |\mathcal{V}^m|^2 + |\mathcal{B}^m|^2 \right) + \varepsilon \int_{\Omega} J |\nabla_{\mathcal{A}} \mathcal{V}^m|^2 \\ &= \int_{\Sigma} \llbracket \mathcal{Q}^m \rrbracket \mathcal{V}^m \cdot \mathcal{N} - \int_{\Sigma} b \cdot \mathcal{N} \llbracket \mathcal{B}^m \rrbracket \cdot \mathcal{V}^m + \sum_{\mathcal{R}} \\ &= - \int_{\Sigma} J^{-1} Z^m \eta \cdot \mathcal{N} \llbracket \partial_3 q \rrbracket \partial_t Z^m \eta \cdot \mathcal{N} + \int_{\Sigma} b_0^\delta \cdot \mathcal{N}_0^\delta J^{-1} Z^m \eta \cdot \mathcal{N} \llbracket \partial_3 b \rrbracket \cdot \partial_t Z^m \eta + \sum_{\mathcal{R}}. \end{aligned} \quad (2.18)$$

Here $\mathcal{N}_0^\delta = \mathcal{N}(\eta_0^\delta)$ and $\sum_{\mathcal{R}}$ denotes terms, whose time integration, after some delicate arguments, can be bounded by $\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m, \delta} + t^{1/2} P(\mathcal{G}_m^\varepsilon(t))$ for $0 < \delta < \delta_0$ with some $\delta_0 > 0$, where

$$\mathcal{M}_0^{m, \delta} := P_\delta \left(\|(\eta_0, p_0, v_0, b_0, \rho_0)\|_m^2 \right). \quad (2.19)$$

It is well known that the geometric symmetry structure for the first term in the right hand side of (2.18) is crucial, which leads to

$$- \int_{\Sigma} J^{-1} Z^m \eta \cdot \mathcal{N} \llbracket \partial_3 q \rrbracket \partial_t Z^m \eta \cdot \mathcal{N} = - \frac{1}{2} \frac{d}{dt} \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \sum_{\mathcal{R}}, \quad (2.20)$$

and this would imply the boundary regularity $|Z^m \eta \cdot \mathcal{N}|_0^2$ if one assumed the Rayleigh–Taylor sign condition (*i.e.*, $\llbracket \partial_3 q \rrbracket > 0$ on Σ). However, there is no such symmetry for the second term in the right hand side of (2.18). Note that this term vanishes when $b_0^\delta \cdot \mathcal{N}_0^\delta = 0$ on Σ , and it seems out of control when $b_0^\delta \cdot \mathcal{N}_0^\delta \neq 0$ on Σ . Our way to overcome this difficulty is to make use of the Cauchy formula (1.17), which implies, by (1.15),

$$b \cdot \nabla_{\mathcal{A}} = J^{-1} J_0^\delta (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla = \rho (\rho_0^\delta)^{-1} b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta}. \quad (2.21)$$

We shall use (2.21) for the term $b \cdot \nabla_{\mathcal{A}} b$ in the second equation of (5.1), which allows one to introduce instead in (2.17),

$$\mathcal{B}^m = Z^m b - Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} b. \quad (2.22)$$

Due to (2.22), the second term in the right hand side of (2.18) is changed to be

$$\begin{aligned} & \int_{\Sigma} b_0^\delta \cdot \mathcal{N}_0^\delta (J_0^\delta)^{-1} Z^m \eta_0^\delta \cdot \mathcal{N}_0^\delta \llbracket \partial_3 b \rrbracket \cdot \partial_t Z^m \eta \\ &= \frac{d}{dt} \int_{\Sigma} b_0^\delta \cdot \mathcal{N}_0^\delta (J_0^\delta)^{-1} Z^m \eta_0^\delta \cdot \mathcal{N}_0^\delta \llbracket \partial_3 b \rrbracket \cdot Z^m \eta + \sum_{\mathcal{R}}. \end{aligned} \quad (2.23)$$

By (2.20) and (2.23), one can then deduce from (2.18), correspondingly, that

$$\|(p, v, b)(t)\|_{0,m}^2 + \varepsilon \int_0^t \|v\|_{1,m}^2 \lesssim |Z^m \eta(t)|_0^2 + \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} P(\mathcal{G}_m^\varepsilon(t)). \quad (2.24)$$

Now to control $|Z^m \eta|_0^2$ in the right hand side of (2.24), our key point here is to use further the Cauchy formula (1.17) in $\|b\|_{0,m}^2$ and then introduce the good unknown

$$\Xi^m := Z^m \eta - Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta. \quad (2.25)$$

These allow one to add $\|(\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla \Xi^m\|_0^2$ to the left hand side of (2.24). Then the boundary regularizing effect of the magnetic field due to that $((\mathcal{A}_0^\delta)^T b_0^\delta)_3 = (J_0^\delta)^{-1} b_0^\delta \cdot \mathcal{N}_0^\delta \neq 0$ near Σ is captured by applying Lemma B.1 to Ξ^m :

$$|\Xi^m|_0^2 \lesssim \left\| (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla \Xi^m \right\|_0 \|\Xi^m\|_0 + \|\Xi^m\|_0^2. \quad (2.26)$$

By (2.26) and Cauchy's inequality, one can then improve (2.24) to be

$$\|(p, v, b)(t)\|_{0,m}^2 + |\eta(t)|_m^2 + \varepsilon \int_0^t \|v\|_{1,m}^2 \leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} P(\mathcal{G}_m^\varepsilon(t)). \quad (2.27)$$

Similarly but in a much simpler way, the rest of highest order tangential energy estimates involving at least one time derivative can be also controlled by the same bound as (2.27).

Now we turn to the derivation of the normal derivatives estimates near the boundary $\Sigma \cup \Sigma_\pm$. For the original ideal MHD (1.10), utilizing again the transversality of the magnetic field near the boundary as in Yanagisawa [29] and Yanagisawa and Matsumura [30], the estimates of normal derivatives of the solution can be derived by expressing them in terms of tangential derivatives. However, for the viscous approximation (2.6), one needs to explore additionally the ODE-in-time structures and certain cancelation related to the viscous term. More precisely, first, by (2.8) and the fourth and second equations in (2.6), one can express $\partial_3 b \cdot \mathcal{N}$ and $\partial_3 v$ as sums of tangential derivatives of p, v, b , denoted by $\sum_{\mathcal{T}}$, up to a multiplication of continuous functions of $\nabla \eta, p, v, b$. Next, one uses the tangential part of the third equation in (2.6) to deduce that

$$\partial_3 b \cdot \tau^\beta + \varepsilon \frac{\mathcal{A}_{k3} \mathcal{A}_{k3}}{(\mathcal{A}^T b)_3^2} \partial_3 (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta = \sum_{\mathcal{T}} + \varepsilon \nabla \partial_\beta v + \sum_{\varepsilon}, \quad \beta = 1, 2. \quad (2.28)$$

where $\tau^\beta, \beta = 1, 2$, are defined in (6.7) and \sum_{ε} denotes the terms that can be ultimately controlled by $\varepsilon \mathcal{G}_m^\varepsilon$. By the fourth and second equations in (2.6), $b \cdot \nabla_{\mathcal{A}} v = \partial_t b - \frac{b}{\gamma p} \partial_t p$ and hence (2.28) can be regarded as an ODE (in time) for $\partial_3 b \cdot \tau^\beta$. On the other hand, the normal part of the third equation in (2.6) yields that

$$\partial_3 p + \partial_3 b \cdot \tau^\beta b \cdot \tau^\beta - \varepsilon \partial_3 \operatorname{div}_{\mathcal{A}} v = \sum_{\mathcal{T}} + \varepsilon \nabla \partial_\beta v + \sum_{\varepsilon}. \quad (2.29)$$

By the second equations in (2.6), $\operatorname{div}_{\mathcal{A}} v = -\frac{1}{\gamma p} \partial_t p$ and hence (2.29) is an ODE for $\partial_3 p$. Therefore, basing on these structures above, by a recursive argument in terms of the numbers of normal derivatives, one can then deduce the desired estimates so that

$$\mathcal{G}_m^\varepsilon(t) \lesssim \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} P(\mathcal{G}_m^\varepsilon(t)) + \varepsilon \mathcal{G}_m^\varepsilon(t). \quad (2.30)$$

It should be noted that it is crucial in deriving (2.30) that (2.28) and (2.29) have an exact cancelation between the two underlined crossing terms; otherwise, they seem out of control. One thus concludes the estimate (2.16) from (2.30) for $0 < \varepsilon \leq \varepsilon_0(\delta)$ and some $T_0 > 0$.

2.3. Notation. The Einstein convention of summing over repeated certain indices will be used, and the repeated over the latin letter i, j, k, ℓ is from 1 to 3 while the repeated over the greek letter β, γ is from 1 to 2. Throughout the paper C denotes for generic positive constants and P for generic polynomials that do not depend on ε, δ , which are allowed to change from line to line. C_δ, P_δ , etc. denote the additional dependence. $A_1 \lesssim A_2$ means that $A_1 \leq CA_2$.

$\mathbb{N} = \{0, 1, 2, \dots\}$ denotes for the collection of non-negative integers. When using space-time differential multi-indices, we write $\mathbb{N}^{1+d} = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial multi-indices, we write \mathbb{N}^d . For $\alpha \in \mathbb{N}^{1+2}$, $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$, for $\alpha \in \mathbb{N}^3$, $Z^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} Z_3^{\alpha_3}$ and for $\alpha \in \mathbb{N}^{1+3}$, $\bar{Z}^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2} Z_3^{\alpha_3}$. Denote the standard commutator

$$[\partial^\alpha, f]g = \partial^\alpha(fg) - f\partial^\alpha g \quad (2.31)$$

and the symmetric commutator

$$[\partial^\alpha, f, g] = \partial^\alpha(fg) - f\partial^\alpha g - \partial^\alpha fg. \quad (2.32)$$

We omit the differential elements dx and $dx_1 dx_2$ of the integrals over Ω_\pm and Σ and also sometimes the differential elements ds of the time integrals.

3. NONLINEAR VISCOUS APPROXIMATION

3.1. Initial data and compatibility conditions of (1.10). For the data $(\eta_0, p_0, v_0, b_0, \rho_0)$ of (1.10) given in Theorem 2.1, one needs to construct the initial data $(\partial_t^j p(0), \partial_t^j v(0), \partial_t^j b(0))$ for $j = 1, \dots, m$ and $\partial_t^j \eta(0)$ for $j = 1, \dots, m-1$ recursively by using

$$\begin{pmatrix} \partial_t^j p(0) \\ \partial_t^j v(0) \\ \partial_t^j b(0) \end{pmatrix} := \partial_t^{j-1} \begin{pmatrix} -\gamma p \operatorname{div}_{\mathcal{A}} v \\ \rho^{-1} (b \cdot \nabla_{\mathcal{A}} b - \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2)) \\ b \cdot \nabla_{\mathcal{A}} v - b \operatorname{div}_{\mathcal{A}} v \end{pmatrix} \Big|_{t=0}, \quad j = 1, \dots, m \quad (3.1)$$

and

$$\partial_t^j \eta(0) := \partial_t^{j-1} v(0), \quad j = 1, \dots, m-1, \quad (3.2)$$

where, as before, $\rho = \rho_0 p_0^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$ and $\mathcal{A} = \mathcal{A}(\eta)$. By the iteration, (3.1) and (3.2) enable one to determine these initial data in terms of $(\eta_0, p_0, v_0, b_0, \rho_0)$ and its spatial derivatives in such a way which is essentially same as that one determines the time derivatives of the solution (η, p, v, b) to (1.10) in terms of (η, p, v, b, ρ) by using (1.10) repeatedly. Moreover, due to the first initial condition in (2.4), it is straightforward to check that

$$\mathcal{E}_m(0) \leq \mathcal{M}_0^m. \quad (3.3)$$

In order for $(\eta_0, p_0, v_0, b_0, \rho_0)$ to be taken as the data for the local well-posedness of (1.10) in our energy functional framework, these data need to satisfy, besides (2.3) and (2.4) (and $\operatorname{div}_{\mathcal{A}_0} b_0 = 0$ in Ω), the following $(m-1)$ -th order compatibility conditions:

$$\left[\partial_t^j p(0) \right] = \left[\partial_t^j v(0) \right] = \Pi_0 \left[\partial_t^j b(0) \right] = 0 \text{ on } \Sigma \text{ and } \partial_t^j v(0) = 0 \text{ on } \Sigma_\pm, \quad j = 0, \dots, m-1, \quad (3.4)$$

where $\Pi_0 = I - |\mathcal{N}_0|^{-2} \mathcal{N}_0 \otimes \mathcal{N}_0$.

Lemma 3.1. *Under the initial conditions (2.3), (2.4) and (3.4), it holds that*

$$\left[\partial_t^j b(0) \right] \cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, \quad j = 0, \dots, m-1 \quad (3.5)$$

and

$$\left[\partial_3 \partial_t^j v(0) \right] = 0 \text{ on } \Sigma, \quad j = 0, \dots, m-2. \quad (3.6)$$

Proof. We prove (3.5) by the induction. First, (3.5) holds for $j = 0$ due to the last condition in (2.3). Now suppose that (3.5) holds for $j = 0, \dots, \ell$ with $\ell \in [0, m - 2]$. Note that by the definitions (3.1) and (3.2), similar to the remarks below (3.2), $\partial_t^{\ell+1}(b \cdot \mathcal{N})(0)$ is determined in terms of $(\eta_0, p_0, v_0, b_0, \rho_0)$ in the same way as the one that $\partial_t^{\ell+1}(b \cdot \mathcal{N})$ is expressed in terms of (η, p, v, b, ρ) by using (1.10) repeatedly. According to (ii) in Proposition 1.1, one thus has

$$\partial_t^{\ell+1}(b \cdot \mathcal{N})(0) = 0. \quad (3.7)$$

Then by (3.7), the last two jump conditions in (3.4) and the induction assumption, one deduces that, recalling the commutator notation (2.31),

$$\left[\partial_t^{\ell+1} b(0) \right] \cdot \mathcal{N}_0 = - \left[\left[\partial_t^{\ell+1}, \mathcal{N} \right] b(0) \right] = 0 \text{ on } \Sigma, \quad (3.8)$$

which yields (3.5).

We now prove (3.6). By the definitions (3.1) and (3.2) again and according to (1.21), similarly as for (3.7), one has

$$b_0 \cdot \mathcal{N}_0 \partial_3 \partial_t^j v(0) = J_0 p_0^{\frac{1}{\gamma}} \partial_t^{j+1} (p^{-\frac{1}{\gamma}} b)(0) - (J_0 \mathcal{A}_0^T b_0)_\beta \partial_\beta \partial_t^j v(0), \quad j \geq 0. \quad (3.9)$$

Then by the jump conditions in (2.3), (3.4) and (3.5), one obtains that for $j = 0, \dots, m - 2$,

$$b_0 \cdot \mathcal{N}_0 \left[\partial_3 \partial_t^j v(0) \right] = J_0 p_0^{\frac{1}{\gamma}} \left[\partial_t^{j+1} (p^{-\frac{1}{\gamma}} b)(0) \right] - (J_0 \mathcal{A}_0^T b_0)_\beta \left[\partial_\beta \partial_t^j v(0) \right] = 0 \text{ on } \Sigma, \quad (3.10)$$

which implies (3.6) since $b_0 \cdot \mathcal{N}_0 \neq 0$ on Σ . \square

3.2. Smoothed initial data and correctors for (2.6). We will construct solutions to (1.10) as the inviscid limit of the corresponding problem for the viscous non-resistive MHD. However, such an approximation scheme is highly technical due to the issue of the boundary conditions and the high order compatibility conditions for the initial data. Our idea here is to regularize (1.10) by the viscous approximate problem (2.6), where the smoothed data $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ and the so-called corrector $\Psi^{\varepsilon, \delta}$ are introduced.

Let $(\eta_0, p_0, v_0, b_0, \rho_0)$ be the data of (1.10) given in Theorem 2.1 and $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ be the smoothed data constructed in Appendix A, with the smoothing parameter $\delta > 0$. Let $(\partial_t^j p^\delta(0), \partial_t^j v^\delta(0), \partial_t^j b^\delta(0))$ for $j = 1, \dots, m$ and $\partial_t^j \eta^\delta(0)$ for $j = 1, \dots, m - 1$ be constructed recursively by using (3.1) and (3.2), with the data $(\eta_0, p_0, v_0, b_0, \rho_0)$ replaced by $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$, ρ replaced by $\rho^\delta = \rho_0^\delta (p_0^\delta)^{-\frac{1}{\gamma}} (p^\delta)^{\frac{1}{\gamma}}$ and \mathcal{A} replaced by $\mathcal{A}^\delta = \mathcal{A}(\eta^\delta)$. Then it follows from the construction in Appendix A and Lemma 3.1 (for the smoothed data correspondingly) that

$$\left[\eta_0^\delta \right] = \left[\partial_3 \eta_0^\delta \right] = 0 \text{ on } \Sigma \text{ and } \eta_{0,3}^\delta = \pm 1 \text{ on } \Sigma_\pm \quad (3.11)$$

and

$$\begin{aligned} \left[\partial_t^j p^\delta(0) \right] &= \left[\partial_t^j v^\delta(0) \right] = \left[\partial_t^j b^\delta(0) \right] = 0 \text{ on } \Sigma \text{ and } \partial_t^j v^\delta(0) = 0 \text{ on } \Sigma_\pm, \\ j = 0, \dots, m - 1 \text{ and } \left[\partial_3 \partial_t^j v^\delta(0) \right] &= 0 \text{ on } \Sigma, \quad j = 0, \dots, m - 2. \end{aligned} \quad (3.12)$$

Moreover,

$$\eta_0^\delta \rightarrow \eta_0 \text{ in } H^m(\Omega_\pm) \cap H^m(\Sigma) \text{ and } (p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta) \rightarrow (p_0, v_0, b_0, \rho_0) \text{ in } H^m(\Omega_\pm) \text{ as } \delta \rightarrow 0, \quad (3.13)$$

and for $0 < \delta < \delta_0$ with some $\delta_0 > 0$ (hereafter),

$$\rho_0^\delta, p_0^\delta, |J_0^\delta| \geq \frac{c_0}{2} > 0 \text{ in } \Omega \text{ and } |b_0^\delta \cdot \mathcal{N}_0^\delta| \geq \frac{c_0}{2} > 0 \text{ on } \Sigma \cup \Sigma_\pm \quad (3.14)$$

and

$$\mathcal{E}_m(\eta^\delta, p^\delta, v^\delta, b^\delta)(0) \leq \mathcal{M}_0^m \text{ and } \mathcal{E}_l(\eta^\delta, p^\delta, v^\delta, b^\delta)(0) \leq \mathcal{M}_0^{m, \delta}, \quad l \geq m + 1. \quad (3.15)$$

