DELAUNAY DECOMPOSITIONS MINIMIZING ENERGY OF WEIGHTED TOROIDAL GRAPHS

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ABSTRACT. Given a weighted toroidal graph, each realization to a Euclidean torus is associated with the Dirichlet energy. By minimizing the energy over all possible Euclidean structures and over all realizations within a fixed homotopy class, one obtains a harmonic map into an optimal Euclidean torus. We show that only with this optimal Euclidean structure, the harmonic map and the edge weights are induced from a weighted Delaunay decomposition.

1. INTRODUCTION

Let $G = (V, E, F)$ be a cell decomposition of a topological torus S and $f : (V, E) \to S$ be a corresponding realization of the graph (V, E). We assume the graph is equipped with some positive edge weights $c : E \to \mathbb{R}_{>0}$ where $c_{ij} = c_{ji}$.

We parameterize the space of Euclidean tori with unit area by the upper half plane $\mathbb{H} \subset \mathbb{C}$ and write S_τ for the corresponding Euclidean torus given by $\tau \in \mathbb{H}$. For every straight-line mapping to a Euclidean torus $h : (V, E) \to S_\tau$, it is associated with the Dirichlet energy

$$E_c(h) := \frac{1}{2} \sum_{ij \in E} c_{ij} \ell_{ij}^2. \tag{1}$$

where ℓ is the edge length.

There are several interpretations of the edge weights and the energy. One is to think of the edges as springs with force constant c. Then the total energy stored in the springs is the Dirichlet energy. There is also an interpretation in terms of electric networks, where the edge weights are regarded as conductance.

In our case, it is a classical result that for every fixed Euclidean torus S_τ and among all mappings $h : (V, E) \to S_\tau$ such that h is homotopic to f, the energy E_c has a minimizer h_τ unique up to translations [14, 6]. The map h_τ has convex faces and is called a Tutte-like embedding [6]. It is also known to be a harmonic map satisfying a discrete Laplace's equation.

One can further minimize the energy by varying the Euclidean structures, i.e. to consider the function

$$\begin{aligned} \mathcal{E}_c : \mathbb{H} \times \mathcal{C}_f &\to \mathbb{R} \\ (\tau, h) &\mapsto E_c(h) \end{aligned}$$

where $\mathcal{C}_f$ denotes the space of mappings h that are homotopic to f. It is known that $\mathcal{E}_c$ has a unique minimizer (τ, h_τ) [12]. Our goal is to explore the relation between this optimal Euclidean structure τ and the edge weights. Specifically, we are interested in a converse question.

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Question A: Given a Euclidean structure τ on the torus and a straight-line embedding $h : V \to S_\tau$ such that all faces are convex, what are the positive edge weights c such that (τ, h) is the minimizer of $\mathcal{E}_c$?

We shall relate the above question to conjugate maps. For every harmonic map $h : (V, E) \to S_\tau$, its lift to the universal cover $\tilde{h} : (\tilde{V}, \tilde{E}) \to \mathbb{C}$ has a conjugate map $\tilde{h}^* : \tilde{F} \to \mathbb{C}$ defined by

$$\tilde{h}^*_{\text{left}(ij)} - \tilde{h}^*_{\text{right}(ij)} = \sqrt{-1}\, c_{ij}(\tilde{h}_j - \tilde{h}_i) \tag{2}$$

where left(ij) is the left face of the oriented edge ij. Alternatively, one can write

$$c_{ij} = \frac{1}{\sqrt{-1}} \frac{\tilde{h}^*_{\text{left}(ij)} - \tilde{h}^*_{\text{right}(ij)}}{\tilde{h}_j - \tilde{h}_i}.$$

The map $\tilde{h}^*$ defines a realization of the dual decomposition of the universal cover $(\tilde{V}^*, \tilde{E}^*, \tilde{F}^*) \cong (\tilde{F}, \tilde{E}, \tilde{V})$ and is harmonic with respect to the edge weights $c^* := 1/c$. The edges of h^* are orthogonal to those of h. The maps h and h^* are said to be reciprocal in the sense of the Maxwell–Cremona correspondence [15]. Since $\tilde{h}$ is doubly periodic, one can deduce that $\tilde{h}^*$ projects to an Euclidean torus $kS_{\tau'}$ for some $k > 0$. Here $kS_{\tau'}$ denotes the scaled copy of the Euclidean torus $S_{\tau'}$ having area k^2.

