Global smooth solutions to the 2D isentropic and irrotational Chaplygin gases with a class of large initial data

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Abstract

In this paper, we establish the global existence of smooth solutions to the 2-dimensional (2D) compressible isentropic irrotational Euler equations for Chaplygin gases with the short pulse initial data introduced by Christodoulou. This is related to the Majda's conjecture on the non-formation of shock waves of solutions from smooth initial data for multi-dimensional nonlinear symmetric systems which are totally linearly degenerate. The main ingredients of our analysis consist of showing the positivity of the inverse foliation density near the outermost conic surface for all time and solving a global Goursat problem inside the outermost cone. To overcome the difficulties due to the slower time decay rate of the solutions to the 2D wave equation and the largeness of the solution, we introduce some new auxiliary energies. It is noted that the methods and results in the paper can be extended to general 2D quasilinear wave equations satisfying corresponding null conditions for short pulse initial data.

Keywords: Totally linearly degenerate, Chaplygin gases, short pulse initial data, null condition, inverse foliation density, Goursat problem

Mathematical Subject Classification: 35L05, 35L72

Contents

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1 Introduction

1.1 The formulation of the problem and main results

The 2-dimensional (2D) compressible isentropic Euler equations for Chaplygin gases are

$$\begin{cases} \partial_t\rho + div(\rho v) = 0, \\ \partial_t(\rho v) + div(\rho v\otimes v) + \nabla p = 0, \end{cases} \tag{1.1}$$

which express the conservations of the mass and momentum respectively. Where $\nabla = (\partial_1, \partial_2) = (\partial_{x_1}, \partial_{x_2})$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}^2$, $v=(v_1,v_2)$, ρ, p stand for the velocity, density, pressure respectively, and the equation of state is given by

$$p(\rho) = P_0 - \frac{B}{\rho} \tag{1.2}$$

with P_0 and B being positive constants.

Away from vacuum, i.e., $\rho > 0$, the system can be symmetrized as

$$A_0\partial_t V + A_1\partial_1 V + A_2 V = 0 \tag{1.3}$$

with

$$V = \begin{pmatrix} v_1 \\ v_2 \\ p \end{pmatrix}, A_0 = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{\rho}{B} \end{pmatrix}, A_1 = \begin{pmatrix} \rho v_1 & 0 & 1 \\ 0 & \rho v_1 & 0 \\ 1 & 0 & \frac{\rho v_1}{B} \end{pmatrix}, A_2 = \begin{pmatrix} \rho v_2 & 0 & 0 \\ 0 & \rho v_2 & 1 \\ 0 & 1 & \frac{\rho v_2}{B} \end{pmatrix}. \tag{1.4}$$

For any $\omega = (\omega^1, \omega^2)\in\mathbb{S}^1$, $A_0^{-1}(\omega^1A_1 + \omega^2A_2)$ has three real distinct eigenvalues

$$\lambda_1(V,\omega) = \omega^1v_1 + \omega^2v_2 - c(\rho), \quad \lambda_2(V,\omega) = \omega^1v_1+\omega^2v_2, \quad \lambda_3(V,\omega) = \omega^1v_1+\omega^2v_2+c(\rho),$$

with the corresponding right eigenvectors $r_1(V,\omega) = (\omega^1, \omega^2, -\rho c(\rho))^T$, $r_2(V,\omega) = (-\omega^2, \omega^1, 0)^T$, $r_3(V,\omega) = (\omega^1, \omega^2, \rho c(\rho))^T$ respectively, where $c(\rho) = \sqrt{p'(\rho)} = \frac{\sqrt{B}}{\rho}$ is the local sound speed. It is then easy to verify that for any $\omega \in \mathbb{S}^1$,

$$\nabla_V \lambda_i(V,\omega) \cdot r_i(V,\omega) \equiv 0, \quad i = 1, 2, 3. \tag{1.5}$$

Thus the 2D compressible isentropic Euler system for Chaplygin gases is the prototype example of multidimensional quasilinear symmetric hyperbolic systems with totally linearly degenerate structures. Our main concern is the global in time smooth solutions to such a system for general smooth initial data, in particular, the non-formation of shock waves for smooth solutions in finite time. This is closely related to the following conjecture formulated clearly by Majda in [31]:

Conjecture. *For any given n-dimensional nonlinear symmetric hyperbolic system with totally linearly degenerate structures, a smooth solution, say in $C([0,T], H^s_{ul}(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}_{ul}(\mathbb{R}^n))$, $T > 0$, $s > \frac{n}{2} + 1$, will exist globally in time in general when the solution runs out of the domain of definition of the Cauchy problem in finite time. In particular, the shock wave formation does not occur in general for any smooth initial data.*

As pointed out in [31] by Majda, such a conjecture is plausible physically and its resolution would both elucidate the nonlinear nature of the condition requiring linear degeneracy of each characteristic field and may isolate the fashion in which the shock wave formation arises in quasilinear hyperbolic systems. However, although some important progress have been achieved in the case $n = 1$ [24, 27, 28], yet this conjecture has been far from being solved in multi-dimensions, even for small data except the special cases [8, 16, 18, 19]. In this paper, we will resolve this conjecture for the physically important system in 2D, (1.1), for a class of irrotational "short pulse" initial data.

Thus we supplement (1.1) with the irrotational initial data

$$\big(\rho, v)(0, \cdot\,) = (\bar{\rho} + \rho_0(\,\cdot\,), v_0(\,\cdot\,)\big) = \big(\bar{\rho} + \rho_0(\,\cdot\,), v_1^0(\,\cdot\,), v_2^0(\,\cdot\,)\big)$$

such that $\bar{\rho} + \rho_0(x) > 0$ and rot $v_0(x) = \partial_2 v_1^0 - \partial_1 v_2^0 \equiv 0$ with $\bar{\rho}$ being a positive constant which can be normalized so that $c(\bar{\rho}) = 1$. Then the irrotationality, rot $v(t, \cdot\,) = (\partial_2 v_1 - \partial_1 v_2)(t, \cdot) \equiv 0$, holds true as long as the solution remains smooth. Hence there exists a potential function ϕ such that $v = \nabla\phi$, and the Bernoulli's law, $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + h(\rho) = 0$, holds with the enthalpy $h(\rho)$ satisfying $h'(\rho) = \frac{c^2(\rho)}{\rho}$ and $h(\bar{\rho}) = 0$. Therefore, for smooth irrotational flows, (1.1) is equivalent to

$$\begin{aligned}\sum_{\alpha,\beta=0}^{2} g^{\alpha\beta}(\partial\phi)\partial^2_{\alpha\beta}\phi \equiv &- \partial_t^2\phi + \triangle\phi - 2\sum_{i=1}^{2}\partial_i\phi\partial_t\partial_i\phi + 2\partial_t\phi\triangle\phi \\ &- \sum_{i,j=1}^{2}\partial_i\phi\partial_j\phi\partial^2_{ij}\phi + |\nabla\phi|^2\triangle\phi = 0\end{aligned} \tag{1.6}$$

with $x^0 = t$, $\partial_0 = \partial_t$, $\partial = (\partial_0, \partial_1, \partial_2)$ and $\Delta = \partial_1^2 + \partial_2^2$. We will study the global smooth solution to equation (1.6) with the initial data of the "short pulse" form as

$$(\phi, \partial_t\phi)|_{t=1} = (\delta^{2-\varepsilon_0/2}\phi_0, \delta^{1-\varepsilon_0/2}\phi_1)(\frac{r-1}{\delta}, \omega), \tag{1.7}$$

where $r = |x| = \sqrt{x_1^2 + x_2^2}$, $\omega = \frac{x}{r} \in \mathbb{S}^1$, $\varepsilon_0 \in (0, \frac{1}{8}]$, $(\phi_0, \phi_1)(s, \omega)$ are any fixed smooth functions with compact supports in $(-1, 0)$ for the variable s. Furthermore, it is required that

$$(\partial_t + \partial_r)^l\phi|_{t=1} = O(\delta^{2-l\varepsilon_0/2}), \ l = 1, 2, \tag{1.8}$$

and

$$(\partial_t + \partial_r)^k \nabla\!\!\!/^j \partial^i \phi|_{t=1} = O(\delta^{2-\varepsilon_0-i}), \quad 0 \le k \le 2, \tag{1.9}$$

where $\nabla\!\!\!/$ stands for the derivative on $\mathbb{S}^1$. Note that although the short pulse initial data $(\phi, \partial_t\phi)|_{t=1}$ in (1.7) do not have the uniform boundedness (smallness) in $H^{\frac{11}{4}+}(\mathbb{R}^2) \times H^{\frac{7}{4}+}(\mathbb{R}^2)$ -norm (independent of δ) required for the well-posedness of regular solutions for 2D quasilinear wave equations in [1–3, 21, 35] (see Remarks 1.1-1.3), yet the conditions (1.7)-(1.9) imply the hyperbolicity of the equation (1.6) and the suitably stronger smallness of the directional derivatives $\partial_t + \partial_r$ of ϕ up to the second order. This is essential for our global existence of smooth solutions to the Cauchy problem (1.6)-(1.7).

The main result of this paper is stated as follows:

Theorem 1.1. *Assume that* (1.8)-(1.9) *hold. Then there exists a suitably small positive constant δ_0 such that for all $\delta \in (0, \delta_0]$, the Cauchy problem,* (1.6)-(1.7)*, admits a global smooth solution $\phi \in C^\infty([1, +\infty) \times \mathbb{R}^2)$. Furthermore, it holds that for all time $t \ge 1$,*

$$|\nabla\phi(t, x)| \le C\delta^{1-\varepsilon_0} t^{-1/2},\ x \in \mathbb{R}^2, \tag{1.10}$$

where $C > 0$ is a constant independent of δ and ε_0.

Some comments on the main result and a brief review of some closely related literature are given in the following remarks.

Remark 1.1. *The initial data of form* (1.9) *are essentially the "short pulse data" first introduced by D. Christodoulou in [5], where it is shown that the formation of black holes in vacuum spacetime is due to the condensation of the gravitational waves for the 3D Einstein equations in general relativity (see also [23]). Note that properties* (1.8)-(1.9) *hold true for given smooth function ϕ_0 and the choice of $\phi_1(\frac{r-1}{\delta}, \omega) = -\partial_s\phi_0(\frac{r-1}{\delta}, \omega) - \frac{\delta}{2}\phi_0(\frac{r-1}{\delta}, \omega)$. In addition, a large class of short pulse initial data with the properties* (1.7)-(1.9) *can be found for general second order quasilinear wave equations (see [12]). Roughly speaking, the "short pulse data" can be regarded as some suitable extensions of a class of "large" symmetric data, for which the smallness restrictions are imposed on angular directions and along the "good" direction tangent to outgoing light cone $\{t = r\}$, but the largeness is kept at least for the second order "bad" directional derivatives $\partial_t - \partial_r$. This provides a powerful framework to study effectively the blowup or the global existence of smooth solutions to the multi-dimensional hyperbolic systems or the second order quasilinear wave equations with short pulse data by virtue of the corresponding knowledge from the 1D cases, see [5, 6, 11–13, 17, 23, 29, 30, 32, 33, 36, 37] and the references therein. It is noted that for short pulse data* (1.7) *and sufficiently small $\delta > 0$, although both $||\phi||_{L^\infty}$ and $||\partial\phi||_{L^\infty}$ are small at $t = 1$, yet the initial data are still regarded as "large" in the sense that $|\partial^2\phi|$ may be large, and in fact,*

$$||\phi(1, \cdot)||_{H^s(\mathbb{R}^2)} = O(\delta^{\frac{5-\varepsilon_0}{2}-s}) \to +\infty \text{ as } \delta \to 0^+ \text{ for } s \ge \frac{11}{4}. \tag{1.11}$$

Remark 1.2. *Theorem 1.1 implies in particular the uniform (independent of δ) local in time well-posedness for the Cauchy problem* (1.6)-(1.7)*, which does not follow from the known results [21, 22, 35]. Indeed, for the Cauchy problem of general 2D quasilinear wave equation with smooth coefficients,*

$$\begin{cases} \sum_{\alpha,\beta=0}^{2} g^{\alpha\beta}(w, \partial w)\partial^2_{\alpha\beta} w = 0, \\ (w(1, x), \partial_t w(1, x)) = (w_0(x), w_1(x)) \in (H^s(\mathbb{R}^2), H^{s-1}(\mathbb{R}^2)), \end{cases} \tag{1.12}$$

the local in time well-posedness of solution $w \in C([1, T], H^s(\mathbb{R}^2)) \cap C^1([1, T], H^{s-1}(\mathbb{R}^2))$ with $s > \frac{11}{4}$ has been established in [35]. However, such a theory cannot be applied to (1.6)-(1.7) *to yield the local well-posedness of smooth solution with time interval independent of δ due to* (1.11).

Remark 1.3. *Consider the second order quasilinear wave equations of the form*

$$\sum_{\alpha,\beta=0}^{2} g^{\alpha\beta}(\partial w)\partial_{\alpha\beta}^2 w = 0 \tag{1.13}$$

with $g^{\alpha\beta}(\partial w)$ being smooth such that for small $|\partial w|$,

$$\begin{aligned} g^{\alpha\beta}(\partial w) =& c_0^{\alpha\beta} + \sum_{\gamma_1=0}^{2} c^{\alpha\beta,\gamma_1}\partial_{\gamma_1} w + \sum_{\gamma_1,\gamma_2=0}^{2} c^{\alpha\beta,\gamma_1\gamma_2}\partial_{\gamma_1} w\partial_{\gamma_2} w + \sum_{\gamma_1,\gamma_2,\gamma_3=0}^{2} c^{\alpha\beta,\gamma_1\gamma_2\gamma_3}\partial_{\gamma_1} w\partial_{\gamma_2} w\partial_{\gamma_3} w \\ &+ \cdots + \sum_{\gamma_1,\gamma_2,\ldots,\gamma_k=0}^{2} c^{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_k}\partial_{\gamma_1} w\partial_{\gamma_2} w\cdots\partial_{\gamma_k} w + O(|\partial w|^{k+1}), \end{aligned}$$

where $k \geq 2$, $\sum_{\alpha,\beta=0}^{2} c_0^{\alpha\beta}\partial_{\alpha\beta}^2 = \Box = -\partial_t^2 + \Delta$, and $c^{\alpha\beta,\gamma_1}$, $c^{\alpha\beta,\gamma_1\gamma_2}$, ..., $c^{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_k}$ are constants. Then the l-th null condition ($l \leq k$) is defined to be

$$\sum_{\alpha,\beta,\gamma_1,\ldots,\gamma_l=0}^{2} c^{\alpha\beta,\gamma_1\cdots\gamma_l}\omega_\alpha\omega_\beta\omega_{\gamma_1}\cdots\omega_{\gamma_l} = 0 \, \textit{for} \, \omega_0 = -1, \ (\omega_1,\omega_2) \in \mathbb{S}^1. \tag{1.14}$$

If one supplements (1.13) *with small data, i.e.,*

$$(w, \partial_t w)(1, x) = \varepsilon(w_0, w_1)(x) \tag{1.15}$$

with $(w_0, w_1) \in C_0^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$ being sufficiently small, then the global well-posedness of the solution to (1.13) *and* (1.15) *has been studied extensively and it is known that the small data solution exists globally if both the first and second null conditions hold, while it blows up in general if otherwise, see [1–3]. However, such a theory does not apply to the Cauchy problem* (1.6)-(1.7) *since though both the first and second null conditions are satisfied for* (1.6)*, yet the smallness condition* (1.15) *can not be true by the short pulse data* (1.7) *due to* (1.11)*. It should be noted that recently in [13], we have considered the global well-posedness problem for* (1.13) *with short pulse data of the form*

$$(w, \partial_t w)(1, x) = (\delta^{2-\varepsilon_0}\phi_0, \delta^{1-\varepsilon_0}\phi_1)(\frac{r-1}{\delta}, \omega), \tag{1.16}$$

where $\phi_0(s,\omega)$ and $\phi_1(s,\omega)$ are smooth functions supported in $(-1, 0)$ for the variable s, and shown that there exists an optimal constant $\varepsilon_k^* \in (0, 1)$, ($\varepsilon_k^* \to 1^+$ as $k \to \infty$), such that the smooth solution to (1.13) *and* (1.16) *exists globally for $\varepsilon_0 \in (0, \varepsilon_k^*)$ if the k-th null condition and all $c^{\alpha\beta,\gamma_1} = c^{\alpha\beta,\gamma_1\gamma_2} = \ldots = c^{\alpha\beta,\gamma_1\gamma_2\cdots\gamma_{k-1}} = 0$ hold, and blows up in finite time if the k-th order null condition fails and $\varepsilon_0 \in [\varepsilon_k^*, 1)$.*

Remark 1.4. *Theorem 1.1 implies that Majda's conjecture is solved for the 2D compressible Euler system for Chaplygin gases,* (1.1) *with small and irrotational data. Yet such a conjecture is still open for general small data of the form $(\rho, v)(0, x) = (\bar{\rho} + \varepsilon\rho_0(x), \varepsilon v_0(x))$ with $\varepsilon > 0$ being small without the irrotationality assumption unless some symmetries are assumed, see [8, 16, 18, 19] and the references therein.*

Remark 1.5. *In [11], we have also established the global existence of smooth solutions for the 3D compressible isentropic Euler equations with irrotational and short pulse initial data of the form* (1.7)-(1.9)*. Compared with the analysis for 3D in [11], due to the slower time-decay rate of the solutions in 2D, more involved and technical analysis is needed in this paper.*

Remark 1.6. *Note that for the short pulse data* (1.7) *satisfying* (1.8)-(1.9)*, it holds that* $|\nabla\phi|$ *is small for small* $\delta > 0$*, which is necessary to ensure the hyperbolicity of* (1.6)*. This is in contrast to the case in [32], where the global smooth solution* $\varphi = (\varphi^1, \cdots, \varphi^N)$ *has been established for the following 3D semi-linear wave system*

$$\Box\varphi^i = Q^i(\nabla\varphi, \nabla\varphi),\ i = 1, \cdots, N \tag{1.17}$$

with some short pulse data, where $Q^i(\nabla\varphi, \nabla\varphi)$ *are quadratic forms in* $\nabla\varphi$ *and satisfy the first null condition. In [32], the short pulse data for* (1.17) *are required to satisfy, instead of* (1.9),

$$|(\partial_t + \partial_r)^k \not\nabla^l \partial^m \varphi|_{t=1} \lesssim \delta^{1/2-m},\ \forall k \le N_0 \tag{1.18}$$

for some sufficiently large N_0*, where* $\not\nabla$ *stands for any derivative on* $\mathbb{S}^2$*. Note that though* (1.18) *imposes no restriction on the size of* $\nabla\varphi$ *since the semi-linear wave equation* (1.17) *is always hyperbolic with respect to the time, yet* N_0 *is required to be sufficiently large. It should be emphasized that one cannot require* (1.9) *holds for large* k *since it follows from* (1.7) *and* (1.6) *that* $\partial_{t,x}^q \phi|_{t=1} = O(\delta^{2-\varepsilon_0/2-|q|})$ *hold such that* (1.9) *seems to be over-determined for* $k \ge 3$*. Furthermore, our energies for* (1.6) *are different from the ones in [32].*

