

Global Well-posedness and Regularity of Weak Solutions to the Prandtl's System

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Abstract

We continue our study on the global solution to the two-dimensional Prandtl's system for unsteady boundary layers in the class considered by Oleinik provided that the pressure is favorable. First, by using a different method from [13], we gave a direct proof of existence of a global weak solution by a direct BV estimate. Then we prove the uniqueness and continuous dependence on data of such a weak solution to the initial-boundary value problem. Finally, we show the smoothness of the weak solutions and then the global existence of smooth solutions.

§1 Introduction

In this paper, we continue our study on the initial-boundary value problem for the two-dimensional unsteady Prandtl's system:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x P = \nu\partial_x^2 u, & 0 < x < L, & y > 0 \\ \partial_x u + \partial_y v = 0 \\ u|_{t=0} = u_0(x, y), & u|_{y=0} = 0 \\ v|_{y=0} = v_0(x, t), & u|_{x=0} = u_1(y, t) \\ u(x, y, t) \rightarrow U(x, t), & y \rightarrow +\infty \end{cases} \quad (1.1)$$

with ν being a fixed positive constant, and the pressure p determined by the Bernoulli's law:

$$\partial_t U + U\partial_x U + \partial_x P = 0. \quad (1.2)$$

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The physical situation described by problem (1.1) with (1.2) corresponds to a plane unsteady flow of viscous incompressible fluid in the presence of an arbitrary injection and removal of fluid across the boundaries. Thus, one may assume that

$$U(x, t) > 0, \quad u_0(x, t) > 0, \quad u_1(y, t) > 0, \quad \text{and} \quad v_0(x, t) \leq 0 \quad (1.3)$$

As in Oleinik [8, 9], we assume that the data are in the monotone class in the sense that

$$\partial_y u_0(x, y) > 0, \quad \partial_y u_1(t, y) > 0 \quad (1.4)$$

Under the further assumption that the pressure is favorable, i.e.,

$$\partial_x P(x, t) \leq 0 \quad \text{for} \quad t > 0, \quad 0 < x \leq L, \quad (1.5)$$

we have shown the existence of a global weak solution to the Cauchy problem (1.1) by a splitting method in [13]. The main purpose in this paper is to show that such a weak solution is in fact unique and depend continuously on the data. Furthermore, such a solution in fact is a classical solution in the sense that it is smooth in the interior.

To formulate the problem, as in [8], we introduce the following Crocco transformation:

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u(x, y, t)}{U(x, t)}, \quad w(\tau, \xi, \eta) = \frac{\partial_y u(x, y, t)}{U(x, t)}. \quad (1.6)$$

Then the initial-boundary value problem (1.1) is transformed into the following initial-boundary value problem:

$$\begin{cases} \partial_\tau w^{-1} + \eta U \partial_\xi w^{-1} + A \partial_\eta w^{-1} - B w^{-1} = -\nu \partial_\eta^2 w \\ \quad \text{on} \quad Q = \{(\xi, \eta, \tau), \quad 0 < \tau < \infty, \quad 0 < \xi < L, \quad 0 < \eta < 1\} \\ w|_{\tau=0} = \frac{\partial_y u_0}{U} \equiv w_0, \quad w|_{\eta=1} = 0 \\ w|_{\xi=0} = w_1 \equiv \frac{\partial_y u_1(y, \tau)}{U(0, \tau)}, \quad (\nu w \partial_\eta w - u_0 w)|_{\eta=0} = \frac{\partial_x P}{U}, \end{cases} \quad (1.7)$$

where

$$\begin{cases} A = (1 - \eta^2) \partial_x U + (1 - \eta) \frac{\partial_t U}{U}, \quad \text{and} \\ B = \eta \partial_x U + \frac{\partial_t U}{U}. \end{cases} \quad (1.8)$$

Set

$$Q_T = \{(\xi, \eta, \tau) \mid 0 < \tau < T, \quad 0 < \xi < L, \quad 0 < \eta < 1\}$$

A weak solution to the initial-boundary value problem (1.7) can be defined as follows:

Definition 1.1 A function $w \in BV(Q_T) \cap L^\infty(Q_T)$ is said to be a weak solution to the problem (1.7) if the following conditions are satisfied:

(i) There exists a positive constant $C = C(T)$ such that

$$C^{-1}(1 - \eta) \leq w(\tau, \xi, \eta) \leq C(1 - \eta), \quad \forall (\xi, \eta, \tau) \in Q_T, \quad \text{and} \quad (1.9)$$

$$(1 - \eta)^{\frac{1}{2}} \partial_\eta w \in L^2(Q_T), \quad (1.10)$$

(ii) $w_{\eta\eta}$ is a locally bounded measure in Q_T , and

$$\iint_{Q_T} (1 - \eta)^2 d|w_{\eta\eta}| < \infty, \quad (1.11)$$

where $d|w_{\eta\eta}|$ denotes the variation of $w_{\eta\eta}$.

(iii) The boundary conditions in (1.7) are satisfied in the sense of trace, i.e.,

$$\begin{cases} \gamma w(\xi, \eta, \tau)|_{\tau=0} = w_0(\xi, \eta) & \text{a.e. on } \bar{Q}_T \cap \{\tau = 0\}, \\ \gamma w(\xi, \eta, \tau)|_{\eta=1} = 0 & \text{a.e. on } \bar{Q}_T \cap \{\eta = 1\}, \\ \gamma w(\xi, \eta, \tau)|_{\xi=0} = w_1(\eta, \tau) & \text{a.e. on } \bar{Q}_T \cap \{\xi = 0\}, \\ \gamma w_\eta(\xi, \eta, \tau)|_{\eta=0} = w_0(\xi, \tau) + \frac{\partial_x P}{U} \frac{1}{\gamma u} \Big|_{\eta=0} & \text{a.e. on } \bar{Q}_T \cap \{\eta = 0\}, \end{cases} \quad (1.12)$$

where γw and γw_η are the traces of w and w_η on the corresponding boundaries respectively.

(iv) For any $\varphi \in C^1(\bar{Q}_T)$ with $\varphi|_{\tau=0} = \varphi|_{\xi=0} = \varphi|_{\xi=L} = 0$, the following identity holds:

$$\begin{aligned} & - \int_{\Omega} w^{-1} \varphi (1 - \eta)^2 d\xi d\eta |_{\tau=\tau} + \iint_{Q_T} w^{-1} (1 - \eta)^2 \partial_\tau \varphi d\xi d\eta d\tau \\ & + \nu \iint_{Q_T} ((1 - \eta)^2 \varphi)_\eta w_\eta d\xi d\eta d\tau + \iint_{Q_T} (\eta U \varphi)_\xi (1 - \eta)^2 w^{-1} d\xi d\eta d\tau \\ & + \iint_{Q_T} ((1 - \eta)^2 A \varphi)_\eta w^{-1} d\xi d\eta d\tau + \iint_{Q_T} (1 - \eta)^2 \varphi B w^{-1} d\xi d\eta d\tau \\ & + \int_0^\tau \int_0^L v_0(\xi, \tau) \varphi(\xi, \eta = 0, \tau) d\xi d\tau = 0 \quad \forall \tau \in (0, T). \end{aligned} \quad (1.13)$$

The main results in this paper can be summarized in the following theorem:

Theorem 1.2 Assume that the data satisfy condition (1.3) and (1.4), and that the pressure is favorable, i.e., (1.5) holds. Then,

- (i) There exists a weak solution $w \in BV(Q_T) \cap L^\infty(Q_T)$ in the sense of Definition 1.1;
- (ii) Such a weak solution is unique and depends continuously on the initial and boundary data in L^1 -norms;
- (iii) Such a weak solution is smooth in Q_T for any $T > 0$.

Remark: It should be remarked that the existence of a weak solution has been proved by the first two authors in [13], in this paper, we give a different proof by a direct BV estimate, which yields also some additional estimates (1.10) and (1.11) that are important for the structure of such weak solutions. The key new ingredients of the current paper are the uniqueness and the regularity of a weak solution. Some new ideas are required due to the strong degeneracy of the equation in (1.7). Moreover, the well-posedness we proved in the BV class in Proposition 3.5, in particular the estimate (3.35) is of independent interests and optimal, and the Poincaré type inequality we introduced for the proof of Hölder regularity of weak solutions is also of independent interests.

Finally, we mention that there have been increasing activities in recent years on the studies of the unsteady Prandtl's system in both two and three dimensions with substantial results for general initial data, ill-posedness, instability and different methods, we refer to [22-32] and references therein for these further developments.

§2 Existence

In this section, we will give an alternative proof of existence of weak solutions to the initial-boundary value problem (1.7) by a vanishing viscosity method, which is somewhat simpler than the approach given by the first two authors in [13].

For the convenience of presentation, we may rewrite the problem (1.7) as (we use notations independent of those in §1).

$$\begin{cases} \partial_t u - u^2 \partial_y^2 u + a \partial_x u + b \partial_y u + cu = 0 \\ u(x, y, t)|_{t=0} = u_0(x, y) \\ u(x, y, t)|_{x=0} = u_1(y, t) \\ u(x, y, t)|_{y=1} = 0, \quad u \partial_y u|_{y=0} = \left(v_0 u + \frac{\partial_x P}{U} \right) \Big|_{y=0} \end{cases} \quad (2.1)$$

with

$$\begin{cases} a(x, y, t) = yU(x, t), \quad b(x, y, t) = A = (1 - y^2) \partial_x U(x, t) + (1 - y) \frac{\partial_t U}{U}, \\ c = c(x, y, t) = B = (1 - y) \partial_x U(x, t) - \frac{\partial_x P}{U}. \end{cases} \quad (2.2)$$

We also use the notations

$$\begin{aligned} Q_T &= \{(x, y, t) | 0 < x < L, \quad 0 < y < 1, \quad 0 < t < T\}, \\ \Omega &= \{(x, y) | 0 < x < L, \quad 0 < y < 1\}. \end{aligned}$$

Then our main results in this section are:

Theorem 2.1 Assume that the pressure is favorable, i.e., (1.9) holds, and there exists a positive constant C_0 such that for all $0 < x < L$, $0 < y < 1$, $0 < t < T$, the initial and boundary condition

$$C_0^{-1}(1 - y) < u_0(x, y) < C_0(1 - y), \quad C_0^{-1}(1 - y) < u_1(y, t) < C_0(1 - y). \quad (2.3)$$

Then there exists a weak solution to the initial-boundary value problem (2.1) in the sense given in Definition 1.1, i.e., there exists a $u \in BV(Q_T) \cap L^\infty(Q_T)$ with the properties:

(i) It holds that

$$(1 - y)^{\frac{\alpha-1}{2}} \partial_y u \in L^2(Q_T), \quad \text{and} \quad (2.4)$$

$$C_1^{-1}(1 - y) \leq u(x, y, t) \leq C_1(1 - y), \quad (x, y, t) \in Q_T \quad (2.5)$$

for some positive constants $C_1 = C_1(T, \Omega)$ and $\alpha > 0$.

(ii) u_{yy} is a locally bounded measure in Q_T in the sense that

$$\iint_{Q_T} (1 - y)^\alpha d|u_{yy}| < +\infty, \quad (2.6)$$

where $d|u_{yy}|$ denotes the variation of u_{yy} , and $\alpha > 0$ is a constant.

(iii) And

$$\begin{cases} \gamma u(x, y, t)|_{t=0} = u_0(x, y) & \text{a.e. on } \Omega, \\ \gamma u(x, y, t)|_{y=1} = 0 & \text{a.e. on } \bar{Q}_T \cap \{y = 1\}, \\ \gamma u(x, y, t)|_{x=0} = u_1(y, t) & \text{a.e. on } \bar{Q}_T \cap \{x = 0\}, \\ \gamma u_y(x, y, t)|_{y=0} = \left(v_0 + \frac{\partial_x P}{U} \frac{1}{\gamma u} \right) \Big|_{y=0} & \text{a.e. on } \bar{Q}_T \cap \{y = 0\}, \end{cases} \quad (2.7)$$

where γu and γu_y are the traces of u and $\partial_y u$ on the corresponding boundaries respectively.

(iv) For any $\varphi \in C^1(\bar{Q}_T)$ with $\varphi|_{t=0} = \varphi|_{x=0} = \varphi|_{x=L} = 0$, the following identity holds:

$$\begin{aligned} & - \int_{\Omega} u^{-1} \varphi (1-y)^2 dx dy \Big|_{t=t} + \iint_{Q_T} [u^{-1} (1-y)^2 \partial_t \varphi + ((1-y)^2 \varphi)_y u_y] dx dy dt \\ & + \iint_{Q_T} [(a\varphi)_x (1-y)^2 \frac{1}{u} + ((1-y)^2 b\varphi)_y \frac{1}{u} + (1-y)^2 \varphi c \frac{1}{u}] dx dy dt \\ & + \int_0^t \int_0^L v_0(x, t) \varphi(x, y, t) dx dt \Big|_{y=0} = 0 \quad \text{for any } t \in (0, T). \end{aligned} \quad (2.8)$$

We will prove the above theorem by studying the following regularized problem:

$$\begin{cases} \partial_t u - (u + \varepsilon)^2 \partial_y^2 u + (a + \varepsilon) \partial_x u + b \partial_y u + cu = 0 \\ u|_{t=0} = u_0^\varepsilon, \quad u|_{x=0} = u_1^\varepsilon \\ u^\varepsilon|_{y=1} = 0, \quad (u + \varepsilon) \partial_y u|_{y=0} = \left(v_0^\varepsilon (u + \varepsilon) + \frac{\partial_x P}{U} \right) \Big|_{y=0} \end{cases} \quad (2.9)$$

where u_0^ε , u_1^ε , and v_0^ε are some suitable regularization of u_0 , u_1 , and v_0 respectively, and $\varepsilon \in (0, 1)$ is a constant.

It follows from the standard theory for initial-boundary value problems for ultra-parabolic equations that for each fixed $\varepsilon > 0$, there exists a unique classical solution u^ε to the problem (2.9). Our main strategy is to pass the limit as $\varepsilon \rightarrow 0^+$ in (2.9) to obtain a solution to the problem (2.1). To this end, some a priori estimates are needed. We start with the super-norm estimates.

Lemma 2.2 Assume that (1.5) holds and that there exists a positive constant C_0 such that the initial and boundary data

$$C_0^{-1}(1-y) \leq u_i^\varepsilon \leq C_0(1-y), \quad \text{on } Q_T, \quad i = 0, 1. \quad (2.10)$$

Then there exists a positive constant $C_1 = C_1(Q_T)$ such that the solution

$$C_1^{-1}(1-y) \leq u^\varepsilon(x, y, t) \leq C_1(1-y), \quad \forall (x, y, t) \in Q_T. \quad (2.11)$$

Proof: First, it follows from the assumptions (1.3), (1.5), (2.10), and the standard maximum principle argument that

$$u^\varepsilon(x, y, t) \geq 0 \quad \text{on } Q_T. \quad (2.12)$$

Next, we estimate the upper bound on u^ε . For any given $\delta > 0$, consider

$$\theta(x, y, t) = \frac{u^\varepsilon}{1-y+\delta}$$

Then $\theta(x, y, t)$ solves the following initial-boundary value problem:

$$\begin{cases} \partial_t \theta - (u^\varepsilon + \varepsilon)\theta_{yy} + (a + \varepsilon)\partial_x \theta + \left(b - \frac{2(u^\varepsilon + \varepsilon)^2}{1 - y + \delta}\right)\theta_y + \left(c - \frac{b}{1 - y + \delta}\right)\theta = 0 \\ \theta|_{t=0} = \frac{u_0^\varepsilon}{1 - y + \delta}, \quad \theta|_{x=0} = \frac{u_1^\varepsilon}{1 - y + \delta}, \quad \theta|_{y=1} = 0 \\ (u^\varepsilon + \varepsilon) \left[\partial_y \theta - \frac{\theta}{1 - y + \delta} \right] \Big|_{y=0} = \left[v_0 \frac{u^\varepsilon + \varepsilon}{1 - y + \delta} + \frac{\partial_x P}{(1 - y + \delta)U} \right] \Big|_{y=0} \end{cases} \quad (2.13)$$

Due to the specific structure of b and c ((2.2)), the coefficient $c(x, y, t) - \frac{b(x, y, t)}{1 - y + \delta}$ admits a uniform bounded,

$$\left| c(x, y, t) - \frac{b(x, y, t)}{1 - y + \delta} \right| \leq \tilde{c}_2$$

where \tilde{c}_2 is independent of δ and ε . Thus, standard maximum principle argument leads to

$$0 \leq \theta(x, y, t) \leq e^{\tilde{c}_2 T} \max\{\theta|_{t=0}, \theta|_{x=0}, \theta|_{y=0}, \theta|_{y=1}\}. \quad (2.14)$$