Question B: Given edge weights c and any Euclidean torus τ, the harmonic map $h_\tau : (V, E) \to S_\tau$ defines a conjugate harmonic map $\tilde{h}^* : (\tilde{V}^*, \tilde{E}^*) \to \mathbb{C}$ over the universal cover. Does $\tilde{h}^*$ project to the same torus S_τ up to scaling? It is shown in [3] that this only holds for some unique Euclidean structure.

We relate the two questions above by applying discrete harmonic conjugates, which is also known as the response matrix in electric networks [11].

Theorem 1.1. *Let $c : E \to \mathbb{R}_{>0}$ be positive edge weights. Then there exists a unique $\tau_c \in \mathbb{H}$ and $k_c > 0$ such that the following holds:*

(1) (τ_c, h_{τ_c}) is the minimizer of $\mathcal{E}_c$ over the space $\mathbb{H} \times \mathcal{C}_f$ with value k_c.
(2) The conjugate map of $h_\tau : V \to S_\tau$ projects to kS_τ for some $k > 0$ if and only if $\tau = \tau_c$ and $k = k_c$.

In other words, the scaled conjugate map $h^\dagger_{\tau_c} := h^*_{\tau_c}/k_{\tau_c}$ projects to the same torus as h_{τ_c}. Both the maps h_{τ_c} and $h^\dagger_{\tau_c}$ are respectively unique up to translations. However, if we fix a choice for h_{τ_c}, then there is a unique choice for $h^\dagger_{\tau_c}$ such that h_{τ_c} is a realization of a weighted Delaunay decomposition with vertex weights defined consistently, and $h^\dagger_{\tau_c}$ is the dual Voronoi diagram [3, Theorem 4.4].

Corollary 1.2. *Let $c : E \to \mathbb{R}_{>0}$ be positive edge weights. Then (τ, h_τ) is the minimizer of the energy $\mathcal{E}_c$ with value k if and only if the realization of the cell decomposition (V, E, F) under h_τ is a weighted Delaunay decomposition and $h^\dagger := h^*_\tau/k$ is the dual Voronoi decomposition. Furthermore, the edge weighs satisfy for every $ij \in E$*

$$c_{ij} = \frac{k}{\sqrt{-1}} \frac{\tilde{h}^\dagger_{\mathit{left}(ij)} - \tilde{h}^\dagger_{\mathit{right}(ij)}}{\tilde{h}_j - \tilde{h}_i} > 0 \tag{3}$$

Here left(ij) denotes the left face of the oriented edge $\{ij\}$ while right(ij) denotes the right face.

A weighted Delaunay decomposition is also known as a power diagram. It is a generalization of the classical Delaunay decomposition where the distance function for the Voronoi cells are modified by vertex weights. One can show that the dual edge is always perpendicular to the primal edge, which motivates the $\sqrt{-1}$ in equation (2). In the case where the vertex weights are constant, one obtains the classical Delaunay decomposition and equation (3) yields the famous cotangent weights from the finite element discretization [17]. Corollary 1.2 provides an answer to Question A and there are embeddings with convex faces that cannot be minimizers of $\mathcal{E}_c$ for any positive edge weights c (See Example 1).

On the other hand, one observes that the minimal Dirichlet energy k_c is intrinsic to the weighted toroidal graph. It is natural to expect that k_c can be expressed in terms of the edge weights without involving the embedding h as in equation (1). We first state the formula and then explain the notations.

Theorem 1.3. *Let $c : E \to \mathbb{R}_{>0}$ be positive edge weights. Then the minimal Dirichlet energy is given by*

$$k_c := \min \mathcal{E}_c = \sqrt{\frac{\det_0(\tilde{d}^T C \tilde{d})}{\det_0(d^T C d)}}$$

In the formula, we fix an arbitrary orientation for the edges and d denotes the $E \times V$-incidence matrix, where d^T is its transpose. C is the $E \times E$ diagonal matrix consisting of edge weights. Thus, $\Delta := d^T C d$ is a $V \times V$-matrix that represents the discrete Laplace operator: for every $g : V \to \mathbb{R}$,

$$(\Delta g)_i = \sum c_{ij}(f_j - f_i).$$

Furthermore, $\det_0$ denotes the product of nonzero eigenvalues of the corresponding matrix. $\tilde{d}$ is a $E \times (V+2)$-matrix. Its first V columns are those of d, which can be regarded as a basis of the space of exact 1-forms. The last two columns of $\tilde{d}$ represent closed 1-forms with nontrivial periods, whose integrals along γ_1 and γ_2 are respectively $(1, 0)$ and $(0, 1)$.

In fact, Theorem 1.1 and Theorem 1.3 hold for edge weights with arbitrary signs as long as the energy functional is positive definite (Section 7). However, Corollary 1.2 has to be interpreted differently in this case, since the Delaunay condition is equivalent to the edge weights being positive.