Next, define the corrector $\Psi^{\varepsilon, \delta}$ such that

$$\partial_t^j ((\rho^\delta)^{-1} \Psi^{\varepsilon, \delta})(0) = -\partial_t^j ((\rho^\delta)^{-1} \varepsilon \Delta_{\mathcal{A}^\delta} v^\delta)(0), \quad j = 0, \dots, m - 3 \text{ and } m - 1 \quad (3.16)$$

and

$$\partial_t^{m-2} ((\rho^\delta)^{-1} \Psi^{\varepsilon, \delta})(0) = -\partial_t^{m-2} ((\rho^\delta)^{-1} \varepsilon \Delta_{\mathcal{A}^\delta} v^\delta)(0) + (\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0), \quad (3.17)$$

where $(\partial_t^{m-1}v^\delta(0))^\varepsilon := \phi_\varepsilon * \mathcal{E}_\Omega(\partial_t^{m-1}v^\delta(0))$ for ϕ_ε the standard mollifier in \mathbb{R}^3 and \mathcal{E}_Ω the Sobolev extension operator. The existence of such $\Psi^{\varepsilon,\delta}$ is standard (see [15]). Note that the classical properties of mollifiers (see [1]) imply

$$\left\| (\partial_t^{m-1}v^\delta(0))^\varepsilon - \partial_t^{m-1}v^\delta(0) \right\|_k \lesssim \varepsilon^l \left\| \partial_t^{m-1}v^\delta(0) \right\|_{k+l}, \quad \forall k, l \geq 0. \quad (3.18)$$

By the definition of $\Psi^{\varepsilon,\delta}$, one deduces from (3.15), (3.14) and (3.18) that

$$\begin{aligned} \sup_{[0,\infty)} \sum_{j=0}^m \left\| \partial_t^j \Psi^{\varepsilon,\delta} \right\|_{m-j}^2 &\lesssim \varepsilon^2 \mathcal{M}_0^{m,\delta} + \left\| (\partial_t^{m-1}v^\delta(0))^\varepsilon - \partial_t^{m-1}v^\delta(0) \right\|_m^2 \\ &\lesssim \varepsilon^2 \mathcal{M}_0^{m,\delta} + \varepsilon^2 \left\| \partial_t^{m-1}v^\delta(0) \right\|_{m+1}^2 \leq \varepsilon^2 \mathcal{M}_0^{m,\delta}. \end{aligned} \quad (3.19)$$

Now take $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ and $\Psi^{\varepsilon,\delta}$ in the above as the ones for the viscous approximate problem (2.6). One then constructs the corresponding data $(\partial_t^j p^{\varepsilon,\delta}(0), \partial_t^j v^{\varepsilon,\delta}(0), \partial_t^j b^{\varepsilon,\delta}(0))$ for $j = 1, \dots, m$ and $\partial_t^j \eta^{\varepsilon,\delta}(0)$ for $j = 1, \dots, m-1$ recursively by

$$\begin{pmatrix} \partial_t^j p^{\varepsilon,\delta}(0) \\ \partial_t^j v^{\varepsilon,\delta}(0) \\ \partial_t^j b^{\varepsilon,\delta}(0) \end{pmatrix} := \partial_t^{j-1} \begin{pmatrix} -\gamma p^{\varepsilon,\delta} \operatorname{div}_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \\ (\rho^{\varepsilon,\delta})^{-1} (b^{\varepsilon,\delta} \cdot \nabla_{\mathcal{A}^{\varepsilon,\delta}} b^{\varepsilon,\delta} - \nabla_{\mathcal{A}^{\varepsilon,\delta}} (p^{\varepsilon,\delta} + \frac{1}{2}|b^{\varepsilon,\delta}|^2) \\ + \varepsilon \Delta_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} + \Psi^{\varepsilon,\delta}) \\ b^{\varepsilon,\delta} \cdot \nabla_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} - b^{\varepsilon,\delta} \operatorname{div}_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \end{pmatrix} \Big|_{t=0}, \quad j = 1, \dots, m \quad (3.20)$$

and

$$\partial_t^j \eta^{\varepsilon,\delta}(0) := \partial_t^{j-1} v^{\varepsilon,\delta}(0), \quad j = 1, \dots, m-1, \quad (3.21)$$

where $\rho^{\varepsilon,\delta} = \rho_0^\delta (p_0^\delta)^{-\frac{1}{\gamma}} (p^{\varepsilon,\delta})^{\frac{1}{\gamma}}$ and $\mathcal{A}^{\varepsilon,\delta} = \mathcal{A}(\eta^{\varepsilon,\delta})$. By comparing (3.1)–(3.2) (with superscript δ added) and (3.20)–(3.21), due to (3.16) and (3.17), one has

$$\begin{aligned} (\partial_t^j p^{\varepsilon,\delta}(0), \partial_t^j b^{\varepsilon,\delta}(0)) &= (\partial_t^j p^\delta(0), \partial_t^j b^\delta(0)), \quad j = 0, \dots, m-1, \\ \partial_t^j v^{\varepsilon,\delta}(0) &= \partial_t^j v^\delta(0), \quad j = 0, \dots, m-2, \quad \partial_t^{m-1} v^{\varepsilon,\delta}(0) = (\partial_t^{m-1} v^\delta(0))^\varepsilon \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} \partial_t^m p^{\varepsilon,\delta}(0) &= \partial_t^m p^\delta(0) - \gamma p_0^\delta \operatorname{div}_{\mathcal{A}_0^\delta} \left((\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right), \\ \partial_t^m v^{\varepsilon,\delta}(0) &= \partial_t^m v^\delta(0) + \varepsilon (\rho_0^\delta)^{-1} \Delta_{\mathcal{A}_0^\delta} \left((\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right), \\ \partial_t^m b^{\varepsilon,\delta}(0) &= \partial_t^m b^\delta(0) + b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \left((\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right) \\ &\quad - b_0^\delta \operatorname{div}_{\mathcal{A}_0^\delta} \left((\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right). \end{aligned} \quad (3.23)$$

Then by (3.22), (3.23), (3.15) and (3.18), one obtains

$$\begin{aligned} \mathcal{E}_m(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})(0) &\leq \mathcal{M}_0^m + \mathcal{M}_0^m \left\| (\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right\|_1^2 \\ &\quad + \mathcal{M}_0^m \varepsilon^2 \left\| (\partial_t^{m-1} v^\delta(0))^\varepsilon - \partial_t^{m-1} v^\delta(0) \right\|_2^2 \\ &\leq \mathcal{M}_0^m + \mathcal{M}_0^m \varepsilon^2 \left\| \partial_t^{m-1} v^\delta(0) \right\|_2^2 \leq \mathcal{M}_0^m + \varepsilon^2 \mathcal{M}_0^{m,\delta}. \end{aligned} \quad (3.24)$$

Moreover, (3.22) and (3.12) imply in particular the following $(m-1)$ -th order compatibility conditions for (2.6):

$$\begin{aligned} \left[\partial_t^j p^{\varepsilon,\delta}(0) \right] &= \left[\partial_t^j v^{\varepsilon,\delta}(0) \right] = \left[\partial_3 \partial_t^j v^{\varepsilon,\delta}(0) \right] = \left[\partial_t^j b^{\varepsilon,\delta}(0) \right] = 0 \text{ on } \Sigma \\ \text{and } \partial_t^j v^{\varepsilon,\delta}(0) &= 0 \text{ on } \Sigma_\pm, \quad j = 0, \dots, m-1. \end{aligned} \quad (3.25)$$

It is worth noting that one more compatibility condition other than those in (3.12), that is, $\left[\partial_3 \partial_t^{m-1} v^{\varepsilon,\delta}(0) \right] = 0$ on Σ , is included in (3.25), which is necessary for the local well-posedness

of the viscous approximation (2.6) in the functional framework below in the next subsection; this is exactly the reason why the last two terms in (3.17) have been added.

3.3. Local well-posedness of (2.6). Now we turn to the local well-posedness of (2.6). For an integer $m \geq 3$, define the energy functionals

$$\mathbb{E}_{2m} := \sum_{j=0}^m \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b) \right\|_{2m-2j}^2 + \|\eta\|_{2m+1}^2 \quad (3.26)$$

and

$$\mathbb{D}_{2m} := \sum_{j=0}^{m+1} \left\| \partial_t^j v \right\|_{2m-2j+1}^2, \quad (3.27)$$

where $\|\cdot\|_{-1}$ denotes the norm of $(H_0^1(\Omega))^*$. The existence and uniqueness of solutions to (2.6) can be stated as follows.

Theorem 3.2. *Let $m \geq 3$ be an integer, $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$ be the smoothed data constructed in Appendix A and $\Phi^{\varepsilon, \delta}$ be the corrector defined according to (3.16) and (3.17). For each $\varepsilon, \delta > 0$, there exist a $T_0^{\varepsilon, \delta} > 0$ and a unique solution $(\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta})$ to (2.6) on $[0, T_0^{\varepsilon, \delta}]$ satisfying*

$$\sup_{[0, T_0^{\varepsilon, \delta}]} \mathbb{E}_{2m}(\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta}) + \int_0^{T_0^{\varepsilon, \delta}} \mathbb{D}_{2m}(\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta}) \leq P_{\varepsilon, \delta}(\mathcal{M}_0^m). \quad (3.28)$$

Moreover, it holds that for $t \in [0, T_0^{\varepsilon, \delta}]$,

$$\rho^{\varepsilon, \delta}, p^{\varepsilon, \delta}, |\mathcal{J}^{\varepsilon, \delta}| \geq \frac{C_0}{4} > 0 \text{ in } \Omega \text{ and } |b^{\varepsilon, \delta} \cdot \mathcal{N}^{\varepsilon, \delta}| \geq \frac{C_0}{4} > 0 \text{ on } \Sigma \cup \Sigma_\pm, \quad (3.29)$$

$$\mathcal{J}^{\varepsilon, \delta} \operatorname{div}_{\mathcal{A}^{\varepsilon, \delta}} b^{\varepsilon, \delta} = J_0^\delta \operatorname{div}_{\mathcal{A}_0^\delta} b_0^\delta \text{ in } \Omega, \quad (3.30)$$

and

$$\llbracket \eta^{\varepsilon, \delta} \rrbracket = \llbracket \partial_3 \eta^{\varepsilon, \delta} \rrbracket = 0 \text{ on } \Sigma. \quad (3.31)$$

Proof. Consider first the following modified problem of (2.6):

$$\begin{cases} \partial_t \eta^{\varepsilon, \delta} = v^{\varepsilon, \delta} & \text{in } \Omega \\ \frac{1}{\gamma p^{\varepsilon, \delta}} \partial_t p^{\varepsilon, \delta} + \operatorname{div}_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = 0 & \text{in } \Omega \\ \rho^{\varepsilon, \delta} \partial_t v^{\varepsilon, \delta} + \nabla_{\mathcal{A}^{\varepsilon, \delta}} (p^{\varepsilon, \delta} + \frac{1}{2} |b^{\varepsilon, \delta}|^2) - \varepsilon \Delta_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = b^{\varepsilon, \delta} \cdot \nabla_{\mathcal{A}^{\varepsilon, \delta}} b^{\varepsilon, \delta} + \Psi^{\varepsilon, \delta} & \text{in } \Omega \\ \partial_t b^{\varepsilon, \delta} + b^{\varepsilon, \delta} \operatorname{div}_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = b^{\varepsilon, \delta} \cdot \nabla_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} & \text{in } \Omega \\ \llbracket v^{\varepsilon, \delta} \rrbracket = 0, \quad \llbracket \nabla_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} \rrbracket \mathcal{N}^{\varepsilon, \delta} = 0 & \text{on } \Sigma \\ v^{\varepsilon, \delta} = 0 & \text{on } \Sigma_\pm \\ (\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta})|_{t=0} = (\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta). \end{cases} \quad (3.32)$$

Since (1.15) and (1.17) hold also for the solution to (3.32), one can eliminate $p^{\varepsilon, \delta}$ (and $\rho^{\varepsilon, \delta}$) and $b^{\varepsilon, \delta}$ from (3.32) to reformulate it equivalently as

$$\begin{cases} \partial_t \eta^{\varepsilon, \delta} = v^{\varepsilon, \delta} & \text{in } \Omega \\ \rho_0^\delta J_0^\delta (\mathcal{J}^{\varepsilon, \delta})^{-1} \partial_t v^{\varepsilon, \delta} - \varepsilon \Delta_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} = ((\mathcal{J}^{\varepsilon, \delta})^{-1} J_0^\delta (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla)^2 \eta^{\varepsilon, \delta} + \Psi^{\varepsilon, \delta} \\ \quad - \nabla_{\mathcal{A}^{\varepsilon, \delta}} (p_0^\delta (J_0^\delta)^\gamma (\mathcal{J}^{\varepsilon, \delta})^{-\gamma} + \frac{1}{2} |(\mathcal{J}^{\varepsilon, \delta})^{-1} J_0^\delta (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla \eta^{\varepsilon, \delta}|^2) & \text{in } \Omega \\ \llbracket v^{\varepsilon, \delta} \rrbracket = 0, \quad \llbracket \nabla_{\mathcal{A}^{\varepsilon, \delta}} v^{\varepsilon, \delta} \rrbracket \mathcal{N}^{\varepsilon, \delta} = 0 & \text{on } \Sigma \\ v^{\varepsilon, \delta} = 0 & \text{on } \Sigma_\pm \\ (\eta^{\varepsilon, \delta}, v^{\varepsilon, \delta})|_{t=0} = (\eta_0^\delta, v_0^\delta). \end{cases} \quad (3.33)$$

For any fixed $\varepsilon, \delta > 0$, similarly as the compressible Navier–Stokes equations, the right hand side of the second equation in (3.33) can be easily controlled by the viscosity term within a local time interval, and so the local solution $(\eta^{\varepsilon, \delta}, v^{\varepsilon, \delta})$ to (3.33), in the functional of $\mathbb{E}_{2m} + \int_0^t \mathbb{D}_{2m}$, on $[0, T_0^{\varepsilon, \delta}]$ for some $T_0^{\varepsilon, \delta} > 0$ can be constructed by using the same scheme; the conditions (3.25) imply in particular that the corresponding $(m-1)$ -th order compatibility conditions for (3.33)

hold. We shall omit the details and refer to, for instance, [9] for the references. Then defining $p^{\varepsilon,\delta}$ (and $\rho^{\varepsilon,\delta}$) and $b^{\varepsilon,\delta}$ by (1.15) and (1.17), respectively, one sees that $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$ solves (3.32) on $[0, T_0^{\varepsilon,\delta}]$ and satisfies the estimate (3.28). The estimate (3.29) follows from the fundamental theorem of calculus, (3.14) and (3.28), by restricting $T_0^{\varepsilon,\delta}$ smaller if necessary, while the identity (3.30) follows by using (i) in Proposition 1.1.

Now to conclude the theorem, it remains to prove the following jump conditions:

$$\llbracket p^{\varepsilon,\delta} \rrbracket = \llbracket b^{\varepsilon,\delta} \rrbracket = \llbracket \partial_3 v^{\varepsilon,\delta} \rrbracket = \llbracket \eta^{\varepsilon,\delta} \rrbracket = \llbracket \partial_3 \eta^{\varepsilon,\delta} \rrbracket = 0 \text{ on } \Sigma. \quad (3.34)$$

Recall that $\{\partial_1 \eta^{\varepsilon,\delta}, \partial_2 \eta^{\varepsilon,\delta}, \mathcal{N}^{\varepsilon,\delta}\}$ is a basis of \mathbb{R}^3 . By the first jump condition in (3.32), one has that $\llbracket \eta^{\varepsilon,\delta} \rrbracket = 0$ on Σ by the first equation in (3.32) since $\llbracket \eta_0^\delta \rrbracket = 0$ on Σ and that

$$\llbracket \nabla_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \rrbracket \partial_\beta \eta^{\varepsilon,\delta} = \llbracket (\partial_\beta \eta^{\varepsilon,\delta})^T \mathcal{A}^{\varepsilon,\delta} (\nabla v^{\varepsilon,\delta})^T \rrbracket = \llbracket \partial_\beta v^{\varepsilon,\delta} \rrbracket = 0 \text{ on } \Sigma, \quad \beta = 1, 2. \quad (3.35)$$

This together with the second jump condition in (3.32) implies

$$\llbracket \nabla_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \rrbracket = 0 \text{ on } \Sigma. \quad (3.36)$$

Now, by the fourth equation in (3.32) and (3.36), one obtains

$$\partial_t \llbracket b^{\varepsilon,\delta} \rrbracket + \llbracket b^{\varepsilon,\delta} \rrbracket \operatorname{div}_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} = \llbracket b^{\varepsilon,\delta} \rrbracket \cdot \nabla_{\mathcal{A}^{\varepsilon,\delta}} v^{\varepsilon,\delta} \text{ on } \Sigma, \quad (3.37)$$

which implies $\llbracket b^{\varepsilon,\delta} \rrbracket = 0$ on Σ since $\llbracket b_0^\delta \rrbracket = 0$ on Σ . Similarly, one has $\llbracket p^{\varepsilon,\delta} \rrbracket = 0$ on Σ since $\llbracket p_0^\delta \rrbracket = 0$ on Σ . One thus concludes (3.34) similarly as in the proof of Proposition 1.2. \square

4. TRANSPORTED ESTIMATES

Now we turn to derive the uniform-in- (ε, δ) estimates for the solution $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$ to (2.6) on $[0, T_0^{\varepsilon,\delta}]$ constructed in Theorem 3.2. It should be noted that the first condition in (3.29) will be always used without mentioning explicitly. For notational simplification, we will keep only the ε -dependence of the functionals such as $\mathcal{G}_m^\varepsilon, \mathcal{E}_m^\varepsilon, \mathcal{D}_m^\varepsilon, \bar{\mathcal{D}}_m^\varepsilon$ and the δ -dependence on the data $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta), \mathcal{A}_0^\delta, J_0^\delta, \mathcal{N}_0^\delta$, but suppress the dependence of the solution on ε, δ . We may assume that $T_0^{\varepsilon,\delta} \leq 1$ and restrict $t \in [0, T_0^{\varepsilon,\delta}]$ in the following.