Since $\theta|_{t=0} \leq C_0$, $\theta|_{x=0} \leq C_0$, and $\theta|_{y=1} = 0$, it suffices to estimate the $\max_{y=0} \theta(x, y, t)$. At the maximum, $\partial_y \theta \leq 0$, so the boundary condition at $y = 0$ yields

$$\frac{u^\varepsilon + \varepsilon}{1 + \delta} \theta \leq |v_0| \frac{u^\varepsilon + \varepsilon}{1 + \delta} + \frac{|\partial_x P|}{(1 + \delta)|U|},$$

which implies

$$\theta^2 \leq |v_0|(1 + \theta) + \frac{|\partial_x P|}{|U|}.$$

Consequently,

$$\max_{y=0} \theta(x, y, t) \leq \tilde{c}_3$$

where \tilde{c}_3 is a positive constant independent of ε and δ . Thus (2.14) shows that

$$0 \leq \theta(x, y, t) \leq e^{\tilde{c}_2 T} \max(c_0, \tilde{c}_3) = \tilde{c}_4,$$

and so

$$0 \leq u^\varepsilon(x, y, t) \leq \tilde{c}_4(1 - y + \delta).$$

Since δ is arbitrary, the desired upper bound in (2.11) follows. Finally, we obtain the lower bound estimate. Set

$$\begin{cases} \theta(x, y, t) = e^{\alpha t} u^\varepsilon - \mu_0 \varphi(y) e^{-\beta t}, & \text{with} \\ \varphi(y) = e^{-\gamma(1-y)^2} (1 - y) \end{cases} \quad (2.15)$$

where α, β, γ are large and μ_0 is small to be determined. Define also operator

$$\mathcal{L} = \partial_t - (u^\varepsilon + \varepsilon)^2 \partial_y^2 + (a + \varepsilon)\partial_x + b\partial_y + c.$$

Then direct calculations yield

$$\begin{aligned} \mathcal{L}\theta &= \alpha\theta + \mu_0 \varphi(y) e^{-\beta t} \left[\alpha + \beta - (u^\varepsilon + \varepsilon)(6\gamma + 4\gamma(1 - y)^2) - b \frac{(2\gamma(y-1)^2 - 1)}{(1-y)} + c \right], \\ \theta|_{t=0} &= u_0^\varepsilon - \mu_0 \varphi(y), \quad \theta|_{y=1} = 0, \\ \theta|_{x=0} &= u_1^\varepsilon e^{\alpha t} - \mu_0 \varphi(y) e^{-\beta t} = e^{\alpha t} [u_1^\varepsilon - \mu_0 e^{-(\alpha+\beta)t} \varphi(y)], \\ \partial_y \theta|_{y=0} &= e^{\alpha t} \left(v_0 + \frac{\partial_x P}{(u^\varepsilon + \varepsilon)U} \right) \Big|_{y=0} - \mu_0 e^{-\gamma} (2\gamma - 1) e^{-\beta t}. \end{aligned}$$

We now fix γ so that $\gamma > \frac{1}{2}$. Thus, $\partial_y \theta|_{y=0} < 0$. Then one can choose μ_0 so small such that $\theta|_{t=0} \geq 0$, and

$$\theta|_{x=0} \geq e^{\alpha t} [u_1^\varepsilon - \mu_0 \phi(y)] \geq 0$$

by the assumption (2.10).

Due to the upper bound estimate on u^ε , one can choose α and β big enough so that

$$\alpha + \beta - (u^\varepsilon + \varepsilon)(6\gamma + 4\gamma(1-y)^2) - b \frac{(2\gamma(y-1)^2 - 1)}{(1-y)} + c > 0.$$

Hence

$$\mathcal{L}\theta - \alpha\theta > 0 \quad \text{in} \quad Q_T.$$

It follows by the maximum principle that $\theta \geq 0$ on Q_T . As a consequence, we have shown that

$$u^\varepsilon(x, y, t) \geq C_1^{-1}(1-y)$$

where $C_1 = C_1(Q_T)$ is a positive constant independent of ε . The proof of Lemma 2.2 is completed. \square

Next, we derive the uniform total variation estimate on the approximation solutions. For any given scalar function ψ , we set

$$\nabla\psi = \text{grad } \psi = (\partial_x\psi, \partial_y\psi, \partial_t\psi)^t, \quad \text{and} \quad (2.16)$$

$$\tilde{\nabla}\psi = (\partial_x\psi, \partial_t\psi)^t. \quad (2.17)$$

Then, our next key uniform estimate is the following total variation estimate on u^ε .

Proposition 2.3 There exists a positive constant $C_2 = C_2(Q_T)$ independent of ε , such that

$$\int_{Q_T} |\nabla u^\varepsilon(x, y, t)| dx dy dt \leq C_2. \quad (2.18)$$

This proposition will follow from the following two lemmas directly.

Lemma 2.4 For any given constant $\alpha > -1$, there exists a positive constant $C_3 = C_3(\alpha)$ such that

$$\int_{Q_T} (1-y)^\alpha |\partial_y u^\varepsilon| dx dy dt \leq C_3, \quad \text{and} \quad (2.19)$$

$$\int_{Q_T} (1-y)^\alpha |\partial_y u^\varepsilon|^2 dx dy dt \leq C_3. \quad (2.20)$$

Proof: Note that (2.19) follows immediately from (2.20) by the Hölder inequality. So it suffices to show that (2.20) holds. Setting $g(u) = \ln(u + \varepsilon) + \varepsilon(u + \varepsilon)^{-1}$, multiplying the equation in

(2.9) by $(1-y)^\alpha g'(u^\varepsilon)$, and integrating the resulting equation over Q_T , one can obtain after integration by parts that

$$\begin{aligned}
& \int_{\Omega} g(u^\varepsilon)(1-y)^\alpha \Big|_0^T dx dy \\
= & - \int_{Q_T} (1-y)^\alpha (\partial_y u^\varepsilon)^2 dx dy dt + \alpha \int_{Q_T} (1-y)^{\alpha-1} u^\varepsilon \partial_y u^\varepsilon dx dy dt \\
& + \int_{Q_T} g(u^\varepsilon) \partial_y ((1-y)^\alpha b) dx dy dt + \int_{Q_T} (1-y)^\alpha \partial_x a g(u^\varepsilon) dx dy dt \\
& - \int_{Q_T} c(1-y)^\alpha \frac{(u^\varepsilon)^2}{(u^\varepsilon + \varepsilon)^2} dx dy dt + \int_0^T \int_0^L (1-y)^\alpha u^\varepsilon \partial_y u^\varepsilon dx dt \Big|_0^1 \\
& - \int_0^T \int_0^L (1-y)^\alpha b g(u^\varepsilon) dx dt \Big|_0^1 - \int_0^T \int_0^1 (1-y)^\alpha (a + \varepsilon) g(u^\varepsilon) dy dt \Big|_0^L
\end{aligned} \tag{2.21}$$

First, it holds that

$$\begin{aligned}
& \left| \alpha \int_{Q_T} (1-y)^{\alpha-1} u^\varepsilon \partial_y u^\varepsilon dx dy dt \right| \\
\leq & \frac{1}{2} \int_{Q_T} (1-y)^\alpha (\partial_y u^\varepsilon)^2 dx dy dt + \frac{\alpha^2}{2} \int_{Q_T} (1-y)^{\alpha-2} (u^\varepsilon)^2 dx dy dt
\end{aligned} \tag{2.22}$$

Next, due to Lemma 2.2, one has that

$$|(1-y)^\alpha g(u^\varepsilon)| \leq C \{(1-y)^\alpha (|lu(1-y)| + (1-y)^2 + 1)\} \quad \text{for } (x, y, t) \in Q_T \tag{2.23}$$

with a positive constant $C = C(Q_T)$. As a consequence, one can derive that

$$\begin{aligned}
& \left| \int_{\Omega} g(u^\varepsilon)(1-y)^\alpha dx dy \Big|_0^T \right| + \left| \int_{Q_T} g(u^\varepsilon) \partial_y (b(1-y)^\alpha) dx dy dt \right| \\
+ & \left| \int_{Q_T} (1-y)^\alpha \partial_x a g(u^\varepsilon) dx dy dt \right| + \left| \int_{Q_T} c(1-y)^\alpha \frac{(u^\varepsilon)^2}{(u^\varepsilon + \varepsilon)^2} dx dy dt \right| \\
+ & \left| \int_0^T \int_0^1 (1-y)^\alpha (a + \varepsilon) g(u^\varepsilon) dy dt \Big|_0^L \right| \leq C
\end{aligned} \tag{2.24}$$

with $C = C(Q_T)$ independent of ε , where we have used the fact that $|b| \leq C(1-y)$. Finally, it follows from the boundary conditions in (2.9) and the structures of b that

$$\begin{aligned}
& \left| \int_0^T \int_0^L (1-y)^\alpha u^\varepsilon \partial_y u^\varepsilon dx dt \Big|_0^1 - \int_0^T \int_0^L (1-y)^\alpha b g(u^\varepsilon) dx dt \Big|_0^1 \right| \\
= & \left| \int_0^T \int_0^L \left(\frac{u^\varepsilon}{\varepsilon + u^\varepsilon} (u^\varepsilon + \varepsilon) \partial_y u^\varepsilon \right) \Big|_0 dx dt - \int_0^T \int_0^L (b g(u^\varepsilon)) \Big|_0 dx dt \right| \\
\leq & \int_0^T \int_0^L \left(\frac{u^\varepsilon}{u^\varepsilon + \varepsilon} |v_0^\varepsilon(u^\varepsilon + \varepsilon)| + \frac{|\partial_x P|}{U} \right) (x, 0, t) dx dt + \tilde{C} \\
\leq & C
\end{aligned} \tag{2.25}$$

with positive constants C and \tilde{C} independent of ε . Now, the desired estimate (2.20) follows from (2.21) - (2.22), and (2.24) - (2.25). This completes the proof of Lemma 2.4. \square

Next, we estimate $\tilde{\nabla} u^\varepsilon$. In fact, we have

Lemma 2.5 There exists a positive constant $C_4 = C_4(\Omega_T)$ independent of ε such that

$$\sup_{0 \leq t \leq T} \int_0^L \int_0^1 |\tilde{\nabla} u^\varepsilon(x, y, t)| dx dy \leq C_4. \quad (2.26)$$

Proof: For any $\delta > 0$, set

$$S_\delta(\theta) = \begin{cases} 1, & \theta > \delta, \\ \frac{\theta}{\delta}, & |\theta| \leq \delta, \\ -1, & \theta < -\delta, \end{cases} \quad (2.27)$$

and for any $\xi \in \mathbb{R}^2$,

$$I_\delta(\xi) = \int_0^{|\xi|} S_\delta(s) ds. \quad (2.28)$$

Then it can be checked easily that

$$\nabla^2 I_\delta(\xi) \geq 0, \quad \partial_{\xi_i} I_\delta(\xi) = S_\delta(|\xi|) \frac{\xi_i}{|\xi|}, \quad (2.29)$$

and

$$\lim_{\delta \rightarrow 0^+} \xi \cdot \nabla^2 I_\delta(\xi) = 0, \quad \text{except } |\xi| = 0. \quad (2.30)$$

To show (2.26), it is more convenient to rewrite the equation in (2.9) in terms of new dependent variable

$$V = \frac{1}{u^\varepsilon + \varepsilon} \quad (2.31)$$

which satisfies

$$\partial_t V - \partial_y \left(\frac{V_y}{V^2} \right) + (a + \varepsilon) \partial_x V + b \partial_y V - \frac{C u^\varepsilon}{u^\varepsilon + \varepsilon} V = 0.$$

Thus,

$$\partial_t \tilde{\nabla} V - \partial_y^2 \left(\frac{\tilde{\nabla} V}{V^2} \right) + \tilde{\nabla}((a + \varepsilon)V_x) + \tilde{\nabla}(bV_y) - \tilde{\nabla} \left(\frac{C u^\varepsilon}{u^\varepsilon + \varepsilon} V \right) = 0. \quad (2.32)$$

Taking inner product of (2.32) with $(1 + \varepsilon - y)^2 S_\delta(\tilde{\nabla} V) \frac{\tilde{\nabla} V}{|\tilde{\nabla} V|}$ and integrating the resulting equation over Ω , one can get after integration by parts that

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 dx dy \\
= & \int_{\Omega} [\partial_y^2 \left(\frac{\tilde{\nabla}V}{V^2} \right) \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 - \tilde{\nabla}((a+\varepsilon)V_x) \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2] dx dy \\
& - \int_{\Omega} [\tilde{\nabla}(bV_y) \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 + \tilde{\nabla} \left(\frac{Cu^{\varepsilon}}{u^{\varepsilon}+\varepsilon} V \right) \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2] dx dy \\
= & \left\{ \int_0^L \left[\partial_y \left(\frac{\tilde{\nabla}V}{V^2} \right) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 - b(1+\varepsilon-y)^2 \tilde{\nabla}V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) \right. \right. \\
& \quad \left. \left. + \frac{\tilde{\nabla}V}{V^2} \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) 2(1+\varepsilon-y) \right] \Big|_0^1 dx \right\} - \left\{ \int_0^1 (a+\varepsilon) I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 dy \Big|_0^L \right\} \\
& + \left\{ - \int_{\Omega} \frac{\tilde{\nabla}V_y}{V^2} \nabla_{\tilde{\nabla}V}^2 I_{\delta}(\tilde{\nabla}V)(\tilde{\nabla}V_y)(1+\varepsilon-y)^2 dx dy \right\} \\
& + \left\{ \int_{\Omega} \left[2 \left(\frac{\tilde{\nabla}V}{V^2} \right) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) - V_x(\tilde{\nabla}a) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 + \partial_x a I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 \right. \right. \\
& \quad \left. \left. + \partial_y(b(1+\varepsilon-y)^2) \tilde{\nabla}V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) + \frac{Cu^{\varepsilon}}{u^{\varepsilon}+\varepsilon} \tilde{\nabla}V \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 \right. \right. \\
& \quad \left. \left. + \varepsilon(-C)V \tilde{\nabla}V \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 \right] dx dy \right\} \\
& + \left\{ \int_{\Omega} \left[2 \frac{\partial_y V}{V^3} \tilde{\nabla}V \cdot \nabla_{\tilde{\nabla}V}^2 I_{\delta}(\tilde{\nabla}V)(\tilde{\nabla}V_y)(1+\varepsilon-y)^2 - 2 \frac{\tilde{\nabla}V}{V^2} \nabla_{\tilde{\nabla}V}^2 I_{\delta}(\tilde{\nabla}V)(\tilde{\nabla}V_y)(1+\varepsilon-y) \right. \right. \\
& \quad \left. \left. + b(1+\varepsilon-y)^2 \tilde{\nabla}V \cdot \nabla_{\tilde{\nabla}V}^2 I_{\delta}(\tilde{\nabla}V)(\tilde{\nabla}V_y) \right] dx dy \right\} \\
& + \left\{ \int_{\Omega} \tilde{\nabla}C \frac{u^{\varepsilon}}{u^{\varepsilon}+\varepsilon} V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 dx dy \right\} \\
& + \left\{ - \int_{\Omega} (\tilde{\nabla}b)V_y \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 dx dy \right\} \\
\equiv & \sum_{i=1}^7 I_i.
\end{aligned} \tag{2.33}$$

We now estimate each I_i , ($1 \leq i \leq 7$) respectively as follows:

First, due to the facts that $\tilde{\nabla}V|_{y=1} = 0$ and ξ and $\partial_{\xi} I_{\delta} \geq 0$, one has

$$\begin{aligned}
I_1 & \equiv \int_0^L \left[\partial_y \left(\frac{\tilde{\nabla}V}{V^2} \right) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon-y)^2 - b(1+\varepsilon-y)^2 \tilde{\nabla}V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) \right. \\
& \quad \left. + \frac{\tilde{\nabla}V}{V^2} \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) 2(1+\varepsilon-y) \right] \Big|_0^1 dx \\
& = - \int_0^L \left[\partial_y \left(\frac{\tilde{\nabla}V}{V^2} \right) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V)(1+\varepsilon)^2 - b(1+\varepsilon)^2 \tilde{\nabla}V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) \right. \\
& \quad \left. + \frac{\tilde{\nabla}V}{V^2} \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) 2(1+\varepsilon) \right] (x, 0, t) dx dt \\
& \leq \int_0^L (1+\varepsilon)^2 \left[\partial_y \left(\frac{\tilde{\nabla}V}{V^2} \right) \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) - b \tilde{\nabla}V \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) \right] (x, 0, t) dx \\
& = +(1+\varepsilon)^2 \int_0^L \left[\left(\tilde{\nabla}V_0 + \tilde{\nabla} \left(\frac{\partial_x P}{U} \right) V \right) \cdot \partial_{\tilde{\nabla}V} I_{\delta}(\tilde{\nabla}V) \right] (x, 0, t) dx \\
& \leq \tilde{C},
\end{aligned} \tag{2.34}$$

here and from now on $\tilde{C} = \tilde{C}(Q_T)$ is a generic positive constant independent ε and δ . Note

that in the derivation of (2.34), one has used Lemma 2.2, the boundary condition at $y = 0$ in (2.9), and the fact that $b(x, 0, t) = -\frac{\partial_x P(x, t)}{U(x, t)}$. Next, at $x = 0$, the equation and boundary condition imply that

$$((a + \varepsilon)|\partial_x V|)(x_1 = 0, y, t) \leq \frac{|\partial_t u_1^\varepsilon|}{(u_1^\varepsilon + \varepsilon)^2} + |\partial_y^2 u_1^\varepsilon| + b(0, y, t) \frac{\partial_y u_1^\varepsilon}{(u_1^\varepsilon + \varepsilon)^2} + |C| \frac{1}{(u_1^\varepsilon + \varepsilon)}.$$

