ON REGULAR SOLUTIONS FOR THREE-DIMENSIONAL FULL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DEGENERATE VISCOSITIES AND FAR FIELD VACUUM

QIN DUAN, ZHOUPING XIN, AND SHENGGUO ZHU

Abstract. In this paper, the Cauchy problem for the three-dimensional (3-D) full compressible Navier-Stokes equations (**CNS**) with zero thermal conductivity is considered. First, when shear and bulk viscosity coefficients both depend on the absolute temperature θ in a power law (θ^ν with $\nu > 0$) of Chapman-Enskog, based on some elaborate analysis of this system's intrinsic singular structures, we identify one class of initial data admitting a local-in-time regular solution with far field vacuum in terms of the mass density ρ, velocity u and entropy S. Furthermore, it is shown that within its life span of such a regular solution, the velocity stays in an inhomogeneous Sobolev space, i.e., $u \in H^3(\mathbb{R}^3)$, S has uniformly finite lower and upper bounds in the whole space, and the laws of conservation of total mass, momentum and total energy are all satisfied. Note that due to the appearance of the vacuum, the momentum equations are degenerate both in the time evolution and viscous stress tensor, and the physical entropy for polytropic gases behaves singularly, which make the study on corresponding well-posedness challenging. For proving the existence, we first introduce an enlarged reformulated structure by considering some new variables, which can transfer the degeneracies of the full **CNS** to the possible singularities of some special source terms related with S, and then carry out some singularly weighted energy estimates carefully designed for this reformulated system.

Contents

Date: February 11, 2022.
2010 *Mathematics Subject Classification.* 35Q30, 35A09, 35A01, 35B44, 35B40, 76N10.
Key words and phrases. Compressible Navier-Stokes equations, three-dimensions, degenerate viscosity, far field vacuum, local-in-time existence, asymptotic behavior.

1. Introduction

The motion of a compressible viscous, heat-conductive, and Newtonian polytropic fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following full **CNS**:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{T}, \\ (\rho \mathcal{E})_t + \mathrm{div}(\rho \mathcal{E} u + P u) = \mathrm{div}(u\mathbb{T}) + \mathrm{div}(\kappa \nabla \theta). \end{cases} \tag{1.1}$$

Here and throughout, $\rho \geq 0$ denotes the mass density, $u = (u^{(1)}, u^{(2)}, u^{(3)})^\top$ the fluid velocity, P the pressure of the fluid, θ the absolute temperature, $\mathcal{E} = e + \frac{1}{2}|u|^2$ the specific total energy, e the specific internal energy, $x = (x_1, x_2, x_3)^\top \in \Omega$ the Eulerian spatial coordinate and finally $t \geq 0$ the time coordinate. The equation of state for polytropic fluids satisfies

$$P = R\rho\theta = (\gamma - 1)\rho e = Ae^{S/c_v}\rho^\gamma, \ \ e = c_v\theta, \ \ c_v = \frac{R}{\gamma - 1}, \tag{1.2}$$

where R is the gas constant, A is a positive constant, c_v is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent and S is the entropy. $\mathbb{T}$ is the viscosity stress tensor given by

$$\mathbb{T} = 2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}_3, \tag{1.3}$$

where $D(u) = \frac{\nabla u + (\nabla u)^\top}{2}$ is the deformation tensor, $\mathbb{I}_3$ is the 3×3 identity matrix, μ is the shear viscosity coefficient, and $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient. κ denotes the coefficient of thermal conductivity.

In the theory of gas dynamics, the compressible Navier-Stokes equations can be derived from the Boltzmann equations through the Chapman-Enskog expansion, cf. Chapman-Cowling [6] and Li-Qin [29]. Under some proper physical assumptions, the viscosity coefficients (μ, λ) and the coefficient of thermal conductivity κ are not constants but functions of the absolute temperature θ such as:

$$\mu(\theta) = r_1 \theta^{\frac{1}{2}} F(\theta), \quad \lambda(\theta) = r_2 \theta^{\frac{1}{2}} F(\theta), \quad \kappa(\theta) = r_3 \theta^{\frac{1}{2}} F(\theta) \tag{1.4}$$

for some constants r_i $(i = 1, 2, 3)$. Actually for the cut-off inverse power force models, if the intermolecular potential varies as $r^{-\Upsilon}$, where r is intermolecular distance, then

$$F(\theta) = \theta^{\varpi} \quad \text{with} \quad \varpi = \frac{2}{\Upsilon} \in [0, \infty) \tag{1.5}$$

in (1.4). In particular, for Maxwellian molecules, $\Upsilon = 4$ and $\varpi = \frac{1}{2}$; for rigid elastic spherical molecules, $\Upsilon = \infty$ and $\varpi = 0$; while for ionized gas, $\Upsilon = 1$ and $\varpi = 2$.

In the current paper, we will consider the following case:

$$\mu(\theta) = \alpha\theta^\nu, \ \ \lambda(\theta) = \beta\theta^\nu, \ \ \kappa = 0, \tag{1.6}$$

i.e., the thermal conductivity vanishes, where (α,β,ν) are all constants satisfying

$$\alpha>0,\quad 2\alpha+3\beta\geq 0\quad\text{and}\quad 0<(\gamma-1)\nu<1. \tag{1.7}$$

In terms of (ρ,u,S), it follows from (1.2) and (1.6) that (1.1) can be rewritten as

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)+\nabla P=A^\nu R^{-\nu}\operatorname{div}(\rho^\delta e^{\frac{S}{c_v}\nu}Q(u)),\\ P\big(S_t+u\cdot\nabla S\big)=A^\nu R^{1-\nu}\rho^\delta e^{\frac{S}{c_v}\nu}H(u),\end{cases} \tag{1.8}$$

where $\delta=(\gamma-1)\nu$, and

$$Q(u)=\alpha(\nabla u+(\nabla u)^\top)+\beta\operatorname{div}u\mathbb{I}_3,\quad H(u)=\operatorname{div}(uQ(u))-u\cdot\operatorname{div}Q(u). \tag{1.9}$$

Let $\Omega=\mathbb{R}^3$. We study the local-in-time well-posedness of smooth solutions (ρ,u,S) with finite total mass and finite total energy to the Cauchy problem (1.8) with (1.2), (1.7), (1.9), and the following initial data and far field behavior:

$$(\rho,u,S)|_{t=0}=(\rho_0(x)>0,u_0(x),S_0(x))\quad\text{for}\quad x\in\mathbb{R}^3, \tag{1.10}$$

$$(\rho,u,S)(t,x)\to(0,0,\bar{S})\quad\text{as}\quad |x|\to\infty\quad\text{for}\quad t\geq 0, \tag{1.11}$$

where $\bar{S}$ is some constant.

Throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{aligned}&\|f\|_s=\|f\|_{H^s(\mathbb{R}^3)},\quad |f|_p=\|f\|_{L^p(\mathbb{R}^3)},\quad \|f\|_{m,p}=\|f\|_{W^{m,p}(\mathbb{R}^3)},\\ &|f|_{C^k}=\|f\|_{C^k(\mathbb{R}^3)},\quad \|f\|_{X_1\cap X_2}=\|f\|_{X_1}+\|f\|_{X_2},\\ &D^{k,r}=\{f\in L^1_{loc}(\mathbb{R}^3):|f|_{D^{k,r}}=|\nabla^k f|_r<\infty\},\\ &D^1_*=\{f\in L^6(\mathbb{R}^3):|f|_{D^1_*}=|\nabla f|_2<\infty\},\quad D^k=D^{k,2},\\ &|f|_{D^{k,r}}=\|f\|_{D^{k,r}(\mathbb{R}^3)},\quad |f|_{D^1_*}=\|f\|_{D^1_*(\mathbb{R}^3)},\quad \int f=\int_{\mathbb{R}^3}f\mathrm{d}x,\\ &X([0,T];Y(\mathbb{R}^3))=X([0,T];Y),\quad \|(f,g)\|_X=\|f\|_X+\|g\|_X.\end{aligned}$$

A detailed study of homogeneous Sobolev spaces can be found in Galdi [11].

Under the assumption that (μ,λ,κ) are all constants, when $\inf_x\rho_0(x)>0$, the local well-posedness of classical solutions to the Cauchy problem of (1.1) follows from the standard symmetric hyperbolic-parabolic structure which satisfies the well-known Kawashima's condition, cf. [16,17,19,35,36,39]. However, such an approach fails in the presence of the vacuum due to some new difficulties, for example, the degeneracy of the time evolution in the momentum equations:

$$\underbrace{\rho(u_t+u\cdot\nabla u)}_{\textit{Degenerate time evolution operator}}+\nabla P=\operatorname{div}(2\mu D(u)+\lambda\operatorname{div}u\mathbb{I}_3). \tag{1.12}$$

Generally vacuum will appear in the far field under some physical requirements such as finite total mass and total energy in the whole space $\mathbb{R}^3$. One of the main issues in the presence of vacuum is to understand the behavior of the fluids velocity, temperature and entropy near the vacuum. For general initial data containing vacuum, the local well-posedness of strong solutions to the Cauchy problem of the 3-D full **CNS** was first obtained by Cho-Kim [7] in a homogeneous Sobolev space in terms of (ρ, u, θ) under suitable initial compatibility conditions, which has been extended recently to be the global-in-time ones with small energy but large oscillations by Huang-Li [15], Wen-Zhu [40] and so on. It should be noticed that, in [7, 15, 40], the solution was established in some homogeneous space, that is, $\sqrt{\rho}u$ rather than u itself has the $L^\infty([0,T];L^2)$ regularity, the regularities obtained for (ρ, u, θ) do not provide any information about the entropy near the vacuum, and in general the solutions do not lie in inhomogeneous Sobolev spaces and guarantee the boundedness of the entropy near the vacuum. In fact, if ρ_0 has compact support, Li-Wang-Xin [24] prove that any non-trivial classical solutions with finite energy to the Cauchy problem of (1.1) with constant viscosities and heat conduction do not exist in general in the standard inhomogeneous Sobolev space for any short time, which indicates in particular that the homogeneous Sobolev space is crucial as studying the well-posedness (even locally in time) for the Cauchy problem of the **CNS** in the presence of such kind of vacuum. However, when the initial density vanishes only at far fields with a slow decay rate, recently in Li-Xin [25–27], for the Cauchy problem of the **CNS** with constant viscosities and heat conduction, it is shown that the uniform boundedness of S and the L^2 regularity of u can be propagated within the solution's life span. Despite these important progress, it remains unclear whether the laws of conservation of momentum and total energy hold for the solutions obtained in [7, 15, 25–27, 40]. We also refer the readers to [9, 10, 13, 14, 18, 20, 25, 26, 33, 41] and the references therein for some other related results on global existence of weak or strong solutions.

In contrast to the fruitful development of the classical setting, the treatment on the physical case (1.4)-(1.5) is lacking due to some new difficulties introduced in such relations, which lead to strong degeneracy and nonlinearity both in viscosity and heat conduction besides the degeneracy in the time evolution. Recently, for the Cauchy problem of the isentropic system (i.e., $(1.8)_1$-$(1.8)_2$ with $S(t,x)$ being constant), by introducing an elaborate (linear) elliptic approach on the singularly weighted regularity estimates for u and a symmetric hyperbolic system with singularity for some density related quantity, Xin-Zhu [43] identifies a class of initial data admitting one unique 3-D local regular solution with far field vacuum and finite energy to the Cauchy problem of (1.8) for the case $0<\delta<1$ in some inhomogeneous Sobolev spaces. Indeed, the momentum equations for isentropic flows can be written as

$$\underbrace{\rho(u_t + u\cdot\nabla u)}_{\textit{Degenerate time evolution operator}} + \nabla P = \underbrace{\operatorname{div}(\rho^\delta Q(u))}_{\textit{Degenerate elliptic operator}}. \tag{1.13}$$

Since the coefficients of the time evolution and the viscous stress tensor near the vacuum are powers of ρ, it is easy to compare the order of the degeneracy of these two operators near the vacuum, which enable us to select the dominant operator to control the behavior of u and lead to the "hyperbolic-strong singular elliptic" coupled structure in [43] and the " quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure in Xin-Zhu [42]. Some other interesting results on the well-posedness with vacuum to the isentropic **CNS** with degenerate viscosities can also be found in [2–5, 8, 12, 23, 28, 30, 31, 34, 38, 45] and the references therein.

Since e and S are fundamental dynamical variables for viscous compressible fluids, it is of great importance to study the corresponding theory for the non-isentropic

CNS. Yet, as indicated even in the constant (μ,λ,κ) case [25–27], this is a subtle and difficult problem in the presence of vacuum. Indeed, for considering the well-posedenss of classical solutions with vacuum to the full **CNS** (1.1)-(1.3) with degenerate viscosities of the form (1.4)-(1.5) with $\kappa=0$, the coefficients' structures of the time evolution operator and the viscous stress tensor are different, and the entropy plays important roles and satisfies a highly degenerate nonlinear transport equation near the vacuum, which cause substantial difficulties in the analysis and make it difficult to adapt the method for the isentropic case in [43]. It should be pointed out tha due to the physical requirements on θ and S near the vacuum, it is of more advantages to formulate the **CNS** (1.1)-(1.3) in terms of (ρ,u,S) instead of (ρ,u,θ) in contrast to [7,15,40] as illustrated below. Due to the equation of state for polytropic fluids (1.2), one has in the fluids region that $\rho>0$,

$$\theta = AR^{-1}\rho^{\gamma-1}e^{S/c_v}, \tag{1.14}$$

which implies that $(1.1)_2$-$(1.1)_3$ can be rewritten into

$$\begin{cases} \underbrace{\rho(u_t+u\cdot\nabla u)}_{\textit{Degenerate time evolution operator}} +\nabla P = \underbrace{A^\nu R^{-\nu}\mathrm{div}(\rho^\delta e^{\frac{S}{c_v}\nu}Q(u))}_{\textit{Degenerate elliptic operator}},\\ \underbrace{P\big(S_t+u\cdot\nabla S\big)}_{\textit{Degenerate time evolution operator}} = \underbrace{A^\nu R^{1-\nu}\rho^\delta e^{\frac{S}{c_v}\nu}H(u)}_{\textit{Strong nonlinearity}}, \end{cases} \tag{1.15}$$

Thus, if S has uniform boundedness in $\mathbb{R}^d$, then it is still possible to compare the orders of the degeneracy of the time evolution and viscous stress tensor near the vacuum via the powers of ρ, and then to choose proper structures to control the behaviors of the physical quantities. However, due to the high degeneracy in the time evolution operator of the entropy equation $(1.15)_2$, the physical entropy for polytropic gases behaves singularly in the presence of vacuum, and it is thus a challgenge to study its dynamics. It is worth pointing out that, when vacuum appears, the system formulated in terms of (ρ,u,θ) is not equivalent to the one formulated in terms of (ρ,u,S), since the boundedness and regularities of (ρ,θ) cannot provide those of S near the vacuum. In fact, for the constant (μ,λ,κ) case, most of the current progress on the well-posedness theory with vacuum [7,9,10,15, 40] to (1.1) are based on the formulation in terms of (ρ,u,θ) except [25–27], where the De Giorgi type iteration is carried out to the entropy equation for establishing the lower and upper bounds of the entropy. However, since the assumption that (μ,λ,κ) are all constants plays a key role in the analysis of [25–27], it seems difficult to adapt their arguments to the degenerate system (1.8). As far as we know, there have no any results on the boundedness and regularities of the entropy near the vacuum for flows with degenerate viscosities in the existing literatures.

Note that with the constraints (1.6)-(1.7), the momentum equations and the entropy equation $(1.8)_2$-$(1.8)_3$ in the fluids region can be formally rewritten as

$$\begin{cases} u_t+u\cdot\nabla u+\dfrac{A\gamma}{\gamma-1}e^{\frac{S}{c_v}}\nabla\rho^{\gamma-1}+A\rho^{\gamma-1}\nabla e^{\frac{S}{c_v}}+A^\nu R^{-\nu}\rho^{\delta-1}e^{\frac{S}{c_v}\nu}Lu\\ =A^\nu R^{-\nu}\dfrac{\delta}{\delta-1}\nabla\rho^{\delta-1}\cdot Q(u)e^{\frac{S}{c_v}\nu}+A^\nu R^{-\nu}\rho^{\delta-1}\nabla e^{\frac{S}{c_v}\nu}\cdot Q(u),\\ S_t+u\cdot\nabla S=A^{\nu-1}R^{1-\nu}\rho^{\delta-\gamma}e^{\frac{S}{c_v}(\nu-1)}H(u), \end{cases} \tag{1.16}$$

where L is the Lamé operator defined by

$$Lu \triangleq -\alpha\triangle u - (\alpha+\beta)\nabla \mathtt{div} u.$$

Then, to establish the existences of classical solutions with vacuum to (1.8), one would encounter some essential difficulties as follows:

(1) first, the right hand side of $(1.16)_1$ contains some strong singularities as:

$$\nabla\rho^{\delta-1}\cdot Q(u)e^{\frac{S}{c_v}\nu} \quad \text{and} \quad \rho^{\delta-1}\nabla e^{\frac{S}{c_v}\nu}\cdot Q(u),$$

whose controls require certain singularly weighted energy estimates which are highly non-trivial due to the singularities in the entropy equation $(1.16)_2$. Furthermore, the second term is more singular than the first one.

(2) second, even in the case that the uniform boundedness of the entropy S can be obtained, the coefficient $A^\nu R^{-\nu}\rho^{\delta-1}e^{\frac{S}{c_v}\nu}$ in front of the Lamé operator L will tend to ∞ as $\rho\to 0$ in the far filed. Then it is necessary to show that the term $\rho^{\delta-1}e^{\frac{S}{c_v}\nu}Lu$ is well defined in some Sobolev space near the vacuum.

(3) at last but more importantly, the time evolution equation $(1.16)_2$ for the entropy S also contains a strong singularity as:

$$A^{\nu-1}R^{1-\nu}\rho^{\delta-\gamma}e^{\frac{S}{c_v}(\nu-1)}H(u),$$

and its singularity can be measured in the level of $\rho^{\delta-\gamma}$ near the vacuum. This singularity will be the main obstacle preventing one from getting the uniform boundedness of the entropy and high order regularities, thus whose analysis becomes extremely crucial.

Therefore, the four quantities

$$(\rho^{\gamma-1},\ \nabla\rho^{\delta-1},\ \rho^{\delta-1}Lu,\ e^{\frac{S}{c_v}})$$

will play significant roles in our analysis on the higher order regularities of (u, S). Due to this observation, we first introduce a proper class of solutions called regular solutions to the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11).

Definition 1.1. *Let $T>0$ be a finite constant. The triple (ρ,u,S) is called a regular solution to the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11) in $[0,T]\times\mathbb{R}^3$ if (ρ,u,S) satisfies this problem in the sense of distributions and:*

(1) $\rho>0,\ \ \rho^{\gamma-1}\in C([0,T];D^1_*\cap D^3),\ \ \nabla\rho^{\delta-1}\in C([0,T];L^\infty\cap D^2)$;

(2) $u\in C([0,T];H^3)\cap L^2([0,T];H^4),\quad \rho^{\frac{\delta-1}{2}}\nabla u\in C([0,T];L^2),$
$\rho^{\delta-1}\nabla u\in L^\infty([0,T];D^1_*),\quad \rho^{\delta-1}\nabla^2 u\in L^\infty([0,T];H^1)\cap L^2([0,T];D^2)$;

(3) $S\in L^\infty([0,T]\times\mathbb{R}^3),\quad e^{\frac{S}{c_v}}-e^{\frac{\bar S}{c_v}}\in C([0,T];D^1_*\cap D^3)$.

Remark 1.1. *First, it follows from Definition 1.1 that $\nabla\rho^{\delta-1}\in L^\infty$, which means that the vacuum occurs if and only in the far field.*

Second, we introduce some physical quantities to be used in this paper:

$$m(t)=\int\rho(t,\cdot)\quad \textit{(total mass)},$$

$$\mathbb{P}(t)=\int\rho(t,\cdot)u(t,\cdot)\quad \textit{(momentum)},$$

$$\begin{aligned}
E_k(t) =& \frac{1}{2}\int \rho(t,\cdot)|u(t,\cdot)|^2 \quad \textit{(total kinetic energy)},\\
E_p(t) =& \frac{1}{\gamma-1}\int P(\rho(t,\cdot),S(t,\cdot)) \quad \textit{(potential energy)},\\
E(t) =& E_k(t)+E_p(t) \quad \textit{(total energy)}.
\end{aligned}$$

It then follows from Definition 1.1 that a regular solution satisfies the conservation of total mass, momentum and total energy (see Lemma 3.14). Note that the conservation of momentum or total energy is not clear for strong solutions with far field vacuum to the flows of constant (μ,λ,κ) obtained in [7, 15, 25–27, 40]. *In this sense, the definition of regular solutions above is consistent with the physical background of the compressible Navier-Stokes equations.*

The regular solutions select (ρ,u,S) in a physically reasonable way when the density approaches the vacuum at far fields. With the help of the regularity assumptions in Definition 1.1, (1.8) can be reformulated into a system consisting of: a transport equation for the density, a special quasi-linear parabolic system with some singular source terms for the velocity, and a transport equation with one singular source term for the entropy. Furthermore, the coefficients in front of the Lamé operator L will tend to ∞ as $\rho\to 0$ at far fields. It will be shown that the problem becomes trackable by studying a suitably designed enlarged system through an elaborate linearization and approximation process.

The first main result in this paper can be stated as follows.

Theorem 1.1. *Let parameters $(\gamma,\nu,\alpha,\beta)$ satisfy*

$$\gamma>1,\quad 0<\delta=(\gamma-1)\nu<1,\quad \alpha>0,\quad 2\alpha+3\beta\geq 0,\quad 4\gamma+3\delta\leq 7. \tag{1.17}$$

If the initial data (ρ_0,u_0,S_0) satisfy

$$\begin{aligned}
&\rho_0>0,\quad \rho_0^{\gamma-1}\in D^1_*\cap D^3,\quad \nabla\rho_0^{\delta-1}\in L^q\cap D^{1,3}\cap D^2,\\
&\nabla\rho_0^{\frac{\delta-1}{2}}\in L^6,\quad u_0\in H^3,\quad S_0-\bar{S}\in D^1_*\cap D^3,
\end{aligned} \tag{1.18}$$

for some $q\in(3,\infty)$, and the compatibility conditions:

$$\begin{aligned}
&\nabla u_0=\rho_0^{\frac{1-\delta}{2}}g_1, && Lu_0=\rho_0^{1-\delta}g_2,\\
&\nabla(\rho_0^{\delta-1}Lu_0)=\rho_0^{\frac{1-\delta}{2}}g_3, && \nabla^2 e^{\frac{S_0}{c_v}}=\rho_0^{\frac{1-\delta}{2}}g_4,
\end{aligned} \tag{1.19}$$

for some functions $(g_1,g_2,g_3,g_4)\in L^2$, then there exist a time $T_*>0$ and a unique regular solution (ρ,u,S) in $[0,T_*]\times\mathbb{R}^3$ to the Cauchy problem (1.8) *with* (1.2) *and* (1.9)-(1.11)*, and the following additional regularities hold:*

$$\begin{aligned}
&\rho_t^{\gamma-1}\in C([0,T_*];H^2),\quad u_t\in C([0,T_*];H^1)\cap L^2([0,T_*];D^2),\\
&t^{\frac{1}{2}}u\in L^\infty([0,T_*];D^4),\quad t^{\frac{1}{2}}u_t\in L^\infty([0,T_*];D^2)\cap L^2([0,T_*];D^3),\\
&u_{tt}\in L^2([0,T_*];L^2),\quad t^{\frac{1}{2}}u_{tt}\in L^\infty([0,T_*];L^2)\cap L^2([0,T_*];D^1_*),\\
&\nabla\rho^{\delta-1}\in C([0,T_*];L^q\cap D^{1,3}\cap D^2),\quad S-\bar{S}\in C([0,T_*];D^1_*\cap D^3),\\
&S_t\in L^\infty([0,T_*];L^\infty\cap L^3\cap D^1_*\cap D^2),\quad S_{tt}\in L^2([0,T_*];D^1_*).
\end{aligned} \tag{1.20}$$

Moreover, it holds that

(1) (ρ, u, S) *is a classical solution to the problem* (1.8) *with* (1.2) *and* (1.9)-(1.11) (or the problem (1.1)-(1.3) with (1.6) and (1.10)-(1.11)) in $(0, T_*] \times \mathbb{R}^3$;

(2) if one assumes $m(0) < \infty$ additionally, then (ρ, u, S) satisfies the conservation of total mass, momentum and total energy:

$$m(t) = m(0), \quad \mathbb{P}(t) = \mathbb{P}(0), \quad E(t) = E(0) \quad \text{for} \quad t \in [0, T_*].$$

Remark 1.2. (1.18)-(1.19) *identifies a class of admissible initial data that provide unique solvability to the problem* (1.8) *with* (1.2) *and* (1.9)-(1.11). *Such initial data include*

$$\rho_0(x) = \frac{1}{(1+|x|^2)^{\varkappa}}, \quad u_0(x) \in C_0^3(\mathbb{R}^3), \quad S_0 = \bar{S} + f(x), \tag{1.21}$$

for some $f(x) \in D_*^1 \cap D^3$, *where*

$$\frac{1}{4(\gamma-1)} < \varkappa < \frac{1-3/q}{2(1-\delta)} \quad \textit{and} \quad \frac{3}{2} + \frac{\delta}{2} < \gamma + \delta \leq 2. \tag{1.22}$$

Note that, when μ, λ *and* κ *are all constants, in order to obtain the uniform boundedness of the entropy in the whole space, in* [25–27], *it is required that the initial density vanishes only at far fields with a rate no more than* $O(\frac{1}{|x|^2})$, *i.e.,*

$$\max\left\{\frac{1}{4(\gamma-1)}, \frac{3}{4}\right\} < \varkappa \leq 1 \quad \textit{and} \quad \gamma > \frac{5}{4}. \tag{1.23}$$

It follows from (1.22)-(1.23) *that, different from the constant viscous flows, the requirement on the decay rate of* ρ_0 *of the degenerate viscous flow depends on the value of the parameter* δ. *If* δ *is sufficient close to* 1, *then* $\varkappa > 1$ *in* (1.21) *is still admissible, which means that the range of initial data can be enlarged substantially.*

Remark 1.3. *The compatibility conditions* (1.19) *are also necessary for the existence of regular solutions* (ρ, u, S) *obtained in Theorem 1.1. In particular,*

- $\nabla u_0 = \rho_0^{\frac{1-\delta}{2}} g_1$ *plays a key role in the derivation of* $\rho^{\frac{1-\delta}{2}} \nabla u \in L^\infty([0, T_*]; L^2)$;
- $Lu_0 = \rho_0^{1-\delta} g_2$ *is crucial in the derivation of* $u_t \in L^\infty([0, T_*]; L^2)$, *which will be used in the uniform estimates for* $|u|_{D^2}$;
- *while*
$$\nabla(\rho_0^{\delta-1} Lu_0) = \rho_0^{\frac{1-\delta}{2}} g_3 \quad \textit{and} \quad \nabla^2 e^{\frac{S_0}{c_v}} = \rho_0^{\frac{1-\delta}{2}} g_4$$
are indispensible in the derivation of $\rho^{\frac{\delta-1}{2}} \nabla u_t \in L^\infty([0, T_*]; L^2)$, *which will be used in the uniform estimates for* $|u|_{D^3}$.

A natural and important question is whether the local solution obtained in Theorem 1.1 can be extended globally in time. In contrast to the classical theory for both constant and degenerate viscous flows [20, 21, 33, 38, 40], we show the following somewhat surprising phenomenon that such an extension is impossible if the velocity field decays to zero as $t \to \infty$, the laws of conservation of total mass and momentum are both satisfied, and the initial total momentum is non-zero. First, based on the physical quantities introduced above, we define a solution class $D(T)$ as follows:

Definition 1.2. *Let* $T > 0$ *be a positive time. For the Cauchy problem* (1.1)-(1.3) *and* (1.10)-(1.11), *a classical solution* (ρ, u, S) *is said to be in* $D(T)$ *if* (ρ, u, S) *satisfies the following conditions:*

- *$m(t)$, $\mathbb{P}(t)$ and $E_k(t)$ all belong to $L^\infty([0,T])$;*
- *Conservation of total mass: $\frac{d}{dt}m(t)=0$ for any $t\in[0,T]$;*
- *Conservation of momentum: $\frac{d}{dt}\mathbb{P}(t)=0$ for any $t\in[0,T]$.*

Then one has:

Theorem 1.2. *Assume $m(0)>0$, $|\mathbb{P}(0)|>0$, and the parameters $(\gamma,\mu,\lambda,\kappa)$ satisfy*

$$\gamma\geq 1,\quad \mu\geq 0,\quad 2\mu+3\lambda\geq 0,\quad \kappa\geq 0. \tag{1.24}$$

Then for the Cauchy problem (1.1)-(1.3) *and* (1.10)-(1.11)*, there is no classical solution $(\rho,u,S)\in D(\infty)$ with*

$$\limsup_{t\to\infty}|u(t,\cdot)|_\infty=0. \tag{1.25}$$

As an immediate corollary of Theorems 1.1-1.2, one can obtain that

Corollary 1.1. *For the Cauchy problem* (1.8) *with* (1.2) *and* (1.9)-(1.11)*, if one assumes $0<m(0)<\infty$ and $|\mathbb{P}(0)|>0$ additionally, then there is no global regular solution (ρ,u,S) with the regularities in Theorem 1.1 satisfying* (1.25)*.*

Moreover, as a corollary of Theorem 1.2, Remark 1.1 and [7,40], one can show that for the **CNS** with constant (μ,λ,κ), there exists a global classical solution which preserves the conservation of total mass, but not the conservation of momentum for all the time $t\in(0,\infty)$. Indeed, consider the Cauchy problem of (1.1)-(1.3) with the following initial data and far field behavior:

$$(\rho,u,\theta)|_{t=0}=(\rho_0(x),u_0(x),\theta_0(x))\quad\text{for}\quad x\in\mathbb{R}^3, \tag{1.26}$$

$$(\rho,u,\theta)(t,x)\to(0,0,0)\quad\text{as}\quad |x|\to\infty\quad\text{for}\quad t\geq 0. \tag{1.27}$$

For simplicity, C_0 will denote a positive constant that depends only on fixed constants μ, λ, κ, R, γ and the initial data (ρ_0,u_0,θ_0).

Corollary 1.2. *Let μ,λ and κ all be constants satisfying*

$$\mu>0,\quad 2\mu+3\lambda\geq 0,\quad \kappa>0.$$

For two constants $\overline{\rho}>0$ and $\overline{\theta}>0$, suppose that the initial data (ρ_0,u_0,θ_0) satisfies

$$\begin{aligned}&0\leq\inf\rho_0\leq\sup\rho_0\leq\overline{\rho},\quad \rho_0\in L^1\cap H^2\cap W^{2,p},\\&0\leq\inf\theta_0\leq\sup\theta_0\leq\overline{\theta},\quad (u_0,\theta_0)\in D^1_*\cap D^2,\end{aligned} \tag{1.28}$$

for some $p\in(3,6)$, and the compatibility conditions

$$\begin{cases}-\mu\triangle u_0-(\mu+\lambda)\nabla divu_0+\nabla P(\rho_0,\theta_0)=\rho_0^{\frac12}g_5,\\ \kappa\triangle\theta_0+\frac{\mu}{2}|\nabla u_0+(\nabla u_0)^\top|^2+\lambda(divu_0)^2=\rho_0^{\frac12}g_6,\end{cases} \tag{1.29}$$

for some $g_i\in L^2$ $(i=5,6)$. Then there exists a positive constant ζ depending on μ, λ, κ, R, γ, $\bar\rho$ and $\bar\theta$ such that if

$$m(0)\leq\zeta, \tag{1.30}$$

the Cauchy problem (1.1)-(1.3) *with* (1.26)-(1.27) *has a unique global classical solution* (ρ, u, θ) *in* $(0,\infty)\times\mathbb{R}^3$ *satisfying, for any* $0<\tau<T<\infty$,

$$m(t)=m(0), \quad t\in[0,\infty); \tag{1.31}$$

$$0\le \rho(t,x)\le 2\overline{\rho}, \quad \theta(t,x)\ge 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3; \tag{1.32}$$

$$\sup_{t\in[0,T]}\int (\rho|\dot{u}|^2+|\nabla u|^2+|\nabla\theta|^2)(t,\cdot)\le C_0; \tag{1.33}$$

$$\rho\in C([0,T];H^2\cap W^{2,p}), \quad \rho_t\in C([0,T];H^1); \tag{1.34}$$

$$(u,\theta)\in C([0,T];D^1_*\cap D^2)\cap L^2([0,T];D^3)\cap L^\infty([\tau,T];D^3); \tag{1.35}$$

$$(u_t,\theta_t)\in L^2([0,T];D^1_*)\cap L^\infty([0,T];D^1_*\cap D^2), \tag{1.36}$$

for $\dot{u}=u_t+u\cdot\nabla u$, *and the following large-time behavior*

$$\lim_{t\to\infty}\int \big(\rho|\theta|^2+|\nabla u|^2+|\nabla\theta|^2\big)(t,\cdot)=0. \tag{1.37}$$

Furthermore, if $m(0)>0$ *and* $|\mathbb{P}(0)|>0$, *then the solution obtained above cannot preserve the conservation of the momentum for all the time* $t\in(0,\infty)$.

Remark 1.4. *Note that for the regular solution* (ρ,u,S) *obtained in Theorem 1.1,* u *stays in the inhomogeneous Sobolev space* H^3 *instead of the homogenous one* $D^1_*\cap D^2$ *in* [7,40] *for flows with constant viscosity and heat conductivity coefficients.*

Based on the conclusions obtained in Theorem 1.1 and [24], *there is a natural question that whether the conclusion obtained in* [24] *can be applied to the degenerate system considered here? Due to strong degeneracy near the vacuum in* $(1.1)_2-(1.1)_3$, *such questions are not easy and will be discussed in the future work Xin-Zhu* [44].

Remark 1.5. *The framework established in this paper is applicable to other physical dimensions, say 1 and 2, with some minor modifications. This is clear from the analysis carried out in the following sections.*

The rest of this paper is organised as follows. In Section 2, we first reformulate the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11) into a specifically chosen enlarged form, which makes the problem trackable through an elaborate linearization and approximation process. Then we outline the main strategy to establish the well-posedness theory. Section 3 is devoted to proving the local-in-time well-posedness theory stated in Theorem 1.1, which can be achieved in five steps:

(1) construct global approximate solutions away from the vacuum for a specially designed linearized problem with an artificial viscosity

$$\sqrt{\rho^{\delta-1}+\epsilon^2}e^{\frac{S}{c_v}\nu}Lu$$

in the momentum equations and $\inf_{x\in\mathbb{R}^3}\rho_0^{\gamma-1}=\frac{\gamma-1}{A\gamma}\eta$ for some positive constants $\epsilon>0$ and $\eta>0$;

(2) establish the a priori estimates independent of both ϵ and η;

(3) then pass to the limit $\epsilon\to 0$ to recover the solution of the corresponding linearized problem away from the vacuum with only physical viscosities;

(4) prove the unique solvability away from the vacuum of the reformulated nonlinear problem through a standard iteration process;

(5) finally take the limit $\eta \to 0$ to recover the solution of the reformulated nonlinear problem with physical viscosities and far field vacuum.