We begin with the transported estimates. Let $m \geq 4$. Define

$$\mathfrak{F}_m^\varepsilon := \sum_{j=0}^{m-1} \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b) \right\|_{m-j-1}^2 + \|\eta\|_m^2 + \varepsilon \|\eta\|_{1,m}^2 + \varepsilon^2 \|\eta\|_{m+1}^2. \quad (4.1)$$

Proposition 4.1. *For $t \in [0, T_0^{\varepsilon,\delta}]$ with $T_0^{\varepsilon,\delta} \leq 1$, it holds that*

$$\mathfrak{F}_m^\varepsilon(t) \lesssim \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t \mathcal{G}_m^\varepsilon(t). \quad (4.2)$$

Proof. It follows directly from the fundamental theorem of calculus, the definitions of $\mathcal{G}_m^\varepsilon$ and $\mathfrak{F}_m^\varepsilon$ and using $\partial_t \eta = v$ that

$$\mathfrak{F}_m^\varepsilon(t) \leq \mathfrak{F}_m^\varepsilon(0) + t \mathcal{G}_m^\varepsilon(t). \quad (4.3)$$

Note that, by (3.24) and (3.15),

$$\mathfrak{F}_m^\varepsilon(0) \lesssim \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \varepsilon \left\| \eta_0^\delta \right\|_{m+1}^2 \leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta}. \quad (4.4)$$

Then (4.2) follows. \square

Denote

$$\Lambda_\infty^\varepsilon(t) := P \left(\sup_{[0,t]} \mathfrak{F}_m^\varepsilon \right). \quad (4.5)$$

Then by (4.2),

$$\Lambda_\infty^\varepsilon(t) \leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t P(\mathcal{G}_m^\varepsilon(t)). \quad (4.6)$$

5. TANGENTIAL ENERGY ESTIMATES

In this section, we will derive the energy evolution estimates for the highest order tangential derivatives of (p, v, b) and the boundary regularity of η on Σ . It is more convenient to use the following equations derived from (2.6):

$$\begin{cases} \frac{1}{\gamma p} \partial_t q - \frac{1}{\gamma p} b \cdot \partial_t b + \operatorname{div}_{\mathcal{A}} v = 0 & \text{in } \Omega \\ \rho \partial_t v + \nabla_{\mathcal{A}} q - b \cdot \nabla_{\mathcal{A}} b - \varepsilon \Delta_{\mathcal{A}} v = \Psi^{\varepsilon, \delta} & \text{in } \Omega \\ \partial_t b - \frac{b}{\gamma p} \partial_t q + \frac{b}{\gamma p} b \cdot \partial_t b - b \cdot \nabla_{\mathcal{A}} v = 0 & \text{in } \Omega, \end{cases} \quad (5.1)$$

where $q = p + \frac{1}{2}|b|^2$ is the total pressure. Recall the boundary conditions:

$$[[q]] = [[b]] = [[v]] = [[\partial_3 v]] = [[\eta]] = [[\partial_3 \eta]] = 0 \text{ on } \Sigma, \quad v = 0 \text{ and } \eta_3 = \pm 1 \text{ on } \Sigma_{\pm}. \quad (5.2)$$

One considers first the estimates of the highest order tangential spatial derivatives. Let Z^m be any Z^α for $\alpha \in \mathbb{R}^3$ with $|\alpha| = m$. To commute Z^m with each term in (5.1), it is useful to establish the following general expressions. Recall the commutator notations (2.31) and (2.32). For $i = 1, 2, 3$, one has

$$Z^m(\partial_i^{\mathcal{A}} f) = \partial_i^{\mathcal{A}} Z^m f + \mathcal{A}_{i3} [Z^m, \partial_3] f + Z^m \mathcal{A}_{ij} \partial_j f + [Z^m, \mathcal{A}_{ij}, \partial_j f] \quad (5.3)$$

and $Z \mathcal{A}_{ij} = -\mathcal{A}_{i\ell} Z \partial_\ell \eta_k \mathcal{A}_{kj}$ implies

$$\begin{aligned} Z^m \mathcal{A}_{ij} \partial_j f &= -\mathcal{A}_{i\ell} \partial_\ell Z^m \eta_k \mathcal{A}_{kj} \partial_j f - \mathcal{A}_{i3} [Z^m, \partial_3] \eta_k \mathcal{A}_{kj} \partial_j f - [Z^{m-1}, \mathcal{A}_{i\ell} \mathcal{A}_{kj}] Z \partial_\ell \eta_k \partial_j f \\ &= -\partial_i^{\mathcal{A}} (Z^m \eta \cdot \nabla_{\mathcal{A}} f) + Z^m \eta \cdot \nabla_{\mathcal{A}} (\partial_i^{\mathcal{A}} f) - \mathcal{A}_{i3} [Z^m, \partial_3] \eta_k \mathcal{A}_{kj} \partial_j f \\ &\quad - [Z^{m-1}, \mathcal{A}_{i\ell} \mathcal{A}_{kj}] Z \partial_\ell \eta_k \partial_j f. \end{aligned} \quad (5.4)$$

It then holds that

$$Z^m(\partial_i^{\mathcal{A}} f) = \partial_i^{\mathcal{A}} (Z^m f - Z^m \eta \cdot \nabla_{\mathcal{A}} f) + \mathcal{C}_i^m(f), \quad (5.5)$$

where

$$\begin{aligned} \mathcal{C}_i^m(f) &= \mathcal{A}_{i3} [Z^m, \partial_3] f + Z^m \eta \cdot \nabla_{\mathcal{A}} (\partial_i^{\mathcal{A}} f) - \mathcal{A}_{i3} [Z^m, \partial_3] \eta_k \mathcal{A}_{kj} \partial_j f \\ &\quad - [Z^{m-1}, \mathcal{A}_{i\ell} \mathcal{A}_{kj}] Z \partial_\ell \eta_k \partial_j f + [Z^m, \mathcal{A}_{ij}, \partial_j f]. \end{aligned} \quad (5.6)$$

It was first observed by Alinhac [1] that the highest order term of η will be cancelled when one uses the good unknown $Z^m f - Z^m \eta \cdot \nabla_{\mathcal{A}} f$, which allows one to perform high order energy estimates.

Lemma 5.1. *It holds that*

$$\|\mathcal{C}^m(f)\|_0 \leq P(\|\eta\|_m) \|f\|_m. \quad (5.7)$$

Proof. First, using Sobolev's embedding theorem, one has that since $m \geq 4$,

$$\begin{aligned} &\|\mathcal{A}_{i3} [Z^m, \partial_3] f\|_0 + \|Z^m \eta \cdot \nabla_{\mathcal{A}} (\partial_i^{\mathcal{A}} f)\|_0 + \|\mathcal{A}_{i3} [Z^m, \partial_3] \eta_k \mathcal{A}_{kj} \partial_j f\|_0 \\ &\leq \|\mathcal{A}_{i3}\|_{L^\infty} \|[Z^m, \partial_3] f\|_0 + \|Z^m \eta\|_0 \|\nabla_{\mathcal{A}} (\partial_i^{\mathcal{A}} f)\|_{L^\infty} + \|[Z^m, \partial_3] \eta_k\|_0 \|\mathcal{A}_{i3} \mathcal{A}_{kj} \partial_j f\|_{L^\infty} \\ &\leq P(\|\eta\|_m) \|f\|_m. \end{aligned} \quad (5.8)$$

By the standard commutator estimates, one obtains

$$\|[Z^m, \mathcal{A}_{ij}, \partial_j f]\|_0 \lesssim \|Z \mathcal{A}\|_{0, m-2} \|Z \nabla f\|_{L^\infty} + \|Z \mathcal{A}\|_{L^\infty} \|Z \nabla f\|_{0, m-2} \leq P(\|\eta\|_m) \|f\|_m \quad (5.9)$$

and

$$\|[Z^{m-1}, \mathcal{A}_{i\ell} \mathcal{A}_{kj}] Z \partial_\ell \eta_k \partial_j f\|_0 \leq \|[Z^{m-1}, \mathcal{A}_{i\ell} \mathcal{A}_{kj}] Z \partial_\ell \eta_k\|_0 \|\nabla f\|_{L^\infty} \leq P(\|\eta\|_m) \|f\|_3. \quad (5.10)$$

Then (5.7) follows. \square

Define the good unknowns:

$$\mathcal{V}^m = Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v, \quad \mathcal{Q}^m = Z^m q - Z^m \eta \cdot \nabla_{\mathcal{A}} q \quad (5.11)$$

and

$$\mathcal{B}^m = Z^m b - Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} b. \quad (5.12)$$

Lemma 5.2. *It holds that*

$$\begin{cases} \frac{1}{\gamma p} \partial_t \mathcal{Q}^m - \frac{1}{\gamma p} b \cdot \partial_t \mathcal{B}^m + \operatorname{div}_{\mathcal{A}} \mathcal{V}^m = F^{1,m} & \text{in } \Omega \\ \rho \partial_t \mathcal{V}^m + \nabla_{\mathcal{A}} \mathcal{Q}^m - b \cdot \nabla_{\mathcal{A}} \mathcal{B}^m - \varepsilon \Delta_{\mathcal{A}} \mathcal{V}^m = F^{2,m} + Z^m \Psi^{\varepsilon, \delta} & \text{in } \Omega \\ \partial_t \mathcal{B}^m - \frac{b}{\gamma p} \partial_t \mathcal{Q}^m + \frac{b}{\gamma p} b \cdot \partial_t \mathcal{B}^m - b \cdot \nabla_{\mathcal{A}} \mathcal{V}^m = F^{3,m} & \text{in } \Omega, \end{cases} \quad (5.13)$$

where $F^{1,m}$, $F^{2,m}$ and $F^{3,m}$ are defined by (5.22)–(5.24), respectively, and that

$$\begin{aligned} \llbracket \mathcal{Q}^m \rrbracket &= -J^{-1} Z^m \eta \cdot \mathcal{N} \llbracket \partial_3 q \rrbracket, \quad \llbracket \mathcal{B}^m \rrbracket = -(J_0^\delta)^{-1} Z^m \eta_0^\delta \cdot \mathcal{N}_0^\delta \llbracket \partial_3 b \rrbracket, \\ \llbracket \mathcal{V}^m \rrbracket &= 0, \quad \llbracket \partial_3 \mathcal{V}^m \rrbracket = -Z^m \eta \cdot \llbracket \partial_3 (\nabla_{\mathcal{A}} v) \rrbracket \text{ on } \Sigma \text{ and } \mathcal{V}^m = 0 \text{ on } \Sigma_{\pm}. \end{aligned} \quad (5.14)$$

Moreover,

$$\|F^{1,m}\|_0^2 + \|F^{2,m}\|_0^2 + \|F^{3,m}\|_0^2 \leq \Lambda_\infty^\varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon). \quad (5.15)$$

Proof. First, (5.5) and (5.11) imply

$$Z^m \operatorname{div}_{\mathcal{A}} v = \operatorname{div}_{\mathcal{A}} \mathcal{V}^m + \mathcal{C}_i^m(v_i), \quad (5.16)$$

$$Z^m \nabla_{\mathcal{A}} q = \nabla_{\mathcal{A}} \mathcal{Q}^m + \mathcal{C}^m(q), \quad (5.17)$$

$$\begin{aligned} Z^m \Delta_{\mathcal{A}} v &= Z^m \partial_i^{\mathcal{A}} \partial_i^{\mathcal{A}} v = \partial_i^{\mathcal{A}} (Z^m \partial_i^{\mathcal{A}} v - Z^m \eta \cdot \nabla_{\mathcal{A}} \partial_i^{\mathcal{A}} v) + \mathcal{C}_i^m(\partial_i^{\mathcal{A}} v) \\ &= \Delta_{\mathcal{A}} \mathcal{V}^m + \partial_i^{\mathcal{A}} (\mathcal{C}_i^m(v) - Z^m \eta \cdot \nabla_{\mathcal{A}} \partial_i^{\mathcal{A}} v) + \mathcal{C}_i^m(\partial_i^{\mathcal{A}} v) \end{aligned} \quad (5.18)$$

and

$$Z^m (b \cdot \nabla_{\mathcal{A}} v) = [Z^m, b] \cdot \nabla_{\mathcal{A}} v + b \cdot Z^m (\nabla_{\mathcal{A}} v) = [Z^m, b] \cdot \nabla_{\mathcal{A}} v + b \cdot \nabla_{\mathcal{A}} \mathcal{V}^m + b \cdot \mathcal{C}^m(v). \quad (5.19)$$

Using the Cauchy formula (1.17) and (1.15), one has

$$b \cdot \nabla_{\mathcal{A}} \equiv \mathcal{A}^T b \cdot \nabla = J^{-1} J_0^\delta (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla = \rho (\rho_0^\delta)^{-1} b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta}. \quad (5.20)$$

Then denoting $\mathcal{C}_0^{m,\delta}$ for \mathcal{C}^m in (5.6) with η replaced by η_0^δ , by (5.20), (5.5) and (5.12), one obtains

$$\begin{aligned} Z^m (b \cdot \nabla_{\mathcal{A}} b) &= [Z^m, \rho (\rho_0^\delta)^{-1} b_0^\delta] \cdot \nabla_{\mathcal{A}_0^\delta} b + \rho (\rho_0^\delta)^{-1} b_0^\delta \cdot Z^m (\nabla_{\mathcal{A}_0^\delta} b) \\ &= [Z^m, \rho (\rho_0^\delta)^{-1} b_0^\delta] \cdot \nabla_{\mathcal{A}_0^\delta} b + b \cdot \nabla_{\mathcal{A}} \mathcal{B}^m + \rho (\rho_0^\delta)^{-1} b_0^\delta \cdot \mathcal{C}_0^{m,\delta}(b). \end{aligned} \quad (5.21)$$

Hence, applying Z^m to (5.1) implies (5.13) with

$$\begin{aligned} F^{1,m} &= -[Z^m, \frac{1}{\gamma p}] \partial_t q + [Z^m, \frac{1}{\gamma p} b] \cdot \partial_t b - \frac{1}{\gamma p} \partial_t (Z^m \eta \cdot \nabla_{\mathcal{A}} q) \\ &\quad + \frac{1}{\gamma p} b \cdot (Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \partial_t b) - \mathcal{C}_i^m(v_i), \end{aligned} \quad (5.22)$$

$$\begin{aligned} F^{2,m} &= -[Z^m, \rho] \partial_t v - \rho \partial_t (Z^m \eta \cdot \nabla_{\mathcal{A}} v) - \mathcal{C}^m(q) + [Z^m, \rho (\rho_0^\delta)^{-1} b_0^\delta] \cdot \nabla_{\mathcal{A}_0^\delta} b \\ &\quad + \rho (\rho_0^\delta)^{-1} b_0^\delta \cdot \mathcal{C}_0^{m,\delta}(b) + \varepsilon \partial_i^{\mathcal{A}} (\mathcal{C}_i^m(v) - Z^m \eta \cdot \nabla_{\mathcal{A}} \partial_i^{\mathcal{A}} v) + \varepsilon \mathcal{C}_i^m(\partial_i^{\mathcal{A}} v) \end{aligned} \quad (5.23)$$

and

$$\begin{aligned} F^{3,m} &= [Z^m, \frac{b}{\gamma p}] \partial_t q - [Z^m, \frac{b}{\gamma p} b] \cdot \partial_t b - Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \partial_t b + \frac{b}{\gamma p} \partial_t (Z^m \eta \cdot \nabla_{\mathcal{A}} q) \\ &\quad - \frac{b}{\gamma p} b \cdot (Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \partial_t b) + [Z^m, b] \cdot \nabla_{\mathcal{A}} v + b \cdot \mathcal{C}^m(v). \end{aligned} \quad (5.24)$$

Next, (5.2) and (5.11)–(5.12) imply

$$\llbracket \mathcal{Q}^m \rrbracket = -Z^m \eta \cdot \llbracket \nabla_{\mathcal{A}} q \rrbracket = -Z^m \eta_j \mathcal{A}_{j3} \llbracket \partial_3 q \rrbracket = -J^{-1} Z^m \eta \cdot \mathcal{N} \llbracket \partial_3 q \rrbracket \text{ on } \Sigma, \quad (5.25)$$

$$\llbracket \mathcal{B}^m \rrbracket = -Z^m \eta_0^\delta \cdot \llbracket \nabla_{\mathcal{A}_0^\delta} b \rrbracket = -(J_0^\delta)^{-1} Z^m \eta_0^\delta \cdot \mathcal{N}_0^\delta \llbracket \partial_3 b \rrbracket \text{ on } \Sigma, \quad (5.26)$$

$$\llbracket \mathcal{V}^m \rrbracket = 0, \quad \llbracket \partial_3 \mathcal{V}^m \rrbracket = -\llbracket \partial_3 (Z^m \eta \cdot \nabla_{\mathcal{A}} v) \rrbracket = -Z^m \eta \cdot \llbracket \partial_3 (\nabla_{\mathcal{A}} v) \rrbracket \text{ on } \Sigma \quad (5.27)$$

and

$$\mathcal{V}^m = -Z^m \eta_j \mathcal{A}_{j3} \partial_3 v = -J^{-1} Z^m \eta_3 \partial_3 v = 0 \text{ on } \Sigma_{\pm}. \quad (5.28)$$

Thus (5.14) follows.

Now one estimates $F^{1,m}$, $F^{2,m}$ and $F^{3,m}$. It follows similarly as for (5.7) and the definitions (2.13) and (2.15) of $\mathfrak{E}_m^\varepsilon$ and $\mathfrak{D}_m^\varepsilon$ that, since $\partial_t \eta = v$ and $m \geq 4$,

$$\|F^{1,m}\|_0^2 + \|F^{3,m}\|_0^2 \leq \Lambda_\infty^\varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon) \quad (5.29)$$

and

$$\|F^{2,m}\|_0^2 \leq \Lambda_\infty^\varepsilon \left(\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon + \varepsilon^2 \|\eta\|_{m+1}^2 + \varepsilon^2 \|v\|_{m+1}^2 \right) \leq \Lambda_\infty^\varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon). \quad (5.30)$$

One then concludes (5.15). \square

Now we state the estimates of the highest order tangential spatial derivatives.

Proposition 5.3. *For $t \in [0, T_0^{\varepsilon, \delta}]$ with $T_0^{\varepsilon, \delta} \leq 1$, it holds that*

$$\|(p, v, b)(t)\|_{0,m}^2 + |\eta(t)|_m^2 + \varepsilon \int_0^t \|v\|_{1,m}^2 \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\varepsilon(t) \right). \quad (5.31)$$

Proof. Taking the $L^2(\Omega)$ inner product of the equations in (5.13) with $J\mathcal{Q}^m$, $J\mathcal{V}^m$ and $J\mathcal{B}^m$, respectively, adding the resulting together, and then integrating by parts in t and by a simple combination, one deduces that, since $\partial_t(\rho J) = 0$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega J \left(\frac{1}{\gamma p} |\mathcal{Q}^m - b \cdot \mathcal{B}^m|^2 + \rho |\mathcal{V}^m|^2 + |\mathcal{B}^m|^2 \right) \\ & + \int_\Omega J \operatorname{div}_{\mathcal{A}}(\mathcal{Q}^m \mathcal{V}^m) - \int_\Omega J b \cdot \nabla_{\mathcal{A}}(\mathcal{B}^m \cdot \mathcal{V}^m) - \varepsilon \int_\Omega J \Delta_{\mathcal{A}} \mathcal{V}^m \cdot \mathcal{V}^m \\ & = \int_\Omega \left(\frac{1}{2} \partial_t \left(\frac{J}{\gamma p} \right) (|\mathcal{Q}^m|^2 + |b \cdot \mathcal{B}^m|^2) + \frac{J}{\gamma p} \partial_t b \cdot \mathcal{B}^m b \cdot \mathcal{B}^m + \frac{1}{2} \partial_t J |\mathcal{B}^m|^2 - \partial_t \left(\frac{Jb}{\gamma p} \right) \cdot \mathcal{B}^m \mathcal{Q}^m \right) \\ & + \int_\Omega J (F^{1,m} \mathcal{Q}^m + F^{2,m} \cdot \mathcal{V}^m + F^{3,m} \cdot \mathcal{B}^m) + \int_\Omega J Z^m \Psi^{\varepsilon, \delta} \cdot \mathcal{V}^m. \end{aligned} \quad (5.32)$$

By the definitions of \mathcal{Q}^m , \mathcal{B}^m and \mathcal{V}^m ,

$$\begin{aligned} & \int_\Omega \left(\frac{1}{2} \partial_t \left(\frac{J}{\gamma p} \right) (|\mathcal{Q}^m|^2 + |b \cdot \mathcal{B}^m|^2) + \frac{J}{\gamma p} \partial_t b \cdot \mathcal{B}^m b \cdot \mathcal{B}^m + \frac{1}{2} \partial_t J |\mathcal{B}^m|^2 - \partial_t \left(\frac{Jb}{\gamma p} \right) \cdot \mathcal{B}^m \mathcal{Q}^m \right) \\ & \leq \Lambda_\infty^\varepsilon \left(\|\mathcal{Q}^m\|_0^2 + \|\mathcal{B}^m\|_0^2 + \|\mathcal{B}^m\|_0 \|\mathcal{Q}^m\|_0 \right) \leq \Lambda_\infty^\varepsilon \mathfrak{E}_m^\varepsilon, \end{aligned} \quad (5.33)$$

and by (5.15) and (3.19),

$$\begin{aligned} & \int_\Omega J (F^{1,m} \mathcal{Q}^m + F^{2,m} \cdot \mathcal{V}^m + Z^m \Psi^{\varepsilon, \delta} \cdot \mathcal{V}^m + F^{3,m} \cdot \mathcal{B}^m) \\ & \leq \Lambda_\infty^\varepsilon \left(\|F^{1,m}\|_0 \|\mathcal{Q}^m\|_0 + \|F^{2,m}\|_0 \|\mathcal{V}^m\|_0 + \|Z^m \Psi^{\varepsilon, \delta}\|_0 \|\mathcal{V}^m\|_0 + \|F^{3,m}\|_0 \|\mathcal{B}^m\|_0 \right) \\ & \leq \Lambda_\infty^\varepsilon \left(\sqrt{\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon} \sqrt{\mathfrak{E}_m^\varepsilon} + \varepsilon \mathcal{M}_0^{m,\delta} \sqrt{\mathfrak{E}_m^\varepsilon} \right). \end{aligned} \quad (5.34)$$