Thus, it follows that

$$\begin{aligned} -I_2 &\equiv \int_0^1 (a + \varepsilon) I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 \Big|_0^L dy \\ &\leq \int_0^1 \left[(a + \varepsilon) I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 \right] (0, y, t) dy \\ &\leq \tilde{C}, \end{aligned} \quad (2.35)$$

where we have used the assumption (2.3) and the regularity assumption on u_1^ε . Next, (2.29) implies that

$$I_3 \equiv - \int_\Omega \frac{\tilde{\nabla} V_y}{V^2} \nabla_{\tilde{\nabla} V}^2 I_\delta(\tilde{\nabla} V) (\tilde{\nabla} V_y) (1 + \varepsilon - y)^2 dx dy \leq 0. \quad (2.36)$$

Next, it follows from the structures of a , b , and c and Lemma 2.2 that

$$\begin{aligned} I_4 &\equiv \int_\Omega \left[2 \left(\frac{\tilde{\nabla} V}{V^2} \right) \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) - V_x(\tilde{\nabla} a) \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 \right. \\ &\quad + \partial_x a I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 + \partial_y (b(1 + \varepsilon - y)^2) \tilde{\nabla} V \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) \\ &\quad \left. + \left\{ \frac{C u^\varepsilon}{u^\varepsilon + \varepsilon} \tilde{\nabla} V \cdot \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) - C \varepsilon V \tilde{\nabla} V \cdot \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) \right\} (1 + \varepsilon - y)^2 \right] dx dy \\ &\leq \tilde{C} \int_\Omega (1 + \varepsilon - y)^2 I_\delta(\tilde{\nabla} V) dx dy. \end{aligned} \quad (2.37)$$

On the other hand, (2.30) implies that

$$\begin{aligned} I_5 &\equiv \int_\Omega \tilde{\nabla} V \cdot \nabla_{\tilde{\nabla} V}^2 I_\delta(\tilde{\nabla} V) (\tilde{\nabla} V_y) \left[\left\{ 2 \frac{\partial_y V}{V^3} - \frac{2}{V^2(1 + \varepsilon - y)} + b \right\} (1 + \varepsilon - y)^2 \right] dx dy \\ &\rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0^+. \end{aligned} \quad (2.38)$$

Next, it follows from Lemma 2.2 that

$$I_6 \equiv \int_\Omega \tilde{\nabla} C \frac{u^\varepsilon}{u^\varepsilon + \varepsilon} V \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 dx dy \leq \tilde{C}. \quad (2.39)$$

Finally, one has

$$\begin{aligned} I_7 &\equiv - \int_\Omega (\tilde{\nabla} b) V_y \partial_{\tilde{\nabla} V} I_\delta(\tilde{\nabla} V) (1 + \varepsilon - y)^2 dx dy \\ &\leq \tilde{C} \int_\Omega |V_y| (1 + \varepsilon - y)^2 dx dy \\ &\leq \tilde{C} \int_\Omega |\partial_y u^\varepsilon(x, y, t)| dx dy. \end{aligned} \quad (2.40)$$

Taking limit as $\delta \rightarrow 0^+$ in (2.33) and using (2.34) - (2.40), we get

$$\begin{aligned} &\frac{d}{dt} \int_\Omega |\tilde{\nabla} V(x, y, t)| (1 + \varepsilon - y)^2 dx dy \\ &\leq \tilde{C} + \tilde{C} \int_\Omega |\tilde{\nabla} V(x, y, t)| (1 + \varepsilon - y)^2 dx dy + \tilde{C} \int_\Omega |\partial_y u^\varepsilon(x, y, t)| dx dy \end{aligned}$$

which yields immediately the desired estimate (2.26) due to Lemma 2.4 and Lemma 2.2. Thus Lemma 2.5 is proved. \square

As an immediate consequence of Lemma 2.4 and Lemma 2.5, we have proved Proposition 2.3. An immediate corollary of Proposition 2.3 and its proof is that

Corollary 2.6 For any $\alpha > 0$, there exists a positive constant $C_5 = C_5(\Omega_T, \alpha)$ independent of ε such that

$$\iint_{Q_T} (1-y)^\alpha |u_{yy}^\varepsilon| dx dy dt \leq C_5 \quad (2.41)$$

Proof: In the proof of Lemma 2.5, if using the weight $(1 + \varepsilon - y)^\alpha$ instead of $(1 + \varepsilon - y)^2$, one can derive similarly that

$$\sup_{0 \leq t \leq T} \int_{\Omega} (1 + \varepsilon - y)^\alpha |\tilde{\nabla} V(x, y, t)| dx dy \leq C \quad (2.42)$$

which, together with the equation in (2.9), implies (2.41). The proof of Corollary 2.6 is complete. \square

Now we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1: It follows from Lemma 2.2, Proposition 2.3, Lemma 2.4 and Lemma 2.5 that there exist a subsequence of $\{u^\varepsilon\}$, which solves (2.9) and will be still denoted by u^ε , and a function $u \in L^\infty(Q_T) \cap BV(Q_T)$ such that as $\varepsilon \rightarrow 0^+$ and $\alpha > 0$,

$$u^\varepsilon \rightarrow u \quad \text{a.e. in } Q_T, \quad (1-y)^{\frac{\alpha-1}{2}} u_y^\varepsilon \rightharpoonup (1-y)^{\frac{\alpha-1}{2}} u_y \quad \text{weakly in } Q_T, \quad (2.43)$$

and $(1-y)^\alpha u_{yy}^\varepsilon \rightarrow (1-y)^\alpha u_{yy}$ is measure on Q_T . Furthermore, the limiting function $u(x, y, t)$ satisfies the requirements (2.4), (2.5) and (2.6). Next, we show that $u(x, y, t)$ satisfies the equality (2.8). For any $\varphi \in C^1(\bar{Q}_T)$ with $\varphi|_{t=0} = \varphi|_{x=0} = \varphi|_{x=L} = 0$ and $\alpha > 0$, one has from (2.9) that

$$\begin{aligned} & - \int_{\Omega} (u^\varepsilon + \varepsilon)^{-1} (1-y)^\alpha \varphi|_t^t dx dy + \int_{Q_t} (u^\varepsilon + \varepsilon)^{-1} (1-y)^\alpha \partial_t \varphi dx dy dt \\ & + \int_{Q_t} (u^\varepsilon + \varepsilon)^{-1} \partial_x ((1-y)^\alpha (a + \varepsilon) \varphi) dx dy dt + \int_{Q_t} ((1-y)^\alpha \varphi)_y u_y^\varepsilon dx dy dt \\ & + \int_{Q_t} ((1-y)^\alpha b \varphi)_y (u^\varepsilon + \varepsilon)^{-1} dx dy dt + \int_0^t \int_0^L \left[\varphi \left(u_y^\varepsilon + \frac{b}{u^\varepsilon + \varepsilon} \right) \right] (x, 0, t) dx dt \\ & + \int_{Q_t} (1-y)^\alpha \varphi c u^\varepsilon (u^\varepsilon + \varepsilon)^{-2} dx dy dt \\ & = 0. \end{aligned} \quad (2.44)$$

Note that

$$[u_y^\varepsilon + b(u^\varepsilon + \varepsilon)^{-1}](x, y = 0, t) = v_0(x, t)$$

due to the boundary condition at $y = 0$ in (2.9) and the fact

$$b(x, y = 0, t) = -(\partial_x P(x, t))U(x, t).$$

Consequently, (2.44) becomes

$$\begin{aligned}
& - \int_{\Omega} (u^\varepsilon + \varepsilon)^{-1} (1-y)^\alpha \varphi|^t dx dy + \int_{Q_t} (u^\varepsilon + \varepsilon)^{-1} (1-y)^\alpha \partial_t \varphi dx dy dt \\
& + \iint_{Q_t} [(1-y)^\alpha (u^\varepsilon + \varepsilon)^{-1} \partial_x((a + \varepsilon)\varphi) + ((1-y)^\alpha \varphi_y - \alpha(1-y)^{\alpha-1} \varphi) u_y^\varepsilon] dx dy dt \\
& + \int_0^t \int_0^L v_0(x, t) \varphi(x, 0, t) dx dt + \iint_{Q_t} (1-y)^\alpha \varphi c u^\varepsilon (u^\varepsilon + \varepsilon)^{-2} dx dy dt \\
& = 0.
\end{aligned}$$

Now, (2.8) follows from passing to the limit $\varepsilon \rightarrow 0^+$ in the above identity and using the convergence in (2.43). Now we verify the boundary conditions. First, it follows from $u \in BV(Q_T)$ that $\gamma u|_{y=1}$ exists. Then (2.5) for u implies $\gamma u|_{y=1} = 0$. Next, we consider the boundary condition at $y = 0$. Observe that $\gamma u_y|_{y=0}$ exists due to (2.6).

Define $\theta \in C_c^\infty[0, 1)$ with the properties that $0 \leq \theta \leq 1$, $\theta(0) = 1$, $\theta(y) \equiv 0$ for $y \in [\frac{1}{2}, 1)$, and $\int_0^1 \theta(y) dy = 1$. Let $\psi(x, t)$ be any function in $C^1([0, L] \times [0, T])$ such that $\psi(0, t) = \psi(L, t) = \psi(x, t = 0) = 0$. Set

$$\varphi(x, y, t) = \eta_\delta(y) \psi(x, t), \quad \text{for } 0 < \delta < 1, \quad (2.45)$$

where

$$\eta_\delta(y) = - \int_y^1 \theta_\delta(s) ds, \quad \theta_\delta(y) = \delta^{-1} \theta\left(\frac{y}{\delta}\right).$$

Clearly,

$$\eta'_\delta(y) = \theta_\delta(y) \geq 0, \quad \eta_\delta(0) = -1, \quad \eta_\delta(y) = 0 \quad \text{for } y \geq \frac{1}{2}\delta, \quad \text{and } |\eta_\delta(y)| \leq 1. \quad (2.46)$$

Using φ in (2.45) as a test function in (2.8), one gets

$$\begin{aligned}
0 = & - \int_{\Omega} u^{-1} \varphi (1-y)^\alpha dx dy|^t + \iint_{Q_t} u^{-1} (1-y)^\alpha \eta_\delta(y) \partial_t \psi dx dy dt \\
& + \iint_{Q_t} (-\alpha) (1-y)^{\alpha-1} \varphi u_y dx dy dt + \iint_{Q_t} (1-y)^\alpha \theta_\delta(y) \psi u_y dx dy dt \\
& + \iint_{Q_t} (a\varphi)_x (1-y)^\alpha u^{-1} dx dy dt + \iint_{Q_t} (1-y)^\alpha b \theta_\delta \psi u^{-1} dx dy dt \\
& + \iint_{Q_t} ((1-y)^\alpha b)_y \varphi u^{-1} dx dy dt + \int_0^t \int_0^L v_0(x, t) \varphi(x, 0, t) dx dt \\
& + \iint_{Q_t} (1-y)^\alpha \varphi c u^{-1} dx dy dt.
\end{aligned}$$

Taking limit as $\delta \rightarrow 0^+$ in the above identity and taking into account of (2.46) and Lebesgue dominant convergence theorem, one gets that

$$0 = \int_0^t \int_0^L \left\{ \gamma u_y|_{y=0}(x, t) + \left[\left(\frac{-\partial_x P}{U} \right) \left(\frac{1}{\gamma u|_{y=0}} \right) \right] (x, t) - U_0(x, t) \right\} \psi(x, t) dx dt$$

It follows that

$$\gamma u_y|_{y=0} = v_0 + \frac{\partial_x P}{U} \frac{1}{\gamma u|_{y=0}}$$

which is the desired boundary condition at $y = 0$. Finally, we verify that $\gamma u|_{x=0}(y, t) = u_1(y, t)$, a.e.. Indeed, let $\psi(y, t) \in C^1([0, 1] \times [0, T])$ and $\eta(x) \in C^1[0, L]$ be arbitrary such that $\eta(0) = 1$ and $\eta(L) = 0$. Furthermore, set $\varphi(x, y, t) = \eta(x) \psi(y, t)$. Then one has

$$\begin{aligned}
& \iint_{Q_T} \varphi(x, y, t) \partial_x u(x, y, t) dx dy dt \\
&= \lim_{\varepsilon \rightarrow 0^+} \iint_{Q_T} \varphi(x, y, t) \partial_x u^\varepsilon(x, y, t) dx dy dt \\
&= \lim_{\varepsilon \rightarrow 0^+} \left[- \int_0^T \int_0^1 u_1^\varepsilon(y, t) \psi(y, t) dy dt - \iint_{Q_T} \partial_x \varphi(x, y, t) u^\varepsilon(x, y, t) dx dy dt \right] \\
&= - \int_0^T \int_0^1 u_1(y, t) \psi(y, t) dy dt - \iint_{Q_T} \partial_x \varphi(x, y, t) u(x, y, t) dx dy dt \\
&= - \int_0^T \int_0^1 u_1(y, t) \psi(y, t) dy dt + \int_0^T \int_0^1 \gamma u|_{x=0}(y, t) \psi(y, t) dy dt \\
&\quad + \iint_{Q_T} \varphi(x, y, t) \partial_x u(x, y, t) dx dy dt
\end{aligned}$$

Consequently,

$$\int_0^T \int_0^1 \gamma u|_{x=0}(y, t) \psi(y, t) dy dt = \int_0^T \int_0^1 u_1(y, t) \psi(y, t) dy dt$$

so the desired conclusion follows. Thus we have shown that $u(x, y, t)$ is indeed a weak solution to the initial-boundary value problem (2.1). This completes the proof of Theorem 2.1. \square

In the next section, we will show that such a weak solution to (2.1) is unique.

§3 Uniqueness And Continuous Dependence Of The Weak Solution

In this section, we will study the uniqueness and continuous dependence of weak solutions to the initial-boundary value problem (1.7) defined in Definition 1.1. The main strategy is show a L^1 -contraction for weak solutions to (1.7) motivated by the pioneering work of Krushkov [6]. We start with some basic structure of a weak solution to (1.7). Let u be a weak solution to (1.7). Then it follows from the definition that $u \in BV(Q_T) \cap L^\infty(Q_T)$, $\partial_y u \in L^2(Q_T)$, $\partial_y^2 u \in \mathcal{M}_{loc}(Q_T)$, and hence $\partial_y u \in BV_y^{loc}(Q_T)$, here and in what follows, $BV_y^{loc}(Q_T)$ denotes the set of functions $w \in L_{loc}^1(Q_T)$ with $\partial_y w$ being a local Random measure on Q_T ([12]). Let Γ_u be the set of approximate jumps of u in Q_T with $\nu = (\nu_x, \nu_y, \nu_t)$ being a normal on Γ_u . Denote by $w^\pm(P)$ be the limiting values of w in the direction $\pm \nu$ for $P \in \Gamma_u$. We also denote by H the 2-dimensional Hausdorff measure. Then our first result in this section is on the jump-conditions for a weak solution u .

Proposition 3.1 Let u be a weak solution to the initial-boundary value problem (1.7). Then H -almost everywhere on Γ_u , one has that

$$\left(\frac{1}{u^+} - \frac{1}{u^-} \right) (\nu_t + a\nu_x) = 0, \quad \nu_y = 0 \quad \text{on } \Gamma_u. \tag{3.1}$$

Proof: Set $v = \frac{1}{u}$. Then it follows from (1.13) in the Definition 1.1 that

$$\partial_t v + \partial_y^2 u + \partial_x(av) + \partial_y(bv) - (\partial_x a + \partial_y b + c)v = 0 \quad (3.2)$$

in the sense of distribution. Since $v \in BV^{loc}(Q_T)$. Thus (3.2) is in fact a measure equality. Let S be any H -measurable subset of Γ_u . Then S is ∇v -measurable. Then it follows from (3.2) that S is $\partial_y^2 u$ -measurable. Furthermore, (3.2) implies that

$$\int_S (v^+ - v^-) \nu_t dH + \int_S (v^+ - v^-) a \nu_x dH + \int_S (v^+ - v^-) b \nu_y dH = \int_S \partial_y^2 u. \quad (3.3)$$

For any given subset E such that $\bar{E} \subset \Omega_T$, we denote by

$$E^{(x,t)} = \{y \in (0,1); \text{ such that } (x,y,t) \in E\}.$$

Then one can obtain

$$\begin{aligned} \int_S u_{yy} &= \int \chi_S u_{yy} = \int_0^T \int_0^L dx dt \int_0^1 \chi_S(x, \cdot, t) u_{yy}(x, \cdot, t) \\ &= \int_0^T \int_0^L dx dt \int_{S^{(x,t)}} u_{yy}(x, \cdot, t) = \int_0^T \int_0^L \sum_{y \in S^{(x,t)}} (u_y^r(x, y, t) - u_y^l(x, y, t)) dx dt \end{aligned}$$

where χ_S is the characteristic function of S , u_y^r and u_y^l denote the right and left approximate limits of $u_y(x, \cdot, t)$ respectively. Now arguing in a similar way as in the proof of Theorem 3.4.1 in [12] (pp. 308 - 309), one obtains

$$\int_S u_{yy} = \int_S (u_y^r(x, y, t) - u_y^l(x, y, t)) |\nu_y| dH. \quad (3.4)$$

On the other hand, since $\partial_y u \in L^2(Q_T)$, so

$$0 = \int_S \partial_y u = \int_S (u^+ - u^-) \nu_y dH. \quad (3.5)$$

Since S is an arbitrary subset of Γ_u , it hence follows from (3.5) that $\nu_y = 0$ H -almost everywhere on Γ_u . This together with (3.4) shows that

$$\int_S u_{yy} = 0.$$

Consequently,

$$\int_S (u^+ - u^-) (\nu_t + a \nu_x) dH = 0$$

which implies (3.1) since S is an arbitrary subset of Γ_u . Hence the proof of Proposition 3.1 is completed. □

We remark here that as an immediate consequence, one can show that for a weak solution u , to (1.7), $\bar{\Gamma}_u$ cannot contain a 2-dimensional open set. However, we do not use this remark later.