The global non-existence results stated in Theorem 1.2 and Corollary 1.1, and the non-conservation of momentum stated in Corollary 1.2 are proved in Section 4. Finally, for convenience of readers, we provide one appendix to list some basic facts which have been used frequently in this paper.

2. Reformulation and main strategy

In this section, we first reformulate the highly degenerate system (1.8) into an enlarged trackable system, and then sketch the main strategies of our analysis.

2.1. Reformulation.

Set $\delta=(\gamma-1)\nu$, and

$$\phi=\frac{A\gamma}{\gamma-1}\rho^{\gamma-1},\quad l=e^{\frac{S}{c_v}},\quad \psi=\frac{\delta}{\delta-1}\nabla\rho^{\delta-1},\quad n=\rho^{2-\delta-\gamma}. \tag{2.1}$$

It follows from (1.8) with (1.2) and (1.9)-(1.11) that

$$\begin{cases} \phi_t+u\cdot\nabla\phi+(\gamma-1)\phi\mathrm{div}u=0,\\ u_t+u\cdot\nabla u+a_1\phi\nabla l+l\nabla\phi+a_2l^\nu\phi^{2\iota}Lu\\ =a_2\phi^{2\iota}\nabla l^\nu\cdot Q(u)+a_3l^\nu\psi\cdot Q(u),\\ l_t+u\cdot\nabla l=a_4l^\nu n\phi^{4\iota}H(u),\\ \psi_t+\sum_{k=1}^3A_k(u)\partial_k\psi+B(u)\psi+\delta a\phi^{2\iota}\nabla\mathrm{div}u=0, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} a_1=&\frac{\gamma-1}{\gamma},\quad a_2=a\Big(\frac{A}{R}\Big)^\nu,\quad a_3=\Big(\frac{A}{R}\Big)^\nu,\\ a_4=&\frac{A^{\nu-1}a^2(\gamma-1)}{R^\nu},\quad \iota=\frac{\delta-1}{2(\gamma-1)},\quad a=\Big(\frac{A\gamma}{\gamma-1}\Big)^{\frac{1-\delta}{\gamma-1}}, \end{aligned} \tag{2.3}$$

$A_k(u)=(a^k_{ij})_{3\times 3}$ for $i,\ j,\ k=1,\ 2,\ 3$, are symmetric with

$$a^k_{ij}=u^{(k)}\quad \text{for } i=j;\quad \text{otherwise } a^k_{ij}=0,$$

and $B(u)=(\nabla u)^\top+(\delta-1)\mathrm{div}u\mathbb{I}_3$.

(2.2) will be regarded as a system for the new variable (ϕ,u,l,ψ), which is equivalent to the (1.8) with (1.2) and (1.9)-(1.11) if one imposes the initial data

$$\begin{aligned} &(\phi,u,l,\psi)|_{t=0}=(\phi_0,u_0,l_0,\psi_0)\\ =&\Big(\frac{A\gamma}{\gamma-1}\rho_0^{\gamma-1}(x),u_0(x),e^{S_0(x)/c_v},\frac{\delta}{\delta-1}\nabla\rho_0^{\delta-1}(x)\Big)\quad \text{for}\quad x\in\mathbb{R}^3, \end{aligned} \tag{2.4}$$

and far filed behavior:

$$(\phi,u,l,\psi)\to(0,0,\bar{l},0)\ \text{ as }\ |x|\to\infty\ \text{ for }\quad t\geq 0, \tag{2.5}$$

where $\bar{l}=e^{\frac{\bar{S}}{c_v}}>0$ is some positive constant.

Note that the reformulated system (2.2) consists of

- one scalar transport equation $(2.2)_1$ for ϕ;
- one *singular parabolic* system $(2.2)_2$ for the velocity u;
- one scalar transport equation $(2.2)_3$ but with a possible singular first order term for l;
- one symmetric hyperbolic system $(2.2)_4$ but with a possible singular second order term for ψ.

Such a structure is important in establishing the local well-posedness of the problem (2.2)-(2.5) and thus to solve the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11) locally in time. We will establish the following theorem.

Theorem 2.1. *Let* (1.17) *hold. If the initial data* $(\phi_0, u_0, l_0, \psi_0)$ *satisfies:*

$$\begin{aligned}&\phi_0>0,\quad \phi_0\in D^1_*\cap D^3,\quad u_0\in H^3,\quad l_0-\bar l\in D^1_*\cap D^3,\\&\inf_{x\in\mathbb{R}^3} l_0>0,\quad \psi_0\in L^q\cap D^{1,3}\cap D^2,\quad \nabla\phi_0^\iota\in L^6,\end{aligned}\tag{2.6}$$

for some $q\in(3,\infty)$, *and the compatibility conditions:*

$$\begin{aligned}\nabla u_0=&\phi_0^{-\iota}g_1,\qquad Lu_0=\phi_0^{-2\iota}g_2,\\ \nabla(\phi_0^{2\iota}Lu_0)=&\phi_0^{-\iota}g_3,\qquad \nabla^2 l_0=\phi_0^{-\iota}g_4,\end{aligned}\tag{2.7}$$

for some $(g_1,g_2,g_3,g_4)\in L^2$, *then there exist a time* $T^*>0$ *and a unique strong solution* $(\phi,u,l,\psi=\frac{a\delta}{\delta-1}\nabla\phi^{2\iota})$ *in* $[0,T^*]\times\mathbb{R}^3$ *to the Cauchy problem* (2.2)-(2.5), *such that*

$$\begin{aligned}&\phi\in C([0,T^*];D^1_*\cap D^3),\quad \psi\in C([0,T^*];L^q\cap D^{1,3}\cap D^2),\\&u\in C([0,T^*];H^3)\cap L^2([0,T^*];H^4),\quad \phi^{2\iota}\nabla u\in L^\infty([0,T^*];D^1_*),\\&\phi^{2\iota}\nabla^2 u\in C([0,T^*];H^1)\cap L^2([0,T^*];D^2),\\&\phi^\iota\nabla u\in C([0,T^*];L^2),\quad \phi^\iota\nabla u_t\in L^\infty([0,T^*];L^2),\\&u_t\in C([0,T^*];H^1)\cap L^2([0,T^*];D^2),\quad (\phi^{2\iota}\nabla^2 u)_t\in L^2([0,T^*];L^2),\\&u_{tt}\in L^2([0,T^*];L^2),\quad t^{\frac12}u\in L^\infty([0,T^*];D^4),\\&t^{\frac12}u_t\in L^\infty([0,T^*];D^2)\cap L^2([0,T^*];D^3),\quad l-\bar l\in C([0,T^*];D^1_*\cap D^3),\\&\inf_{(t,x)\in[0,T^*]\times\mathbb{R}^3} l>0,\quad l_t\in C([0,T^*];D^1_*\cap D^2),\quad l_{tt}\in L^2([0,T^*];D^1_*).\end{aligned}\tag{2.8}$$

Remark 2.1. *In Theorem 2.1,* $(\phi,u,l,\psi=\frac{a\delta}{\delta-1}\nabla\phi^{2\iota})$ *in* $[0,T^*]\times\mathbb{R}^3$ *is called a strong solution to the Cauchy problem* (2.2)-(2.5), *if it satisfies* (2.2)-(2.5) *in the sense of distributions, and satisfies the equations* (2.2)-(2.3) *a.e.* $(t,x)\in(0,T^*]\times\mathbb{R}^3$.

2.2. **Main strategy.** Now we sketch the main strategy for the proof of Theorem 2.1.

2.2.1. *Closed energy estimates based on the singular structure introduced.* We now formally indicate how to obtain closed energy estimates based on the singular structure described above.

Note first that the velocity u can be controlled by the following equations:

$$u_t+u\cdot\nabla u+a_1\phi\nabla l+l\nabla\phi+\underbrace{a_2\phi^{2\iota}l^\nu Lu}_{\textbf{SSE}}=\underbrace{a_2\phi^{2\iota}\nabla l^\nu\cdot Q(u)+a_3 l^\nu\psi\cdot Q(u)}_{\textbf{SSS}},$$

where **SSE** denotes the elliptic operator which is strongly singular near the vacuum, while **SSS** represents the source terms which are also highly singular near the vacuum. Due to (1.6), $\kappa = 0$, so the entropy is expected to be bounded below uniformly such that $l = e^{\frac{S}{c_v}}$ and $\phi^{2\iota}$ with $\iota < 0$ should have uniformly positive lower bounds in the whole space. Then for this quasi-linear parabolic system, one can find formally that even though the coefficients $a_2\phi^{2\iota}l^{\nu}$ in front of Lamé operator Lu will tend to ∞ as $\rho \to 0$ in the far filed, yet this structure could give a better a priori estimate on u in H^3 than those of [7, 30, 31, 42] if one can control the possible singular term ψ in $L^q \cap D^{1,3} \cap D^2$, $l - \bar{l}$ in $D^1_* \cap D^3$ and the first-order product term $\phi^{2\iota}\nabla l^{\nu} \cdot Q(u)$ with singular coefficient in proper spaces successfully.

On the one hand, $(2.2)_4$ implies that the subtle term ψ could be controlled by a symmetric hyperbolic system with a possible singular higher order term $\delta a\phi^{2\iota}\nabla \mathrm{div} u$, while l is governed by $(2.2)_3$, which is a scalar transport equation with a possible singular term $a_4 l^{\nu} n\phi^{4\iota}H(u)$. Thus in order to close the desired estimates, one needs to control both $\phi^{2\iota}\nabla \mathrm{div} u$ in $L^q \cap D^{1,3} \cap D^2$ and $l^{\nu} n\phi^{4\iota}H(u)$ in H^2.

First, the necessary estimates on $\phi^{2\iota}\nabla \mathrm{div} u$ can be obtained by regarding the momentum equations as the following inhomogeneous Lamé equations:

$$\begin{aligned} a_2 L(\phi^{2\iota}u) =& - l^{-\nu}(u_t + u \cdot \nabla u + a_1\phi\nabla l + l\nabla\phi) + a_3\psi \cdot Q(u) \\ & + a_2 l^{-\nu}\phi^{2\iota}\nabla l^{\nu} \cdot Q(u) - \frac{\delta - 1}{\delta}\Big(\frac{A}{R}\Big)^{\nu} G(\psi, u) = W, \end{aligned}$$

where

$$G(\psi, u) = \alpha\psi \cdot \nabla u + \alpha \mathrm{div}(u \otimes \psi) + (\alpha + \beta)\big(\psi \mathrm{div} u + \psi \cdot \nabla u + u \cdot \nabla\psi\big).$$

In fact, one has

$$\begin{aligned} |\phi^{2\iota}\nabla^2 u|_{D^1_*} \le & C(|\psi|_\infty |\nabla^2 u|_2 + |\phi^{2\iota}\nabla^3 u|_2) \\ \le & C(|\psi|_\infty |\nabla^2 u|_2 + |\nabla\psi|_3|\nabla u|_6 + |\nabla^2\psi|_2|u|_\infty + |W|_{D^1_*}), \end{aligned} \tag{2.9}$$

for some constant $C > 0$ independent of the lower bound of ϕ provided that

$$\phi^{2\iota}u \to 0 \qquad \text{as} \qquad |x| \to \infty,$$

which can be verified in an approximation process from non-vacuum flows to the flow with far field vacuum. Similar calculations can be done for $|\phi^{2\iota}\nabla^2 u|_{D^2}$.

Next, we turn to the estimates on $l^{\nu} n\phi^{4\iota}H(u)$, which are more complicated and depend on the estimates of n and $\phi^{4\iota}|\nabla u|^2$. An observation used here is that the initial assumption (1.18) and the definition of n in (2.1) imply that

$$n(0, x) \in L^\infty \cap D^{1,q} \cap D^{1,6} \cap D^{2,3} \cap D^3.$$

It is easy to check that n can be controlled by the following hyperbolic equation:

$$n_t + u \cdot \nabla n + (2 - \delta - \gamma)n \mathrm{div} u = 0, \tag{2.10}$$

which, along with the expected regularities of u, implies that

$$n(t, x) \in L^\infty \cap D^{1,q} \cap D^{1,6} \cap D^{2,3} \cap D^3$$

within the solution's life span. In fact, it follows from the assumption (1.17) that the upper bound of n depends on that of ϕ, and the estimates on its space derivatives can be obtained from those of ψ via the relation $n = (a\phi^{2\iota})^{\frac{2-\delta-\gamma}{\delta-1}}$, which, along with the equation (2.10) and the weighted estimates on u, yields the estimates on its time derivatives. While $\phi^{4\iota}|\nabla u|^2$ can be controlled by using the weighted estimates on u including $|\phi^\iota \nabla u|_2$, $|\phi^\iota \nabla u_t|_2$, $|\phi^{2\iota}\mathrm{div} u|_\infty$, $|\phi^{2\iota}\nabla^2 u|_{D^2}$, $|\phi^{2\iota}\nabla^2 u_t|_2$ and so on.

Finally, in order to deal with the high singularity and nonlinearity of the source term $\phi^{2\iota}\nabla l^\nu \cdot Q(u)$ in the time evolution equations $(2.2)_2$ of u, one still needs one singular weighted estimate on the entropy: $|\phi^{\frac{\iota}{2}}\nabla l|_6$, which can be obtained from the transport structure of the equation $(2.2)_3$ of l. It is worth pointing out that for achieving this key estimate, we require that $4\gamma + 3\delta \leq 7$ in (2.1).

2.2.2. *An elaborate linearization of the nonlinear singular problem.* To prove Theorem 2.1, it is crucial to carry out the strategy of energy estimates discussed above for suitably chosen approximate solutions while are constructed by an elaborate linear scheme. In §3.1, we design an elaborate linearization (3.1) of the nonlinear one (2.2)-(2.5) based on a careful analysis on the structure of the nonlinear system (2.2), and the global approximate solutions for this linearized problem when $\phi(0,x) = \phi_0$ has positive lower bound η are established. The choice of the linear scheme for this problem needs to be careful due to the appearance of the far field vacuum. Some necessary structures should be preserved in order to establish the desired a priori estimates as mentioned above. For the problem (2.2)-(2.5), a crucial point is how to deal with the estimates on ψ. According to the analysis in the above paragraphs, we need to keep the two factors $\phi^{2\iota}$ and $\nabla\mathrm{div} u$ of the source term $\delta a \phi^{2\iota}\nabla \mathrm{div} u$ in equations $(2.2)_4$ in the same step. Then let $v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3$ be a known vector, g and w be known real (scalar) functions satisfying $(v(0,x), g(0,x), w(0,x)) = (u_0, \phi_0^{2\iota}, l_0)$ and (3.3). A natural linearization of the system (2.2) seems to be

$$
\begin{cases}
\phi_t + v\cdot\nabla\phi + (\gamma-1)\phi\mathrm{div} v = 0,\\
u_t + v\cdot\nabla v + a_1\phi\nabla l + l\nabla\phi + a_2\phi^{2\iota}l^\nu Lu\\
= a_2 g\nabla l^\nu \cdot Q(v) + a_3 l^\nu \psi\cdot Q(v),\\
l_t + v\cdot\nabla l = a_4 w^\nu n g^2 H(v),\\
\psi_t + \sum_{k=1}^3 A_k(v)\partial_k\psi + B(v)\psi + \delta a g\nabla\mathrm{div} v = 0.
\end{cases}
\tag{2.11}
$$

However, it should be noted that, in (2.11), the important relationship

$$
\psi = \frac{a\delta}{\delta-1}\nabla\phi^{2\iota}
$$

between ψ and ϕ cannot be guaranteed due to the term $g\nabla\mathrm{div} v$ in $(2.11)_4$. Then one would encounter the following difficulties in deriving L^2 estimate for u:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u|_2^2 + a_2\alpha|l^{\frac{\nu}{2}}\phi^\iota\nabla u|_2^2 + a_2(\alpha+\beta)|l^{\frac{\nu}{2}}\phi^\iota \mathrm{div} u|_2^2\\
=&-\int \big(v\cdot\nabla v + a_1\phi\nabla l + l\nabla\phi + a_2 l^\nu \underbrace{\nabla\phi^{2\iota}}_{\neq\frac{\delta-1}{a\delta}\psi}\cdot Q(u)\big)\cdot u\\
&+\int \big(-a_2\phi^{2\iota}\nabla l^\nu\cdot Q(u) + a_2 g\nabla l^\nu\cdot Q(v) + a_3 l^\nu\psi\cdot Q(v)\big)\cdot u.
\end{aligned}
\tag{2.12}
$$

Since $\nabla\phi^{2\iota}$ does not coincide with $\frac{\delta-1}{a\delta}\psi$ in (2.11), it seems extremely difficult to control the term $a_2 l^\nu\nabla\phi^{2\iota}\cdot Q(u)$ in the above energy estimates. In order to overcome this difficulty, in (3.1), we first linearize the equation for $h = \phi^{2\iota}$ as:

$$h_t + v \cdot \nabla h + (\delta - 1) g \mathrm{div} v = 0, \tag{2.13}$$

and then use h to define $\psi = \frac{a\delta}{\delta-1}\nabla h$ again. Here, it should be pointed out that, due to the term $(\delta - 1)g\mathrm{div}v$ in (2.13), the relation $h = \phi^{2\iota}$ between h and ϕ no longer exists in the linear problem. The linear equations for u are chosen as

$$\begin{aligned} & u_t + v \cdot \nabla v + a_1 \phi \nabla l + l \nabla \phi + a_2 \sqrt{h^2 + \epsilon^2} l^\nu L u \\ =& a_2 g \nabla l^\nu \cdot Q(v) + a_3 l^\nu \psi \cdot Q(v), \end{aligned}$$

for any positive constant $\epsilon > 0$. Here the appearance of ϵ is used to compensate the lack of lower bound of h. It follows from (2.13) and the relation $\psi = \frac{a\delta}{\delta-1}\nabla h$ that

$$\psi_t + \sum_{k=1}^{3} A_k(v)\partial_k \psi + (\nabla v)^\top \psi + a\delta\big(g\nabla \mathrm{div} v + \nabla g \mathrm{div} v\big) = 0,$$

which turns out to be the right structure to ensure the desired estimates on ψ.

Another subtle issue is that to linearize the equation $(2.2)_3$ for the entropy, one would face the problem how to define n (by ϕ or h?) since the relation $h = \phi^{2\iota}$ does not hold for the linearized scheme above. Here, in order to make full use of the estimates on ψ and the singular weighted estimates on u, we will define n as

$$n = (ah)^{\frac{2-\delta-\gamma}{\delta-1}}.$$

Then in §3.2, the uniform a priori estimates independent for the lower bound (ϵ, η) of the solutions (ϕ, u, l, h) to the linearized problem (3.1) are established. Based on these uniform estimates, one can obtain the local-in-time well-posedness of the regular solution with far field vacuum by passing to the limit $\epsilon \to 0$, an iteration process that connects the linear approximation problems to the nonlinear one, and finally passing to the limit as $\eta \to 0$.

3. Local-in-time well-posedness with far field vacuum

In this section, the proof for the local-in-time well-posedness of strong solutions with far field vacuum stated in Theorem 2.1 will be given.

3.1. **Linearization.** Let $T > 0$ be some positive time. In order to solve the nonlinear problem (2.2)-(2.5), we consider the following linearized problem for $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})$ in $[0, T] \times \mathbb{R}^3$:

$$\begin{cases} \phi_t^{\epsilon,\eta} + v \cdot \nabla \phi^{\epsilon,\eta} + (\gamma - 1)\phi^{\epsilon,\eta} \mathrm{div} v = 0, \\ u_t^{\epsilon,\eta} + v \cdot \nabla v + a_1 \phi^{\epsilon,\eta} \nabla l^{\epsilon,\eta} + l^{\epsilon,\eta} \nabla \phi^{\epsilon,\eta} + a_2 (l^{\epsilon,\eta})^\nu \sqrt{(h^{\epsilon,\eta})^2 + \epsilon^2} L u^{\epsilon,\eta} \\ = a_2 g \nabla (l^{\epsilon,\eta})^\nu \cdot Q(v) + a_3 (l^{\epsilon,\eta})^\nu \psi^{\epsilon,\eta} \cdot Q(v), \\ l_t^{\epsilon,\eta} + v \cdot \nabla l^{\epsilon,\eta} = a_4 w^\nu n^{\epsilon,\eta} g^2 H(v), \\ h_t^{\epsilon,\eta} + v \cdot \nabla h^{\epsilon,\eta} + (\delta - 1) g \mathrm{div} v = 0, \\ (\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})|_{t=0} = (\phi_0^\eta, u_0^\eta, l_0^\eta, h_0^\eta) \\ = (\phi_0 + \eta, u_0, l_0, (\phi_0 + \eta)^{2\iota}) \quad \text{for} \quad x \in \mathbb{R}^3, \\ (\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta}) \to (\eta, 0, \bar{l}, \eta^{2\iota}) \quad \text{as } |x| \to \infty \quad \text{for} \quad \mathrm{t} \geq 0, \end{cases} \tag{3.1}$$

where ϵ and η are positive constants,

$$\psi^{\epsilon,\eta} = \frac{a\delta}{\delta-1}\nabla h^{\epsilon,\eta}, \quad n^{\epsilon,\eta} = (ah^{\epsilon,\eta})^b, \quad b = \frac{2-\delta-\gamma}{\delta-1} \le 0, \tag{3.2}$$

$v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3$ is a given vector, g and w are given real functions satisfying $w \ge 0$, $(v(0,x), g(0,x), w(0,x)) = (u_0(x), h_0(x) = (\phi_0^\eta)^{2\iota}(x), l_0(x))$, and

$$\begin{aligned}
& g \in L^\infty \cap C([0,T]\times\mathbb{R}^3), \quad \nabla g \in C([0,T]; L^q \cap D^{1,3} \cap D^2),\\
& g_t \in C([0,T]; H^2), \quad \nabla g_{tt} \in L^2([0,T]; L^2),\\
& v \in C([0,T]; H^3) \cap L^2([0,T]; H^4), \quad t^{\frac12} v \in L^\infty([0,T]; D^4),\\
& v_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2), \quad v_{tt} \in L^2([0,T]; L^2),\\
& t^{\frac12} v_t \in L^\infty([0,T]; D^2) \cap L^2([0,T]; D^3),\\
& t^{\frac12} v_{tt} \in L^\infty([0,T]; L^2) \cap L^2([0,T]; D^1_*), \quad w \in C([0,T]; D^1_* \cap D^3),\\
& w_t \in C([0,T]; D^1_* \cap D^2), \quad \nabla w_{tt} \in L^2([0,T]; L^2).
\end{aligned} \tag{3.3}$$

For the sake of clarity, we declare that the functions (ϕ_0, u_0, l_0) and the constant $\bar{l} = e^{\frac{\bar{S}}{c_v}} > 0$ shown above in problem (3.1) is exactly the ones in (2.4)-(2.5), and also define here

$$\psi_0 = \frac{a\delta}{\delta-1}\nabla \phi_0^{2\iota}.$$

It follows from the classical theory [19, 22, 32], at least when η and ϵ are positive, that the following global well-posedness of (3.1) in $[0,T]\times\mathbb{R}^3$ holds.

Lemma 3.1. *Let* (1.17) *hold,* $\eta > 0$ *and* $\epsilon > 0$ *and* (ϕ_0, u_0, l_0, h_0) *satisfy* (2.6)-(2.7). *Then for any time* $T > 0$*, there exists a unique strong solution* $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})$ *in* $[0,T]\times\mathbb{R}^3$ *to* (3.1) *such that*

$$\begin{aligned}
& \phi^{\epsilon,\eta} - \eta \in C([0,T]; D^1_* \cap D^3), \quad \phi^{\epsilon,\eta}_t \in C([0,T]; H^2),\\
& h^{\epsilon,\eta} \in L^\infty \cap C([0,T]\times\mathbb{R}^3), \quad \nabla h^{\epsilon,\eta} \in C([0,T]; H^2),\\
& h^{\epsilon,\eta}_t \in C([0,T]; H^2), \quad u^{\epsilon,\eta} \in C([0,T]; H^3) \cap L^2([0,T]; H^4),\\
& u^{\epsilon,\eta}_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2), \quad u^{\epsilon,\eta}_{tt} \in L^2([0,T]; L^2),\\
& t^{\frac12} u^{\epsilon,\eta} \in L^\infty([0,T]; D^4), \quad t^{\frac12} u^{\epsilon,\eta}_t \in L^\infty([0,T]; D^2) \cap L^2([0,T]; D^3),\\
& t^{\frac12} u^{\epsilon,\eta}_{tt} \in L^\infty([0,T]; L^2) \cap L^2([0,T]; D^1_*), \quad l^{\epsilon,\eta} - \bar{l} \in C([0,T]; D^1_* \cap D^3),\\
& l^{\epsilon,\eta}_t \in C([0,T^*]; D^1_* \cap D^2), \quad \nabla l^{\epsilon,\eta}_{tt} \in L^2([0,T^*]; L^2).
\end{aligned} \tag{3.4}$$

Now we are going to derive the uniform a priori estimates, independent of (ϵ, η), for the strong solution $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})$ to (3.1) obtained in Lemma 3.1.

3.2. **A priori estimates independent of** (ϵ, η)**.** For any fixed $\eta \in (0,1]$, since

$$(\phi_0^\eta, u_0^\eta, l_0^\eta, h_0^\eta) = (\phi_0 + \eta, u_0, l_0, (\phi_0+\eta)^{2\iota}),$$

(ϕ_0, u_0, l_0, h_0) satisfy (2.6)-(2.7) and $\psi_0 = \frac{a\delta}{\delta-1}\nabla\phi_0^{2\iota}$, there exists a constant $c_0 > 0$ independent of η such that

$$\begin{aligned}&2+\eta+\bar{l}+\|\phi_0^\eta-\eta\|_{D_*^1\cap D^3}+\|u_0^\eta\|_3+\|\nabla h_0^\eta\|_{L^q\cap D^{1,3}\cap D^2}+|\nabla(h_0^\eta)^{\frac{1}{2}}|_6\\&|(h_0^\eta)^{-1}|_\infty+|g_1^\eta|_2+|g_2^\eta|_2+|g_3^\eta|_2+|g_4^\eta|_2+\|l_0^\eta-\bar{l}\|_{D_*^1\cap D^3}+|(l_0^\eta)^{-1}|_\infty\leq c_0,\end{aligned} \tag{3.5}$$

where

$$g_1^\eta=(\phi_0^\eta)^\iota\nabla u_0^\eta,\quad g_2^\eta=(\phi_0^\eta)^{2\iota}Lu_0^\eta,\quad g_3^\eta=(\phi_0^\eta)^\iota\nabla((\phi_0^\eta)^{2\iota}Lu_0^\eta),\quad g_4^\eta=(\phi_0^\eta)^\iota\nabla^2 l_0^\eta.$$

Remark 3.1. *First, it follows from the definition of g_2^η, $\phi_0^\eta > \eta$ and the far field behavior as in* $(3.1)_6$ *that*

$$\begin{cases}L((\phi_0^\eta)^{2\iota}u_0^\eta)=g_2^\eta-\dfrac{\delta-1}{a\delta}G(\psi_0^\eta,u_0^\eta),\\(\phi_0^\eta)^{2\iota}u_0^\eta\longrightarrow 0\quad as\quad |x|\longrightarrow\infty,\end{cases} \tag{3.6}$$

where $\psi_0^\eta=\frac{a\delta}{\delta-1}\nabla(\phi_0^\eta)^{2\iota}=\frac{a\delta}{\delta-1}\nabla h_0^\eta$, *and*

$$G=\alpha\psi_0^\eta\cdot\nabla u_0^\eta+\alpha\, div(u_0^\eta\otimes\psi_0^\eta)+(\alpha+\beta)(\psi_0^\eta\, div u_0^\eta+\psi_0^\eta\cdot\nabla u_0^\eta+u_0^\eta\cdot\nabla\psi_0^\eta). \tag{3.7}$$

Then it follows from the standard elliptic theory and (3.5) *that*

$$\begin{aligned}|(\phi_0^\eta)^{2\iota}u_0^\eta|_{D^2}\leq& C(|g_2^\eta|_2+|G(\psi_0^\eta,u_0^\eta)|_2)\leq C_1<\infty,\\|(\phi_0^\eta)^{2\iota}\nabla^2u_0^\eta|_2\leq& C(|(\phi_0^\eta)^{2\iota}u_0^\eta|_{D^2}+|\nabla\psi_0^\eta|_3|u_0^\eta|_6+|\psi_0^\eta|_\infty|\nabla u_0^\eta|_2)\leq C_1,\end{aligned} \tag{3.8}$$

where $C_1 > 0$ is a generic constant independent of (ϵ,η). Due to $\nabla^2\phi_0^{2\iota}\in L^3$ and (3.8), *it holds that*

$$|(\phi_0^\eta)^\iota\nabla^2\phi_0^\eta|_2+|(\phi_0^\eta)^\iota\nabla(\psi_0^\eta\cdot Q(u_0^\eta))|_2\leq C_1, \tag{3.9}$$

where one has used the fact that

$$\begin{aligned}|\phi_0^\iota\nabla^2\phi_0|_2\leq& C_1(|\phi_0|_6|\phi_0|_\infty^{-\iota}|\nabla^2\phi_0^{2\iota}|_3+|\nabla\phi_0^\iota|_6|\nabla\phi_0|_3)\leq C_1,\\|(\phi_0^\eta)^\iota\nabla^2\phi_0^\eta|_2=&\Big|\phi_0^\iota\nabla^2\phi_0\frac{\phi_0^{-\iota}}{(\phi_0+\eta)^{-\iota}}\Big|_2\leq|\phi_0^\iota\nabla^2\phi_0|_2\leq C_1.\end{aligned}$$

Second, it follows from the initial compatibility condition

$$\nabla((\phi_0^\eta)^{2\iota}Lu_0^\eta)=(\phi_0^\eta)^{-\iota}g_3^\eta\in L^2,$$

that formally,

$$\begin{cases}L((\phi_0^\eta)^{2\iota}u_0^\eta)=\triangle^{-1}div((\phi_0^\eta)^{-\iota}g_3^\eta)-\dfrac{\delta-1}{a\delta}G(\psi_0^\eta,u_0^\eta),\\(\phi_0^\eta)^{2\iota}u_0^\eta\longrightarrow 0\quad as\quad |x|\longrightarrow\infty.\end{cases} \tag{3.10}$$

Thus the standard elliptic theory yields

$$\begin{aligned}|(\phi_0^\eta)^{2\iota}u_0^\eta|_{D^3}\leq& C(|\phi_0^\eta)^{-\iota}g_3^\eta|_2+|G(\psi_0^\eta,u_0^\eta)|_{D^1})\leq C_1<\infty,\\|(\phi_0^\eta)^{2\iota}\nabla^3u_0^\eta|_2\leq& C(|(\phi_0^\eta)^{2\iota}u_0^\eta|_{D^3}+|\nabla\psi_0^\eta|_3|\nabla u_0^\eta|_6\\&+|\psi_0^\eta|_\infty|\nabla^2u_0^\eta|_2+|\nabla^2\psi_0^\eta|_2|u_0^\eta|_\infty)\leq C_1.\end{aligned} \tag{3.11}$$

Actually, the rigorous proof for (3.10) *can be obtained by a standard smoothing process of the initial data, which is omitted here.*

Now let T be a positive fixed constant, and assume that there exist some time $T^* \in (0,T]$ and constants $c_i (i = 1, \cdots, 5)$ such that

$$1 < c_0 \le c_1 \le c_2 \le c_3 \le c_4 \le c_5, \tag{3.12}$$

and

$$\begin{aligned}
&\sup_{0\le t\le T^*} \|\nabla g(t)\|^2_{L^q\cap D^{1,3}\cap D^2} \le c_1^2, \quad \sup_{0\le t\le T^*} \|w(t)-\bar{l}\|^2_{D^1_*\cap D^3} \le c_1^2,\\
&\inf_{[0,T_*]\times\mathbb{R}^3} w(t,x) \ge c_1^{-1}, \quad \sup_{0\le t\le T^*} \|v(t)\|_1^2 + \int_0^{T^*} (|v|^2_{D^2} + |v_t|_2^2)\mathrm{d}t \le c_2^2,\\
&\sup_{0\le t\le T^*} (|v|^2_{D^2} + |v_t|_2^2 + |g\nabla^2 v|_2^2)(t) + \int_0^{T^*} (|v|^2_{D^3} + |v_t|^2_{D^1_*})\mathrm{d}t \le c_3^2,\\
&\sup_{0\le t\le T^*} (|v|^2_{D^3} + |\sqrt{g}\nabla v_t|_2^2 + |\nabla v_t|_2^2)(t) + \int_0^{T^*} (|v|^2_{D^4} + |v_t|^2_{D^2} + |v_{tt}|_2^2)\mathrm{d}t \le c_4^2,\\
&\sup_{0\le t\le T^*} (|g_t|^2_{D^1_*} + |g\nabla^2 v|^2_{D^1_*})(t) + \int_0^{T^*} (|(g\nabla^2 v)_t|_2^2 + |g\nabla^2 v|^2_{D^2})\mathrm{d}t \le c_4^2,\\
&\sup_{0\le t\le T^*} (|g_t|^2_\infty + |w_t|_3^2 + |w_t|^2_{D^1_*})(t) \le c_4^2,\\
&\operatorname{ess}\sup_{0\le t\le T^*} (t|v|^2_{D^4} + t|\nabla^2 v_t|_2^2 + t|g\nabla^2 v_t|_2^2)(t) + \int_0^{T^*} |g_{tt}|^2_{D^1_*}\mathrm{d}t \le c_5^2,\\
&\sup_{0\le t\le T^*} (|w_t|^2_\infty + |w_t(t)|^2_{D^2}) + \int_0^{T^*} |w_{tt}|^2_{D^1_*}\mathrm{d}t \le c_5^2,\\
&\operatorname{ess}\sup_{0\le t\le T^*} t|v_{tt}(t)|_2^2 + \int_0^{T^*} t(|v_{tt}|^2_{D^1_*} + |\sqrt{g}v_{tt}|^2_{D^1_*} + |v_t|^2_{D^3})\mathrm{d}t \le c_5^2.
\end{aligned} \tag{3.13}$$

T^* and $c_i (i = 1, \cdots, 5)$ will be determined later, and depend only on c_0 and the fixed constants $(A, R, c_v, \alpha, \beta, \gamma, \delta, T)$. Hereinafter, $M(c) \ge 1$ will denote a generic continuous and increasing function on $[0, \infty)$, and $C \ge 1$ will denote a generic positive constant. Both $M(c)$ and C depend only on fixed constants $(A, R, c_v, \alpha, \beta, \gamma, \delta, T)$, and may be different from line to line. Moreover, in the rest of §3.2, without causing ambiguity, we simply drop the superscript ϵ and η in $(\phi_0^\eta, u_0^\eta, l_0^\eta, h_0^\eta, \psi_0^\eta)$, $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta}, \psi^{\epsilon,\eta})$, and $(g_1^\eta, g_2^\eta, g_3^\eta, g_4^\eta)$.