Now we turn to estimate the left hand side of (5.32). First, integrating by parts over Ω_{\pm} and using (5.14), one obtains

$$\begin{aligned} -\varepsilon \int_\Omega J \Delta_{\mathcal{A}} \mathcal{V}^m \cdot \mathcal{V}^m & = \varepsilon \int_\Sigma \llbracket \nabla_{\mathcal{A}} \mathcal{V}^m \rrbracket \mathcal{N} \cdot \mathcal{V}^m + \varepsilon \int_\Omega J |\nabla_{\mathcal{A}} \mathcal{V}^m|^2 \\ & = -\varepsilon \int_\Sigma J^{-1} |\mathcal{N}|^2 Z^m \eta \cdot \llbracket \partial_3(\nabla_{\mathcal{A}} v_i) \rrbracket \mathcal{V}_i^m + \varepsilon \int_\Omega J |\nabla_{\mathcal{A}} \mathcal{V}^m|^2. \end{aligned} \quad (5.35)$$

By the trace theory, one gets

$$\varepsilon \int_\Sigma J^{-1} |\mathcal{N}|^2 Z^m \eta \cdot \llbracket \partial_3(\nabla_{\mathcal{A}} v_i) \rrbracket \mathcal{V}_i^m \leq \varepsilon \Lambda_\infty^\varepsilon |Z^m \eta|_0 |\mathcal{V}^m|_0 \leq \varepsilon \Lambda_\infty^\varepsilon \sqrt{\mathfrak{E}_m^\varepsilon} \|\mathcal{V}^m\|_1. \quad (5.36)$$

Next, one estimates the most delicate remaining two terms. Integrating by parts over Ω_{\pm} and using (5.14), one has

$$-\int_{\Omega} J \operatorname{div}_{\mathcal{A}}(\mathcal{Q}^m \mathcal{V}^m) = \int_{\Sigma} \llbracket \mathcal{Q}^m \rrbracket \mathcal{V}^m \cdot \mathcal{N} = -\int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} \mathcal{V}^m \cdot \mathcal{N}. \quad (5.37)$$

By the definition of \mathcal{V}^m and since $\partial_t \eta = v$, integrating by parts in t yields

$$\begin{aligned} & -\int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} \mathcal{V}^m \cdot \mathcal{N} = -\int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} (Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v) \cdot \mathcal{N} \\ & = -\int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} Z^m \partial_t \eta \cdot \mathcal{N} + \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} (Z^m \eta \cdot \nabla_{\mathcal{A}} v) \cdot \mathcal{N} \\ & = -\frac{1}{2} \frac{d}{dt} \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \frac{1}{2} \int_{\Sigma} \partial_t (\llbracket \partial_3 q \rrbracket J^{-1}) |Z^m \eta \cdot \mathcal{N}|^2 \\ & \quad + \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} Z^m \eta \cdot \partial_t \mathcal{N} + \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} Z^m \eta \cdot \mathcal{N} (Z^m \eta \cdot \nabla_{\mathcal{A}} v) \cdot \mathcal{N} \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \Lambda_{\infty}^{\varepsilon} |Z^m \eta|_0^2 \\ & \leq -\frac{1}{2} \frac{d}{dt} \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \Lambda_{\infty}^{\varepsilon} \mathfrak{E}_m^{\varepsilon}. \end{aligned} \quad (5.38)$$

Similarly, by using the Piola identity, Proposition 1.1 and (5.14), one deduces

$$\begin{aligned} & \int_{\Omega} J b \cdot \nabla_{\mathcal{A}}(\mathcal{B}^m \cdot \mathcal{V}^m) = -\int_{\Sigma} b \cdot \mathcal{N} \llbracket \mathcal{B}^m \rrbracket \cdot \mathcal{V}^m - \int_{\Omega} J \operatorname{div}_{\mathcal{A}} b \mathcal{B}^m \cdot \mathcal{V}^m \\ & = -\int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \mathcal{B}^m \rrbracket \cdot (Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v) - \int_{\Omega} J_0^{\delta} \operatorname{div}_{\mathcal{A}_0^{\delta}} b_0^{\delta} \mathcal{B}^m \cdot \mathcal{V}^m \\ & = \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot (Z^m \partial_t \eta - Z^m \eta \cdot \nabla_{\mathcal{A}} v) - \int_{\Omega} J_0^{\delta} \operatorname{div}_{\mathcal{A}_0^{\delta}} b_0^{\delta} \mathcal{B}^m \cdot \mathcal{V}^m \\ & = \frac{d}{dt} \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot Z^m \eta - \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 \partial_t b \rrbracket \cdot Z^m \eta \\ & \quad - \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot Z^m \eta \cdot \nabla_{\mathcal{A}} v - \int_{\Omega} J_0^{\delta} \operatorname{div}_{\mathcal{A}_0^{\delta}} b_0^{\delta} \mathcal{B}^m \cdot \mathcal{V}^m \\ & \leq \frac{d}{dt} \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot Z^m \eta + \Lambda_{\infty}^{\varepsilon} \left(|Z^m \eta_0^{\delta}|_0 |Z^m \eta|_0 + \|\mathcal{B}^m\|_0 \|\mathcal{V}^m\|_0 \right) \\ & \leq \frac{d}{dt} \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot Z^m \eta + \Lambda_{\infty}^{\varepsilon} \left(\sqrt{\mathfrak{E}_m^{\varepsilon}(0)} \sqrt{\mathfrak{E}_m^{\varepsilon}} + \mathfrak{E}_m^{\varepsilon} \right). \end{aligned} \quad (5.39)$$

As a consequence of the estimates (5.33)–(5.39), one deduces from (5.32) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} J \left(\frac{1}{\gamma p} |\mathcal{Q}^m - b \cdot \mathcal{B}^m|^2 + \rho |\mathcal{V}^m|^2 + |\mathcal{B}^m|^2 \right) + \varepsilon \int_{\Omega} J |\nabla_{\mathcal{A}} \mathcal{V}^m|^2 \\ & \leq \frac{d}{dt} \mathcal{I}_m + \Lambda_{\infty}^{\varepsilon} \left(\sqrt{\mathfrak{E}_m^{\varepsilon}(0)} + \mathfrak{E}_m^{\varepsilon} + \mathfrak{D}_m^{\varepsilon} \sqrt{\mathfrak{E}_m^{\varepsilon}} + \varepsilon \mathcal{M}_0^{m,\delta} \sqrt{\mathfrak{E}_m^{\varepsilon}} + \varepsilon \sqrt{\mathfrak{E}_m^{\varepsilon}} \|\mathcal{V}^m\|_1 \right), \end{aligned} \quad (5.40)$$

where

$$\begin{aligned} \mathcal{I}_m & := -\frac{1}{2} \int_{\Sigma} \llbracket \partial_3 q \rrbracket J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \int_{\Sigma} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} (J_0^{\delta})^{-1} Z^m \eta_0^{\delta} \cdot \mathcal{N}_0^{\delta} \llbracket \partial_3 b \rrbracket \cdot Z^m \eta \\ & \leq \Lambda_{\infty}^{\varepsilon} \left(|Z^m \eta|_0^2 + |Z^m \eta_0^{\delta}|_0 |Z^m \eta|_0 \right). \end{aligned} \quad (5.41)$$

Integrating (5.40) directly in time, by (5.41), (3.15) and using the following inequality

$$\|\nabla f\|_0^2 \lesssim \int_{\Omega} J |\nabla_{\mathcal{A}} f|^2, \quad (5.42)$$

Poincaré's inequality, Cauchy's inequality and the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
& \|(\mathcal{Q}^m, \mathcal{V}^m, \mathcal{B}^m)(t)\|_0^2 + \varepsilon \int_0^t \|\mathcal{V}^m\|_1^2 \\
& \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon^2 \mathcal{M}_0^{m,\delta} + |Z^m \eta(t)|_0^2 + \int_0^t \sqrt{\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon} \sqrt{\mathfrak{E}_m^\varepsilon} \right) \\
& \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon^2 \mathcal{M}_0^{m,\delta} + |Z^m \eta(t)|_0^2 + t \mathcal{G}_m^\varepsilon(t) + \sqrt{\mathcal{G}_m^\varepsilon(t)} \int_0^t \sqrt{\mathfrak{D}_m^\varepsilon} \right) \\
& \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon^2 \mathcal{M}_0^{m,\delta} + |Z^m \eta(t)|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon(t) \right). \tag{5.43}
\end{aligned}$$

By the definitions of good unknowns again, (5.43) and (4.2), one deduces

$$\begin{aligned}
& \|(Z^m p, Z^m v, Z^m b)(t)\|_0^2 + \varepsilon \int_0^t \|Z^m v\|_1^2 \\
& \leq \Lambda_\infty^\varepsilon(t) \left(\mathfrak{F}_m^\varepsilon(t) + \|(\mathcal{Q}^m, \mathcal{V}^m, \mathcal{B}^m)(t)\|_0^2 + \|Z^m \eta(t)\|_0^2 + \varepsilon \int_0^t \left(\|\mathcal{V}^m\|_1^2 + \|Z^m \eta\|_1^2 \right) \right) \\
& \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + |Z^m \eta(t)|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon(t) \right). \tag{5.44}
\end{aligned}$$

The key point here now is to use the term $\|(Z^m p, Z^m b)\|_0^2$ in the left hand side of (5.44) to control the term $|Z^m \eta|_0^2$ in the right hand side. Recalling the Cauchy formula (1.17), then by (5.44) and (4.2), one has

$$\begin{aligned}
& \left\| Z^m (b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta) \right\|_0^2 = \left\| Z^m (\rho_0^\delta \rho^{-1} b) \right\|_0^2 \leq \Lambda_\infty^\varepsilon \left(\mathfrak{F}_m^\varepsilon + \|(Z^m p, Z^m b)\|_0^2 \right) \\
& \leq \Lambda_\infty^\varepsilon \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + |Z^m \eta|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon \right). \tag{5.45}
\end{aligned}$$

Introducing further the good unknown

$$\Xi^m := Z^m \eta - Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta, \tag{5.46}$$

then by (5.5) and (5.46), one obtains

$$\begin{aligned}
Z^m (b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta) &= \left[Z^m, b_0^\delta \right] \cdot \nabla_{\mathcal{A}_0^\delta} \eta + b_0^\delta \cdot Z^m \nabla_{\mathcal{A}_0^\delta} \eta \\
&= \left[Z^m, b_0^\delta \right] \cdot \nabla_{\mathcal{A}_0^\delta} \eta + b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \Xi^m + b_0^\delta \cdot \mathcal{C}_0^{m,\delta}(\eta). \tag{5.47}
\end{aligned}$$

Hence, (5.47) and (5.45) imply that, similarly as for (5.7),

$$\begin{aligned}
\left\| b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \Xi^m \right\|_0^2 &\leq \left\| Z^m (b_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta) \right\|_0^2 + \left\| \left[Z^m, b_0^\delta \right] \cdot \nabla_{\mathcal{A}_0^\delta} \eta \right\|_0^2 + \left\| b_0^\delta \cdot \mathcal{C}_0^{m,\delta}(\eta) \right\|_0^2 \\
&\leq \Lambda_\infty^\varepsilon \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + |Z^m \eta|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon(t) \right). \tag{5.48}
\end{aligned}$$

Now, since $((\mathcal{A}_0^\delta)^T b_0^\delta)_3 \neq 0$ near Σ , it is crucial to apply Proposition B.1 with $\bar{B} = (\mathcal{A}_0^\delta)^T b_0^\delta$ and $f = \Xi^m$ to have

$$|\Xi^m|_0^2 \lesssim \left\| (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla \Xi^m \right\|_0 \|\Xi^m\|_0 + \|\Xi^m\|_0^2. \tag{5.49}$$

Then by (5.45), (5.49), (5.48), (3.15) and using Cauchy's inequality, one has that for any $\epsilon > 0$,

$$\begin{aligned}
|Z^m \eta|_0^2 &\leq |\Xi^m|_0^2 + \left| Z^m \eta_0^\delta \cdot \nabla_{\mathcal{A}_0^\delta} \eta \right|_0^2 \\
&\lesssim \left\| (\mathcal{A}_0^\delta)^T b_0^\delta \cdot \nabla \Xi^m \right\|_0 \|\Xi^m\|_0 + \|\Xi^m\|_0^2 + \Lambda_\infty^\varepsilon \left| Z^m \eta_0^\delta \right|_0^2 \\
&\lesssim \sqrt{\Lambda_\infty^\varepsilon \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + |Z^m \eta|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon(t) \right)} \|\Xi^m\|_0 + \|\Xi^m\|_0^2 + \Lambda_\infty^\varepsilon \mathcal{M}_0^m \\
&\leq \epsilon |Z^m \eta|_0^2 + C_\epsilon \Lambda_\infty^\varepsilon \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \|\Xi^m\|_0^2 + t^{1/2} \mathcal{G}_m^\varepsilon \right) \\
&\leq \epsilon |Z^m \eta|_0^2 + C_\epsilon \Lambda_\infty^\varepsilon \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\varepsilon \right), \tag{5.50}
\end{aligned}$$

which implies, by taking $\epsilon > 0$ sufficiently small,

$$|Z^m \eta|_0^2 \leq \Lambda_\infty^\epsilon \left(\mathcal{M}_0^m + \epsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\epsilon \right). \quad (5.51)$$

Therefore, by (5.51), one obtains from (5.44) that

$$\|(Z^m p, Z^m v, Z^m b)(t)\|_0^2 + \epsilon \int_0^t \|Z^m v\|_1^2 \leq \Lambda_\infty^\epsilon(t) \left(\mathcal{M}_0^m + \epsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\epsilon(t) \right). \quad (5.52)$$

This gives (5.31) by recalling (4.2). \square

Now we consider the estimates of the rest of highest order tangential derivatives. Let $\alpha \in \mathbb{N}^{1+3}$ be with $|\alpha| = m$ and $\alpha_0 \geq 1$, and one applies \bar{Z}^α to (5.1) to find

$$\begin{cases} \frac{1}{\gamma p} \partial_t \bar{Z}^\alpha q - \frac{1}{\gamma p} b \cdot \partial_t \bar{Z}^\alpha b + \operatorname{div}_{\mathcal{A}} \bar{Z}^\alpha v = F^{1,\alpha} & \text{in } \Omega \\ \rho \partial_t \bar{Z}^\alpha v + \nabla_{\mathcal{A}} \bar{Z}^\alpha q - b \cdot \nabla_{\mathcal{A}} \bar{Z}^\alpha b - \epsilon \Delta_{\mathcal{A}} \bar{Z}^\alpha v = F^{2,\alpha} + \bar{Z}^\alpha \Psi^{\epsilon,\delta} & \text{in } \Omega \\ \partial_t \bar{Z}^\alpha b - \frac{b}{\gamma p} \partial_t \bar{Z}^\alpha q + \frac{b}{\gamma p} b \cdot \partial_t \bar{Z}^\alpha b - b \cdot \nabla_{\mathcal{A}} \bar{Z}^\alpha v = F^{3,\alpha} & \text{in } \Omega, \end{cases} \quad (5.53)$$

where

$$F^{1,\alpha} = - \left[\bar{Z}^\alpha, \frac{1}{\gamma p} \right] \partial_t q + \left[\bar{Z}^\alpha, \frac{1}{\gamma p} b \right] \cdot \partial_t b - [\bar{Z}^\alpha, \operatorname{div}_{\mathcal{A}}] v, \quad (5.54)$$

$$F^{2,\alpha} = - [\bar{Z}^\alpha, \rho] \partial_t v - [\bar{Z}^\alpha, \nabla_{\mathcal{A}}] q + [\bar{Z}^\alpha, b \cdot \nabla_{\mathcal{A}}] b + \epsilon [\bar{Z}^\alpha, \Delta_{\mathcal{A}}] v \quad (5.55)$$

and

$$F^{3,\alpha} = \left[\bar{Z}^\alpha, \frac{b}{\gamma p} \right] \partial_t q - \left[\bar{Z}^\alpha, \frac{b}{\gamma p} b \right] \cdot \partial_t b + [\bar{Z}^\alpha, b \cdot \nabla_{\mathcal{A}}] v. \quad (5.56)$$

Lemma 5.4. *It holds that*

$$\|F^{1,\alpha}\|_0^2 + \|F^{2,\alpha}\|_0^2 + \|F^{3,\alpha}\|_0^2 \leq \Lambda_\infty^\epsilon (\mathfrak{E}_m^\epsilon + \mathfrak{D}_m^\epsilon). \quad (5.57)$$

Proof. (5.57) follows similarly as for (5.15). \square

Now we estimate such derivatives.