Motivated by the arguments of Krushkov [6], we analyze the “entropy” $|v - k|$ in order to prove the uniqueness and continuous dependence of weak solutions to (1.7). We define a C^1 -approximation of the

sign function as for any $\delta > 0$,

$$S_\delta(\theta) = \begin{cases} 1, & \theta > \delta, \\ \frac{2\theta}{\delta} - \frac{\theta^2}{\delta^2}, & 0 \leq \theta \leq \delta, \\ \frac{2\theta}{\delta} - \frac{\theta^2}{\delta^2}, & -\delta \leq \theta < 0, \\ -1, & \theta < -\delta, \end{cases} \quad (3.6)$$

and $I_\delta(\theta) = \int_0^\theta S_\delta(\xi) d\xi$. Clearly $S_\delta \in C^1(\mathbb{R}^1)$ and $I_\delta \in C^2(\mathbb{R}^1)$.

Now, for any given functions of $f(s)$ and $w(x, y, t)$, $\hat{f}(v)$ will denote the functional superposition defined by Volport in [12], and $\bar{w}(x, y, t)$ will denote the symmetric mean of $w(x, \cdot, t)$ as in [12]. For any given constant k , $\hat{I}'_\delta(v - k)$ is bounded and measurable with respect to ∇v , and consequently bounded and measurable with respect to u_{yy} due to (3.2). Thus for any smooth test function $\varphi \in C_0^2(Q_T)$, one has from (3.2) and the *BV*-calculus in [12] that

$$\begin{aligned} & \int_{Q_T} \left(\phi \{ \partial_t I_\delta(v - k) + a \partial_x I_\delta(v - k) + b \partial_y I_\delta(v - k) \} + c \phi \hat{I}'_\delta(v - k) v \right) \\ &= \int_{Q_T} \left(-\phi \hat{I}'_\delta(v - k) \partial_y^2 u \right) \end{aligned} \quad (3.7)$$

Note that the singular set of v has H -measure zero, and on the set of approximate continuous points of v , $\hat{I}'_\delta(v - k) = \bar{I}'_\delta(v - k) = I'_\delta(\bar{v} - k)$. It follows from Proposition 3.1, (3.6), and the *BV*-calculus that

$$\begin{aligned} \int_{Q_T} \left(-\phi \hat{I}'_\delta(v - k) \partial_y^2 u \right) &= \int_{Q_T} \left(-\phi \bar{I}'_\delta(v - k) \partial_y^2 u \right) + \int_{Q_T} -\phi \left(\hat{I}'_\delta(v - k) - \bar{I}'_\delta(v - k) \right) \partial_y^2 u \\ &= \int_{Q_T} (-\phi) \partial_y \left(\hat{I}'_\delta(v - k) \partial_y u \right) + \int_{Q_T} \phi \widehat{I''}_\delta(v - k) \partial_y v \overline{\partial_y u} dx dy \\ &= \int_{Q_T} \partial_y \phi \hat{I}'_\delta(v - k) \partial_y u dx dy dt + \int_{Q_T} \phi I''_\delta(v - k) \partial_y v \partial_y u dx dy dt \end{aligned} \quad (3.8)$$

where we have used the fact that $\partial_y u \in L^2(Q_T)$ and $\partial_y v \in L^2_{loc}(Q_T)$, which follows from the definition. Thus, it follows from (3.7), (3.8) and Proposition 3.1 again that

$$\begin{aligned} & \int_{Q_T} \left\{ I_\delta(v - k) (\partial_t \phi + \partial_x(a\phi) + \partial_y(b\phi)) - I'_\delta(v - k) v c \phi \right\} dx dy dt \\ &= - \int_{Q_T} \partial_y \phi \hat{I}'_\delta(v - k) \partial_y u dx dy dt - \int_{Q_T} \phi I''_\delta(v - k) \partial_y v \partial_y u dx dy dt \end{aligned} \quad (3.9)$$

Consequently, we have shown the following lemma.

Lemma 3.2 Let u be a weak solution to (2.1) and $v = u^{-1}$. Then for any constant k and $\phi \in C_0^\infty(Q_T)$, one has the identity

$$\begin{aligned} & \int_{Q_T} \left\{ |v - k| (\partial_t \phi + \partial_x(a\phi) + \partial_y(b\phi)) - \operatorname{sgn}(v - k) v c \phi \right\} dx dy dt \\ &= - \int_{Q_T} (\partial_y \phi) \operatorname{sgn}(v - k) \partial_y u dx dy dt - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \phi I''_\delta(v - k) \partial_y v \partial_y u dx dy dt \end{aligned} \quad (3.10)$$

As an immediate consequence of this lemma, one can have

Corollary 3.3 Let u be a weak solution to (2.1) and $v = u^{-1}$. Then for any constant k , and $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$, it holds that

$$\int_{Q_T} \{ |v - k|(\partial_t \psi + \partial_x(a\psi) + \partial_y(b\psi)) + \partial_y \psi \operatorname{sgn}(v - k) \partial_y u - \operatorname{sgn}(v - k) v c \psi \} dx dy dt \geq 0. \quad (3.11)$$

Now let u_1 and u_2 be two weak solutions to (2.1) with their corresponding initial and boundary data respectively. Set $v_i = (u_i)^{-1}$. Then our key estimate in the argument for continuous dependence is the following proposition:

Proposition 3.4 Let u_1 and u_2 be two weak solutions to (2.1). For any $\phi \in C_0^\infty(Q_T)$, it holds that

$$\int_{Q_T} \{ |v_1 - v_2|(\partial_t \phi + \partial_x(a\phi) + \partial_y(b\phi) - c\phi) - \partial_y \phi \partial_y |u_1 - u_2| \} dx dy dt \geq 0. \quad (3.12)$$

Proof: This proposition will be proved by modifying the doubling-variable argument of Krushkov [6] based on Lemma 3.2. Denote by $\mathbf{x} = (x, y, t)$ and $\bar{\mathbf{x}} = (\bar{x}, \bar{y}, \bar{t})$. Now for any nonnegative function $\psi = \psi(\mathbf{x}, \bar{\mathbf{x}}) \in C^\infty$ such that

$$\operatorname{supp} \Psi(\cdot, \bar{\mathbf{x}}) \subset\subset Q_T, \quad \forall \bar{\mathbf{x}} \in Q_T, \quad \operatorname{supp} \Psi(\mathbf{x}, \cdot) \subset\subset Q_T, \quad \forall \mathbf{x} \in Q_T, \quad (3.13)$$

it follows from (3.10) in Lemma 3.2 that

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} \{ |v_1(X) - v_2(\bar{X})|(\partial_t \Psi + \partial_x(a(X)\Psi) + \partial_y(b(X)\Psi) - c(X)v_1(X)) \operatorname{sgn}(v_1(X) - v_2(\bar{X}))\Psi \\ & \quad + \partial_y \Psi \operatorname{sgn}(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) \} dX d\bar{X} \\ &= - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \Psi I_\delta''(v_1(X) - v_2(\bar{X})) \partial_y v_1(X) \partial_y u_1(X) dX d\bar{X}, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} \{ |v_2(\bar{X}) - v_1(X)|(\partial_t \Psi + \partial_x(a(\bar{X})\Psi) + \partial_{\bar{y}}(b(\bar{X})\Psi)) - c(\bar{X})v_2(\bar{X}) \operatorname{sgn}(v_2(\bar{X}) - v_1(X))\Psi \\ & \quad + \partial_{\bar{y}} \Psi \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X}) \} d\bar{X} dX \\ &= - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \Psi I_\delta''(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} v_2(\bar{X}) \partial_{\bar{y}} u_2(\bar{X}) d\bar{X} dX, \end{aligned} \quad (3.15)$$

Adding (3.14) to (3.15) shows that

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} \{ |v_1(X) - v_2(\bar{X})|((\partial_t \Psi + \partial_t \Psi) + \partial_x(a(X)\Psi) + \partial_x(a(\bar{X})\Psi) + \partial_y(b(X)\Psi) + \partial_{\bar{y}}(b(\bar{X})\Psi) \\ & \quad - c(X)\Psi) + (c(X) - c(\bar{X}))\Psi v_2(\bar{X}) \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) \} dX d\bar{X} \\ & \quad + \int_{Q_T} \int_{Q_T} \{ \partial_y \Psi \operatorname{sgn}(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + \partial_{\bar{y}} \Psi \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X}) \} dX d\bar{X} \\ &= - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \Psi I_\delta''(v_2(\bar{X}) - v_1(X)) (\partial_y v_1(X) \partial_y u_1(X) + \partial_{\bar{y}} v_2(\bar{X}) \partial_{\bar{y}} u_2(\bar{X})) dX d\bar{X}, \end{aligned} \quad (3.16)$$

where we have used the property that $I_\delta''(\xi)$ is an even function. We will estimate each term in (3.16) respectively for a special choice of the test function Ψ . Let $\rho \in C_0^\infty(\mathbb{R}^1)$, $\rho(\xi) \geq 0$, $\operatorname{supp} \rho(\cdot) \subset (-1, 1)$, and $\int_{\mathbb{R}^1} \rho(\xi) d\xi = 1$. For any $h > 0$ sufficiently small, set $\rho_h(\xi) = \frac{1}{h} \rho(\frac{\xi}{h})$.

Set

$$\Psi(X, \bar{X}) = \frac{1}{8} \phi\left(\frac{x+\bar{x}}{2}, \frac{y+\bar{y}}{2}, \frac{t+\bar{t}}{2}\right) \rho_h\left(\frac{x-\bar{x}}{2}\right) \rho_h\left(\frac{y-\bar{y}}{2}\right) \rho_h\left(\frac{t-\bar{t}}{2}\right), \forall (X, \bar{X}) \in Q_T \times Q_T \quad (3.17)$$

which clearly satisfies the requirements in (3.13). In the following, for the convenience of notations, we will use $\partial_1, \partial_2, \partial_3$ to denote $\partial_x, \partial_y, \partial_t$ respectively, and

$$\tilde{\rho}_h(X, \bar{X}) = \frac{1}{8} \rho_h\left(\frac{x-\bar{x}}{2}\right) \rho_h\left(\frac{y-\bar{y}}{2}\right) \rho_h\left(\frac{t-\bar{t}}{2}\right).$$

Since

$$\partial_t \Psi + \partial_{\bar{t}} \Psi = \partial_3 \phi\left(\frac{X+\bar{X}}{2}\right) \tilde{\rho}_h(X, \bar{X}),$$

so it follows from the Lebesgue dominant convergence theorem as in [6] that

$$\int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (\partial_t \Psi + \partial_{\bar{t}} \Psi) dX d\bar{X} \longrightarrow \int_{Q_T} |v_1(X) - v_2(X)| (\partial_t \phi(X)) dX, \quad h \rightarrow 0^+. \quad (3.18)$$

Next, one has

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (\partial_x(a(X)\Psi) + \partial_{\bar{x}}(a(\bar{X})\Psi)) dX d\bar{X} \\ &= \int_{Q_T} \int_{Q_T} (\partial_x a(X) + \partial_{\bar{x}} a(\bar{X})) |v_1(X) - v_2(\bar{X})| \Psi dX d\bar{X} \\ & \quad + \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| \frac{1}{2} (a(X) + a(\bar{X})) \partial_1 \phi\left(\frac{X+\bar{X}}{2}\right) \tilde{\rho}_h\left(\frac{X-\bar{X}}{2}\right) dX d\bar{X} \\ & \quad + \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| \frac{1}{2} (a(X) - a(\bar{X})) \left(\frac{\rho_h'}{\rho_h}\right)\left(\frac{x-\bar{x}}{2}\right) \Psi dX d\bar{X} \\ &\equiv \sum_{i=1}^3 I_i. \end{aligned} \quad (3.19)$$

Clearly,

$$\begin{aligned} I_1 &\equiv \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (\partial_x a(X) + \partial_{\bar{x}} a(\bar{X})) \Psi dX d\bar{X} \\ &\longrightarrow 2 \int_{Q_T} |v_1(X) - v_2(X)| (\partial_x a(X)) \phi(X) dX, \quad \text{as } h \rightarrow 0^+, \end{aligned} \quad (3.20)$$

$$\begin{aligned} I_2 &\equiv \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| \frac{1}{2} (a(X) + a(\bar{X})) \partial_1 \phi\left(\frac{X+\bar{X}}{2}\right) \tilde{\rho}_h\left(\frac{X-\bar{X}}{2}\right) dX d\bar{X} \\ &\longrightarrow \int_{Q_T} |v_1(X) - v_2(X)| a(X) \partial_x \phi(X) dX, \quad \text{as } h \rightarrow 0^+. \end{aligned} \quad (3.21)$$

We rewrite I_3 as

$$\begin{aligned} I_3 &\equiv \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| \frac{1}{2} (a(X) - a(\bar{X})) \left(\frac{\rho_h'}{\rho_h}\right)\left(\frac{x-\bar{x}}{2}\right) \Psi dX d\bar{X} \\ &= \int_{Q_T} \int_{Q_T} \frac{1}{2} \{ |v_1(X) - v_2(X)| + |v_1(\bar{X}) - v_2(\bar{X})| \\ & \quad + [(|v_1(X) - v_2(\bar{X})| - |v_1(X) - v_2(X)|) + (|v_1(X) - v_2(\bar{X})| - |v_1(\bar{X}) - v_2(\bar{X})|)] \} \\ & \quad \left[\frac{1}{2} (a(X) - a(\bar{X})) \right] \left(\frac{\rho_h'}{\rho_h}\right)\left(\frac{x-\bar{x}}{2}\right) \Psi dX d\bar{X} \\ &\equiv \sum_{i=1}^3 I_{3i}. \end{aligned} \quad (3.22)$$

Since,

$$\begin{aligned}
I_{31} &= \int_{Q_T} \int_{Q_T} \frac{1}{2} |v_1(X) - v_2(X)| \frac{1}{2} (a(X) - a(\bar{X})) \left(\frac{\rho'_h}{\rho_h} \right) \left(\frac{x - \bar{x}}{2} \right) \Psi dX d\bar{X} \\
&= \int_{Q_T} \int_{Q_T} \frac{1}{2} |v_1(X) - v_2(X)| \frac{1}{2} (a(X) - a(\bar{X})) \partial_1 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X} \\
&\quad - \frac{1}{2} \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(X)| \partial_{\bar{x}} a(\bar{X}) \Psi d\bar{X} dX,
\end{aligned}$$

so,

$$I_{31} \longrightarrow -\frac{1}{2} \int_{Q_T} |v_1(X) - v_2(X)| \partial_x a(X) \phi(X) dX \quad \text{as } h \rightarrow 0^+. \quad (3.23)$$

Similarly,

$$\begin{aligned}
I_{32} &\equiv \int_{Q_T} \int_{Q_T} \frac{1}{2} |v_1(\bar{X}) - v_2(\bar{X})| \frac{1}{2} (a(X) - a(\bar{X})) \left(\frac{\rho'_h}{\rho_h} \right) \left(\frac{x - \bar{x}}{2} \right) \Psi dX d\bar{X} \\
&\longrightarrow -\frac{1}{2} \int_{Q_T} |v_1(X) - v_2(X)| \partial_x a(X) \phi(X) dX \quad \text{as } h \rightarrow 0^+.
\end{aligned} \quad (3.24)$$