1.1. **Related work.**

It is common in computer graphics [6, 7] and discrete conformal geometry [4, 5, 16, 9, 19] to consider edge weights defined by equation (3) from a given Delaunay decomposition. On the one hand, it includes the cotangent-weight Laplacian which is obtained from the finite element discretization [17]. On the other hand, the corresponding discrete harmonic functions are related to deformations of circle patterns [13].

Delaunay decompositions minimize energy in several other ways. An example is Rippa's theorem [18]. For that setting, vertex positions are fixed while combinatorics are allowed to change. It is in contrast to our setting where combinatorics and edge weights are fixed but vertex positions are allowed to move.

Realizations of graphs to other surfaces that minimize the Dirichlet energy have been considered [20]. For a finite planar graph, one obtains the classical Tutte embedding. For hyperbolic surfaces, one also obtain a harmonic map to an optimal hyperbolic surface [8]. However, its connection to Delaunay decomposition remains unclear.

The energy that we consider is naturally associated to an electric network where the edges weights play the role of conductance. Our main tool is to consider discrete harmonic conjugates, which is also called the response matrix in electric networks. If we regard the weights as those in dimer models, there are related realizations [10].

We focus on edge weights where the energy functional is positive definite (Definition 7.1). However, it is also interesting to consider edge weights where the energy functional fails to be positive definite [2, 13].

2. The space of marked Euclidean tori

We assume that γ_1, γ_2 are the generators of the fundamental group and parameterize the space of Euclidean tori with unit area by the upper half plane $\mathbb{H} \subset \mathbb{C}$ such that for each $\tau \in \mathbb{H}$, the Euclidean torus S_τ is obtained as a quotient of $\mathbb{R}^2 \cong \mathbb{C}$ by translations and has holonomy

$$\rho_{\gamma_1}(z) = z + \frac{1}{\sqrt{\operatorname{Im} \tau}}, \quad \rho_{\gamma_2}(z) = z + \frac{\tau}{\sqrt{\operatorname{Im} \tau}}.$$

For every mapping $h : (V, E) \to S_\tau$ where edges are realized as straight lines, we consider its lift to the universal cover $\tilde{h} : (\tilde{V}, \tilde{E}) \to \mathbb{C}$. It satisfies

$$\tilde{h} \circ \gamma_1 = \tilde{h} + \frac{1}{\sqrt{\operatorname{Im} \tau}}, \quad \tilde{h} \circ \gamma_2 = \tilde{h} + \frac{\tau}{\sqrt{\operatorname{Im} \tau}} \tag{4}$$

In terms of the lift, the energy can be expressed as

$$E_c(h) := \frac{1}{2} \sum_{ij \in E} c_{ij} |\tilde{h}_j - \tilde{h}_i|^2.$$

The map h is harmonic if it is a critical point of the energy under variation of vertex positions, equivalently for $i \in \tilde{V}$

$$\sum_j c_{ij} (\tilde{h}_j - \tilde{h}_i) = 0.$$

In the following sections, we shall interpret $\operatorname{Re} \tilde{h}$ and $\operatorname{Im} \tilde{h}$ as integrals of harmonic 1-forms with specific periods.

3. Discrete harmonic 1-forms

A discrete 1-form is a function on oriented edges $\omega : \vec{E} \to \mathbb{R}$ such that $\omega_{ij} = -\omega_{ji}$ for every edge ij. A discrete 1-form is closed if its summation over the boundary of every oriented face is zero. For example, in the case of a triangulation, ω is closed if for every triangle $\{ijk\}$,

$$\omega_{ij} + \omega_{jk} + \omega_{ki} = 0.$$

For a closed discrete 1-form ω on a torus, one can consider its periods

$$\sum_{\gamma_1} \omega = A, \quad \sum_{\gamma_2} \omega = B$$

where the summation is over an edge path homotopic to γ_k. Because ω is closed, the summation is independent of the path chosen. A 1-form ω is exact if there exists $f : V \to \mathbb{R}$ such that

$$\omega_{ij} = f_j - f_i.$$

One can show that a 1-form is exact if and only if it is closed with vanishing periods, i.e. $(A, B) = (0, 0)$ in the case of tori.

The orientation of the primal edges naturally induces an orientation for the dual edges. Given an oriented edge ij, the dual edge $*ij$ is oriented from right face to the left face of ij. In this way, we say a 1-form ω is co-closed if it is a closed 1-form with respect to the dual decomposition, i.e. for every vertex $i \in V$

$$\sum_j \omega_{ij} = 0$$

where the summation is over the neighboring vertices of i.