Remark 1.7. *We have generalized the main ideas and techniques in this paper to study the Cauchy problem for general 4D quasilinear wave equations of the form*

$$\sum_{\alpha,\beta=0}^{4} g^{\alpha\beta}(w, \partial w)\partial_{\alpha\beta}^2 w = 0 \tag{1.19}$$

with short pulse initial data (1.16)*, where* (1.19) *satisfies the first null condition, and* $\phi_0(s,\omega)$ *and* $\phi_1(s,\omega)$ *are smooth functions on* $\mathbb{R} \times \mathbb{S}^3$ *with compact support in* $(-1, 0)$ *for the variable* s *satisfying*

$$(\partial_t + \partial_r)^k \Omega^l \partial^q w|_{t=1} = O(\delta^{2-\varepsilon_0-|q|}) \text{ for } 0 \le k \le 3,\ \Omega \in \{x^i\partial_j - x^j\partial_i, 1 \le i < j \le 4\}.$$

We have shown in [12] that there exists an optimal constant $\varepsilon^* \in (0, 1)$ *such that for* $\varepsilon_0 \in (0, \varepsilon^*)$*, such a problem has a global smooth solution.*

Remark 1.8. *There have been many very interesting recent works on global well-posedness or finite blowup of smooth solutions to various nonlinear multi-dimensional wave equations with some short pulse initial data. In particular, in [33], Miao and Yu studied the following 3D problem*

$$\begin{cases} -(1 + 3G''(0)(\partial_t\phi)^2)\partial_t^2\phi + \Delta\phi = 0, \\ (\phi, \partial_t\phi)(-2, x) = (\delta^{3/2}\phi_0, \delta^{1/2}\phi_1)(\dfrac{r-2}{\delta}, \omega), \end{cases} \tag{1.20}$$

where $G''(0)$ *is a non-zero constant,* $\omega \in \mathbb{S}^2$*,* $\delta > 0$ *is suitably small, and* $\phi_0(s,\omega)$ *and* $\phi_1(s,\omega)$ *are given smooth functions supported in* $(0, 1]$ *with respect to* s*, and proved the shock formation before time* $t = -1$ *due to the genuinely nonlinear structure in the equation* (1.20) *which leads to the compression of incoming characteristic conic surfaces. This is one of main motivations of our works. Moreover, some systematic results along this line have been obtained in [11, 14, 29] for general quasilinear wave equations of the form* $\sum_{\alpha,\beta=0}^{3} g^{\alpha\beta}(\partial w)\partial_{\alpha\beta}^2 w = 0$ *with short pulse initial data* (1.16) *(for* $0 < \varepsilon_0 < 1$*).*

Remark 1.9. *Note that for the short pulse data* (1.7)*, the corresponding initial data* (ρ, v) *for* (1.1) *are small perturbations of the non-vacuum uniform state* $(\bar\rho, 0)$*. For general large initial data, one cannot expect the global existence of smooth solution to* (1.1) *in general. Indeed, even in 1D case, such a global* *existence of smooth solution fails for general large data as shown by the following example. Consider the following Cauchy problem of the 1D compressible isentropic Euler equation for Chaplygin gases*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0 \\ (\rho, v)(0, x) = \big(1, v_0(x)\big) \end{cases} \tag{1.21}$$

where $p(\rho) = 2 - \frac{1}{\rho}$, $v_0(x) \in C_0^\infty(-1, 1)$ *and* $v_0(x) \not\equiv 0$. *In terms of Lagrange coordinate* (s, m)*:*

$$s = t, \quad m = \int_{\eta(t,0)}^{x} \rho(t, y) dy,$$

where $\eta(t, y)$ *is the particular path through* y*,* (1.21) *becomes*

$$\begin{cases} \partial_s \rho + \rho^2 \partial_m v = 0, \\ \partial_s v + \partial_m p(\rho) = 0, \\ \rho(0, m) = 1, \quad v(0, m) = v_0(m). \end{cases}$$

Then the special volume $V = \frac{1}{\rho}$ *solves the following problem*

$$\begin{cases} \Box V = (\partial_s^2 - \partial_m^2) V = 0, \\ (V, \partial_s V)(0, m) = (1, v_0'(m)), \end{cases}$$

whose unique solution is

$$V(s, m) = 1 + \frac{1}{2}\big(v_0(m + s) - v_0(m - s)\big). \tag{1.22}$$

It follows from (1.22) *that one can choose* v_0 *such that* $v_0(m_2) - v_0(m_1) = -2$ *for* $m_1, m_2 \in (-1, 1)$ *with* $m_1 < 0 < m_2$. *Hence for such initial data, there exists a* $s^* \le \frac{m_2 - m_1}{2}$ *such that*

$$0 < V(s, m),\ s < s^*, \min V(s^*, m) = 0,$$

which implies $\max \rho(s, m) \to \infty$ *as* $s \to s^*_-$*, so concentration occurs at* s^**. Note that even for such an example, the Majda's conjecture still holds true since the singularity is the density concentration, not formation of shocks. For more results on finite time blow up of smooth solutions to* n*-dimensional quasilinear wave equations with general large data (not the short pulse data), we refer to [34] and the references therein.*

Remark 1.10. *Finally, we make some brief comments on the analysis of the proof of Theorem 1.1, the details are given in the next subsection. Our analysis is strongly motivated by the geometric approach initiated by D. Christodoulou in order to study the formation of shocks for multi-dimensional hyperbolic systems and the second order wave equations with the genuinely nonlinear conditions, see also [6, 17, 30, 33, 36, 37]. In the seminal work [4], Christodoulou introduced the "inverse foliation density"* μ *to measure the compression of the outgoing characteristic surfaces, and proved the finite time formation of shocks for 3D relativistic Euler equations with small initial data by developing a geometric approach which has been applied and refined to study shock formation for other important problems, see [4–6, 17, 29,33,36,37], where the key is to show that* μ *is positive away from the shock and approaches* 0^+ *near the blowup curve in finite time based on the genuinely nonlinear conditions. In this paper, the characteristic fields for Chaplygin gases are linearly degenerate, so it is possible to exclude the possibility of finite time* *collapse of the outgoing characteristic surfaces. Then as in [4,5,33,36], we may choose a similar inverse foliation density μ on a domain, $A_{2\delta} = \{(t,x) : t \geq 1+2\delta, 0 \leq t-r \leq 2\delta\}$, near the outermost outgoing conic surface $C_0 = \{t = r\}$. The first main step is to show that there exists a positive constant C such that $\mu \geq C > 0$ on $A_{2\delta}$ for all time, which can be established simultaneously with suitable apriori time-decay estimate on the solution ϕ to* (1.6) *on $A_{2\delta}$. To this end and to overcome the difficulties due to the slow time decay rates of solutions to th 2D wave equation, we introduce some new auxiliary energies and take full advantages that the equation* (1.6) *is totally linearly degenerate and satisfies both the first and second null conditions, which are rather different from the analysis in [4, 31]. Then we can obtain the global smooth solution ϕ of* (1.6) *with suitable time-decay in $A_{2\delta}$. The second main step to prove Theorem 1.1 is to solve a Goursat problem for* (1.6) *in the domain $B_{2\delta} = \{(t,x) : t \geq 1+\delta, t-r \geq 2\delta\}$ inside the outermost outgoing cone $\{r \leq t\}$. To this end, we first derive some delicate estimates on $\tilde{C}_{2\delta} = \{(t,x) : t \geq 1+2\delta, t-r = 2\delta\}$ which is the lateral boundary of $B_{2\delta}$ (see the end of Section 10 for details). Then we can establish the global weighted energies of ϕ in $B_{2\delta}$ by making use of both the first and second null conditions satisfied by* (1.6). *It should be noted that* (1.7) *is the short pulse data, it seems difficult to adopt the ideas in [8, 32] where the data are small. Finally, Theorem 1.1 follows from these two main steps.*

1.2 Sketch for the proof of Theorem 1.1

Since the proof of Theorem 1.1 is rather lengthy, for convenience of the reader, we will outline the main steps and ideas of the analysis in this subsection. Recall some notations in Remark 1.10: $C_0 = \{(t,x) : t \geq 1+2\delta, t = r\}$ is the outermost outgoing conic surface; $A_{2\delta} = \{(t,x) : t \geq 1+2\delta, 0 \leq t-r \leq 2\delta\}$ is a domain containing C_0; and $B_{2\delta} = \{(t,x) : t \geq 1+\delta, t-r \geq 2\delta\}$ is a conic domain inside C_0 with the lateral boundary given by $\tilde{C}_{2\delta} = \{(t,x) : t \geq 1+2\delta, t-r = 2\delta\}$. Then the proof of Theorem 1.1 consists of three parts: the local existence of solution ϕ for $t \in [1, 1+2\delta]$, the global existence of the solution in $A_{2\delta}$, and the global existence in $B_{2\delta}$. The main ideas are sketched as follows.

1.2.1 Local existence of the solution for $t \in [1, 1+2\delta]$

At first, by the energy method and the special structure of equation (1.6), one can obtain the local existence of the smooth solution ϕ to (1.6) with (1.7)-(1.9) for $t \in [1, 1+2\delta]$. Furthermore, following basic estimates on $\phi(1+2\delta, x)$ can be derived:

$$|L^a \partial^\alpha \Omega^\kappa \phi(1+2\delta, x)| \lesssim \delta^{2-|\alpha|-\varepsilon_0} \quad \text{for } r \in [1-2\delta, 1+2\delta], \tag{1.23}$$

$$|\underline{L}^a \partial^\alpha \Omega^\kappa \phi(1+2\delta, x)| \lesssim \delta^{2-|\alpha|-\varepsilon_0} \quad \text{for } r \in [1-3\delta, 1+\delta], \tag{1.24}$$

where $a \in \mathbb{N}_0$ can be chosen as large as needed, $\alpha \in \mathbb{N}_0^3$, $\kappa \in \mathbb{N}_0$, $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, and $\Omega = x^1\partial_2 - x^2\partial_1$. Note that (1.23) and (1.24) yield some better smallness property of ϕ along certain directional derivatives L or $\underline{L}$ in different space domains than that on the whole hypersurface $\{(t,x) : t = 1+2\delta\}$ (i.e., $|L^a \partial^\alpha \Omega^\kappa \phi(1+2\delta, x)| \lesssim \delta^{4-|a|-|\alpha|-\varepsilon_0}$ and $|\underline{L}^a \partial^\alpha \Omega^\kappa \phi(1+2\delta, x)| \lesssim \delta^{2-|a|-|\alpha|-\varepsilon_0}$ can be obtained for $x \in \mathbb{R}^2$ and $a \geq 2$). Based on these crucial estimates together with the totally linearly degenerate structure of (1.6), we can obtain the desired delicate estimates of ϕ near C_0, which is one of the key points for proving the global existence of ϕ in $A_{2\delta}$.

1.2.2 Global existence of the solution in $A_{2\delta}$

Motivated by the strategy of D. Christodoulou in [4], we start to construct the solution ϕ of (1.6) in $A_{2\delta}$. Note that although the main results in [4] concern the finite time blowup of smooth solutions to the 3-D compressible Euler equations, which seem to be different from our goal to establish global smooth solution to (1.6)-(1.7), yet we can still make full use of the idea in [4] to verify whether the outgoing characteristic surfaces collapse since the intersection of characteristic surfaces will lead to the formation of singularities of smooth solutions and further correspond to the formation of shock waves, as indicated in other studies for global existence of smooth solutions to the compressible Euler equation [7, 38–41]. In this process, one of the key elements is to derive the suitable time-decay rates of the solution ϕ in $A_{2\delta}$ under some suitable coordinate transformation. Due to the slower time-decay rate of solutions to the 2-D wave equation, we need to introduce some new auxiliary energies and carry out some delicate analysis on the related nonlinear forms by utilizing the distinguished characters of the resulting new 2-D quasilinear wave equations satisfying the first and second null conditions as well as the totally linearly degenerate condition. This is rather different from those in [4] and [36] for the 3D small data solution problem with the genuinely nonlinear condition.

As in [4] or [36], for a given smooth solution ϕ to (1.6), one can study the related eikonal equation $\sum\limits_{\alpha,\beta=0}^{2} g^{\alpha\beta}(\partial\phi)\partial_\alpha u\partial_\beta u = 0$ with the initial data $u(1+2\delta, x) = 1+2\delta - r$. Then the inverse foliation density $\mu = -(\sum_{\alpha,\beta=0}^{2} g^{\alpha\beta}\partial_\alpha u\partial_\beta t)^{-1}$ can be defined. Under the suitable bootstrap assumptions on $\partial\phi$ (see $(\star)$ in Section 4) and with the help of the totally linearly degeneracy of (1.6), then μ satisfies $\mathring{L}\mu = O(\delta^{1-\varepsilon_0}t^{-3/2}\mu)$, where $\mathring{L} = -\mu\sum\limits_{\alpha,\beta=0}^{2} g^{\alpha\beta}\partial_\alpha u\partial_\beta$ is a vector field approximating $\partial_t + \partial_r$. Thus $\mu \sim 1$ can be derived. The positivity of μ means that the outgoing characteristic conic surfaces never intersect as long as the smooth solution ϕ exists. Set $\varphi = (\varphi_0, \varphi_1, \varphi_2) := \partial\phi = (\partial_0\phi, \partial_1\phi, \partial_2\phi)$. Then it follows from (1.6) that

$$\mu\Box_g\varphi_\gamma = F_\gamma(\varphi, \partial\varphi), \quad \gamma = 0, 1, 2, \tag{1.25}$$

where $g = g_{\alpha\beta}(\varphi)dx^\alpha dx^\beta$ is the Lorentzian metric, $(g_{\alpha\beta}(\varphi))$ is the inverse matrix of $(g^{\alpha\beta}(\varphi))$, $\Box_g = \frac{1}{\sqrt{|\det g|}}\sum\limits_{\alpha,\beta=0}^{2}\partial_\alpha(\sqrt{|\det g|}g^{\alpha\beta}\partial_\beta)$, and F_γ are smooth functions in their arguments. To study the quasilinear wave system (1.25), we first focus on its linearization

$$\mu\Box_g\Psi = \Phi. \tag{1.26}$$

Note that the function μ never goes to 0 in the paper, which means that the defined energies and the fluxes of Ψ (see (6.20)-(6.23) in Section 6) do not contain the degenerate factors. As in [12], it is crucial to derive the global time-decay rate of Ψ since $\Psi = \Psi_k = Z^k\varphi$ will be chosen in (1.26), here Z stands for one of some first order vector fields. In this case, by computing the commutator $[\mu\Box_g, Z^k]$, there will appear the quantities containing the $(k+2)$-th order derivatives of φ in the expression Φ of (1.26). For examples, $\not\nabla Z^{k-1}\lambda$ and $\not\nabla^2 Z^{k-1}\mu$ will occur in Φ, where $\lambda = g(\mathscr{D}_X\mathring{L}, X)$ is the second fundamental form of $S_{s,u}$ with $\mathscr{D}$ being the Levi-Civita connection of g and $X = \frac{\partial}{\partial\vartheta}$ (ϑ is the extended local coordinate of $\theta \in \mathbb{S}^1$ which is described by $\mathring{L}\vartheta = 0$ and $\vartheta|_{t=1+2\delta} = \theta$). Following the analogous procedures in Section 3-Section 11 of [12] and by the much involved analysis to the 2D problem (1.6) with (1.7), we can eventually obtain $|\varphi| \lesssim \delta^{1-\varepsilon_0}t^{-1/2}$ and further close the basic bootstrap assumptions. However, compared with the corresponding treatments for the global 4D problem in [12] or the 3D problem before the shock formation in [4, 36], it is more difficult and complicated to derive the global weighted energy estimates for the 2D system (1.26) with the suitable time-decay rates due to the lower space dimensions. In order to obtain the global weighted energy of Ψ, one can proceed as usual to compute $\int_{D^{t,u}}\mu\Box_g\Psi(V\Psi)$ through appropriate choices of some first order vector field V, where $D^{t,u} =$

$\{(\mathfrak{t}', u', \vartheta) : 1 \le \mathfrak{t}' < \mathfrak{t}, 0 \le u' \le u, 0 \le \vartheta \le 2\pi\}$. We will choose the vector field V as, respectively,

$$J_1 = -\varrho^{2m} \sum_{\alpha,\kappa,\beta=0}^{2} g^{\alpha\kappa} Q_{\kappa\beta} \mathring{L}^\beta \partial_\alpha,$$

$$J_2 = -\sum_{\alpha,\kappa,\beta=0}^{2} g^{\alpha\kappa} Q_{\kappa\beta} \mathring{\underline{L}}^\beta \partial_\alpha,$$

$$J_3 = \sum_{\alpha=0}^{2} \big(\frac{1}{2}\varrho^{2m-1}\Psi \mathscr{D}^\alpha \Psi - \frac{1}{4}\Psi^2 \mathscr{D}^\alpha(\varrho^{2m-1})\big)\partial_\alpha,$$

where $m \in (\frac{1}{2}, \frac{3}{4})$ is a fixed constant, $\varrho = \mathfrak{t} - u$, $\mathring{\underline{L}} = -\mu(\mathring{L} + 2\sum_{\nu=0}^{2} g^{\nu 0}\partial_\nu)$, and Q is the *energy-momentum tensor field* of Ψ given as

$$Q_{\alpha\beta} = (\partial_\alpha \Psi)(\partial_\beta \Psi) - \frac{1}{2} g_{\alpha\beta} \sum_{\nu,\lambda=0}^{2} g^{\nu\lambda}(\partial_\nu \Psi)(\partial_\lambda \Psi).$$

It should be pointed out that the weight ϱ acts as the time $\mathfrak{t}$ in J_1 and J_3 since $\rho \sim \mathfrak{t}$, and the requirement of $m > \frac{1}{2}$ in J_1 is due to the slow time-decay rate of solutions to the 2-D wave equation such that some related integrals are convergent (see (9.22) in Section 9, where $\int_{t_0}^{\mathfrak{t}} \tau^{-1/2-m} d\tau$ will be uniformly bounded for $\mathfrak{t}$). However, once $m > \frac{1}{2}$ is chosen, then $\int_{D^{\mathfrak{t},u}} \mu \Box_g \Psi(V\Psi)$ with $V = J_1$ and J_2 will contain the non-negative integral $(m - \frac{1}{2}) \int_{D^{\mathfrak{t},u}} \mu\varrho^{2m-1}|\not{d}\Psi|^2$ which cannot be controlled by the corresponding energies and fluxes on the left hand side of the resulting inequality. To overcome this crucial difficulty, we introduce a new vector field J_3. Thanks to the special structure of J_3 and by some technical manipulations, we can eventually obtain a new term $(m-1)\int_{D^{\mathfrak{t},u}} \mu\varrho^{2m-1}|\not{d}\Psi|^2$ instead of $(m - \frac{1}{2}) \int_{D^{\mathfrak{t},u}} \mu\varrho^{2m-1}|\not{d}\Psi|^2$ on the right hand side of the related energy inequality, which has the desired sign for $m < 1$. Meanwhile, in the estimate of $\int_{D^{\mathfrak{t},u}} \mu \Box_g \Psi(J_3\Psi)$, it is necessary to restrict $m < \frac{3}{4}$ since the integral $\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2} d\tau$ is required to be uniformly bounded (see (6.32)). In addition, it is noted that for the shock formation problem of the 3-D potential flow equation for polytropic gases with small initial data, the vector fields J_1 with $m = 1$ and J_2 are chosen in the related energy estimates, see Subsection 10.2 of [36].