Note that

$$\begin{aligned}
&|(|v_1(X) - v_2(\bar{X})| - |v_1(X) - v_2(X)|) + (|v_1(X) - v_2(\bar{X})| - |v_1(\bar{X}) - v_2(\bar{X})|)| \\
&\leq |v_1(X) - v_1(\bar{X})| + |v_2(X) - v_2(\bar{X})|
\end{aligned}$$

and $a(\cdot)$ is smooth. It thus follows from the Lebesgue dominant convergence theorem that as $h \rightarrow 0^+$,

$$\begin{aligned}
I_{33} &\equiv \int_{Q_T} \int_{Q_T} [(|v_1(X) - v_2(\bar{X})| - |v_1(X) - v_2(X)|) + (|v_1(X) - v_2(\bar{X})| - |v_1(\bar{X}) - v_2(\bar{X})|)] \\
&\quad \frac{1}{2} (a(X) - a(\bar{X})) \left(\frac{\rho'_h}{\rho_h} \right) \left(\frac{x - \bar{x}}{2} \right) \Psi dX d\bar{X} \longrightarrow 0.
\end{aligned} \quad (3.25)$$

Consequently, we have shown from (3.22) - (3.25) that

$$I_3 \longrightarrow - \int_{Q_T} |v_1(X) - v_2(X)| \partial_x a(X) \phi(X) dX \quad \text{as } h \rightarrow 0^+.$$

This, together with (3.20) and (3.21), shows that

$$\begin{aligned}
&\int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (\partial_x (a(X)\Psi) + \partial_{\bar{x}} (a(\bar{X})\Psi)) dX d\bar{X} \\
&\longrightarrow \int_{Q_T} |v_1(X) - v_2(X)| \partial_x (a(X)\phi(X)) dX \quad \text{as } h \rightarrow 0^+.
\end{aligned} \quad (3.26)$$

Similarly, one can show that

$$\begin{aligned}
&\int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (\partial_y (b(X)\Psi) + \partial_{\bar{y}} (b(\bar{X})\Psi)) dX d\bar{X} \\
&\longrightarrow \int_{Q_T} |v_1(X) - v_2(X)| \partial_y (b(X)\phi(X)) dX \quad \text{as } h \rightarrow 0^+.
\end{aligned} \quad (3.27)$$

Next, one has

$$\begin{aligned}
&\int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (-c(X)\Psi) dX d\bar{X} \\
&\longrightarrow - \int_{Q_T} |v_1(X) - v_2(X)| c(X)\phi(X) dX \quad \text{as } h \rightarrow 0^+,
\end{aligned} \quad (3.28)$$

and

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} |v_1(X) - v_2(\bar{X})| (c(X) - c(\bar{X})) \Psi v_2(\bar{X}) \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) dX d\bar{X} \\ \longrightarrow 0 & \quad \text{as } h \rightarrow 0^+. \end{aligned} \quad (3.29)$$

Finally, we treat the last term on the left hand side of (3.16). Note that

$$\begin{aligned} & \int_{Q_T} \int_{Q_T} \{ \partial_y \Psi \operatorname{sgn}(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + \partial_{\bar{y}} \Psi \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X}) \} dX d\bar{X} \\ = & \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \{ \partial_y \Psi I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + \partial_{\bar{y}} \Psi I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X}) \} dX d\bar{X} \\ = & \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \partial_2 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X} \\ & + \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) - I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \left(\frac{\rho'_h}{\rho_h} \right) \left(\frac{y - \bar{y}}{2} \right) \Psi dX d\bar{X} \\ \equiv & J_1 + J_2 \end{aligned} \quad (3.30)$$

Using the fact that $\partial_y u_1(\cdot) \in L^2(Q_T)$ and $\partial_{\bar{y}} u_2(\cdot) \in L^2(Q_T)$, one can show easily that

$$\begin{aligned} J_1 &= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \partial_2 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X} \\ &= \int_{Q_T} \int_{Q_T} (\operatorname{sgn}(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + \operatorname{sgn}(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \partial_2 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X} \\ &= \int_{Q_T} \int_{Q_T} (-\partial_y |u_1(X) - u_2(\bar{X})| - \partial_{\bar{y}} |u_2(\bar{X}) - u_1(X)|) \\ & \quad \frac{1}{2} \partial_2 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X}, \end{aligned}$$

and so,

$$J_1 \longrightarrow -\frac{1}{2} \int_{Q_T} \partial_y |u_1(X) - u_2(X)| \partial_y \phi(X) dX \quad \text{as } h \rightarrow 0^+. \quad (3.31)$$

It remains to estimate J_2 . It follows from the fact that $\partial_y u_2(\cdot) \in L^2(Q_T)$, $\partial_y v_i(\cdot) \in L^2_{loc}(Q_T)$, $\partial_{\bar{y}} u_i(\cdot) \in L^2(Q_T)$, $\partial_{\bar{y}} v_i(\cdot) \in L^2(Q_T)$, and Proposition 3.1 that

$$\begin{aligned} J_2 &= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) - I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \left(\frac{\rho'_h}{\rho_h} \right) \left(\frac{y - \bar{y}}{2} \right) \Psi dX d\bar{X} \\ &= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I'_\delta(v_1(X) - v_2(\bar{X})) \partial_y u_1(X) + I'_\delta(v_2(\bar{X}) - v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \\ & \quad \frac{1}{2} \partial_2 \phi \left(\frac{X + \bar{X}}{2} \right) \tilde{\rho}_h \left(\frac{X - \bar{X}}{2} \right) dX d\bar{X} \\ & \quad + \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I''_\delta(v_1(X) - v_2(\bar{X})) (-\partial_{\bar{y}} u_2(\bar{X})) \partial_y u_1(X) \\ & \quad \quad + I''_\delta(v_2(\bar{X}) - v_1(X)) (-\partial_y v_1(X)) \partial_{\bar{y}} u_2(\bar{X})) \Psi dX d\bar{X} \\ \equiv & J_1 + J_{22}. \end{aligned} \quad (3.32)$$

To treat J_{22} , we can estimate it by using the facts that $\partial_y u_i \in L^2(Q_T)$, $\partial_{\bar{y}} v_i \in L^2(Q_T)$, $\partial_y v_i \in L^2_{loc}(Q_T)$, and $\partial_{\bar{y}} v_i \in L^2_{loc}(Q_T)$ for $i = 1, 2$, as

$$\begin{aligned}
J_{22} &= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} (I''_{\delta} (v_1(X) - v_2(\bar{X})) (-\partial_{\bar{y}} v_2(\bar{X}) \partial_y u_1(X) - \partial_y v_1(X) \partial_{\bar{y}} u_2(\bar{X})) \Psi dX d\bar{X} \\
&= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} I''_{\delta} (v_1(X) - v_2(\bar{X})) \left(\frac{v_2(\bar{X})}{v_1(X)} + \frac{v_1(X)}{v_2(\bar{X})} \right) (\partial_y l_n u_1(X)) (\partial_{\bar{y}} l_n u_2(\bar{X})) \Psi dX d\bar{X} \\
&\leq \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} I''_{\delta} (v_1(X) - v_2(\bar{X})) \left(\frac{v_2(\bar{X})}{v_1(X)} + \frac{v_1(X)}{v_2(\bar{X})} \right) \\
&\quad \frac{1}{2} [(\partial_y l_n u_1(X))^2 + (\partial_{\bar{y}} l_n u_2(\bar{X}))^2] \Psi dX d\bar{X},
\end{aligned} \tag{3.33}$$

Finally, we rewrite the integral on the right hand side of (3.16) as

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \Psi I''_{\delta} (v_2(\bar{X}) - v_1(X)) (\partial_y v_1(X) \partial_y u_1(X) + \partial_{\bar{y}} v_2(\bar{X}) \partial_{\bar{y}} u_2(\bar{X})) dX d\bar{X} \\
&= \int_{Q_T} \int_{Q_T} I''_{\delta} (v_2(\bar{X}) - v_1(X)) [(\partial_y l_n u_1(X))^2 + (\partial_{\bar{y}} l_n u_2(\bar{X}))^2] \Psi dX d\bar{X} \Psi dX d\bar{X},
\end{aligned}$$

Thus,

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} \Psi I''_{\delta} (v_2(\bar{X}) - v_1(X)) (\partial_y v_1(X) \partial_y u_1(X) + \partial_{\bar{y}} v_2(\bar{X}) \partial_{\bar{y}} u_2(\bar{X})) dX d\bar{X} \\
& \quad - \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} I''_{\delta} (v_1(X) - v_2(\bar{X})) \left(\frac{v_2(\bar{X})}{v_1(X)} + \frac{v_1(X)}{v_2(\bar{X})} \right) \\
& \quad \quad \frac{1}{2} [(\partial_y l_n u_1(X))^2 + (\partial_{\bar{y}} l_n u_2(\bar{X}))^2] \Psi dX d\bar{X} \\
&= \lim_{\delta \rightarrow 0^+} \int_{Q_T} \int_{Q_T} I''_{\delta} (v_1(X) - v_2(\bar{X})) \left(-\frac{\Psi}{2} \right) \frac{(v_1(X) - v_2(\bar{X}))^2}{v_1(X) v_2(\bar{X})} \\
& \quad \quad [(\partial_y l_n u_1(X))^2 + (\partial_{\bar{y}} l_n u_2(\bar{X}))^2] dX d\bar{X} \\
&= 0,
\end{aligned} \tag{3.34}$$

where in the last step, we have used the definition of I_{δ} and the Lebesgue's dominant convergence theorem.

Now, passing to the limit $h \rightarrow 0^+$ in (3.16), and using the estimates (3.18), (3.26) - (3.29), (3.30) - (3.34), we have derived the desired estimate (3.12). This completes the proof of Proposition 3.4. \square

Now, we are in the position to prove the uniqueness and continuous dependence of weak solutions to the initial-boundary value problem (2.1).

Proposition 3.5 Let $u_1(x, y, t)$ and $u_2(x, y, t)$ be two weak solutions to the problem (2.1) with corresponding initial data $(u_{10}(x, y), u_{20}(x, y))$ and boundary data $(u_{11}(y, t), u_{21}(y, t))$ and $(v_{01}(x, t), v_{02}(x, t))$ respectively. Then it holds that for almost all $t \in [0, T]$,

$$\begin{aligned}
& \int_0^L \int_0^1 |u_1(x, y, t) - u_2(x, y, t)| dy dx \\
& \leq c_6 \left\{ \int_0^1 \int_0^L |u_{10}(x, y) - u_{20}(x, y)| dx dy + \int_0^t \int_0^1 |u_{11}(y, s) - u_{21}(y, s)| dy ds \right. \\
& \quad \left. + \int_0^t \int_0^L |v_{01}(x, s) - v_{02}(x, s)| dx ds \right\},
\end{aligned} \tag{3.35}$$

where $c_6 = c_6(Q, T)$ is a positive constant.

Proof: This proposition follows from Proposition 3.4 by choosing appropriate test function ϕ . Indeed, let $\rho = \rho(\xi)$ be the mollifying function defined in the proof of Proposition 3.4. Fix any $t \in (0, T]$. For $\varepsilon \in (0, \frac{1}{4} \min(t, L, 1))$, we set

$$\phi_{0\varepsilon}(\tau) = \int_{\tau-t+2\varepsilon}^{\tau-2\varepsilon} \rho_\varepsilon(s) ds, \quad \phi_{1\varepsilon}(x) = \int_{x-l+2\varepsilon}^{x-2\varepsilon} \rho_\varepsilon(s) ds, \quad \phi_{2\varepsilon}(y) = \int_{y-1+2\varepsilon}^{y-2\varepsilon} \rho_\varepsilon(s) ds.$$

Then, clearly, $\phi_{0\varepsilon} \in c_0^\infty(0, t)$, $\phi_{1\varepsilon} \in c_0^\infty(0, L)$, $\phi_{2\varepsilon} \in c_0^\infty(0, 1)$, and

$$0 \leq \phi_{0\varepsilon}(\tau), \phi_{1\varepsilon}(x), \phi_{2\varepsilon}(y) \leq 1, \quad \forall (t, x, y) \in (0, t) \times [0, L] \times [0, 1]. \quad (3.36)$$

Set,

$$\phi_\varepsilon(x, y, \tau) = (1-y)^2 \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) \phi_{2\varepsilon}(y), \quad \forall (x, y, \tau) \in Q_t. \quad (3.37)$$

Then,

$$\phi_\varepsilon \in c_0^\infty(Q_t), \quad \text{and} \quad 0 \leq \phi_\varepsilon(x, y, \tau) \leq 1, \quad \forall (x, y, \tau) \in Q_t. \quad (3.38)$$

It then follows from (3.12) in Proposition 3.4 that

$$\begin{aligned} & I(u_1, u_2, \phi_\varepsilon) \\ & \equiv \int_{Q_t} \{ |v_1(x, y, \tau) - v_2(x, y, \tau)| (\partial_{\bar{t}} \phi_\varepsilon + \partial_x(a \phi_\varepsilon) + \partial_y(b \phi_\varepsilon) - c \phi_\varepsilon) - \partial_y \phi_\varepsilon \partial_y |u_1 - u_2| \} dx dy d\tau \quad (3.39) \\ & \geq 0 \end{aligned}$$

with $v_i = (u_i)^{-1}$. We can rewrite $I(u_1, u_2, \phi_\varepsilon)$ as

$$\begin{aligned} I(u_1, u_2, \phi_\varepsilon) &= \int_{Q_t} (1-y)^2 |v_1 - v_2| \phi'_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) \phi_{2\varepsilon}(y) dx dy d\tau \\ &\quad + \int_{Q_t} (1-y)^2 a |v_1 - v_2| \phi_{0\varepsilon}(\tau) \phi'_{1\varepsilon}(x) \phi_{2\varepsilon}(y) dx dy d\tau \\ &\quad + \int_{Q_t} b |v_1 - v_2| \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) \left((1-y)^2 \phi_{2\varepsilon}(y) \right)_y dx dy d\tau \\ &\quad - \int_{Q_t} (\partial_y |u_1 - u_2|) \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) \left((1-y)^2 \phi_{2\varepsilon}(y) \right)_y dx dy d\tau \quad (3.40) \\ &\quad + \int_{Q_t} (\partial_x a + \partial_y b - c) |v_1 - v_2| \phi_\varepsilon dx dy d\tau \\ &\equiv \sum_{i=1}^5 K_i^\varepsilon \end{aligned}$$

Each term on the right hand above can be estimated as follows. First, since

$$\begin{aligned} K_1^\varepsilon &\equiv \int_{Q_t} (1-y)^2 |v_1 - v_2| \phi'_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) \phi_{2\varepsilon}(y) dx dy d\tau \\ &= \int_{Q_t} (1-y)^2 |v_1 - v_2| (\rho_\varepsilon(\tau - 2\varepsilon) - \rho_\varepsilon(\tau - t + 2\varepsilon)) \phi_{1\varepsilon}(x) \phi_{2\varepsilon}(y) dx dy d\tau, \end{aligned}$$

it follows that as $\varepsilon \rightarrow 0^+$

$$K_1^\varepsilon \longrightarrow - \int_0^L \int_0^1 (1-y)^2 |v_1 - v_2| \Big|_{\tau=0}^{\tau=t} dy dx \equiv K_1. \quad (3.41)$$

Similarly, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned}
K_2^\varepsilon &\equiv \int_{Q_t} (1-y)^2 |v_1 - v_2| \phi_{0\varepsilon}(\tau) \phi'_{1\varepsilon}(x) \phi_{2\varepsilon}(y) dx dy d\tau \\
&\longrightarrow - \int_0^t \int_0^1 (1-y)^2 a |v_1 - v_2| \Big|_{x=0}^{x=L} dy d\tau \\
&\equiv K_2,
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
K_3^\varepsilon &\equiv \int_{Q_t} b |v_1 - v_2| \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) ((1-y)^2 \phi_{2\varepsilon}(y))_y dy dx d\tau \\
&= \int_{Q_t} b |v_1 - v_2| \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) [2(y-1) \phi_{2\varepsilon}(y) \\
&\quad + (1-y)^2 (\rho_\varepsilon(y-2\varepsilon) - \rho_\varepsilon(y-1+2\varepsilon))] dy dx d\tau \\
&\longrightarrow \int_{Q_t} b |v_1 - v_2| 2(y-1) dx dy d\tau + \int_0^t \int_0^L (\gamma b \gamma |v_1 - v_2|) \Big|_{y=0} dx dt \\
&\equiv K_3,
\end{aligned} \tag{3.43}$$

here and what follows, by $\gamma w|_\theta$, we mean the trace of w on θ .