For every 1-form ω, there is an associated 1-form $*\omega$ defined by $*\omega = c\omega$. The map sending ω to $*\omega$ is a discrete analogue of the Hodge star operator.

We call ω a harmonic 1-form on the primal decomposition (V, E, F) if ω is closed and $*\omega = c\omega$ is co-closed. The co-closeness implies for every vertex $i \in V$

$$\sum_j *\omega = \sum_j c_{ij}\omega_{ij} = 0.$$

One can check that $*\omega$ is a harmonic 1-form with respect to the dual cell decomposition and edge weights $c^* := 1/c$. We call $*\omega$ the harmonic conjugate of ω.

It is known [1, Theorem 3.9] that the space of discrete harmonic 1-forms on a torus is parameterized by the period $(A, B) \in \mathbb{R}^2$.

4. Response matrix over the period space

For any edge weights c, we consider a map on the period space

$$\begin{aligned} L : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (A, B) &\mapsto (\tilde{A}, \tilde{B}) \end{aligned}$$

which maps the periods of a harmonic 1-form on (V, E, F) to the periods of the conjugate harmonic 1-form, which is also called the response matrix in the setting of electric networks. Namely, for every $(A, B) \in \mathbb{R}^2$, there exists a unique harmonic 1-form $\omega : \vec{E} \to \mathbb{R}$ such that

$$\sum_{\gamma_1} \omega = A, \quad \sum_{\gamma_2} \omega = B.$$

Its conjugate would have periods

$$\sum_{\gamma_1} *\omega = \tilde{A}, \quad \sum_{\gamma_2} *\omega = \tilde{B}.$$

Then we define $L(A, B) = (\tilde{A}, \tilde{B})$. We shall relate it to the Dirichlet energy and derive an explicit form.

Definition 4.1. *We define a skew symmetric bilinear form over $\mathbb{R}^2$. For any $(A, B), (\tilde{A}, \tilde{B}) \in \mathbb{R}^2$,*

$$\{(A, B), (\tilde{A}, \tilde{B})\} := A\tilde{B} - B\tilde{A}.$$

Given a closed 1-form ω and a co-closed 1-form $\tilde{\omega}$ we can consider their product

$$\sum_{ij\in E} \omega_{ij}\tilde{\omega}_{ij}$$

Because of the closeness and the co-closeness, the summation can be rewritten as the product of integrals along the boundary of a fundamental domain, which is analogous to Stokes' theorem.

Proposition 4.2. [1] *Suppose ω is a closed 1-form on the primal decomposition (V,E,F) with periods (A,B) and $\tilde{\omega}$ is co-closed, i.e. a closed 1-form on the dual decomposition (V^*,E^*,F^*), with periods $(\tilde{A},\tilde{B})$. Then*

$$\sum_{ij} \omega_{ij}\tilde{\omega}_{ij} = \{(A,B),(\tilde{A},\tilde{B})\}$$

For any harmonic 1-form ω and $\tilde{\omega}$ on the primal graph, we take the product

$$\sum c_{ij}\omega_{ij}\tilde{\omega}_{ij} = \sum \omega * \tilde{\omega} = \sum *\omega\, \tilde{\omega}$$

Notice that ω is closed on the primal graph while $*\omega := c\omega$ is co-closed. So we can apply Proposition 4.2.

Corollary 4.3. *Suppose $\omega, \tilde{\omega} : \vec{E} \to \mathbb{R}$ are harmonic 1-forms on the primal decomposition with periods $(A,B),(\tilde{A},\tilde{B})$ respectively. Then*

$$\sum c_{ij}\omega_{ij}\tilde{\omega}_{ij} = \{(A,B),L(\tilde{A},\tilde{B})\} = \{(\tilde{A},\tilde{B}),L(A,B)\} = -\{L(A,B),(\tilde{A},\tilde{B})\} \tag{5}$$

Now we are able to deduce the operator L.

Proposition 4.4. *For any edge weight $c : E \to \mathbb{R}_{>0}$, in terms of the standard basis of $\mathbb{R}^2$, the operator L over the period space has the matrix form*

$$L = \begin{pmatrix} k_c \frac{\operatorname{Re}\tau_c}{\operatorname{Im}\tau_c} & -k_c \frac{1}{\operatorname{Im}\tau_c} \\ k_c \frac{|\tau_c|^2}{\operatorname{Im}\tau_c} & -k_c \frac{\operatorname{Re}\tau_c}{\operatorname{Im}\tau_c} \end{pmatrix}$$

for some $k_c > 0$ and $\tau_c \in \mathbb{H}$.