Proposition 5.5. *For $t \in [0, T_0^{\epsilon,\delta}]$ with $T_0^{\epsilon,\delta} \leq 1$, it holds that*

$$\begin{aligned} & \sum_{j=1}^m \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b)(t) \right\|_{0,m-j}^2 + \epsilon \int_0^t \sum_{j=1}^m \left\| \partial_t^j v \right\|_{1,m-j}^2 \\ & \leq \Lambda_\infty^\epsilon(t) \left(\mathcal{M}_0^m + \epsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\epsilon(t) \right). \end{aligned} \quad (5.58)$$

Proof. Let $\alpha \in \mathbb{N}^{1+3}$ be with $|\alpha| = m$ and $\alpha_0 \geq 1$. Taking the $L^2(\Omega)$ inner product of the equations in (5.53) for such α with $J\bar{Z}^\alpha q$, $J\bar{Z}^\alpha v$ and $J\bar{Z}^\alpha b$, respectively, similarly as for (5.32), one obtains

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} J \left(\frac{1}{\gamma p} |\bar{Z}^\alpha q - b \cdot \bar{Z}^\alpha b|^2 + \rho |\bar{Z}^\alpha v|^2 + |\bar{Z}^\alpha b|^2 \right) \\ & + \int_{\Omega} J \operatorname{div}_{\mathcal{A}} (\bar{Z}^\alpha q \cdot \bar{Z}^\alpha v) - \int_{\Omega} J b \cdot \nabla_{\mathcal{A}} (\bar{Z}^\alpha b \cdot \bar{Z}^\alpha v) - \epsilon \int_{\Omega} J \Delta_{\mathcal{A}} \bar{Z}^\alpha v \cdot \bar{Z}^\alpha v \\ & = \int_{\Omega} \left(\frac{1}{2} \partial_t \left(\frac{J}{\gamma p} \right) (|\bar{Z}^\alpha q|^2 + |b \cdot \bar{Z}^\alpha b|^2) + \frac{J}{\gamma p} \partial_t b \cdot \bar{Z}^\alpha b b \cdot \bar{Z}^\alpha b + \frac{1}{2} \partial_t J |\bar{Z}^\alpha b|^2 - \partial_t \left(\frac{Jb}{\gamma p} \right) \cdot \bar{Z}^\alpha b \bar{Z}^\alpha q \right) \\ & + \int_{\Omega} J (F^{1,\alpha} \cdot \bar{Z}^\alpha q + F^{2,\alpha} \cdot \bar{Z}^\alpha v + F^{3,\alpha} \cdot \bar{Z}^\alpha b) + \int_{\Omega} \bar{Z}^\alpha \Psi^{\epsilon,\delta} \cdot \bar{Z}^\alpha v. \end{aligned} \quad (5.59)$$

Integrating by parts over Ω_{\pm} and using the boundary conditions in (5.2), one gets

$$\begin{aligned} & - \int_{\Omega} J \operatorname{div}_{\mathcal{A}} (\bar{Z}^\alpha q \cdot \bar{Z}^\alpha v) + \int_{\Omega} J b \cdot \nabla_{\mathcal{A}} (\bar{Z}^\alpha b \cdot \bar{Z}^\alpha v) \\ & = - \int_{\Omega} J \operatorname{div}_{\mathcal{A}} b \bar{Z}^\alpha b \cdot \bar{Z}^\alpha v \leq \Lambda_\infty^\epsilon \|\bar{Z}^\alpha b\|_0 \|\bar{Z}^\alpha v\|_0 \leq \Lambda_\infty^\epsilon \mathfrak{E}_m^\epsilon \end{aligned} \quad (5.60)$$

and

$$-\varepsilon \int_{\Omega} J \Delta_{\mathcal{A}} \bar{Z}^{\alpha} v \cdot \bar{Z}^{\alpha} v = \varepsilon \int_{\Omega} J |\nabla_{\mathcal{A}} \bar{Z}^{\alpha} v|^2. \quad (5.61)$$

One has directly

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{2} \partial_t \left(\frac{J}{\gamma p} \right) \left(|\bar{Z}^{\alpha} q|^2 + |b \cdot \bar{Z}^{\alpha} b|^2 \right) + \frac{J}{\gamma p} \partial_t b \cdot \bar{Z}^{\alpha} b b \cdot \bar{Z}^{\alpha} b + \frac{1}{2} \partial_t J |\bar{Z}^{\alpha} b|^2 - \partial_t \left(\frac{J b}{\gamma p} \right) \cdot \bar{Z}^{\alpha} b \bar{Z}^{\alpha} q \right) \\ & \leq \Lambda_{\infty}^{\varepsilon} \left(\|\bar{Z}^{\alpha} q\|_0^2 + \|\bar{Z}^{\alpha} b\|_0^2 + \|\bar{Z}^{\alpha} q\|_0 \|\bar{Z}^{\alpha} b\|_0 \right) \leq \Lambda_{\infty}^{\varepsilon} \mathfrak{E}_m^{\varepsilon}. \end{aligned} \quad (5.62)$$

By (5.57) and (3.19), one has directly

$$\begin{aligned} & \int_{\Omega} J (F^{1,\alpha} \cdot \bar{Z}^{\alpha} q + F^{2,\alpha} \cdot \bar{Z}^{\alpha} v + \bar{Z}^{\alpha} \Psi^{\varepsilon,\delta} \cdot \bar{Z}^{\alpha} v + F^{3,\alpha} \cdot \bar{Z}^{\alpha} b) \\ & \leq \Lambda_{\infty}^{\varepsilon} \left(\|F^{1,\alpha}\|_0 \|\bar{Z}^{\alpha} q\|_0 + \|F^{2,\alpha}\|_0 \|\bar{Z}^{\alpha} v\|_0 + \|\bar{Z}^{\alpha} \Psi^{\varepsilon,\delta}\|_0 \|\bar{Z}^{\alpha} v\|_0 + \|F^{3,\alpha}\|_0 \|\bar{Z}^{\alpha} b\|_0 \right) \\ & \leq \Lambda_{\infty}^{\varepsilon} \left(\sqrt{\mathfrak{E}_m^{\varepsilon} + \mathfrak{D}_m^{\varepsilon}} \sqrt{\mathfrak{E}_m^{\varepsilon}} + \varepsilon \mathcal{M}_0^{m,\delta} \sqrt{\mathfrak{E}_m^{\varepsilon}} \right). \end{aligned} \quad (5.63)$$

As a consequence of the estimates (5.60)–(5.63), one deduces from (5.59) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} J \left(\frac{1}{\gamma p} |\bar{Z}^{\alpha} q - b \cdot \bar{Z}^{\alpha} b|^2 + \rho |\bar{Z}^{\alpha} v|^2 + |\bar{Z}^{\alpha} b|^2 \right) + \varepsilon \int_{\Omega} J |\nabla_{\mathcal{A}} \bar{Z}^{\alpha} v|^2 \\ & \leq \Lambda_{\infty}^{\varepsilon} \left(\sqrt{\mathfrak{E}_m^{\varepsilon} + \mathfrak{D}_m^{\varepsilon}} \sqrt{\mathfrak{E}_m^{\varepsilon}} + \varepsilon \mathcal{M}_0^{m,\delta} \sqrt{\mathfrak{E}_m^{\varepsilon}} \right). \end{aligned} \quad (5.64)$$

Integrating (5.64) directly in time, by (5.42), (3.19) and using Poincaré's inequality and Hölder's inequality, one obtains

$$\begin{aligned} \left\| (\bar{Z}^{\alpha} p, \bar{Z}^{\alpha} v, \bar{Z}^{\alpha} b)(t) \right\|_0^2 + \varepsilon \int_0^t \|\bar{Z}^{\alpha} v\|_1^2 & \leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_{\infty}^{\varepsilon}(t) \int_0^t \sqrt{\mathfrak{E}_m^{\varepsilon} + \mathfrak{D}_m^{\varepsilon}} \sqrt{\mathfrak{E}_m^{\varepsilon}} \\ & \leq \mathcal{M}_0^m + \varepsilon^2 \mathcal{M}_0^{m,\delta} + \Lambda_{\infty}^{\varepsilon}(t) t^{1/2} \mathcal{G}_m^{\varepsilon}(t). \end{aligned} \quad (5.65)$$

This gives (5.58) by summing over such α and recalling (4.2). \square

6. NORMAL DERIVATIVES ESTIMATES

In this section, we will estimate the normal derivatives of (p, v, b) . In light of the tangential estimates in Section 5, it suffices to derive the estimates near the boundary $\Sigma \cup \Sigma_{\pm}$, and thus the analysis in the following will be carried out mostly near $\Sigma \cup \Sigma_{\pm}$, without mentioning explicitly. The fact that $(\mathcal{A}^T b)_3 = J^{-1} b_0^{\delta} \cdot \mathcal{N}_0^{\delta} \neq 0$ in such region will be used crucially.

Proposition 6.1. *For $t \in [0, T_0^{\varepsilon,\delta}]$ with $T_0^{\varepsilon,\delta} \leq 1$, it holds that*

$$\mathfrak{E}_m^{\varepsilon}(t) + \int_0^t \bar{\mathfrak{D}}_m^{\varepsilon} \leq \Lambda_{\infty}^{\varepsilon}(t) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^{\varepsilon}(t) \right). \quad (6.1)$$

Proof. First, (3.30) implies

$$\partial_3 b \cdot \mathcal{N} = -J \mathcal{A}_{i\beta} \partial_{\beta} b_i + J_0^{\delta} \operatorname{div}_{\mathcal{A}_0^{\delta}} b_0^{\delta}. \quad (6.2)$$

Then by (3.14) and (3.15), one obtains that for $j = 0, \dots, m-1$,

$$\left\| \partial_t^j \partial_3 b \cdot \mathcal{N} \right\|_{0, m-j-1}^2 \leq \Lambda_{\infty}^{\varepsilon} \left(\mathfrak{F}_m^{\varepsilon} + \left\| \partial_t^j b \right\|_{0, m-j}^2 \right) + \mathcal{M}_0^m. \quad (6.3)$$

Next, the second equation and fourth equation in (2.6) imply

$$b \cdot \nabla_{\mathcal{A}} v = \partial_t b - \frac{b}{\gamma p} \partial_t p. \quad (6.4)$$

Thus, since $(\mathcal{A}^T b)_3 \neq 0$ near $\Sigma \cup \Sigma_{\pm}$, one obtains

$$\partial_3 v = \frac{1}{(\mathcal{A}^T b)_3} (b \cdot \nabla_{\mathcal{A}} v - (\mathcal{A}^T b)_{\beta} \partial_{\beta} v) = \frac{1}{(\mathcal{A}^T b)_3} \left(\partial_t b - \frac{b}{\gamma p} \partial_t p - (\mathcal{A}^T b)_{\beta} \partial_{\beta} v \right). \quad (6.5)$$

One then deduces that for $j = 0, \dots, m-1$,

$$\left\| \partial_3 \partial_t^j v \right\|_{0, m-j-1}^2 \leq \Lambda_\infty^\varepsilon \left(\mathfrak{F}_m^\varepsilon + \left\| (\partial_t^{j+1} p, \partial_t^{j+1} b) \right\|_{0, m-j-1}^2 + \left\| \partial_t^j v \right\|_{0, m-j}^2 \right). \quad (6.6)$$

This, together with (6.3), Propositions 4.1, 5.3 and 5.5, yields (6.1). \square

Next, in terms of the orthogonal base:

$$\left\{ \tau^1 = \frac{\partial_1 \eta}{|\partial_1 \eta|}, \tau^2 = \frac{\tilde{\tau}}{|\tilde{\tau}|}, n = \frac{\mathcal{N}}{|\mathcal{N}|} \right\} \text{ with } \tilde{\tau} = \partial_2 \eta - \frac{\partial_1 \eta \cdot \partial_2 \eta}{|\partial_1 \eta|^2} \partial_1 \eta, \quad (6.7)$$

one has the following expressions for $\partial_3 b \cdot \tau^\beta$, $\beta = 1, 2$, and $\partial_3 p$.

Lemma 6.2. *Let $\alpha \in \mathbb{N}^{1+3}$ with $|\alpha| \leq m-1$. It holds that near $\Sigma \cup \Sigma_\pm$,*

$$\partial_3 \partial^\alpha b \cdot \tau^\beta + \varepsilon \frac{|\mathcal{N}|^2}{(J\mathcal{A}^T b)_3^2} \partial_3 \partial^\alpha (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta = G_{\tau^\beta}^\alpha, \quad \beta = 1, 2 \quad (6.8)$$

and

$$\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta - \varepsilon \partial_3 \partial^\alpha \operatorname{div}_{\mathcal{A}} v = G_n^\alpha. \quad (6.9)$$

Here $G_{\tau^\beta}^\alpha$ and G_n^α are defined by (6.17) and (6.24), respectively, which satisfy

$$\begin{aligned} \|G_{\tau^\beta}^\alpha\|_0^2 + \|G_n^\alpha\|_0^2 &\leq \Lambda_\infty^\varepsilon \left(\left\| \partial^\alpha \Psi^{\varepsilon, \delta} \right\|_0^2 + \mathfrak{F}_m^\varepsilon + \|(\partial^\alpha p, \partial^\alpha b)\|_{0,1}^2 + \|\partial^\alpha \partial_t v\|_0^2 \right. \\ &\quad \left. + \varepsilon^2 (\|\partial^\alpha v\|_{1,1}^2 + \mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon) \right). \end{aligned} \quad (6.10)$$

Proof. One first projects the third equation in (2.6) along τ^β , $\beta = 1, 2$. For this, one writes

$$\Delta_{\mathcal{A}} v = \mathcal{A}_{k3} \mathcal{A}_{k3} \partial_3^2 v + \mathcal{A}_{k3} \partial_3 \mathcal{A}_{k3} \partial_3 v + \sum_{i+j < 6} \mathcal{A}_{ki} \partial_i (\mathcal{A}_{kj} \partial_j v) \quad (6.11)$$

and by the first identity in (6.5),

$$\partial_3^2 v = \frac{\partial_3 (b \cdot \nabla_{\mathcal{A}} v)}{(\mathcal{A}^T b)_3} + \partial_3 \left(\frac{1}{(\mathcal{A}^T b)_3} \right) b \cdot \nabla_{\mathcal{A}} v - \partial_3 \left(\frac{(\mathcal{A}^T b)_\beta \partial_\beta v}{(\mathcal{A}^T b)_3} \right), \quad (6.12)$$

then the third equation in (2.6) yields

$$(\mathcal{A}^T b)_3 \partial_3 b + \varepsilon \frac{\mathcal{A}_{k3} \mathcal{A}_{k3}}{(\mathcal{A}^T b)_3} \partial_3 (b \cdot \nabla_{\mathcal{A}} v) = \nabla_{\mathcal{A}} q + \mathfrak{f}, \quad (6.13)$$

where

$$\begin{aligned} \mathfrak{f} := & - (\mathcal{A}^T b)_\beta \partial_\beta b + \rho \partial_t v - \varepsilon \sum_{i+j < 6} \mathcal{A}_{ki} \partial_i (\mathcal{A}_{kj} \partial_j v) - \varepsilon \mathcal{A}_{k3} \partial_3 \mathcal{A}_{k3} \partial_3 v \\ & - \varepsilon \mathcal{A}_{k3} \mathcal{A}_{k3} \left(\partial_3 \left(\frac{1}{(\mathcal{A}^T b)_3} \right) b \cdot \nabla_{\mathcal{A}} v - \partial_3 \left(\frac{(\mathcal{A}^T b)_\beta \partial_\beta v}{(\mathcal{A}^T b)_3} \right) \right) - \Psi^{\varepsilon, \delta}. \end{aligned} \quad (6.14)$$

Hence, one deduces from (6.13) that, by (1.12),

$$\partial_3 b \cdot \tau^\beta + \varepsilon \frac{|\mathcal{N}|^2}{(J\mathcal{A}^T b)_3^2} \partial_3 (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta = G_{\tau^\beta}, \quad (6.15)$$

where

$$G_{\tau^\beta} = \frac{1}{(\mathcal{A}^T b)_3} (\nabla_{\mathcal{A}} q + \mathfrak{f}) \cdot \tau^\beta. \quad (6.16)$$

Note that $\nabla_{\mathcal{A}} q \cdot \tau^\beta$ involves only $\partial_1 q, \partial_2 q$. Applying ∂^α to (6.15) then yields (6.8) with

$$G_{\tau^\beta}^\alpha = \partial^\alpha G_{\tau^\beta} - \left[\partial^\alpha, \tau^\beta \right] \cdot \partial_3 b - \varepsilon \left[\partial^\alpha, \frac{|\mathcal{N}|^2}{(J\mathcal{A}^T b)_3^2} \tau^\beta \right] \cdot \partial_3 (b \cdot \nabla_{\mathcal{A}} v). \quad (6.17)$$

Now one projects the third equation in (2.6) along n . First, the third equation in (2.6) yields

$$\mathcal{A}_{i3} \partial_3 p + \mathcal{A}_{i3} \partial_3 b_j b_j - b_k \mathcal{A}_{k3} \partial_3 b_i - \varepsilon \mathcal{A}_{j3} \partial_3 \partial_j^A v_i = \mathfrak{g}_i, \quad (6.18)$$

where

$$\mathfrak{g}_i := \varepsilon \mathcal{A}_{j\beta} \partial_\beta \partial_j^A v_i + b_k \mathcal{A}_{k\beta} \partial_\beta b_i - \mathcal{A}_{i\beta} \partial_\beta p - \mathcal{A}_{i\beta} \partial_\beta b_j b_j - \rho \partial_t v_i + \Psi_i^{\varepsilon, \delta}. \quad (6.19)$$

Note that

$$\mathcal{A}_{i3} \mathcal{A}_{i3} \partial_3 b_j b_j - \mathcal{A}_{i3} b_k \mathcal{A}_{k3} \partial_3 b_i = \mathcal{A}_{i3} \mathcal{A}_{i3} (\partial_3 b \cdot b - \partial_3 b \cdot n b \cdot n) = \mathcal{A}_{i3} \mathcal{A}_{i3} \partial_3 b \cdot \tau^\beta b \cdot \tau^\beta \quad (6.20)$$

and

$$\begin{aligned} -\varepsilon \mathcal{A}_{i3} \mathcal{A}_{j3} \partial_3 \partial_j^A v_i &= -\varepsilon \mathcal{A}_{j3} \partial_i^A \partial_j^A v_i + \varepsilon \mathcal{A}_{j3} \mathcal{A}_{i\beta} \partial_\beta \partial_j^A v_i \\ &= -\varepsilon \mathcal{A}_{j3} \mathcal{A}_{j3} \partial_3 \operatorname{div}_{\mathcal{A}} v - \varepsilon \mathcal{A}_{j\beta} \mathcal{A}_{j\beta} \partial_\beta \operatorname{div}_{\mathcal{A}} v + \varepsilon \mathcal{A}_{j3} \mathcal{A}_{i\beta} \partial_\beta \partial_j^A v_i. \end{aligned} \quad (6.21)$$

Hence, one deduces from (6.18) that, by (6.20) and (6.21),

$$\partial_3 p + \partial_3 b \cdot \tau^\beta b \cdot \tau^\beta - \varepsilon \partial_3 \operatorname{div}_{\mathcal{A}} v = G_n, \quad (6.22)$$

where

$$G_n = \frac{1}{\mathcal{A}_{i3} \mathcal{A}_{i3}} (\mathfrak{g}_j \mathcal{A}_{j3} + \varepsilon \mathcal{A}_{j\beta} \mathcal{A}_{j\beta} \partial_\beta \operatorname{div}_{\mathcal{A}} v - \varepsilon \mathcal{A}_{j3} \mathcal{A}_{i\beta} \partial_\beta \partial_j^A v_i). \quad (6.23)$$

Applying ∂^α to (6.22) then yields (6.9) with

$$G_n^\alpha = \partial^\alpha G_n - \left[\partial^\alpha, b \cdot \tau^\beta \tau^\beta \right] \cdot \partial_3 b. \quad (6.24)$$

The estimate (6.10) follows directly from these expressions of $G_{\tau^\beta}^\alpha$ and G_n^α . \square

Proposition 6.3. For $t \in [0, T_0^{\varepsilon, \delta}]$ with $T_0^{\varepsilon, \delta} \leq 1$, it holds that

$$\int_0^t \mathfrak{D}_m^\varepsilon \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m, \delta} + t \mathcal{G}_m^\varepsilon(t) + \varepsilon \mathcal{G}_m^\varepsilon(t) \right). \quad (6.25)$$

Proof. Fix first $k = 0, \dots, m-1$ and then $\ell = 0, \dots, m-k-1$. Let $\alpha \in \mathbb{N}^{1+3}$ be with $|\alpha| \leq m-1$ such that $\alpha_0 = k$ and $\alpha_3 \leq \ell$. First, multiplying both sides of (6.8) by $(J\mathcal{A}^T b)_3 |\mathcal{N}|^{-1}$ and taking the square to integrate in $\tilde{\Omega}$, some region near $\Sigma \cup \Sigma_\pm$, one has

$$\begin{aligned} \mathcal{I}_\tau^\alpha + \sum_{\beta=1,2} \int_{\tilde{\Omega}} \left(\frac{(J\mathcal{A}^T b)_3^2}{|\mathcal{N}|^2} \left| \partial_3 \partial^\alpha b \cdot \tau^\beta \right|^2 + \varepsilon^2 \frac{|\mathcal{N}|^2}{(J\mathcal{A}^T b)_3^2} \left| \partial_3 \partial^\alpha (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta \right|^2 \right) \\ = \sum_{\beta=1,2} \int_{\tilde{\Omega}} \frac{(J\mathcal{A}^T b)_3^2}{|\mathcal{N}|^2} |G_{\tau^\beta}^\alpha|^2, \end{aligned} \quad (6.26)$$

where

$$\mathcal{I}_\tau^\alpha = 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta \partial_3 \partial^\alpha (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta. \quad (6.27)$$