3.2.1. *The a priori estimates for ϕ.* In the rest of §3.2, let (ϕ, u, l, h) be the unique classical solution to (3.1) in $[0,T]\times\mathbb{R}^3$ obtained in Lemma 3.1.

Lemma 3.2.

$$\begin{aligned}
&\|\phi(t)-\eta\|_{D^1_*\cap D^3} \le Cc_0, \quad |\phi_t(t)|_2 \le Cc_0c_2, \quad |\phi_t(t)|_{D^1_*} \le Cc_0c_3,\\
&|\phi_t(t)|_{D^2} \le Cc_0c_4, \quad |\phi_{tt}(t)|_2 \le Cc_4^3, \quad \int_0^t \|\phi_{tt}(s)\|_1^2 ds \le Cc_0^2c_4^2,
\end{aligned} \tag{3.14}$$

for $0 \le t \le T_1 = \min\{T^*, (1 + Cc_4)^{-6}\}$.

Proof. First, it follows directly from $(3.1)_1$ that, for $0 \le t \le T_1$,

$$|\phi|_\infty \le |\phi_0|_\infty \exp\Big(C\int_0^t |\mathrm{div} v|_\infty \mathrm{d}s\Big) \le Cc_0. \tag{3.15}$$

Second, the standard energy estimates for transport equations, (3.13) and (3.15) yield that, for $0 \le t \le T_1$,

$$\|\phi-\eta\|_{D^1_*\cap D^3} \le C(\|\phi_0-\eta\|_{D^1_*\cap D^3}+\eta\int_0^t \|\nabla v\|_3 \mathrm{d}s)\exp\Big(\int_0^t C\|v\|_4\mathrm{d}s\Big)\le Cc_0.$$

This, together with $(3.1)_1$, yields that for $0 \le t \le T_1$,

$$\begin{cases}|\phi_t(t)|_2 \le C(|v|_3|\nabla\phi|_6+|\phi|_\infty|\nabla v|_2)\le Cc_0c_2,\\ |\phi_t(t)|_{D^1_*}\le C(|v|_\infty|\nabla^2\phi|_2+|\nabla\phi|_6|\nabla v|_3+|\phi|_\infty|\nabla^2 v|_2)\le Cc_0c_3,\\ |\phi_t(t)|_{D^2}\le C\|v\|_3(\|\nabla\phi\|_2+|\phi|_\infty)\le Cc_0c_4.\end{cases} \tag{3.16}$$

Similarly, it follows from

$$\phi_{tt}=-v_t\cdot\nabla\phi-v\cdot\nabla\phi_t-(\gamma-1)\phi_t\mathrm{div}v-(\gamma-1)\phi\mathrm{div}v_t$$

and (3.13) that for $0 \le t \le T_1$,

$$\begin{aligned}|\phi_{tt}|_2 \le & C(|v_t|_3|\nabla\phi|_6+|v|_\infty|\nabla\phi_t|_2+|\nabla v|_\infty|\phi_t|_2+|\phi|_\infty|\nabla v_t|_2)\le Cc_4^3,\\ \int_0^t\|\phi_{tt}\|_1^2\mathrm{d}s\le & \int_0^t(\|v_t\cdot\nabla\phi\|_1+\|v\cdot\nabla\phi_t\|_1+\|\phi_t\mathrm{div}v\|_1+\|\phi\mathrm{div}v_t\|_1)^2\mathrm{d}s\le Cc_0^2c_4^2.\end{aligned}$$

The proof of Lemma 3.2 is complete. □

3.2.2. *The a priori estimates for ψ.* The following estimates for ψ are needed to deal with the degenerate elliptic operator.

Lemma 3.3. For $t\in[0,T_1]$ and $q>3$, it holds that

$$\begin{aligned}&|\psi(t)|_\infty^2+\|\psi(t)\|^2_{L^q\cap D^{1,3}\cap D^2}\le Cc_0^2,\quad |\psi_t(t)|_2\le Cc_3^2,\\ &|h_t(t)|^2_\infty\le Cc_3^3c_4,\quad |\psi_t(t)|^2_{D^1_*}+\int_0^t(|\psi_{tt}|_2^2+|h_{tt}|_6^2)\mathrm{d}s\le Cc_4^4.\end{aligned} \tag{3.17}$$

Proof. It follows from $\psi=\frac{a\delta}{\delta-1}\nabla h$ and $(3.1)_4$ that

$$\psi_t+\sum_{k=1}^3 A_k(v)\partial_k\psi+B^*(v)\psi+a\delta(g\nabla\mathrm{div}v+\nabla g\mathrm{div}v)=0, \tag{3.18}$$

with $B^*(v)=(\nabla v)^\top$ and $A_k(v)$ defined in (2.2).

First, multiplying (3.18) by $q\psi|\psi|^{q-2}$ and integrating over $\mathbb{R}^3$ yield that

$$\begin{aligned}\frac{d}{dt}|\psi|_q^q \le & C(|\nabla v|_\infty|\psi|_q^q+|\mathrm{div}v|_\infty|\nabla g|_q|\psi|_q^{q-1}+|g\nabla^2v|_q|\psi|_q^{q-1})\\ \le & C(|\nabla v|_\infty|\psi|_q^q+|\mathrm{div}v|_\infty|\nabla g|_q|\psi|_q^{q-1}+\|g\nabla^2 v\|_2|\psi|_q^{q-1}).\end{aligned} \tag{3.19}$$

According to (3.13), one can obtain that

$$\int_0^t\|g\nabla^2v\|_2\mathrm{d}s\le t^{\frac12}\Big(\int_0^t\|g\nabla^2v\|_2^2\mathrm{d}s\Big)^{\frac12}\le c_4t^{\frac12},$$

which, together with (3.19) and Gronwall's inequality, yields that

$$|\psi(t)|_q\le Cc_0 \quad \text{for} \quad 0\le t\le T_1.$$

Second, set $\varsigma=(\varsigma_1,\varsigma_2,\varsigma_3)^\top$ ($|\varsigma|=1$ and $\varsigma_i=0,1$). Applying ∂_x^ς to (3.18), multiplying by $3|\partial_x^\varsigma\psi|\partial_x^\varsigma\psi$ and then integrating over $\mathbb{R}^3$, one can get

$$\frac{d}{dt}|\partial_x^\varsigma\psi|_3^3\leq\Big(\sum_{k=1}^3|\partial_k A_k(v)|_\infty+|B^*(v)|_\infty\Big)|\partial_x^\varsigma\psi|_3^3+C|\Theta_\varsigma|_3|\partial_x^\varsigma\psi|_3^2, \tag{3.20}$$

where

$$\Theta_\varsigma=\partial_x^\varsigma(B^*\psi)-B^*\partial_x^\varsigma\psi+\sum_{k=1}^3\big(\partial_x^\varsigma(A_k\partial_k\psi)-A_k\partial_k\partial_x^\varsigma\psi\big)+a\delta\partial_x^\varsigma\big(g\nabla\mathrm{div}v+\nabla g\mathrm{div}v\big).$$

On the other hand, for $|\varsigma|=2$ and $\varsigma_i=0,1,2$, applying ∂_x^ς to (3.18), multiplying by $2\partial_x^\varsigma\psi$ and then integrating over $\mathbb{R}^3$ lead to

$$\frac{d}{dt}|\partial_x^\varsigma\psi|_2^2\leq\Big(\sum_{k=1}^3|\partial_k A_k(v)|_\infty+|B^*(v)|_\infty\Big)|\partial_x^\varsigma\psi|_2^2+C|\Theta_\varsigma|_2|\partial_x^\varsigma\psi|_2. \tag{3.21}$$

For $|\varsigma|=1$, it is easy to obtain

$$|\Theta_\varsigma|_3\leq C\big(|\nabla^2 v|_3(|\psi|_\infty+|\nabla g|_\infty)+|\nabla v|_\infty(|\nabla\psi|_3+|\nabla^2 g|_3)+|\nabla(g\nabla^2 v)|_3\big). \tag{3.22}$$

Similarly, for $|\varsigma|=2$, one has

$$\begin{aligned}|\Theta_\varsigma|_2\leq&C\big(|\nabla v|_\infty(|\nabla^2\psi|_2+|\nabla^3 g|_2)+|\nabla^2 v|_6(|\nabla\psi|_3+|\nabla^2 g|_3)\big)\\&+C|\nabla^3 v|_2(|\psi|_\infty+|\nabla g|_\infty)+C|g\nabla\mathrm{div}v|_{D^2}.\end{aligned} \tag{3.23}$$

It follows from (3.20)-(3.23) and the Gagliardo-Nirenberg inequality

$$|\psi|_\infty\leq C|\psi|_q^\xi|\nabla\psi|_6^{1-\xi}\leq C|\psi|_q^\xi|\nabla^2\psi|_2^{1-\xi}\quad\text{with}\quad\xi=\frac{q}{6+q},$$

that

$$\frac{d}{dt}\|\psi(t)\|_{D^{1,3}\cap D^2}\leq Cc_4\|\psi(t)\|_{D^{1,3}\cap D^2}+C|g\nabla\mathrm{div}v|_{D^2}+Cc_4^2,$$

which, along with Gronwall's inequality, implies that for $0\leq t\leq T_1$,

$$\|\psi(t)\|_{D^{1,3}\cap D^2}\leq\Big(c_0+Cc_4^2t+C\int_0^t|g\nabla\mathrm{div}v|_{D^2}\mathrm{d}s\Big)\exp(Cc_4t)\leq Cc_0. \tag{3.24}$$

Next, due to (3.18), it holds that for $0\leq t\leq T_1$,

$$\begin{cases}|\psi_t(t)|_2\leq C\big(|\nabla v|_2|\psi|_{D^{1,3}}+|\nabla v|_2|\psi|_\infty+|g\nabla^2 v|_2+|\nabla g|_\infty|\nabla v|_2\big)\leq Cc_3^2,\\[2ex]|\nabla\psi_t(t)|_2\leq C\big(\|v\|_3(\|\psi\|_{L^q\cap D^{1,3}\cap D^2}+\|\nabla g\|_{L^q\cap D^{1,3}\cap D^2})+|g\nabla^2 v|_{D^1_*}\big)\leq Cc_4^2.\end{cases}$$

Similarly, via the relation

$$\psi_{tt}=-\nabla(v\cdot\psi)_t-a\delta\big(g\nabla\mathrm{div}v+\nabla g\mathrm{div}v\big)_t,$$

for $0\leq t\leq T_1$, one gets

$$\begin{aligned}\int_0^t|\psi_{tt}|_2^2\mathrm{d}s\leq&C\int_0^t\Big(|v_t|_6^2|\nabla\psi|_3^2+|\nabla v|_\infty^2|\psi_t|_2^2+|v|_\infty^2|\nabla\psi_t|_2^2+|\psi|_\infty^2|\nabla v_t|_2^2\\&+|(g\nabla\mathrm{div}v)_t|_2^2+|\nabla g|_\infty^2|\nabla v_t|_2^2+|\nabla v|_\infty^2|\nabla g_t|_2^2\Big)\mathrm{d}s\leq Cc_4^4.\end{aligned} \tag{3.25}$$

Finally, it follows from Gagliardo-Nirenberg inequality and (3.13) that

$$\begin{aligned}|g\mathrm{div}v|_\infty \le & C|g\mathrm{div}v|_{D^1}^{\frac{1}{2}}|g\mathrm{div}v|_{D^2}^{\frac{1}{2}} \le C\big(|\nabla g|_\infty|\nabla v|_2+|g\nabla^2 v|_2\big)^{\frac{1}{2}}\\ &\cdot\big(|\nabla^2 g|_2|\nabla v|_\infty+|\nabla g|_\infty|\nabla^2 v|_2+|g\nabla^2 v|_{D^1_*}\big)^{\frac{1}{2}} \le Cc_3^{\frac{3}{2}}c_4^{\frac{1}{2}}.\end{aligned} \tag{3.26}$$

Then, together with $(3.1)_4$, it yields that for $0\le t\le T_1$,

$$|h_t(t)|_\infty \le C(|v|_\infty|\psi|_\infty+|g\mathrm{div}v|_\infty)\le Cc_3^{\frac{3}{2}}c_4^{\frac{1}{2}},$$

$$\int_0^t |h_{tt}|_6^2\mathrm{d}s \le C\int_0^t \big(|v|_\infty|\psi_t|_6+|v_t|_6|\psi|_\infty+|g_t|_\infty|\nabla v|_6+|g\nabla v_t|_6\big)^2\mathrm{d}s\le Cc_4^4,$$

where one has used the fact that

$$\begin{aligned}|g\nabla v_t|_6 \le & C\big(|\nabla g|_\infty|\nabla v_t|_2+|g\nabla^2 v_t|_2)\\ \le & C\big(|\nabla g|_\infty|\nabla v_t|_2+|(g\nabla^2 v)_t|_2+|g_t|_\infty|\nabla^2 v|_2).\end{aligned} \tag{3.27}$$

The proof of Lemma 3.3 is complete. □

3.2.3. *The a priori estimates for h-related auxiliary variables.* Set

$$\varphi=h^{-1}\quad\text{and}\qquad n=(ah)^b=a^b h^{\frac{2-\delta-\gamma}{\delta-1}}.$$

Lemma 3.4. For $t\in[0,T_1]$ and $q>3$, it holds that

$$\begin{aligned}h(t,x)>\frac{1}{2c_0},\quad \frac{2}{3}\eta^{-2\iota}<\varphi(t,x)<2|\varphi_0|_\infty\le & 2c_0,\\ |\nabla\sqrt{h}(t)|_6\le Cc_0,\quad \|n(t)\|_{L^\infty\cap D^{1,q}\cap D^{1,6}\cap D^{2,3}\cap D^3}\le & M(c_0),\\ |n_t(t)|_\infty\le M(c_0)c_4^2,\quad |n_t(t)|_6\le & M(c_0)c_3^2,\\ |\nabla n_t(t)|_3\le M(c_0)c_4^2,\quad |\nabla n_t(t)|_6\le & M(c_0)c_4^2.\end{aligned} \tag{3.28}$$

Proof. **Step 1:** Estimates on φ. Note that

$$\varphi_t+v\cdot\nabla\varphi-(\delta-1)g\varphi^2\mathrm{div}v=0. \tag{3.29}$$

Let $X(t;x)$ be the particle path defined by

$$\begin{cases}\frac{d}{ds}X(t;x)=v(s,X(t;x)), & 0\le t\le T;\\ X(0;x)=x, & x\in\mathbb{R}^3\end{cases} \tag{3.30}$$

Then

$$\varphi(t,X(t;x))=\varphi_0(x)\Big(1+(1-\delta)\varphi_0(x)\int_0^t g\mathrm{div}v(s,X(s;x))\mathrm{d}s\Big)^{-1}. \tag{3.31}$$

This, along with (3.26), implies that

$$\frac{2}{3}\eta^{-2\iota}<\varphi(t,x)<2|\varphi_0|_\infty\le 2c_0\quad\text{for}\quad [t,x]\in[0,T_1]\times\mathbb{R}^3. \tag{3.32}$$

Step 2: Estimates on $\nabla\sqrt{h}$. It follows from $(3.1)_4$ that

$$(\sqrt{h})_t+v\cdot\nabla\sqrt{h}+\frac{1}{2}(\delta-1)h^{-\frac{1}{2}}g\mathrm{div}v=0, \tag{3.33}$$

which implies

$$(\nabla\sqrt{h})_t+\nabla(v\cdot\nabla\sqrt{h})+\frac{1}{2}(\delta-1)(\nabla h^{-\frac{1}{2}}g\mathrm{div}v+h^{-\frac{1}{2}}\nabla(g\mathrm{div}v))=0. \tag{3.34}$$

Multiplying (3.34) by $6|\nabla\sqrt{h}|^4\nabla\sqrt{h}$ and integrating with respect to x over $\mathbb{R}^3$ yield

$$\begin{aligned}\frac{d}{dt}|\nabla\sqrt{h}|_6^6\le&C|\nabla v|_\infty|\nabla\sqrt{h}|_6^6+C\big(|\varphi|_\infty^{\frac{1}{2}}(|g\nabla^2v|_6+|\nabla g|_\infty|\nabla v|_6)|\nabla\sqrt{h}|_6^5\\&+|\varphi|_\infty^{\frac{3}{2}}|g\mathrm{div}v|_6|\psi|_\infty|\nabla\sqrt{h}|_6^5\big).\end{aligned}\tag{3.35}$$

Integrating (3.35) with respect to t and using (3.13), (3.32) and Lemma 3.3 lead to

$$|\nabla\sqrt{h}(t)|_6\le Cc_0\quad\text{for}\quad 0\le t\le T_1. \tag{3.36}$$

Step 3: Estimates on n. Since $n=(ah)^b$, then

$$n_t+v\cdot\nabla n+(2-\delta-\gamma)a^bh^{b-1}g\mathrm{div}v=0. \tag{3.37}$$

Then it follows from Lemma 3.3, (3.32) and (3.36) that for $0\le t\le T_1$,

$$\begin{aligned}&|n|_\infty\le a^b|\varphi|_\infty^{-b}\le M(c_0),\quad |\nabla n|_q=a^b|bh^{b-1}\nabla h|_q\le M(c_0),\\&|\nabla n|_6=2a^b|h^{b-\frac{1}{2}}\nabla\sqrt{h}|_6\le M(c_0),\\&|\nabla^2n|_6\le C(|h^{b-1}\nabla^2h|_6+|h^{b-\frac{3}{2}}\nabla h\cdot\nabla\sqrt{h}|_6)\le M(c_0),\\&|\nabla^3n|_2\le C(|h^{b-1}\nabla^3h|_2+|h^{b-\frac{3}{2}}\nabla^2h\cdot\nabla\sqrt{h}|_2+|h^{b-\frac{3}{2}}|\nabla\sqrt{h}|^3|_2)\le M(c_0),\\&|\nabla^2n|_3\le C(|h^{b-1}\nabla^2h|_3+|h^{b-1}|\nabla\sqrt{h}|^2|_3)\le M(c_0),\\&|n_t|_\infty\le C(|v|_\infty|\nabla n|_\infty+|\varphi^{1-b}|_\infty|g\mathrm{div}v|_\infty)\le M(c_0)c_4^2,\end{aligned}$$

and

$$\begin{aligned}|n_t|_6\le&C(|v|_\infty|\nabla n|_6+|\varphi^{1-b}|_\infty|g\mathrm{div}v|_6)\le M(c_0)c_3^2,\\|\nabla n_t|_3\le&C(|\nabla v\cdot\nabla n|_3+|v\cdot\nabla^2n|_3+|h^{b-1}\nabla g\mathrm{div}v|_3)\\&+C(|h^{b-\frac{3}{2}}\nabla\sqrt{h}g\mathrm{div}v|_3+|h^{b-1}g\nabla\mathrm{div}v|_3)\le M(c_0)c_4^2,\\|\nabla n_t|_6\le&C(|\nabla v\cdot\nabla n|_6+|v\cdot\nabla^2n|_6+|h^{b-1}\nabla g\mathrm{div}v|_6)\\&+C(|h^{b-1}\nabla hg\mathrm{div}v|_6+|h^{b-1}g\nabla\mathrm{div}v|_6)\le M(c_0)c_4^2.\end{aligned}$$

The proof of Lemma 3.4 is complete. □

3.2.4. *The a priori estimates for l.* We now turn to the estimates involving the entropy.

Lemma 3.5. *Set $T_2=\min\{T_1,(1+Cc_4)^{-20-2\nu}\}$. Then for $t\in[0,T_2]$,*

$$\begin{aligned}&c_0^{-1}\le l(t,x)\le M(c_0),\quad \|l-\bar{l}\|_{D_*^1\cap D^3}\le M(c_0),\\&|l_t|_\infty\le M(c_0)c_3^{3+\nu}c_4,\quad |l_t|_3\le M(c_0)c_3^{4+\nu},\quad |l_t|_{D_*^1}\le M(c_0)c_3^{6+\nu}c_4^{\frac{1}{2}},\\&|\nabla^2l_t|_2\le M(c_0)c_3^{8+\nu}c_4^{\frac{3}{2}},\quad \int_0^t|l_{tt}|_6^2dt\le M(c_0)c_4^{6+2\nu},\\&\int_0^t|\nabla l_{tt}|_2^2dt\le M(c_0)c_4^{10+2\nu},\quad |h^{\frac{1}{4}}\nabla l|_6\le M(c_0).\end{aligned}\tag{3.38}$$

Proof. **Step 1:** It follows from the equation for entropy, $(3.1)_3$ that

$$\frac{dl(t,X(t,x))}{dt}=a_4w^\nu ng^2H(v)=a_4w^\nu ng^2(2\alpha|Dv|^2+\beta|\mathrm{div}v|^2)\geq 0,$$

which yields that

$$l(t,X(t,x))\geq l_0(x)\geq c_0^{-1}\quad\text{for}\quad 0\leq t\leq T_1. \tag{3.39}$$

Step 2: Set $\varsigma=(\varsigma_1,\varsigma_2,\varsigma_3)^\top$ with ς_i nonnegative integer and $|\varsigma|=\varsigma_1+\varsigma_2+\varsigma_3$. Applying ∂_x^ς to $(3.1)_3$, multiplying by $2\partial_x^\varsigma l$ and then integrating over $\mathbb{R}^3$, one has

$$\frac{d}{dt}|\partial_x^\varsigma l|_2^2\leq|\nabla v|_\infty|\partial_x^\varsigma l|_2^2+|\Lambda_\varsigma|_2|\partial_x^\varsigma l|_2, \tag{3.40}$$

where

$$\Lambda_\varsigma=\partial_x^\varsigma(a_4w^\nu ng^2H(v))-(\partial_x^\varsigma(v\cdot\nabla l)-v\cdot\nabla\partial_x^\varsigma l),$$

which will be estimated according to $|\varsigma|$.

For $|\varsigma|=1$, direct calculations show that

$$\begin{aligned}|\Lambda_\varsigma|_2\leq&C|w|_\infty^\nu(|\nabla n|_6|g\nabla v|_6^2+|n|_\infty|g\nabla v|_\infty(|\nabla g|_\infty|\nabla v|_2+|g\nabla^2v|_2))\\&+C(|w^{\nu-1}|_\infty|n|_\infty|g\nabla v|_6^2|\nabla w|_6+|\nabla v|_\infty|\nabla l|_2).\end{aligned} \tag{3.41}$$

Next, for $|\varsigma|=2$, it holds that

$$\begin{aligned}|\Lambda_\varsigma|_2\leq&C(|w^\nu n\nabla^2(g^2H(v))|_2+|\nabla(w^\nu n)\nabla(g^2H(v))|_2\\&+|\nabla^2(w^\nu n)g^2H(v)|_2+|\partial_x^\varsigma(v\cdot\nabla l)-v\cdot\nabla\partial_x^\varsigma l|_2)=C\sum_{i=1}^4A_i.\end{aligned} \tag{3.42}$$

By direct calculations, Sobolev inequalities and Lemma 5.2, one can obtain

$$\begin{aligned}A_1\leq&C|w^\nu|_\infty|n|_\infty|\nabla^2(g^2H(v))|_2\\\leq&C|w^\nu|_\infty|n|_\infty(|g\nabla v|_\infty|\nabla^2g|_3|\nabla v|_6+|g\nabla v|_\infty|\nabla g|_\infty|\nabla^2v|_2\\&+|\nabla g|_\infty^2|\nabla v|_\infty|\nabla v|_2+|g\nabla v|_\infty|g\nabla^3v|_2+|g\nabla^2v|_4^2),\\A_2\leq&C|w^\nu|_\infty|\nabla n|_\infty|g\nabla v|_\infty(|\nabla g|_\infty|\nabla v|_2+|g\nabla^2v|_2)\\&+C|w^{\nu-1}|_\infty|n|_\infty|\nabla w|_\infty|g\nabla v|_\infty(|g\nabla^2v|_2+|\nabla g|_\infty|\nabla v|_2),\\A_3\leq&C|w^\nu|_\infty|\nabla^2n|_6|g\nabla v|_6^2+C|w^{\nu-1}|_\infty|g\nabla v|_6^2(|n|_\infty|\nabla^2w|_6+|\nabla n|_\infty|\nabla w|_6)\\&+C|w^{\nu-2}|_\infty|n|_\infty|\nabla w|_4^2|g\nabla v|_\infty^2,\\A_4\leq&C(|\nabla^2v|_3+|\nabla v|_\infty)|\nabla^2l|_2.\end{aligned} \tag{3.43}$$

Similarly, for $|\varsigma|=3$, one has

$$\begin{aligned}|\Lambda_\varsigma|_2\leq&C(|\nabla^3(w^\nu n)(g^2H(v))|_2+|w^\nu n\nabla^3(g^2H(v))|_2\\&+|\nabla^2(w^\nu n)\nabla(g^2H(v))|_2+|\nabla(w^\nu n)\nabla^2(g^2H(v))|_2\\&+|\partial_x^\varsigma(v\cdot\nabla l)-v\cdot\nabla\partial_x^\varsigma l|_2=C\sum_{i=1}^5B_i,\end{aligned} \tag{3.44}$$

and each B_i $(i=1,...,5)$ can be estimated as follows:

$$
\begin{aligned}
B_1 \leq & |g\nabla v|_\infty^2 |\nabla^3(w^\nu n)|_2 \\
\leq & C|g\nabla v|_\infty^2 (|w^\nu|_\infty |\nabla^3 n|_2 + |w^{\nu-1}|_\infty |\nabla^2 n|_6 |\nabla w|_3 \\
& + |w^{\nu-1}|_\infty |n|_\infty |\nabla^3 w|_2 + |w^{\nu-1}|_\infty |\nabla n|_6 |\nabla^2 w|_3 \\
& + |w^{\nu-2}|_\infty |\nabla n|_6 |\nabla w|_6^2 + |w^{\nu-2}|_\infty |n|_\infty |\nabla w|_6 |\nabla^2 w|_3 \\
& + |w^{\nu-3}|_\infty |n|_\infty |\nabla w|_6^3), \\
B_2 \leq & |w^\nu|_\infty |n|_\infty |\nabla^3 (g^2 H(v))|_2 \\
\leq & C|w^\nu|_\infty |n|_\infty \big(|g\nabla v|_\infty |g\nabla^4 v|_2 + |g\nabla^3 v|_6 |g\nabla^2 v|_3 \\
& + |\nabla g|_\infty^2 |\nabla^2 v|_2 |\nabla v|_\infty + |g\nabla v|_\infty |\nabla^2 g|_6 |\nabla^2 v|_3 \\
& + |g\nabla v|_\infty |\nabla^3 g|_2 |\nabla v|_\infty + |\nabla g|_\infty |\nabla^2 g|_3 |\nabla v|_6 |\nabla v|_\infty \\
& + |g\nabla^3 v|_2 |\nabla g|_\infty |\nabla v|_\infty + |g\nabla^2 v|_6 |\nabla g|_\infty |\nabla^2 v|_3\big), \\
B_3 \leq & |\nabla^2(w^\nu n)|_3 |\nabla (g^2 H(v))|_6 \\
\leq & C\big(|w^\nu|_\infty |\nabla^2 n|_3 + |n|_\infty (|w^{\nu-1}|_\infty |\nabla^2 w|_3 + |w^{\nu-2}|_\infty |\nabla w|_6^2) \\
& + |w^{\nu-1}|_\infty |\nabla n|_\infty |\nabla w|_3\big) \cdot (|g\nabla v|_\infty |\nabla g|_\infty |\nabla v|_6 + |g\nabla v|_\infty |g\nabla^2 v|_6), \\
B_4 \leq & |\nabla(w^\nu n)|_\infty |\nabla^2 (g^2 H(v))|_2 \\
\leq & C(|w^{\nu-1}|_\infty |\nabla w|_\infty |n|_\infty + |w^\nu|_\infty |\nabla n|_\infty) \cdot (|g\nabla v|_\infty |\nabla^2 g|_3 |\nabla v|_6 \\
& + |\nabla g|_\infty^2 |\nabla v|_4^2 + |g\nabla v|_\infty |g\nabla^3 v|_2 + |g\nabla^2 v|_4^2 + |g\nabla v|_\infty |\nabla g|_\infty |\nabla^2 v|_2), \\
B_5 \leq & C(|\nabla^3 v|_2 + |\nabla^2 v|_3 + |\nabla v|_\infty)\|\nabla l\|_2.
\end{aligned}
\tag{3.45}
$$

(3.13) implies that

$$
\begin{aligned}
|w^\nu|_\infty \leq Cc_1^\nu, \quad |w^{\nu-1}|_\infty \leq Cc_1^{\nu+1}, \quad |w^{\nu-2}|_\infty \leq & Cc_1^{\nu+2}, \\
|w^{\nu-3}|_\infty \leq Cc_1^{\nu+3}, \quad |\nabla w|_\infty + \|\nabla w\|_2 \leq & Cc_1,
\end{aligned}
\tag{3.46}
$$

which, along with (3.40)-(3.45), (3.13) and (3.28), yields that

$$
\frac{d}{dt}\|\nabla l\|_2 \leq C\|v\|_3 \|\nabla l\|_2 + M(c_0)(c_4^{\nu+10} + c_4^{\nu+2}|g\nabla^4 v|_2).
$$

It then follows from Gronwall's inequality and (3.13) again that

$$
\|\nabla l\|_2 \leq (\|\nabla l_0\|_2 + M(c_0)c_4^{\nu+10}(t+t^{\frac{1}{2}}))\exp(Cc_4 t) \leq M(c_0) \quad \text{for } 0 \leq t \leq T_2. \tag{3.47}
$$

Second, according to $(3.1)_3$, for $0 \leq t \leq T_2$, it holds that

$$
\begin{aligned}
|l_t|_\infty \leq C(|v|_\infty |\nabla l|_\infty + |w^\nu|_\infty |n|_\infty |g\nabla v|_\infty^2) \leq M(c_0)c_3^{3+\nu}c_4, \\
|l_t|_3 \leq C(|v|_\infty |\nabla l|_3 + |w^\nu|_\infty |n|_\infty |g\nabla v|_6^2) \leq M(c_0)c_3^{4+\nu}.
\end{aligned}
\tag{3.48}
$$

It follows from (3.41) and $(3.1)_3$ that for $|\varsigma|=1$,

$$
|\partial_x^\varsigma l_t|_2 \leq C(|v|_\infty |\nabla^2 l|_2 + |\varLambda_\varsigma|_2) \leq M(c_0)c_3^{6+\nu}c_4^{\frac{1}{2}}. \tag{3.49}
$$

Similarly, for $|\varsigma|=2$, from (3.42)-(3.43), one can obtain

$$
|\partial_x^\varsigma l_t|_2 \leq C(|v|_\infty |\nabla^3 l|_2 + |\varLambda_\varsigma|_2) \leq M(c_0)c_3^{8+\nu}c_4^{\frac{3}{2}}. \tag{3.50}
$$

On the other hand, since

$$l_{tt} = -(v\cdot\nabla l)_t + a_4(w^\nu n g^2 H(v))_t, \tag{3.51}$$

one gets

$$\begin{aligned}|l_{tt}|_6 \le & C(|v_t|_6|\nabla l|_\infty + |v|_\infty|\nabla l_t|_6 + |w^\nu|_\infty|g\nabla v|_\infty(|n_t|_\infty|g\nabla v|_6\\ & + |n|_\infty|g_t|_\infty|\nabla v|_6 + |n|_\infty|g\nabla v_t|_6) + |w^{\nu-1}|_\infty|n|_\infty|w_t|_6|g\nabla v|^2_\infty),\end{aligned}$$

which implies that

$$\int_0^t |l_{tt}|_6^2 \mathrm{d}s \le M(c_0)c_4^{6+2\nu} \quad \text{for} \quad 0\le t\le T_2. \tag{3.52}$$

Due to (3.51), one has that for $|\varsigma|=1$,

$$\partial_x^\varsigma l_{tt} = -\partial_x^\varsigma\big((v\cdot\nabla l)_t\big) + \partial_x^\varsigma(a_4 w^\nu n g^2 H(v))_t. \tag{3.53}$$

It follows from Lemma 3.4, (3.47)-(3.48), (3.49)-(3.50) and (3.13) that

$$\begin{aligned}|\partial_x^\varsigma\big((v\cdot\nabla l)_t\big)|_2 \le & C(|v_t|_6|\nabla^2 l|_3 + |\nabla v_t|_2|\nabla l|_\infty + |\nabla v|_\infty|\nabla l_t|_2 + |v|_\infty|\nabla^2 l_t|_2)\\ \le & M(c_0)|\nabla v_t|_2 + M(c_0)c_4^{11+\nu},\\ |\partial_x^\varsigma(w^\nu n g^2 H(v))_t|_2 \le & C(|\partial_x^\varsigma((w^\nu)_t n g^2 H(v))|_2 + |\partial_x^\varsigma(w^\nu n_t g^2 H(v))|_2\\ & + |\partial_x^\varsigma(w^\nu n g g_t H(v))|_2 + |\partial_x^\varsigma(w^\nu n g^2 H(v)_t)|_2) = C\sum_{i=1}^4 K_i,\end{aligned}$$

and each K_i $(i=1,...,4)$ can be estimated as follows:

$$\begin{aligned}K_1 \le & C\big(|n|_\infty|g\nabla v|^2_\infty(|w^{\nu-1}|_\infty|\nabla w_t|_2 + |w^{\nu-2}|_\infty|w_t|_3|\nabla w|_6)\\ & + |w^{\nu-1}|_\infty|w_t|_3|\nabla n|_6|g\nabla v|^2_\infty + |w^{\nu-1}|_\infty|n|_\infty|\nabla g|_\infty|w_t|_3|g\nabla v|_\infty|\nabla v|_6\\ & + |w^{\nu-1}|_\infty|n|_\infty|w_t|_3|g\nabla v|_\infty|g\nabla^2 v|_6\big),\\ K_2 \le & C\big(|w^{\nu-1}|_\infty|n_t|_\infty|g\nabla v|^2_\infty|\nabla w|_2 + |w^\nu|_\infty|g\nabla v|_\infty|g\nabla v|_6|\nabla n_t|_3\\ & + |w^\nu|_\infty|n_t|_\infty|\nabla g|_\infty|g\nabla v|_\infty|\nabla v|_2 + |w^\nu|_\infty|n_t|_\infty|g\nabla v|_\infty|g\nabla^2 v|_2\big),\\ K_3 \le & C\big(|w^{\nu-1}|_\infty|n|_\infty|g\nabla v|_\infty|g_t|_\infty|\nabla w|_3|\nabla v|_6 + |w^\nu|_\infty|g\nabla v|_\infty|g_t|_\infty|\nabla n|_6|\nabla v|_3\\ & + |w^\nu|_\infty|n|_\infty(|g_t|_\infty|\nabla g|_\infty|\nabla v|^2_4 + |g\nabla v|_\infty|\nabla v|_\infty|\nabla g_t|_2 + |g\nabla v|_\infty|g_t|_\infty|\nabla^2 v|_2)\big),\\ K_4 \le & C\big(|w^{\nu-1}|_\infty|n|_\infty|g\nabla v|_\infty|g\nabla v_t|_6|\nabla w|_3 + |w^\nu|_\infty|g\nabla v|_6|g\nabla v_t|_6|\nabla n|_6\\ & + |w^\nu|_\infty|n|_\infty(|g\nabla v|_\infty|\nabla g|_\infty|\nabla v_t|_2 + |g\nabla v_t|_6|g\nabla^2 v|_3 + |g\nabla v|_\infty|g\nabla^2 v_t|_2)\big),\end{aligned}$$

which yields that

$$|\partial_x^\varsigma(w^\nu n g^2 H(v))_t|_2 \le M(c_0)c_4^{8+\nu} + M(c_0)c_4^{4+\nu}|g\nabla^2 v_t|_2 + M(c_0)c_4^{5+\nu}|\nabla v_t|_2.$$

These and (3.53) yield

$$\int_0^t |\nabla l_{tt}|_2^2 \mathrm{d}s \le M(c_0)c_4^{10+2\nu} \quad \text{for} \quad 0\le t\le T_2. \tag{3.54}$$

Step 3: The estimate on $|h^{\frac14}\nabla l|_6$. Applying $h^{\frac14}\nabla$ to $(3.1)_3$ yields

$$(h^{\frac14}\nabla l)_t - (h^{\frac14})_t\nabla l + h^{\frac14}\nabla(v\cdot\nabla l) = a_4 h^{\frac14}\nabla(w^\nu n g^2 H(v)). \tag{3.55}$$

Denoting $h^{\frac{1}{4}}\nabla l = z$, multiplying $6|z|^4 z$ on both side of (3.55), integrating over $\mathbb{R}^3$, and integration by part, one has

$$\begin{aligned}\frac{d}{dt}|z|_6^6 \le & C\int \big|\big((h^{\frac{1}{4}})_t\nabla l - h^{\frac{1}{4}}\nabla(v\cdot\nabla l) + a_4 h^{\frac{1}{4}}\nabla(w^\nu n g^2 H(v))\big)\cdot |z|^4 z\big| \\ \le & C(|h_t|_\infty|\varphi|_\infty + |\nabla v|_\infty + |v|_\infty|\psi|_\infty|\varphi|_\infty)|z|_6^6 + J_*,\end{aligned} \tag{3.56}$$

where $J_* = C\int |h^{\frac{1}{4}}\nabla(w^\nu n g^2 H(v))\cdot |z|^4 z|$.