Proof. In terms of the standard basis of $\mathbb{R}^2$, we write the L operator as a 2×2-matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Equation (5) implies for any column vectors $U, V \in \mathbb{R}^2$,

$$U^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} LV = -U^t L^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} V$$

and we deduce that $a = -d$.

Furthermore, since the energy is always non-negative, Equation (5) implies for any nonzero column vector $U \in \mathbb{R}^2$,

$$0 < U^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} LU = U^t \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix} U$$

It is deduced that $\begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}$ is positive definite. Thus

$$\det \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix} = -bc - a^2 > 0$$

$$c > 0$$
$$b < 0$$

We define

$$\begin{aligned} k_c :=& \sqrt{-bc - a^2} > 0 \\ \operatorname{Re}\tau_c :=& -\frac{a}{b} \\ \operatorname{Im}\tau_c :=& -\frac{k}{b} > 0 \end{aligned}$$

and the claim follows. □

5. Proof of Theorem 1.1

We investigate harmonic maps to the torus from the information of the operator L. Throughout the section, we assume τ_c, k_c to be the one given in Proposition 4.4. Theorem 1.1 is a restatement of Proposition 5.1 and 5.2.

Proposition 5.1. *Let $\tau \in \mathbb{H}$ represents any Euclidean torus and $h_\tau : V \to S_\tau$ be the corresponding harmonic map. Then its energy satisfies*

$$\mathcal{E}_c(\tau, h_\tau) = E_c(h_\tau) \geq k_c = \mathcal{E}_c(\tau_c, h_{\tau_c})$$

and (τ_c, h_{τ_c}) is the unique minimizer of $\mathcal{E}_c$.

Proof. We abbreviate h_τ as h and write $\tilde{h}$ the lift of h to the universal cover. For every oriented edge ij, we consider

$$\omega_{ij} := \operatorname{Re}(\tilde{h}_{\tilde{j}} - \tilde{h}_{\tilde{i}}), \quad \eta_{ij} := \operatorname{Im}(\tilde{h}_{\tilde{j}} - \tilde{h}_{\tilde{i}}).$$

which defines harmonic 1-forms on the torus (V, E, F). Equation (4) implies that they have periods

$$\begin{aligned} \sum_{\gamma_1} \omega = \frac{1}{\sqrt{\operatorname{Im}\tau}}, &\quad \sum_{\gamma_2} \omega = \frac{\operatorname{Re}\tau}{\sqrt{\operatorname{Im}\tau}} \\ \sum_{\gamma_1} \eta = 0, &\quad \sum_{\gamma_2} \eta = \frac{\operatorname{Im}\tau}{\sqrt{\operatorname{Im}\tau}} \end{aligned}$$

We compute its energy using Proposition 4.2

$$\begin{aligned} E_c(h_\tau) &= \frac{1}{2}\left(\sum_{ij} c_{ij}|\omega|^2 + \sum_{ij} c_{ij}|\eta|^2\right) \\ &= \frac{1}{2}\{(\frac{1}{\sqrt{\operatorname{Im}\tau}}, \frac{\operatorname{Re}\tau}{\sqrt{\operatorname{Im}\tau}}), L(\frac{1}{\sqrt{\operatorname{Im}\tau}}, \frac{\operatorname{Re}\tau}{\sqrt{\operatorname{Im}\tau}})\} + \frac{1}{2}\{(0, \frac{\operatorname{Im}\tau}{\sqrt{\operatorname{Im}\tau}}), L(0, \frac{\operatorname{Im}\tau}{\sqrt{\operatorname{Im}\tau}})\} \\ &= \frac{k_c}{2\operatorname{Im}\tau_c \operatorname{Im}\tau}(|\tau_c|^2 + |\tau|^2 - 2\operatorname{Re}\tau_c \operatorname{Re}\tau) \\ &= \frac{k_c}{2\operatorname{Im}\tau_c \operatorname{Im}\tau}((\operatorname{Re}\tau_c - \operatorname{Re}\tau)^2 + |\operatorname{Im}\tau_c|^2 + |\operatorname{Im}\tau|^2) \\ &= \frac{k_c|\tau - \tau_c|^2}{2\operatorname{Im}\tau_c \operatorname{Im}\tau} + k_c \\ &\geq k_c \end{aligned}$$

The equality holds if and only if $\tau = \tau_c$. □

Proposition 5.2. *For any $\tau \in \mathbb{H}$, the conjugate map of $h_\tau : V \to S_\tau$ projects to kS_τ for some $k > 0$ if and only if $(\tau, h_\tau) = (\tau_c, h_{\tau_c})$ and $k = k_c$.*