1.2.3 Global existence of the solution in $B_{2\delta}$

It follows from (1.23) and (1.24) that for $r \in [1 - 3\delta, 1 + \delta]$, $\partial^\alpha \phi$ admit the smallness of the higher order $\delta^{2-|\alpha|-\varepsilon_0}$ at time $t = 1 + 2\delta$. In addition, note that the outgoing characteristic cones of (1.6) starting from $\{t = 1 + 2\delta, 1 - 2\delta \le r \le 1 + \delta\}$ are almost straight, and contain $\tilde{C}_{2\delta}$. By the properties (1.23) and (1.24), we can prove that on $\tilde{C}_{2\delta}$, the solution ϕ and its derivatives satisfy $|\partial^\alpha \phi| \lesssim \delta^{2-\varepsilon_0} t^{-1/2}$ with the better smallness $O(\delta^{2-\varepsilon_0})$. Based on such "good" smallness of ϕ on $\tilde{C}_{2\delta}$, we will solve the global Goursat problem of (1.6) in the conic domain $B_{2\delta}$. To this end, we intend to establish the global weighted energy estimates for the solution ϕ in $B_{2\delta}$ and make use of the Klainerman-Sobolev inequality to get the time-decay rates for $\partial^\alpha \phi$. However, since the classical Klainerman-Sobolev inequality holds generally on the whole space (see Proposition 6.5.1 in [21]), we need a modified Klainerman-Sobolev inequality for $B_{2\delta}$ whose lateral boundary is $\tilde{C}_{2\delta}$. Using this together with the bootstrap energy assumptions, and a careful analysis on the weighted energy estimates with the ghost weight $W = e^{2(1+t-r)^{-1/2}}$, we can obtain the large time behaviours of the solution ϕ up to the forth order derivatives (see Proposition 11.1), for examples, $(\omega^i\partial_t+\partial_i)\phi=O(\delta^{25/16-\varepsilon_0})t^{-3/2}(1+t-r)$, $(\omega^i\partial_t+\partial_i)\partial\phi=O(\delta^{9/8-\varepsilon_0})t^{-3/2}(1+t-r)$, $(\omega^i\partial_t+\partial_i)\partial^2\phi=O(\delta^{5/16-\varepsilon_0})t^{-3/2}(1+t-r)$ and $(\omega^i\partial_t+\partial_i)\partial^3\phi=O(\delta^{-9/16-\varepsilon_0})t^{-3/2}(1+t-r)$ with $i=1,2$ (which imply that $\partial^4\phi$ may be large). On the other hand, the bootstrap assumptions can not be closed directly since the corresponding higher order energies of ϕ may grow in time as $t^{2\iota}$ (here $\iota>0$ is some positive constant, see Theorem 11.1). To overcome this difficulty, we turn to studying the nonlinear equation for the error $\dot\phi=\phi-\phi_a$, where ϕ_a is the solution to the 2D free wave equation $\Box\phi_a=0$ with the initial data $(\phi(1+2\delta,x),\partial_t\phi(1+2\delta,x))$, and obtain the uniformly controllable energy estimates for $\dot\phi$ by a delicate analysis. Based on this, the bootstrap energy assumptions of ϕ can be closed, and the global existence of ϕ with $|\partial\phi|\le C\delta^{25/16-\varepsilon_0}t^{-1/2}$ inside $B_{2\delta}$ is established.

1.3 Organization of the paper

Our paper is organized as follows. In Section 2, we prove the local existence of the solution ϕ for $1\le t\le 1+2\delta$ by the energy method, moreover, some desired smallness properties of ϕ are obtained. In Section 3, we first list some preliminary knowledge in the differential geometry, such as the definitions of the optical function, the inverse foliation density μ, the deformation tensor, the null frame and some norms of smooth functions. Then the equation for μ is derived, and some elementary calculations for the covariant derivatives of the null frame and for the deformation tensors are given. In Section 4, the crucial bootstrap assumptions $(\star)$ in $A_{2\delta}$ are listed, meanwhile, under assumptions $(\star)$ we derive some estimates on several quantities which will be extensively used in subsequent sections. In Section 5, under assumptions $(\star)$, the L^∞ estimates for the higher order derivatives of φ in $A_{2\delta}$ are established. In Section 6, we carry out the energy estimates for the linearized equation $\mu\Box_g\Psi=\Phi$ and define some suitable higher order weighted energies and fluxes as in [36]. In Section 7, under assumptions $(\star)$, we derive the higher order L^2 estimates of several key quantities. In Section 8, L^2 estimates on the highest order derivatives of trλ and $\not{\Delta}\mu$ are established, where trλ is the trace of the second fundamental form λ, and $\not{\Delta}$ is the Laplacian operator on $\mathbb{S}^1$. In Section 9, we deal with the error terms appeared in the energy inequalities of Section 6. Based on all the estimates in Section 4-Section 9, in Section 10 we complete the bootstrap argument and further establish the global existence of the solution ϕ to (1.6) in $A_{2\delta}$. In addition, in the end of Section 10, we derive the delicate estimates of ϕ on $\tilde C_{2\delta}$, which will play an important role in solving the global Goursat problem inside $B_{2\delta}$. Finally, we establish the global existence of the solution ϕ in $B_{2\delta}$ and complete the proof of Theorem 1.1 in Section 11.

1.4 Notations

Through the whole paper, unless stated otherwise, Greek indices $\{\alpha,\beta,\cdots\}$, corresponding to the spacetime coordinates, are chosen in $\{0,1,2\}$; Latin indices $\{i,j,k,\cdots\}$, corresponding to the spatial coordinates, are $\{1,2\}$; and we use the Einstein summation convention to sum over repeated upper and lower indices. In addition, the convention $f\lesssim g$ means that there exists a generic positive constant C such that $f\le Cg$.

Since (1.6) is a nonlinear wave equation, it is natural to introduce the inverse spacetime metric $(g^{\alpha\beta})$ as follows:

$$g^{00}=-1,\quad g^{i0}=g^{0i}=-\partial_i\phi,\quad g^{ij}=c\delta_{ij}-\partial_i\phi\partial_j\phi,\tag{1.27}$$

while $(g_{\alpha\beta})$ represents the corresponding metric:

$$g_{00}=-1+c^{-1}|\nabla\phi|^2,\quad g_{i0}=g_{0i}=-c^{-1}\partial_i\phi,\quad g_{ij}=c^{-1}\delta_{ij},\tag{1.28}$$

where $c=1+2\partial_t\phi+|\nabla\phi|^2$, and δ_{ij} is the usual Kronecker symbol.

Finally, the following notations will be used throughout the paper:

$$
\begin{aligned}
&t_0 = 1 + 2\delta,\\
&L = \partial_t + \partial_r,\\
&\underline{L} = \partial_t - \partial_r,\\
&\Omega = \epsilon_i^j x^i \partial_j,\\
&S = t\partial_t + r\partial_r = \frac{t-r}{2}\underline{L} + \frac{t+r}{2}L,\\
&H_i = t\partial_i + x^i\partial_t = \omega^i\big(\frac{r-t}{2}\underline{L} + \frac{t+r}{2}L\big) + \frac{t\omega_\perp^i}{r}\Omega,\\
&\Sigma_t = \{(t',x) : t' = t, x \in \mathbb{R}^2\},
\end{aligned}
$$

where $\epsilon_1^2 = 1$, $\epsilon_2^1 = -1$, $\epsilon_i^i = 0$ and $\omega_\perp = (-\omega^2, \omega^1)$.

2 Local existence of the smooth solution ϕ

In this section, we use the energy method to prove the local existence of the smooth solution ϕ to (1.6) with (1.7) for $1 \le t \le t_0$. Furthermore, we derive the important smallness estimates of $\phi(t_0, x)$ on some special spatial domains. The main result is:

Theorem 2.1. *Under the assumptions* (1.8) *and* (1.9) *on* (ϕ_0, ϕ_1)*, when* $\delta > 0$ *is suitably small, the Cauchy problem,* (1.6)-(1.7)*, admits a local smooth solution* $\phi \in C^\infty([1, t_0] \times \mathbb{R}^2)$*. Moreover, for* $a \in \mathbb{N}_0$*,* $b \in \mathbb{N}_0$*,* $q \in \mathbb{N}_0^3$ *and* $k \in \mathbb{N}_0$*, it holds that*

(i)

$$|L^a \partial^q \Omega^k \phi(t_0, x)| \lesssim \delta^{2-|q|-\varepsilon_0}, \quad r \in [1-2\delta, 1+2\delta], \tag{2.1}$$

$$|\underline{L}^a \partial^q \Omega^k \phi(t_0, x)| \lesssim \delta^{2-|q|-\varepsilon_0}, \quad r \in [1-3\delta, 1+\delta]. \tag{2.2}$$

(ii)

$$|\partial^q \Omega^k \phi(t_0, x)| \lesssim \begin{cases} \delta^{2-\varepsilon_0}, & as \quad |q| \le 2, \\ \delta^{4-|q|-\varepsilon_0}, & as \quad |q| > 2, \end{cases} \quad r \in [1-3\delta, 1+\delta]. \tag{2.3}$$

(iii)

$$|\underline{L}^a L^b \Omega^k \phi(t_0, x)| \lesssim \delta^{2-\varepsilon_0}, \quad r \in [1-2\delta, 1+\delta]. \tag{2.4}$$

Proof. Although the proof is rather analogous to that of Theorem 3.1 in [12], due to the different structures between the general 4D quasilinear wave equation satisfying the first null condition in [12] and the 2D quasilinear equation (1.6) fulfilling the first and second null conditions, we still give the details for the reader's convenience.

Denote by Z_g any fixed vector field in $\{S, H_i, i = 1, 2\}$. Suppose that for $1 \le t \le t_0$ and $N_0 \in \mathbb{N}_0$ with $N_0 \ge 6$,

$$|\partial^q \Omega^k Z_g^a \phi| \le \delta^{3/2-|q|} \quad (|q| + k + a \le N_0, \quad a \le 2). \tag{2.5}$$

Define the following energy for (1.6) and for $n \in \mathbb{N}_0$,

$$M_n(t) = \sum_{|q|+k+a \le n} \delta^{2|q|} \|\partial \partial^q \Omega^k Z_g^a \phi(t, \cdot)\|^2_{L^2(\mathbb{R}^2)}.$$

Let $w=\delta^{|q|}\partial^q\Omega^k Z_g^a\phi\quad(|q|+k+a\le 2N_0-2)$. It follows from (1.27), (1.6) and integration by parts that

$$\begin{aligned}&\int_1^t\int_{\Sigma_\tau}(\partial_t w g^{\alpha\beta}\partial^2_{\alpha\beta}w)(\tau,x)dxd\tau\\ =&\frac{1}{2}\int_{\Sigma_t}\big(-(\partial_t w)^2-g^{ij}\partial_i w\partial_j w\big)(t,x)dx-\frac{1}{2}\int_{\Sigma_1}\big(-(\partial_t w)^2-g^{ij}\partial_i w\partial_j w\big)(1,x)dx\\ &+\int_1^t\int_{\Sigma_\tau}\big(-\partial_i g^{0i}(\partial_t w)^2-(\partial_i g^{ij})\partial_j w\partial_t w+\frac{1}{2}(\partial_t g^{ij})\partial_i w\partial_j w\big)(\tau,x)dxd\tau\end{aligned}\tag{2.6}$$

with

$$\begin{aligned}g^{\alpha\beta}\partial^2_{\alpha\beta}w=&\delta^{|q|}\sum_{\substack{\Sigma_{i=1}^2(q_i,k_i,a_i)\le(q,k,a)\\(q_2,k_2,a_2)<(q,k,a)}}(\partial\partial^{q_1}\Omega^{k_1}Z_g^{a_1}\phi)(\partial^2\partial^{q_2}\Omega^{k_2}Z_g^{a_2}\phi)\\ &+\delta^{|q|}\sum_{\substack{\Sigma_{i=1}^3(q_i,k_i,a_i)\le(q,k,a)\\(q_3,k_3,a_3)<(q,k,a)}}(\partial\partial^{q_1}\Omega^{k_1}Z_g^{a_1}\phi)(\partial\partial^{q_2}\Omega^{k_2}Z_g^{a_2}\phi)(\partial^2\partial^{q_3}\Omega^{k_3}Z_g^{a_3}\phi),\end{aligned}\tag{2.7}$$

where we have neglected the unnecessary constant coefficients in (2.7).

It follows from the assumption (2.5) and (2.6) that

$$\begin{aligned}&\int_{\Sigma_t}\big((\partial_t w)^2+|\nabla w|^2\big)(t,x)dx\\ \lesssim&\int_{\Sigma_1}\big((\partial_t w)^2+|\nabla w|^2\big)(1,x)dx+\int_1^t\int_{\Sigma_\tau}\delta^{-1/2}\big((\partial_t w)^2+|\nabla w|^2\big)(\tau,x)dxd\tau\\ &+\int_1^t\int_{\Sigma_\tau}|(\partial_t w g^{\alpha\beta}\partial^2_{\alpha\beta}w)|(\tau,x)dxd\tau.\end{aligned}\tag{2.8}$$

Using the bootstrap assumption (2.5) to estimate (2.7) and substituting the resulting estimates into (2.8), one can get from the Gronwall's inequality that for $1\le t\le t_0$,

$$M_{2N_0-3}(t)\lesssim M_{2N_0-3}(1)e^{\delta^{-1/2}(t-1)}\lesssim\delta^{3-2\varepsilon_0}.$$

Next, it follows from the following Sobolev's imbedding theorem on the circle $\mathbb{S}^1_r$ (with center at the origin and radius r):

$$|w(t,x)|\lesssim\frac{1}{\sqrt{r}}\|\Omega^{\le 1}w\|_{L^2(\mathbb{S}^1_r)},$$

together with $r\sim 1$ for $t\in[1,t_0]$ and $(t,x)\in\text{supp }w$ that

$$|\partial^q\Omega^k Z_g^a\phi(t,x)|\lesssim\|\Omega^{\le1}\partial^q\Omega^k Z_g^a\phi\|_{L^2(\mathbb{S}^1_r)}\lesssim\delta^{1/2}\|\partial\Omega^{\le1}\partial^q\Omega^k Z_g^a\phi\|_{L^2(\Sigma_t)}\lesssim\delta^{2-|q|-\varepsilon_0},\tag{2.9}$$

when $|q|+k+a\le N_0$ and $a\le 2$, where $N_0\ge 6$ has been used. Therefore, (2.5) can be closed for suitably small $\delta>0$ and $\varepsilon_0<\frac{1}{2}$.

This, together with $L=(t+r)^{-1}(S+\omega^i H_i)$, yields

$$|L^a\partial^q\Omega^k\phi(t,x)|\lesssim|Z_g^a\partial^q\Omega^k\phi(t,x)|\lesssim\delta^{2-|q|-\varepsilon_0}\tag{2.10}$$

with $|q|+k+a\le N_0$ and $a\le 2$.

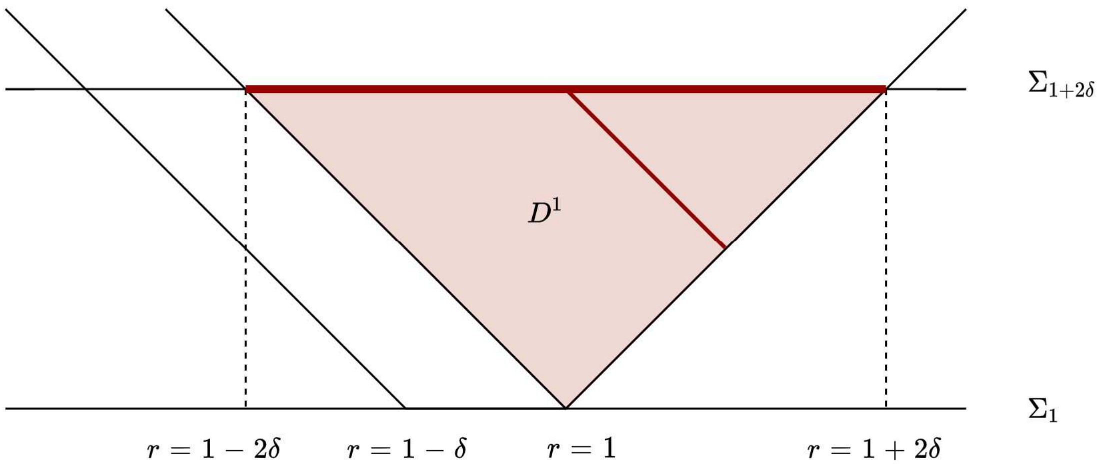

Figure 1. Space-time domain $D^1 = \{(t,r) : 1 \le t \le t_0, 2 - t \le r \le t\}$

Now we start to improve the L^∞ estimate of $\phi(t_0, x)$ on some special domains. Note that (1.6) can be rewritten as

$$L\underline{L}\phi = \frac{1}{\Theta}\Big\{\frac{1}{2r}L\phi - \frac{1}{2r}\underline{L}\phi + \frac{1}{r^2}\Omega^2\phi + \frac{1}{2}\underline{L}^2\phi \cdot L\phi + \frac{1}{2}L^2\phi \cdot \underline{L}\phi - \frac{1}{2r}(\underline{L}\phi)^2 + G_1 + G_2\Big\}, \quad (2.11)$$

where $\Theta = 1 + \frac{1}{2}\underline{L}\phi$, G_1 and G_2 are quadratic and cubic nonlinearities in the first and second order derivatives of ϕ, which have better smallness due to (2.10).