Next, we deal with K_4^ε . Note that

$$\begin{aligned}
K_4^\varepsilon &\equiv - \int_{Q_t} (\partial_y |u_1 - u_2|) \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) ((1-y)^2 \phi_{2\varepsilon}(y))_y dx dy d\tau \\
&= - \int_{Q_t} (\partial_y |u_1 - u_2|) \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) (1-y)^2 (\rho_\varepsilon(y-2\varepsilon) - \rho_\varepsilon(y-1+2\varepsilon)) dx dy d\tau \\
&\quad + \int_{Q_t} (\partial_y |u_1 - u_2|) \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) 2(y-1) \phi_{2\varepsilon}(y) dx dy d\tau \\
&\equiv K_{41}^\varepsilon + K_{42}^\varepsilon.
\end{aligned}$$

It is clear that

$$K_{41}^\varepsilon \longrightarrow - \int_0^t \int_0^L \gamma (\partial_y |u_1 - u_2|) \Big|_{y=0} dx d\tau \quad \text{as } \varepsilon \rightarrow 0^+.$$

And since,

$$\begin{aligned}
K_{42}^\varepsilon &\equiv - \int_{Q_t} (\partial_y |u_1 - u_2|) \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) 2(y-1) \phi_{2\varepsilon}(y) dx dy dt \\
&= + \int_{Q_t} |u_1 - u_2| \phi_{0\varepsilon}(\tau) \phi_{1\varepsilon}(x) (2\phi_{2\varepsilon}(y) + 2(y-1) (\rho_\varepsilon(y-2\varepsilon) - \rho_\varepsilon(y-1+2\varepsilon))) dy dx d\tau,
\end{aligned}$$

so as $\varepsilon \rightarrow 0^+$,

$$K_{42}^\varepsilon \longrightarrow 2 \int_{Q_t} |u_1 - u_2| dy dx d\tau - 2 \int_0^t \int_0^L \gamma |u_1 - u_2| \Big|_{y=0} dx dt.$$

Consequently, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned}
K_4^\varepsilon &\longrightarrow 2 \int_{Q_t} |u_1 - u_2| dx dy d\tau - 2 \int_0^t \int_0^L \gamma |u_1 - u_2| \Big|_{y=0} dx dt - \int_0^t \int_0^L \gamma (\partial_y |u_1 - u_2|) \Big|_{y=0} dx dt \\
&\equiv K_4
\end{aligned} \tag{3.44}$$

Finally, one has that as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned}
K_5^\varepsilon &\equiv \int_{Q_t} (\partial_x a + \partial_y b - c) |v_1 - v_2| \phi_\varepsilon \, dx \, dy \, d\tau \\
&\longrightarrow \int_{Q_t} (\partial_x a + \partial_y b - c) |v_1 - v_2| (1-y)^2 \, dx \, dy \, d\tau \\
&= K_5.
\end{aligned} \tag{3.45}$$

It follows from (3.39) - (3.45) that

$$\begin{aligned}
0 &\leq K_1 + K_2 + K_3 + K_4 + K_5 \\
&= - \left\{ \int_0^L \int_0^1 (1-y)^2 \gamma |v_1 - v_2| \Big|_{\tau=t} \, dx \, dy + \int_0^t \int_0^1 (1-y)^2 (a\gamma |v_1 - v_2|) \Big|_{x=L} \, dy \, d\tau \right. \\
&\quad \left. + 2 \int_0^t \int_0^L \gamma |u_1 - u_2| \Big|_{y=0} \, dx \, d\tau \right\} + \int_0^L \int_0^1 (1-y)^2 \gamma |v_1 - v_2| \Big|_{\tau=0} \, dx \, dy \\
&\quad + \int_0^t \int_0^1 (1-y)^2 (a\gamma |v_1 - v_2|) \Big|_{x=0} \, dy \, dt + \int_{Q_t} b 2(y-1) |v_1 - v_2| \, dx \, dy \, d\tau \\
&\quad + 2 \int_{Q_t} |u_1 - u_2| \, dx \, dy \, d\tau + \int_{Q_t} (\partial_x a + \partial_y b - c) |v_1 - v_2| (1-y)^2 \, dx \, dy \, d\tau \\
&\quad + \int_0^t \int_0^L (b\gamma |v_1 - v_2| - \gamma(\partial_y |u_1 - u_2|)) \Big|_{y=0} \, dx \, d\tau
\end{aligned} \tag{3.46}$$

On the other hand, it follows from the boundary condition in (2.1) that

$$(b\gamma |v_1 - v_2| - \gamma(\partial_y |u_1 - u_2|)) \Big|_{y=0} = \overline{-\operatorname{sgn}(u_1 - u_2)} (v_{01} - v_{02}) \quad \text{a.e.}$$

Thus,

$$\int_0^t \int_0^L (b\gamma |v_1 - v_2| - \gamma(\partial_y |u_1 - u_2|)) \Big|_{y=0} \, dx \, dt \leq \int_0^t \int_0^L |v_{01} - v_{02}| \, dx \, d\tau. \tag{3.47}$$

Noting that

$$\begin{aligned}
c_0 |u_1 - u_2| &\leq (1-y)^2 |v_1 - v_2| \leq c_0^{-1} |u_1 - u_2|, \quad \text{and} \\
|b(x, y, t)(1-y) |v_1 - v_2|| &\leq c_0^{-1} |u_1 - u_2|,
\end{aligned}$$

one gets from (3.46) and (3.47) that

$$\begin{aligned}
&\int_0^L \int_0^1 |u_1(x, y, t) - u_2(x, y, t)| \, dy \, dx + \int_0^t \int_0^1 |u_1(L, y, t) - u_2(L, y, t)| \, dy \, dt \\
&\quad + \int_0^t \int_0^L |u_1(x, 0, t) - u_2(x, 0, t)| \, dx \, dt \\
&\leq \tilde{c}_0 \left\{ \int_0^L \int_0^1 |u_{10}(x, y) - u_{20}(x, y)| \, dy \, dx + \int_0^t \int_0^1 |u_1(0, y, t) - u_2(0, y, s)| \, dy \, dt \right. \\
&\quad \left. + \int_0^t \int_0^L |v_{01} - v_{02}|(x, t) \, dx \, dt + \int_{Q_t} |u_1 - u_2| \, dx \, dy \, d\tau \right\}
\end{aligned} \tag{3.48}$$

This yields the desired estimate (3.35) immediately. So the proof of Proposition 3.5 is completed. \square

As a consequence of Proposition 3.5, we conclude that

Theorem 3.6 The initial-boundary value problem (2.1) has a unique weak solution u as defined in Theorem 2.1. Furthermore, such a weak solution depends on its initial and boundary data continuously in L^1 -norm.

In next section, we will study the regularity of the weak solution to the initial-boundary value problem (2.1).

§4 Interior Regularity

We now turn to establish the interior smoothness of the unique weak solution obtained in the previous sections. The main result of this section can be stated as follows:

Theorem 4.1 *Under the same assumptions as in Theorem 1.1, the unique weak solution to (1.7) is smooth in the interior of Ω_T .*

The proof of the theorem is based on studies of the regularity theory for a class of ultra-parabolic equations with rough coefficients, which is rather subtle and complicated. To illustrate our main ideas and make the presentation clear, we will only present the proof of the desired result for the case that the corresponding Euler flow is uniform, e.g.,

$$U(x, t) \equiv 1 \tag{4.1}$$

In this case, the problem (2.1) becomes

$$\begin{cases} \partial_t u - u^2 \partial_y^2 u + y \partial_x u = 0 \\ u(x, y, t = 0) = u_0(x, y), \quad u(x = 0, y, t) = u_1(y, t) \\ u(x, y = 1, t = 0) = 0, \quad \partial_y u(x, y = 0, t) = U_0(x, t) \end{cases} \tag{4.2}$$

It follows from Theorem 4.1 that (4.2) has a uniqueness weak solution $u \in BV(Q_T) \cap L^\infty(Q_T)$ with the properties in (2.3)-(2.8). We will study the interior regularity in

$$Q_T = \{(x, y, t) | 0 < t < T, 0 < x < L, 0 < y < 1\}.$$

For any fixed point $(x_0, y_0, t_0) \in Q_T$, there exists a positive constant $\delta > 0$ such that

$$\bar{B}_\delta = \bar{B}_\delta(x_0, y_0, t_0) \triangleq \{(x, y, t) | |x - x_0| \leq \delta^3, |y - y_0| \leq \delta, |t - t_0| < \delta^2\} \subset Q_T \tag{4.3}$$

One can study the regularity of $u(x, y, t)$ in B_δ . To this end, one can set $w(x, y, t) = u^{-1}(x, y, t)$. Then (4.2) implies that

$$\partial_t w - \partial_y(a \partial_y w) + y \partial_x w = 0 \quad \text{on} \quad B_\delta. \tag{4.4}$$

with

$$a = u^2(x, y, t) \tag{4.5}$$

By shifting and rescaling the independent variables

$$(\tilde{x}, \tilde{y}, \tilde{t}) = \left(\frac{x - x_0}{\delta^3}, \frac{y - y_0}{\delta}, \frac{t - t_0}{\delta^2} \right), \quad (4.6)$$

one obtains from (4.4) that

$$\partial_{\tilde{t}} w - \partial_{\tilde{y}}(a \partial_{\tilde{y}} w) + (\tilde{y} + y_0) \partial_{\tilde{x}} w = 0 \quad \text{on} \quad \tilde{B}_1(0, 0, 0) \quad (4.7)$$

with

$$\tilde{B}_1(0, 0, 0) = \{(\tilde{x}, \tilde{y}, \tilde{t}) \mid |\tilde{x}| < 1, |\tilde{y}| < 1, |\tilde{t}| < 1\} \quad (4.8)$$

The equation (4.7) can be simplified further by introduce the following transformation

$$\tau = \tilde{t}, \quad \xi = \tilde{x} - y_0 \tilde{t}, \quad \eta = \tilde{y}, \quad (4.9)$$

such that

$$\partial_{\tau} w - \partial_{\eta}(a \partial_{\eta} w) + \eta \partial_{\xi} w = 0 \quad \text{on} \quad \tilde{\tilde{B}}_1(0, 0, 0) \quad (4.10)$$

with $\tilde{\tilde{B}}_1(0, 0, 0) = \{(\tau, \xi, \eta) \mid |\xi + y_0 \tau| < 1, |\eta| < 1, |\tau| < 1\}$.

Choose constant $\delta_1 > 0$ so that

$$\bar{B}_{\delta_1}(0, 0, 0) \equiv \{(\tau, \xi, \eta) \mid |\xi| \leq \delta_1^3, |\eta| \leq \delta_1, |\tau|^2 \leq \delta_1^2\} \subset \tilde{\tilde{B}}_1(0, 0, 0) \quad (4.11)$$

Rescale again by

$$(\tilde{\xi}, \tilde{\eta}, \tilde{\tau}) = \left(\frac{\xi}{\delta_1^3}, \frac{\eta}{\delta_1}, \frac{\tau}{\delta_1^2} \right) \quad (4.12)$$

so that one can get from (4.10) that

$$\partial_{\tilde{\tau}} w - \partial_{\tilde{\eta}}(a \partial_{\tilde{\eta}} w) + \tilde{\eta} \partial_{\tilde{\xi}} w = 0 \quad \text{on} \quad \tilde{B}_1(0, 0, 0) \quad (4.13)$$

with

$$\tilde{B}_1(0, 0, 0) = \{(\tilde{\xi}, \tilde{\eta}, \tilde{\tau}) \mid |\tilde{\xi}| < 1, |\tilde{\eta}| < 1, |\tilde{\tau}| < 1\} \quad (4.14)$$

Thus we will study the equation (4.13). For notational convenience, we will use u for w , (x, y, t) for $(\tilde{\xi}, \tilde{\eta}, \tilde{\tau})$, and B_1 for \tilde{B}_1 . Hence consider

$$\partial_t u - \partial_y(a \partial_y u) + y \partial_x u = 0 \quad \text{on} \quad B_1(0, 0, 0) \quad (4.15)$$

with the assumption that

$$a \in L^\infty, \Lambda^{-1} \leq a \leq \Lambda \quad \text{on} \quad B_1(0, 0, 0) \quad (4.16)$$

for a positive constant Λ .

We will modify the idea of Krushkov in [5] on the C^α -regularity theory of weak solutions to uniform parabolic equations to analyze the regularity of weak solutions to the ultra parabolic equation (4.15) with the assumption (4.15). In particular, we introduce a "weak" form of Poincaré inequality. The key step is

to establish suitable oscillation estimates. To this end, one needs the following notations. For any given constants $r \in (0, 1], \alpha, \beta \in (0, 1)$, set

$$\begin{aligned} B_1 &= B_r(0, 0, 0) = \{(x, y, t) \mid |x| < r^3, |y| < r, |t| < r^2\}, \\ B_r^\pm &\triangleq B_r \cap \{t \gtrless 0\}, \\ C_r &= \{(x, y) \mid |x| < r^3, |y| < r\}, \\ H_{t,h} &\triangleq H_{t,h}^u = \{(x, y) \mid (x, y) \in C_{\beta r}, u(x, y, t) \geq h\} \end{aligned}$$

for any given weak solution u to (4.15), and any fixed $t \in (-r^2, 0)$ and $h \in (0, 1)$. Then the following estimates holds.

Proposition 4.2 *Let $u(x, y, t)$ be a non-negative weak solution to (4.15) in B_1^- with the property that*

$$\text{mes}\{(x, y, t) \in B_r^-, u(x, y, t) \geq 1\} \geq \frac{1}{2} \text{mes} B_r^-. \quad (4.17)$$

Then there exist constants $\alpha, \beta, h_1 \in (0, 1)$, $\alpha \ll 1$, $\beta \approx 1$, depending only on Λ , such that for almost all $t \in (-\alpha r^2, 0)$, $0 < h \leq h_1$, it holds that

$$\text{mes} H_{t,h} \geq \frac{1}{11} \text{mes} C_{\beta r} \quad (4.18)$$

Proof: For $h \in (0, \frac{1}{2})$, set

$$V(x, y, t) = \ln^+ \frac{1}{u(x, y, t) + h^{\frac{9}{8}}} = (-\ln(u + h^{\frac{9}{8}}))^+, \quad (4.19)$$

where, and in the following, f^+ denote the positive part of the function f .

Then it can be checked easily that V is a non-negative weak solution to the following equation

$$\partial_t V - \partial_y(a \partial_y V) + a(\partial_y V)^2 + y \partial_x V = 0 \quad \text{on } B_1^-. \quad (4.20)$$

Let $\chi(s)$ be a smooth cut-off function such that

$$\begin{cases} \chi \in C^\infty([0, \infty)), 0 \leq \chi \leq 1 & \text{for } 0 \leq s \leq \beta r, \chi(s) = 0 & \text{for } s \geq r \\ |\chi'(s)| \leq \frac{2}{(1-\beta)r} & \text{for all } s \geq 0 \end{cases} \quad (4.21)$$

It then follows from (4.20) that for almost all $t, \tau \in (0 - r^2, 0)$, $\tau \leq t$, it holds that

$$\begin{aligned} & \int_{C_r} \chi^2(|y|) V(x, y, t) dx dy + \int_\tau^t \int_{C_r} |\partial_y V|^2 a \chi^2(|y|) dx dy ds \\ &= \int_{C_r} \chi^2(|y|) V(x, y, \tau) dx dy - \int_\tau^t \int_{C_r} (2\chi \partial_y \chi) a \partial_y V dx dy ds - \int_\tau^t \int_{-r}^r \chi^2 y V|_{-r^3}^{r^3} dy ds \end{aligned} \quad (4.22)$$

Cauchy-Schwartz inequality yields

$$\begin{aligned} - \int_\tau^t \int_{C_r} (2\chi \partial_y \chi) a \partial_y V dx dy ds &\leq \frac{1}{2} \int_\tau^t \int_{C_r} a \chi^2 |\partial_y V|^2 dx dy ds + 4 \int_\tau^t \int_{C_r} a |\partial_y \chi|^2 dx dy ds \\ &\leq \frac{1}{2} \int_\tau^t \int_{C_r} a \chi^2 |\partial_y V|^2 dx dy ds + \frac{16\Lambda}{(1-\beta)^2 r^2} (t-\tau) \text{mes} C_r \\ &\leq \frac{1}{2} \int_\tau^t \int_{C_r} a \chi^2 |\partial_y V|^2 dx dy ds + \frac{16\Lambda}{\beta^4 (1-\beta)^2} \text{mes} C_{\beta r}. \end{aligned}$$

Due to the definition of V , one gets by direct calculations that

$$- \int_\tau^t \int_{-r}^r \chi^2 y V|_{-r^3}^{r^3} dy ds \leq (t-\tau) r^2 \ln h^{-\frac{9}{8}} \leq \frac{1}{4} \beta^{-4} \ln h^{-\frac{9}{8}} \text{mes} C_{\beta r}$$

These together with (4.22) yield

$$\begin{aligned} & \int_{C_r} \chi^2(|y|)V(x, y, t)dx dy + \frac{1}{2} \int_{\tau}^t \int_{C_r} a\chi^2|\partial_y V|^2 dx dy ds \\ & \leq \int_{C_r} \chi^2(|y|)V(x, y, \tau)dx dy + \frac{1}{4}\beta^{-4}lnh^{-\frac{9}{8}}\text{mes } C_{\beta\gamma} + \frac{16\Lambda}{\beta^4(1-\beta)^2}\text{mes } C_{\beta\gamma}. \end{aligned} \quad (4.23)$$

To estimate the last term on the right hand side above, one can define

$$\mu(t) = \text{mes} \{(x, y) \in C_r, u(x, y, t) \geq 1\} \quad \text{for any } t \in [-r^2, 0]. \quad (4.24)$$