Proof. We abbreviate h_τ as h and define ω, η as in the proof of Proposition 5.1. We consider the conjugates $*\omega$ and $*\eta$ which have periods

$$(\sum_{\gamma_1} *\omega, \sum_{\gamma_2} *\omega) = L(\frac{1}{\sqrt{\operatorname{Im}\tau}}, \frac{\operatorname{Re}\tau}{\sqrt{\operatorname{Im}\tau}}) = \frac{k_c}{\operatorname{Im}\tau_c\sqrt{\operatorname{Im}\tau}}(\operatorname{Re}\tau_c - \operatorname{Re}\tau, |\tau_c|^2 - \operatorname{Re}\tau_c \operatorname{Re}\tau)$$

$$(\sum_{\gamma_1} *\eta, \sum_{\gamma_2} *\eta) = L(0, \frac{\operatorname{Im}\tau}{\sqrt{\operatorname{Im}\tau}}) = \frac{k_c}{\operatorname{Im}\tau_c\sqrt{\operatorname{Im}\tau}}(-\operatorname{Im}\tau, -\operatorname{Re}\tau_c \operatorname{Im}\tau)$$

By Equation (2), the conjugate harmonic map $\tilde{h}^* : \tilde{V} \to \mathbb{C}$ is in the form

$$\tilde{h}^*_{\text{left}(ij)} - \tilde{h}^*_{\text{right}(ij)} = - * \eta_{ij} + \sqrt{-1} * \omega_{ij}$$

and hence

$$\tilde{h}^* \circ \gamma_1 - \tilde{h}^* = \frac{k_c}{\sqrt{\operatorname{Im}\tau_c}}(\sqrt{\frac{\operatorname{Im}\tau}{\operatorname{Im}\tau_c}} + \sqrt{-1}\,\frac{\operatorname{Re}\tau_c - \operatorname{Re}\tau}{\sqrt{\operatorname{Im}\tau_c \operatorname{Im}\tau}})$$

$$\tilde{h}^* \circ \gamma_2 - \tilde{h}^* = \frac{k_c}{\sqrt{\operatorname{Im}\tau_c}}(\frac{\operatorname{Im}\tau \operatorname{Re}\tau_c}{\operatorname{Im}\tau_c} + \sqrt{-1}\,\frac{|\tau_c|^2 - \operatorname{Re}\tau_c \operatorname{Re}\tau}{\operatorname{Im}\tau_c})$$

Thus $\tilde{h}^*$ projects to kS_τ, i.e. in the form

$$\tilde{h}^* \circ \gamma_1 - \tilde{h}^* = \frac{k}{\sqrt{\operatorname{Im}\tau}}$$

$$\tilde{h}^* \circ \gamma_2 - \tilde{h}^* = \frac{k\tau}{\sqrt{\operatorname{Im}\tau}}$$

for some $k > 0$ if and only if $\tau = \tau_c$ and $k = k_c$. □

6. Response matrix L from Laplace operator Δ

In the section, we shall express the response matrix $L : \mathbb{R}^2 \to \mathbb{R}^2$ in terms of the discrete Laplacian and prove Theorem 1.3.

We first define an incidence matrix d. We fix an arbitrary orientation for every edge e so that e_+ and e_- represent the head and the tail of the oriented edge e. Then we define $d : \mathbb{R}^V \to \mathbb{R}^E$ by

$$df(e) = f(e_+) - f(e_-).$$

We further define

$$\Delta := d^T C d$$

where $C : \mathbb{R}^E \to \mathbb{R}^E$ is the diagonal matrix of the corresponding edge weights and d^T is the transpose of d. One can show that the operator $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ satisfies for every vertex i

$$(\Delta f)_i = \sum_j c_{ij}(f_j - f_i)$$

and Δ is the so called discrete Laplacian. It is semi-positive definite and the kernel consists of constant functions.