Acting the operator L^2 on both sides of (2.11) yields an expression of $\underline{L}L^3\phi$ by $\underline{L}L = L\underline{L}$ and direct computations. It can be checked easily that the worst terms in the expression of $\underline{L}L^3\phi$ are $\frac{1}{2\Theta}(L^4\phi)(\underline{L}\phi)$ and $\frac{1}{2\Theta}(\underline{L}^2\phi)(L^3\phi)$. Then one can use (2.10) to get $|\underline{L}L^3\phi| \lesssim \delta^{1-2\varepsilon_0}$. Using this together with the vanishing property of ϕ on C_0, one can integrate $\underline{L}L^3\phi$ along integral curves of $\underline{L}$ to show that for $(t,r) \in D^1$ (see Figure 1),

$$|L^3\phi(t,x)| \lesssim \delta^{2-2\varepsilon_0}. \quad (2.12)$$

Similarly, it holds that for $(t,r) \in D^1$,

$$|L^3\partial^q\Omega^k\phi(t,x)| \lesssim \delta^{2-|q|-2\varepsilon_0} \quad \text{for } |q| + k \le N_0 - 4. \quad (2.13)$$

Substituting (2.10) and (2.13) into the expression of $\underline{L}L^3\phi$ again, and noting that the worst term in the expression of $\underline{L}L^3\phi$ becomes $-\frac{1}{2r\Theta}L^2\underline{L}\phi$, one then further gets $|\underline{L}L^3\phi| \lesssim \delta^{1-\varepsilon_0}$ for $(t,r) \in D^1$ by (2.10). Hence the following improved smallness estimate holds:

$$|L^3\phi(t_0,x)| \lesssim \delta^{2-\varepsilon_0} \quad \text{for } 1 - 2\delta \le r \le 1 + 2\delta.$$

Similar arguments show that

$$|L^3\partial^q\Omega^k\phi(t_0,x)| \lesssim \delta^{2-|q|-\varepsilon_0}, \quad \text{for } |q| + k \le N_0 - 5 \text{ and } 1 - 2\delta \le r \le 1 + 2\delta.$$

Analogously, for $r \in [1 - 2\delta, 1 + 2\delta]$, an induction argument yields that

$$|L^2L^a\partial^q\Omega^k\phi(t_0,x)| \lesssim \delta^{2-|q|-\varepsilon_0}, \quad 3a + |q| + k \le N_0 - 2. \quad (2.14)$$

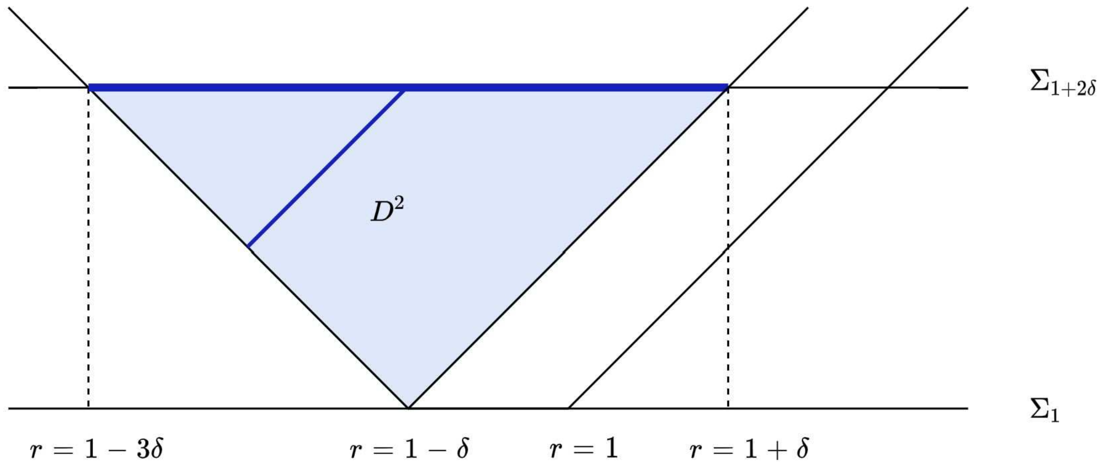

Figure 2. Space domain for $1 - 3\delta \leq r \leq 1 + \delta$ **on** $\Sigma_{1+2\delta}$

Similarly, by the expression of $L\underline{L}^a\partial^q\Omega^\kappa\phi$, integrating along integral curves of L yields that for $r \in [1 - 3\delta, 1 + \delta]$ (see Figure 2),

$$|\underline{L}^a\partial^q\Omega^k\phi(t_0, x)| \lesssim \delta^{2-|q|-\varepsilon_0}, \quad 2a + |q| + k \leq N_0 - 1. \tag{2.15}$$

Furthermore, since $\partial_t = \frac{1}{2}(L + \underline{L})$ and $\partial_i = \frac{\omega^i}{2}(L - \underline{L}) + \frac{\omega^i_\perp}{r}\Omega$, so (2.15), (2.11) and (2.10) imply that when $|q| + k \leq N_0 - 3$ and $r \in [1 - 3\delta, 1 + \delta]$,

$$|\partial^q\Omega^k\phi(t_0, x)| \lesssim \begin{cases} \delta^{2-\varepsilon_0}, & \text{as} \quad |q| \leq 2, \\ \delta^{4-|q|-\varepsilon_0}, & \text{as} \quad |q| > 2. \end{cases} \tag{2.16}$$

Next we prove the final estimate (2.4). Note that (2.16) implies that on the surface Σ_{t_0} with $r \in [1 - 2\delta, 1 + \delta]$, $|\underline{L}^{\leq 2}\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0}$ and $|L^{\leq 2}\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0}$ for $k \leq N_0 - 5$, and hence $|L\underline{L}\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0}$ by (2.11). Furthermore, we claim that

$$|\underline{L}^a L^b\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0}, \quad \text{for } 3a + 3b + k \leq N_0 - 1. \tag{2.17}$$

(2.17) can be proved by induction. Indeed, assume that (2.17) holds for $a + b \leq n_0$ with $n_0 \in \mathbb{N}_0$ satisfying $3n_0 + k \leq N_0 - 1$. One needs to verify the estimate in (2.17) for $3(n_0 + 1) + k \leq N_0$ and $a + b = n_0$. If $a \geq 1$, by (2.11) and the induction assumption, one can get that

$$\begin{aligned} &|\underline{L}^a L^{b+1}\Omega^k\phi| = |\underline{L}^{a-1}L^b\Omega^k(L\underline{L}\phi)| \\ \lesssim &\delta^{2-\varepsilon_0} + \delta^{2-\varepsilon_0}|\underline{L}^{a+1}L^b\Omega^{\leq k}\phi| + \delta^{2-\varepsilon_0}|\underline{L}^{a-1}L^{b+2}\Omega^{\leq k}\phi|. \end{aligned} \tag{2.18}$$

This together with an induction argument yields

$$|\underline{L}^a L^{b+1}\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0} + \delta^{2-\varepsilon_0}|\underline{L}^{a+1}L^b\Omega^{\leq k}\phi|. \tag{2.19}$$

If $b \geq 1$, similar to the proof of (2.19), one has

$$|\underline{L}^{a+1}L^b\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0} + \delta^{2-\varepsilon_0}|\underline{L}^a L^{b+1}\Omega^{\leq k}\phi|. \tag{2.20}$$

Combining (2.19) with (2.20) yields

$$|\underline{L}^a L^{b+1}\Omega^k\phi| + |\underline{L}^{a+1}L^b\Omega^k\phi| \lesssim \delta^{2-\varepsilon_0},$$

which means (2.4). Therefore, the proof of Theorem 2.1 is finished. □

3 Some preliminaries

3.1 The related geometry and definitions

In this subsection, we give some preliminaries on the related geometry and definitions, which will be utilized as basic tools later on. It is assumed that a smooth solution ϕ to (1.6) is given. The *"optical function"* corresponding to (1.6) can be introduced as in [4] (see also Definition 3.4 of [36]).

Definition 3.1 (Optical function)**.** *A C^1 function $u(t,x)$ is called the optical function of problem* (1.6) *if u satisfies the eikonal equation*

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{3.1}$$

Choose the initial data $u(t_0, x) = 1 + 2\delta - r$ and pose the condition $\partial_t u > 0$ for (3.1). For a given optical function, the *inverse foliation density* μ of the outgoing cones is defined as

$$\mu := -\frac{1}{g^{\alpha\beta}\partial_\alpha u \partial_\beta t} \quad (= -\frac{1}{g^{\alpha 0}\partial_\alpha u}). \tag{3.2}$$

We will show that $\mu \geq C > 0$ as long as the smooth solution ϕ to (1.6) exists. We adopt most of terminologies and definitions introduced by Christodoulou in [4] (see also [36]).

Note that

$$\tilde{L} = -g^{\alpha\beta}\partial_\alpha u \partial_\beta$$

is a tangent vector field for the outgoing light cone $\{u = C\}$. In addition, $\tilde{L}$ is geodesic and $\tilde{L}t = \mu^{-1}$. Then it is natural to rescale $\tilde{L}$ as

$$\mathring{L} = \mu \tilde{L},$$

which actually approximates to $L = \partial_t + \partial_r$. To obtain the approximate vector field of the incoming light cone, one sets $\tilde{T} = -g^{\nu 0}\partial_\nu - \mathring{L}$, which is near $-\partial_r$ for $t = t_0$. Then in order to define a null frame, one can set

$$T = \mu\tilde{T}, \quad \mathring{\underline{L}} = \mu\mathring{L} + 2T,$$

where $\mathring{L}$ and $\mathring{\underline{L}}$ are two vector fields in the null frame. Finally, the third vector field X in the null frame can be constructed by using $\mathring{L}$. Extending the local coordinate θ on $\mathbb{S}^1$ as

$$\begin{cases} \mathring{L}\vartheta = 0, \\ \vartheta|_{t=t_0} = \theta. \end{cases}$$

Subsequently, let $X = \frac{\partial}{\partial\vartheta}$. Then X is the tangent vector on $S_{t,u}$. Rewrite $X = X^\alpha \partial_\alpha$. Then $X^0 = 0$ holds due to $\frac{\partial t}{\partial \vartheta} = 0$.

Lemma 3.1. *$\{\mathring{L}, \mathring{\underline{L}}, X\}$ constitutes a null frame with respect to the metric $(g_{\alpha\beta})$, and admits the following identities:*

$$g(\mathring{L}, \mathring{L}) = g(\mathring{\underline{L}}, \mathring{\underline{L}}) = g(\mathring{L}, X) = g(\mathring{\underline{L}}, X) = 0, \tag{3.3}$$

$$g(\mathring{L}, \mathring{\underline{L}}) = -2\mu. \tag{3.4}$$

In addition,

$$\mathring{L}t = 1, \quad \mathring{L}u = 0, \tag{3.5}$$

$$\mathring{\underline{L}}t = \mu, \quad \mathring{\underline{L}}u = 2. \tag{3.6}$$

And

$$g(\mathring{L}, T) = -\mu, \quad g(T, T) = \mu^2, \tag{3.7}$$

$$Tt = 0, \quad Tu = 1. \tag{3.8}$$

As in [36] or [33], one can perform the change of coordinates: $(t, x^1, x^2) \longrightarrow (\mathfrak{t}, u, \vartheta)$ near C_0 with

$$\begin{cases} \mathfrak{t} = t, \\ u = u(t, x), \\ \vartheta = \vartheta(t, x). \end{cases} \tag{3.9}$$

Under the new coordinate $(\mathfrak{t}, u, \vartheta)$, we introduce the following subsets (see Figure 3 below):

Definition 3.2. *Set*

$$\begin{aligned} &\Sigma_{\mathfrak{t}}^u := \{(\mathfrak{t}', u', \vartheta) : \mathfrak{t}' = \mathfrak{t}, 0 \le u' \le u\}, \quad u \in [0, 4\delta], \\ &C_u := \{(\mathfrak{t}', u', \vartheta) : \mathfrak{t}' \ge t_0, u' = u\}, \\ &C_u^{\mathfrak{t}} := \{(\mathfrak{t}', u', \vartheta) : t_0 \le \mathfrak{t}' \le \mathfrak{t}, u' = u\}, \\ &S_{\mathfrak{t},u} := \Sigma_{\mathfrak{t}} \cap C_u, \\ &D^{\mathfrak{t},u} := \{(\mathfrak{t}', u', \vartheta) : t_0 \le \mathfrak{t}' < \mathfrak{t}, 0 \le u' \le u\}. \end{aligned}$$

Next, we list some geometric notations which will be used frequently.

Definition 3.3. *For the metric g on the spacetime,*

- *$\underline{g} = (g_{ij})$ is defined as the induced metric of g on $\Sigma_{\mathfrak{t}}$, i.e., $\underline{g}(U, V) = g(U, V)$ for any tangent vectors U and V of $\Sigma_{\mathfrak{t}}$;*
- *$\not{\Pi}_\alpha^\beta := \delta_\alpha^\beta - \delta_\alpha^0 \mathring{L}^\beta + \mathring{L}_\alpha \tilde{T}^\beta$ is the projection tensor field on $S_{\mathfrak{t},u}$ of type $(1,1)$, where δ_α^β is Kronecker delta;*
- *$\not{\xi} = \not{\Pi}\xi$ is the tensor field on $S_{\mathfrak{t},u}$ for any (m, n)-type spacetime tensor field ξ, whose components are*
$$\not{\xi}_{\beta_1 \cdots \beta_n}^{\alpha_1 \cdots \alpha_m} := (\not{\Pi}\xi)_{\beta_1 \cdots \beta_n}^{\alpha_1 \cdots \alpha_m} = \not{\Pi}_{\beta_1}^{\beta_1'} \cdots \not{\Pi}_{\beta_n}^{\beta_n'} \not{\Pi}_{\alpha_1'}^{\alpha_1} \cdots \not{\Pi}_{\alpha_m'}^{\alpha_m} \xi_{\beta_1' \cdots \beta_n'}^{\alpha_1' \cdots \alpha_m'}.$$
In particular, $\not{g} = (\not{g}_{\alpha\beta})$ is the induced metric of g on $S_{\mathfrak{t},u}$;
- *$(\not{g}^{XX})$ is defined as the inverse of $\not{g}_{XX}$ with $\not{g}_{XX} = g(X, X)$;*
- *$\mathscr{D}$ and $\not{\nabla}$ denote the Levi-Civita connection of g and $\not{g}$, respectively;*
- *$\Box_g := g^{\alpha\beta} \mathscr{D}_{\alpha\beta}^2$, $\not{\Delta} := \not{g}^{XX} \not{\nabla}_X^2$;*
- *$\mathcal{L}_V \xi$ is the Lie derivative of ξ with respect to V and $\not{\mathcal{L}}_V \xi := \not{\Pi}(\mathcal{L}_V \xi)$ for any tensor field ξ and vector V;*
- *For any (m, n)-type spacetime tensor field ξ,*

$$|\xi|^2 := g_{\alpha_1 \alpha_1'} \cdots g_{\alpha_m \alpha_m'} g^{\beta_1 \beta_1'} \cdots g^{\beta_n \beta_n'} \xi_{\beta_1 \cdots \beta_n}^{\alpha_1 \cdots \alpha_m} \xi_{\beta_1' \cdots \beta_n'}^{\alpha_1' \cdots \alpha_m'};$$

- *divU* $:= \mathscr{D}_\alpha U^\alpha$ *for any vectorfield* U*;* $\text{d}\!\!\!/\text{iv}Y := \nabla\!\!\!\!/\,_X Y^X$ *and* $\text{d}\!\!\!/\text{iv}\kappa := \nabla\!\!\!\!/\,^X \kappa_X$ *are angular divergence for any vectorfield* Y *and one form* κ *on* $S_{\mathfrak{t},u}$*.*

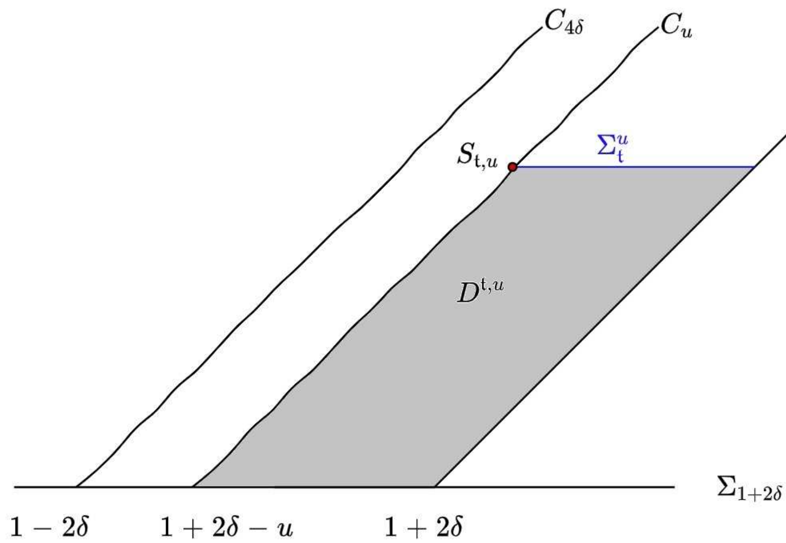

Figure 3. The indications of some domains

Under the frame $\{\mathring{L}, \underline{\mathring{L}}, X\}$, *the second fundamental forms* λ and $\tilde{\theta}$ can be defined as

$$\lambda_{XX} = g(\mathscr{D}_X \mathring{L}, X), \quad \tilde{\theta}_{XX} = g(\mathscr{D}_X \tilde{T}, X). \tag{3.10}$$

At the same time, define *one-form tensors* ζ and ξ as

$$\zeta_X = g(\mathscr{D}_X \mathring{L}, \tilde{T}), \quad \xi_X = -g(\mathscr{D}_X T, \mathring{L}). \tag{3.11}$$

Then $\mu\zeta_X = -X\mu + \xi_X$. For any vector field V, denote its associate *deformation tensor* by

$$^{(V)}\pi_{\alpha\beta} = g(\mathscr{D}_\alpha V, \partial_\beta) + g(\mathscr{D}_\beta V, \partial_\alpha). \tag{3.12}$$

On the initial hypersurface $\Sigma_{t_0}^{4\delta}$, one has that $\tilde{T}^i = -\dfrac{x^i}{r} + O(\delta^{1-\varepsilon_0})$, $\mathring{L}^0 = 1$, $\mathring{L}^i = \dfrac{x^i}{r} + O(\delta^{2-\varepsilon_0})$ and $\lambda_{XX} = \dfrac{1}{r}\not{g}_{XX} + O(\delta^{1-\varepsilon_0})$. Note that on Σ_{t_0}, r is just $t_0 - u$. For $\mathfrak{t} \geq t_0$, we define the "*error vectors*" with the components being

$$\begin{aligned} \check{L}^0 &:= 0, \\ \check{L}^i &:= \mathring{L}^i - \frac{x^i}{\varrho}, \\ \check{T}^i &:= \tilde{T}^i + \frac{x^i}{\varrho}, \\ \check{\lambda}_{XX} &:= \lambda_{XX} - \frac{1}{\varrho}\not{g}_{XX}, \end{aligned} \tag{3.13}$$

where $\varrho := \mathfrak{t} - u$.

Note that ϑ is the coordinate on $S_{\mathfrak{t},u}$. Then under the new coordinate system $(\mathfrak{t}, u, \vartheta)$, one has $\mathring{L} = \frac{\partial}{\partial \mathfrak{t}}$. In addition, it follows from (3.8) that $T = \frac{\partial}{\partial u} - \eta^X X$ for some smooth function η^X. And moreover, a similar analysis as for Lemma 3.66 of [36] gives that

Lemma 3.2. *In domain $D^{\mathfrak{t},u}$, the Jacobian determinant of map $(\mathfrak{t},u,\vartheta)\to(x^0,x^1,x^2)$ is*

$$\det\frac{\partial(x^0,x^1,x^2)}{\partial(\mathfrak{t},u,\vartheta)}=\mu(\det\underline{g})^{-1/2}\sqrt{\not{g}_{XX}}. \tag{3.14}$$

Remark 3.1. *It follows from (3.14) that if the metrics $\underline{g}$ and $\not{g}$ are regular, that is, $\det\underline{g}>0$ and $\not{g}_{XX}>0$, then the transformation of coordinates between $(\mathfrak{t},u,\vartheta)$ and (x^0,x^1,x^2) makes sense as long as $\mu>0$.*

On $\mathbb{S}^1$, one is used to applying the standard rotation vector field $\Omega=\epsilon_i^j x^i\partial_j$ as the tangent derivative. In order to project Ω on $S_{\mathfrak{t},u}$, as in (3.39b) of [36], one can denote by

$$R:=\not{\Pi}\Omega,\quad \not{d}:=\not{\Pi}d$$

the rotation vectorfield and differential of $S_{\mathfrak{t},u}$, respectively. Then

$$R=(\not{\Pi}\Omega)^i\partial_i=(\not{\Pi}_j^i\Omega^j)\partial_i=(\delta_j^k-g_{ja}\tilde{T}^a\tilde{T}^k)\Omega^j\partial_k=\Omega-g_{ja}\tilde{T}^a\Omega^j\tilde{T}. \tag{3.15}$$

Set

$$\upsilon:=g_{ja}\tilde{T}^a\Omega^j=g_{ij}\tilde{T}^i\Omega^j. \tag{3.16}$$

Then one has

$$R=\Omega-\upsilon\tilde{T}.$$

For domains with $\mu>0$, we give some definitions of related integrations and norms, which will be utilized repeatedly in subsequent sections.