Note that $n = (ah)^b$ and $\frac{1}{4} + b \le 0$ due to (1.17). One can get

$$\begin{aligned}J_* \le & C|\varphi|_\infty^{-\frac{1}{4}-b}(|w^{\nu-1}|_\infty|\nabla w|_6|g\nabla v|_\infty^2 + |w^\nu|_\infty|g\nabla v|_\infty|\nabla g|_6|\nabla v|_\infty \\ & + |w^\nu|_\infty|g\nabla v|_\infty|g\nabla^2 v|_6 + |\varphi|_\infty|\psi|_\infty|w^\nu|_\infty|g\nabla v|_\infty|g\nabla v|_6)|z|_6^5.\end{aligned} \tag{3.57}$$

Note that (3.5) implies

$$|h_0^{\frac{1}{4}}\nabla l_0|_6 \le C|\nabla(\phi_0^{\frac{\iota}{2}}\nabla l_0)|_2 \le C(|\phi_0^{-\frac{\iota}{2}}|_\infty|\phi_0^\iota\nabla^2 l_0|_2 + |\nabla\phi_0^{\frac{\iota}{2}}\cdot\nabla l_0|_2) \le M(c_0). \tag{3.58}$$

It follows from (3.56)-(3.58) and Gronwall's inequality that

$$|h^{\frac{1}{4}}\nabla l|_6 \le M(c_0) \quad \text{for} \quad 0 \le t \le T_2. \tag{3.59}$$

The proof of Lemma 3.5 is complete. □

3.2.5. *The equivalence of g and h in a short time.*

Lemma 3.6. *It holds that*

$$\widetilde{C}^{-1} \le gh^{-1} \le \widetilde{C} \tag{3.60}$$

for $0 \le t \le T_2$, where $\widetilde{C}$ is a suitable constant.

Proof. Set $gh^{-1} = y(t,x)$. Then a simple computation shows

$$y_t + yh^{-1}h_t = g_t\varphi; \quad y(0,x) = 1. \tag{3.61}$$

Thus

$$y(t,x) = \exp\big(-\int_0^t h_s h^{-1}\mathrm{d}s\big)\big(1 + \int_0^t g_s\varphi\exp\big(\int_0^s h_\tau h^{-1}\mathrm{d}\tau\big)\mathrm{d}s\big), \tag{3.62}$$

which, along with Lemmas 3.3-3.4 and (3.13), yields (3.60).

The proof of Lemma 3.6 is complete. □

3.2.6. *The a priori estimates for u.* Based on the estimates of ϕ, h and l obtained above, now we are ready to derive the lower order energy estimates for u.

Lemma 3.7. For $t \in [0, T_2]$, it holds that

$$\begin{aligned}|\sqrt{h}\nabla u(t)|_2^2 + \|u(t)\|_1^2 + \int_0^t(\|\nabla u\|_1^2 + |u_t|_2^2)\mathrm{d}s \le & M(c_0), \\ (|u|_{D^2}^2 + |h\nabla^2 u|_2^2 + |u_t|_2^2)(t) + \int_0^t(|u|_{D^3}^2 + |u_t|_{D_*^1}^2)\mathrm{d}s \le & M(c_0)c_2^3 c_3.\end{aligned} \tag{3.63}$$

Proof. **Step 1:** Estimate on $|u|_2$. It follows from $(3.1)_2$ that

$$l^{-\nu}(u_t+v\cdot\nabla v+a_1\phi\nabla l+l\nabla\phi)+a_2\sqrt{h^2+\epsilon^2}Lu=a_2gl^{-\nu}\nabla l^\nu\cdot Q(v)+a_3\psi\cdot Q(v). \quad (3.64)$$

Multiplying (3.64) by u and integrating over $\mathbb{R}^3$, one can obtain by integration by parts, Gagliardo-Nirenberg inequality, Hölder's inequality and Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|l^{-\frac{\nu}{2}}u|_2^2+a_2\alpha|(h^2+\epsilon^2)^{\frac{1}{4}}\nabla u|_2^2+a_2(\alpha+\beta)|(h^2+\epsilon^2)^{\frac{1}{4}}\mathrm{div}u|_2^2\\
=&-\int l^{-\nu}(v\cdot\nabla v+a_1\phi\nabla l+l\nabla\phi-a_2\nabla l^\nu\cdot gQ(v)-a_3l^\nu\psi\cdot Q(v))\cdot u\\
&+\frac{1}{2}\int(l^{-\nu})_t|u|^2-a_2\int\nabla\sqrt{h^2+\epsilon^2}\cdot Q(u)\cdot u\\
\le& C\big(|l^{-\frac{\nu}{2}}|_\infty(|v|_\infty|\nabla v|_2+|\nabla l|_2|\phi|_\infty+|l|_\infty|\nabla\phi|_2)+|l^{\frac{\nu}{2}-1}|_\infty|g\nabla v|_\infty|\nabla l|_2\\
&+|l^{\frac{3}{2}\nu}|_\infty|\psi|_\infty|\nabla v|_2\big)|l^{-\frac{\nu}{2}}u|_2+C|l^{-1}|_\infty|l_t|_\infty|l^{-\frac{\nu}{2}}u|_2^2\\
&+C|\psi|_\infty|l^{\frac{\nu}{2}}|_\infty|\varphi|_\infty^{\frac{1}{2}}|\sqrt{h}\nabla u|_2|l^{-\frac{\nu}{2}}u|_2\\
\le& M(c_0)c_4^{4+\nu}|l^{-\frac{\nu}{2}}u|_2^2+M(c_0)c_4^4+\frac{1}{2}a_2\alpha|\sqrt{h}\nabla u|_2^2,
\end{aligned}\quad (3.65)$$

which, along with Gronwall's inequalty and Lemma 3.5, yields that for $0\le t\le T_2$,

$$\begin{aligned}
&|u|_2^2+|l^{-\frac{\nu}{2}}u|_2^2+\int_0^t|\sqrt{h}\nabla u|_2^2\mathrm{d}s\\
\le& M(c_0)(|u_0|_2^2+c_4^4t)\exp\left(M(c_0)c_4^{4+\nu}t\right)\le M(c_0).
\end{aligned}\quad (3.66)$$

Step 2: Estimate on $|\nabla u|_2$. Multiplying (3.64) by u_t , integrating over $\mathbb{R}^3$ and integration by parts, one gets by the Gagliardo-Nirenberg inequality, Hölder's inequality, Young's inequality and Lemmas 3.2-3.5 that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(a_2\alpha|(h^2+\epsilon^2)^{\frac{1}{4}}\nabla u|_2^2+a_2(\alpha+\beta)|(h^2+\epsilon^2)^{\frac{1}{4}}\mathrm{div}u|_2^2)+|l^{-\frac{\nu}{2}}u_t|_2^2\\
=&-\int l^{-\nu}\big(v\cdot\nabla v+a_1\phi\nabla l+l\nabla\phi-a_2g\nabla l^\nu\cdot Q(v)-a_3l^\nu\psi\cdot Q(v)\big)\cdot u_t\\
&+\frac{1}{2}\int a_2\frac{h}{\sqrt{h^2+\epsilon^2}}h_t(\alpha|\nabla u|^2+(\alpha+\beta)|\mathrm{div}u|^2)\\
&-\int a_2\nabla\sqrt{h^2+\epsilon^2}\cdot Q(u)\cdot u_t\\
\le& C|l^{-\frac{\nu}{2}}|_\infty(|v|_\infty|\nabla v|_2+|\nabla l|_2|\phi|_\infty+|l|_\infty|\nabla\phi|_2+|g\nabla v|_\infty|l^{\nu-1}|_\infty|\nabla l|_2\\
&+|\psi|_\infty|l^\nu|_\infty|\nabla v|_2)|l^{-\frac{\nu}{2}}u_t|_2+C|h_t|_\infty|\varphi|_\infty|\sqrt{h}\nabla u|_2^2\\
&+C|l^{\frac{\nu}{2}}|_\infty|\psi|_\infty|l^{-\frac{\nu}{2}}u_t|_2|\varphi|_\infty^{\frac{1}{2}}|\sqrt{h}\nabla u|_2\\
\le& M(c_0)c_4^4|\sqrt{h}\nabla u|_2^2+M(c_0)c_4^4+\frac{1}{2}|l^{-\frac{\nu}{2}}u_t|_2^2,
\end{aligned}$$

which, along with Gronwall's inequality and (3.5), implies that for $0 \le t \le T_2$,

$$\begin{aligned} &|\sqrt{h}\nabla u|_2^2 + |\nabla u|_2^2 + \int_0^t \big(|l^{-\frac{\nu}{2}} u_t|_2^2 + |u_t|_2^2\big)\mathrm{d}s \\ \le& M(c_0)(1 + c_4^4 t)\exp\big(M(c_0)c_4^4 t\big) \le M(c_0). \end{aligned} \tag{3.67}$$

By the definitions of the Lamé operator L and ψ, it holds that

$$\begin{aligned} &a_2 L(\sqrt{h^2+\epsilon^2}u) = a_2\sqrt{h^2+\epsilon^2}Lu - a_2 G(\nabla\sqrt{h^2+\epsilon^2}, u) \\ =& l^{-\nu}\mathcal{H} - a_2 G(\nabla\sqrt{h^2+\epsilon^2}, u), \end{aligned} \tag{3.68}$$

where

$$\mathcal{H} = -u_t - v\cdot\nabla v - l\nabla\phi - a_1\phi\nabla l + a_2 g\nabla l^\nu \cdot Q(v) + a_3 l^\nu \psi\cdot Q(v). \tag{3.69}$$

Next, for giving the L^2 estimate of $\nabla^2 u$, we consider the L^2 estimates of

$$(\mathcal{H}, \widetilde{G} = G(\nabla\sqrt{h^2+\epsilon^2}, u)).$$

It follows from (3.7), (3.13), (3.66)-(3.67), (3.69) and Lemmas 3.2-3.5 that

$$\begin{aligned} |\mathcal{H}|_2 \le& C(|u_t|_2 + |v|_6|\nabla v|_3 + |l|_\infty|\nabla\phi|_2 + |\phi|_\infty|\nabla l|_2 \\ &+ |g\nabla l^\nu\cdot Q(v)|_2 + |l^\nu|_\infty|\psi|_\infty|\nabla v|_2) \\ \le& M(c_0)(|u_t|_2 + c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}), \\ |\widetilde{G}|_2 \le& |\nabla\sqrt{h^2+\epsilon^2}|_\infty|\nabla u|_2 + |\nabla^2\sqrt{h^2+\epsilon^2}|_3|u|_6) \le M(c_0), \end{aligned} \tag{3.70}$$

where one has used the facts that

$$\begin{aligned} |\nabla v|_3 \le& |\nabla v|_2^{\frac{1}{2}}|\nabla v|_6^{\frac{1}{2}} \le C|\nabla v|_2^{\frac{1}{2}}|\nabla^2 v|_2^{\frac{1}{2}} \le Cc_2^{\frac{3}{2}}c_3^{\frac{1}{2}}, \\ |\nabla l^\nu\cdot gQ(v)|_2 =& |(\nabla l^\nu\cdot gQ(v))(0,x) + \int_0^t (\nabla l^\nu\cdot gQ(v))_s \mathrm{d}s|_2 \\ \le& |\nabla l_0^\nu\cdot h_0 Q(u_0)|_2 + t^{\frac{1}{2}}\Big(\int_0^t |(\nabla l^\nu\cdot gQ(v))_s|_2^2\mathrm{d}s\Big)^{\frac{1}{2}} \\ \le& |l_0^{\nu-1}|_\infty|h_0^{\frac{1}{4}}\nabla l_0|_6|h_0^{\frac{3}{4}}\nabla u_0|_3 \\ &+ Ct^{\frac{1}{2}}\Big(\int_0^t \big(|g_t|_\infty^2|\nabla l^\nu|_\infty^2|\nabla v|_2^2 + |\nabla l^\nu|_3^2|g\nabla v_t|_6^2\big)\mathrm{d}s\Big)^{\frac{1}{2}} \\ &+ Ct^{\frac{1}{2}}\Big(\int_0^t |g\nabla v|_\infty^2\big(|\nabla l_t|_2^2|l^{\nu-1}|_\infty^2 + |(l^{\nu-1})_t|_\infty^2|\nabla l|_2^2\big)\mathrm{d}s\Big)^{\frac{1}{2}} \\ \le& |l_0^{\nu-1}|_\infty|h_0^{\frac{1}{4}}\nabla l_0|_6|\sqrt{h_0}\nabla u_0|_2^{\frac{1}{2}}|h_0\nabla u_0|_6^{\frac{1}{2}} \\ &+ M(c_0)c_4^{8.5+\nu}(t + t^{\frac{1}{2}}) \le M(c_0), \\ |\nabla\sqrt{h^2+\epsilon^2}|_\infty \le& C|\nabla h|_\infty \le C|\psi|_\infty \le Cc_0, \\ |\nabla^2\sqrt{h^2+\epsilon^2}|_3 \le& C(|\nabla\sqrt{h}|_6^2 + |\nabla^2 h|_3) \le C(|\nabla\sqrt{h}|_6^2 + |\nabla\psi|_3) \le Cc_0^2. \end{aligned} \tag{3.71}$$

Then it follows from (3.66)-(3.70), the classical theory for elliptic equations and Lemmas 3.3-3.5 that

$$\begin{aligned}
|\sqrt{h^2+\epsilon^2}u|_{D^2} \leq & C(|l^{-\nu}\mathcal{H}|_2+|G(\nabla\sqrt{h^2+\epsilon^2},u)|_2)\\
\leq & C(|l^{-\nu}|_\infty|\mathcal{H}|_2+|G(\nabla\sqrt{h^2+\epsilon^2},u)|_2)\\
\leq & M(c_0)(|u_t|_2+c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}),\\
|\sqrt{h^2+\epsilon^2}\nabla^2 u|_2 \leq & C(|\sqrt{h^2+\epsilon^2}u|_{D^2}+|\nabla\psi|_3|u|_6+|\psi|_\infty|\nabla u|_2\\
& +M(c_0)|\psi|_\infty^2|u|_2|\varphi|_\infty)\leq C|\sqrt{h^2+\epsilon^2}u|_{D^2}+M(c_0).
\end{aligned} \tag{3.72}$$

Finally, it follows from (3.67), (3.72) and Lemma 3.4 that

$$\int_0^t (|h\nabla^2 u|_2^2+|\nabla^2 u|_2^2)\mathrm{d}s\leq M(c_0)\quad \text{for}\quad 0\leq t\leq T_2.$$

Step 3: Estimate on $|u|_{D^2}$. Applying ∂_t to $(3.1)_2$ yields

$$\begin{aligned}
u_{tt}+a_2 l^\nu\sqrt{h^2+\epsilon^2}Lu_t = & -(v\cdot\nabla v)_t-(l\nabla\phi)_t-a_1(\phi\nabla l)_t\\
& -a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu-a_2\frac{h}{\sqrt{h^2+\epsilon^2}}l^\nu h_t Lu\\
& +(a_2 g\nabla l^\nu\cdot Q(v)+a_3 l^\nu\psi\cdot Q(v))_t.
\end{aligned} \tag{3.73}$$

Multiplying (3.73) by $l^{-\nu}u_t$, integrating over $\mathbb{R}^3$ and integration by part lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|l^{-\frac{\nu}{2}}u_t|_2^2+a_2\alpha|(h^2+\epsilon^2)^{\frac{1}{4}}\nabla u_t|_2^2+a_2(\alpha+\beta)|(h^2+\epsilon^2)^{\frac{1}{4}}\mathrm{div}u_t|_2^2\\
=&\int l^{-\nu}\Big(-(v\cdot\nabla v)_t-(l\nabla\phi)_t-a_1(\phi\nabla l)_t-a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu\\
&+(a_2 g\nabla l^\nu\cdot Q(v)+a_3 l^\nu\psi\cdot Q(v))_t\Big)\cdot u_t-\int a_2\frac{h}{\sqrt{h^2+\epsilon^2}}h_t Lu\cdot u_t\\
&-\int a_2\nabla\sqrt{h^2+\epsilon^2}\cdot Q(u_t)\cdot u_t+\frac{1}{2}\int (l^{-\nu})_t|u_t|^2\\
\leq & C\Big(|l^{-\frac{\nu}{2}}|_\infty(|v|_\infty|\nabla v_t|_2+|v_t|_2|\nabla v|_\infty+|l_t|_\infty|\nabla\phi|_2+|\nabla l_t|_2|\phi|_\infty\\
&+|\phi_t|_\infty|\nabla l|_2+|l|_\infty|\nabla\phi_t|_2)+|l^{\frac{\nu}{2}-1}|_\infty|l_t|_\infty|\sqrt{h^2+\epsilon^2}\nabla^2 u|_2\\
&+|l^{\frac{\nu}{2}-2}|_\infty|g\nabla v|_\infty|l_t|_\infty|\nabla l|_2+|l^{\frac{\nu}{2}-1}|_\infty(|g_t|_\infty|\nabla v|_\infty|\nabla l|_2+|g\nabla v|_\infty|\nabla l_t|_2)\\
&+|l^{\frac{\nu}{2}}|_\infty(|\psi_t|_2|\nabla v|_\infty+|\psi|_\infty|\nabla v_t|_2)+|l^{\frac{\nu}{2}-1}|_\infty|\psi|_\infty|l_t|_\infty|\nabla v|_2\Big)|l^{-\frac{\nu}{2}}u_t|_2\\
&+C|l^{-1}|_\infty|\sqrt{g}\nabla v_t|_2|\nabla l|_3(|\sqrt{h}\nabla u_t|_2+|l^{\frac{\nu}{2}}|_\infty|\varphi|_\infty^{\frac{1}{2}}|\psi|_\infty|l^{-\frac{\nu}{2}}u_t|_2)\\
&+C|l^{-1}|_\infty|l_t|_\infty|l^{-\frac{\nu}{2}}u_t|_2^2\\
&+C|l^{\frac{\nu}{2}}|_\infty(|h_t|_\infty|\nabla^2 u|_2+|\varphi|_\infty^{\frac{1}{2}}|\psi|_\infty|\sqrt{h}\nabla u_t|_2)|l^{-\frac{\nu}{2}}u_t|_2,
\end{aligned} \tag{3.74}$$

where one has used

$$\begin{aligned}&\int |l^{-\nu} g\nabla l^{\nu}\cdot Q(v_t)\cdot u_t| = \nu\int \Big|l^{-1}\frac{\sqrt{g}}{\sqrt{h}}\nabla l\cdot \sqrt{g}Q(v_t)\cdot \sqrt{h}u_t\Big|\\ \le& C|l^{-1}|_\infty|\sqrt{g}\nabla v_t|_2|\nabla l|_3|\sqrt{h}u_t|_6\\ \le& C|l^{-1}|_\infty|\sqrt{g}\nabla v_t|_2|\nabla l|_3(|\sqrt{h}\nabla u_t|_2+|l^{\frac{\nu}{2}}|_\infty|\varphi|_\infty^{\frac12}|\psi|_\infty|l^{-\frac{\nu}{2}}u_t|_2).\end{aligned} \tag{3.75}$$

Integrating (3.74) over (τ,t) ($\tau\in(0,t)$), one can get by using (3.13), Lemmas 3.2-3.5 and Young's inequality that

$$\begin{aligned}&\frac12|l^{-\frac{\nu}{2}}u_t(t)|_2^2+\frac{a_2\alpha}{2}\int_\tau^t|\sqrt{h}\nabla u_t|_2^2\mathrm{d}s\\ \le&\frac12|l^{-\frac{\nu}{2}}u_t(\tau)|_2^2+M(c_0)c_4^{8+2\nu}\int_0^t|l^{-\frac{\nu}{2}}u_t|^2\mathrm{d}s+M(c_0)c_4^9t+M(c_0).\end{aligned} \tag{3.76}$$

Due to $(3.1)_2$, it holds that

$$\begin{aligned}|u_t(\tau)|_2\le& C(|v|_\infty|\nabla v|_2+|\phi|_\infty|\nabla l|_2+|\nabla\phi|_2|l|_\infty+|l|_\infty^\nu|(h+\epsilon)Lu|_2\\ &+|l^{\nu-1}|_\infty|g\nabla v|_\infty|\nabla l|_2+|\psi|_\infty|l^\nu|_\infty|\nabla v|_2)(\tau).\end{aligned} \tag{3.77}$$

It follows from this, (3.3), (3.5), (3.8) and Lemma 3.1 that

$$\begin{aligned}\limsup_{\tau\to0}|u_t(\tau)|_2\le& C(|u_0|_\infty|\nabla u_0|_2+|\phi_0|_\infty|\nabla l_0|_2+|\nabla\phi_0|_2|l_0|_\infty+|\psi_0|_\infty|l_0^\nu|_\infty|\nabla u_0|_2\\ &+|l_0^\nu|_\infty(|g_2|_2+|Lu_0|_2)+|l_0^{\nu-1}|_\infty|\phi_0^{2\iota}\nabla u_0|_\infty|\nabla l_0|_2)\le M(c_0).\end{aligned}$$

Letting $\tau\to0$ in (3.76) and using Gronwall's inequality give that for $0\le t\le T_2$,

$$\begin{aligned}&|u_t(t)|_2^2+\int_0^t\big(|\sqrt{h}\nabla u_t(s)|_2^2+|\nabla u_t(s)|_2^2\big)\mathrm{d}s\\ \le&(M(c_0)c_4^9t+M(c_0))\exp\big(M(c_0)c_4^{8+2\nu}t\big)\le M(c_0),\end{aligned} \tag{3.78}$$

which, along with (3.72), yields that for $0\le t\le T_2$,

$$|\sqrt{h^2+\epsilon^2}u(t)|_{D^2}+|h\nabla^2u(t)|_2+|u(t)|_{D^2}\le M(c_0)c_2^{\frac32}c_3^{\frac12}. \tag{3.79}$$

Next, for giving the L^2 estimate of ∇^3u, we consider the L^2 estimates of

$$(\nabla\mathcal{H},\nabla\widetilde{G}=\nabla G(\nabla\sqrt{h^2+\epsilon^2},u)).$$

It follows from (3.7), (3.13), (3.66)-(3.67), (3.69), (3.71), (3.79) and Lemmas 3.2-3.5 that

$$\begin{aligned}|\mathcal{H}|_{D_*^1}\le& C(|u_t|_{D_*^1}+|v|_\infty|\nabla^2v|_2+|\nabla v|_6|\nabla v|_3+|l|_\infty|\nabla^2\phi|_2\\ &+|\nabla\phi|_2|\nabla l|_\infty+|\phi|_\infty|\nabla^2 l|_2+|\nabla g|_\infty|\nabla l^\nu|_\infty|\nabla v|_2\\ &+|\nabla^2l^\nu|_3|g\nabla v|_6+|\nabla l^\nu|_\infty|g\nabla^2v|_2\\ &+|\nabla l^\nu|_\infty|\psi|_\infty|\nabla v|_2+|l^\nu|_\infty|\nabla\psi|_3|\nabla v|_6\\ &+|l^\nu|_\infty|\psi|_\infty|\nabla^2v|_2)\le M(c_0)(|u_t|_{D_*^1}+c_3^2),\\ |\widetilde{G}|_{D_*^1}\le& C(|\nabla\sqrt{h^2+\epsilon^2}|_\infty|\nabla^2u|_2+|\nabla^2\sqrt{h^2+\epsilon^2}|_3|\nabla u|_6\\ &+|\nabla^3\sqrt{h^2+\epsilon^2}|_2|u|_\infty)\le M(c_0)c_3^2,\end{aligned} \tag{3.80}$$

where one has used the fact that

$$
\begin{aligned}
|\nabla^3\sqrt{h^2+\epsilon^2}|_2 \leq & M(c_0)(|\nabla\sqrt{h}|_6^3+|\nabla^2 h|_3|\nabla\sqrt{h}|_6+|\nabla^3 h|_2)\\
\leq & M(c_0)(|\nabla\sqrt{h}|_6^3+|\nabla\psi|_3|\nabla\sqrt{h}|_6+|\nabla^2\psi|_2)\leq M(c_0).
\end{aligned}
\tag{3.81}
$$

It follows from (3.66)-(3.68), (3.70), (3.79)-(3.80), the classical theory for elliptic equations and Lemmas 3.3-3.5 that

$$
\begin{aligned}
|\sqrt{h^2+\epsilon^2}u(t)|_{D^3} \leq & C|l^{-\nu}\mathcal{H}|_{D^1_*}+C|G(\nabla\sqrt{h^2+\epsilon^2},u)|_{D^1_*}\\
\leq & C(|l^{-\nu}|_\infty|\mathcal{H}|_{D^1_*}+|\nabla l^{-\nu}|_\infty|\mathcal{H}|_2\\
& +|G(\nabla\sqrt{h^2+\epsilon^2},u)|_{D^1_*})\\
\leq & M(c_0)(|u_t|_{D^1_*}+c_3^2),\\
|\sqrt{h^2+\epsilon^2}\nabla^3 u(t)|_2 \leq & C(|\sqrt{h^2+\epsilon^2}u(t)|_{D^3}+\|\psi\|_{L^\infty\cap D^{1,3}\cap D^2}\|u\|_2)\\
& +C(1+\|\psi\|^3_{L^\infty\cap D^{1,3}\cap D^2}\|u\|_1)(1+|\varphi|_\infty^2)\\
\leq & M(c_0)(|\sqrt{h^2+\epsilon^2}u(t)|_{D^3}+c_3^2).
\end{aligned}
\tag{3.82}
$$

Finally, it follows from (3.78), (3.82) and Lemma 3.4 that

$$
\int_0^t(|h\nabla^3 u|_2^2+|h\nabla^2 u|^2_{D^1_*}+|u|^2_{D^3})\mathrm{d}s\leq M(c_0)\quad \text{for}\quad 0\leq t\leq T_2. \tag{3.83}
$$

The proof of Lemma 3.7 is complete. □

Next, we estimate the higher order derivatives of the velocity u.

Lemma 3.8. For $t\in[0,T_2]$, it holds that

$$
\begin{aligned}
(|\sqrt{h}\nabla u_t|_2^2+|u_t|^2_{D^1_*}+|u|^2_{D^3}+|h\nabla^2 u|^2_{D^1_*})(t)\leq M(c_0)c_3^4,&\\
\int_0^t(|u_{tt}|_2^2+|u_t|^2_{D^2})\mathrm{d}s\leq M(c_0),&\\
\int_0^t(|h\nabla^2 u_t|_2^2+|u|^2_{D^4}+|h\nabla^2 u|^2_{D^2}+|(h\nabla^2 u)_t|_2^2)\mathrm{d}s\leq M(c_0).&
\end{aligned}
\tag{3.84}
$$

Proof. Multiplying (3.73) by $l^{-\nu}u_{tt}$ and integrating over $\mathbb{R}^3$ lead to

$$
\frac{1}{2}\frac{d}{dt}(a_2\alpha|(h^2+\epsilon^2)^{\frac{1}{4}}\nabla u_t|_2^2+a_2(\alpha+\beta)|(h^2+\epsilon^2)^{\frac{1}{4}}\mathrm{div}u_t|_2^2)+|l^{-\frac{\nu}{2}}u_{tt}|_2^2=\sum_{i=1}^4 I_i,
\tag{3.85}
$$

where I_i, $i=1,2,3,4$, are given and estimated as follows:

$$
\begin{aligned}
I_1=&\int l^{-\nu}\Big(-(v\cdot\nabla v)_t-(l\nabla\phi)_t-a_1(\phi\nabla l)_t\\
&-a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu-a_2l^\nu\frac{h}{\sqrt{h^2+\epsilon^2}}h_tLu\Big)\cdot u_{tt}\\
\leq& C|l^{-\frac{\nu}{2}}|_\infty\Big(|v|_\infty|\nabla v_t|_2+|v_t|_2|\nabla v|_\infty+|l_t|_\infty|\nabla\phi|_2+|\nabla l_t|_2|\phi|_\infty\\
&+|\phi_t|_\infty|\nabla l|_2+|l|_\infty|\nabla\phi_t|_2\\
&+|l^{\nu-1}|_\infty|l_t|_\infty|\sqrt{h^2+\epsilon^2}\nabla^2u|_2+|l^\nu|_\infty|h_t|_\infty|\nabla^2u|_2\Big)|l^{-\frac{\nu}{2}}u_{tt}|_2,\\
I_2=&\int l^{-\nu}\big(a_2g\nabla l^\nu\cdot Q(v)+a_3l^\nu\psi\cdot Q(v)\big)_t\cdot u_{tt}\\
\leq& C|l^{-\frac{\nu}{2}}|_\infty\big(|(\nabla l^\nu)_t|_2|g\nabla v|_\infty+|g_t|_\infty|\nabla l^\nu|_3|\nabla v|_6\\
&+|h^{\frac14}\nabla l^\nu|_6|h^{-\frac14}g^{\frac14}|_\infty|g^{\frac34}\nabla v_t|_3+|l^\nu|_\infty|\psi|_\infty|\nabla v_t|_2\\
&+|l^\nu|_\infty|\psi_t|_2|\nabla v|_\infty+|(l^\nu)_t|_\infty|\psi|_\infty|\nabla v|_2\big)|l^{-\frac{\nu}{2}}u_{tt}|_2,\\
I_3+I_4=&-\int a_2\nabla\sqrt{h^2+\epsilon^2}Q(u_t)\cdot u_{tt}\\
&+\frac12\int a_2\frac{h}{\sqrt{h^2+\epsilon^2}}h_t(\alpha|\nabla u_t|^2+(\alpha+\beta)|\mathrm{div}u_t|^2)\\
\leq& C(|l^{\frac{\nu}{2}}|_\infty|\varphi|_\infty^{\frac12}|\psi|_\infty|\sqrt{h}\nabla u_t|_2|l^{-\frac{\nu}{2}}u_{tt}|_2+|h_t|_\infty|\sqrt{h}\nabla u_t|_2^2|\varphi|_\infty).
\end{aligned}
\tag{3.86}
$$

Integrating (3.85) over (τ,t) and combining (3.86) yield that for $0\leq t\leq T_2$,

$$
\begin{aligned}
&|\sqrt{h}\nabla u_t(t)|_2^2+\int_\tau^t|l^{-\frac{\nu}{2}}u_{tt}|_2^2\mathrm{d}s\\
\leq& C|(h^2+\epsilon^2)^{\frac14}\nabla u_t(\tau)|_2^2+M(c_0)c_4^4\int_0^t|\sqrt{h}\nabla u_t|_2^2\mathrm{d}s+M(c_0)(c_4^{17+2\nu}t+1),
\end{aligned}
\tag{3.87}
$$

where one has used the fact (3.60) and

$$
|g^{\frac34}\nabla v_t|_3\leq C|\sqrt{g}\nabla v_t|_2^{\frac12}|g\nabla v_t|_6^{\frac12}. \tag{3.88}
$$

Due to $(3.1)_2$, one gets

$$
\begin{aligned}
|\sqrt{h}\nabla u_t(\tau)|_2\leq&(|\sqrt{h}\nabla(v\cdot\nabla v+a_1\phi\nabla l+l\nabla\phi+a_2\sqrt{h^2+\epsilon^2}l^\nu Lu\\
&-a_2\nabla l^\nu\cdot gQ(v)-a_3l^\nu\psi\cdot Q(v))|_2)(\tau).
\end{aligned}
\tag{3.89}
$$