By (6.4), one has

$$\begin{aligned} \mathcal{I}_\tau^\alpha &= 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta \partial_3 \partial^\alpha \left(\partial_t b - \frac{b}{\gamma p} \partial_t p \right) \cdot \tau^\beta \\ &= \varepsilon \frac{d}{dt} \sum_{\beta=1,2} \int_{\tilde{\Omega}} \left| \partial_3 \partial^\alpha b \cdot \tau^\beta \right|^2 - 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta \partial_3 \partial^\alpha b \cdot \partial_t \tau^\beta \\ &\quad - 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta \frac{1}{\gamma p} \partial_3 \partial^\alpha \partial_t p - 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta \left[\partial_3 \partial^\alpha, \frac{b}{\gamma p} \right] \partial_t p \cdot \tau^\beta \\ &\geq \varepsilon \frac{d}{dt} \sum_{\beta=1,2} \int_{\tilde{\Omega}} \left| \partial_3 \partial^\alpha b \cdot \tau^\beta \right|^2 - \underline{\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta \frac{1}{\gamma p} \partial_3 \partial^\alpha \partial_t p - \varepsilon \Lambda_\infty^\varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon)}. \end{aligned} \quad (6.28)$$

Next, take the square of (6.9) and integrate in $\tilde{\Omega}$ to have

$$\mathcal{I}_n^\alpha + \int_{\tilde{\Omega}} \left| \partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta \right|^2 + \varepsilon^2 \left| \partial_3 \partial^\alpha \operatorname{div}_{\mathcal{A}} v \right|^2 = \int_{\tilde{\Omega}} |G_n^\alpha|^2, \quad (6.29)$$

where

$$\mathcal{I}_n^\alpha = -2\varepsilon \int_{\tilde{\Omega}} (\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta) \partial_3 \partial^\alpha \operatorname{div}_{\mathcal{A}} v. \quad (6.30)$$

By the second equation in (2.6), one has

$$\partial^\alpha \operatorname{div}_{\mathcal{A}} v = -\frac{1}{\gamma p} \partial_t \partial^\alpha p - \left[\partial^\alpha, \frac{1}{\gamma p} \right] \partial_t p. \quad (6.31)$$

Hence,

$$\begin{aligned} \mathcal{I}_n^\alpha &= 2\varepsilon \int_{\tilde{\Omega}} (\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta) \frac{1}{\gamma p} \partial_3 \partial_t \partial^\alpha p \\ &\quad + 2\varepsilon \int_{\tilde{\Omega}} (\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta) \left(\partial_3 \left(\frac{1}{\gamma p} \right) \partial_t \partial^\alpha p + \partial_3 \left(\left[\partial^\alpha, \frac{1}{\gamma p} \right] \partial_t p \right) \right) \\ &= \varepsilon \frac{d}{dt} \int_{\tilde{\Omega}} \frac{1}{\gamma p} |\partial_3 \partial^\alpha p|^2 - \varepsilon \int_{\tilde{\Omega}} \partial_t \left(\frac{1}{\gamma p} \right) |\partial_3 \partial^\alpha p|^2 + 2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta \frac{1}{\gamma p} \partial_3 \partial_t \partial^\alpha p \\ &\quad + 2\varepsilon \int_{\tilde{\Omega}} (\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta) \left(\partial_3 \left(\frac{1}{\gamma p} \right) \partial_t \partial^\alpha p + \partial_3 \left(\left[\partial^\alpha, \frac{1}{\gamma p} \right] \partial_t p \right) \right) \\ &\geq \varepsilon \frac{d}{dt} \int_{\tilde{\Omega}} \frac{1}{\gamma p} |\partial_3 \partial^\alpha p|^2 + \underbrace{2\varepsilon \int_{\tilde{\Omega}} \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta \frac{1}{\gamma p} \partial_3 \partial^\alpha \partial_t p}_{\text{exact cancellation}} - \varepsilon \Lambda_\infty^\varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon). \end{aligned} \quad (6.32)$$

One can see that it is crucial that there is an exact cancelation for the two underlined terms in (6.28) and (6.32), which themselves seem out of control. Hence, combining (6.28) and (6.32) and by (6.10), one gets

$$\begin{aligned} &\varepsilon \frac{d}{dt} \int_{\tilde{\Omega}} \left(\sum_{\beta=1,2} |\partial_3 \partial^\alpha b \cdot \tau^\beta|^2 + \frac{1}{\gamma p} |\partial_3 \partial^\alpha p|^2 \right) \\ &\quad + \int_{\tilde{\Omega}} \left(\frac{(J\mathcal{A}^T b)_3^2}{|\mathcal{N}|^2} \sum_{\beta=1,2} |\partial_3 \partial^\alpha b \cdot \tau^\beta|^2 + |\partial_3 \partial^\alpha p + \partial_3 \partial^\alpha b \cdot \tau^\beta b \cdot \tau^\beta|^2 \right) \\ &\quad + \varepsilon^2 \int_{\tilde{\Omega}} \left(\frac{|\mathcal{N}|^2}{(J\mathcal{A}^T b)_3^2} \sum_{\beta=1,2} |\partial_3 \partial^\alpha (b \cdot \nabla_{\mathcal{A}} v) \cdot \tau^\beta|^2 + |\partial_3 \partial^\alpha \operatorname{div}_{\mathcal{A}} v|^2 \right) \\ &\leq \Lambda_\infty^\varepsilon \left(\sum_{\beta=1,2} \|G_{\tau^\beta}^\alpha\|_0^2 + \|G_n^\alpha\|_0^2 + \varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon) \right) \\ &\leq \Lambda_\infty^\varepsilon \left(\|\partial^\alpha \Psi^{\varepsilon, \delta}\|_0^2 + \mathfrak{F}_m^\varepsilon + \|(\partial^\alpha p, \partial^\alpha b)\|_{0,1}^2 + \|\partial^\alpha \partial_t v\|_0^2 + \varepsilon^2 \|\partial^\alpha v\|_{1,1}^2 + \varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon) \right). \end{aligned} \quad (6.33)$$

Integrating (6.33) in time and using (6.2),

$$\partial_3 v \cdot \tau^\beta = \frac{1}{(\mathcal{A}^T b)_3} \left((b \cdot \nabla_{\mathcal{A}}) v \cdot \tau^\beta - (\mathcal{A}^T b)_\gamma \partial_\gamma v \cdot \tau^\beta \right) \quad (6.34)$$

and

$$\partial_3 v \cdot \mathcal{N} = J \operatorname{div}_{\mathcal{A}} v - J \mathcal{A}_{i\gamma} \partial_\gamma v_i, \quad (6.35)$$

one can deduce that, by (3.19) and (4.2),

$$\begin{aligned} &\int_0^t \left(\|(\partial_3 \partial^\alpha b, \partial_3 \partial^\alpha p)\|_{L^2(\tilde{\Omega})}^2 + \varepsilon^2 \|\partial^\alpha v\|_{H^2(\tilde{\Omega})}^2 \right) \\ &\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m, \delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\|(\partial^\alpha p, \partial^\alpha b)\|_{0,1}^2 + \|\partial^\alpha \partial_t v\|_0^2 + \varepsilon^2 \|\partial^\alpha v\|_{1,1}^2 + \varepsilon (\mathfrak{E}_m^\varepsilon + \mathfrak{D}_m^\varepsilon) \right). \end{aligned} \quad (6.36)$$

Consequently, summing (6.36) over such α and by (6.1), one obtains

$$\int_0^t \left(\|(\partial_t^k p, \partial_t^k b)\|_{\ell+1, m-1-k-\ell}^2 + \varepsilon^2 \|\partial_t^k v\|_{\ell+2, m-1-k-\ell}^2 \right)$$

$$\begin{aligned}
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{\ell, m-k-\ell}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{\ell+1, m-k-\ell}^2 \right) \\
&\quad + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t^{k+1} v \right\|_{m-1-k}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.37}
\end{aligned}$$

Now taking $k = 1, \dots, m-1$ in (6.37), by (6.4) and similarly as in Proposition 6.1, one deduces that for $\ell = 0, \dots, m-k-1$,

$$\begin{aligned}
&\int_0^t \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{\ell+1, m-1-k-\ell}^2 + \left\| \partial_t^{k-1} v \right\|_{\ell+2, m-1-k-\ell}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{\ell+2, m-1-k-\ell}^2 \right) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{\ell, m-k-\ell}^2 + \left\| \partial_t^{k-1} v \right\|_{\ell+1, m-k-\ell}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{\ell+1, m-k-\ell}^2 \right) \\
&\quad + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t^{k+1} v \right\|_{m-1-k}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.38}
\end{aligned}$$

A suitable linear combination of (6.38) for $\ell = 0, \dots, m-k-1$ yields that

$$\begin{aligned}
&\int_0^t \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{m-k}^2 + \left\| \partial_t^{k-1} v \right\|_{m-k+1}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{m-k+1}^2 \right) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{0, m-k}^2 + \left\| \partial_t^{k-1} v \right\|_{1, m-k}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{1, m-k}^2 \right) \\
&\quad + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t^{k+1} v \right\|_{m-1-k}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t^{k+1} v \right\|_{m-1-k}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.39}
\end{aligned}$$

Combining (6.39) for $k = 1, \dots, m-1$ suitably again implies

$$\begin{aligned}
&\int_0^t \sum_{k=1}^{m-1} \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{m-k}^2 + \left\| \partial_t^{k-1} v \right\|_{m-k+1}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{m-k+1}^2 \right) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| \partial_t^m v \right\|_0^2 + \left\| \partial_t^{m-1} v \right\|_1^2 \right) + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.40}
\end{aligned}$$

On the other hand, taking $k = 0$ in (6.37) yields that for $\ell = 0, \dots, m-1$,

$$\begin{aligned}
&\int_0^t \left(\left\| (p, b) \right\|_{\ell+1, m-1-\ell}^2 + \varepsilon^2 \left\| v \right\|_{\ell+2, m-1-\ell}^2 \right) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| (p, b) \right\|_{\ell, m-\ell}^2 + \varepsilon^2 \left\| v \right\|_{\ell+1, m-\ell}^2 \right) \\
&\quad + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t v \right\|_{m-1}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.41}
\end{aligned}$$

Thus, a suitable linear combination of (6.41) for $\ell = 0, \dots, m-1$ yields

$$\begin{aligned}
&\int_0^t \left(\left\| (p, b) \right\|_m^2 + \varepsilon^2 \left\| v \right\|_{m+1}^2 \right) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left(\left\| (p, b) \right\|_{0, m}^2 + \varepsilon^2 \left\| v \right\|_{1, m}^2 + \left\| \partial_t v \right\|_{m-1}^2 \right) + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)) \\
&\leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) \int_0^t \left\| \partial_t v \right\|_{m-1}^2 + \Lambda_\infty^\varepsilon(t) (t\mathcal{G}_m^\varepsilon(t) + \varepsilon\mathcal{G}_m^\varepsilon(t)). \tag{6.42}
\end{aligned}$$

Consequently, it follows from (6.40) and (6.42) that

$$\begin{aligned} & \int_0^t \left(\sum_{k=0}^{m-2} \left\| \partial_t^k v \right\|_{m-k}^2 + \sum_{k=0}^{m-1} \left(\left\| (\partial_t^k p, \partial_t^k b) \right\|_{m-k}^2 + \varepsilon^2 \left\| \partial_t^k v \right\|_{m-k+1}^2 \right) \right) \\ & \leq \mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + \Lambda_\infty^\varepsilon(t) (t \mathcal{G}_m^\varepsilon(t) + \varepsilon \mathcal{G}_m^\varepsilon(t)). \end{aligned} \quad (6.43)$$

This gives (6.25). \square

7. (ε, δ) -INDEPENDENT LOCAL WELL-POSEDNESS OF (2.6)

Now we can conclude the following (ε, δ) -independent estimates.

Theorem 7.1. *There exist positive constants T_0 and δ_0 with the property that for each $0 < \delta < \delta_0$, one can find an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ so that for $0 < \varepsilon \leq \varepsilon_0$, it holds that*

$$\mathcal{G}_m^\varepsilon(T_0) \leq \mathcal{M}_0^m \quad (7.1)$$

and that for $t \in [0, T_0]$,

$$\rho, p, |J| \geq \frac{c_0}{4} > 0 \text{ in } \Omega \text{ and } |b \cdot \mathcal{N}| \geq \frac{c_0}{4} > 0 \text{ on } \Sigma \cup \Sigma_\pm. \quad (7.2)$$

Proof. Note that by the definitions of \mathcal{M}_0^m and $\mathcal{M}_0^{m,\delta}$, for each $0 < \delta < \delta_0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ so that for $0 < \varepsilon \leq \varepsilon_0$,

$$\varepsilon \mathcal{M}_0^{m,\delta} \leq \mathcal{M}_0^m. \quad (7.3)$$

Then by the definition of $\mathcal{G}_m^\varepsilon(t)$, (6.1), (6.25) and (4.6), one deduces that for $t \in [0, T_0^{\varepsilon,\delta}]$ with $T_0^{\varepsilon,\delta} \leq 1$,

$$\begin{aligned} \mathcal{G}_m^\varepsilon(t) & \leq \Lambda_\infty^\varepsilon(t) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\varepsilon(t) + \varepsilon \mathcal{G}_m^\varepsilon(t) \right) \\ & \leq (\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t P(\mathcal{G}_m^\varepsilon(t))) \left(\mathcal{M}_0^m + \varepsilon \mathcal{M}_0^{m,\delta} + t^{1/2} \mathcal{G}_m^\varepsilon(t) + \varepsilon \mathcal{G}_m^\varepsilon(t) \right) \\ & \leq \mathcal{M}_0^m \left(1 + t^{1/2} P(\mathcal{G}_m^\varepsilon(t)) + \varepsilon \mathcal{G}_m^\varepsilon(t) \right). \end{aligned} \quad (7.4)$$

Therefore, for $0 < \varepsilon \leq \varepsilon_0$ by restricting ε_0 smaller if necessary, it holds that

$$\mathcal{G}_m^\varepsilon(t) \leq \mathcal{M}_0^m \left(1 + t^{1/2} P(\mathcal{G}_m^\varepsilon(t)) \right). \quad (7.5)$$

From (7.5), there is a T_0 , depending on \mathcal{M}_0^m but not on ε, δ , such that

$$\mathcal{G}_m^\varepsilon(T_0) \leq \mathcal{M}_0^m. \quad (7.6)$$

This yields (7.1). The estimate (7.2) follows from the fundamental theorem of calculus, (3.14) and (7.1), by restricting T_0 smaller if necessary. \square

We now present the

Proof of Theorem 2.4. With the estimates (7.1) and (7.2) of Theorem 7.1 in hand, following that for the compressible Navier–Stokes equations (see for instance [9] for the references), one can show routinely that

$$\sup_{[0, T_0]} \mathbb{E}_{2m} + \int_0^{T_0} \mathbb{D}_{2m} \leq P_\varepsilon(\mathcal{G}_m^\varepsilon(T_0), \mathbb{E}_{2m}(0)) \leq P_{\varepsilon,\delta}(\mathcal{M}_0^m). \quad (7.7)$$

The point here is that although the bound of $\mathbb{E}_{2m}(t)$ depends on ε, δ , it does not depend on $t \in [0, T_0]$. Now we indicate the dependence of the solutions to (2.6) on ε, δ by $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$. Hence, by a standard continuity argument and the local well-posedness recorded in Theorem 3.2, the unique solution $(\eta^{\varepsilon,\delta}, p^{\varepsilon,\delta}, v^{\varepsilon,\delta}, b^{\varepsilon,\delta})$ above exists actually on $[0, T_0]$ and satisfies the estimate (2.16). The proof of Theorem 2.4 is thus completed. \square

8. LOCAL WELL-POSEDNESS OF (1.10)

Now we present the

Proof of Theorem 2.1. By Theorem 2.4, one can easily pass to the limit as first $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$ in (2.6) to find that the limit (η, p, v, b) of $(\eta^{\varepsilon, \delta}, p^{\varepsilon, \delta}, v^{\varepsilon, \delta}, b^{\varepsilon, \delta})$, up to extraction of a subsequence, solves (1.10) on $[0, T_0]$, where the constraint that $\operatorname{div}_{\mathcal{A}} b = 0$ in Ω is recovered by using (i) in Proposition 1.1 as $\operatorname{div}_{\mathcal{A}_0} b_0 = 0$ in Ω . Moreover, (η, p, v, b) satisfies

$$\sup_{[0, T_0]} \mathfrak{E}_m^0 + \int_0^{T_0} \mathfrak{D}_m^0 \leq \mathcal{M}_0^m. \quad (8.1)$$

To improve the estimates, we revisit those estimates in Sections 5 and 6 with setting $\varepsilon = 0$ to obtain

$$\mathfrak{E}_m^0(t) \leq \Lambda_\infty^0(t) \left(\mathcal{M}_0^m + t \sup_{[0, t]} \mathcal{E}_m \right) \quad (8.2)$$

and

$$\mathfrak{D}_m^0 \leq \Lambda_\infty^0 \mathfrak{E}_m^0, \quad (8.3)$$

respectively. Then combining (8.2) and (8.3) yields

$$\mathcal{E}_m(t) \leq \Lambda_\infty^0 \left(\mathcal{M}_0^m + t \sup_{[0, t]} \mathcal{E}_m \right) \leq \mathcal{M}_0^m + tP(\sup_{[0, t]} \mathcal{E}_m). \quad (8.4)$$

From (8.4), there is a \tilde{T}_0 , depending on \mathcal{M}_0^m , such that

$$\mathcal{E}_m(t) \leq \mathcal{M}_0^m, \quad \forall t \in [0, T_0]. \quad (8.5)$$

This yields (2.5) by resetting T_0 .