Definition 3.4 (Integrations and norms)**.** *For any continuous function f, set*

$$\begin{aligned}
&\int_{S_{\mathfrak{t},u}}f:=\int_{S_{\mathfrak{t},u}}fd\nu_{\not{g}}:=\int_{\mathbb{S}^1}f(\mathfrak{t},u,\vartheta)\sqrt{\not{g}_{XX}(\mathfrak{t},u,\vartheta)}d\vartheta, \qquad \|f\|^2_{L^2(S_{\mathfrak{t},u})}:=\int_{S_{\mathfrak{t},u}}|f|^2,\\
&\int_{C_u^{\mathfrak{t}}}f:=\int_{t_0}^{\mathfrak{t}}\int_{S_{\tau,u}}f(\tau,u,\vartheta)d\nu_{\not{g}}d\tau, \qquad \|f\|^2_{L^2(C_u^{\mathfrak{t}})}:=\int_{C_u^{\mathfrak{t}}}|f|^2,\\
&\int_{\Sigma_{\mathfrak{t}}^u}f:=\int_0^u\int_{S_{\mathfrak{t},u'}}f(\mathfrak{t},u',\vartheta)d\nu_{\not{g}}du', \qquad \|f\|^2_{L^2(\Sigma_{\mathfrak{t}}^u)}:=\int_{\Sigma_{\mathfrak{t}}^u}|f|^2,\\
&\int_{D^{\mathfrak{t},u}}f:=\int_{t_0}^{\mathfrak{t}}\int_0^u\int_{S_{\tau,u'}}f(\tau,u',\vartheta)d\nu_{\not{g}}du'd\tau, \quad \|f\|^2_{L^2(D^{\mathfrak{t},u})}:=\int_{D^{\mathfrak{t},u}}|f|^2.
\end{aligned}$$

For reader's convenience, the notation of contractions is recalled as follows:

Definition 3.5 (Contraction)**.** *If Θ is a $(0,2)$-type spacetime tensor, κ is a one form, U and V are vector fields, the contraction of Θ with respect to U and V is then defined as*

$$\Theta_{UV}:=\Theta_{\alpha\beta}U^\alpha V^\beta,$$

and the contraction of κ with respect to U is

$$\kappa_U:=\kappa_\alpha U^\alpha.$$

Definition 3.6. *If ξ is a $(0,2)$-type tensor on $S_{\mathfrak{t},u}$, then the trace of ξ is defined as*

$$tr\xi:=\not{g}^{XX}\xi_{XX}.$$

3.2 Basic equalities in the null frames

In this subsection, with the help of λ, $\tilde{\theta}$, ζ and ξ defined in Section 3.1, we will derive some basic equalities in the frame $\{\mathring{L}, \underline{\mathring{L}}, X\}$ or $\{T, \underline{\mathring{L}}, X\}$.

Set

$$G_{\alpha\beta}^{\gamma} := \frac{\partial g_{\alpha\beta}}{\partial \varphi_\gamma} \quad \text{and} \quad G_{\alpha\beta}^{\gamma\nu} := \frac{\partial G_{\alpha\beta}^{\gamma}}{\partial \varphi_\nu}.$$

For any vector fields $U = U^\alpha \dfrac{\partial}{\partial x^\alpha}$ and $V = V^\alpha \dfrac{\partial}{\partial x^\alpha}$, as in Definition 3.5, one can define $G_{UV}^{\gamma} := G_{\alpha\beta}^{\gamma} U^\alpha V^\beta$ and $G_{UV}^{\gamma\nu} := G_{\alpha\beta}^{\gamma\nu} U^\alpha V^\beta$. Direct computations yield

$$\begin{aligned}
&G_{\mathring{L}\mathring{L}}^{\gamma} = -2c^{-1}\mathring{L}^\gamma, \quad G_{\mathring{L}\tilde{T}}^{0} = 2c^{-1}, \quad G_{\mathring{L}\tilde{T}}^{a} = 2c^{-1}\varphi_a - c^{-1}\tilde{T}^a,\\
&G_{\tilde{T}\tilde{T}}^{0} = -2c^{-1}, \quad G_{\tilde{T}\tilde{T}}^{a} = -2c^{-1}\varphi_a, \quad G_{\mathring{L}X}^{0} = 0, \quad G_{\mathring{L}X}^{a} = -c^{-1}\not{d}_X x^a,\\
&G_{\tilde{T}X}^{\gamma} = 0, \quad G_{XX}^{0} = -2c^{-1}\not{g}_{XX}, \quad G_{XX}^{a} = -2c^{-1}\varphi_a\not{g}_{XX}
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&G_{\mathring{L}\mathring{L}}^{00} = 8c^{-2}, \quad G_{\mathring{L}\mathring{L}}^{0a} = 4c^{-2}(\varphi_a + \mathring{L}^a), \quad G_{\mathring{L}\mathring{L}}^{ab} = 4c^{-2}(\varphi_a\mathring{L}^b + \varphi_b\mathring{L}^a),\\
&G_{XX}^{00} = 8c^{-2}\not{g}_{XX}, \quad G_{XX}^{0a} = 8c^{-2}\varphi_a\not{g}_{XX}, \quad G_{XX}^{ab} = 2c^{-2}(4\varphi_a\varphi_b - c\delta_{ab})\not{g}_{XX},\\
&G_{\mathring{L}X}^{00} = 0, \quad G_{\mathring{L}X}^{0a} = 2c^{-2}\not{d}_X x^a, \quad G_{\mathring{L}X}^{ab} = 2c^{-2}(\varphi_a\not{d}_X x^b + \varphi_a\not{d}_X x^a),\\
&G_{\mathring{L}\tilde{T}}^{00} = -8c^{-2}, \quad G_{\mathring{L}\tilde{T}}^{0a} = 2c^{-2}(\tilde{T}^a - 4\varphi_a), \quad G_{\mathring{L}\tilde{T}}^{ab} = 2c^{-2}(c\delta_{ab} - 4\varphi_a\varphi_b + \varphi_a\tilde{T}^b + \varphi_b\tilde{T}^a),\\
&G_{\tilde{T}\tilde{T}}^{00} = 8c^{-2}, \quad G_{\tilde{T}\tilde{T}}^{0a} = 8c^{-2}\varphi_a, \quad G_{\tilde{T}\tilde{T}}^{ab} = 2c^{-2}(4\varphi_a\varphi_b - c\delta_{ab}), \quad G_{\tilde{T}X}^{\alpha\beta} = 0,
\end{aligned} \tag{3.18}$$

where $\not{d}_X f = X^i \not{d}_i f$ and $\not{d}_i f = \not{\Pi}_i^\alpha (d_\alpha f) = \not{\Pi}_i^\alpha(\partial_\alpha f)$ for any smooth function f.

Based on (3.17), μ satisfies the following transport equation.

Lemma 3.3. *μ satisfies*

$$\mathring{L}\mu = c^{-1}\mu\big(\mathring{L}^\alpha \mathring{L}\varphi_\alpha - \mathring{L}c\big). \tag{3.19}$$

Proof. Note that $\tilde{L} = -g^{\alpha\beta}\partial_\alpha u \partial_\beta$ is geodesic, i.e., $\mathscr{D}_{\tilde{L}}\tilde{L} = 0$. Then it holds that

$$\tilde{L}^\alpha(\partial_\alpha \tilde{L}^\beta)\partial_\beta + \tilde{L}^\alpha\tilde{L}^\beta\Gamma_{\alpha\beta}^{\gamma}\partial_\gamma = 0, \tag{3.20}$$

where $\Gamma_{\alpha\beta}^{\gamma}$ are the Christoffel symbols. Since $\tilde{L}^0 = \mu^{-1}$ and $\mathring{L} = \mu\tilde{L}$, taking the component relative to t in (3.20) yields

$$\begin{aligned}
\mathring{L}\mu &= \mu\tilde{L}\mu = \mu\mathring{L}^\alpha\mathring{L}^\beta\Gamma_{\alpha\beta}^{0}\\
&= \frac{1}{2}G_{\mathring{L}\mathring{L}}^{\alpha}(T\varphi_\alpha) - \frac{1}{2}\mu G_{\mathring{L}\mathring{L}}^{\alpha}(\mathring{L}\varphi_\alpha) - \mu G_{\tilde{T}\mathring{L}}^{\alpha}\mathring{L}\varphi_\alpha.
\end{aligned} \tag{3.21}$$

Since $T\varphi_\alpha = \mu\sum_i \varphi_i\partial_i\varphi_\alpha - \mu\mathring{L}\varphi_\alpha + \mu\partial_t\varphi_\alpha$ and $\partial_\alpha\varphi_\beta = \partial_\beta\varphi_\alpha$, one can get from (3.17) that

$$G_{\mathring{L}\mathring{L}}^{\alpha}(T\varphi_\alpha) = -2c^{-1}\mu\tilde{T}^a(\mathring{L}\varphi_a). \tag{3.22}$$

Substituting (3.22) into (3.21) and applying (3.17) yield (3.19) immediately. □

Remark 3.2. *The importance of the expression* (3.19) *should be emphasized here. Due to the special structure of equation* (1.6)*, $\mathring{L}\mu$ is just the combination of $\mathring{L}\varphi_\alpha$ (note that $\mathring{L}c = 2\mathring{L}\varphi_0 + 2\sum_{i=1}^{2}\varphi_i\mathring{L}\varphi_i$). With the smallness and some suitable time-decay rate of $\mathring{L}\varphi_\alpha$, we will be able to show that $\mu \geq C$ for some positive constant C (see* (4.2) *in Section 4). This is in contrast to the cases in [4], [33] and [36], where the expressions of $\mathring{L}\mu$ contain the "bad" factor $T\varphi_\alpha$ which leads to $\mu \to 0+$ in finite time (e.g. (2.36) in [33]).*

Note that the quantity "deformation tensor" defined in (3.12) will occur in the subsequent energy estimates. It is necessary to check the components of ${}^{(V)}\pi$ in the null frame $\{\mathring{L}, \mathring{\underline{L}}, X\}$.

Let ${}^{(V)}\not\pi_{UX} = {}^{(V)}\pi_{UX}$ for $U \in \{\mathring{L}, \mathring{\underline{L}}, X\}$. Following the computations for Proposition 7.7 of [36], one can have

(1) for $V = T$,

$$\begin{aligned} &{}^{(T)}\pi_{\mathring{L}\mathring{L}} = 0, \quad {}^{(T)}\pi_{T\tilde{T}} = 2T\mu, \quad {}^{(T)}\pi_{\mathring{L}T} = -T\mu, \quad {}^{(T)}\not\pi_{TX} = 0, \\ &{}^{(T)}\not\pi_{\mathring{L}X} = -\not d_X\mu - 2c^{-1}\mu\tilde{T}^a\not d_X\varphi_a, \quad {}^{(T)}\not\pi_{XX} = 2\mu\tilde{\theta}_{XX}; \end{aligned} \tag{3.23}$$

(2) for $V = \mathring{L}$,

$$\begin{aligned} &{}^{(\mathring{L})}\pi_{\mathring{L}\mathring{L}} = 0, \quad {}^{(\mathring{L})}\pi_{T\tilde{T}} = 2\mathring{L}\mu, \quad {}^{(\mathring{L})}\pi_{\mathring{L}T} = -\mathring{L}\mu, \quad {}^{(\mathring{L})}\not\pi_{\mathring{L}X} = 0, \\ &{}^{(\mathring{L})}\not\pi_{TX} = \not d_X\mu + 2c^{-1}\mu\tilde{T}^a\not d_X\varphi_a, \quad {}^{(\mathring{L})}\not\pi_{XX} = 2\lambda_{XX}; \end{aligned} \tag{3.24}$$

(3) for $V = R$,

$$\begin{aligned} &{}^{(R)}\pi_{\mathring{L}\mathring{L}} = 0, \quad {}^{(R)}\pi_{T\tilde{T}} = 2R\mu, \quad {}^{(R)}\pi_{\mathring{L}T} = -R\mu, \\ &{}^{(R)}\not\pi_{\mathring{L}X} = -R^X\check{\lambda}_{XX} + g_{aj}\epsilon_i^j\check{L}^i\not d_Xx^a + \upsilon c^{-1}\tilde{T}^a\not d_X\varphi_a - \frac{1}{2}c^{-1}R^X\not g_{XX}(\mathring{L}c) - \frac{1}{2}c^{-1}\upsilon\not d_Xc \\ &{}^{(R)}\not\pi_{TX} = \mu R^X\check{\lambda}_{XX} + \upsilon\not d_X\mu + \mu g_{aj}\epsilon_i^j\check{T}^i\not d_Xx^a + \frac{1}{2}c^{-1}\mu R^X\not g_{XX}(\mathring{L}c) - c^{-1}\mu(R\varphi_a)\not d_Xx^a \\ &\qquad + \frac{1}{2}c^{-1}\mu\upsilon\not d_Xc, \\ &{}^{(R)}\not\pi_{XX} = 2\upsilon\lambda_{XX} + c^{-1}\upsilon(\mathring{L}c)\not g_{XX} - 2c^{-1}\upsilon(\not d_Xx^a)\not d_X\varphi_a - c^{-1}(Rc)\not g_{XX}. \end{aligned} \tag{3.25}$$

As seen in (3.19) and (3.23)-(3.25), the components of $\mathring{L}$ and $\tilde{T}$ appear frequently. In view of $\tilde{T}^i = \varphi_i - \mathring{L}^i$, one can find the equations for $\mathring{L}^i$ and $\check{L}^i$ under the actions of the derivatives in the null frame $\{T, \mathring{L}, X\}$.

Lemma 3.4. *It holds that*

$$\mathring{L}\mathring{L}^i = -c^{-1}\mathring{L}^\alpha(\mathring{L}\varphi_\alpha)\tilde{T}^i, \tag{3.26}$$

$$\mathring{L}(\varrho\check{L}^i) = \varrho\mathring{L}\mathring{L}^i = -c^{-1}\varrho\mathring{L}^\alpha(\mathring{L}\varphi_\alpha)\tilde{T}^i, \tag{3.27}$$

$$\not d_X\mathring{L}^i = (tr\lambda)\not d_Xx^i + \frac{1}{2}c^{-1}(\mathring{L}c)\not d_Xx^i - \frac{1}{2}c^{-1}(\not d_Xc)\tilde{T}^i + c^{-1}\tilde{T}^a(\not d_X\varphi_a)\tilde{T}^i, \tag{3.28}$$

$$\not d_X\check{L}^i = (tr\check{\lambda})\not d_Xx^i + \frac{1}{2}c^{-1}(\mathring{L}c)\not d_Xx^i - \frac{1}{2}c^{-1}(\not d_Xc)\tilde{T}^i + c^{-1}\tilde{T}^a(\not d_X\varphi_a)\tilde{T}^i, \tag{3.29}$$

where $\not d^Xf = \not g^{XX}\not d_Xf = \not g^{XX}(Xf)$ for any smooth function f.

Proof. As for the proof of Proposition 4.7 in [36], (3.26)-(3.29) can be obtained directly by using (3.17). □

In addition, it follows from (3.10)-(3.11), (3.17) and Lemma 3.4 that

$$\zeta_X = c^{-1}\tilde{T}^a(\not{d}_X\varphi_a), \qquad \xi_X = c^{-1}\mu\tilde{T}^a(\not{d}_X\varphi_a) + \not{d}_X\mu, \tag{3.30}$$

$$\tilde{\theta}_{XX} = -\frac{1}{2}c^{-1}\mu^{-1}(Tc)\not{g}_{XX} - \frac{1}{2}c^{-1}(\mathring{L}c)\not{g}_{XX} + c^{-1}(\not{d}_X\varphi_a)\not{d}_X x^a - \lambda_{XX}, \tag{3.31}$$

which have been given in Lemma 5.1 of [36] in terms of G_{XY}. Then (3.30) implies that

$$T\mathring{L}^i = -c^{-1}(\tilde{T}^a\mathring{L}\varphi_a)T^i + (\not{d}_X\mu)(\not{d}^X x^i) + c^{-1}\mu(\tilde{T}^a\not{d}_X\varphi_a)\not{d}^X x^i + \frac{1}{2}c^{-1}\mu(\not{d}^X c)(\not{d}_X x^i). \tag{3.32}$$

For later analysis, one also needs the following connection coefficients in the new frames, which are given in Lemma 5.1 and Lemma 5.3 of [36].