It then follows from the assumption (4.17) and the definition (4.24) that

$$\int_{-r^2}^{\circ} \mu(t)d\tau = \text{mes} \{(x, y, t) \in B_r^-, u(x, y, t) \geq 1\} \geq \frac{1}{2} \text{mes } B_r^- = \frac{1}{2}r^2 \text{mes } C_r$$

which implies that for any constant $\alpha \in (0, \frac{1}{2})$,

$$\int_{-r^2}^{-\alpha r^2} \mu(t)dt \geq \frac{1}{2}r^2 \text{mes } C_r - \int_{-\alpha r^2}^{\circ} \mu(t)dt \geq (\frac{1}{2} - \alpha)r^2 \text{mes } C_r.$$

Thus there exists $\tau \in (-r^2, -\alpha r^2)$ such that

$$\mu(\tau) \geq (\frac{1}{2} - \alpha)(1 - \alpha)^{-1} \text{mes } C_r. \quad (4.25)$$

Note that the definition of V and (4.25) imply that

$$\begin{aligned} \int_{C_r} V(x, y, \tau)dx dy &= \int_{C_r \cap \{u \leq 1\}} V(x, y, \tau)dx dy \\ &\leq (lnh^{-\frac{9}{8}})\text{mes} \{C_r \cap \{u \leq 1\}\} \\ &= lnh^{-\frac{9}{8}}(\text{mes } C_r - \mu(\tau)) \leq \frac{1}{2}(1 - \alpha)^{-1} \text{mes } C_r lnh^{-\frac{9}{8}}. \end{aligned} \quad (4.26)$$

We now turn to estimate the first term on the left hand side of (4.23). Note that for $h \in (0, \frac{1}{2})$, $V(x, y, t) \geq ln\frac{1}{h+h^{\frac{9}{8}}}$ for all $(x, y) \notin H_{t,h}$. It holds that

$$\int_{C_r} \chi^2 V(x, y, t)dx dy \geq \int_{C_{\beta r}} V(x, y, t)dx dy \geq ln\frac{1}{h+h^{\frac{9}{8}}}\text{mes}(C_{\beta r} \setminus H_{t,h}). \quad (4.27)$$

It follows from (4.23), (4.26) and (4.27) that

$$\text{mes}(C_{\beta r} \setminus H_{t,h}) \leq \frac{lnh^{-\frac{9}{8}}}{ln\frac{1}{h+h^{\frac{9}{8}}}} \left(\frac{1}{2}(1 - \alpha)^{-1}\beta^{-4} + \frac{1}{4}\beta^{-4} + \frac{16\Lambda}{\beta^4(1-\beta)^2} \frac{1}{lnh^{-\frac{9}{8}}} \right) \text{mes } C_{\beta r} \quad (4.28)$$

Since $\lim_{h \rightarrow 0^+} \frac{lnh^{-\frac{9}{8}}}{ln\frac{1}{h+h^{\frac{9}{8}}}} = \frac{9}{8}$, so there exist constants $\alpha, \beta, h_1 \in (0, 1)$ with property that $\alpha \ll 1, \beta \approx 1$, and $h_1 = h_1(s)$ suitably small such that for $h \in (0, h_1)$,

$$\frac{lnh^{-\frac{9}{8}}}{ln\frac{1}{h+h^{\frac{9}{8}}}} \left(\frac{1}{2}(1 - \alpha)^{-1}\beta^{-4} + \frac{1}{4}\beta^{-4} + \frac{16\Lambda}{\beta^4(1-\beta)^2} \frac{1}{lnh^{-\frac{9}{8}}} \right) \leq \frac{10}{11}.$$

This and (4.28) yield the desired estimate (4.18). The proof of Proposition 4.2 is complete.

The next key element of our analysis is a "weak" form of Poincaré's inequality based on the fundamental solution of the equation (4.15) with $a \equiv 1$. This kind of "weak" form of Poincaré's inequality is needed for non-negative sub-solutions to (4.15) defined as follows:

Definition 4.3 u is said to be a weak subsolution to (4.15) on Ω if $u, y\partial_x u + \partial_t u \in L^2_{loc}(\Omega)$, $u \in L^\infty_{loc}(\Omega)$, and for any $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, it holds that

$$-\int_{\Omega}^1 (a\partial_\varphi \varphi \partial_y u + (\partial_t u + y\partial_x u)\varphi) dx dy dt \geq 0. \quad (4.29)$$

Consider the following basic ultra-parabolic equation

$$\mathcal{L}_0 u \equiv \partial_t u - \partial_y^2 u + y\partial_x u = 0 \quad (4.30)$$

Set $z = (x, y, t)$ and $\zeta = (\xi, \eta, \tau)$. Then the fundamental solution of \mathcal{L}_0 can be constructed in [14, 15, 10] as

$$\Gamma_0(z, \zeta) = \Gamma_0(\zeta^{-1}oz, 0) = \begin{cases} \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\left\{-\frac{(y-\eta)^2}{t-\tau} - \frac{3}{(t-\tau)^2}(x-\xi - \frac{(t-\tau)}{2}(y+\eta)^2)\right\} & t > \tau \\ 0 & t \leq \tau, \end{cases} \quad (4.31)$$

where ζoz is the left translation of z and ζ in the group law associated with \mathcal{L}_0 . It is known that

$$\int_{\mathbb{R}^2} \Gamma_0(z, \zeta) dx dy = \int_{\mathbb{R}^2} \Gamma_0(z, \zeta) d\xi d\eta = 1 \quad \text{for } t > \tau, \quad (4.32)$$

and

$$\Gamma_0(\delta_\mu oz, 0) = \mu^{-4} \Gamma_0(z, 0), \quad \forall z \neq 0, \quad \mu > 0 \quad (4.33)$$

where δ_μ is the dilations associated with \mathcal{L}_0 . To establish a ‘‘weak’’ form of the Poincaré’s inequality for non-negative weak subsolutions to (4.15), one need to construct suitable test functions. To this end, one chooses a cut-off function $\chi \in C^\infty([0, \infty))$ with the properties that $0 < \chi(s) < s$ for $0 \leq s < r$, $\chi(s) = 0$ for all $s > r$, $\chi(s) \equiv 1$ for $0 \leq s \leq \theta^{\frac{1}{6}} r$, and

$$\begin{cases} 0 \leq -\chi'(s) \leq \frac{2}{(1-\theta^{\frac{1}{6}})r}, |\chi''(s)| \leq \frac{C}{r^2}, & \text{for all } s \in \mathbb{R}_+^1, \\ |\chi'(s)| \geq C(\beta_1, \beta_2)r^{-1} > 0 & \text{for all } s \in [\beta_1 r, \beta_2 r], \quad \theta^{\frac{1}{6}} < \beta_1 < \beta_2 < 1, \end{cases} \quad (4.34)$$

where $\theta \in (0, 2^{-6})$ is a positive constant to be chosen and C and $C(\beta_1, \beta_2)$ are fixed positive constants. Set

$$Q_\theta^- = \left\{ (x, y, t) \mid -r^2 \leq t \leq 0, \quad |y| \leq \frac{r}{\theta}, \quad |x| \leq \frac{r^3}{\theta} \right\} \quad (4.35)$$

and

$$\phi(x, y, t) = \phi_0(x, y, \tau)\phi_1(x, y, t) \quad (4.36)$$

with

$$\phi_0(x, y, t) = \chi([\theta^2 x^2 - 6tr^4]^{\frac{1}{6}}), \quad \phi_1(x, y, t) = \chi(\theta|y|) \quad (4.37)$$

Then the following elementary facts can be verified by direct computations:

Lemma 4.4 *It holds that*

1.

$$(-\partial_x - \partial_t)\phi_0(z) \leq 0 \quad \text{for } z \in Q_\theta^-; \quad (4.38)$$

2.

$$\phi(z) \equiv 1 \quad \text{on} \quad B_{\theta r}^-; \quad (4.39)$$

3.

$$\text{supp } \phi \cap \{(x, y, t) | t \leq 0\} \subset Q_\theta^-; \quad (4.40)$$

4. There exists a positive constant $\alpha_1 \in (0, \min(\alpha, \frac{1}{12}))$ such that

$$\{(x, y, t) | -\alpha_1 r^2 \leq t \leq 0, (x, y) \in C_{\beta r}\} \subseteq \text{supp } \phi, \quad (4.41)$$

5. Assume that $\alpha_1 > \theta$. Then

$$0 < \phi_0(z) < 1 \quad \text{on} \quad \{(x, y, t) | -\alpha_1 r \leq t \leq -\theta r^2, (x, y) \in C_{\beta r}\}. \quad (4.42)$$

Now we choose θ and α_1 such that all the requirements in Lemma 4.4 are satisfied. Thus (4.38)-(4.42) hold true. Then we have the following key inequality of Poincaré type for any non-negative weak subsolutions to (4.15).

Proposition 4.5 *Let w be a non-negative weak subsolution to (4.15) in B_1^- . Then there exists a uniform positive constant C such that for any $r \in (0, \theta)$, it holds that*

$$\int_{B_{r/\theta}^-} |(w(z) - I_0)^+|^2 dz \leq C\theta^2 r^2 \int_{B_{r/\theta}^-} |\partial_y w(z)|^2 dz, \quad (4.43)$$

where $I_0 = \sup_{B_{\theta r}^-} I_1(z)$ with

$$I_1(z) = \int_{B_{r/\theta}^-} \partial_\eta \phi(\zeta) \partial_\eta \Gamma_0(z, \zeta) w(\zeta) d\zeta + \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) |\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)| w(\zeta) d\zeta. \quad (4.44)$$

Proof: Since $\Gamma_0(z, \zeta)$ is the fundamental solution to (4.40), so a standard approximation [10] implies that

$$(w\phi)(z) = \int_{\mathbb{R}^{2+1}} [\partial_\eta \Gamma_0 \partial_\eta (w\phi) + \Gamma_0 (\partial_\tau + \eta \partial_\xi)(w\phi)] d\zeta. \quad (4.45)$$

It follows from this, Lemma 4.4 and $Q_\theta^- \subseteq B_{r/\theta}^-$ that for $z \in B_{\theta r}^-$,

$$\begin{aligned} w(z) &= \int_{B_{r/\theta}^-} [\partial_\eta \Gamma_0 \partial_\eta (w\phi) + \Gamma_0 (\partial_\tau + \eta \partial_\xi)(w\phi)] d\zeta \\ &= I_1(z) + \int_{B_{r/\theta}^-} [-\Gamma_0 \partial_\eta \phi \partial_\eta w + \phi \partial_\eta \Gamma_0 \partial_\eta w + \phi \Gamma_0 (\partial_\tau w + \eta \partial_\xi w)] d\zeta \\ &= I_1(z) + I_2(z) + I_3(z), \end{aligned} \quad (4.46)$$

where $I_1(z)$ is given by (4.44), while I_2 and I_3 are given by

$$I_2(z) = \int_{B_{r/\theta}^-} [-(1 + a(\zeta)) \Gamma_0(z, \zeta) \partial_\eta \phi(\zeta) \partial_\eta w(\zeta) + (1 - a(\zeta)) \phi(\zeta) \partial_\eta \Gamma_0(z, \zeta) \partial_\eta w(\zeta)] d\zeta, \quad (4.47)$$

$$I_3(z) = \int_{B_{r/\theta}^-} [a(\zeta) \partial_\eta w(\zeta) \partial_\eta (\Gamma_0(z, \zeta) \phi(\zeta)) + \phi(\zeta) \Gamma_0(z, \zeta) (\partial_\tau w + \eta \partial_\xi w)] d\zeta. \quad (4.48)$$

First, note that by approximation if necessary, one can take $\phi(\zeta)\Gamma_0(z, \zeta)$ as a non-negative test function (4.29), thus it follows from Definition 4.3 that $I_3(z) \leq 0$ (as in the proof of Lemma 2.5 in [10]). This and (4.46) imply that for any $z \in B_{\theta r}^-$,

$$0 \leq (w(z) - I_0)^+ \leq (w(z) - I_1(z))^+ \leq I_2(z)^+. \quad (4.49)$$

It remains to estimate $I_2(z) \equiv I_{21}(z) + I_{22}(z)$ as follows. Note that for $z \in B_{\theta r}^-$,

$$\begin{aligned} I_{22}(z) &= \int_{B_{r/\theta}^-} ((1 - a(\zeta))\phi(\zeta)\partial_\eta w(\zeta))\partial_\eta \Gamma_0(z, \zeta) d\zeta \\ &= \int_{\mathbb{R}^3} [(1 - a(\zeta))\phi(\zeta)\chi_{\tau \leq 0}] \partial_\eta w(\zeta) \partial_\eta \Gamma_0(z, \zeta) d\zeta \end{aligned}$$

where $\chi_{\tau \leq 0}$ is the characteristic function on the set $\{\tau \leq 0\}$. It follows from (4.40) and Pascucci-Polidoro's estimate (Corollary 2.2 in [10]) that

$$\begin{aligned} \|I_{22}\|_{L^3(B_{\theta r}^-)} &\leq C \|[(1 - a(\zeta))\chi_{\tau \leq 0}\phi(\zeta)]\partial_\eta w(\zeta)\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|\partial_\eta w\|_{L^2(B_{r/\theta}^-)}, \end{aligned}$$

which implies that

$$\|I_{22}\|_{L^2(B_{\theta r}^-)} \leq C\theta r \|\partial_\eta w\|_{L^2(B_{r/\theta}^-)}. \quad (4.50)$$

Next, note that for $z \in B_{\theta r}^-$,

$$\begin{aligned} I_{21}(z) &= - \int_{B_{r/\theta}^-} (1 + a)\Gamma_0(z, \zeta)\partial_\eta w(\zeta)\partial_\eta \phi(\zeta) d\zeta \\ &= - \int_{\mathbb{R}^{2+1}} \Gamma_0(z, \zeta)((1 + a)\partial_\eta w(\zeta)\partial_\eta \phi(\zeta)\chi_{\tau \leq 0}) d\zeta. \end{aligned}$$

It then follows from the Pascucci-Polidoro estimate [10], (4.40) and (4.34) that

$$\begin{aligned} \|I_{21}\|_{L^6(B_{\theta r}^-)} &\leq \|\Gamma_0((1 + a)\partial_\eta w\partial_\eta \phi\chi_{\tau \leq 0})\|_{L^6(\mathbb{R}^{2+1})} \\ &\leq C \|(1 + a)\chi_{\tau \leq 0}\partial_\eta w\partial_\eta \phi\|_{L^2(\mathbb{R}^{2+1})} \\ &\leq C \|\chi_{\tau \leq 0}\partial_\eta \phi\partial_\eta w\|_{L^2(B_{r/\theta}^-)} \leq \frac{C}{r} \|\partial_\eta w\|_{L^2(B_{r/\theta}^-)}, \end{aligned}$$

which yields that

$$\|I_{21}\|_{L^2(B_{\theta r}^-)} \leq C\theta^2 r \|\partial_\eta w\|_{L^2(B_{r/\theta}^-)}. \quad (4.51)$$

Thus, the desired estimate (4.43) follows from (4.49), (4.50) and (4.51). This completes the proof of Proposition 4.5.

Proposition 4.2 will be applied to a special class of non-negative weak solutions of (4.15). Indeed, let u be a non-negative weak solution to (4.15) in B_1^- . Set

$$w(z) = ln^+ \frac{h}{h^{\frac{\alpha}{8}} + u(z)}. \quad (4.52)$$

Then it can be checked that $w(z)$ is a non-negative weak subsolution to (4.15). We will apply Proposition 4.5 to estimate w . To this end, one needs to derive a key estimate on I_0 which is a ‘‘mean value’’ of w .

Lemma 4.6 *Let u be a non-negative weak solution to (4.15) in B_1^- satisfying the assumptions in Proposition 4.1, and $w(z)$ be defined by (4.52). Then there exist uniform positive constants $h_0(\leq h_1)$, θ , λ_0 (independent of u) such that*

$$\lambda_0 < 1, \quad |I_0| \leq \lambda_0 \ln(h^{-\frac{1}{8}}) \quad \text{for all } r < \theta, 0 < h \leq h_0, \quad (4.53)$$

with I_0 defined in Proposition 4.5.

Proof: Since u is a non-negative weak solution to (4.15), it can be checked directly that $w(z)$ given by (4.52) is a non-negative weak subsolution to (4.15) by Definition 4.1. Furthermore, following the same arguments in Lemma 4.4 and Proposition 4.5, one can check easily that Proposition 4.5 applies to w . So let I_0 be the corresponding ‘‘mean value’’ of w defined in Proposition 4.5. Then

$$\begin{aligned} I_0 &= \sup_{B_{\theta r}^-} I_1(z) \quad \text{with} \\ I_1(z) &= \int_{B_{r/\theta}^-} [-\Gamma_0(z, \zeta) \partial_\eta^2 \phi(\zeta) w(\zeta)] d\zeta + \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (\partial_\tau \phi + \eta \partial_\zeta \phi) w(\zeta) d\zeta \equiv I_{11} + I_{12} \end{aligned} \quad (4.54)$$

where one has integrated by parts and used Lemma 4.4 and the structure of $\partial_\eta \phi(\xi)$.