Notice that each column vector of d represents a closed discrete 1-form that is exact. The column vectors form a basis of the space of exact 1-forms on the cell decomposition $G = (V, E, F)$. To get a basis of the space of closed 1-forms, we consider two more closed 1-forms m_1 and m_2 that have nontrivial periods

$$\sum_{\gamma_1} m_1 = 1, \sum_{\gamma_2} m_1 = 0$$
$$\sum_{\gamma_1} m_2 = 0, \sum_{\gamma_2} m_2 = 1$$

Thus, every closed 1-form with periods

$$\sum_{\gamma_1} \omega = A, \sum_{\gamma_2} \omega = B$$

can be expressed as

$$\omega = df + \begin{pmatrix} | & | \\ m_1 & m_2 \\ | & | \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} =: df + M \begin{pmatrix} A \\ B \end{pmatrix}$$

for some $f \in \mathbb{R}^V$ unique up to constants. The closed 1-form ω is harmonic if

$$0 = d^T C\omega = d^T C df + d^T CM \begin{pmatrix} A \\ B \end{pmatrix} = \Delta f + d^T CM \begin{pmatrix} A \\ B \end{pmatrix}$$

To obtain a solution in terms of f uniquely, we can pick a vertex o and demand $f_o = 0$. We write $d_{\bar{o}}$ as the submatrix of d with the column corresponding to vertex o removed and $\Delta_{\bar{o}\bar{o}}$ as the submatrix of Δ with the column and the row corresponding to vertex o removed. One can show that $\Delta_{\bar{o}\bar{o}}$ is invertible. The values of f at vertices other than o can be obtained via

$$f_{\bar{o}} = -\Delta_{\bar{o}\bar{o}}^{-1} d_{\bar{o}}^T CM \begin{pmatrix} A \\ B \end{pmatrix} \tag{6}$$

Now we can compute the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ in terms of the edge weights.

Proposition 6.1.

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (-M^T C d_{\bar{o}} \Delta_{\bar{o}\bar{o}}^{-1} d_{\bar{o}}^T CM + M^T CM)$$

Proof. To compute the formula, we need further notations on the dual graph. Since the orientation of the primal edges is chosen, it naturally induces an orientation of the dual edges. Namely, a dual edge $*e$ is oriented from the right face of e to the left. We then define the incidence matrix $d_* : \mathbb{R}^F \to \mathbb{R}^E$ similarly. The columns of d_* form a basis of the space of exact 1-forms on the dual graph. We further define a $|E|\times 2$ matrix M_* such that its columns m_{*1} and m_{*2} represent closed 1-forms on the dual graph having nontrivial periods

$$\sum_{\gamma_1} m_{*1} = 1, \sum_{\gamma_2} m_{*1} = 0$$
$$\sum_{\gamma_1} m_{*2} = 0, \sum_{\gamma_2} m_{*2} = 1$$

For a harmonic 1-form ω, it is closed and $*\omega$ is co-closed. So we have

$$Cdf + CM \begin{pmatrix} A \\ B \end{pmatrix} = C\omega = d_* g + M_* \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix}$$

for some $f : V \to \mathbb{R}$ and $g : F \to \mathbb{R}$. Applying M^T to both sides from the left yields

$$M^T C df + M^T C M \begin{pmatrix} A \\ B \end{pmatrix} = (M^T d_*) g + M^T M_* \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = 0 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix}$$

where Proposition 4.2 is applied to $M^T d_*$ and $M^T M_*$ since the rows of M^T represent closed 1-forms on the primal graph while the columns of d_* and M_* represents co-closed 1-forms. Because $f_o = 0$, we have from Equation (6)

$$\begin{aligned} \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (M^T C d_{\bar{o}} f_{\bar{o}} + M^T C M \begin{pmatrix} A \\ B \end{pmatrix}) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (-M^T C d_{\bar{o}} \Delta_{\bar{o}\bar{o}}^{-1} d_{\bar{o}}^T C M + M^T C M) \begin{pmatrix} A \\ B \end{pmatrix} \end{aligned}$$

and thus

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (-M^T C d_{\bar{o}} \Delta_{\bar{o}\bar{o}}^{-1} d_{\bar{o}}^T C M + M^T C M)$$

□

Proof of Theorem 1.3. From Proposition 4.4, the minimal energy is k_c. We know

$$k_c^2 = \det L = \det(-M^T C d_{\bar{o}} \Delta_{\bar{o}\bar{o}}^{-1} d_{\bar{o}}^T C M + M^T C M) \tag{7}$$