Now for the uniqueness, consider two solutions (η^i, p^i, v^i, b^i) , $i = 1, 2$ to (1.10) on $[0, T_0]$ with the same data $(\eta_0, p_0, v_0, b_0, \rho_0)$ which satisfy the estimate (2.5). Denote $\mathcal{A}^i = \mathcal{A}(\eta^i)$, $\mathcal{N}^i = \mathcal{N}(\eta^i)$ and $J^i = J(\eta^i)$, $i = 1, 2$. Denote $\bar{f} = f^1 - f^2$ for the difference of each unknown f , and define

$$\bar{\mathcal{E}} := \sum_{j=0}^3 \left\| (\partial_t^j \bar{p}, \partial_t^j \bar{v}, \partial_t^j \bar{b}) \right\|_{3-j}^2 + \|\bar{\eta}\|_3^2 + |\bar{\eta}|_3^2. \quad (8.6)$$

To estimate the differences, one uses again the equations (5.1) with setting $\varepsilon = 0$ for each solution. Define again the corresponding good unknowns as

$$\mathcal{V}^i = Z^3 v^i - Z^3 \eta^i \cdot \nabla_{\mathcal{A}^i} v^i, \quad \mathcal{Q}^i = Z^3 q^i - Z^3 \eta^i \cdot \nabla_{\mathcal{A}^i} q^i \quad (8.7)$$

and

$$\mathcal{B}^i = Z^3 b^i - Z^3 \eta_0 \cdot \nabla_{\mathcal{A}_0} b^i. \quad (8.8)$$

Hence,

$$\bar{\mathcal{V}} = Z^3 \bar{v} - Z^3 \bar{\eta} \cdot \nabla_{\mathcal{A}^1} v^1 - Z^3 \eta^2 \cdot \overline{\nabla_{\mathcal{A}^1} v^1}, \quad \bar{\mathcal{Q}} = Z^3 \bar{q} - Z^3 \bar{\eta} \cdot \nabla_{\mathcal{A}^1} q^1 - Z^3 \eta^2 \cdot \overline{\nabla_{\mathcal{A}^1} q^1} \quad (8.9)$$

and

$$\bar{\mathcal{B}} = Z^3 \bar{b} - Z^3 \eta_0 \cdot \nabla_{\mathcal{A}_0} \bar{b}. \quad (8.10)$$

Note that

$$\begin{aligned} \llbracket \bar{\mathcal{Q}} \rrbracket &= -(J^1)^{-1} Z^3 \bar{\eta} \cdot \mathcal{N}^1 \llbracket \partial_3 q^1 \rrbracket - Z^3 \eta^2 \cdot \overline{J^{-1} \mathcal{N} \llbracket \partial_3 q \rrbracket}, \\ \llbracket \bar{\mathcal{B}} \rrbracket &= -J_0^{-1} Z^3 \eta_0 \cdot \mathcal{N}_0 \llbracket \partial_3 \bar{b} \rrbracket, \quad \llbracket \bar{\mathcal{V}} \rrbracket = 0 \text{ on } \Sigma, \quad \bar{\mathcal{V}} = 0 \text{ on } \Sigma_\pm. \end{aligned} \quad (8.11)$$

One can proceed as in the proof of Proposition 5.3 to get, by (2.5),

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} J^1 \left(\frac{1}{\gamma p^1} |\bar{\mathcal{Q}} - b^1 \cdot \bar{\mathcal{B}}|^2 + \rho^1 |\bar{\mathcal{V}}|^2 + |\bar{\mathcal{B}}|^2 \right) \\ & + \int_{\Omega} J^1 \operatorname{div}_{\mathcal{A}^1} (\bar{\mathcal{Q}} \bar{\mathcal{V}}) - \int_{\Omega} J^1 b^1 \cdot \nabla_{\mathcal{A}^1} (\bar{\mathcal{B}} \cdot \bar{\mathcal{V}}) \leq \mathcal{M}_0^m \bar{\mathcal{E}}. \end{aligned} \quad (8.12)$$

By (8.11) and since $\partial_t \bar{\eta} = \bar{v}$, one obtains

$$\begin{aligned}
& - \int_{\Omega} J^1 \operatorname{div}_{\mathcal{A}^1}(\bar{\mathcal{Q}}\bar{\mathcal{V}}) = \int_{\Sigma} \llbracket \bar{\mathcal{Q}} \rrbracket \bar{\mathcal{V}} \cdot \mathcal{N}^1 \\
& = - \int_{\Sigma} ((J^1)^{-1} Z^3 \bar{\eta} \cdot \mathcal{N}^1 \llbracket \partial_3 q^1 \rrbracket + Z^3 \eta^2 \cdot \overline{J^{-1} \mathcal{N} \llbracket \partial_3 q \rrbracket}) (Z^3 \bar{v} - Z^3 \bar{\eta} \cdot \nabla_{\mathcal{A}^1} v^1 - Z^3 \eta^2 \cdot \overline{\nabla_{\mathcal{A}} v}) \cdot \mathcal{N}^1 \\
& \leq - \int_{\Sigma} ((J^1)^{-1} Z^3 \bar{\eta} \cdot \mathcal{N}^1 \llbracket \partial_3 q^1 \rrbracket + Z^3 \eta^2 \cdot \overline{J^{-1} \mathcal{N} \llbracket \partial_3 q \rrbracket}) Z^3 \partial_t \bar{\eta} \cdot \mathcal{N}^1 + \mathcal{M}_0^m \bar{\mathcal{E}} \\
& \leq - \frac{d}{dt} \int_{\Sigma} \left(\frac{1}{2} (J^1)^{-1} \llbracket \partial_3 q^1 \rrbracket |Z^3 \bar{\eta} \cdot \mathcal{N}^1|^2 + Z^3 \eta^2 \cdot \overline{J^{-1} \mathcal{N} \llbracket \partial_3 q \rrbracket} Z^3 \bar{\eta} \cdot \mathcal{N}^1 \right) + \mathcal{M}_0^m \bar{\mathcal{E}} \tag{8.13}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} J^1 b^1 \cdot \nabla_{\mathcal{A}^1}(\bar{\mathcal{B}} \cdot \bar{\mathcal{V}}) = - \int_{\Sigma} b^1 \cdot \mathcal{N}^1 \llbracket \bar{\mathcal{B}} \rrbracket \cdot \bar{\mathcal{V}} \\
& = - \int_{\Sigma} b_0 \cdot \mathcal{N}_0 \llbracket \bar{\mathcal{B}} \rrbracket \cdot (Z^3 \bar{v} - Z^3 \bar{\eta} \cdot \nabla_{\mathcal{A}^1} v^1 - Z^3 \eta^2 \cdot \overline{\nabla_{\mathcal{A}} v}) \\
& \leq \int_{\Sigma} b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^3 \eta_0 \cdot \mathcal{N}_0 \llbracket \partial_3 \bar{b} \rrbracket \cdot Z^3 \partial_t \bar{\eta} + \mathcal{M}_0^m \bar{\mathcal{E}} \\
& \leq \frac{d}{dt} \int_{\Sigma} b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^3 \eta_0 \cdot \mathcal{N}_0 \llbracket \partial_3 \bar{b} \rrbracket \cdot Z^3 \bar{\eta} + \mathcal{M}_0^m \bar{\mathcal{E}}. \tag{8.14}
\end{aligned}$$

By plugging (8.13) and (8.14) into (8.12) and then integrating in time, using the fundamental theorem of calculus and the fact that the two solutions take the same initial data, one deduces that

$$\|(\bar{\mathcal{Q}}, \bar{\mathcal{V}}, \bar{\mathcal{B}})(t)\|_0^2 \leq \mathcal{M}_0^m \left(|Z^3 \bar{\eta}(t)|_0^2 + \int_0^t \bar{\mathcal{E}} \right), \tag{8.15}$$

which yields

$$\|(Z^3 \bar{p}, Z^3 \bar{v}, Z^3 \bar{b})(t)\|_0^2 \leq \mathcal{M}_0^m \left(|Z^3 \bar{\eta}(t)|_0^2 + \int_0^t \bar{\mathcal{E}} \right). \tag{8.16}$$

Recalling the Cauchy formula (1.17), one has

$$\bar{b} = \rho^1 \rho_0^{-1} \mathcal{A}_0^T b_0 \cdot \nabla \bar{\eta} + \bar{\rho} \rho_0^{-1} \mathcal{A}_0^T b_0 \cdot \nabla \eta^2. \tag{8.17}$$

Then as in the proof of Proposition 5.3, one can deduce from (8.16) that

$$\|(\bar{p}, \bar{v}, \bar{b})(t)\|_{0,3}^2 + |\bar{\eta}(t)|_3^2 \leq \mathcal{M}_0^m \int_0^t \bar{\mathcal{E}}. \tag{8.18}$$

On the other hand, one can proceed as in the proof of Propositions 4.1, 5.5, 6.1, 6.3 to estimate the other terms in $\bar{\mathcal{E}}$ and conclude that

$$\bar{\mathcal{E}}(t) \leq \mathcal{M}_0^m \int_0^t \bar{\mathcal{E}}. \tag{8.19}$$

This and the Gronwall lemma imply $\bar{\mathcal{E}}(t) = 0$ since $\bar{\mathcal{E}}(0) = 0$, and hence the uniqueness follows. The proof of Theorem 2.4 is thus completed. \square

APPENDIX A. SMOOTHING INITIAL DATA

We now elaborate the smoothing process that regularizes the data $(\eta_0, p_0, v_0, b_0, \rho_0)$ of (1.10) given in Theorem 2.1 to produce the smooth data $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta)$, with the smoothing parameter $\delta > 0$.

Let $\Lambda_\delta, \phi_\delta$ be the standard mollifiers in \mathbb{R}^2 and \mathbb{R}^3 , respectively, and \mathcal{E}_{Ω_\pm} be the Sobolev extension operator. Let $\tilde{\rho}_0^\delta := \phi_\delta * \mathcal{E}_{\Omega_\pm} \rho_{0,\pm}$ in Ω_\pm . Define $\tilde{\eta}_0^\delta$ as the solution to (see [15])

$$\begin{cases} -\Delta^2 \tilde{\eta}_0^\delta = -\Delta^2(\phi_\delta * \mathcal{E}_{\Omega_\pm} \eta_{0,\pm}) & \text{in } \Omega_\pm \\ \tilde{\eta}_0^\delta = \Lambda_\delta * \eta_0, \quad \partial_3 \tilde{\eta}_0^\delta = \Lambda_\delta * \partial_3 \eta_0 & \text{on } \Sigma \cup \Sigma_\pm, \end{cases} \tag{A.1}$$

$\tilde{p}_0^\delta := \mathcal{S}_{\Omega_\pm} p_{0,\pm}$ in Ω_\pm as the solution to

$$\begin{cases} -\Delta \tilde{p}_0^\delta = -\Delta(\phi_\delta * \mathcal{E}_{\Omega_\pm} p_{0,\pm}) & \text{in } \Omega_\pm \\ \tilde{p}_0^\delta = \Lambda_\delta * p_0 & \text{on } \Sigma \cup \Sigma_\pm, \end{cases} \quad (\text{A.2})$$

$\tilde{v}_0^\delta := \mathcal{S}_{\Omega_\pm} v_{0,\pm}$ in Ω_\pm and $\tilde{b}_0^\delta := \mathcal{S}_{\Omega_\pm} b_{0,\pm}$ in Ω_\pm . Then $(\tilde{\eta}_0^\delta, \tilde{p}_0^\delta, \tilde{v}_0^\delta, \tilde{b}_0^\delta, \tilde{\rho}_0^\delta) \in C^\infty(\Omega_\pm)$, and by (2.3) and (3.4) with $j = 0$, one has

$$\left[\tilde{p}_0^\delta \right] = \left[\tilde{v}_0^\delta \right] = \left[\tilde{b}_0^\delta \right] = \left[\tilde{\eta}_0^\delta \right] = \left[\partial_3 \tilde{\eta}_0^\delta \right] = 0 \text{ on } \Sigma, \quad \tilde{v}_0^\delta = 0 \text{ and } \tilde{\eta}_{0,3}^\delta = \pm 1 \text{ on } \Sigma_\pm. \quad (\text{A.3})$$

By the standard elliptic theory, it is straightforward to verify that

$$\tilde{\eta}_0^\delta \rightarrow \eta_0 \text{ in } H^m(\Omega_\pm) \cap H^m(\Sigma) \text{ and } (\tilde{p}_0^\delta, \tilde{v}_0^\delta, \tilde{b}_0^\delta, \tilde{\rho}_0^\delta) \rightarrow (p_0, v_0, b_0, \rho_0) \text{ in } H^m(\Omega_\pm) \text{ as } \delta \rightarrow 0. \quad (\text{A.4})$$

Moreover, by (2.4) and (A.4), one obtains that for $0 < \delta < \delta_0$ with some $\delta_0 > 0$,

$$\tilde{\rho}_0^\delta, \tilde{p}_0^\delta, |\tilde{J}_0^\delta| \geq \frac{c_0}{2} > 0 \text{ in } \Omega \text{ and } |\tilde{b}_0^\delta \cdot \tilde{\mathcal{N}}_0^\delta| \geq \frac{c_0}{2} > 0 \text{ on } \Sigma \cup \Sigma_\pm, \quad (\text{A.5})$$

where $\tilde{J}_0^\delta = J(\tilde{\eta}_0^\delta)$ and $\tilde{\mathcal{N}}_0^\delta = \mathcal{N}(\tilde{\eta}_0^\delta)$. Note that the smoothed data $(\tilde{\eta}_0^\delta, \tilde{p}_0^\delta, \tilde{v}_0^\delta, \tilde{b}_0^\delta, \tilde{\rho}_0^\delta)$ satisfies the zero-th order compatibility conditions for (1.10) (cf. (3.4) for $j = 0$), but it may not satisfy any higher order compatibility conditions (cf. (3.4) for $j \geq 1$). To adjust this, the idea is to add corrections to $(\tilde{p}_0^\delta, \tilde{v}_0^\delta, \tilde{b}_0^\delta)$ so that the higher order compatibility conditions hold. We may assume $\Omega = \mathbb{T}^2 \times \mathbb{R}$ for simplicity; in such case, one only needs to add corrections to $(\tilde{p}_{0,+}^\delta, \tilde{v}_{0,+}^\delta, \tilde{b}_{0,+}^\delta)$ and keep $(\tilde{p}_{0,-}^\delta, \tilde{v}_{0,-}^\delta, \tilde{b}_{0,-}^\delta)$ unchanged.

To simplify the notations, we denote $U := (p, v, b)$ and define

$$\partial_t^j U^T := \partial_t^{j-1} \begin{pmatrix} -\gamma p \operatorname{div}_{\mathcal{A}} v \\ \rho^{-1} (b \cdot \nabla_{\mathcal{A}} b - \nabla_{\mathcal{A}} (p + \frac{1}{2}|b|^2)) \\ b \cdot \nabla_{\mathcal{A}} v - b \operatorname{div}_{\mathcal{A}} v \end{pmatrix}, \quad j \geq 1, \quad (\text{A.6})$$

recursively, where $\mathcal{A} = \mathcal{A}(\eta)$ and $\partial_t^j \eta = \partial_t^{j-1} v$ and $\partial_t^j \rho = -\partial_t^{j-1} (\rho \operatorname{div}_{\mathcal{A}} v)$ for $j \geq 1$ have been assumed. One finds that $\partial_t^j U$ depends on $\nabla \eta, \rho, \tilde{\nabla}^j U, \partial_3^j U$, $j \geq 1$, where $\tilde{\nabla}^j$ denotes a collection of ∂^α for $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq j$ and $\alpha_3 < j$. Then we define

$$\mathbf{U}_{\nabla \eta, \rho}^j(\tilde{\nabla}^j U, \partial_3^j U) := \partial_t^j U^T, \quad j \geq 0. \quad (\text{A.7})$$

Set

$$\mathbf{V}_{\nabla \eta, \rho}^j(\tilde{\nabla}^j U, \partial_3^j U) := \begin{pmatrix} \mathbf{U}_p^j \\ \mathbf{U}_v^j \cdot \tau^1 \\ \mathbf{U}_v^j \cdot \tau^2 \\ \mathbf{U}_v^j \cdot n \\ \mathbf{U}_b^j \cdot \tau^1 \\ \mathbf{U}_b^j \cdot \tau^2 \end{pmatrix}, \quad j \geq 0 \text{ and } \mathbf{W}_{\nabla \eta}^j(\partial_3^j U) := \begin{pmatrix} \partial_3^j p \\ \partial_3^j v \cdot \tau^1 \\ \partial_3^j v \cdot \tau^2 \\ \partial_3^j v \cdot n \\ \partial_3^j b \cdot \tau^1 \\ \partial_3^j b \cdot \tau^2 \end{pmatrix}, \quad j \geq 1, \quad (\text{A.8})$$

where $(\tau^1, \tau^2, n) = (\tau^1, \tau^2, n)(\eta)$ and \mathbf{U}_p^j denotes the p -component (the first component) of \mathbf{U}^j , etc. By (A.6)–(A.8), it holds that

$$\mathbf{V}_{\nabla \eta, \rho}^j(\tilde{\nabla}^j U, \partial_3^j U) = (\mathbb{E}_{\nabla \eta, \rho}(U))^j \mathbf{W}_{\nabla \eta}^j(\partial_3^j U) + \mathbb{F}_{\nabla \eta, \rho}^j(\tilde{\nabla}^j U), \quad j \geq 1, \quad (\text{A.9})$$

where

$$\mathbb{E}_{\nabla \eta, \rho}(U) := \begin{pmatrix} 0 & 0 & 0 & -\gamma p J^{-1} |\mathcal{N}| & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^{-1} J^{-1} b \cdot \mathcal{N} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^{-1} J^{-1} b \cdot \mathcal{N} \\ -\rho^{-1} J^{-1} |\mathcal{N}| & 0 & 0 & 0 & -\rho^{-1} J^{-1} |\mathcal{N}| b \cdot \tau^1 & -\rho^{-1} J^{-1} |\mathcal{N}| b \cdot \tau^2 \\ 0 & J^{-1} b \cdot \mathcal{N} & 0 & J^{-1} |\mathcal{N}| b \cdot \tau^1 & 0 & 0 \\ 0 & 0 & J^{-1} b \cdot \mathcal{N} & J^{-1} |\mathcal{N}| b \cdot \tau^2 & 0 & 0 \end{pmatrix}. \quad (\text{A.10})$$

Here for simplicity we have not written out the explicit form of $\mathbb{F}_{\nabla \eta, \rho}^j(\tilde{\nabla}^j U)$ but only state its dependences, and the key feature here is that it does not depend on $\partial_3^j U$ when $j \geq 1$, that is, we separate the dependences of $\mathbf{V}_{\nabla \eta, \rho}^j$ on $\partial_3^j U$ and $\tilde{\nabla}^j U$.