Lemma 3.5. *The covariant derivatives in the frame* $\{T, \mathring{L}, X\}$ *are*

$$\begin{aligned}
&\mathscr{D}_{\mathring{L}}\mathring{L} = (\mu^{-1}\mathring{L}\mu)\mathring{L}, \quad \mathscr{D}_T\mathring{L} = -\mathring{L}\mu\mathring{L} + \xi^X X, \quad \mathscr{D}_X\mathring{L} = -\zeta_X\mathring{L} + tr\lambda X,\\
&\mathscr{D}_{\mathring{L}}T = -\mathring{L}\mu\mathring{L} - \mu\zeta^X X, \quad \mathscr{D}_T T = \mu\mathring{L}\mu\mathring{L} + (\mu^{-1}T\mu + \mathring{L}\mu)T - \mu(\not{d}^X\mu)X,\\
&\mathscr{D}_X T = \mu\zeta_X\mathring{L} + \mu^{-1}\xi_X T + (\mu tr\tilde{\theta})X,\\
&\mathscr{D}_X X = \not{\nabla}_X X + (\tilde{\theta}_{XX} + \lambda_{XX})\mathring{L} + \mu^{-1}\lambda_{XX}T.
\end{aligned} \tag{3.33}$$

Lemma 3.6. *The covariant derivatives in the frame* $\{\mathring{\underline{L}}, \mathring{L}, X\}$ *are*

$$\begin{aligned}
&\mathscr{D}_{\mathring{\underline{L}}}\mathring{L} = -\mathring{L}\mu\mathring{L} + 2\xi^X X, \quad \mathscr{D}_{\mathring{L}}\mathring{\underline{L}} = -2\mu\zeta^X X,\\
&\mathscr{D}_{\mathring{\underline{L}}}\mathring{\underline{L}} = (\mu^{-1}\mathring{\underline{L}}\mu + \mathring{L}\mu)\mathring{\underline{L}} - (2\mu\not{d}^X\mu)X.
\end{aligned} \tag{3.34}$$

Next we compute the equation for φ_γ under the action of the covariant wave operator $\Box_g$. With the metric g given in (1.28), one has

$$\begin{aligned}
\Box_g\varphi_\gamma =&\frac{1}{\sqrt{|\mathrm{det}g|}}\partial_\alpha\big(\sqrt{|\mathrm{det}g|}g^{\alpha\beta}\partial_\beta\varphi_\gamma\big)\\
=&-c^{-1}g^{\alpha\beta}\partial_\alpha c(\partial_\beta\varphi_\gamma) + g^{\alpha\beta}\partial^2_{\alpha\beta}\varphi_\gamma\\
&-\sum_i\partial_i\varphi_i(\partial_t + \sum_j\varphi_j\partial_j)\varphi_\gamma + \sum_i\partial_i\varphi_\gamma(\partial_t + \sum_j\varphi_j\partial_j)\varphi_i.
\end{aligned} \tag{3.35}$$

Differentiating (1.6) with respect to the variable x^γ yields

$$g^{\alpha\beta}\partial^2_{\alpha\beta}\varphi_\gamma = 2\sum_i\partial_i\varphi_\gamma(\partial_t + \sum_j\varphi_j\partial_j)\varphi_i - 2\sum_i\partial_i\varphi_i(\partial_t + \sum_j\varphi_j\partial_j)\varphi_\gamma. \tag{3.36}$$

Substituting (3.36) into (3.35) yields

$$\begin{aligned}
\Box_g\varphi_\gamma =&-c^{-1}g^{\alpha\beta}(\partial_\alpha c)(\partial_\beta\varphi_\gamma)\\
&-3\sum_i\partial_i\varphi_i(\partial_t + \sum_j\varphi_j\partial_j)\varphi_\gamma + 3\sum_i\partial_i\varphi_\gamma(\partial_t + \sum_j\varphi_j\partial_j)\varphi_i.
\end{aligned} \tag{3.37}$$

Note that

$$g^{\alpha\beta} = -\mathring{L}^\alpha\mathring{L}^\beta - \tilde{T}^\alpha\mathring{L}^\beta - \mathring{L}^\alpha\tilde{T}^\beta + (\not{d}_X x^\alpha)(\not{d}^X x^\beta), \tag{3.38}$$

$$\partial_i = c^{-1}\mu^{-1}\tilde{T}^i T + c^{-1}(\not{d}^X x^i)X, \tag{3.39}$$

$$\partial_t + \sum_j \varphi_j \partial_j = \mathring{L} + \mu^{-1}T. \tag{3.40}$$

Then in the frame $\{T, \mathring{L}, X\}$, (3.37) can be rewritten as

$$\begin{aligned}\mu\Box_g\varphi_\gamma =& c^{-1}\big\{\frac{1}{2}\mu\mathring{L}c + Tc - 3\tilde{T}^i T\varphi_i - \frac{3}{2}\mu\tilde{T}^i\mathring{L}\varphi_i - \frac{3}{2}\mu\not{d}^X x^i(\not{d}_X\varphi_i)\big\}\mathring{L}\varphi_\gamma\\ &+ \frac{1}{2}c^{-1}\big\{\mathring{L}c + 3\tilde{T}^i\mathring{L}\varphi_i - 3\not{d}^X x^i(\not{d}_X\varphi_i)\big\}\underline{\mathring{L}}\varphi_\gamma\\ &+ c^{-1}\big\{-\mu\not{d}^X c + 3\mu(\mathring{L}\varphi_i)\not{d}^X x^i + 3(T\varphi_i)\not{d}^X x^i\big\}\not{d}_X\varphi_\gamma,\end{aligned} \tag{3.41}$$

where $\gamma = 0, 1, 2$.

For later use, we list in the following lemmas some identities involving commutators, which are given in Lemmas 4.10, 8.9 and 8.11 of [36].

Lemma 3.7. *In the frame $\{\mathring{L}, T, R\}$, it holds that*

$$\begin{aligned}[]
[\mathring{L}, R] &= {}^{(R)}\not{\pi}_{\mathring{L}}{}^X X,\\
[\mathring{L}, T] &= {}^{(T)}\not{\pi}_{\mathring{L}}{}^X X,\\
[T, R] &= {}^{(R)}\not{\pi}_T{}^X X,
\end{aligned} \tag{3.42}$$

where ${}^{(R)}\not{\pi}_{\mathring{L}}{}^X = \not{g}^{XX}\,{}^{(R)}\not{\pi}_{\mathring{L}X}$ *and* ${}^{(R)}\not{\pi}_{\mathring{L}X} = {}^{(R)}\pi_{\mathring{L}X} = {}^{(R)}\pi_{\alpha\beta}\mathring{L}^\alpha X^\beta$.

Lemma 3.8. *For any vector field $Z \in \{\mathring{L}, T, R\}$,*

1. *if f is a smooth function, then*

$$\big([\not\nabla^2, \not{\mathcal{L}}_Z]f\big)_{XX} = \frac{1}{2}\not\nabla_X\big(tr^{(Z)}\not{\pi}\big)\not{d}_X f;$$

2. *if Θ is a one-form on $S_{t,u}$, then*

$$([\not\nabla_X, \not{\mathcal{L}}_Z]\Theta)_X = \frac{1}{2}\not\nabla_X\big(tr^{(Z)}\not{\pi}\big)\Theta_X;$$

3. *if Θ is a $(0,2)$-type tensor on $S_{t,u}$, then*

$$\begin{aligned}([\not\nabla_X, \not{\mathcal{L}}_Z]\Theta)_{XX} &= \not\nabla_X\big(tr^{(Z)}\not{\pi}\big)\Theta_{XX},\\ ([\not\nabla_X, \not{\mathcal{L}}_Z]\not\nabla\Theta)_{XXX} &= \frac{3}{2}\not\nabla_X\big(tr^{(Z)}\not{\pi}\big)\not\nabla_X\Theta_{XX},\end{aligned}$$

where ${}^{(Z)}\not{\pi}_{XX} = {}^{(Z)}\pi_{XX}$.

4 Bootstrap assumptions on $\partial\phi$ near C_0 and some related estimates

To show the global existence of a solution to (1.6) near C_0, we will utilize a bootstrap argument. To this end, for any given smooth solution ϕ to (1.6), we make the following assumptions in $D^{\mathfrak{t},u} \subset A_{2\delta}$:

$$\begin{aligned}
&\delta^l \|\mathring{L} Z^\alpha \varphi_\gamma\|_{L^\infty(\Sigma_{\mathfrak{t}}^u)} \le M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2},\\
&\delta^l \|\not{d} Z^\alpha \varphi_\gamma\|_{L^\infty(\Sigma_{\mathfrak{t}}^u)} \le M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2},\\
&\delta^l \|\mathring{\underline{L}} Z^\alpha \varphi_\gamma\|_{L^\infty(\Sigma_{\mathfrak{t}}^u)} \le M\delta^{-\varepsilon_0}\mathfrak{t}^{-1/2},\\
&\|\not{\nabla}^2 \varphi_\gamma\|_{L^\infty(\Sigma_{\mathfrak{t}}^u)} \le M\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2},\\
&\|\varphi_\gamma\|_{L^\infty(\Sigma_{\mathfrak{t}}^u)} \le M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2},
\end{aligned} \tag{$\star$}$$

where $|\alpha| \le N$, N is a fixed large positive integer, M is some positive number to be suitably chosen later (at least double bounds of the corresponding quantities on time t_0), $Z \in \{\varrho\mathring{L}, T, R\}$, and l is the number of T included in Z^α.

We now give a rough estimate of μ under assumptions ($\star$). Note that $1 = g_{ij}\tilde{T}^i\tilde{T}^j = \big(1 + O(M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2})\big)\sum_{i=1}^{2}|\tilde{T}^i|^2$ by (3.7). This means

$$|\tilde{T}^i|, |\mathring{L}^i| \le 1 + O(M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}) \tag{4.1}$$

due to $\mathring{L}^i = \varphi_i - \tilde{T}^i$. This, together with ($\star$) and (3.19), implies $|\mathring{L}\mu| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\mu$. When $\delta > 0$ is small, by integrating $\mathring{L}\mu$ along integral curves of $\mathring{L}$ and noting $\mu = \frac{1}{\sqrt{c}} = 1 + O(\delta^{1-\varepsilon_0})$ on Σ_{t_0}, one can get directly that

$$\mu = 1 + O(M\delta^{1-\varepsilon_0}). \tag{4.2}$$

To improve the assumptions ($\star$), we may rewrite (3.41) in the frame $\{\mathring{\underline{L}}, \mathring{L}, X\}$ as

$$\mathring{L}\mathring{\underline{L}}\varphi_\gamma + \frac{1}{2\varrho}\mathring{\underline{L}}\varphi_\gamma = H_\gamma, \tag{4.3}$$

here one has used the fact that $\mu\Box_g\varphi_\gamma = -\mathscr{D}^2_{\mathring{L}\mathring{\underline{L}}}\varphi_\gamma + \mu\not{g}^{XX}\mathscr{D}^2_X\varphi_\gamma = \mathring{L}\mathring{\underline{L}}\varphi_\gamma - 2\mu\zeta^X\not{d}_X\varphi_\gamma + \mu\not{\Delta}\varphi_\gamma - \mu(\mathrm{tr}\tilde{\theta} + \mathrm{tr}\lambda)\mathring{L}\varphi_\gamma - \mathrm{tr}\lambda T\varphi_\gamma$ by (3.33) and (3.34). In addition, by (3.30)-(3.31),

$$\begin{aligned}
H_\gamma =& \mu\not{\Delta}\varphi_\gamma + \big\{-\frac{1}{2}c^{-1}Tc + 3c^{-1}\tilde{T}^aT\varphi_a + \frac{3}{2}c^{-1}\mu\tilde{T}^a\mathring{L}\varphi_a + \frac{1}{2}c^{-1}\mu(\not{d}^Xx^a)\not{d}_X\varphi_a\\
&+\frac{1}{2}\mu\mathrm{tr}\check{\lambda} + \frac{\mu}{2\varrho}\big\}\mathring{L}\varphi_\gamma + \big\{-\frac{1}{2}\mathrm{tr}\check{\lambda} - \frac{1}{2}c^{-1}\mathring{L}c - \frac{3}{2}c^{-1}\tilde{T}^a\mathring{L}\varphi_a + \frac{3}{2}c^{-1}(\not{d}^Xx^a)\not{d}_X\varphi_a\big\}\mathring{\underline{L}}\varphi_\gamma\\
&-c^{-1}\mu\big\{3\tilde{T}^a + \mathring{L}^\alpha\big\}(\not{d}^X\varphi_\alpha)\not{d}_X\varphi_\gamma.
\end{aligned} \tag{4.4}$$

It follows from the expression of H_γ that when there are some terms containing factors $T\varphi_\alpha$ or $\mathring{\underline{L}}\varphi_\alpha$ which are not small "enough" and have slow decay rate in time, then there will appear always some accompanying factors $\mathring{L}\varphi_\gamma$ or $\mathrm{tr}\check{\lambda}$ with the "good" smallness and fast time-decay rate (see $-\frac{1}{2}c^{-1}(Tc)(\mathring{L}\varphi_\gamma)$ and $-\frac{1}{2}\mathrm{tr}\check{\lambda}(\mathring{\underline{L}}\varphi_\gamma)$ in H_γ). This implies that H_γ may possess some desired "good" properties for our analysis later.

Unless stated otherwise, from now on to Section 9, the pointwise estimates for the corresponding quantities are all made inside the domain $D^{\mathfrak{t},u}$.

It follows from (4.3) that $\mathring{\underline{L}}\varphi_\gamma$ can be estimated by integrating (4.3) along integral curves of $\mathring{L}$. To this end, we start with the estimates of $\not{d}x^i$, $\check{\lambda}$ and so on in H_γ.

Note that $|\not{d}x^i|^2 = \not{g}^{ab}\not{d}_a x^i \not{d}_b x^i = c - (\tilde{T}^i)^2$. Then $(\star)$ and (4.1) imply that

$$|\not{d}x^i| \lesssim 1. \tag{4.5}$$

To estimate $\check{\lambda}$, one needs to study the structure equation of λ by utilizing the Riemann curvature $\mathscr{R}$ of the metric g, which is defined as follows (see the Definition 11.1 of [36]):

$$\mathscr{R}_{WXYZ} := -g(\mathscr{D}^2_{WX}Y - \mathscr{D}^2_{XW}Y, Z), \tag{4.6}$$

where W, X, Y and Z are vector fields and $\mathscr{D}^2_{WX}Y := W^\alpha X^\beta \mathscr{D}_\alpha \mathscr{D}_\beta Y$. Due to (3.17) and (3.18), one has

Lemma 4.1. *Let $\mathscr{R}$ be the Riemann curvature tensor defined as* (4.6). *Then in the frame $\{\mathring{L}, T, X\}$, $\mathscr{R}_{TX\mathring{L}X}$ and $\mathscr{R}_{\mathring{L}X\mathring{L}X}$ have the following forms:*

$$\begin{aligned}\mathscr{R}_{TX\mathring{L}X} =& -\frac{1}{2}c^{-1}\not{d}_X x^a(\not{d}_X T\varphi_a) + c^{-1}\not{g}_{XX}(\mathring{L}T\varphi_0 + \varphi_a\mathring{L}T\varphi_a)\\ &- c^{-1}\mu\{\not{\nabla}^2_X\varphi_0 + (\varphi_a - \frac{1}{2}\tilde{T}^a)\not{\nabla}^2_X\varphi_a\} + \frac{1}{2}c^{-1}(\tilde{T}^a\not{d}_X\varphi_a)\not{d}_X\mu\\ &- \frac{1}{2}c^{-1}\mu\not{d}_X x^a(\not{d}_X\varphi_a)tr\lambda + \frac{1}{2}c^{-1}(Tc)\lambda_{XX} - \frac{1}{2}c^{-1}(\tilde{T}^a T\varphi_a)\lambda_{XX}\\ &+ c^{-2}f_1(\mathring{L}^i,\varphi)\begin{pmatrix}\mu\mathring{L}\varphi\\ T\varphi\end{pmatrix}\begin{pmatrix}\not{g}_{XX}\mathring{L}\varphi\\ \not{d}_X x^a\cdot\not{d}_X\varphi_a\end{pmatrix} + c^{-2}f_2(\mathring{L}^i,\varphi)\begin{pmatrix}\mu\not{d}_X\varphi\cdot\not{d}_X\varphi\\ \not{g}^{XX}\mu(\not{d}_X x^a\not{d}_X\varphi_a)^2\end{pmatrix},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathscr{R}_{\mathring{L}X\mathring{L}X} =& -c^{-1}\not{d}_X x^a(\not{d}_X\mathring{L}\varphi_a) + c^{-1}\not{g}_{XX}(\mathring{L}^2\varphi_0 + \varphi_a\mathring{L}^2\varphi_a) + c^{-1}\mathring{L}^\gamma(\not{\nabla}^2_X\varphi_\gamma)\\ &+ c^{-1}tr\lambda(\not{d}_X x^a)\not{d}_X\varphi_a - c^{-1}\lambda_{XX}(\tilde{T}^a\mathring{L}\varphi_a) + c^{-2}f(\mathring{L}^i,\varphi)\begin{pmatrix}\not{d}_X\varphi\cdot\not{d}_X\varphi\\ \not{g}^{XX}(\not{d}_X x^a\not{d}_X\varphi_a)^2\\ \not{g}_{XX}(\mathring{L}\varphi)(\mathring{L}\varphi)\\ (\not{d}_X x^a\not{d}_X\varphi_a)\mathring{L}\varphi\end{pmatrix},\end{aligned} \tag{4.8}$$

where f, f_1 and f_2 are generic smooth functions with respect to their arguments and $\begin{pmatrix}A_1\\ \vdots\\ A_n\end{pmatrix}\begin{pmatrix}B_1\\ \vdots\\ B_m\end{pmatrix}$ *stands for those terms which are of forms A_iB_j $(1\le i\le n, 1\le j\le m)$.*

Proof. Note that $\mathscr{R}$ is a $(0,4)$-type tensor field. Taking $W = \partial_\kappa$, $X = \partial_\lambda$, $Y = \partial_\alpha$ and $Z = \partial_\beta$ in (4.6), and applying $\mathscr{D}_\alpha\partial_\beta = g^{\gamma\nu}\Gamma_{\alpha\gamma\beta}\partial_\nu$, one gets

$$\mathscr{R}_{\kappa\lambda\alpha\beta} = \partial_\lambda\Gamma_{\alpha\beta\kappa} - \partial_\kappa\Gamma_{\lambda\beta\alpha} - g^{\delta\gamma}\Gamma_{\alpha\gamma\kappa}\Gamma_{\lambda\delta\beta} + g^{\delta\gamma}\Gamma_{\lambda\gamma\alpha}\Gamma_{\kappa\delta\beta}$$

with the Christoffel symbols

$$\Gamma_{\alpha\beta\kappa} = \frac{1}{2}\{G^\gamma_{\beta\kappa}\partial_\alpha\varphi_\gamma + G^\gamma_{\alpha\beta}\partial_\kappa\varphi_\gamma - G^\gamma_{\alpha\kappa}\partial_\beta\varphi_\gamma\}.$$

Then

$$\begin{aligned}\mathscr{R}_{\kappa\lambda\alpha\beta} =&\frac{1}{2}\{G^{\nu}_{\beta\kappa}\mathscr{D}^2_{\lambda\alpha}\varphi_\nu - G^{\nu}_{\alpha\kappa}\mathscr{D}^2_{\lambda\beta}\varphi_\nu - G^{\nu}_{\beta\lambda}\mathscr{D}^2_{\kappa\alpha}\varphi_\nu + G^{\nu}_{\alpha\lambda}\mathscr{D}^2_{\kappa\beta}\varphi_\nu\}\\
&-\frac{1}{4}g^{\delta\gamma}G^{\nu}_{\beta\kappa}G^{\sigma}_{\lambda\alpha}(\partial_\delta\varphi_\sigma)(\partial_\gamma\varphi_\nu)+\frac{1}{4}g^{\delta\gamma}G^{\nu}_{\beta\lambda}G^{\sigma}_{\kappa\alpha}(\partial_\delta\varphi_\sigma)(\partial_\gamma\varphi_\nu)\\
&-\frac{1}{4}g^{\delta\gamma}\big(G^{\nu}_{\delta(\beta}\partial_{\lambda)}\varphi_\nu\big)\big(G^{\sigma}_{\gamma(\kappa}\partial_{\alpha)}\varphi_\sigma\big)+\frac{1}{4}g^{\delta\gamma}\big(G^{\nu}_{\delta(\beta}\partial_{\kappa)}\varphi_\nu\big)\big(G^{\sigma}_{\gamma(\lambda}\partial_{\alpha)}\varphi_\sigma\big)\\
&+\frac{1}{2}\{G^{\nu\sigma}_{\beta\kappa}(\partial_\lambda\varphi_\sigma)(\partial_\alpha\varphi_\nu)-G^{\nu\sigma}_{\alpha\kappa}(\partial_\lambda\varphi_\sigma)(\partial_\beta\varphi_\nu)-G^{\nu\sigma}_{\lambda\beta}(\partial_\kappa\varphi_\sigma)(\partial_\alpha\varphi_\nu)\\
&\quad+G^{\nu\sigma}_{\lambda\alpha}(\partial_\kappa\varphi_\sigma)(\partial_\beta\varphi_\nu)\},\end{aligned}$$

where $G^{\gamma}_{\kappa(\lambda}\partial_{\alpha)}\varphi_\gamma := G^{\gamma}_{\kappa\lambda}\partial_\alpha\varphi_\gamma + G^{\gamma}_{\kappa\alpha}\partial_\lambda\varphi_\gamma$. It follows from (3.38) that

$$g^{\delta\gamma}(\partial_\delta\varphi_b)(\partial_\gamma\varphi_a) = (\not{d}^X\varphi_a)(\not{d}_X\varphi_b) - (\mathring{L}\varphi_a)(\mathring{L}\varphi_b) - \mu^{-1}(T\varphi_a)(\mathring{L}\varphi_b) - \mu^{-1}(T\varphi_b)(\mathring{L}\varphi_a).$$