It follows from (3.3), (3.5), Lemma 3.1 and Remark 3.1 that

$$\begin{aligned}
&\limsup_{\tau\to 0}|\sqrt{h}\nabla u_t(\tau)|_2\\
\le& C(|\sqrt{h_0}\nabla(u_0\cdot\nabla u_0)|_2+|\sqrt{h_0}l_0\nabla^2\phi_0|_2+|\sqrt{h_0}\nabla^2 l_0\phi_0|_2\\
&+|\sqrt{h_0}\nabla l_0\cdot\nabla\phi_0|_2+|\sqrt{h_0}\nabla(\sqrt{h_0^2+\epsilon^2}l_0^\nu Lu_0)|_2\\
&+|\sqrt{h_0}\nabla(\nabla l_0^\nu\cdot h_0Q(u_0))|_2+|\sqrt{h_0}\nabla(l_0^\nu\psi_0\cdot Q(u_0))|_2)\\
\le& C(|\phi_0^\iota u_0|_6|\nabla^2u_0|_3+|\nabla u_0|_\infty|\phi_0^\iota\nabla u_0|_2+|l_0|_\infty|\phi_0^\iota\nabla^2\phi_0|_2\\
&+|\nabla^2 l_0\phi_0^{\iota+1}|_2+|\nabla l_0|_3|\phi_0^\iota\nabla\phi_0|_6+|l_0^\nu|_\infty(|\nabla\psi_0|_3|\phi_0^\iota\nabla u_0|_6\\
&+|\psi_0|_\infty|\phi_0^\iota\nabla^2u_0|_2)+|l_0^{\nu-1}|_\infty|\psi_0|_\infty|\phi_0^\iota\nabla u_0|_6|\nabla l_0|_3\\
&+|\sqrt{h_0}\nabla(\sqrt{h_0^2+\epsilon^2}l_0^\nu Lu_0)|_2+|\sqrt{h_0}\nabla(\nabla l_0^\nu\cdot h_0Q(u_0))|_2).
\end{aligned}\tag{3.90}$$

(3.5) and (3.9) imply

$$|\phi_0^\iota\nabla^2 l_0|_2=|g_4|_2\le c_0,\quad |\phi_0^\iota\nabla^2\phi_0|_2\le M(c_0).\tag{3.91}$$

Note that

$$\begin{aligned}
|\sqrt{h_0}\nabla(\sqrt{h_0^2+\epsilon^2}l_0^\nu Lu_0)|_2\le&|\sqrt{h_0}l_0^\nu\nabla(\sqrt{h_0^2+\epsilon^2}Lu_0)|_2\\
&+|\sqrt{h_0}\sqrt{h_0^2+\epsilon^2}Lu_0\otimes\nabla l_0^\nu|_2=|J_1|_2+|J_2|_2,
\end{aligned}\tag{3.92}$$

which will be estimated as follows. For J_1, it holds that

$$\begin{aligned}
J_1=&\sqrt{h_0}l_0^\nu\Big(\sqrt{h_0^2+\epsilon^2}\nabla Lu_0+\frac{h_0}{\sqrt{h_0^2+\epsilon^2}}Lu_0\otimes\nabla h_0\Big)\\
=&l_0^\nu\Big(\frac{h_0}{\sqrt{h_0^2+\epsilon^2}}g_3+\epsilon^2\nabla Lu_0\frac{\sqrt{h_0}}{\sqrt{h_0^2+\epsilon^2}}\Big),
\end{aligned}$$

which yields

$$|J_1|_2\le C|l_0^\nu|_\infty(|g_3|_2+\epsilon^2|\varphi_0|_\infty^{\frac12}|\nabla^3u_0|_2).\tag{3.93}$$

For J_2, one gets from (3.5) that

$$\begin{aligned}
|J_2|_2\le& C(|h_0^{\frac32}l_0^{\nu-1}Lu_0\otimes\nabla l_0|_2+\epsilon|h_0^{\frac12}l_0^{\nu-1}Lu_0\otimes\nabla l_0|_2)\\
\le& C|l_0^{\nu-1}|_\infty|h_0^{\frac32}Lu_0|_6|\nabla l_0|_3+C|g_2|_2|\phi_0^{-\iota}|_\infty|l_0^{\nu-1}|_\infty|\nabla l_0|_\infty\\
\le& C|l_0^{\nu-1}|_\infty|\nabla(h_0^{\frac32}Lu_0)|_2|\nabla l_0|_3+C|g_2|_2|\phi_0^{-\iota}|_\infty|l_0^{\nu-1}|_\infty|\nabla l_0|_\infty\le M(c_0).
\end{aligned}\tag{3.94}$$

On the other hand, it follows from (3.5) and Remark 3.1 that

$$\begin{aligned}
&|\sqrt{h_0}\nabla(\nabla l_0^\nu\cdot h_0Q(u_0))|_2\\
\le& C(|\sqrt{h_0}\nabla^2l_0^\nu|_2|h_0\nabla u_0|_\infty+|\sqrt{h_0}\nabla l_0^\nu|_6(|h_0\nabla^2u_0|_3+|\psi_0|_\infty|\nabla u_0|_3))\\
\le& C(|\sqrt{h_0}\nabla^2l_0^\nu|_2|h_0\nabla u_0|_6^{\frac12}|\nabla(h_0\nabla u_0)|_6^{\frac12}\\
&+(|\sqrt{h_0}\nabla^2l_0^\nu|_2+|\varphi_0^{\frac12}\psi_0|_\infty|\nabla l_0^\nu|_2)(|h_0\nabla^2u_0|_3+|\psi_0|_\infty|\nabla u_0|_3))\le M(c_0).
\end{aligned}$$

Hence, combining all the estimates with (3.90) yields

$$\limsup_{\tau\to 0}|\sqrt{h}\nabla u_t(\tau)|_2 \leq M(c_0), \tag{3.95}$$

which implies also

$$\limsup_{\tau\to 0}|\sqrt{\epsilon}\nabla u_t(\tau)|_2 \leq \limsup_{\tau\to 0}\sqrt{\epsilon}|\varphi|_\infty^{\frac{1}{2}}|\sqrt{h}\nabla u_t(\tau)|_2 \leq M(c_0).$$

Letting $\tau \to 0$, one gets from (3.87) and Gronwall's inequality that for $0 \leq t \leq T_2$,

$$\begin{aligned}
&|\sqrt{h}\nabla u_t(t)|_2^2 + |\nabla u_t(t)|_2^2 + \int_0^t |u_{tt}(s)|_2^2 \mathrm{d}s\\
\leq & M(c_0)(1 + c_4^{17+2\nu} t)\exp(M(c_0)c_4^4 t) \leq M(c_0),
\end{aligned} \tag{3.96}$$

which, along with (3.82), yields

$$|\sqrt{h^2+\epsilon^2}u|_{D^3} + |\sqrt{h^2+\epsilon^2}\nabla^3 u|_2 + |h\nabla^2 u|_{D^1} + |\nabla^3 u|_2 \leq M(c_0)c_3^2. \tag{3.97}$$

Note that (3.73) gives

$$\begin{aligned}
a_2 L(\sqrt{h^2+\epsilon^2}u_t) =& a_2\sqrt{h^2+\epsilon^2}Lu_t - a_2 G(\nabla\sqrt{h^2+\epsilon^2}, u_t)\\
=& l^{-\nu}\mathcal{G} - a_2 G(\nabla\sqrt{h^2+\epsilon^2}, u_t),
\end{aligned} \tag{3.98}$$

with

$$\begin{aligned}
\mathcal{G} =& - u_{tt} - (v\cdot\nabla v)_t - (l\nabla\phi)_t - a_1(\phi\nabla l)_t - a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu\\
&- a_2\frac{h}{\sqrt{h^2+\epsilon^2}}h_t l^\nu Lu + (a_2 g\nabla l^\nu \cdot Q(v) + a_3 l^\nu\psi\cdot Q(v))_t.
\end{aligned} \tag{3.99}$$

Next, for giving the L^2 estimates of $(\nabla^2 u_t, \nabla^4 u)$, we consider the L^2 estimates of

$$(\mathcal{G}, \widehat{G} = G(\nabla\sqrt{h^2+\epsilon^2}, u_t), \nabla^2\mathcal{H}).$$

In fact, it follows from (3.7), (3.13), (3.69), (3.71), (3.96)-(3.97), (3.99) and Lemmas 3.2-3.7 that

$$\begin{aligned}
|\mathcal{G}|_2 \leq& C(|u_{tt}|_2 + \|v\|_2|\nabla v_t|_2 + \|l\|_{L^\infty\cap D^1\cap D^2}\|\phi_t\|_1 + \|\phi\|_2\|l_t\|_{L^3\cap D^1}\\
&+ |(l^\nu)_t|_\infty|\sqrt{h^2+\epsilon^2}Lu|_2 + |l^\nu|_\infty|h_t|_\infty|\nabla^2 u|_2 + |g_t|_\infty|\nabla l^\nu|_2|\nabla v|_\infty\\
&+ |g\nabla v|_\infty|\nabla(l^\nu)_t|_2 + |l^{\nu-1}|_\infty|g^{\frac{1}{4}}\nabla l|_6|g^{\frac{3}{4}}\nabla v_t|_3\\
&+ |(l^\nu)_t|_3|\psi|_\infty|\nabla v|_6 + |l^\nu|_\infty|\psi_t|_2|\nabla v|_\infty + |l^\nu|_\infty|\psi|_\infty|\nabla v_t|_2\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{8.5+\nu} + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}}),\\
|\mathcal{H}|_{D^2} \leq& C(|u_t|_{D^2} + \|v\|_2\|\nabla v\|_2 + \|l\|_{L^\infty\cap D^1\cap D^3}\|\nabla\phi\|_2\\
&+ \|\nabla l^\nu\|_2(\|g\nabla v\|_{L^\infty\cap D^1\cap D^2} + \|\nabla g\|_{L^\infty\cap D^2}\|\nabla v\|_2)\\
&+ \|l^\nu\|_{L^\infty\cap D^1\cap D^2}\|\psi\|_{L^q\cap D^{1,3}\cap D^2}\|\nabla v\|_2)\\
\leq& M(c_0)(|u_t|_{D^2} + c_4^2),\\
|\widehat{G}|_2 \leq& C(|\nabla\sqrt{h^2+\epsilon^2}|_\infty|\nabla u_t|_2 + |\nabla^2\sqrt{h^2+\epsilon^2}|_3|u_t|_6) \leq M(c_0),
\end{aligned} \tag{3.100}$$

where one has used (3.60) and

$$|g^{\frac{3}{4}}\nabla v_t|_3 \leq C|\sqrt{g}\nabla v_t|_2^{\frac{1}{2}}|g\nabla v_t|_6^{\frac{1}{2}} \leq Cc_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}}.$$

It follows from (3.68), (3.70), (3.80), (3.96)-(3.98), (3.100), the classical theory for elliptic equations and Lemmas 3.2-3.7 that

$$\begin{aligned}
|\sqrt{h^2+\epsilon^2}u_t|_{D^2} \leq& C|l^{-\nu}\mathcal{G}|_2 + C|G(\nabla\sqrt{h^2+\epsilon^2}, u_t)|_2\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}} + c_4^{8.5+\nu}),\\
|\sqrt{h^2+\epsilon^2}\nabla^2 u_t|_2 \leq& C(|\sqrt{h^2+\epsilon^2}u_t|_{D^2} + |\nabla u_t|_2(|\psi|_\infty + |\nabla\psi|_3)\\
&+|\psi|_\infty^2|u_t|_2|\varphi|_\infty)\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}} + c_4^{8.5+\nu}),\\
|(h\nabla^2 u)_t|_2 \leq& C(|h\nabla^2 u_t|_2 + |h_t|_\infty|\nabla^2 u|_2)\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}} + c_4^{8.5+\nu}),\\
|u|_{D^4} \leq& C|(h^2+\epsilon^2)^{-\frac{1}{2}}l^{-\nu}\mathcal{H}|_{D^2}\\
\leq& M(c_0)(|u_t|_{D^2} + c_4^2)\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}} + c_4^{8.5+\nu}).
\end{aligned} \tag{3.101}$$

$(3.1)_2$ yield that for multi-index $\xi \in \mathbb{Z}^3_+$ with $|\xi| = 2$,

$$\begin{aligned}
&a_2L(\sqrt{h^2+\epsilon^2}\nabla^\xi u) = a_2\sqrt{h^2+\epsilon^2}\nabla^\xi Lu - a_2G(\nabla\sqrt{h^2+\epsilon^2}, \nabla^\xi u)\\
=&\sqrt{h^2+\epsilon^2}\nabla^\xi\big[\big(\sqrt{h^2+\epsilon^2}\big)^{-1}l^{-\nu}\mathcal{H}\big] - a_2G(\nabla\sqrt{h^2+\epsilon^2}, \nabla^\xi u),
\end{aligned} \tag{3.102}$$

which, along with (3.70), (3.80), (3.96)-(3.97), (3.100)-(3.101), the classical theory for elliptic equations and Lemmas 3.2-3.7, implies that

$$\begin{aligned}
|\sqrt{h^2+\epsilon^2}\nabla^2 u(t)|_{D^2} \leq& C|\sqrt{h^2+\epsilon^2}\nabla^\xi\big[\big(\sqrt{h^2+\epsilon^2}\big)^{-1}l^{-\nu}\mathcal{H}\big]|_2\\
&+C(|\psi|_\infty|u|_{D^3} + |\nabla\psi|_3|\nabla^2 u|_6 + |\nabla^2 u|_2|\psi|_\infty^2|\varphi|_\infty)\\
\leq& M(c_0)(|u_t|_{D^2} + c_4^2)\\
\leq& M(c_0)(|u_{tt}|_2 + c_4^{\frac{1}{2}}|g\nabla v_t|_{D^1_*}^{\frac{1}{2}} + c_4^{8.5+\nu}).
\end{aligned} \tag{3.103}$$

At last, it follows from (3.13), (3.96), (3.101), (3.103) and Lemma 3.4 that

$$\int_0^{T_2}(|h\nabla^2 u_t|_2^2 + |u_t|_{D^2}^2 + |u|_{D^4}^2 + |h\nabla^2 u|_{D^2}^2 + |(h\nabla^2 u)_t|_2^2)\mathrm{d}t \leq M(c_0). \tag{3.104}$$

The proof of Lemma 3.8 is complete. □

Finally, we derive the time weighted estimates for the velocity u.

Lemma 3.9. *Set* $T_3=\min\{T_2,(1+Cc_5)^{-20-2\nu}\}$*. Then for* $t\in[0,T_3]$,

$$\begin{aligned} t|u_t(t)|^2_{D^2}+t|h\nabla^2u_t(t)|_2^2+t|u_{tt}(t)|_2^2+t|u(t)|^2_{D^4}(t)\leq M(c_0)c_4^{19+2\nu},\\ \int_0^t s(|u_{tt}|^2_{D^1_*}+|u_t|^2_{D^3}+|\sqrt{h}u_{tt}|^2_{D^1_*})ds\leq M(c_0)c_4^{19+2\nu}. \end{aligned}\tag{3.105}$$

Proof. Differentiating (3.73) with respect to t yields

$$\begin{aligned} &u_{ttt}+a_2\sqrt{h^2+\epsilon^2}l^\nu Lu_{tt}\\ =&-2v_t\cdot\nabla v_t-v_{tt}\cdot\nabla v-v\cdot\nabla v_{tt}-2a_1\phi_t\nabla l_t-a_1\phi\nabla l_{tt}-a_1\phi_{tt}\nabla l\\ &-2l_t\nabla\phi_t-l_{tt}\nabla\phi-l\nabla\phi_{tt}+a_3l^\nu\psi_{tt}\cdot Q(v)+a_3(l^\nu)_{tt}\psi\cdot Q(v)\\ &+a_3l^\nu\psi\cdot Q(v_{tt})+2a_3(l^\nu)_t\psi_t\cdot Q(v)+2a_3l^\nu\psi_t\cdot Q(v_t)+2a_3(l^\nu)_t\psi\cdot Q(v_t)\\ &+a_2g(\nabla l^\nu)_{tt}\cdot Q(v)+a_2\nabla l^\nu\cdot(gQ(v))_{tt}\\ &+2a_2(\nabla l^\nu)_t\cdot(gQ(v))_t-a_2(l^\nu)_{tt}\sqrt{h^2+\epsilon^2}Lu\\ &-2a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu_t-a_2l^\nu\frac{h}{\sqrt{h^2+\epsilon^2}}h_{tt}Lu\\ &-a_2l^\nu\frac{\epsilon^2h_t^2}{(h^2+\epsilon^2)^{\frac32}}Lu-2a_2\frac{h}{\sqrt{h^2+\epsilon^2}}(h_tl^\nu Lu_t+(l^\nu)_th_tLu). \end{aligned}\tag{3.106}$$

Multiplying (3.106) by $l^{-\nu}u_{tt}$ and integrating over $\mathbb{R}^3$ give

$$\frac12\frac{d}{dt}|l^{-\frac{\nu}{2}}u_{tt}|_2^2+a_2\alpha|(h^2+\epsilon^2)^{\frac14}\nabla u_{tt}|_2^2+a_2(\alpha+\beta)|(h^2+\epsilon^2)^{\frac14}\mathrm{div}u_{tt}|_2^2=\sum_{i=1}^4H_i,\tag{3.107}$$

where H_i, $i=1,2,3,4$, are given and estimated as follows.

$$\begin{aligned} H_1=&\int l^{-\nu}\big(-2v_t\cdot\nabla v_t-v_{tt}\cdot\nabla v-v\cdot\nabla v_{tt}-2a_1\phi_t\nabla l_t-a_1\phi\nabla l_{tt}\\ &-a_1\phi_{tt}\nabla l-2l_t\nabla\phi_t-l_{tt}\nabla\phi-l\nabla\phi_{tt}\big)\cdot u_{tt}\\ \leq&C|l^{-\frac{\nu}{2}}|_\infty\big(|\nabla v_t|_6|v_t|_3+|\nabla v|_\infty|v_{tt}|_2+|v|_\infty|\nabla v_{tt}|_2\\ &+|\phi_{tt}|_2|\nabla l|_\infty+|\phi_t|_6|\nabla l_t|_3+|\phi|_\infty|\nabla l_{tt}|_2\\ &+|l_t|_\infty|\nabla\phi_t|_2+|l_{tt}|_6|\nabla\phi|_3+|l|_\infty|\nabla\phi_{tt}|_2)|l^{-\frac{\nu}{2}}u_{tt}|_2,\\ H_2=&\int l^{-\nu}(a_3l^\nu\psi_{tt}\cdot Q(v)+a_3(l^\nu)_{tt}\psi\cdot Q(v)+a_3l^\nu\psi\cdot Q(v_{tt})\\ &+2a_3(l^\nu)_t\psi_t\cdot Q(v)+2a_3l^\nu\psi_t\cdot Q(v_t)+2a_3(l^\nu)_t\psi\cdot Q(v_t))\cdot u_{tt}\\ \leq&C|l^{-\frac{\nu}{2}}|_\infty(|l^\nu|_\infty|\psi_{tt}|_2|\nabla v|_\infty+|(l^\nu)_{tt}|_6|\psi|_\infty|\nabla v|_3\\ &+|l^\nu|_\infty|\psi|_\infty|\nabla v_{tt}|_2+|\psi_t|_3|(l^\nu)_t|_\infty|\nabla v|_6\\ &+|\psi|_\infty|(l^\nu)_t|_\infty|\nabla v_t|_2+|l^\nu|_\infty|\psi_t|_3|\nabla v_t|_6)|l^{-\frac{\nu}{2}}u_{tt}|_2, \end{aligned}\tag{3.108}$$

$$
\begin{aligned}
H_3 =&\int l^{-\nu}(a_2 g(\nabla l^\nu)_{tt}\cdot Q(v)+a_2\nabla l^\nu\cdot (gQ(v))_{tt}\\
&+2a_2(\nabla l^\nu)_t\cdot (gQ(v))_t)\cdot u_{tt}\\
\leq& C|l^{-\frac{\nu}{2}}|_\infty(|(\nabla l^\nu)_{tt}|_2|g\nabla v|_\infty+|(\nabla l^\nu)_t|_3|(g\nabla v)_t|_6)|l^{-\frac{\nu}{2}}u_{tt}|_2\\
&+M(c_0)\int \big(|\nabla l^\nu\cdot g\nabla v_{tt}\cdot u_{tt}|+|\nabla l^\nu\cdot g_{tt}\nabla v\cdot u_{tt}|\\
&+|\nabla l^\nu\cdot g_t\nabla v_t\cdot u_{tt}|\big)\\
\leq& C|l^{-\frac{\nu}{2}}|_\infty(|(\nabla l^\nu)_{tt}|_2|g\nabla v|_\infty+|(\nabla l^\nu)_t|_3|(g\nabla v)_t|_6)|l^{-\frac{\nu}{2}}u_{tt}|_2\\
&+M(c_0)(|\nabla l|_3|g_{tt}|_6|\nabla v|_\infty|l^{-\frac{\nu}{2}}u_{tt}|_2+|\nabla l|_3|g_t|_\infty|\nabla v_t|_6|l^{-\frac{\nu}{2}}u_{tt}|_2)\\
&+M(c_0)\int |\nabla l\cdot g\nabla v_{tt}\cdot u_{tt}|.
\end{aligned}
\tag{3.109}
$$

In order to estimate $\int |\nabla l\cdot g\nabla v_{tt}\cdot u_{tt}|$, one uses Lemma 3.5-3.6 to get

$$
\begin{aligned}
&\int |\nabla l\cdot g\nabla v_{tt}\cdot u_{tt}|=\int |h^{\frac14}\nabla l\sqrt{g}\cdot\nabla v_{tt}\cdot\frac{\sqrt{g}}{\sqrt{h}}h^{\frac14}u_{tt}|\\
\leq&|h^{\frac14}\nabla l|_6|\frac{\sqrt{g}}{\sqrt{h}}|_\infty|\sqrt{g}\nabla v_{tt}|_2|h^{\frac14}u_{tt}|_3\leq M(c_0)|\sqrt{g}\nabla v_{tt}|_2|h^{\frac14}u_{tt}|_3.
\end{aligned}
\tag{3.110}
$$

Note also that

$$
|h^{\frac14}u_{tt}|_3\leq |u_{tt}|_2^{\frac12}|\sqrt{h}u_{tt}|_6^{\frac12}. \tag{3.111}
$$

It then follows from the Sobolev and Hölder inequalities, and Lemmas 3.3-3.4 that

$$
\begin{aligned}
&\int |\nabla l\cdot g\nabla v_{tt}\cdot u_{tt}|\leq M(c_0)|\sqrt{g}\nabla v_{tt}|_2|\sqrt{h}u_{tt}|_6^{\frac12}|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac12}\\
\leq& M(c_0)|\sqrt{g}\nabla v_{tt}|_2(|\sqrt{h}\nabla u_{tt}|_2^{\frac12}+|\varphi|_\infty^{\frac14}|\psi|_\infty^{\frac12}|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac12})|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac12}\\
\leq& M(c_0)(|\sqrt{g}\nabla v_{tt}|_2|\sqrt{h}\nabla u_{tt}|_2^{\frac12}|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac12}+|\sqrt{g}\nabla v_{tt}|_2|\varphi|_\infty^{\frac14}|\psi|_\infty^{\frac12}|l^{-\frac{\nu}{2}}u_{tt}|_2)\\
\leq&\frac{a_2\alpha}{4}|\sqrt{h}\nabla u_{tt}|_2^2+M(c_0)|\sqrt{g}\nabla v_{tt}|_2^{\frac43}|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac23}+M(c_0)|\sqrt{g}\nabla v_{tt}|_2|l^{-\frac{\nu}{2}}u_{tt}|_2.
\end{aligned}
\tag{3.112}
$$

Hence,

$$
\begin{aligned}
H_3\leq& M(c_0)\Big(\big(|\nabla l_{tt}|_2+|l_t|_3|\nabla l_t|_6+|l_{tt}|_6|\nabla l|_3+|l_t|_\infty^2|\nabla l|_2)|g\nabla v|_\infty\\
&+(|\nabla l_t|_3+|l_t|_\infty|\nabla l|_3)|(g\nabla v)_t|_6\Big)|l^{-\frac{\nu}{2}}u_{tt}|_2+\frac{a_2\alpha}{4}|\sqrt{h}\nabla u_{tt}|_2^2\\
&+M(c_0)(|\nabla l|_3|g_{tt}|_6|\nabla v|_\infty|l^{-\frac{\nu}{2}}u_{tt}|_2+|\nabla l|_3|g_t|_\infty|\nabla v_t|_6|l^{-\frac{\nu}{2}}u_{tt}|_2)\\
&+M(c_0)(|\sqrt{g}\nabla v_{tt}|_2^{\frac43}|l^{-\frac{\nu}{2}}u_{tt}|_2^{\frac23}+|\sqrt{g}\nabla v_{tt}|_2|l^{-\frac{\nu}{2}}u_{tt}|_2),
\end{aligned}
\tag{3.113}
$$

where one has used

$$
\begin{aligned}
|(\nabla l^\nu)_{tt}|_2\leq& M(c_0)(|\nabla l_{tt}|_2+|l_t|_3|\nabla l_t|_6+|l_{tt}|_6|\nabla l|_3+|l_t|_\infty^2|\nabla l|_2),\\
|(\nabla l^\nu)_t|_3\leq& M(c_0)(|\nabla l_t|_3+|l_t|_\infty|\nabla l|_3).
\end{aligned}
$$

Similarly, one can also obtain that

$$\begin{aligned}
H_4=&-\int l^{-\nu}\big(a_2\frac{hl^\nu}{\sqrt{h^2+\epsilon^2}}\nabla h\cdot Q(u_{tt})+a_2(l^\nu)_{tt}\sqrt{h^2+\epsilon^2}Lu\\
&+2a_2(l^\nu)_t\sqrt{h^2+\epsilon^2}Lu_t+a_2l^\nu\frac{h}{\sqrt{h^2+\epsilon^2}}h_{tt}Lu+a_2l^\nu\frac{\epsilon^2h_t^2}{(h^2+\epsilon^2)^{\frac{3}{2}}}Lu\\
&+2a_2\frac{h}{\sqrt{h^2+\epsilon^2}}(h_tl^\nu Lu_t+(l^\nu)_th_tLu)\big)\cdot u_{tt}+\frac{1}{2}\int(l^{-\nu})_t|u_{tt}|^2\\
\le& C|l^{-\frac{\nu}{2}}|_\infty\big(|l^\nu|_\infty|\psi|_\infty|\sqrt{h}\nabla u_{tt}|_2|\varphi|_\infty^{\frac{1}{2}}+|(l^\nu)_{tt}|_6|\sqrt{h^2+\epsilon^2}\nabla^2u|_3\\
&+|(l^\nu)_t|_\infty|\sqrt{h^2+\epsilon^2}\nabla^2u_t|_2+|l^\nu|_\infty|h_{tt}|_6|\nabla^2u|_3\\
&+|l^\nu|_\infty|h_t|^2_\infty|\varphi|^3_\infty|\nabla^2u|_2+|l^\nu|_\infty|h_t|_\infty|\varphi|_\infty|h\nabla^2u_t|_2\\
&+|(l^\nu)_t|_\infty|h_t|_\infty|\nabla^2u|_2\big)|l^{-\frac{\nu}{2}}u_{tt}|_2+M(c_0)|l_t|_\infty|l^{-\frac{\nu}{2}}u_{tt}|_2^2.
\end{aligned}\tag{3.114}$$

Multiplying (3.107) by t and integrating over (τ,t), one can obtain from above estimates on H_i, (3.13) and Lemmas 3.2-3.8 that

$$\begin{aligned}
&t|l^{-\frac{\nu}{2}}u_{tt}(t)|_2^2+\frac{a_2\alpha}{4}\int_\tau^t s|\sqrt{h}\nabla u_{tt}|_2^2\mathrm{d}s\\
\le&\tau|l^{-\frac{\nu}{2}}u_{tt}(\tau)|_2^2+M(c_0)c_4^{19+2\nu}(1+t)+M(c_0)c_5^{12+2\nu}\int_\tau^t s|l^{-\frac{\nu}{2}}u_{tt}|_2^2\mathrm{d}s.
\end{aligned}\tag{3.115}$$

It follows from (3.96) and Lemma 5.5 that there exists a sequence s_k such that

$$s_k\longrightarrow 0,\quad \text{and}\quad s_k|u_{tt}(s_k,x)|_2^2\longrightarrow 0,\quad \text{as}\quad k\longrightarrow\infty.$$

Taking $\tau=s_k$ and letting $k\to\infty$ in (3.115), one can get by Gronwall's inequality that

$$t|u_{tt}(t)|_2^2+\int_0^t s|\sqrt{h}\nabla u_{tt}|_2^2\mathrm{d}s+\int_0^t s|\nabla u_{tt}|_2^2\mathrm{d}s\le M(c_0)c_4^{19+2\nu},\tag{3.116}$$

for $0\le t\le T_3=\min\{T_2,(1+Cc_5)^{-20-2\nu}\}$.

It follows from (3.101) and (3.116) that

$$t^{\frac{1}{2}}|\nabla^2u_t(t)|_2+t^{\frac{1}{2}}|h\nabla^2u_t(t)|_2+t^{\frac{1}{2}}|\nabla^4u(t)|_2\le M(c_0)c_4^{9.5+\nu}.\tag{3.117}$$

Next, for giving the L^2 estimate of ∇^3u_t, we consider the L^2 estimates of

$$(\nabla\mathcal{G},\nabla\widehat{G}=\nabla G(\nabla\sqrt{h^2+\epsilon^2},u_t)).$$

It follows from (3.7), (3.71), (3.81), (3.99) and Lemmas 3.2-3.8 that

$$\begin{aligned}|\mathcal{G}|_{D^1_*} \le & C(|u_{tt}|_{D^1_*}+\|\nabla v\|_2|\nabla v_t|_2+|v|_\infty|\nabla^2 v_t|_2\\&+\|l\|_{L^\infty\cap D^1\cap D^3}\|\phi_t\|_2+\|l_t\|_{L^3\cap D^1_*\cap D^2}\|\phi\|_3\\&+\|l^{\nu-1}\|_{1,\infty}\|l_t\|_{L^\infty\cap D^2}(\|\sqrt{h^2+\epsilon^2}Lu\|_1+|\psi|_\infty|\nabla^2 u|_2)\\&+(1+|\psi|_\infty)(1+|\varphi|_\infty)\|h_t\|_{L^\infty\cap D^2}\|l^\nu\|_{1,\infty}\|\nabla^2 u\|_1\\&+\|g_t\|_{L^\infty\cap D^1}\|\nabla l^\nu\|_2\|\nabla v\|_2+\|\nabla l^\nu\|_2(|\nabla g|_\infty|\nabla v_t|_2+|g\nabla^2 v_t|_2)\\&+(|g\nabla v|_\infty+|\nabla g|_\infty\|\nabla v\|_2+\|g\nabla^2 v\|_1)\|l_t\|_{D^1_*\cap D^2}\|l^{\nu-1}\|_{L^\infty\cap D^1\cap D^3}\\&+\|l^{\nu-1}\|_{1,\infty}\|l_t\|_{L^\infty\cap D^1_*}\|\psi\|_{L^\infty\cap D^{1,3}}\|\nabla v\|_2\\&+\|l^\nu\|_{1,\infty}\|\psi_t\|_1\|\nabla v\|_2+\|l^\nu\|_{1,\infty}\|\psi\|_{L^\infty\cap D^{1,3}}\|\nabla v_t\|_1)\\ \le & M(c_0)(|\nabla u_{tt}|_2+c_4|v_t|_{D^2}+|g\nabla^2 v_t|_2+c_4^{11.5+\nu}),\\ |\widehat{G}|_{D^1_*}\le & C(|\nabla\sqrt{h^2+\epsilon^2}|_\infty|\nabla^2 u_t|_2+|\nabla^2\sqrt{h^2+\epsilon^2}|_3|\nabla u_t|_6\\&+|\nabla^3\sqrt{h^2+\epsilon^2}|_2|u_t|_\infty)\le M(c_0)(|u_t|_{D^2}+c_4^2).\end{aligned}\tag{3.118}$$

Hence (3.98), (3.118), the classical theory for elliptic equations and Lemmas 3.2-3.8 yield that for $0\le t\le T_3$,

$$\begin{aligned}|\sqrt{h^2+\epsilon^2}u_t|_{D^3}\le & C|l^{-\nu}\mathcal{G}|_{D^1_*}+C|G(\nabla\sqrt{h^2+\epsilon^2},u_t)|_{D^1_*}\\ \le & M(c_0)(|\nabla u_{tt}|_2+c_4(|u_t|_{D^2}+|v_t|_{D^2})+|g\nabla^2 v_t|_2+c_4^{11.5+\nu}),\\ |\sqrt{h^2+\epsilon^2}\nabla^3 u_t(t)|_2\le & C(|\sqrt{h^2+\epsilon^2}u_t|_{D^3}+|u_t|_\infty|\nabla^2\psi|_2+|\nabla u_t|_6|\nabla\psi|_3\\&+|\nabla^2 u_t|_2|\psi|_\infty+|\nabla u_t|_2\|\psi\|^2_{L^\infty\cap D^{1,3}\cap D^2}|\varphi|_\infty+|u_t|_2|\psi|^3_\infty|\varphi|^2_2)\\ \le & C|\sqrt{h^2+\epsilon^2}u_t|_{D^3}+M(c_0)(|u_t|_{D^2}+c_4^2),\end{aligned}$$

which, along with (3.84), (3.116)-(3.117) and Lemma 3.4, implies that

$$\int_0^{T_3} s(|\sqrt{h^2+\epsilon^2}u_t|^2_{D^3}+|h\nabla^3 u_t|_2^2+|\nabla^3 u_t|_2^2)\mathrm{d}s\le M(c_0)c_4^{19+2\nu}.$$

The proof of Lemma 3.9 is complete. □

It then follows from Lemmas 3.2-3.9 that for $0\le t\le T_3=\min\{T^*,(1+Cc_5)^{-20-2\nu}\}$,

$$\begin{gathered}\|(\phi-\eta)(t)\|^2_{D^1_*\cap D^3}+\|\phi_t(t)\|_2^2+|\phi_{tt}(t)|_2^2+\int_0^t\|\phi_{tt}\|_1^2\mathrm{d}s\le Cc_4^6,\\ \|\psi(t)\|^2_{L^q\cap D^{1,3}\cap D^2}\le M(c_0),\ \ |\psi_t(t)|_2\le Cc_3^2,\quad |h_t(t)|^2_\infty\le Cc_3^2c_4,\\ h(t,x)>\frac{1}{2c_0},\quad |\psi_t(t)|^2_{D^1_*}+\int_0^t(|\psi_{tt}|_2^2+|h_{tt}|_6^2)\mathrm{d}s\le Cc_4^4,\\ \frac{2}{3}\eta^{-2\iota}<\varphi,\ \ |h^{\frac14}\nabla l(t)|_6\le M(c_0),\quad c_0^{-1}\le|l(t)|_\infty\le M(c_0),\\ \|(l-\bar{l})(t)\|^2_{D^1_*\cap D^3}\le M(c_0),\quad |l_t(t)|^2_\infty\le M(c_0)c_4^{8+2\nu},\quad |l_t(t)|_3^2\le M(c_0)c_3^{8+2\nu},\\ |l_t(t)|^2_{D^1_*}\le M(c_0)c_3^{12+2\nu}c_4,\ \ |\nabla^2 l_t(t)|_2^2+\int_0^t|\nabla l_{tt}|_2^2\mathrm{d}s\le M(c_0)c_4^{19+2\nu},\end{gathered}$$

$$
\begin{aligned}
|\sqrt{h}\nabla u(t)|_2^2+\|u(t)\|_1^2+\int_0^t \left(\|\nabla u\|_1^2+|u_t|_2^2\right)\mathrm{d}s \leq& M(c_0),\\
(|u|_{D^2}^2+|h\nabla^2 u|_2^2+|u_t|_2^2)(t)+\int_0^t (|u|_{D^3}^2+|h\nabla^2 u|_{D_*^1}^2+|u_t|_{D_*^1}^2)\mathrm{d}s \leq& M(c_0)c_2^3c_3,\\
(|u_t|_{D_*^1}^2+|\sqrt{h}\nabla u_t|_2^2+|u|_{D^3}^2+|h\nabla^2 u|_{D_*^1}^2)(t)+\int_0^t |u_t|_{D^2}^2\mathrm{d}s \leq& M(c_0)c_3^4,\\
\int_0^t (|u_{tt}|_2^2+|u|_{D^4}^2+|h\nabla u|_{D_*^1}^2+|h\nabla^2 u|_{D^2}^2+|(h\nabla^2 u)_t|_2^2)\mathrm{d}s \leq& M(c_0),\\
t|u_t(t)|_{D^2}^2+t|h\nabla^2 u_t(t)|_2^2+t|u_{tt}(t)|_2^2+t|u(t)|_{D^4}^2(t) \leq& M(c_0)c_4^{19+2\nu},\\
\int_0^t s(|u_{tt}|_{D_*^1}^2+|u_t|_{D^3}^2+|\sqrt{h}u_{tt}|_{D_*^1}^2)\mathrm{d}s \leq& M(c_0)c_4^{19+2\nu}.
\end{aligned}
$$