Now we construct the smoothed data that satisfies the higher order compatibility conditions as follows. First, define $U_0^{\delta,(0)} := (p_0^{\delta,(0)}, v_0^{\delta,(0)}, b_0^{\delta,(0)}) = (\tilde{p}_0^\delta, \tilde{v}_0^\delta, \tilde{b}_0^\delta)$ as the zero-th order smooth approximation of $U_0 := (p_0, v_0, b_0)$. Note that by (A.3), $(\tilde{\eta}_0^\delta, U_0^{\delta,(0)}, \tilde{\rho}_0^\delta)$ satisfies the zero-th order compatibility conditions:

$$\left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta}^0(U_0^{\delta,(0)}) \right] = 0 \text{ on } \Sigma, \quad (\text{A.11})$$

and by (A.5), $\mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_0^{\delta,(0)})$ (here, unlike the other terms, we have specified the subscript “+” in $\tilde{\rho}_{0,+}^\delta$ as $\tilde{\rho}_0^\delta$ may not be continuous across the interface Σ , etc.) is non-singular on Σ :

$$\det \mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_0^{\delta,(0)}) = \gamma \tilde{p}_0^\delta |\tilde{\mathcal{N}}_0^\delta|^2 (\tilde{\rho}_{0,+}^\delta)^{-3} (\tilde{J}_0^\delta)^{-6} |\tilde{b}_0^\delta \cdot \tilde{\mathcal{N}}_0^\delta|^4 > 0 \text{ on } \Sigma. \quad (\text{A.12})$$

Next, suppose that $j \geq 0$ and the j -th order smooth approximation $U_0^{\delta,(j)} := (p_0^{\delta,(j)}, v_0^{\delta,(j)}, b_0^{\delta,(j)})$ of U_0 has been constructed so that $(\tilde{\eta}_0^\delta, U_0^{\delta,(j)}, \tilde{\rho}_0^\delta)$ satisfies the j -th order compatibility conditions:

$$\left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^\ell(\tilde{\nabla}^\ell U_0^{\delta,(j)}, \partial_3^\ell U_0^{\delta,(j)}) \right] = 0, \quad \ell = 0, \dots, j \quad (\text{A.13})$$

and

$$\det \mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_0^{\delta,(j)}) > 0 \text{ on } \Sigma. \quad (\text{A.14})$$

Define

$$\tilde{\Phi}^{\delta,(j+1)} = - \left(\mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_{0,+}^{\delta,(j)}) \right)^{-(j+1)} \left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_0^{\delta,(j)}, \partial_3^{j+1} U_0^{\delta,(j)}) \right] \text{ on } \Sigma \quad (\text{A.15})$$

and

$$\Phi^{\delta,(j+1)} := \begin{pmatrix} \tilde{\Phi}_1^{\delta,(j+1)} \\ \tilde{\Phi}_2^{\delta,(j+1)} \tilde{\tau}_0^{1,\delta} + \tilde{\Phi}_3^{\delta,(j+1)} \tilde{\tau}_0^{2,\delta} + \tilde{\Phi}_4^{\delta,(j+1)} \tilde{n}_0^\delta \\ \tilde{\Phi}_5^{\delta,(j+1)} \tilde{\tau}_0^{1,\delta} + \tilde{\Phi}_6^{\delta,(j+1)} \tilde{\tau}_0^{2,\delta} \end{pmatrix} \text{ on } \Sigma, \quad (\text{A.16})$$

where $(\tilde{\tau}_0^{1,\delta}, \tilde{\tau}_0^{2,\delta}, \tilde{n}_0^\delta) = (\tau^1, \tau^2, n)(\tilde{\eta}_0^\delta)$. Then we construct the $(j+1)$ -th order smooth approximation $U_0^{\delta,(j+1)} := (p_0^{\delta,(j+1)}, v_0^{\delta,(j+1)}, b_0^{\delta,(j+1)})$ of U_0 such that

$$\partial_3^\ell U_{0,+}^{\delta,(j+1)} = \partial_3^\ell U_{0,+}^{\delta,(j)}, \quad \ell = 0, \dots, j \text{ and } \partial_3^{j+1} U_{0,+}^{\delta,(j+1)} = \partial_3^{j+1} U_{0,+}^{\delta,(j)} + \Phi^{\delta,(j+1)} \text{ on } \Sigma. \quad (\text{A.17})$$

The existence of such $U_0^{\delta,(j+1)}$ is standard (see [15]). It then follows that

$$\left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^\ell(\tilde{\nabla}^\ell U_0^{\delta,(j+1)}, \partial_3^\ell U_0^{\delta,(j+1)}) \right] = \left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^\ell(\tilde{\nabla}^\ell U_0^{\delta,(j)}, \partial_3^\ell U_0^{\delta,(j)}) \right] = 0 \text{ on } \Sigma, \quad \ell = 0, \dots, j, \quad (\text{A.18})$$

$$\det \mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_{0,+}^{\delta,(j+1)}) = \det \mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_{0,+}^{\delta,(j)}) > 0 \text{ on } \Sigma, \quad (\text{A.19})$$

$$\mathbf{W}_{\nabla \tilde{\eta}_0^\delta}^{j+1}(\partial_3^{j+1} U_{0,+}^{\delta,(j+1)}) = \mathbf{W}_{\nabla \tilde{\eta}_0^\delta}^{j+1}(\partial_3^{j+1} U_{0,+}^{\delta,(j)}) + \tilde{\Phi}^{\delta,(j+1)} \text{ on } \Sigma \quad (\text{A.20})$$

and

$$\mathbb{F}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_{0,+}^{\delta,(j+1)}) = \mathbb{F}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_{0,+}^{\delta,(j)}) \text{ on } \Sigma. \quad (\text{A.21})$$

By (A.9), (A.20) and (A.15), one has

$$\begin{aligned} & \left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_0^{\delta,(j+1)}, \partial_3^{j+1} U_0^{\delta,(j+1)}) \right] \\ &= \left[\left(\mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}(U_0^{\delta,(j+1)}) \right)^{j+1} \mathbf{W}_{\nabla \tilde{\eta}_0^\delta}^{j+1}(\partial_3^{j+1} U_0^{\delta,(j+1)}) \right] + \left[\mathbb{F}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_0^{\delta,(j+1)}) \right] \\ &= \left[\left(\mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}(U_0^{\delta,(j)}) \right)^{j+1} \mathbf{W}_{\nabla \tilde{\eta}_0^\delta}^{j+1}(\partial_3^{j+1} U_0^{\delta,(j+1)}) \right] + \left[\mathbb{F}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_0^{\delta,(j)}) \right] \\ &= \left[\mathbf{V}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_0^\delta}^{j+1}(\tilde{\nabla}^{j+1} U_0^{\delta,(j)}, \partial_3^{j+1} U_0^{\delta,(j)}) \right] + \left(\mathbb{E}_{\nabla \tilde{\eta}_0^\delta, \tilde{\rho}_{0,+}^\delta}(U_{0,+}^{\delta,(j)}) \right)^{j+1} \tilde{\Phi}^{\delta,(j+1)} = 0. \end{aligned} \quad (\text{A.22})$$

This together with (A.18) implies that $(\tilde{\eta}_0^\delta, U_0^{\delta,(j+1)}, \tilde{\rho}_0^\delta)$ satisfies the $(j+1)$ -th order compatibility conditions. Consequently, one can construct recursively the smooth data that satisfies any high order compatibility conditions.

Now the desired smoothed data is constructed as $(\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta, \rho_0^\delta) := (\tilde{\eta}_0^\delta, U_0^{\delta, (m-1)}, \tilde{\rho}_0^\delta)$. Note that by (A.4), $\eta_0^\delta \rightarrow \eta_0$ in $H^m(\Omega_\pm) \cap H^m(\Sigma)$ and $\rho_0^\delta \rightarrow \rho_0$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$, however, in general, $(p_0^\delta, v_0^\delta, b_0^\delta)$ does not converge to (p_0, v_0, b_0) . But if $(\eta_0, p_0, v_0, b_0, \rho_0)$ satisfy the $(m-1)$ -th order compatibility conditions:

$$\left[\mathbf{V}_{\nabla \eta_0, \rho_0}^\ell (\tilde{\nabla}^\ell U_0, \partial_3^\ell U_0) \right] = 0, \quad \ell = 0, \dots, m-1, \quad (\text{A.23})$$

then $(p_0^\delta, v_0^\delta, b_0^\delta) \rightarrow (p_0, v_0, b_0)$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$. Indeed, we claim that $U_0^{\delta, (j)} \rightarrow U_0$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$ for $j = 0, \dots, m-1$. We prove the claim by induction on j . First, for $j = 0$, no corrections have been added and so it is direct to have that $U_0^{\delta, (0)} \rightarrow U_0$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$. Next, suppose that $j \in [0, m-2]$ and that $U_0^{\delta, (\ell)} \rightarrow U_0$ in $H^m(\Omega_\pm)$ for $\ell = 0, \dots, j$ as $\delta \rightarrow 0$ have been proved. Then by the definition of the correction $\Phi^{\delta, (j+1)}$ (cf. (A.15) and (A.16)), the induction assumption and (A.23) with $\ell = j+1$, one deduces that $\Phi^{\delta, (j+1)} \rightarrow 0$ in $H^{m-j-3/2}(\Sigma)$ as $\delta \rightarrow 0$. So by the definition of $U_0^{\delta, (j+1)}$ (cf. (A.17)), $U_0^{\delta, (j+1)} - U_0^{\delta, (j)} \rightarrow 0$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$, which together with the induction assumption again implies that $U_0^{\delta, (j+1)} \rightarrow U_0$ in $H^m(\Omega_\pm)$ as $\delta \rightarrow 0$. The claim is thus proved, and the construction of the smoothed data is completed.

APPENDIX B. ANISOTROPIC TRACE ESTIMATES

Lemma B.1. *Assume that $\bar{B} \in \mathbb{R}^3$ with $|\bar{B}_3| \geq \theta > 0$ for $0 \leq |x_3| \leq \iota$. Then it holds that*

$$|f|_0^2 \leq C_{\bar{B}, \theta, \iota} \left(\|\bar{B} \cdot \nabla f\|_0 \|f\|_0 + \|f\|_0^2 \right), \quad (\text{B.1})$$

where $C_{\bar{B}, \theta, \iota}$ is a positive constant depending on the C^1 norm of \bar{B} , θ and ι .

Proof. Without loss of generality, assume that $\bar{B}_3 \geq \theta > 0$ for $0 \leq x_3 \leq \iota$. Set $\tilde{B} := \bar{B}/\bar{B}_3 = (\tilde{B}_h, 1)$ and define the 2D trajectory $Y = Y(x_1, x_2; s)$ by

$$\begin{cases} \frac{d}{ds} Y(x_1, x_2; s) = \tilde{B}_h(Y(x_1, x_2; s), s) \\ Y(x_1, x_2; 0) = (x_1, x_2). \end{cases} \quad (\text{B.2})$$

Then for any $0 \leq y_3 \leq \iota$, by the fundamental theorem of calculus,

$$\begin{aligned} f^2(x_1, x_2, 0) &= f^2(Y(x_1, x_2; 0), 0) = f^2(Y(x_1, x_2; y_3), y_3) - \int_0^{y_3} \frac{d}{ds} (f^2(Y(x_1, x_2; s), s)) ds \\ &= f^2(Y(x_1, x_2; y_3), y_3) - 2 \int_0^{y_3} (\tilde{B} \cdot \nabla f f)(Y(x_1, x_2; s), s) ds. \end{aligned} \quad (\text{B.3})$$

Integrating over $(x_1, x_2) \in \mathbb{T}^2$ and using the Fubini theorem, one has

$$|f|_0^2 = \int_{\mathbb{T}^2} f^2(Y(x_1, x_2; y_3), y_3) dx_1 dx_2 - 2 \int_0^{y_3} \int_{\mathbb{T}^2} (\tilde{B} \cdot \nabla f f)(Y(x_1, x_2; s), s) dx_1 dx_2 ds. \quad (\text{B.4})$$

For any fixed $0 \leq s \leq \iota$, by a change of variables $(y_1, y_2) = Y(x_1, x_2; s)$, with the Jacobian $J(x_1, x_2; s) := \det \nabla Y(x_1, x_2; s)$ given by

$$J(x_1, x_2; s) = \exp \left(- \int_0^s (\partial_1 \tilde{B}_1 + \partial_2 \tilde{B}_2)(Y(x_1, x_2; \tau), \tau) d\tau \right), \quad (\text{B.5})$$

one has that for any $0 \leq s \leq \iota$,

$$C_{\tilde{B}, \iota}^{-1} \leq J(x_1, x_2; s) \leq C_{\tilde{B}, \iota} := \exp \left(\left\| \partial_1 \tilde{B}_1 + \partial_2 \tilde{B}_2 \right\|_{L^\infty} \iota \right). \quad (\text{B.6})$$

Hence, for any $0 \leq y_3 \leq \iota$, it holds that

$$\begin{aligned} \int_{\mathbb{T}^2} f^2(Y(x_1, x_2, y_3), y_3) dx_1 dx_2 &= \int_{\mathbb{T}^2} f^2(y_1, y_2, y_3) J^{-1}(y_1, y_2; y_3) dy_1 dy_2 \\ &\leq C_{\tilde{B}, \iota} \int_{\mathbb{T}^2} f^2(y_1, y_2, y_3) dy_1 dy_2 \end{aligned} \quad (\text{B.7})$$

and similarly, by the Cauchy-Schwarz inequality, one has

$$\begin{aligned}
& -2 \int_0^{y_3} \int_{\mathbb{T}^2} (\tilde{B} \cdot \nabla f f)(Y(x_1, x_2; s), s) dx_1 dx_2 ds \\
& \leq 2 \left(\int_0^{y_3} \int_{\mathbb{T}^2} (\tilde{B} \cdot \nabla f)^2(Y(x_1, x_2; s), s) dx_1 dx_2 ds \right)^{1/2} \left(\int_0^{y_3} \int_{\mathbb{T}^2} f^2(Y(x_1, x_2; s), s) dx_1 dx_2 ds \right)^{1/2} \\
& \leq 2C_{\tilde{B}, \iota} \left(\int_0^{y_3} \int_{\mathbb{T}^2} (\tilde{B} \cdot \nabla f)^2(y_1, y_2, s) dy_1 dy_2 ds \right)^{1/2} \left(\int_0^{y_3} \int_{\mathbb{R}^2} f^2(y_1, y_2, s) dy_1 dy_2 ds \right)^{1/2} \\
& \leq 2C_{\tilde{B}, \iota} \left\| \tilde{B} \cdot \nabla f \right\|_0 \|f\|_0
\end{aligned} \tag{B.8}$$

Combining (B.4), (B.7) and (B.8) yields

$$|f|_0^2 \leq C_{\tilde{B}, \iota} \int_{\mathbb{T}^2} f^2(y_1, y_2, y_3) dy_1 dy_2 + 2C_{\tilde{B}, \iota} \left\| \tilde{B} \cdot \nabla f \right\|_0 \|f\|_0. \tag{B.9}$$

Integrating (B.9) over $y_3 \in (0, \iota)$, one has

$$\iota |f|_0^2 \leq C_{\tilde{B}, \iota} \int_0^\iota \int_{\mathbb{T}^2} f^2(y_1, y_2, y_3) dy_1 dy_2 dy_3 + 2C_{\tilde{B}, \iota} \left\| \tilde{B} \cdot \nabla f \right\|_0 \|f\|_0, \tag{B.10}$$

which implies, recalling $\tilde{B} = \bar{B}/\bar{B}_3$,

$$\begin{aligned}
|f|_0^2 & \leq C_{\tilde{B}, \iota} \iota^{-1} \|f\|_0^2 + 2C_{\tilde{B}, \iota} \left\| \bar{B}_3^{-1} \bar{B} \cdot \nabla f \right\|_0 \|f\|_0 \\
& \leq C_{\tilde{B}, \iota} \iota^{-1} \|f\|_0^2 + 4C_{\tilde{B}, \iota} \theta^{-1} \left\| \bar{B} \cdot \nabla f \right\|_0 \|f\|_0.
\end{aligned} \tag{B.11}$$

This gives (B.1) by redefining the constant $C_{\bar{B}, \theta, \iota}$. \square

REFERENCES

- [1] S. Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations* **14** (1989), no. 2, 173–230.
- [2] A. Blokhin, Y. Trakhinin. Stability of strong discontinuities in fluids and MHD. *Handbook of Mathematical Fluid Dynamics, Vol. I*, North-Holland, Amsterdam, 2002, pp. 545–652.
- [3] A.-L. Cauchy. Théorie de la propagation des ondes à la surface d'une fluide pesant d'une fluide pesant d'une profondeur indéfinie. *Oeuvres Complètes d'Augustin Cauchy*, Série I, Tome I, 5–318, 1882.
- [4] G.-Q. Chen, Y.-G. Wang. Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **187** (2008), no. 3, 369–408.
- [5] C. H. A. Cheng, D. Coutand, S. Shkoller. On the motion of vortex sheets with surface tension in three-dimensional Euler equations with vorticity. *Comm. Pure Appl. Math.* **61** (2008), no. 12, 1715–1752.
- [6] J.-F. Coulombel, A. Morando, P. Secchi, P. Trebeschi. A priori estimates for 3D incompressible current-vortex sheets. *Comm. Math. Phys.* **311** (2012), no. 1, 247–275.
- [7] J.-F. Coulombel, P. Secchi. The stability of compressible vortex sheets in two space dimensions. *Indiana Univ. Math. J.* **53** (2004), no. 4, 941–1012.
- [8] J.-F. Coulombel, P. Secchi. Nonlinear compressible vortex sheets in two space dimensions. *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 1, 85–139.
- [9] I. V. Denisova, V. A. Solonnikov. Local and global solvability of free boundary problems for the compressible Navier–Stokes equations near equilibria. *Handbook of mathematical analysis in mechanics of viscous fluids*, Springer, Cham, 2018, pp. 1947–2035.
- [10] D. Ebin. Ill-posedness of the Rayleigh–Taylor and Helmholtz problems for incompressible fluids. *Comm. Partial Differential Equations* **13** (1988), no. 10, 1265–1295.
- [11] J. A. Fejer, J. W. Miles. On the stability of a plane vortex sheet with respect to three-dimensional disturbances. *J. Fluid Mech.* **15** (1963), 335–336.
- [12] J. Goedbloed, S. Poedts. Principles of magnetohydrodynamics with applications to laboratory and astrophysical plasmas. Cambridge University Press, Cambridge, (2004).
- [13] Y. Guo, I. Tice. Compressible, inviscid Rayleigh–Taylor instability. *Indiana Univ. Math. J.* **60** (2011), no. 2, 677–712.
- [14] L. D. Landau, E. M. Lifshiz. Electrodynamics of continuous media. Course of Theoretical Physics Vol. 8. Pergamon Press, Oxford, (1984).
- [15] J. L. Lions, E. Magenes. Nonhomogeneous Boundary Value Problems and Applications. Springer-Verlag, Berlin, 1972.

- [16] J. W. Miles. On the disturbed motion of a plane vortex sheet. *J. Fluid Mech.* **4** (1958), 538–552.
- [17] A. Morando, Y. Trakhinin, P. Trebeschi. Well-posedness of the linearized problem for MHD contact discontinuities. *J. Differential Equations* **258** (2015), no. 7, 2531–2571.
- [18] A. Morando, Y. Trakhinin, P. Trebeschi. Local existence of MHD contact discontinuities. *Arch. Ration. Mech. Anal.* **228** (2018), no. 2, 691–742.
- [19] A. Morando, P. Trebeschi. Two-dimensional vortex sheets for the nonisentropic Euler equations: linear stability. *J. Hyperbolic Differ. Equ.* **5** (2008), no. 3, 487–518.
- [20] A. Morando, P. Trebeschi, T. Wang. Two-dimensional vortex sheets for the nonisentropic Euler equations: nonlinear stability. *J. Differential Equations* **266** (2019), no. 9, 5397–5430.
- [21] J. Shatah, C. Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.* **61** (2008), no. 6, 848–876.
- [22] J. Shatah, C. Zeng. Local well-posedness for fluid interface problems. *Arch. Ration. Mech. Anal.* **199** (2011), no. 2, 653–705.
- [23] B. Stevens. Short-time structural stability of compressible vortex sheets with surface tension. *Arch. Ration. Mech. Anal.* **222** (2016), no. 2, 603–730.
- [24] Y. Sun, W. Wang, Z. Zhang. Nonlinear stability of the current-vortex sheet to the incompressible MHD equations. *Comm. Pure Appl. Math.* **71** (2018), no. 2, 356–403.
- [25] S. I. Syrovatskij. The stability of tangential discontinuities in a magnetohydrodynamic medium. *Zh. Ehksper. Teor. Fiz.* **24** (1953), no. 6, 622–629.
- [26] S. I. Syrovatskij. Instability of tangential discontinuities in a compressible medium. *Zh. Ehksper. Teor. Fiz.* **27** (1954), 121–123.
- [27] Y. Trakhinin. Existence of compressible current-vortex sheets: variable coefficients linear analysis. *Arch. Ration. Mech. Anal.* **177** (2005), no. 3, 331–366.
- [28] Y. Trakhinin. The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **191** (2009), no. 2, 245–310.
- [29] T. Yanagisawa. The initial boundary value problem for the equations of ideal Magneto-Hydrodynamics. *Hokkaido Math. J.* **16** (1987), no. 3, 295–314.
- [30] T. Yanagisawa, A. Matsumura. The fixed boundary value problems for the equations of ideal magnetohydrodynamics with a perfectly conducting wall condition. *Comm. Math. Phys.* **136** (1991), no. 1, 119–140.

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, FUJIAN 361005, CHINA
E-mail address, Y. J. Wang: yanjin.wang@xmu.edu.cn

THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, NT, HONG KONG
E-mail address, Z. P. Xin: zpxin@ims.cuhk.edu.hk