Contracting $\mathscr{R}_{\kappa\lambda\alpha\beta}$ with $T^\kappa(\not{d}_X x^\lambda)\mathring{L}^\alpha(\not{d}_X x^\beta)$ leads to

$$\begin{aligned}\mathscr{R}_{TX\mathring{L}X} =&\frac{1}{2}\{\mu G^{\nu}_{X\tilde{T}}\mathscr{D}^2_{X\mathring{L}}\varphi_\nu - \mu G^{\nu}_{\mathring{L}\tilde{T}}\mathscr{D}^2_{X}\varphi_\nu - G^{\nu}_{XX}\mathscr{D}^2_{\mathring{L}T}\varphi_\nu + G^{\nu}_{X\mathring{L}}\mathscr{D}^2_{XT}\varphi_\nu\}\\
&+c^{-2}f(\not{d}\vec{x},\mathring{L}^a,\varphi,\not{g})\begin{pmatrix}\mu\mathring{L}\varphi\\ T\varphi\\ \mu\not{d}\varphi\end{pmatrix}\begin{pmatrix}\mathring{L}\varphi\\ \not{d}\varphi\end{pmatrix}.\end{aligned}\tag{4.9}$$

In addition, (3.31), (3.30) and Lemma 3.5 imply that

$$\begin{aligned}\mathscr{D}^2_{XT}\varphi_\gamma =&\not{d}_X T\varphi_\gamma - (\mathscr{D}_X T)\varphi_\gamma\\
=&\not{d}_X T\varphi_\gamma - (\mu^{-1}T\varphi_\gamma)\not{d}_X\mu + (\mu\not{d}_X\varphi_\gamma)\mathrm{tr}\lambda + c^{-2}f(\not{d}\vec{x},\mathring{L}^a,\varphi,\not{g})\begin{pmatrix}\mu\mathring{L}\varphi\\ \mu\not{d}\varphi\\ T\varphi\end{pmatrix}\begin{pmatrix}\mathring{L}\varphi\\ \not{d}\varphi\end{pmatrix}.\end{aligned}\tag{4.10}$$

Similarly,

$$\mathscr{D}^2_{\mathring{L}T}\varphi_\gamma = \mathring{L}T\varphi_\gamma + c^{-2}f(\not{d}\vec{x},\mathring{L}^a,\varphi,\not{g})\begin{pmatrix}\mu\mathring{L}\varphi\\ \mu\not{d}\varphi\end{pmatrix}\begin{pmatrix}\mathring{L}\varphi\\ \not{d}\varphi\end{pmatrix},\tag{4.11}$$

$$\mu\mathscr{D}^2_{X}\varphi_\gamma = \mu\not{\nabla}^2_X\varphi_\gamma - \lambda_{XX}(T\varphi_\gamma) + c^{-2}f(\not{d}\vec{x},\mathring{L}^a,\varphi,\not{g})\begin{pmatrix}\mu\mathring{L}\varphi\\ \mu\not{d}\varphi\\ T\varphi\end{pmatrix}\begin{pmatrix}\mathring{L}\varphi\\ \not{d}\varphi\end{pmatrix}.\tag{4.12}$$

Substituting (4.10)-(4.12) into (4.9) yields (4.7).

(4.8) follows by an analogous argument for (4.7), the details are omitted here. However, it should be emphasized that the terms containing factors $T\varphi_\alpha\mathring{L}\varphi_\beta$ and $T\varphi_\alpha\not{d}\varphi_\beta$ in (4.7) disappear in (4.8) due to the special structure of (1.6). □

Note that in Lemma 4.1, $\mathscr{R}_{TX\mathring{L}X}$ and $\mathscr{R}_{\mathring{L}X\mathring{L}X}$ are obtained after contracting $\mathscr{R}_{\kappa\lambda\alpha\beta}$ with respect to the corresponding vectorfields. If using (4.6) directly, e.g. $\mathscr{R}_{\mathring{L}X\mathring{L}X} = g\big(\mathscr{D}_X(\mathscr{D}_{\mathring{L}}\mathring{L}) - \mathscr{D}_{\mathring{L}}(\mathscr{D}_X\mathring{L}), X\big)$, then one can get the following structure equations for λ_{XX} and $\check{\lambda}_{XX}$ with the help of Lemmas 4.1.

Lemma 4.2. *The second fundamental form* λ *and its "error" form* $\check{\lambda}$*, defined in* (3.10) *and* (3.13) *respectively, satisfy the following structure equations:*

$$\begin{aligned}\mathring{L}\lambda_{XX} =& c^{-1}\not{d}_X x^a(\not{d}_X \mathring{L}\varphi_a) - c^{-1}\not{g}_{XX}\big(\mathring{L}^2\varphi_0 + \varphi_a \mathring{L}^2\varphi_a\big) - c^{-1}\mathring{L}^\gamma(\not{\nabla}^2_X\varphi_\gamma) + (tr\lambda)\lambda_{XX}\\ &- c^{-1}tr\lambda(\not{d}_X x^a)\not{d}_X\varphi_a - \frac{1}{2}c^{-1}(\mathring{L}c)\lambda_{XX} + c^{-2}f(\mathring{L}^i,\varphi)\begin{pmatrix}(\not{d}_X\varphi)(\not{d}_X\varphi)\\ \not{g}^{XX}(\not{d}_X x^a\not{d}_X\varphi_a)^2\\ \not{g}^{XX}(\mathring{L}\varphi)(\mathring{L}\varphi)\\ (\not{d}_X x^a\not{d}_X\varphi_a)\mathring{L}\varphi\end{pmatrix},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\not{\mathcal{L}}_T\lambda_{XX} =& \not{\nabla}^2_X\mu + c^{-1}\Big\{\frac{1}{2}\not{d}_X x^a(\not{d}_X T\varphi_a) - \not{g}_{XX}(\mathring{L}T\varphi_0 + \varphi_a\mathring{L}T\varphi_a) + \mu\not{\nabla}^2_X\varphi_0\\ &+ \mu(\varphi_a + \frac{1}{2}\tilde{T}^a)\not{\nabla}^2_X\varphi_a\Big\} - \mu(tr\lambda)\lambda_{XX} - \frac{1}{2}c^{-1}(\tilde{T}^aT\varphi_a + Tc)\lambda_{XX}\\ &+ \frac{1}{2}c^{-1}\mu(\not{d}_X x^a\not{d}_X\varphi_a)tr\lambda + \frac{3}{2}c^{-1}(\tilde{T}^a\not{d}_X\varphi_a)\not{d}_X\mu\\ &+ c^{-2}f_1(\mathring{L}^i,\varphi)\begin{pmatrix}\mu\mathring{L}\varphi\\ T\varphi\end{pmatrix}\begin{pmatrix}\not{g}_{XX}\mathring{L}\vec{\varphi}\\ \not{d}_X x^a\cdot\not{d}\varphi_a\end{pmatrix} + c^{-2}f_2(\mathring{L}^i,\varphi)\begin{pmatrix}\mu(\not{d}_X\varphi)(\not{d}_X\varphi)\\ \not{g}^{XX}\mu(\not{d}_X x^a\not{d}\varphi_a)^2\end{pmatrix},\end{aligned} \tag{4.14}$$

and hence,

$$\begin{aligned}\mathring{L}\check{\lambda}_{XX} =& c^{-1}\not{d}_X x^a(\not{d}_X\mathring{L}\varphi_a) - c^{-1}\not{g}_{XX}\big(\mathring{L}^2\varphi_0 + \varphi_a\mathring{L}^2\varphi_a\big) - c^{-1}\mathring{L}^\gamma(\not{\nabla}^2_X\varphi_\gamma) - \frac{1}{2}c^{-1}(\mathring{L}c)\check{\lambda}_{XX}\\ &- c^{-1}(tr\check{\lambda})(\not{d}_X x^a)\not{d}_X\varphi_a - \frac{1}{2}\varrho^{-1}c^{-1}(\mathring{L}c)\not{g}_{XX} - \varrho^{-1}c^{-1}(\not{d}_X x^a)\not{d}_X\varphi_a + (tr\check{\lambda})\check{\lambda}_{XX}\\ &+ c^{-2}f(\mathring{L}^i,\varphi)\begin{pmatrix}(\not{d}_X\varphi)(\not{d}_X\varphi)\\ \not{g}^{XX}(\not{d}_X x^a\not{d}_X\varphi_a)^2\\ \not{g}^{XX}(\mathring{L}\varphi)(\mathring{L}\varphi)\\ (\not{d}_X x^a\not{d}_X\varphi_a)\mathring{L}\varphi\end{pmatrix},\end{aligned} \tag{4.15}$$

$$\begin{aligned}\not{\mathcal{L}}_T\check{\lambda}_{XX} =& \not{\nabla}^2_X\mu + c^{-1}\Big\{\frac{1}{2}\not{d}_X x^a(\not{d}_X T\varphi_a) - \not{g}_{XX}(\mathring{L}T\varphi_0 + \varphi_a\mathring{L}T\varphi_a) + \mu\not{\nabla}^2_X\varphi_0 + \mu(\varphi_a + \frac{1}{2}\tilde{T}^a)\not{\nabla}^2_X\varphi_a\Big\}\\ &- \mu(tr\check{\lambda})\check{\lambda}_{XX} - \frac{1}{2}c^{-1}(\tilde{T}^aT\varphi_a + Tc)\lambda_{XX} + \frac{1}{2}c^{-1}\mu(\not{d}_X x^a\not{d}_X\varphi_a)tr\lambda + \frac{3}{2}c^{-1}(\tilde{T}^a\not{d}_X\varphi_a)\not{d}_X\mu\\ &+ \frac{c^{-1}}{\varrho}(Tc)\not{g}_{XX} + \frac{c^{-1}\mu}{\varrho}(\mathring{L}c)\not{g}_{XX} - \frac{2c^{-1}\mu}{\varrho}\not{d}_X x^a\not{d}_X\varphi_a + \frac{\mu-1}{\varrho^2}\not{g}_{XX}\\ &+ c^{-2}f_1(\mathring{L}^i,\varphi)\begin{pmatrix}\mu\mathring{L}\varphi\\ T\varphi\end{pmatrix}\begin{pmatrix}\not{g}_{XX}\mathring{L}\varphi\\ \not{d}_X x^a\cdot\not{d}\varphi_a\end{pmatrix} + c^{-2}f_2(\mathring{L}^i,\varphi)\begin{pmatrix}\mu(\not{d}_X\varphi)(\not{d}_X\varphi)\\ \not{g}^{XX}\mu(\not{d}_X x^a\not{d}\varphi_a)^2\end{pmatrix}.\end{aligned} \tag{4.16}$$

Proof. 1. It follows from (4.6) that

$$\mathscr{R}_{\mathring{L}X\mathring{L}X} = g\big(\mathscr{D}_X(\mathscr{D}_{\mathring{L}}\mathring{L}) - \mathscr{D}_{\mathring{L}}(\mathscr{D}_X\mathring{L}), X\big). \tag{4.17}$$

Substituting the expressions for $\mathscr{D}_{\mathring{L}}\mathring{L}$ and $\mathscr{D}_X\mathring{L}$ in Lemma 3.5 into (4.17) and noting that $\lambda_{XX} = g(\mathscr{D}_X\mathring{L}, X) = g(\mathscr{D}_{\mathring{L}}X, X)$, one can get

$$\mathscr{R}_{\mathring{L}X\mathring{L}X} = \mu^{-1}(\mathring{L}\mu)\lambda_{XX} - \mathring{L}\lambda_{XX} + (\mathrm{tr}\lambda)\lambda_{XX}. \tag{4.18}$$

This, together with (4.8) and (3.19), yields (4.13). On the other hand, (4.15) follows directly from $\check{\lambda}_{XX} = \lambda_{XX} - \frac{1}{\varrho}\not{g}_{XX}$ and $\mathring{L}\not{g}_{XX} = 2\lambda_{XX}$.

2. Using the definition of Lie derivatives, and by some detailed computations, one has

$$\not{\mathcal{L}}_T\lambda_{XX}=\mathcal{L}_T\lambda_{XX}=g(\mathscr{D}_X\mathscr{D}_T\mathring{L},X)+g(\mathscr{D}_X\mathring{L},\mathscr{D}_TX)-g(X,\mathscr{D}_{[T,X]}\mathring{L})-\mathscr{R}_{TX\mathring{L}X},\quad(4.19)$$

where

$$\begin{aligned}&g(\mathscr{D}_X\mathscr{D}_T\mathring{L},X)=g\big(\mathscr{D}_X(-\mathring{L}\mu\mathring{L}+\xi^XX),X\big)\\=&\not\nabla_X^2\mu+\not\nabla_X(\mu\zeta_X)-(\mathring{L}\mu)\lambda_{XX}\end{aligned}\quad(4.20)$$

and

$$\begin{aligned}g(\mathscr{D}_X\mathring{L},\mathscr{D}_TX)=&\zeta_X\xi_X+g(\mathscr{D}_{\Pi\mathscr{D}_TX}\mathring{L},X)\\=&\zeta_X\xi_X+g(\mathscr{D}_{[T,X]}\mathring{L},X)+g(\mathscr{D}_{\not\Pi\mathscr{D}_XT}\mathring{L},X)\\=&\zeta_X\xi_X+g(\mathscr{D}_{[T,X]}\mathring{L},X)+\mu\mathrm{tr}\tilde{\theta}g(\mathscr{D}_X\mathring{L},X),\end{aligned}\quad(4.21)$$

here (3.10), (3.11) and (3.33) have been used repeatedly. Substituting (4.20) and (4.21) into (4.19) yields

$$\not{\mathcal{L}}_T\lambda_{XX}=\not\nabla_X^2\mu+\not\nabla_X(\mu\zeta_X)-(\mathring{L}\mu)\lambda_{XX}+\zeta_X\xi_X+\mu\mathrm{tr}\tilde{\theta}g(\mathscr{D}_X\mathring{L},X)-\mathscr{R}_{TX\mathring{L}X}.$$

This leads to (4.14) with the help of (3.30), (3.19), (3.31), (4.7) and (3.10). Similarly for (4.15), one can use $\check{\lambda}_{XX}=\lambda_{XX}-\frac{1}{\varrho}\not{g}_{XX}$ and $\not{\mathcal{L}}_T\not{g}_{XX}=2\mu\tilde{\theta}_{XX}$ to obtain (4.16).

□

Based on $\mathring{L}\check{\lambda}$ in (4.15), the estimate of $\check{\lambda}$ could be achieved by integrating along integral curves of $\mathring{L}$.

Proposition 4.1. *Under the assumptions* $(\star)$ *with* $\delta>0$ *suitably small, it holds that*

$$|\check{\lambda}|=|tr\check{\lambda}|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2},\quad(4.22)$$

and

$$|\lambda|=\frac{1}{\varrho}+O(M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}).\quad(4.23)$$

Proof. It follows from $\mathrm{tr}\check{\lambda}=\not{g}^{XX}\check{\lambda}_{XX}$ that

$$\mathring{L}\big(\mathrm{tr}\check{\lambda}\big)=-2(\mathrm{tr}\check{\lambda})^2-\frac{2}{\rho}\mathrm{tr}\check{\lambda}+\not{g}^{XX}\mathring{L}\check{\lambda}_{XX}.\quad(4.24)$$

Substituting (4.15) into (4.24) and using $(\star)$, (4.1) and (4.5) to estimate the right hand side of (4.24) except $\check{\lambda}$ itself, one can get

$$|\mathring{L}\big(\varrho^2\mathrm{tr}\check{\lambda}\big)|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\varrho^2+M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}|\varrho^2\mathrm{tr}\check{\lambda}|+\varrho^{-2}|\varrho^2\mathrm{tr}\check{\lambda}|^2.$$

Thus, for small $\delta>0$, we obtain (4.22) by integrating along integral curves of $\mathring{L}$, which also yields (4.23) directly due to (3.13). □

It follows from Proposition 4.1, (3.28) and (3.29) that for small $\delta>0$,

$$|\not{d}\mathring{L}^i|\lesssim\varrho^{-1},\quad|\not{d}\check{L}^i|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}.\quad(4.25)$$

Note that

$$\mathring{L}\big(\varrho^2|\not{d}\mu|^2\big)=2\varrho^2\big\{-\mathrm{tr}\check{\lambda}|\not{d}\mu|^2+(\not{d}_X\mathring{L}\mu)(\not{d}^X\mu)\big\}.\quad(4.26)$$

Substituting (3.19) into (4.26) and applying $(\star)$, (4.1) and (4.25) give

$$|\mathring{L}\big(\varrho|\not{d}\mu|\big)| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\big(\varrho|\not{d}\mu|\big) + M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}.$$

This implies immediately that

$$|\not{d}\mu| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1}. \tag{4.27}$$

Now we are ready to improve the estimate on $\mathring{\underline{L}}\varphi_\gamma$, which will lead to better estimates for $\mathring{L}\varphi_\gamma$ and some other related quantities independent of M.

Proposition 4.2. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *suitably small, it holds that*

$$|\mathring{\underline{L}}\varphi_\gamma(\mathfrak{t}, u, \vartheta)| + |T\varphi_\gamma(\mathfrak{t}, u, \vartheta)| \lesssim \delta^{-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{4.28}$$

Proof. By $(\star)$, (4.1), (4.5) and (4.22), it follows from (4.3) that

$$|\mathring{L}\mathring{\underline{L}}\varphi_\gamma + \frac{1}{2\varrho}\mathring{\underline{L}}\varphi_\gamma| \lesssim M^2\delta^{1-2\varepsilon_0}\mathfrak{t}^{-2}.$$

This yields

$$|\mathring{L}\big(\varrho^{1/2}\mathring{\underline{L}}\varphi_\gamma\big)(\mathfrak{t}, u, \vartheta)| \lesssim M^2\delta^{1-2\varepsilon_0}\mathfrak{t}^{-3/2}. \tag{4.29}$$

Integrating (4.29) along integral curves of $\mathring{L}$ gives

$$|\varrho^{1/2}\mathring{\underline{L}}\varphi_\gamma(\mathfrak{t}, u, \vartheta) - \varrho_0^{1/2}\mathring{\underline{L}}\varphi_\gamma(t_0, u, \vartheta)| \lesssim M^2\delta^{1-2\varepsilon_0}, \tag{4.30}$$

where $\varrho_0 = t_0 - u$. Then the estimate on $|\mathring{\underline{L}}\varphi_\gamma|$ in (4.28) follows from (4.30) for small δ. This, together with $\mathring{\underline{L}} = \mu\mathring{L} + 2T$, yields the estimate for $|T\varphi_\gamma|$ in (4.28) by $(\star)$. □

To improve the estimate on φ_γ further, we need to treat η first since $T = \frac{\partial}{\partial u} - \eta^X X$.