Therefore, defining the time

$$
T^*=\min\{T,(1+CM(c_0)^{466+125\nu+8\nu^2})^{-20-2\nu}\}, \tag{3.119}
$$

and constants

$$
\begin{aligned}
c_1^2 =& c_2^2=M(c_0)^2, \quad c_3^2=M(c_0)^8,\\
c_4^2 =& M(c_0)^{98+16\nu}, \quad c_5^2=M(c_0)^{932+250\nu+16\nu^2},
\end{aligned} \tag{3.120}
$$

one can arrive at the following desirable estimates for $0\leq t\leq T^*$:

$$
\begin{aligned}
\|(\phi-\eta)(t)\|_{D_*^1\cap D^3}^2+\|\phi_t(t)\|_2^2+|\phi_{tt}(t)|_2^2+\int_0^t \|\phi_{tt}\|_1^2\mathrm{d}s \leq& c_5^2,\\
\|\psi(t)\|_{L^q\cap D^{1,3}\cap D^2}^2+|h^{\frac{1}{4}}\nabla l(t)|_6^2\leq c_1^2, \quad |h_t(t)|_\infty^2+|\psi_t(t)|_2 \leq& c_4^2,\\
|l_t(t)|_\infty^2+|\nabla^2 l_t(t)|_2^2+|\psi_t(t)|_{D_*^1}^2+\int_0^t (|\nabla l_{tt}|_2^2+|\psi_{tt}|_2^2+|h_{tt}|_6^2)\mathrm{d}s \leq& c_5^2,\\
\inf_{[0,T_*]\times\mathbb{R}^3} l(t,x)\geq c_0^{-1}, \quad h(t,x)>\frac{1}{2c_0}, \quad \frac{2}{3}\eta^{-2\iota} <& \varphi,\\
\|(l-\bar{l})(t)\|_{D_*^1\cap D^3}^2\leq c_1^2, \quad |l_t(t)|_3^2+|l_t(t)|_{D_*^1}^2 \leq& c_4^2,\\
|\sqrt{h}\nabla u(t)|_2^2+\|u(t)\|_1^2+\int_0^t \left(\|\nabla u\|_1^2+|u_t|_2^2\right)\mathrm{d}s \leq& c_2^2,\\
(|u|_{D^2}^2+|h\nabla^2 u|_2^2+|u_t|_2^2)(t)+\int_0^t (|u|_{D^3}^2+|h\nabla^2 u|_{D_*^1}^2+|u_t|_{D_*^1}^2)\mathrm{d}s \leq& c_3^2,\\
(|u_t|_{D_*^1}^2+|\sqrt{h}\nabla u_t|_2^2+|u|_{D^3}^2+|h\nabla u|_{D_*^1}^2+|h\nabla^2 u|_{D_*^1}^2)(t)+\int_0^t |u_t|_{D^2}^2\mathrm{d}s \leq& c_4^2,\\
\int_0^t (|u_{tt}|_2^2+|u|_{D^4}^2+|h\nabla^2 u|_{D^2}^2+|(h\nabla^2 u)_t|_2^2)\mathrm{d}s\leq c_4^2, \quad t|h\nabla^2 u_t(t)|_2^2 \leq& c_5^2,\\
t(|u_t|_{D^2}^2+|u_{tt}|_2^2+|u|_{D^4}^2)(t)+\int_0^t s(|u_{tt}|_{D_*^1}^2+|u_t|_{D^3}^2+|\sqrt{h}u_{tt}|_{D_*^1}^2)\mathrm{d}s \leq& c_5^2.
\end{aligned} \tag{3.121}
$$

3.3. **Passing to the limit $\epsilon\to 0$.** With the help of the (ϵ,η)-independent estimates established in (3.121), one can now obtain the local well-posedness of the linearized problem (3.1) with $\epsilon=0$ and $\phi_0^\eta\geq\eta$.

Lemma 3.10. *Let* (1.17) *hold and* $\eta > 0$. *Assume* (ϕ_0, u_0, l_0, h_0) *satisfy* (2.6)-(2.7), *and there exists a positive constant* c_0 *independent of* η *such that* (3.5) *holds. Then there exists a time* $T^* > 0$ *independent of* η, *and a unique strong solution* $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ *in* $[0, T^*] \times \mathbb{R}^3$ *to* (3.1) *with* $\epsilon = 0$ *satisfying* (3.4) *with* T *replaced by* T^*. *Moreover, the uniform estimates (independent of* η*)* (3.121) *hold.*

Proof. The well-posedness of (3.1) with $\epsilon = 0$ can be proved as follows:
Step 1: Existence. First, it follows from Lemmas 3.1–3.9 that for every $\epsilon > 0$ and $\eta > 0$, there exist a time $T^* > 0$ independent of (ϵ, η), and a unique strong solution $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})(t,x)$ in $[0, T^*] \times \mathbb{R}^3$ to the linearized problem (3.1) satisfying the estimates in (3.121), which are independent of (ϵ, η).

Second, by using of the characteristic method and the standard energy estimates for $(3.1)_4$, one can show that for $0 \le t \le T^*$,

$$|h^{\epsilon,\eta}(t)|_\infty + |\nabla h^{\epsilon,\eta}(t)|_2 + |h_t^{\epsilon,\eta}(t)|_2 \le C(A, R, c_v, \eta, \alpha, \beta, \gamma, \delta, T^*, \phi_0, u_0). \tag{3.122}$$

Thus, it follows from (3.121), (3.122) and Lemma 5.3 (see [37]) that for any $R > 0$, there exists a subsequence of solutions (still denoted by) $(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta})$, which converges to a limit $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ as $\epsilon \to 0$ in the following strong sense:

$$(\phi^{\epsilon,\eta}, u^{\epsilon,\eta}, l^{\epsilon,\eta}, h^{\epsilon,\eta}) \to (\phi^\eta, u^\eta, l^\eta, h^\eta) \quad \text{in} \quad C([0,T^*]; H^2(B_R)) \quad \text{as} \quad \epsilon \to 0, \tag{3.123}$$

where B_R is a ball centered at origin with radius R, and in the following weak or weak* sense:

$$\begin{aligned}
(\phi^{\epsilon,\eta} - \eta, u^{\epsilon,\eta}) \rightharpoonup (\phi^\eta - \eta, u^\eta) \quad & \text{weakly* in} \quad L^\infty([0,T^*]; H^3),\\
(\phi_t^{\epsilon,\eta}, \psi^{\epsilon,\eta}, h_t^{\epsilon,\eta}) \rightharpoonup (\phi_t^\eta, \psi^\eta, h_t^\eta) \quad & \text{weakly* in} \quad L^\infty([0,T^*]; H^2),\\
u_t^{\epsilon,\eta} \rightharpoonup u_t^\eta \quad & \text{weakly* in} \quad L^\infty([0,T^*]; H^1),\\
t^{\frac{1}{2}}(\nabla^2 u_t^{\epsilon,\eta}, \nabla^4 u^{\epsilon,\eta}) \rightharpoonup t^{\frac{1}{2}}(\nabla^2 u_t^\eta, \nabla^4 u^\eta) \quad & \text{weakly* in} \quad L^\infty([0,T^*]; L^2),\\
(\phi_{tt}^{\epsilon,\eta}, t^{\frac{1}{2}} u_{tt}^{\epsilon,\eta}) \rightharpoonup (\phi_{tt}^\eta, t^{\frac{1}{2}} u_{tt}^\eta) \quad & \text{weakly* in} \quad L^\infty([0,T^*]; L^2),\\
(h^{\epsilon,\eta}, h_t^{\epsilon,\eta}) \rightharpoonup (h^\eta, h_t^\eta) \quad & \text{weakly* in} \quad L^\infty([0,T^*]; L^\infty),\\
\nabla u^{\epsilon,\eta} \rightharpoonup \nabla u^\eta \quad & \text{weakly in} \quad L^2([0,T^*]; H^3),\\
u_t^{\epsilon,\eta} \rightharpoonup u_t^\eta \quad & \text{weakly in} \quad L^2([0,T^*]; H^2),\\
\phi_{tt}^{\epsilon,\eta} \rightharpoonup \phi_{tt}^\eta \quad & \text{weakly in} \quad L^2([0,T^*]; H^1),\\
(\psi_{tt}^{\epsilon,\eta}, u_{tt}^{\epsilon,\eta}) \rightharpoonup (\psi_{tt}^\eta, u_{tt}^\eta) \quad & \text{weakly in} \quad L^2([0,T^*]; L^2),\\
h_{tt}^{\epsilon,\eta} \rightharpoonup h_{tt}^\eta \quad & \text{weakly in} \quad L^2([0,T^*]; L^6),\\
t^{\frac{1}{2}}(\nabla u_{tt}^{\epsilon,\eta}, \nabla^3 u_t^{\epsilon,\eta}) \rightharpoonup t^{\frac{1}{2}}(\nabla u_{tt}^\eta, \nabla^3 u_t^\eta) \quad & \text{weakly in} \quad L^2([0,T^*]; L^2),\\
l^{\epsilon,\eta} - \bar{l} \rightharpoonup l^\eta - \bar{l} \quad & \text{weakly* in} \quad L^\infty([0,T^*]; D_*^1 \cap D^3),\\
l_t^{\epsilon,\eta} \rightharpoonup l_t^\eta \quad & \text{weakly* in} \quad L^\infty([0,T^*]; D_*^1 \cap D^2),\\
l_{tt}^{\epsilon,\eta} \rightharpoonup l_{tt}^\eta \quad & \text{weakly in} \quad L^2([0,T^*]; D_*^1).
\end{aligned} \tag{3.124}$$

Furthermore, it follows from the lower semi-continuity of weak or weak* convergence that $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ satisfies also the corresponding estimates in (3.121) and (3.122) except those weighted estimates on u^η.

Now, the uniform estimates on $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ obtained above and the convergences in (3.123)-(3.124) imply that

$$\begin{aligned}
\sqrt{h^{\epsilon,\eta}}\nabla u^{\epsilon,\eta} \rightharpoonup \sqrt{h^\eta}\nabla u^\eta \quad & \text{weakly* in } L^\infty([0,T^*];L^2),\\
\sqrt{h^{\epsilon,\eta}}\nabla u_t^{\epsilon,\eta} \rightharpoonup \sqrt{h^\eta}\nabla u_t^\eta \quad & \text{weakly* in } L^\infty([0,T^*];L^2),\\
h^{\epsilon,\eta}\nabla^2 u^{\epsilon,\eta} \rightharpoonup h^\eta\nabla^2 u^\eta \quad & \text{weakly* in } L^\infty([0,T^*];H^1),\\
(h^{\epsilon,\eta}\nabla^2 u^{\epsilon,\eta})_t \rightharpoonup (h^\eta\nabla^2 u^\eta)_t \quad & \text{weakly in } L^2([0,T^*];L^2),\\
h^{\epsilon,\eta}\nabla^2 u^{\epsilon,\eta} \rightharpoonup h^\eta\nabla^2 u^\eta \quad & \text{weakly in } L^2([0,T^*];D_*^1\cap D^2),
\end{aligned} \tag{3.125}$$

which, along with the lower semi-continuity of weak or weak* convergence again, implies that $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ satisfies also the uniform weighted estimates on u^η.

Now we are ready to show that $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ is a weak solution in the sense of distributions to (3.1) with $\epsilon = 0$. First, multiplying $(3.1)_2$ by any given $X(t,x) = (X^{(1)}, X^{(2)}, X^{(3)})^\top \in C_c^\infty([0,T^*)\times\mathbb{R}^3)$ on both sides, and integrating over $[0,t)\times\mathbb{R}^3$ for $t\in(0,T^*]$, one has

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\Big(u^{\epsilon,\eta}\cdot X_t - (v\cdot\nabla)v\cdot X - a_1\phi^{\epsilon,\eta}\nabla l^{\epsilon,\eta}\cdot X - l^{\epsilon,\eta}\nabla\phi^{\epsilon,\eta}\cdot X\Big)\mathrm{d}x\mathrm{d}s\\
=&\int u^{\epsilon,\eta}(t,x)\cdot X(t,x) + a_2\int_0^t\int_{\mathbb{R}^3} l^{\epsilon,\eta}\sqrt{(h^{\epsilon,\eta})^2+\epsilon^2}Lu^{\epsilon,\eta}\cdot X\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_{\mathbb{R}^3}\Big(a_2 g\nabla(l^{\epsilon,\eta})^\nu\cdot Q(v)\cdot X + a_3(l^{\epsilon,\eta})^\nu\psi^{\epsilon,\eta}\cdot Q(v)\cdot X\Big)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.126}$$

It follows from the uniform estimates obtained above and (3.123)-(3.125) that one can take limit $\epsilon\to 0$ in (3.126) to get

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\Big(u^{\eta}\cdot X_t - (v\cdot\nabla)v\cdot X - a_1\phi^{\eta}\nabla l^{\eta}\cdot X - l^{\eta}\nabla\phi^{\eta}\cdot X\Big)\mathrm{d}x\mathrm{d}s\\
=&\int u^{\eta}(t,x)\cdot X(t,x) + a_2\int_0^t\int_{\mathbb{R}^3} l^{\eta}h^\eta Lu^{\eta}\cdot X\mathrm{d}x\mathrm{d}s\\
&-\int_0^t\int_{\mathbb{R}^3}\Big(a_2 g\nabla(l^{\eta})^\nu\cdot Q(v)\cdot X + a_3(l^{\eta})^\nu\psi^{\eta}\cdot Q(v)\cdot X\Big)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.127}$$

Similarly, one can show that $(\phi^\eta, l^\eta, h^\eta)$ satisfies also the equations $(3.1)_1$, $(3.1)_3$-$(3.1)_4$ and the initial data in the sense of distributions. So it is clear that $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ is a weak solution in the sense of distributions to the linearized problem (3.1) with $\epsilon = 0$ satisfying the following regularities

$$\begin{aligned}
&\phi^\eta-\eta\in L^\infty([0,T^*];H^3),\quad h^\eta\in L^\infty([0,T^*]\times\mathbb{R}^3),\\
&(\nabla h^\eta, h_t^\eta)\in L^\infty([0,T^*];H^2),\quad u^\eta\in L^\infty([0,T^*];H^3)\cap L^2([0,T^*];H^4),\\
&u_t^\eta\in L^\infty([0,T^*];H^1)\cap L^2([0,T^*];D^2),\quad u_{tt}^\eta\in L^2([0,T^*];L^2),\\
&t^{\frac12}u^\eta\in L^\infty([0,T^*];D^4),\quad t^{\frac12}u_t^\eta\in L^\infty([0,T^*];D^2)\cap L^2([0,T^*];D^3),\\
&t^{\frac12}u_{tt}^\eta\in L^\infty([0,T^*];L^2)\cap L^2([0,T^*];D_*^1),\\
&l^\eta-\bar l\in L^\infty([0,T^*];D_*^1\cap D^3),\quad l_t^\eta\in L^\infty([0,T^*];D_*^1\cap D^2).
\end{aligned}$$

Therefore, this weak solution $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ is actually a strong one.

Step 2: Uniqueness and time continuity. Since the estimate $h^\eta > \frac{1}{2c_0}$ holds, the uniqueness and the time continuity of the strong solution obtained above can be obtained by the same arguments as in Lemma 3.1, so details are omitted.

Thus the proof of Lemma 3.10 is complete.

□

3.4. **Nonlinear approximation solutions away from vacuum.** In this subsection, we will prove the local well-posedness of the classical solution to the following Cauchy problem under the assumption that $\phi_0^\eta \geq \eta$.

$$\begin{cases} \phi^\eta_t + u^\eta \cdot \nabla \phi^\eta + (\gamma-1)\phi^\eta \text{div} u^\eta = 0, \\ u^\eta_t + u^\eta \cdot \nabla u^\eta + a_1 \phi^\eta \nabla l^\eta + l^\eta \nabla \phi^\eta + a_2 (l^\eta)^\nu h^\eta L u^\eta \\ = a_2 h^\eta \nabla (l^\eta)^\nu \cdot Q(u^\eta) + a_3 (l^\eta)^\nu \psi^\eta \cdot Q(u^\eta), \\ l^\eta_t + u^\eta \cdot \nabla l^\eta = a_4 (l^\eta)^\nu n^\eta (\phi^\eta)^{4\iota} H(u^\eta), \\ h^\eta_t + u^\eta \cdot \nabla h^\eta + (\delta-1)(\phi^\eta)^{2\iota} \text{div} u^\eta = 0, \\ (\phi^\eta, u^\eta, l^\eta, h^\eta)|_{t=0} = (\phi^\eta_0, u^\eta_0, l^\eta_0, h^\eta_0) \\ = (\phi_0 + \eta, u_0, l_0, (\phi_0+\eta)^{2\iota}) \qquad x \in \mathbb{R}^3, \\ (\phi^\eta, u^\eta, l^\eta, h^\eta) \to (\eta, 0, \bar{l}, \eta^{2\iota}) \quad \text{as } |x| \to \infty \quad \text{for} \quad \text{t} \geq 0, \end{cases} \tag{3.128}$$

where $\psi^\eta = \frac{a\delta}{\delta-1} \nabla h^\eta$ and $n^\eta = (ah^\eta)^b$.

Theorem 3.1. *Let* (1.17) *hold and* $\eta > 0$. *Assume that* (ϕ_0, u_0, l_0, h_0) *satisfy* (2.6)-(2.7), *and there exists a positive constant* c_0 *independent of* η *such that* (3.5) *holds. Then there exists a time* $T_* > 0$ *independent of* η, *and a unique strong solution*

$$(\phi^\eta, u^\eta, l^\eta, h^\eta = \phi^{2\iota})$$

in $[0, T_*] \times \mathbb{R}^3$ *to* (3.128) *satisfying* (3.4), *where* T_* *is independent of* η. *Moreover, the uniform estimates (independent of* η*)* (3.121) *hold for* $(\phi^\eta, u^\eta, l^\eta, h^\eta)$ *with* T^* *replaced by* T_*.

The proof is given by an iteration scheme described below.

Let (ϕ^0, u^0, l^0, h^0) be the solution to the following Cauchy problem

$$\begin{cases} U_t + u_0 \cdot \nabla U = 0, \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ Y_t - W \triangle Y = 0, \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ Z_t + u_0 \cdot \nabla Z = 0, \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ W_t + u_0 \cdot \nabla W = 0, \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ (U, Y, Z, W)|_{t=0} = (\phi^\eta_0, u^\eta_0, l^\eta_0, h^\eta_0) \\ = (\phi_0 + \eta, u_0, l_0, (\phi_0+\eta)^{2\iota}) \quad \text{in} \quad \mathbb{R}^3, \\ (U, Y, Z, W) \to (\eta, 0, \bar{l}, \eta^{2\iota}) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad \text{t} \geq 0. \end{cases} \tag{3.129}$$

Choose a time $\bar{T} \in (0, T^*]$ small enough such that

$$
\begin{aligned}
&\sup_{0\leq t\leq \bar{T}} \|\nabla h^0(t)\|^2_{L^q\cap D^{1,3}\cap D^2} \leq c_1^2, \quad \sup_{0\leq t\leq \bar{T}} \|(l^0-\bar{l})(t)\|^2_{D^1_*\cap D^3} \leq c_1^2,\\
&\inf_{[0,T_*]\times\mathbb{R}^3} l^0(t,x) \geq c_0^{-1}, \quad \sup_{0\leq t\leq \bar{T}} \|u^0(t)\|_1^2 + \int_0^{\bar{T}} \big(|u^0|^2_{D^2} + |u^0_t|_2^2\big)\mathrm{d}t \leq c_2^2,\\
&\sup_{0\leq t\leq \bar{T}} (|u^0|^2_{D^2} + |h^0\nabla^2 u^0|_2^2 + |u^0_t|_2^2)(t) + \int_0^{\bar{T}} (|u^0|^2_{D^3} + |u^0_t|^2_{D^1_*})\mathrm{d}t \leq c_3^2,\\
&\sup_{0\leq t\leq \bar{T}} (|u^0_t|^2_{D^1_*} + |h^0_t|^2_{D^1_*} + |u^0|^2_{D^3})(t) + \int_0^{\bar{T}} (|u^0_t|^2_{D^2} + |u^0|^2_{D^4} + |u^0_{tt}|_2^2)\mathrm{d}t \leq c_4^2,\\
&\sup_{0\leq t\leq \bar{T}} |h^0_t(t)|^2_\infty + \int_0^{\bar{T}} (|(h^0\nabla^2 u^0)_t|_2^2 + |h^0\nabla^2 u^0|^2_{D^2})\mathrm{d}t \leq c_4^2,\\
&\sup_{0\leq t\leq \bar{T}} \big(|\sqrt{h^0}\nabla u^0_t|_2^2 + |h^0\nabla^2 u^0|^2_{D^1_*} + |l^0_t|_3^2 + |l^0_t|^2_{D^1_*}\big)(t) \leq c_4^2,\\
&\sup_{0\leq t\leq \bar{T}} \big(|l^0_t|^2_\infty + |\nabla^2 l^0_t|_2^2\big)(t) + \int_0^{\bar{T}} |\nabla l^0_{tt}|_2^2 \mathrm{d}t \leq c_5^2,\\
&\operatorname{ess}\sup_{0\leq t\leq T^*} (t|u^0_t(t)|^2_{D^2} + t|h^0\nabla^2 u^0_t(t)|_2^2 + t|u^0_{tt}(t)|_2^2 + t|u^0(t)|^2_{D^4}) \leq c_5^2,\\
&\int_0^t s(|u^0_{tt}|^2_{D^1_*} + |u^0_t|^2_{D^3} + |\sqrt{h^0}u^0_{tt}|^2_{D^1_*})\mathrm{d}s \leq c_5^2.
\end{aligned}
\tag{3.130}
$$

Proof. **Step 1:** Existence. One starts with the initial iteration $(v, w, g) = (u^0, l^0, h^0)$, and can obtain a classical solution (ϕ^1, u^1, l^1, h^1) to the problem (3.1) with $\epsilon = 0$. Inductively, one constructs approximate sequences $(\phi^{k+1}, u^{k+1}, l^{k+1}, h^{k+1})$ as follows: given (u^k, l^k, h^k) for $k \geq 1$, define $(\phi^{k+1}, u^{k+1}, l^{k+1}, h^{k+1})$ by solving the following problem:

$$
\begin{cases}
\phi^{k+1}_t + u^k\cdot\nabla\phi^{k+1} + (\gamma-1)\phi^{k+1}\mathrm{div}u^k = 0,\\
(l^{k+1})^{-\nu}(u^{k+1}_t + u^k\cdot\nabla u^k + a_1\phi^{k+1}\nabla l^{k+1} + l^{k+1}\nabla\phi^{k+1})\\
\ + a_2 h^{k+1}Lu^{k+1} = a_2(l^{k+1})^{-\nu}h^k\nabla(l^{k+1})^\nu\cdot Q(u^k) + a_3\psi^{k+1}\cdot Q(u^k),\\
l^{k+1}_t + u^k\cdot\nabla l^{k+1} = a_4(l^k)^\nu n^{k+1}(h^k)^2 H(u^k),\\
h^{k+1}_t + u^k\cdot\nabla h^{k+1} + (\delta-1)h^k\mathrm{div}u^k = 0,\\
(\phi^{k+1}, u^{k+1}, l^{k+1}, h^{k+1})|_{t=0} = (\phi^\eta_0, u^\eta_0, l^\eta_0, h^\eta_0)\\
=(\phi_0+\eta, u_0, l_0, (\phi_0+\eta)^{2\iota}) \ \text{ in } \ \mathbb{R}^3,\\
(\phi^{k+1}, u^{k+1}, l^{k+1}, h^{k+1}) \longrightarrow (\eta, 0, \bar{l}, \eta^{2\iota}) \quad \text{as } |x|\to\infty \quad \text{for } \quad \mathrm{t}\geq 0,
\end{cases}
\tag{3.131}
$$

where $\psi^{k+1} = \frac{a\delta}{\delta-1}\nabla h^{k+1}$ and $n^{k+1} = (ah^{k+1})^b$. It follows from Lemma 3.10 and mathematical induction that, by replacing (v, w, g) with (u^k, l^k, h^k), one can solve (3.131) locally in time, and the solution $(\phi^{k+1}, u^{k+1}, l^{k+1}, h^{k+1})$ $(k = 0, 1, 2, \cdots)$

satisfies the uniform estimates (3.121). Moreover, ψ^{k+1} satisfies

$$\psi_t^{k+1}+\nabla(u^k\cdot\psi^{k+1})+(\delta-1)\psi^k\mathrm{div}u^k+a\delta h^k\nabla\mathrm{div}u^k=0. \tag{3.132}$$

To show the strong convergence of (ϕ^k,u^k,l^k,ψ^k), we set

$$\begin{aligned}&\bar\phi^{k+1}=\phi^{k+1}-\phi^k,\quad \bar u^{k+1}=u^{k+1}-u^k,\quad \bar l^{k+1}=l^{k+1}-l^k,\\ &\bar\psi^{k+1}=\psi^{k+1}-\psi^k,\quad \bar h^{k+1}=h^{k+1}-h^k,\quad \bar n^{k+1}=n^{k+1}-n^k.\end{aligned}$$

Then (3.131) and (3.132) yield

$$\begin{cases}\bar\phi_t^{k+1}+u^k\cdot\nabla\bar\phi^{k+1}+\bar u^k\cdot\nabla\phi^k+(\gamma-1)(\bar\phi^{k+1}\mathrm{div}u^k+\phi^k\mathrm{div}\bar u^k)=0,\\ (l^{k+1})^{-\nu}(\bar u_t^{k+1}+u^k\cdot\nabla\bar u^k+\bar u^k\cdot\nabla u^{k-1}+a_1\bar\phi^{k+1}\nabla l^{k+1}+a_1\phi^k\nabla\bar l^{k+1}\\ +\bar l^{k+1}\nabla\phi^{k+1}+l^k\nabla\bar\phi^{k+1})+a_2h^{k+1}L\bar u^{k+1}+a_2\bar h^{k+1}Lu^k\\ =-\big((l^{k+1})^{-\nu}-(l^k)^{-\nu}\big)(u_t^k+u^{k-1}\cdot\nabla u^{k-1}+a_1\phi^k\nabla l^k+l^k\nabla\phi^k)\\ +a_2(l^{k+1})^{-\nu}\Big(h^k\big(\nabla(l^{k+1})^\nu-\nabla(l^k)^\nu\big)\cdot Q(u^k)+h^k\nabla(l^k)^\nu\cdot Q(\bar u^k)\\ +\bar h^k\nabla(l^k)^\nu\cdot Q(u^{k-1})\Big)+a_3\bar\psi^{k+1}\cdot Q(u^k)+a_3\psi^k\cdot Q(\bar u^k)\\ +a_2\big((l^{k+1})^{-\nu}-(l^k)^{-\nu}\big)\big(h^{k-1}\nabla(l^k)^\nu\cdot Q(u^{k-1})\big),\\ \bar l_t^{k+1}+u^k\cdot\nabla\bar l^{k+1}+\bar u^k\cdot\nabla l^k=a_4(l^k)^\nu\big(n^{k+1}(h^k)^2(H(u^k)-H(u^{k-1}))\\ +\bar h^k(h^k+h^{k-1})n^{k+1}H(u^{k-1})+(h^{k-1})^2\bar n^{k+1}H(u^{k-1})\big)\\ +a_4\big((l^k)^\nu-(l^{k-1})^\nu\big)n^k(h^{k-1})^2H(u^{k-1}),\\ \bar\psi_t^{k+1}+\nabla(u^k\cdot\bar\psi^{k+1}+\bar u^k\cdot\psi^k)+(\delta-1)(\bar\psi^k\mathrm{div}u^k+\psi^{k-1}\mathrm{div}\bar u^k)\\ +a\delta(h^k\nabla\mathrm{div}\bar u^k+\bar h^k\nabla\mathrm{div}u^{k-1})=0,\\ (\bar\phi^{k+1},\bar u^{k+1},\bar l^{k+1},\bar\psi^{k+1})|_{t=0}=(0,0,0,0)\quad\text{in}\quad\mathbb{R}^3,\\ (\bar\phi^{k+1},\bar u^{k+1},\bar l^{k+1},\bar\psi^{k+1})\longrightarrow(0,0,0,0)\quad\text{as}\quad|x|\to\infty\quad\text{for}\quad \mathrm{t}\geq0.\end{cases} \tag{3.133}$$

We now give some necessary estimates on $(\bar\phi^{k+1},\bar u^{k+1},\bar l^{k+1},\bar\psi^{k+1},\bar h^{k+1})$ to be used later. We start with $\bar h^{k+1}$, for which one need the following lemma whose proof is given in Remark 3.3 later.

Lemma 3.11.

$$\bar h^{k+1},\ \bar\phi^{k+1},\ \bar\psi^{k+1}\in L^\infty([0,\bar T];H^2(\mathbb{R}^3))\quad \textit{for}\quad k=1,2,....$$

Remark 3.2. *This lemma is helpful to deal with some singular terms of forms of $\infty-\infty$.*

Assume that Lemma 3.11 holds at this moment, one can deduce the uniform estimate for $\bar h^{k+1}$ as follows. It follows from $(3.131)_4$ that

$$\bar h_t^{k+1}+u^k\cdot\nabla\bar h^{k+1}+\bar u^k\cdot\nabla h^k+(\delta-1)(h^k\mathrm{div}u^k-h^{k-1}\mathrm{div}u^{k-1})=0. \tag{3.134}$$

Multiplying (3.134) by $5|\bar{h}^{k+1}|^4\bar{h}^{k+1}$ and using integration by parts yield

$$\begin{aligned}\frac{d}{dt}|\bar{h}^{k+1}(t,x)|_6^6 \le & C(|\nabla u^k|_\infty|\bar{h}^{k+1}|_6^6+|\bar{u}^k|_6|\psi^k|_\infty|\bar{h}^{k+1}|_6^5\\ &+(|h^k\mathrm{div}u^k|_6+|h^{k-1}\mathrm{div}u^{k-1}|_6)|\bar{h}^{k+1}|_6^5),\end{aligned}\tag{3.135}$$

with a generic constant C independent of η and k, which implies

$$|\bar{h}^{k+1}(t,x)|_6\le C \quad \text{for} \quad 0<t\le \bar{T} \quad \text{and} \quad k=0,1,2,.... \tag{3.136}$$

Second, multiplying $(3.133)_4$ by $2\bar{\psi}^{k+1}$ and integrating over $\mathbb{R}^3$ lead to

$$\begin{aligned}\frac{d}{dt}|\bar{\psi}^{k+1}|_2^2 \le & C|\nabla u^k|_\infty|\bar{\psi}^{k+1}|_2^2+C(|\bar{u}^k|_6|\nabla\psi^k|_3+|\psi^k|_\infty|\nabla \bar{u}^k|_2\\ &+|\bar{\psi}^k\mathrm{div}u^k|_2+|h^k\nabla^2\bar{u}^k|_2\\ &+|\psi^{k-1}|_\infty|\nabla\bar{u}^k|_2+|\bar{h}^k|_6|\nabla^2u^{k-1}|_3)|\bar{\psi}^{k+1}|_2,\end{aligned}\tag{3.137}$$

which, along with the fact

$$\begin{aligned}|\bar{\psi}^k\mathrm{div}u^k|_2 = & |\psi^k\mathrm{div}u^k-\psi^{k-1}\mathrm{div}u^k|_2\le C(|\psi^k|_\infty+|\psi^{k-1}|_\infty)|\mathrm{div}u^k|_2,\\ |h^k\nabla^2\bar{u}^k|_2 = & |h^k\nabla^2u^k-h^{k-1}\nabla^2u^{k-1}-\bar{h}^k\nabla^2u^{k-1}|_2\\ \le & C\big(|h^k\nabla^2u^k|_2+|h^{k-1}\nabla^2u^{k-1}|_2+|\bar{h}^k|_6|\nabla^2u^{k-1}|_3\big),\end{aligned}\tag{3.138}$$

the uniform estimates (3.121) for (ϕ^k,u^k,l^k,h^k) $(k=1,2,...)$, (3.136) and Gronwall's inequality that

$$|\bar{\psi}^{k+1}(t,x)|_2+|\bar{\psi}^{k+1}(t,x)|_6\le C \quad \text{for} \quad 0<t\le \bar{T} \quad \text{and} \quad k=0,1,2,.... \tag{3.139}$$

Moreover, (3.137), Young's inequality and the uniform estimates (3.121) for $(\phi^k,u^k,$ $l^k,h^k)$ $(k=1,2,...)$ also imply that

$$\frac{d}{dt}|\bar{\psi}^{k+1}|_2^2 \le C\sigma^{-1}|\bar{\psi}^{k+1}|_2^2+\sigma(|\sqrt{h^k}\nabla\bar{u}^k|_2^2+|\bar{\psi}^k|_2^2+|h^k\nabla^2\bar{u}^k|_2^2), \tag{3.140}$$

where $\sigma\in(0,1)$ is a constant to be determined later.