Recall that the determinant of a block matrix satisfies

$$\det \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} = \det P \det(R - Q^T P^{-1} Q)$$

whenever P is an invertible matrix. Applying this formula to Equation 7 yields

$$k_c^2 = \frac{\det \begin{pmatrix} \Delta_{\bar{o}\bar{o}} & d_{\bar{o}}^T C M \\ M^T C d_{\bar{o}} & M^T C M \end{pmatrix}}{\det \Delta_{\bar{o}\bar{o}}} = \frac{\det(\tilde{d}_{\bar{o}}^T \mathcal{C} \tilde{d}_{\bar{o}})}{\det(d_{\bar{o}}^T \mathcal{C} d_{\bar{o}})}$$

where

$$\tilde{d}_{\bar{o}} = \begin{pmatrix} d_{\bar{o}} & M \end{pmatrix}$$

is a $|E| \times (|V|+1)$ matrix. Notice that both matrices $\tilde{d}^T \mathcal{C} \tilde{d}$ and $d^T \mathcal{C} d$ are semi-positive definite. One has

$$d^T \mathcal{C} d f = 0$$

if and only if $f \in \mathbb{R}^V$ is a constant function. On the other hand

$$\tilde{d}^T \mathcal{C} \tilde{d} \begin{pmatrix} f \\ A \\ B \end{pmatrix} = 0$$

if and only if $f \in \mathbb{R}^V$ is a constant function while $A = B = 0$. Thus

$$k = \sqrt{\frac{\det(\tilde{d}_{\bar{o}}^T \mathcal{C} \tilde{d}_{\bar{o}})}{\det(d_{\bar{o}}^T \mathcal{C} d_{\bar{o}})}} = \sqrt{\frac{\det_0(\tilde{d}^T \mathcal{C} \tilde{d})}{\det_0(d^T \mathcal{C} d)}}$$

and $\det_0$ denotes the product of non-zero eigenvalues. □

Remark 6.2. *The discrete Laplace operator $d^T \mathcal{C} d$ is a sub-matrix of $\tilde{d}^T \mathcal{C} \tilde{d}$. The Schur complement of the block $(d^T \mathcal{C} d)_{\bar{o}\bar{o}}$ of the matrix $(\tilde{d}^T \mathcal{C} \tilde{d})_{\bar{o}\bar{o}}$ is in fact the matrix*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L$$

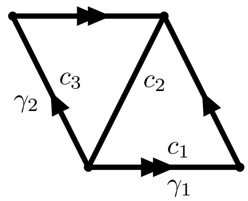

FIGURE 1. One-vertex triangulation of the torus. The vertex has six neighboring edges on the universal cover.

where L is the response matrix that we used.

Example 1. *We consider a one-vertex triangulation of the torus (Figure 1). By computing the L operator on the period space, one finds that the minimal energy is*

$$k_c = \sqrt{c_1c_2 + c_2c_3 + c_3c_1}$$

and the optimal Euclidean structure is given by

$$\tau_c = \frac{-c_2 + \mathbf{i}\sqrt{c_1c_2 + c_2c_3 + c_3c_1}}{c_2 + c_3}$$

where $\mathbf{i} = \sqrt{-1}$. Notice that $\operatorname{Re}\tau_c < 0$ for this weighted graph. In other words, for this graph, any realization to a Euclidean torus S_τ with $\operatorname{Re}\tau > 0$ cannot be a minimizer of $\mathcal{E}_c$ for any positive edge weights c.

7. Weights with arbitrary sign

In Theorem 1.1, the assumption about edge weights being positive is not compulsory. We can allow the edge weights taking negative values as long as the energy functional is positive definite over the space of closed discrete 1-forms.

Definition 7.1. *Let (V, E, F) be a cell decomposition of a torus. The edge weight $c : E \to \mathbb{R}$ is said to be non-degenerate if for every closed 1-form $\omega : \vec{E} \to \mathbb{R}$*

$$E_c(\omega) = \sum_{ij} c_{ij}\omega_{ij}^2 \geq 0$$

and the equality holds if and only if $\omega \equiv 0$

Proposition 7.2. *Suppose the edge weight $c : E \to \mathbb{R}$ is non-degenerate. Then over the period space $L : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined.*

Proof. We first show that if ω is a harmonic 1-form with vanishing periods, then it must be trivial. Observe that ω having vanishing periods implies there exists $f : V \to \mathbb{R}$ such that $\omega_{ij} = f_j - f_i$. Thus

$$\sum c_{ij}\omega_{ij}^2 = \sum_i f_i \sum_j c_{ij}(f_j - f_i) = 0$$

The non-degeneracy of ω yields $\omega \equiv 0$. This also implies the uniqueness of harmonic 1-form with any given periods if exist. Indeed, by dimensional argument, we thus deduce that for any prescribed period (A, B), there exists a unique harmonic 1-form ω such that

$$\sum_{\gamma_1} \omega = A, \quad \sum_{\gamma_1} \omega = B$$

Thus the map $L : \mathbb{R}^2 \to \mathbb{R}^2$ as in Section 4 is well defined. □

For non-degenerate edge weights, one can check Proposition 4.4, 5.1 and 5.2 hold and the proofs remain the same. However, Corollary 1.2 has to be interpreted differently.

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