We start to estimate $I_{11} = \int_{B_{r/\theta}^-} [-\Gamma_0(z, \xi) \partial_\eta^2 \phi(\xi) w(\xi)] d\xi$. Note that for any $z \in B_{\theta r}^-$, $\text{supp}((\partial_\eta^2 \phi(\zeta)) \Gamma_0(z, \zeta) w(\zeta)) \subset B_{r/\theta}^-$. Thus

$$\begin{aligned} |I_{11}| &\leq \int_{B_{r/\theta}^-} |\partial_\eta^2 \phi(\zeta)| w(\zeta) \Gamma_0(z, \zeta) d\zeta \\ &\leq \ln h^{-\frac{1}{8}} \int_{-r^2}^0 \int_{\mathbb{R}^2} |\partial_\eta^2 \phi(\zeta)| \Gamma_0(z, \zeta) d\zeta \\ &\leq \ln h^{-\frac{1}{8}} r^2 \sup_{\text{supp} \phi} |\partial_\eta^2 \phi(\zeta)|, \end{aligned} \quad (4.55)$$

where one has used (4.32) and the fact

$$B_{r/\theta}^- \cap \text{supp} \phi \subset Q_\theta^-$$

due to Lemma 4.4. On the other hand, it follows from (4.36)-(4.37), (4.34), and direct computations that

$$|\partial_\eta^2 \phi(\zeta)| \leq c\theta^2 r^{-2}.$$

This, together with (4.55), yields

$$|I_{11}| \leq c\theta^2 \ln(h^{-\frac{1}{8}}). \quad (4.56)$$

We now turn to the estimate $I_{12} = \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) w(\zeta) d\zeta$.

First, note that for suitably small $\theta > 0$, $z \in B_{r/\theta}^-$, $\phi(z) = 1$. Thus for $z \in B_{r/\theta}^-$,

$$\begin{aligned} 1 &= \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (\partial_\tau + \eta \partial_\xi - \partial_\eta^2) \phi(\zeta) d\zeta \\ &= \int_{B_{r/\theta}^-} \phi_1 \Gamma_0(z, \zeta) (\partial_\tau \phi_0 + \eta \partial_\xi \phi_0) d\zeta + \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (-\partial_\eta^2 \phi(\zeta)) d\zeta. \end{aligned}$$

This and the arguments for (4.55)-(4.56) show that

$$\int_{B_{r/\theta}^-} \phi_1(\zeta) \Gamma_0(z, \zeta) (\partial_\tau \phi_0(\zeta) + \eta \partial_\xi \phi_0(\zeta)) d\zeta = 1 + O(1)\theta^2. \quad (4.57)$$

Next, let α_1, β and h_1 be given in Proposition 4.2 and set

$$S = \left\{ \zeta = (\xi, \eta, \tau) \mid -\alpha_1 r^2 \leq \tau \leq -\frac{\alpha_1}{2} r^2, \quad (\xi, \eta) \in C_{\beta r}, \quad w(\zeta) = 0 \right\} \quad (4.58)$$

Note that if ζ is such that $u(\zeta) \geq h$, then $w(\zeta) = 0$, and u is assumed to satisfy the assumptions in Proposition 4.2. Thus it follows from Proposition 4.2 and the definition of S that there exists a positive constant $c(\alpha_1, \beta) > 0$ such that

$$\text{mes } S \geq c(\alpha_1, \beta) r^6 \quad \text{for } h \leq h_1 \quad (4.59)$$

Furthermore, it follows from the construction of χ ((4.34)) and Lemma 4.4 that for θ suitably small and any $\zeta \in S$, $\phi_1(\zeta) = 1$, $\phi_0(\zeta) > 0$, $\theta^{\frac{1}{6}} r < (\theta^2 |\xi|^2 - 6\tau r^4)^{\frac{1}{6}} < r$, $6r^4 - 2\eta\xi\theta^2 > 3r^4$, and

$$|\chi'([\theta^2 |\xi|^2 - 6\tau r^4]^{\frac{1}{6}})| \geq \frac{c(\alpha_1, \theta)}{r} > 0$$

with a uniform constant $c(\alpha_1, \theta)$ independent of r . Hence

$$\begin{aligned} & \int_S \Gamma_0(z, \zeta) (\partial_\tau \phi_0(\zeta) + \eta \partial_\xi \phi_0(\zeta)) \phi_1(\zeta) d\zeta \\ &= \int_S \Gamma_0(z, \zeta) |\chi'([\theta^2 |\xi|^2 - 6\tau r^4]^{\frac{1}{6}})| \frac{1}{6} [\theta^2 \xi^2 - 6\tau r^4]^{\frac{5}{6}} [6r^4 - 2\eta\xi\theta^2] d\zeta \\ &\geq c(\alpha_1, \theta) \int_S r^{-2} \Gamma_0(z, \zeta) \alpha \zeta \geq \underline{c} > 0, \end{aligned} \quad (4.60)$$

where $\underline{c} = c(\alpha_1, \beta, \theta) > 0$, and one has used (4.59) and the fact that

$$\Gamma_0(z, \zeta) \geq cr^{-4} \quad \text{for } \tau \leq -\frac{\alpha_1}{2} r^2, \quad z \in B_{\theta r}^-$$

if $\theta^2 \leq \frac{1}{4}\alpha_1$. We are now ready to estimate I_{12} . It follows from (4.58) and Lemma 4.4 that

$$\begin{aligned} |I_{12}| &= \left| \int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) w(\zeta) d\zeta \right| \\ &= \int_{B_{r/\theta}^- \setminus S} \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) w(\zeta) d\zeta \\ &\leq \ln h^{-\frac{1}{8}} \int_{B_{r/\theta}^- \setminus S} \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) d\zeta \\ &= \ln h^{-\frac{1}{8}} \left(\int_{B_{r/\theta}^-} \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) d\zeta - \int_S \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) d\zeta \right) \\ &= \ln h^{-\frac{1}{8}} \left(1 + O(1)\theta^2 - \int_S \Gamma_0(z, \zeta) (\partial_\tau \phi(\zeta) + \eta \partial_\xi \phi(\zeta)) d\zeta \right) \\ &\leq (1 - \underline{c} + O(1)\theta^2) \ln h^{-\frac{1}{8}}, \end{aligned} \quad (4.61)$$

where one has used (4.57) and (4.60).

Consequently, it holds that

$$I_0 = \sup_{B_{\theta r}^-} I_1(z) \leq (1 - \underline{c} + O(1)\theta^2) \ln h^{-\frac{1}{8}}$$

for any $r < \theta$, $h \leq h_1$, and θ being suitably small. Set $\lambda_0 = 1 - \underline{c} + O(1)\theta^2$. Then the conclusion in Lemma 4.2 holds. This completes the proof of Lemma 4.6. \square

We are now ready to give the main step in the oscillation estimates.

Lemma 4.7 *Let u be a non-negative weak solution to (4.15) in B_1^- satisfying the assumption in Proposition 4.2. Then there exist positive constants \bar{h} and $\bar{\theta}$ in $(0,1)$, which depend only on λ_0 and Λ , such that*

$$u(z) \geq \bar{h} > 0 \quad \text{on} \quad B_{\bar{\theta}r}^-. \quad (4.62)$$

Proof: Let h_0 and θ be fixed such that the conclusions in Proposition 4.5 and Lemma 4.6 hold. Set

$$w(z) = \ln^+ \left(\frac{h}{u + h^{\frac{9}{8}}} \right), \quad 0 < h \leq h_0$$

It then follows from Proposition 4.5 that

$$f_{B_{\theta r}^-}((w - I_0)^+)^2 dz \leq c \frac{\theta^2 r^2}{|B_{\theta r}^-|} \int_{B_{r/\theta}^-} |\partial_y w(z)|^2 dz. \quad (4.63)$$

Where $f_{\Omega} f dz$ denotes the average of f on Ω . To estimate the integral on the right hand side of (4.63), we note that $w(z) = \ln^+ \left(\frac{1}{\tilde{u} + h^{\frac{9}{8}}} \right)$ with $\tilde{u} = \frac{u}{h}$ being a non-negative solution to (4.15). One then can follow the argument for (4.22) in the proof of Proposition 4.2 that for $\tau < t$ in $(-\tilde{r}^2, 0)$ with $\tilde{r} = \frac{r}{\theta} < 1$, it holds that

$$\begin{aligned} & \int_{c_{\tilde{r}}^-} w(z) dx dy + \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 a |\partial_y w|^2 dz \\ & \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 w(l, y, \tau) dx dy - \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} 2\chi_{\delta} (\partial_y \chi_{\delta}) a \partial_y w dz - \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 y \partial_x w dz \end{aligned} \quad (4.64)$$

where $\delta \in (0, 1)$ so that $(1 + \delta)\tilde{r} < 1$, and $\chi_{\delta} = \chi_{\delta}(|y|)$ and $\chi_{\delta} \in C^{\infty}[0, \infty)$, $0 < \chi_{\delta}(s) \leq 1$, $\chi_{\delta}(s) = 1$ for $0 \leq s \leq \tilde{r}$, $\chi_{\delta}(s) = 0$ for $s > (1 + \delta)\tilde{r}$, and

$$|\chi'_{\delta}(s)| \leq \frac{2}{\delta \tilde{r}} \quad \text{for all } s \geq 0 \quad (4.65)$$

As in the derivation of (4.23), one has from (4.65) that

$$\begin{aligned} \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} 2\chi_{\delta} \partial_y \chi_{\delta} a \partial_y w dz & \leq \frac{1}{2} \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} a \chi_{\delta}^2 (\partial_y w)^2 dz + \frac{16\Lambda}{\delta^2 \tilde{r}^2} (t - \tau) |c_{(1+\delta)\tilde{r}}^-| \\ & \leq \frac{1}{2} \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} a \chi_{\delta}^2 |\partial_y w|^2 dz + \frac{16\Lambda}{\delta^2} (1 + \delta)^4 \tilde{r}^4, \\ - \int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 y \partial_x w dz & = - \int_{\tau}^t \int_{-(1+\delta)\tilde{r}}^{(1+\delta)\tilde{r}} \chi_{\delta}^2 y w \Big|_{-((1+\delta)\tilde{r})^3}^{((1+\delta)\tilde{r})^3} dy ds \\ & \leq (t - \tau) (1 + \delta)^2 \tilde{r}^2 \ln h^{-\frac{1}{8}} \leq (1 + \delta)^2 \tilde{r}^4 \ln h^{-\frac{1}{8}}, \end{aligned}$$

and

$$\begin{aligned} \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 w(x, y, \tau) dx dy & = \int_{c_{(1+\delta)\tilde{r}}^- \cap \{\tilde{u} \leq 1\}} \chi_{\delta}^2 w(x, y, \tau) dx dy \\ & \leq \ln h^{-\frac{1}{8}} \text{mes } c_{(1+\delta)\tilde{r}}^- = (1 + \delta)^4 \tilde{r}^4 \ln h^{-\frac{1}{8}}. \end{aligned}$$

These and (4.64) yield that for τ and t , $\tau < t$ in $(-\tilde{r}^2, 0)$,

$$\int_{\tau}^t \int_{c_{(1+\delta)\tilde{r}}^-} \chi_{\delta}^2 a |\partial_y w(z)|^2 dz \leq \left((1 + \delta)^4 \left(1 + \frac{16\Lambda}{\delta^2} \right) \right) \tilde{r}^4 \ln h^{-\frac{1}{8}}.$$

This, together with the construction of χ_δ and (4.16), implies that

$$\int_{B_{\bar{r}}^-} |\partial_y w(z)|^2 dz \leq C(\Lambda, \delta) \bar{r}^4 \ln h^{-\frac{1}{8}} \quad (4.66)$$

It follows from (4.66) and (4.63) that there exists a positive constant $C = C(\Lambda, \theta, \delta) > 0$ such that

$$f_{B^-}((w - I_0)^+)^2 dz \leq C \frac{\theta^2 r^2}{|B_{\theta r}^-|} \int_{B_{\bar{r}}^-} |\partial_y w|^2 dz \leq C \ln h^{-\frac{1}{8}}. \quad (4.67)$$

Based on (4.67), one can modify the Moser's iteration method for $(w - I_0)^+$ as in [10] to show that there exists a uniform constant $k \in (0, 1)$ such that

$$\sup_{B_{k\theta r}^-} ((w - I_0)^+)^2 \leq C \ln h^{-\frac{1}{8}}. \quad (4.68)$$

Set $\bar{\theta} = k\theta$. Then (4.68) and Lemma 4.6 imply that for all $z \in B_{\bar{\theta}r}^-$,

$$w(z) \leq I_0 + C(\ln h^{-\frac{1}{8}})^2 \leq \lambda_0 \ln h^{-\frac{1}{8}} + C(\ln h^{-\frac{1}{8}})^2. \quad (4.69)$$

Since $\lim_{h \rightarrow 0^+} \frac{\lambda_1 \ln h^{-\frac{1}{8}} + C(\ln h^{-\frac{1}{8}})^{\frac{1}{2}}}{\ln(\frac{1}{2h^{\frac{1}{8}}})} = \lambda_0 < 1$ due to Lemma 4.6, thus there exists a constant $h_2 \leq \min(h_0, 2^{-8})$ such that

$$\lambda_0 \ln h_2^{-\frac{1}{8}} + C(\ln h_2^{-\frac{1}{8}})^{\frac{1}{2}} \leq \ln\left(\frac{1}{2h_0^{\frac{1}{8}}}\right).$$

Consequently, it holds that

$$\max_{B_{\bar{\theta}r}^-} \ln^+\left(\frac{h_2}{u + h_2^{\frac{9}{8}}}\right) \leq \ln\left(\frac{1}{2h_0^{\frac{1}{8}}}\right). \quad (4.70)$$

It follows from (4.70) and the choice of h_2 that

$$\min_{B_{\bar{\theta}r}^-} u \geq h_2^{\frac{10}{9}} \quad (4.71)$$

which yields the desired estimate (4.62). Hence the proof of Lemma 4.7 is completed.

As an immediately consequence of Lemma 4.7, the following desired oscillation estimate holds.

Proposition 4.8 *Let u be a weak solution of (4.15) in B_1^- and $\bar{\theta}$ and r be given as in Lemma 4.7. Then there exists an uniform constant $\bar{\beta}$, $0 < \bar{\beta} < 1$, such that*

$$Osc_{B_{\bar{\theta}r}^-} u \leq \bar{\beta} Osc_{B_r^-} u \quad (4.72)$$

where $Osc_Q f$ denotes the oscillation of f over Q for any domain Ω .

Proof: Since u is bounded, one can assume that $M \equiv \max_{B_r^-} u = -m \equiv -\min_{B_r^-} u$ without loss of generality since otherwise, one may consider $u - \frac{1}{2}(M + m)$. Thus both $1 + \frac{u}{M}$ and $1 - \frac{u}{M}$ are non-negative weak solutions to (4.15), and at least one of them satisfies the main assumption (4.17). We treat the case that $1 - \frac{u}{M}$ satisfies (4.17). Then Lemma 4.7 can be applied to $(1 - \frac{u}{M})$ to get

$$\left(1 - \frac{u}{M}\right)(z) \geq \bar{h} \quad \text{on } B_{\bar{\theta}r}^-,$$

where $\bar{h} \in (0, 1)$ is a constant given in Lemma 4.7. Thus one gets that $\max_{B_{\bar{\theta}r}^-} u \leq (1 - \bar{h})M$. Consequently,

$$\begin{aligned} \text{Osc}_{B_{\bar{\theta}r}^-} u &= \max_{B_{\bar{\theta}r}^-} u - \min_{B_{\bar{\theta}r}^-} u \leq \max_{B_{\bar{\theta}r}^-} u + M \\ &\leq (1 - \bar{h})M + M = (1 - \frac{\bar{h}}{2})2M = (1 - \frac{\bar{h}}{2})\text{Osc}_{B_{\bar{\theta}r}^-} u, \end{aligned}$$

which yields the desired estimate (4.72) with $\bar{\beta} = (1 - \frac{\bar{h}}{2})$. Hence the proof of Proposition (4.72) is completed. \square

It follows from the oscillation estimate (4.72) in Proposition 4.8 and the standard regularity arguments [16] that the following statement holds.

Proposition 4.9 *Let u be a weak solution to (4.15) in B_1^- . Then there exists a positive constant $\delta > 0$ such that u is Hölder continuous on B_δ^- which is a small neighborhood of $z = 0$.*

Now, Proposition 4.9 shows that the weak solution u to (4.2) is Hölder continuous in an interior point in Q_T . Then the standard regularity theory for ultra-parabolic equation [1, 14]. u is in fact C^∞ smooth in Q_T . Thus the proof of Theorem 4.1 is completed.

Remark: In the preparation of the current paper, there are some new developments in Hölder continuity of weak solutions to a class of ultraparabolic equations, which generalized Theorem 4.1 [17,18,19,20,21]. In particular, in [19,21] their results include a different proof of Theorem 4.1 in the general case.

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