Next, multiplying $(3.133)_1$ by $2\bar{\phi}^{k+1}$ and integrating over $\mathbb{R}^3$ give

$$\frac{d}{dt}|\bar{\phi}^{k+1}|_2^2\le C(|\nabla u^k|_\infty|\bar{\phi}^{k+1}|_2+|\bar{u}^k|_6|\nabla\phi^k|_3+|\nabla\bar{u}^k|_2|\phi^k|_\infty)|\bar{\phi}^{k+1}|_2. \tag{3.141}$$

Applying ∂_x^ς $(|\varsigma|=1)$ to $(3.133)_1$, multiplying by $2\partial_x^\varsigma\bar{\phi}^{k+1}$ and integrating over $\mathbb{R}^3$, one gets

$$\begin{aligned}\frac{d}{dt}|\partial_x^\varsigma\bar{\phi}^{k+1}|_2^2 \le & C(|\nabla u^k|_\infty|\nabla\bar{\phi}^{k+1}|_2+|\nabla\phi^k|_\infty|\nabla\bar{u}^k|_2+|\bar{u}^k|_6|\nabla^2\phi^k|_3)|\nabla\bar{\phi}^{k+1}|_2\\ &+C(|\nabla^2u^k|_3|\bar{\phi}^{k+1}|_6+|\phi^k|_\infty|\nabla\mathrm{div}\bar{u}^k|_2)|\nabla\bar{\phi}^{k+1}|_2.\end{aligned}$$

Hence, it holds that for $t\in[0,\bar{T}]$,

$$\frac{d}{dt}\|\bar{\phi}^{k+1}\|_1^2\le C\sigma^{-1}\|\bar{\phi}^{k+1}\|_1^2+\sigma(|\sqrt{h^k}\nabla\bar{u}^k|_2^2+|h^k\nabla^2\bar{u}^k|_2^2), \tag{3.142}$$

where one has used that

$$|\nabla\bar{u}^k|_2\le C|\sqrt{h^k}\nabla\bar{u}^k|_2,\quad |\nabla^2\bar{u}^k|_2\le C|h^k\nabla^2\bar{u}^k|_2.$$

On the other hand, it follows from the definition of $n^k = (ah^k)^b$ and $h^k \geq Cc_0^{-1}$ that

$$
\begin{aligned}
&a^{-b}|\bar{n}^{k+1}|_6 = |(h^{k+1})^b - (h^k)^b|_6 \leq C|\bar{\psi}^{k+1}|_2,\\
&a^{-b}\nabla \bar{n}^{k+1} = b((h^{k+1})^{b-1} - (h^k)^{b-1})\psi^{k+1} + b(h^k)^{b-1}\bar{\psi}^{k+1}.
\end{aligned} \tag{3.143}
$$

Applying derivative ∂_x^ς ($|\varsigma| = 1$) to $(3.133)_3$, multiplying by $2\partial_x^\varsigma \bar{l}^{k+1}$ and integrating over $\mathbb{R}^3$ lead to

$$
\begin{aligned}
&\frac{d}{dt}|\partial_x^\varsigma \bar{l}^{k+1}|_2^2 = -\int \partial_x^\varsigma (u^k \cdot \nabla \bar{l}^{k+1} + \bar{u}^k \cdot \nabla l^k) 2\partial_x^\varsigma \bar{l}^{k+1}\\
&+ a_4 \int \partial_x^\varsigma \Big((l^k)^\nu \big(n^{k+1}(h^k)^2 (H(u^k) - H(u^{k-1}))\\
&+ \bar{h}^k (h^k + h^{k-1}) n^{k+1} H(u^{k-1}) + (h^{k-1})^2 \bar{n}^{k+1} H(u^{k-1})\big)\Big) 2\partial_x^\varsigma \bar{l}^{k+1}\\
&+ a_4 \int \partial_x^\varsigma \Big(((l^k)^\nu - (l^{k-1})^\nu) n^k (h^{k-1})^2 H(u^{k-1})\Big) 2\partial_x^\varsigma \bar{l}^{k+1} \equiv \sum_{i=1}^{3} N_i.
\end{aligned} \tag{3.144}
$$

N_1, N_2 and N_3 can be estimated by (3.121), (3.136), (3.139) and (3.143) as follows:

$$
\begin{aligned}
|N_1| \leq & C|\nabla \bar{l}^{k+1}|_2^2 + C|\sqrt{h^k}\nabla \bar{u}^k|_2 |\nabla \bar{l}^{k+1}|_2,\\
|N_2| \leq & C(|h^k \nabla \bar{u}^k|_6 + |h^k \nabla^2 \bar{u}^k|_2)|\nabla \bar{l}^{k+1}|_2\\
& + C(|\bar{\psi}^k|_2 + |\bar{n}^{k+1}|_6 + |\bar{\psi}^{k+1}|_2)|\nabla \bar{l}^{k+1}|_2\\
\leq & C(|\sqrt{h^k}\nabla \bar{u}^k|_2 + |h^k \nabla^2 \bar{u}^k|_2)|\nabla \bar{l}^{k+1}|_2 + C(|\bar{\psi}^k|_2 + |\bar{\psi}^{k+1}|_2)|\nabla \bar{l}^{k+1}|_2,\\
|N_3| \leq & C|\nabla \bar{l}^k|_2 |\nabla \bar{l}^{k+1}|_2.
\end{aligned}
$$

These, along with (3.143)-(3.144), yield that

$$
\begin{aligned}
\frac{d}{dt}|\nabla \bar{l}^{k+1}|_2^2 \leq & C\sigma^{-1}|\nabla \bar{l}^{k+1}|_2^2 + \sigma(|\sqrt{h^k}\nabla \bar{u}^k|_2^2 + |h^k \nabla^2 \bar{u}^k|_2^2)\\
& + \sigma(|\bar{\psi}^k|_2^2 + |\nabla \bar{l}^k|_2^2 + |\bar{\psi}^{k+1}|_2^2).
\end{aligned} \tag{3.145}
$$

We now estimate $\bar{u}^{k+1}$. Multiplying $(3.133)_2$ by $2\bar{u}^{k+1}$ and integrating over $\mathbb{R}^3$ yield that

$$
\begin{aligned}
&\frac{d}{dt}|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}|_2^2 + a_2\alpha|\sqrt{h^{k+1}}\nabla \bar{u}^{k+1}|_2^2 + a_2(\alpha+\beta)|\sqrt{h^{k+1}}\mathrm{div}\bar{u}^{k+1}|_2^2\\
\leq & C\sigma^{-1}|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}|_2^2 + \sigma(|\sqrt{h^k}\nabla \bar{u}^k|_2^2 + |h^k \nabla^2 \bar{u}^k|_2^2 + |\bar{\psi}^k|_2^2)\\
&+ C(\|\bar{\phi}^{k+1}\|_1^2 + |\bar{\psi}^{k+1}|_2^2 + |\nabla \bar{l}^{k+1}|_2^2).
\end{aligned} \tag{3.146}
$$

Multiplying $(3.133)_2$ by $2\bar{u}_t^{k+1}$ and integrating over $\mathbb{R}^3$ yield that

$$
\begin{aligned}
2|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}_t^{k+1}|_2^2 + & \frac{d}{dt}(a_2\alpha|\sqrt{h^{k+1}}\nabla \bar{u}^{k+1}|_2^2\\
& + a_2(\alpha+\beta)|\sqrt{h^{k+1}}\mathrm{div}\bar{u}^{k+1}|_2^2) \equiv \sum_{i=1}^{8} O_i,
\end{aligned} \tag{3.147}
$$

where, O_i, $i=1,2,\cdots,8$, are defined and estimated as follows:

$$
\begin{aligned}
O_1=&-2\int (l^{k+1})^{-\nu}(u^k\cdot\nabla \bar{u}^k+\bar{u}^k\cdot\nabla u^{k-1}+a_1\bar{\phi}^{k+1}\nabla l^{k+1}\\
&+a_1\phi^k\nabla \bar{l}^{k+1}+\bar{l}^{k+1}\nabla\phi^{k+1}+l^k\nabla\bar{\phi}^{k+1})\cdot\bar{u}^{k+1}_t\\
\le& C(|\sqrt{h^k}\nabla\bar{u}^k|_2+\|\bar{\phi}^{k+1}\|_1+|\nabla \bar{l}^{k+1}|_2)|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_2=&-2\frac{\delta-1}{a\delta}a_2\int \psi^{k+1}\cdot Q(\bar{u}^{k+1})\cdot\bar{u}^{k+1}_t\\
\le& C|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_3=&-2a_2\int \bar{h}^{k+1}Lu^k\cdot\bar{u}^{k+1}_t\le C|\bar{\psi}^{k+1}|_2|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_4=&-2\int \big((l^{k+1})^{-\nu}-(l^k)^{-\nu}\big)(u^k_t+u^{k-1}\cdot\nabla u^{k-1}\\
&+a_1\phi^k\nabla l^k+l^k\nabla\phi^k)\cdot\bar{u}^{k+1}_t\le C|\nabla\bar{l}^{k+1}|_2|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_5=&2a_3\int(\bar{\psi}^{k+1}\cdot Q(u^k)+\psi^k\cdot Q(\bar{u}^k))\cdot\bar{u}^{k+1}_t\\
\le& C(|\bar{\psi}^{k+1}|_2+|\sqrt{h^k}\nabla\bar{u}^k|_2)|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_6=&2a_2\int \big((l^{k+1})^{-\nu}-(l^k)^{-\nu}\big)h^{k-1}\nabla(l^k)^{\nu}\cdot Q(u^{k-1})\cdot\bar{u}^{k+1}_t\\
\le& C|\nabla\bar{l}^{k+1}|_2|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_7=&2a_2\int(l^{k+1})^{-\nu}\Big(h^k\big(\nabla(l^{k+1})^{\nu}-\nabla(l^k)^{\nu}\big)\cdot Q(u^k)+h^k\nabla(l^k)^{\nu}\cdot Q(\bar{u}^k)\\
&+\bar{h}^k\nabla(l^k)^{\nu}\cdot Q(u^{k-1})\Big)\cdot\bar{u}^{k+1}_t\\
\le& C(|\nabla\bar{l}^{k+1}|_2+|\bar{\psi}^k|_2+|(h^k)^{\frac14}\nabla l^k|_6|(h^k)^{\frac34}\nabla\bar{u}^k|_3)|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2\\
\le& C(|\nabla\bar{l}^{k+1}|_2+|\bar{\psi}^k|_2+|h^k\nabla^2\bar{u}^k|_2^{\frac12}|\sqrt{h^k}\nabla\bar{u}^k|_2^{\frac12})|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2,\\
O_8=&a_2\int h^{k+1}_t(\alpha|\nabla\bar{u}^{k+1}|^2+(\alpha+\beta)|\mathrm{div}\bar{u}^{k+1}|^2)\le C|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.148}
$$

It follows from (3.147)-(3.148) and Young's inequality that

$$
\begin{aligned}
&|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}_t|_2^2+\frac{d}{dt}a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2\\
\le& C(|\sqrt{h^k}\nabla\bar{u}^k|_2^2+|\bar{\psi}^k|_2^2+\epsilon_1^{-1}|\sqrt{h^k}\nabla\bar{u}^k|_2^2)+\epsilon_1|h^k\nabla^2\bar{u}^k|_2^2\\
&+C(|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2+\|\bar{\phi}^{k+1}\|_1^2+|\nabla\bar{l}^{k+1}|_2^2+|\bar{\psi}^{k+1}|_2^2),
\end{aligned}
\tag{3.149}
$$

where $\epsilon_1>0$ is a small enough constant.

On the other hand, it follows directly from $(3.133)_2$ that

$$
\begin{aligned}
|h^k\nabla^2\bar{u}^k|_2^2\le& C(|\bar{u}^k_t|_2^2+|\sqrt{h^k}\nabla\bar{u}^k|_2^2+|\sqrt{h^{k-1}}\nabla\bar{u}^{k-1}|_2^2+\|\bar{\phi}^k\|_1^2+|\bar{\psi}^{k-1}|_2^2\\
&+|\nabla\bar{l}^k|_2^2+|\bar{\psi}^k|_2^2+\epsilon_2|h^{k-1}\nabla^2\bar{u}^{k-1}|_2^2+\epsilon_2^{-1}|\sqrt{h^{k-1}}\nabla\bar{u}^{k-1}|_2^2),
\end{aligned}
\tag{3.150}
$$

where $\epsilon_2>0$ is a small constant to be chosen.

Hence (3.140), (3.142), (3.145)-(3.146), (3.149)-(3.150) imply that

$$
\begin{aligned}
&\frac{d}{dt}(\|\bar{\phi}^{k+1}\|_1^2+|\bar{\psi}^{k+1}|_2^2+|\nabla \bar{l}^{k+1}|_2^2+|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}|_2^2\\
&+\upsilon_1 a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2)\\
&+a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2+\upsilon_1|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}_t^{k+1}|_2^2+\upsilon_2|h^k\nabla^2\bar{u}^k|_2^2\\
\leq& C\sigma^{-1}(\|\bar{\phi}^{k+1}\|_1^2+|\nabla \bar{l}^{k+1}|_2^2+|\bar{\psi}^{k+1}|_2^2+|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}|_2^2)\\
&+C\sigma^{-1}\upsilon_1 a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2+C(\sigma+\upsilon_1+\upsilon_1\epsilon_1^{-1}+\upsilon_2)|\sqrt{h^k}\nabla\bar{u}^k|_2^2\\
&+C(\sigma+\upsilon_1+\upsilon_2)(\|\bar{\phi}^k\|_1^2+|\bar{\psi}^k|_2^2+|\nabla\bar{l}^k|_2^2+|\bar{\psi}^{k-1}|_2^2)\\
&+C(\upsilon_2\epsilon_2^{-1}+\upsilon_2)|\sqrt{h^{k-1}}\nabla\bar{u}^{k-1}|_2^2+(6\sigma+\upsilon_1\epsilon_1)|h^k\nabla^2\bar{u}^k|_2^2\\
&+C\upsilon_2\epsilon_2|h^{k-1}\nabla^2\bar{u}^{k-1}|_2^2+C\upsilon_2|(l^k)^{-\frac{\nu}{2}}\bar{u}_t^k|_2^2,
\end{aligned}
\tag{3.151}
$$

where $\upsilon_1>0$, $\upsilon_2>0$ are small constants to be determined later.

Now, define

$$
\begin{aligned}
\varGamma^{k+1}(t,\upsilon_1)=&\sup_{0\leq s\leq t}\|\bar{\phi}^{k+1}\|_1^2+\sup_{0\leq s\leq t}|\bar{\psi}^{k+1}|_2^2+\sup_{0\leq s\leq t}|\nabla\bar{l}^{k+1}|_2^2\\
&+\sup_{0\leq s\leq t}\alpha a_2\upsilon_1|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2+\sup_{0\leq s\leq t}|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}^{k+1}|_2^2,
\end{aligned}
$$

and set $\upsilon_2=8\sigma$, $\sigma=\upsilon_1^{\frac{3}{2}}$, $\epsilon_1=\upsilon_1^{\frac{1}{2}}$, $\epsilon_2=\upsilon_2^{\frac{1}{2}}$ so that

$$
\upsilon_2-6\sigma-\upsilon_1\epsilon_1=\upsilon_1^{\frac{3}{2}}>0.
$$

Then it follows from (3.151) and Gronwall's inequality that

$$
\begin{aligned}
&\varGamma^{k+1}(t,\upsilon_1)+\int_0^t\Big(a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2+\upsilon_1|(l^{k+1})^{-\frac{\nu}{2}}\bar{u}_t^{k+1}|_2^2\\
&+\upsilon_1^{\frac{3}{2}}|h^k\nabla^2\bar{u}^k|_2^2\Big)\mathrm{d}s\\
\leq& C\Big(\int_0^t\Big(a_2\alpha(\sigma+\upsilon_1+\upsilon_1\epsilon_1^{-1}+\upsilon_2)|\sqrt{h^k}\nabla\bar{u}^k|_2^2\\
&+(\upsilon_2\epsilon_2^{-1}+\upsilon_2)|\sqrt{h^{k-1}}\nabla\bar{u}^{k-1}|_2^2\\
&+\upsilon_2\epsilon_2|h^{k-1}\nabla^2\bar{u}^{k-1}|_2^2+\upsilon_2|(l^k)^{-\frac{\nu}{2}}\bar{u}_t^k|_2^2\Big)\mathrm{d}s\\
&+(\sigma+\upsilon_1+\upsilon_2)t(\varGamma^k(t,\upsilon_1)+\varGamma^{k-1}(t,\upsilon_1)\Big)\exp{(C\sigma^{-1}t)}\\
\leq& C\Big(\int_0^t\big(a_2\alpha\upsilon_1^{\frac{1}{2}}|\sqrt{h^k}\nabla\bar{u}^k|_2^2+\upsilon_2^{\frac{1}{2}}|\sqrt{h^{k-1}}\nabla\bar{u}^{k-1}|_2^2\\
&+\upsilon_2^{\frac{3}{2}}|h^{k-1}\nabla^2\bar{u}^{k-1}|_2^2+\upsilon_2|(l^k)^{-\frac{\nu}{2}}\bar{u}_t^k|_2^2\big)\mathrm{d}s\\
&+\upsilon_1 t(\varGamma^k(t,\upsilon_1)+\varGamma^{k-1}(t,\upsilon_1))\Big)\exp{(C\sigma^{-1}t)}.
\end{aligned}
\tag{3.152}
$$

Now one first chooses $\upsilon_1 = \bar{\upsilon} \in (0,1)$ such that $C\bar{\upsilon}^{\frac{1}{2}} \leq \frac{1}{64}$, and then chooses $T_* \in (0,\bar{T}]$ such that

$$
\begin{aligned}
&(T_*+1)\exp(C\bar{\upsilon}^{-\frac{3}{2}}T_*) \leq 2,\\
&C\upsilon_1^{\frac{1}{2}}\exp(C\upsilon_1^{-\frac{3}{2}}T_*) = C\bar{\upsilon}^{\frac{1}{2}}\exp(C\bar{\upsilon}^{-\frac{3}{2}}T_*) \leq \frac{1}{32},\\
&C\upsilon_2^{\frac{1}{2}}\exp(C\upsilon_1^{-\frac{3}{2}}T_*) = C(8\upsilon_1^{\frac{3}{2}})^{\frac{1}{2}}\exp(C\upsilon_1^{-\frac{3}{2}}T_*) \leq \frac{1}{16},\\
&C\upsilon_2^{\frac{3}{2}}\exp(C\upsilon_1^{-\frac{3}{2}}T_*) \leq \frac{\sqrt{2}}{4}\bar{\upsilon}^{\frac{3}{2}},\\
&C\upsilon_2\exp(C\upsilon_1^{-\frac{3}{2}}T_*) \leq \frac{\bar{\upsilon}}{2}, \quad C\upsilon_1 T_*\exp(C\upsilon_1^{-\frac{3}{2}}T_*) \leq \frac{1}{32}.
\end{aligned}
$$

We can get finally that

$$
\sum_{k=1}^{\infty}\Big(\varGamma^{k+1}(T_*,\bar{\upsilon}) + \int_0^{T_*}(a_2\alpha|\sqrt{h^{k+1}}\nabla\bar{u}^{k+1}|_2^2 + \bar{\upsilon}|\bar{u}_t^{k+1}|_2^2 + \bar{\upsilon}^{\frac{3}{2}}|h^k\nabla^2\bar{u}^k|_2^2)\mathrm{d}s\Big) < \infty,
$$

which, along with the local estimates (3.121) independent of k, yields in particular that

$$
\begin{aligned}
\lim_{k\to\infty}(\|\bar{\phi}^{k+1}\|_{s'} + \|\bar{u}^{k+1}\|_{s'} + \|\bar{l}^{k+1}\|_{L^\infty\cap D^1\cap D^{s'}}) =&0,\\
\lim_{k\to\infty}(\|\bar{\psi}^{k+1}\|_{L^\infty\cap L^q} + |\bar{h}^{k+1}|_\infty) =&0.
\end{aligned}
\tag{3.153}
$$

for any $s' \in [1,3)$. Then there exist a subsequence (still denoted by $(\phi^k, u^k, l^k, \psi^k)$) and limit functions $(\phi^\eta, u^\eta, l^\eta, \psi^\eta)$ such that

$$
\begin{aligned}
&(\phi^k-\eta, u^k) \to (\phi^\eta-\eta, u^\eta) \ \text{ in } \ L^\infty([0,T_*]; H^{s'}(\mathbb{R}^3)),\\
&l^k-\bar{l} \to l^\eta-\bar{l} \ \text{ in } \ L^\infty([0,T_*]; L^\infty\cap D^1\cap D^{s'}(\mathbb{R}^3)),\\
&\psi^k \to \psi^\eta \ \text{ in } \ L^\infty([0,T_*]; L^\infty\cap L^q(\mathbb{R}^3)),\\
&h^k \to h^\eta \ \text{ in } \ L^\infty([0,T_*]; L^\infty(\mathbb{R}^3)).
\end{aligned}
\tag{3.154}
$$

Again by virtue of the local estimates (3.121) independent of k, there exists a subsequence (still denoted by $(\phi^k, u^k, l^k, \psi^k)$) converging to the limit $(\phi^\eta, u^\eta, l^\eta, \psi^\eta)$ in the weak or weak* sense. According to the lower semi-continuity of norms, the corresponding estimates in (3.121) for $(\phi^\eta, u^\eta, l^\eta, \psi^\eta)$ still hold except those weighted estimates on u^η.

Next, it remains to show

$$
\psi^\eta = \frac{a\delta}{\delta-1}\nabla(\phi^\eta)^{2\iota}. \tag{3.155}
$$

Set

$$
\psi^* = \psi^\eta - \frac{a\delta}{\delta-1}\nabla(\phi^\eta)^{2\iota}.
$$

Then it follows from $(3.128)_1$ and $(3.128)_4$ that

$$\begin{cases}\psi^*_t+\sum_{k=1}^3 A_k(u^\eta)\partial_k\psi^*+B^*(u^\eta)\psi^*=0,\\ \psi^*|_{t=0}=0 \quad \text{in} \quad \mathbb{R}^3,\\ \psi^*\to 0 \quad \text{as} \quad |x|\to\infty \quad \text{for} \quad \text{t}\geq 0,\end{cases} \tag{3.156}$$

which, together with the standard energy method for symmetric hyperbolic systems, implies that

$$\psi^*=0 \quad \text{for} \quad (t,x)\in[0,T_*]\times\mathbb{R}^3.$$

Thus (3.155) has been verified.

Note also that the following weak convergence holds true:

$$h^k\nabla^2u^k \rightharpoonup h^\eta\nabla^2u^\eta \ \text{ weakly* } \text{ in } \ L^\infty([0,T_*];L^2).$$

Indeed, (3.154) gives

$$\begin{aligned}&\int_0^{T_*}\int_{\mathbb{R}^3}(h^k\nabla^2u^k-h^\eta\nabla^2u^\eta)X\mathrm{d}x\mathrm{d}t\\ =&\int_0^{T_*}\int_{\mathbb{R}^3}\big((h^k-h^\eta)\nabla^2u^k+h^\eta(\nabla^2u^k-\nabla^2u^\eta)\big)X\mathrm{d}x\mathrm{d}t\\ \leq& C(\sup_{0\leq t\leq T_*}|h^k-h^\eta|_\infty+\|\nabla^2u^k-\nabla^2u^\eta\|_{L^\infty([0,T_*];L^2)})T_*\to 0 \ \text{ as } \ k\to\infty\end{aligned}$$

for any test function $X(t,x)\in C_c^\infty([0,T_*)\times\mathbb{R}^3)$, which implies that

$$h^k\nabla^2u^k \rightharpoonup h^\eta\nabla^2u^\eta \ \text{ weakly* } \text{ in } \ L^\infty([0,T_*];L^2).$$

Similarly, one can also obtain that

$$\begin{aligned}&\sqrt{h^k}(\nabla u^k,\nabla u^k_t)\rightharpoonup\sqrt{h^\eta}(\nabla u^\eta,\nabla u^\eta_t) \ \text{ weakly* } \text{ in } \ L^\infty([0,T_*];L^2),\\ &h^k\nabla^2u^k\rightharpoonup h^\eta\nabla^2u^\eta \ \text{ weakly* } \text{ in } \ L^\infty([0,T_*];D^1_*),\\ &h^k\nabla^2u^k\rightharpoonup h^\eta\nabla^2u^\eta \ \text{ weakly } \text{ in } \ L^2([0,T_*];D^1_*\cap D^2),\\ &(h^k\nabla^2u^k)_t\rightharpoonup(h^\eta\nabla^2u^\eta)_t \ \text{ weakly } \text{ in } \ L^2([0,T_*];L^2).\end{aligned} \tag{3.157}$$

Hence the corresponding weighted estimates for the velocity in (3.121) still hold for the limit. Thus, $(\phi^\eta,u^\eta,l^\eta,\psi^\eta)$ is a weak solution in the sense of distributions to the Cauchy problem (3.128).

Step 2: Uniqueness. Let (ϕ_1,u_1,l_1,ψ_1) and (ϕ_2,u_2,l_2,ψ_2) be two strong solutions to the Cauchy problem (3.128) satisfying the estimates in (3.121).

Set

$$\begin{aligned}h_i=&\phi_i^{2\iota}, \quad n_i=(ah_i)^b, \quad i=1,2; \quad \bar h=h_1-h_2,\\ \bar\phi=&\phi_1-\phi_2, \quad \bar u=u_1-u_2, \quad \bar l=l_1-l_2, \quad \bar\psi=\psi_1-\psi_2.\end{aligned}$$

Then it follows from the equations in (3.128) that

$$\begin{cases}\bar\phi_t+u_1\cdot\nabla\bar\phi+\bar u\cdot\nabla\phi_2+(\gamma-1)(\bar\phi\text{div}u_1+\phi_2\text{div}\bar u)=0,\\ \bar u_t+u_1\cdot\nabla\bar u+l_1\nabla\bar\phi+a_1\phi_1\nabla\bar l+a_2l_1^\nu h_1L\bar u\\ =-\bar u\cdot\nabla u_2-a_1\bar\phi\nabla l_2-\bar l\nabla\phi_2-a_2(l_1^\nu h_1-l_2^\nu h_2)Lu_2\\ +a_2(h_1\nabla l_1^\nu\cdot Q(u_1)-h_2\nabla l_2^\nu\cdot Q(u_2))\\ +a_3(l_1^\nu\psi_1\cdot Q(u_1)-l_2^\nu\psi_2\cdot Q(u_2)),\\ \bar l_t+u_1\cdot\nabla\bar l+\bar u\cdot\nabla l_2=a_4(l_1^\nu n_1\phi_1^{4\iota}H(u_1)-l_2^\nu n_2\phi_2^{4\iota}H(u_2)),\\ \bar h_t+u_1\cdot\nabla\bar h+\bar u\cdot\nabla h_2+(\delta-1)(\bar h\text{div}u_2+h_1\text{div}\bar u)=0,\\ \bar\psi_t+\sum_{k=1}^3A_k(u_1)\partial_k\bar\psi+B(u_1)\bar\psi+a\delta(\bar h\nabla\text{div}u_2+h_1\nabla\text{div}\bar u)\\ =-\sum_{l=k}^3A_k(\bar u)\partial_k\psi_2-B(\bar u)\psi_2,\\ (\bar\phi,\bar u,\bar l,\bar h,\bar\psi)|_{t=0}=(0,0,0,0,0)\quad\text{in}\quad\mathbb{R}^3,\\ (\bar\phi,\bar u,\bar l,\bar h,\bar\psi)\longrightarrow(0,0,0,0,0)\quad\text{as}\quad|x|\to\infty\quad\text{for}\quad \text{t}\geq0.\end{cases}\tag{3.158}$$

Set

$$\Phi(t)=\|\bar\phi\|_1^2+|\bar\psi|_2^2+|\nabla\bar l|_2^2+a_2\alpha|\sqrt{h_1}\nabla\bar u|_2^2+|l_1^{-\frac{\nu}{2}}\bar u|_2^2.$$

In a similar way for the derivation of (3.152), one can show that

$$\frac{d}{dt}\Phi(t)+C\big(|\nabla\bar u|_2^2+|l_1^{-\frac{\nu}{2}}\bar u_t|_2^2\big)\leq H(t)\Phi(t),\tag{3.159}$$

with a continuous function $H(t)$ satisfying

$$\int_0^t H(s)\ \mathrm{d}s\leq C\quad\text{for}\quad 0\leq t\leq T_*.$$

It follows from Gronwall's inequality that $\bar\phi=\bar l=0$ and $\bar\psi=\bar u=0$. Thus the uniqueness is obtained.

Step 3. The time-continuity follows easily from the same procedure as in Lemma 3.1.

Thus the proof of Theorem 3.1 is completed.

□

Remark 3.3. *It remains to prove Lemma* 3.11.

Proof. Define $X_R(x)=X(x/R)$, where $X(x)\in C_c^\infty(\mathbb{R}^3)$ is a truncation function satisfying

$$0\leq X(x)\leq1,\quad\text{and}\quad X(x)=\begin{cases}1 & \text{if}\ \ |x|\leq1,\\ 0 & \text{if}\ \ |x|\geq2.\end{cases}\tag{3.160}$$

Set $\bar{h}^{k+1,R}=\bar{h}^{k+1}X_R$. Then (3.134) yields

$$\begin{aligned}&\bar{h}^{k+1,R}_t+u^k\cdot\nabla\bar{h}^{k+1,R}+(\delta-1)(\bar{h}^{k,R}\mathrm{div}u^{k-1}+h^k\mathrm{div}\bar{u}^kX_R)\\=&u^k\bar{h}^{k+1}\cdot\nabla X_R-\frac{\delta-1}{a\delta}\bar{u}^k\cdot\psi^kX_R.\end{aligned}\tag{3.161}$$

Multiplying (3.161) by $2\bar{h}^{k+1,R}$ and integrating over $\mathbb{R}^3$, one can get

$$\begin{aligned}\frac{d}{dt}|\bar{h}^{k+1,R}|_2\leq&C|\nabla u^k|_\infty|\bar{h}^{k+1,R}|_2+C\big(|\bar{h}^k|_\infty|\mathrm{div}u^{k-1}|_2+|h^k|_\infty|\mathrm{div}\bar{u}^k|_2\big)\\&+C\big(|u^k|_2|\bar{h}^{k+1}|_\infty+|\bar{u}^k|_2|\psi^k|_\infty\big)\\\leq&\hat{C}|\bar{h}^{k+1,R}|_2+\hat{C},\end{aligned}\tag{3.162}$$

with $\hat{C}>0$ being a generic constant depending on η, but independent of R. Then Gronwall's inequality yields that

$$|\bar{h}^{k+1,R}(t)|_2\leq\hat{C}\exp{(\hat{C}\bar{T})}\quad\text{for}\quad(t,R)\in[0,\bar{T}]\times[0,\infty).$$

Hence, $\bar{h}^{k+1}\in L^\infty([0,\bar{T}];L^2(\mathbb{R}^3))$, which, along with $\bar{h}^{k+1}=h^{k+1}-h^k$ and

$$\frac{a\delta}{\delta-1}\nabla h^k=\psi^k\in L^\infty([0,\bar{T}];L^q\cap D^{1,3}\cap D^2),$$

implies that $\bar{h}^{k+1}\in L^\infty([0,\bar{T}];H^3(\mathbb{R}^3))$.

Similarly, one can show that $\bar{\phi}^{k+1},\ \bar{\psi}^{k+1}\in L^\infty([0,\bar{T}];H^2(\mathbb{R}^3))$.

□

3.5. **Limit from the non-vaccum flows to the flow with far field vacuum.** Based on the uniform estimates in (3.121), we are ready to prove Theorem 2.1.

Proof. **Step 1:** The locally uniform positivity of ϕ. For any $\eta\in(0,1)$, set

$$\phi^\eta_0=\phi_0+\eta,\quad\psi^\eta_0=\frac{a\delta}{\delta-1}\nabla(\phi_0+\eta)^{2\iota},\quad h^\eta_0=(\phi_0+\eta)^{2\iota}.$$

Then the corresponding initial compatibility conditions can be written as

$$\begin{aligned}&\nabla u_0=(\phi_0+\eta)^{-\iota}g^\eta_1,\quad Lu_0=(\phi_0+\eta)^{-2\iota}g^\eta_2,\\&\nabla((\phi_0+\eta)^{2\iota}Lu_0)=(\phi_0+\eta)^{-\iota}g^\eta_3,\quad\nabla^2l_0=(\phi_0+\eta)^{-\iota}g^\eta_4,\end{aligned}\tag{3.163}$$

where $g^\eta_i(i=1,2,3,4)$ are given as

$$\begin{cases}g^\eta_1=\dfrac{\phi_0^{-\iota}}{(\phi_0+\eta)^{-\iota}}g_1,\quad g^\eta_2=\dfrac{\phi_0^{-2\iota}}{(\phi_0+\eta)^{-2\iota}}g_2,\\g^\eta_3=\dfrac{\phi_0^{-3\iota}}{(\phi_0+\eta)^{-3\iota}}(g_3-\dfrac{\eta\nabla\phi_0^{2\iota}}{\phi_0+\eta}\phi_0^\iota Lu_0),\\g^\eta_4=\dfrac{\phi_0^{-\iota}}{(\phi_0+\eta)^{-\iota}}g_4.\end{cases}$$

It follows from (2.6)-(2.7) that there exists a $\eta_1>0$ such that if $0<\eta<\eta_1$, then

$$\begin{aligned}&1+\eta+\bar{l}+\|\phi^\eta_0-\eta\|_{D^1_*\cap D^3}+\|u_0\|_3+\|(h^\eta_0)^{-1}\|_{L^\infty\cap D^{1,q}\cap D^{2,3}\cap D^3}\\&+\|\psi^\eta_0\|_{L^q\cap D^{1,3}\cap D^2}+|\nabla(h^\eta_0)^{\frac12}|_6+|g^\eta_1|_2+|g^\eta_2|_2+|g^\eta_3|_2+|g^\eta_4|_2\\&+\|l_0-\bar{l}\|_{D^1_*\cap D^3}+|l_0^{-1}|_\infty\leq\bar{c}_0,\end{aligned}\tag{3.164}$$

where $\bar{c}_0$ is a positive constant independent of η. Therefore, it follows from Theorem 3.1 that for initial data $(\phi_0^\eta, u_0^\eta, l_0^\eta, \psi_0^\eta)$, the problem (3.128) admits a unique strong solution $(\phi^\eta, u^\eta, l^\eta, \psi^\eta)$ in $[0, T_*] \times \mathbb{R}^3$ satisfying the local estimate in (3.121) with c_0 replaced by $\bar{c}_0$, and the life span T_* is also independent of η.

Moreover, ϕ^η is uniform positive locally as shown below.

Lemma 3.12. *For any $R_0 > 0$ and $\eta \in (0, 1]$, there esists a constant a_{R_0} independent of η such that*

$$\phi^\eta(t, x) \geq a_{R_0} > 0, \quad \forall (t, x) \in [0, T_*] \times B_{R_0}. \tag{3.165}$$

Proof. It suffices to consider the case that R_0 is sufficiently large.

It follows from (2.6) and Gagliardo-Nirenberg inequality that $\nabla \phi_0^{2\iota} \in L^\infty$. This implies that the initial vacuum does not occur in the interior point but in the far field, and for every $R' > 2$, there exists a constant $C_{R'}$ such that

$$\phi_0^\eta(x) \geq C_{R'} + \eta > 0, \quad \forall\, x \in B_{R'}, \tag{3.166}$$

where $C_{R'}$ is independent of η.

Now, let $x(t; x_0)$ be the particle path starting from x_0 at $t = 0$, i.e.,

$$\begin{cases} \dfrac{d}{dt} x(t; x_0) = u^\eta(t, x(t; x_0)), \\ x(0; x_0) = x_0, \end{cases} \tag{3.167}$$

and $B(t, R')$ be the image of $B_{R'}$ under the flow map (3.167).

It follows from $(3.128)_1$ that

$$\phi^\eta(t, x) = \phi_0^\eta(x_0) \exp\Big(- \int_0^t (\gamma - 1) \mathrm{div} u^\eta(s; x(s; x_0)) \mathrm{d}s \Big). \tag{3.168}$$

It follows from (3.121) that for $0 \leq t \leq T_*$,

$$\int_0^t |\mathrm{div} u^\eta(t, x(t; x_0))| \mathrm{d}s \leq \int_0^t \|\nabla u^\eta\|_2 \mathrm{d}s \leq c_3 T_*^{\frac{1}{2}}. \tag{3.169}$$

This, together with (3.166) and (3.168), yields that for $0 \leq t \leq T_*$,

$$\phi^\eta(t, x) \geq C^*(C_{R'} + \eta) > 0, \quad \forall\, x \in B(t, R'), \tag{3.170}$$

where $C^* = \exp\big(- (\gamma - 1) c_3 T_*^{\frac{1}{2}} \big)$.

On the other hand, it follows from (3.167) and (3.119)-(3.121) that

$$|x_0 - x| = |x_0 - x(t; x_0)| \leq \int_0^t |u^\eta(s, x(s; x_0))| \mathrm{d}s \leq c_3 t \leq 1 \leq R'/2,$$

for all $(t, x) \in [0, T_*] \times B_{R'}$, which implies $B_{R'/2} \subset B(t, R')$. Thus, one can choose

$$R' = 2R_0, \quad \text{and} \quad a_{R_0} = C^* C_{R'}.$$

This Lemma 3.12 is proved. □

Step 2: Taking limit $\eta \to 0^+$. Due to the η-independent estimate (3.121), there exists a subsequence $(\phi^\eta, u^\eta, l^\eta, \psi^\eta)$ converging to a limit (ϕ, u, l, ψ) in weak or weak* sense:

$$\begin{aligned}
&\phi^\eta-\eta\rightharpoonup\phi \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];D^1_*\cap D^3),\\
&u^\eta\rightharpoonup u \ \text{ weakly } \ \text{ in } \ L^2([0,T_*];H^4),\\
&\psi^\eta\rightharpoonup\psi \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];L^q\cap D^{1,3}\cap D^2),\\
&\phi^\eta_t\rightharpoonup\phi_t \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];H^2),\\
&(u^\eta_t,\psi^\eta_t)\rightharpoonup(u_t,\psi_t) \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];H^1),\\
&l^\eta-\bar l\rightharpoonup l-\bar l \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];D^1_*\cap D^3),\\
&l^\eta_t\rightharpoonup l_t \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];L^\infty\cap D^1_*\cap D^2).
\end{aligned}\tag{3.171}$$

Then by the lower semi-continuity of weak convergences, (ϕ,u,l,ψ) satisfies the corresponding estimates as in (3.121) except weighted ones on u.

On the other hand, for any $R>0$, the Aubin-Lions Lemma and Lemma 3.12 imply that there exists a subsequence (still denote by $(\phi^\eta,u^\eta,l^\eta,\psi^\eta)$) satisfying

$$\begin{aligned}
&\phi^\eta-\eta\to\phi \ \text{ in } \ C([0,T_*];D^1_*(B_R)), \quad \psi^\eta\to\psi \ \text{ in } \ C([0,T_*];D^{1,3}(B_R)),\\
&u^\eta\to u \ \text{ in } \ C([0,T_*];H^2(B_R)), \quad l^\eta-\bar l\to l-\bar l \ \text{ in } \ C([0,T_*];D^1_*(B_R)),\\
&\qquad\qquad h^\eta\to h \ \text{ in } \ C([0,T_*];H^2(B_R)).
\end{aligned}\tag{3.172}$$

Also, one can verify that:

$$h=\phi^{2\iota},\quad \psi=\frac{a\delta}{\delta-1}\nabla h=\frac{a\delta}{\delta-1}\nabla\phi^{2\iota},\tag{3.173}$$

by the same argument used in the proof of (3.155).

Furthermore, one has

$$\begin{aligned}
&\int_0^{T_*}\int_{\mathbb{R}^3}(h^\eta\nabla^2u^\eta-h\nabla^2u)X\mathrm{d}x\mathrm{d}t\\
=&\int_0^{T_*}\int_{\mathbb{R}^3}\big((h^\eta-h)\nabla^2u^\eta+h(\nabla^2u^\eta-\nabla^2u)\big)X\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for any test function $X(t,x)\in C^\infty_c([0,T_*]\times\mathbb{R}^3)$. Due to (3.172) and Lemma 3.12, it holds that

$$h^\eta\nabla^2u^\eta\rightharpoonup h\nabla^2u \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];L^2).\tag{3.174}$$

Similarly, one can also get that

$$\begin{aligned}
\sqrt{h^\eta}(\nabla u^\eta,\nabla u^\eta_t)\rightharpoonup\sqrt{h}(\nabla u,\nabla u_t) \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];L^2),\\
h^\eta\nabla^2u^\eta\rightharpoonup h\nabla^2u \ \text{ weakly}^* \ \text{ in } \ L^\infty([0,T_*];D^1_*),\\
h^\eta\nabla^2u^\eta\rightharpoonup h\nabla^2u \ \text{ weakly } \ \text{ in } \ L^2([0,T_*];D^1_*\cap D^2),\\
(h^\eta\nabla^2u^\eta)_t\rightharpoonup(h\nabla^2u)_t \ \text{ weakly } \ \text{ in } \ L^2([0,T_*];L^2).
\end{aligned}\tag{3.175}$$

Hence, the corresponding weighted estimates for u in (3.121) still hold for the limit functions. Furthermore, (ϕ,u,l,ψ) is a weak solution to the Cauchy problem (2.2)-(2.5) in the sense of distributions.

Step 3. The uniqueness follows easily from the same procedure as that for Theorem 3.1.

Step 4: Time continuity. The time continuity of (ϕ,ψ,l) can be obtained by a similar argument as for Lemma 3.1.

For the velocity u, the a priori estimates obtained above and Sobolev embedding theorem imply that

$$u \in C([0,T_*];H^2)\cap C([0,T_*];\text{weak-}H^3) \quad \text{and} \quad \phi^\iota \nabla u \in C([0,T_*];L^2). \tag{3.176}$$

It then follows from $(2.2)_2$ that

$$\phi^{-2\iota}u_t \in L^2([0,T_*];H^2), \quad (\phi^{-2\iota}u_t)_t \in L^2([0,T_*];L^2),$$

which implies that $\phi^{-2\iota}u_t \in C([0,T_*];H^1)$. This and the classical elliptic estimates for

$$a_2 Lu = -l^{-\nu}\phi^{-2\iota}(u_t + u\cdot\nabla u + a_1\phi\nabla l + l\nabla\phi - a_2\phi^{2\iota}\nabla l^\nu \cdot Q(u) - a_3 l^\nu \psi\cdot Q(u))$$

show that $u \in C([0,T_*];H^3)$ immediately.

Finally, note that

$$h\nabla^2 u \in L^\infty([0,T_*];H^1)\cap L^2([0,T_*];D^2) \quad \text{and} \quad (h\nabla^2 u)_t \in L^2([0,T_*];L^2).$$

Thus the classical Sobolev embedding theorem implies that

$$h\nabla^2 u \in C([0,T_*];H^1).$$

Then the time continuity of u_t follows easily. We conclude that (2.8) holds.

In summary, (ϕ,u,l,ψ) is the unique strong solution in $[0,T_*]\times\mathbb{R}^3$ to the Cauchy problem (2.2)-(2.5).

Thus the proof of Theorem 2.1 is complete. □

3.6. **The proof for Theorem 1.1.** Now we are ready to establish the local-in-time well-posedness of regular solutions stated in Theorem 1.1 to the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11).

Proof. **Step 1.** It follows from the initial assumptions (1.18)-(1.19) and Theorem 2.1 that there exists a time $T_*>0$ such that the problem (2.2)-(2.5) has a unique strong solution (ϕ,u,l,ψ) satisfying the regularity (2.8), which implies that

$$\phi \in C^1([0,T_*]\times\mathbb{R}^3), \quad (u,\nabla u)\in C([0,T_*]\times\mathbb{R}^3), \quad l\in C^1([0,T_*]\times\mathbb{R}^3).$$

Set $\rho = (\frac{\gamma-1}{A\gamma}\phi)^{\frac{1}{\gamma-1}}$ with $\rho(0,x)=\rho_0$. According to the relations between (φ,ψ) and ϕ, one can obtain

$$\varphi = a\rho^{1-\delta}, \quad \psi = \frac{\delta}{\delta-1}\nabla\rho^{\delta-1}.$$

Then multiplying $(2.2)_1$ by $\frac{\partial\rho}{\partial\phi}$, $(2.2)_2$ by ρ, and $(2.2)_3$ by $Ac_v\rho^\gamma$ respectively shows that the equations in (1.8) are satisfied.

Hence, we have shown that the triple (ρ,u,S) satisfied the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11) in the sense of distributions and the regularities in Definition 1.1. Moreover, it follows from the continuity equation that $\rho(t,x)>0$ for $(t,x)\in[0,T_*]\times\mathbb{R}^3$. In summary, the Cauchy problem (1.8) with (1.2) and (1.9)-(1.11) has a unique regular solution (ρ,u,S).

Step 2. Now we will show that the regular solution obtained in the above step in fact is also a classical one within its life span.

First, according to the regularities of (ρ,u,S) and the fact

$$\rho(t,x)>0 \quad \text{for} \quad (t,x)\in[0,T_*]\times\mathbb{R}^3,$$

one can obtain that

$$(\rho, \nabla\rho, \rho_t, u, \nabla u, S, S_t, \nabla S) \in C([0, T_*] \times \mathbb{R}^3).$$

Second, by the classical Sobolev embedding theorem:

$$L^2([0, T_*]; H^1) \cap W^{1,2}([0, T_*]; H^{-1}) \hookrightarrow C([0, T_*]; L^2),$$

and the regularity (1.20), one gets that

$$tu_t \in C([0, T_*]; H^2), \quad \text{and} \quad u_t \in C([\tau, T_*] \times \mathbb{R}^3).$$

Finally, it remains to show that $\nabla^2 u \in C([\tau, T_*] \times \mathbb{R}^3)$. Note that the following elliptic system holds

$$\begin{aligned} a_2 Lu = & -l^{-\nu}\phi^{-2\iota}(u_t + u\cdot\nabla u + a_1\phi\nabla l + l\nabla\phi - a_2\phi^{2\iota}\nabla l^\nu \cdot Q(u) - a_3 l^\nu \psi \cdot Q(u)) \\ \equiv & \, l^{-\nu}\phi^{-2\iota}\mathbb{M}. \end{aligned}$$

It follows from the definition of regular solutions and (1.20) directly that

$$tl^{-\nu}\phi^{-2\iota}\mathbb{M} \in L^\infty([0, T_*]; H^2),$$

and

$$\begin{aligned} (tl^{-\nu}\phi^{-2\iota}\mathbb{M})_t = & l^{-\nu}\phi^{-2\iota}\mathbb{M} + t(l^{-\nu})_t\phi^{-2\iota}\mathbb{M} + tl^{-\nu}(\phi^{-2\iota})_t\mathbb{M} \\ & + tl^{-\nu}\phi^{-2\iota}\mathbb{M}_t \in L^2([0, T_*]; L^2), \end{aligned}$$

which, along with the classical Sobolev embedding theorem:

$$L^\infty([0, T_*]; H^2) \cap W^{1,2}([0, T_*]; H^{-1}) \hookrightarrow C([0, T_*]; L^r),$$

for any $r \in [2, 6)$, yields that

$$tl^{-\nu}\phi^{-2\iota}\mathbb{M} \in C([0, T_*]; W^{1,4}), \quad t\nabla^2 u \in C([0, T_*]; W^{1,4}).$$

These and the standard elliptic regularity yield immediately that $\nabla^2 u \in C((0, T_*] \times \mathbb{R}^3)$.

By the way, according to the relation (1.14), one also has that (ρ, u, S) is also a classical solution to the Cauchy problem (1.1)-(1.3) with $\kappa = 0$ and (1.10)-(1.11)) in $(0, T_*] \times \mathbb{R}^3$.

Step 3. We show finally that if one assumes $m(0) < \infty$ additionally, then (ρ, u, S) preserves the conservation of total mass, momentum and total energy within its life span. First, we show that (ρ, u, S) has finite total mass $m(t)$, momentum $\mathbb{P}(t)$ and total energy $E(t)$.

Lemma 3.13. *Under the additional assumption,* $0 < m(0) < \infty$*, it holds that*

$$m(t) + |\mathbb{P}(t)| + E(t) < \infty \quad \textit{for} \quad t \in [0, T_*].$$

Proof. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a non-increasing C^2 function satisfying

$$f(s) = \begin{cases} 1 & s \in [0, \frac{1}{2}], \\ \text{non-negative polynomial} & s \in [\frac{1}{2}, 1), \\ e^{-s} & s \geq 1. \end{cases}$$

It is obvious that there exists a generic constant $C > 0$ such that

$$|f'(s)| \leq Cf(s).$$

For any $R>1$, define $f_R(x)=f(\frac{|x|}{R})$. Then it holds that for any $p\geq 0$,

$$|x|^p f_R(x)\leq C \quad \text{and} \quad \lim_{|x|\to\infty}|x|^p f_R(x)=0. \tag{3.177}$$

According to the regularity of solutions obtained and the definition of f, one can make sure that,

$$\begin{aligned}&\int \big(\rho+|\mathrm{div}(\rho u)|\big) f_R\leq C(R),\\ &\int \Big(\big|\rho u\cdot x f'(\tfrac{|x|}{R})\tfrac{1}{R|x|}\big|+\rho|u|f(\tfrac{|x|}{R})\tfrac{1}{R}\Big)\leq C(R),\end{aligned}$$

for any fixed $R>1$, where $C(R)$ is a positive constant depending on R.

Since the continuity equation $(1.1)_1$ holds everywhere, one can multiply $(1.1)_1$ by $f_R(x)$ and integrate with respect to x to get

$$\frac{d}{dt}\int \rho f_R(x)=-\int \mathrm{div}(\rho u) f_R(x). \tag{3.178}$$

Then it follows from integration by parts that

$$-\int \mathrm{div}(\rho u) f_R(x)=\int \rho u\cdot x f'(\tfrac{|x|}{R})\tfrac{1}{R|x|}\leq C\frac{|u|_\infty}{R}\int \rho f_R(x),$$

which, along with (3.178) and Gronwall's inequality, shows that

$$\text{ess}\sup_{0\leq t\leq T}\int \rho f_R(x)\leq C\int \rho_0 f_R(x),$$

with C a generic constant independent of R. Note that

$$\rho f_R(x)\to\rho \quad \text{as} \quad R\to\infty$$

for all $x\in\mathbb{R}^3$, thus by Fatou's lemma (i.e., Lemma 5.6)

$$\text{ess}\sup_{0\leq t\leq T}\int \rho\leq \text{ess}\sup_{0\leq t\leq T}\liminf_{R\to\infty}\int \rho f_R(x)<\infty. \tag{3.179}$$

Second, based on (3.179), one has

$$\begin{aligned}|\mathbb{P}(t)|=&\Big|\int \rho u\Big|\leq C|\rho|_1|u|_\infty<\infty,\\ E(t)=&\int\Big(\frac{1}{2}\rho|u|^2+\frac{P}{\gamma-1}\Big)\\ \leq& C(|\rho|_1|u|^2_\infty+|\rho|^{\gamma-1}_\infty|\rho|_1|e^{\frac{S}{c_v}}|_\infty)<\infty.\end{aligned} \tag{3.180}$$

The proof of this lemma is complete. □

We are now ready to prove the conservation of total mass, momentum and total energy.

Lemma 3.14. *Under the additional assumption,* $0<m(0)<\infty$*, it holds that*

$$m(t)=m(0), \quad \mathbb{P}(t)=\mathbb{P}(0), \quad E(t)=E(0) \quad \textit{for} \quad t\in[0,T_*].$$

Proof. First, $(1.1)_2$ and the regularity of the solution imply that

$$\mathbb{P}_t = -\int \mathrm{div}(\rho u \otimes u) - \int \nabla P + \int \mathrm{div}\mathbb{T} = 0, \tag{3.181}$$

where one has used the fact that

$$\rho u^{(i)} u^{(j)}, \quad \rho^\gamma e^{\frac{S}{c_v}} \quad \text{and} \quad \rho^\delta e^{\frac{S}{c_v}\nu} \nabla u \in W^{1,1}(\mathbb{R}^3) \quad \text{for} \quad i,\ j = 1,\ 2,\ 3.$$

Second, the energy equation $(1.1)_3$ implies that

$$E_t = -\int \mathrm{div}(\rho \mathcal{E} u + Pu - u\mathbb{T}) = 0, \tag{3.182}$$

where the following facts have been used:

$$\frac{1}{2}\rho|u|^2 u, \quad \rho^\gamma e^{\frac{S}{c_v}} u \quad \text{and} \quad \rho^\delta e^{\frac{S}{c_v}\nu} u \nabla u \in W^{1,1}(\mathbb{R}^3).$$

Similarly, one can show the conservation of the total mass. □

Hence the proof of Theorem 1.1 is complete. □

4. Remarks on the asymptotic behavior of u

4.1. Non-existence of global solutions with L^∞ decay on u.

4.1.1. *Proof of Theorem 1.2.* Now we prove Theorem 1.2. Let $T > 0$ be any constant, and $(\rho, u, S) \in D(T)$. It follows from the definitions of $m(t)$, $\mathbb{P}(t)$ and $E_k(t)$ that

$$|\mathbb{P}(t)| \leq \int \rho(t,x)|u(t,x)| \leq \sqrt{2m(t)E_k(t)},$$

which, together with the definition of the solution class $D(T)$, implies that

$$0 < \frac{|\mathbb{P}(0)|^2}{2m(0)} \leq E_k(t) \leq \frac{1}{2} m(0) |u(t)|_\infty^2 \quad \text{for} \quad t \in [0,T].$$

Then one obtains that there exists a positive constant $C_u = \frac{|\mathbb{P}(0)|}{m(0)}$ such that

$$|u(t)|_\infty \geq C_u \quad \text{for} \quad t \in [0,T].$$

Thus one obtains the desired conclusion as shown in Theorem 1.2.

4.1.2. *Proof of Corollary 1.1.* Let $(\rho, u, S)(t,x)$ defined in $[0,T] \times \mathbb{R}^3$ be the regular solution obtained in Theorem 1.1. It follows from Theorem 1.1 that $(\rho, u, S) \in D(T)$, which, along with Theorem 1.2, yields that Corollary 1.1 holds.

4.2. Non-conservation of momentum for constant viscosities.

However, for flows of constant viscosities and thermal conductivity [7, 40], in Corollary 1.2, we will show that the classical solution exists globally and keeps the conservation of total mass, but can not keep the conservation of momentum for large time for a class of initial data with far field vacuum. This is essentially due to that $\mathbb{T}$ does not belong to $W^{1,1}(\mathbb{R}^3)$.

Set:

$$F \triangleq (2\mu + \lambda)\mathrm{div} u - P, \quad \omega \triangleq \nabla \times u,$$

to be the effective viscous flux and the vorticity respectively. Then F and ω satisfy the following elliptic equations:

$$\triangle F = \mathrm{div}(\rho \dot{u}) \quad \text{and} \quad \mu \triangle \omega = \nabla \times (\rho \dot{u}). \tag{4.1}$$

The proof of Corollary 1.2 is divided into three steps:

Step 1: Local-in-time well-posedness. It follows easily from the initial assumption (1.28)-(1.29) and the arguments used in Appendix B of [40] that there exists a time $T_0 > 0$ and a unique classical solution (ρ, u, θ) in $(0, T_0] \times \mathbb{R}^3$ to the Cauchy problem (1.1)-(1.3) with (1.26)-(1.27) satisfying (1.34)-(1.36), and

$$\rho(t,x) \geq 0, \quad \theta(t,x) \geq 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Step 2: Global-in-time well-posedness. It follows from the local-in-time well-posedenss obtained in Step 1, the smallness assumption (1.30) and the conclusions obtained in Theorems 2.13 and 2.17, Remark 2.14, Lemma 3.1 and Proposition 4.1 of [40] that the Cauchy problem (1.1)-(1.3) with (1.26)-(1.27) has a unique global classical solution (ρ, u, θ) in $(0, \infty) \times \mathbb{R}^3$ satisfying

$$m(t) = m(0) \quad \text{and} \quad 0 \leq E_k(t) \leq \frac{1}{2} m(0) |u(t,\cdot)|_\infty^2 < \infty \quad \text{for} \quad t \in [0,T], \tag{4.2}$$

for arbitrarily large $T > 0$, and (1.32)-(1.37) for any $0 < \tau < T < \infty$.

Step 3: Large time behavior on u. First, by the Gagliardo-Nirenberg inequality in Lemma 5.1 and the standard regularity theory for elliptic systems, one has

$$\begin{aligned} |u|_\infty \leq & C|u|_6^{\frac{1}{2}} |\nabla u|_6^{\frac{1}{2}} \leq C|\nabla u|_2^{\frac{1}{2}} (|F|_6^{\frac{1}{2}} + |\omega|_6^{\frac{1}{2}} + |P|_6^{\frac{1}{2}}) \\ \leq & C|\nabla u|_2^{\frac{1}{2}} (|\rho \dot{u}|_2 + |P|_6)^{\frac{1}{2}} \leq C|\nabla u|_2^{\frac{1}{2}} (|\sqrt{\rho} \dot{u}|_2 + |\nabla \theta|_2)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

which, together with (1.32)-(1.33) and (1.37), implies that

$$\limsup_{t \to \infty} |u(t,x)|_\infty = 0. \tag{4.4}$$

Finally, it follows from (4.2), (4.4), and the proof of Theorem 1.2 that if $m(0) > 0$ and $|\mathbb{P}(0)| > 0$, the law of conservation of momentum of the global solution obtained in Step 2 can not be preserved for all the time $t \in (0, \infty)$.

Thus the proof of Corollary 1.2 is complete.

5. Appendix

For convenience of readers, we list some basic facts which have been used frequently in this paper.

The first one is the well-known Gagliardo-Nirenberg inequality.

Lemma 5.1. [22] *Let function $u \in L^{q_1} \cap D^{1,r}(\mathbb{R}^d)$ for $1 \leq q_1, r \leq \infty$. Suppose also that real numbers ξ and q_2, and natural numbers m, i and j satisfy*

$$\frac{1}{q_2} = \frac{j}{d} + \left(\frac{1}{r} - \frac{i}{d} \right) \xi + \frac{1-\xi}{q_1} \quad \text{and} \quad \frac{j}{i} \leq \xi \leq 1.$$

Then $u \in D^{j,q_2}(\mathbb{R}^d)$, and there exists a constant C depending only on i, d, j, q_1, r and ξ such that

$$\|\nabla^j u\|_{L^{q_2}} \leq C \|\nabla^i u\|_{L^r}^{\xi} \|u\|_{L^{q_1}}^{1-\xi}. \tag{5.1}$$

Moreover, if $j = 0$, $ir < d$ and $q_1 = \infty$, then it is necessary to make the additional assumption that either u tends to zero at infinity or that u lies in $L^s(\mathbb{R}^d)$ for some *finite* $s > 0$*; if* $1 < r < \infty$ *and* $i - j - d/r$ *is a non-negative integer, then it is necessary to assume also that* $\xi \neq 1$*.*

The second one concerns commutator estimates, which can be found in [32].

Lemma 5.2. [32] *Let* r*,* r_1 *and* r_2 *be constants such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \textit{and} \quad 1 \leq r_1,\ r_2,\ r \leq \infty.$$

For $s \geq 1$*, if* $f, g \in W^{s,r_1} \cap W^{s,r_2}(\mathbb{R}^3)$*, then it holds that*

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s\big(|\nabla f|_{r_1}|\nabla^{s-1}g|_{r_2} + |\nabla^s f|_{r_2}|g|_{r_1}\big), \tag{5.2}$$

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s\big(|\nabla f|_{r_1}|\nabla^{s-1}g|_{r_2} + |\nabla^s f|_{r_1}|g|_{r_2}\big), \tag{5.3}$$

where $C_s > 0$ *is a constant depending only on* s*, and* $\nabla^s f$ *(*$s \geq 1$*) is the set of all* $\partial_x^\varsigma f$ *with* $|\varsigma| = s$*. Here* $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3)^\top \in \mathbb{R}^3$ *is a multi-index.*

The third lemma gives some compactness results obtained via the Aubin-Lions Lemma.

Lemma 5.3. [37] *Let* $X_0 \subset X \subset X_1$ *be three Banach spaces. Suppose that* X_0 *is compactly embedded in* X *and* X *is continuously embedded in* X_1*. Then the following statements hold.*

i) *If* J *is bounded in* $L^r([0,T]; X_0)$ *for* $1 \leq r < +\infty$*, and* $\frac{\partial J}{\partial t}$ *is bounded in* $L^1([0,T]; X_1)$*, then* J *is relatively compact in* $L^r([0,T]; X)$*;*

ii) *If* J *is bounded in* $L^\infty([0,T]; X_0)$ *and* $\frac{\partial J}{\partial t}$ *is bounded in* $L^r([0,T]; X_1)$ *for* $r > 1$*, then* J *is relatively compact in* $C([0,T]; X)$*.*

The following lemma is used to improve weak convergence to strong one.

Lemma 5.4. [32] *If the sequence* $\{w_k\}_{k=1}^\infty$ *converges weakly to* w *in a Hilbert space* X*, then it converges strongly to* w *in* X *if and only if*

$$\|w\|_X \geq \lim sup_{k\to\infty}\|w_k\|_X.$$

The following lemma is used to obtain the time-weighted estimates of u.

Lemma 5.5. [1] *If* $f(t,x) \in L^2([0,T]; L^2)$*, then there exists a sequence* s_k *such that*

$$s_k \to 0 \quad \textit{and} \quad s_k|f(s_k,x)|_2^2 \to 0 \quad \textit{as} \quad k \to \infty.$$

Finally, we list the well-known Fatou's lemma.

Lemma 5.6. *Given a measure space* $(V, \mathcal{F}, F)$ *and a set* $X \in \mathcal{F}$*, let* $\{f_k\}$ *be a sequence of* $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$*-measurable non-negative functions* $f_k : X \to [0,\infty]$*. Define a function* $f : X \to [0,\infty]$ *by*

$$f(x) = \liminf_{k\to\infty} f_k(x),$$

for every $x \in X$*. Then* f *is* $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$*-measurable, and*

$$\int_X f(x)\mathrm{d}F \leq \liminf_{k\to\infty} \int_X f_k(x)\mathrm{d}F.$$

Acknowledgement: The research is partially supported by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK-14301421, CUHK-14300917, CUHK-14302819, and CUHK-14300819. Duan's research is also supported in part by National Natural Science Foundation of China under Grant 11771300, Natural Science Foundation of Guandong under Grant 2020A1515010554 and Research Foundation for "Kongque" Talents of Shenzhen. Xin's research is also supported in part by the key project of National Natural Science Foundation of China (No. 12131010) and Guangdong Province Basic and Applied Basic Research Foundation 2020B1515310002. Zhu's research is also supported in part by National Natural Science Foundation of China under Grants 12101395 and 12161141004, The Royal Society-Newton International Fellowships NF170015, and Newton International Fellowships Alumni AL/201021 and AL/211005.

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] J. L. Boldrini, M. A. Rojas-Medar, and E. Fernández-Cara, Semi-Galerkin approximation and regular solutions to the equations of the nonhomogeneous asymmetric fluids, *J. Math. Pures Appl.* **82** (2003), 1499-1525.

[2] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous Shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.* **238** (2003), 211-223.

[3] D. Bresch, B. Desjardins, and C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and Shallow water systems, *Comm. Partial Differential Equations* **28** (2003), 843-868.

[4] D. Bresch, B. Desjardins, and G. Métivier, Recent mathematical results and open problems about Shallow water equations, *Analysis and Simulation of Fluid Dynamics*, 15–31, Adv. Math. Fluid Mech., Birkhäuser, Basel, 2007.

[5] D. Bresch, A. Vasseur, and C. Yu, Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density-dependent viscosities, to appear in J. Eur. Math. Soc., 2021, arXiv:1905.02701.

[6] S. Chapman and T. Cowling, *The Mathematical Theory of Non-Uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, Cambridge University Press, 1990.

[7] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, *J. Differential Equations*, **228** (2006), 377-411.

[8] M. Ding and S. Zhu, Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow with far field vacuum, *J. Math. Pures Appl.* **107** (2017), 288-314.

[9] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid, *Indiana Univ. Math. J.*, **53** (2004), 1705-1738.

[10] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford: Oxford University Press, 2004.

[11] G. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer, New York, 1994.

[12] Y. Geng, Y. Li, and S. Zhu, Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow with vacuum, *Arch. Ration. Mech. Anal.* **234** (2019), 727-775.

[13] D. Hoff, Discontinuous solutions of the Navier–Stokes equations for multidimensional flows of heat-conducting fluids, *Arch. Ration. Mech. Anal.* **139** (1997), 303-354.

[14] D. Hoff and H. Jenssen, Symmetric nonbarotropic flows with large data and forces, *Arch. Ration. Mech. Anal.* **173** (2004), 297-343.

[15] X. Huang and J. Li, Global classical and weak solutions to the three-dimensional full compressible Navier–Stokes system with vacuum and large oscillations, *Arch. Ration. Mech. Anal.* **227** (2018), 995-1059.
[16] N. Itaya, The existence and uniqueness of the solution of the equations describing compressible viscous fluid flow, *Proc. Japan Acad.* **46** (1970), 379-382.
[17] N. Itaya, On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid, *Kōdai Math. Sem. Rep.* **23** (1971), 60-120.
[18] S. Jiang, Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, *Comm. Math. Phys.* **178** (1996), 339-374.
[19] S. Kawashima, Systems of A Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics, Ph.D Thesis, Kyoto University, DOI: 10.14989/doctor.k3193, 1983.
[20] S. Kawashima and T. Nishida, The initial-value problems for the equations of viscous compressible and perfect compressible fluids, *RIMS, Kokyuroku 428, Kyoto University, Nonlinear Functional Analysis*, 1981, 34-59.
[21] P. E. Kloeden, Global existence of classical solutions in the dissipative shallow water equations. *SIAM J. Math. Anal.* **16** (1985), 301-315.
[22] O. Ladyzenskaja, V. Solonnikov, and N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.
[23] H.-L. Li, J. Li, and Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations, *Comm. Math. Phys.* **281** (2008), 401-444.
[24] H.-L. Li, Y. Wang, and Z. Xin, Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **232** (2019), 557-590.
[25] J. Li and Z. Xin, Entropy bounded solutions to the one-dimensional compressible Navier-Stokes equations with zero heat conduction and far field vacuum, *Adv. Math.* **361** (2020), 106923.
[26] J. Li and Z. Xin, Entropy bounded solutions to the one-dimensional heat conductive compressible Navier-Stokes equations with far field vacuum, to appear at *Comm. Pure Appl. Math.*, 2021, arXiv:2002.03372.
[27] J. Li and Z. Xin, Propagation of uniform boundedness of entropy and inhomogeneous regularities for viscous and heat conductive gases with far field vacuum in three dimensions, preprint, 2021, arXiv:2111.14057.
[28] J. Li and Z. Xin, Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities, preprint, 2015, arXiv:1504.06826.
[29] Tatsien Li and T. Qin, *Physics and Partial Differential Equations*, SIAM: Philadelphia, Higher Education Press: Beijing, 2014.
[30] Y. Li, R. Pan, and S. Zhu, On classical solutions to 2D Shallow water equations with degenerate viscosities, *J. Math. Fluid Mech.* **19** (2017), 151-190.
[31] Y. Li, R. Pan, and S. Zhu, On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum, *Arch. Ration. Mech. Anal.* **234** (2019), 1281-1334.
[32] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Applied Mathematical Science **53** Spinger-Verlag: New York, 1984.
[33] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* **20** (1980), 67-104.
[34] A. Mellet and A. Vasseur, On the barotropic compressible Navier-Stokes equations, *Comm. Partial Differential Equations* **32** (2007), 431-452.
[35] J. Nash, Le probleme de Cauchy pour les équations différentielles dún fluide général, *Bull. Soc. Math. France* **90** (1962), 487-491.
[36] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Ration. Mech. Anal.* **3** (1959), 271-288.
[37] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146** (1987), 65-96.
[38] L. Sundbye, Global existence for the Cauchy problem for the viscous Shallow water equations, *Rocky Mountain J. Math.* **28** (1998), 1135-1152.
[39] A. Tani, On the first initial-boundary value problem of compressible viscous fluid motion, *Publ. RIMS, Kyoto Univ.* **13** (1977), 193-253.

[40] H. Wen and C. Zhu, Global solutions to the three-dimensional full compressible Navier-Stokes equations with vacuum at infinity in some classes of large data, *SIAM J. Math. Anal.* **49** (2017), 162-221.

[41] Z. Xin, Blow-up of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure App. Math.* **51** (1998), 229-240.

[42] Z. Xin and S. Zhu, Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum, *Adv. Math.* **393** (2021), 108072.

[43] Z. Xin and S. Zhu, Well-posedness of three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and far field vacuum, *J. Math. Pures Appl.* **152** (2021), 94-144.

[44] Z. Xin and S. Zhu, Remarks on the existence of regular solutions with finite energy to the three-dimensional isentropic compressible Navier-Stokes equations, prepare.

[45] T. Zhang and D. Fang, Compressible flows with a density-dependent viscosity coefficient, *SIAM J. Math. Anal.* **41** (2010), 2453-2488.

(Q. Duan) College of Mathematics and Computational Science, Shenzhen University, Shenzhen, China; The Institute of Mathematical Sciences, Chinese University of Hong Kong, Hong Kong.

Email address: qduan@szu.edu.cn

(Z.P. Xin) The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong.

Email address: zpxin@ims.cuhk.edu.hk

(S. G. Zhu) School of Mathematical Sciences, CMA-Shanghai, and MOE-LSC, Shanghai Jiao Tong University, Shanghai, 200240, China; Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK.

Email address: zhushengguo@sjtu.edu.cn