Lemma 4.3. *Under the assumptions* $(\star)$*, it holds that for* $\delta > 0$ *small,*

$$|\eta| = \sqrt{g_{XX}\eta^X\eta^X} \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}. \tag{4.31}$$

Proof. Note that η is a vector field on $S_{\mathfrak{t},u}$ and one has

$$\mathring{L}\eta^X = [T, \mathring{L}]^X = \big(\mathscr{D}_T\mathring{L} - \mathscr{D}_{\mathring{L}}T\big)^X = \not{d}^X\mu + 2\mu\zeta^X. \tag{4.32}$$

Then, it follows from (3.30) that

$$\mathring{L}(\varrho^{-2}|\eta|^2) = \varrho^{-2}\big\{2\mathrm{tr}\check{\lambda}|\eta|^2 + \not{d}_X\mu\eta^X + 2c^{-1}\mu\tilde{T}^a\not{d}_X\varphi_a\eta^X\big\}.$$

This, together with (4.27), (4.22) and $(\star)$, yields

$$|\mathring{L}(\varrho^{-1}|\eta|)| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}(\varrho^{-1}|\eta|) + M\delta^{1-\varepsilon_0}\mathfrak{t}^{-2}. \tag{4.33}$$

Thus (4.31) follows immediately by integrating (4.33) along integral curves of $\mathring{L}$. □

Based on Proposition 4.2 and Lemma 4.3, we can now improve the estimates on φ_γ and $\mathring{L}\varphi_\gamma$ which are independent of M.

Proposition 4.3. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *suitably small, it holds that*

$$|\varphi_\gamma(\mathfrak{t}, u, \vartheta)| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}, \qquad |\mathring{L}\varphi_\gamma(\mathfrak{t}, u, \vartheta)| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}. \tag{4.34}$$

Proof. First, substituting $T = \frac{\partial}{\partial u} - \eta^X X$ into (4.28) and using $(\star)$ and (4.31) lead to

$$|\frac{\partial}{\partial u}\varphi_\gamma(\mathfrak{t}, u, \vartheta)| \lesssim \delta^{-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{4.35}$$

Integrating (4.35) from 0 to u yields the desired estimate on φ_γ in (4.34) since φ_γ vanishes on C_0.

Next, note that $\mathring{L}\varphi_\gamma$ satisfies the equation

$$\mathring{\underline{L}}\mathring{L}\varphi_\gamma = -\frac{1}{2\varrho}\mathring{\underline{L}}\varphi_\gamma + H_\gamma - \mathring{L}\mu\mathring{L}\varphi_\gamma + 2\eta^A \not{d}_A\varphi_\gamma + 2\mu\zeta^A\not{d}_A\varphi_\gamma$$

due to (3.34) and (4.3). Then $|\mathring{\underline{L}}\mathring{L}\varphi_\gamma| \lesssim \delta^{-\varepsilon_0}\mathfrak{t}^{-3/2}$ follows from (4.4), (3.19) and (3.30). Similarly as for (4.35), one can get

$$|\frac{\partial}{\partial u}\mathring{L}\varphi_\gamma(\mathfrak{t}, u, \vartheta)| \lesssim \delta^{-\varepsilon_0}\mathfrak{t}^{-3/2}. \tag{4.36}$$

Then the estimate for $\mathring{L}\varphi_\gamma$ in (4.34) follows by integrating (4.36) from 0 to u. □

One can now use Propositions 4.2 and 4.3 to measure the errors of the components of the frames.

Lemma 4.4. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *suitably small, the following estimates hold:*

$$|\check{L}^i| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}, \qquad |\check{T}^i| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}, \tag{4.37}$$

$$|\frac{r}{\varrho} - 1| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}, \tag{4.38}$$

$$|R^k - \Omega^k| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{1/2}. \tag{4.39}$$

Proof. First, it follows from (4.1), (4.34), and (3.27) that

$$|\mathring{L}(\varrho\check{L}^i)| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{4.40}$$

Integrating (4.40) along integral curves of $\mathring{L}$ yields the desired estimate on $\check{L}^i$ in (4.37), which implies the estimate on $\check{T}^i$ in (4.37) by $\check{T}^i = \varphi_i - \check{L}^i$.

Next, since $g_{ij}(\check{T}^i - \dfrac{x^i}{\varrho})(\check{T}^j - \dfrac{x^j}{\varrho}) = 1$, then $\frac{r^2}{\varrho^2} = c - \delta_{ij}\check{T}^i\check{T}^j + 2\delta_{ij}\check{T}^i\frac{x^j}{\varrho}$. Thus,

$$\frac{r}{\varrho} - 1 = \frac{2(\varphi_0 + \varphi_i\omega^i) + 2\check{L}^i\varphi_i - \delta_{ij}\check{L}^i\check{L}^j - 2\delta_{ij}\check{L}^i\omega^j}{\sqrt{c + (\delta_{ij}\check{T}^i\omega^j)^2 - \delta_{ij}\check{T}^i\check{T}^j} + 1 - \delta_{ij}\check{T}^i\omega^j}, \tag{4.41}$$

which implies (4.38) due to $(\star)$ and (4.37).

Finally, (4.39) follows directly from (3.16), (4.37) and (4.38).

□

Note that the rotation vector field R behaves just as the scaling operator $r\not{\nabla}$ under the assumptions $(\star)$ and the estimate (4.22) as stated in the following lemma, which is similar to Lemma 12.22 in [36] or Sect.3.3.3 in [33].

Lemma 4.5. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *suitably small, it holds that*

(i) if κ *is a 1-form on* $S_{\mathfrak{t},u}$*, then*

$$(\kappa \cdot R)^2 = \big(1 + O(\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2})\big)r^2|\kappa|^2, \tag{4.42}$$

$$|\not{\mathcal{L}}_R\kappa|^2 = \big(1 + O(\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2})\big)r^2|\not{\nabla}\kappa|^2 + O(\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}M^2)|\kappa|^2; \tag{4.43}$$

(ii) if Θ *is a 2-form on* $S_{\mathfrak{t},u}$*, then*

$$|\not{\mathcal{L}}_R\Theta|^2 = r^2|\not{\nabla}\Theta|^2\big(1 + O(\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2})\big) + |\Theta|^2 O(\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}M^2). \tag{4.44}$$

5 L^∞ estimates for the higher order derivatives of $\partial\phi$ and some related quantities near C_0

Note that although the L^∞ estimates for lower order derivatives of some geometrical quantities (such as (4.22), (4.25) and (4.27)) and the M-independent estimates for the first order derivatives of φ_γ (see (4.28) and (4.34)) obtained in Section 4 are not enough to close the assumptions $(\star)$, yet the method proving (4.34) indicates that it is possible to improve the estimates on φ_γ by making use of the equation (4.3). Thus, in this section, under the assumptions $(\star)$, we will improve the estimates of φ_γ up to the $(N-1)^{th}$ order derivatives. To this end, we start with some preliminary results which involve only the rotational vector fields on $S_{\mathfrak{t},u}$.

Lemma 5.1. *Under the assumptions $(\star)$ with $\delta>0$ suitably small, it holds that for $k\le N-1$,*

$$\begin{aligned}&|\not{\mathcal{L}}_R^{k+1}\not{d}x^i|\lesssim 1,\quad |\not{\mathcal{L}}_R^k\check{\lambda}|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2},\quad |R^{k+1}\mathring{L}^j|\lesssim 1,\\ &|\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2},\quad |\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}_{\mathring{L}}|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2},\\ &|R^{k+1}\check{L}^j|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2},\quad |R^{k+1}\upsilon|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{1/2}.\end{aligned}\tag{5.1}$$

Proof. This will be proved by induction with respect to the number k.

First, for $k=0$, the desired estimates for $\check{\lambda}$, $R\mathring{L}^j$, $R\check{L}^j$, ${}^{(R)}\not{\pi}$, ${}^{(R)}\not{\pi}_{\mathring{L}X}$ and $R\upsilon$ follow easily from (4.22), (4.25), (4.37), (4.42), (3.16) and (3.25), respectively. Since

$$R^2x^i=R(\Omega^i-\upsilon\tilde{T}^i)=\epsilon_j^iRx^j-R(\upsilon\tilde{T}^i),$$

then $|R^2x^i|\lesssim\mathfrak{t}$ by (4.38) and (4.39). This and (4.42) with $\xi=\not{d}Rx^i$ yield

$$|\not{\mathcal{L}}_R\not{d}x^i|=|\not{d}Rx^i|\lesssim r^{-1}|R^2x^i|\lesssim 1.\tag{5.2}$$

Next, assuming that (5.1) holds up to the order $k-1$ $(0\le k\le N-1)$, one needs to show that (5.1) is also true for the number k. This is done by the following three steps:

(1) **Treatments of $\not{\mathcal{L}}_R^k\check{\lambda}$ and $\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}_{\mathring{L}}$**

Using the expression of ${}^{(R)}\not{\pi}_{\mathring{L}X}$ obtained in (3.25) and the induction assumption, one can check that

$$|\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}_{\mathring{L}}|\lesssim\mathfrak{t}|\not{\mathcal{L}}_R^k\check{\lambda}|+\delta^{1-\varepsilon_0}M\mathfrak{t}^{-1/2}.\tag{5.3}$$

Then $|\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}_{\mathring{L}}|$ admits the upper bound once $\mathfrak{t}|\not{\mathcal{L}}_R^k\check{\lambda}|$ is bounded suitably. Thus, it remains to estimate $\not{\mathcal{L}}_R^k\check{\lambda}$. It follows from direct computations that

$$\begin{aligned}\mathring{L}\big(\varrho^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\big)&=-2\varrho^2(\mathrm{tr}\check{\lambda})\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})+\varrho^2\not{g}^{XX}(\not{\mathcal{L}}_{\mathring{L}}\not{\mathcal{L}}_R^k\check{\lambda}_{XX})\\ &=-2\varrho^2(\mathrm{tr}\check{\lambda})\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})+\varrho^2\not{g}^{XX}(\not{\mathcal{L}}_R^k\not{\mathcal{L}}_{\mathring{L}}\check{\lambda}_{XX})\\ &\quad+\sum_{k_1=0}^{k-1}\varrho^2\not{g}^{XX}\not{\mathcal{L}}_R^{k_1}\big({}^{(R)}\not{\pi}_{\mathring{L}X}\not{\nabla}^X\not{\mathcal{L}}_R^{k-k_1-1}\check{\lambda}_{XX}+\not{\mathcal{L}}_R^{k-k_1-1}\check{\lambda}_{XX}\not{\mathrm{div}}{}^{(R)}\not{\pi}_{\mathring{L}}\big),\end{aligned}\tag{5.4}$$

where we have neglected the constant coefficients in the summation above. It then follows from (5.4), (4.15), and the induction assumptions that

$$|\mathring{L}\big(\varrho^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\big)|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}|\varrho^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})|+M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}.\tag{5.5}$$

Integrating the inequality (5.5) along integral curves of $\mathring{L}$ and using (5.3) yield that for small $\delta>0$,

$$|\not{\mathcal{L}}_R^k\check{\lambda}|+\mathfrak{t}^{-1}|\not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi}_{\mathring{L}}|\lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}.$$

(2) **Treatments of $R^{k+1}\check{L}^i$, $R^{k+1}\mathring{L}^i$, $R^{k+1}\upsilon$ and $\not{\mathcal{L}}_R^{k}{}^{(R)}\not\pi$**

Due to (3.29), it holds that

$$\begin{aligned}&R^{k+1}\check{L}^i = R^k(R^X \not{d}_X \check{L}^i)\\ =&R^k\big\{\mathrm{tr}\check{\lambda}(Rx^i) + \frac{1}{2}c^{-1}(\mathring{L}c)Rx^i - \frac{1}{2}c^{-1}(Rc)\tilde{T}^i + c^{-1}\tilde{T}^a(R\varphi_a)\tilde{T}^i\big\}.\end{aligned} \tag{5.6}$$

This implies $|R^{k+1}\check{L}^i| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}$ by ($\star$), the induction assumptions and the estimate on $\not{\mathcal{L}}_R^k\check{\lambda}$ in part (1). Hence $|R^{k+1}\mathring{L}^i| \lesssim 1$ holds for small $\delta > 0$. It follows from this and induction assumptions that the desired estimate for $R^{k+1}\upsilon$ can be obtained by applying (3.16) directly. In addition, $|\not{\mathcal{L}}_R^{k}{}^{(R)}\not\pi| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}$ holds due to (3.25).

(3) **Treatment of $\not{\mathcal{L}}_R^{k+1}\not{d}x^i$**

Note that

$$R^{k+2}x^i = \epsilon^i_j R^{k+1}x^j - R^{k+1}(\upsilon\tilde{T}^i).$$

Then by the induction assumptions, the estimates on $R^{k+1}\upsilon$ and $R^{k+1}\tilde{T}^j (= R^{k+1}\varphi_j - R^{k+1}\mathring{L}^j)$ in part (2), one can get $|R^{k+2}x^j| \lesssim \mathfrak{t}$, and hence $|\not{\mathcal{L}}_R^{k+1}\not{d}x^j| \lesssim 1$ as for (5.2).

□

Based on Lemma 5.1, one has the following estimate on the higher order rotational derivatives of μ.

Proposition 5.1. *Under the same assumptions in Lemma 5.1, it holds that*

$$|R^{k+1}\mu| \lesssim M\delta^{1-\varepsilon_0}, \quad k \le N-1. \tag{5.7}$$

Proof. This will be proved by induction.

When $k = 0$, (5.7) follows from (4.27) and (4.42). Assume that (5.7) holds up to the order $k-1$ ($k \le N-1$). Then (3.19) and (3.42) imply

$$\begin{aligned}\mathring{L}R^{k+1}\mu &= [\mathring{L}, R^{k+1}]\mu + R^{k+1}\mathring{L}\mu\\ &= \sum_{k_1+k_2=k} R^{k_1}\big({}^{(R)}\not\pi_{\mathring{L}}{}^X \not{d}_X R^{k_2}\mu\big) + R^{k+1}\big\{c^{-1}\mu(\mathring{L}^\alpha\mathring{L}\varphi_\alpha - \mathring{L}c)\big\}.\end{aligned} \tag{5.8}$$

Using ($\star$), the induction assumptions and Lemma 5.1, one can estimate the right hand side of (5.8) to get

$$|\mathring{L}R^{k+1}\mu| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}|R^{k+1}\mu| + M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}. \tag{5.9}$$

Integrating (5.9) along integral curves of $\mathring{L}$ yields $|R^{k+1}\mu| \lesssim M\delta^{1-\varepsilon_0}$ for small $\delta > 0$. □

One can conclude from (3.25), Lemma 5.1 and Proposition 5.1 that

$$|\not{\mathcal{L}}_R^{k}{}^{(R)}\not\pi_T| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad k \le N-1. \tag{5.10}$$

Similarly, we can estimate the derivatives containing T as follows.

Proposition 5.2. *Under the assumptions* $(\star)$ *with small* $\delta > 0$*, it holds that for* $k \leq N-1$ *and any operator* $\bar{Z} \in \{T, R\}$*,*

$$
\begin{aligned}
&|\not{\mathcal{L}}_{\bar{Z}}^{k+1;i} \not{d} x^j| \lesssim \delta^{1-i}\mathfrak{t}^{-1}, \quad |\not{\mathcal{L}}_{\bar{Z}}^{k;i}\check{\lambda}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-3/2}, \quad |\not{\mathcal{L}}_{\bar{Z}}^{k+1;i}R| \lesssim M\delta^{2-i-\varepsilon_0}\mathfrak{t}^{-1/2},\\
&|\not{\mathcal{L}}_{\bar{Z}}^{k;i}{}^{(R)}\not{\pi}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\not{\mathcal{L}}_{\bar{Z}}^{k;i}{}^{(R)}\not{\pi}_{\mathring{L}}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2},\\
&|\bar{Z}^{k+1;i}\check{L}^j| \lesssim M\delta^{2-i-\varepsilon_0}\mathfrak{t}^{-1}, \quad |\bar{Z}^{k+1;i}\check{T}^j| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\bar{Z}^{k+1;i}\upsilon| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{1/2},\\
&|\bar{Z}^{k+1;i}\mu| \lesssim M\delta^{1-i-\varepsilon_0}, \quad |\not{\mathcal{L}}_{\bar{Z}}^{k;i}{}^{(R)}\not{\pi}_T| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\not{\mathcal{L}}_{\bar{Z}}^{k;i}{}^{(T)}\not{\pi}| \lesssim M\delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2},
\end{aligned}
\tag{5.11}
$$

where the array $(k; i)$ *means that the number of* $\bar{Z}$ *is* k *and of* T *is* i *(*$i \geq 1$*).*

Proof. We first check the special case of $i = 1$.

1. Taking Lie derivative R^m on both hand sides of (4.16), and using Lemma 5.1, Proposition 5.1 and the assumptions $(\star)$, one can get that for $m \leq N-2$,

$$|\not{\mathcal{L}}_R^m \not{\mathcal{L}}_T \check{\lambda}| \lesssim M\delta^{-\varepsilon_0}\mathfrak{t}^{-3/2}. \tag{5.12}$$

   Note that $[T, R] = {}^{(R)}\not{\pi}_T{}^X X$ in (3.42) and for $k_1 + k_2 \leq m$,

$$\not{\mathcal{L}}_R^{k_1}\not{\mathcal{L}}_T\not{\mathcal{L}}_R^{k_2}\check{\lambda} = \not{\mathcal{L}}_R^{k_1}\not{\mathcal{L}}_R^{k_2}\not{\mathcal{L}}_T\check{\lambda} + \sum_{l_1+l_2=k_2-1}\not{\mathcal{L}}_R^{k_1}\not{\mathcal{L}}_R^{l_1}\not{\mathcal{L}}_{[T,R]}\not{\mathcal{L}}_R^{l_2}\check{\lambda}.$$

   Then $|\not{\mathcal{L}}_{\bar{Z}}^{k;1}\check{\lambda}| \lesssim \delta^{-\varepsilon_0}M\mathfrak{t}^{-3/2}$ follows for $k \leq N-1$ by making use of (5.12), (5.10) and the estimate of $\check{\lambda}$ in Lemma 5.1.

2. Similarly, noting that

$$
\begin{aligned}
&\not{\mathcal{L}}_T \not{d} x^j = (\not{d}\mu)\tilde{T}^j + \mu\not{d}\tilde{T}^j,\\
&T\check{L}^j = T\mathring{L}^j - \frac{\mu-1}{\varrho}\tilde{T}^j - \frac{\check{T}^j}{\varrho},\\
&T\upsilon = \epsilon_a^b g_{jb}\big\{-c^{-1}(Tc)x^a\check{T}^j - \mu\frac{x^a}{\varrho}\check{T}^j + x^a T\check{T}^j\big\},\\
&\not{\mathcal{L}}_T R = [T, R] = {}^{(R)}\not{\pi}_T^X X,
\end{aligned}
$$

   which are due to the definitions, (3.16) and (3.42), one can get the estimates in (5.11) for $\not{\mathcal{L}}_{\bar{Z}}^{k+1;1}\not{d}x^j$, $\not{\mathcal{L}}_{\bar{Z}}^{k+1;1}R$, $\bar{Z}^{k+1;1}\check{L}^j$, $\bar{Z}^{k+1;1}\check{T}^j$, $\bar{Z}^{k+1;1}\upsilon$ by (3.32) and $\check{T}^b = \varphi_b - \check{L}^b$.

3. Now we estimate μ by induction.

   Differentiating (3.19) with respect to T and using (3.42) lead to

$$\mathring{L}T\mu = [\mathring{L}, T]\mu + T\mathring{L}\mu = (-\not{d}^X\mu - 2c^{-1}\mu\tilde{T}^a\not{d}^X\varphi_a)\not{d}_X\mu + T\{c^{-1}\mu(\mathring{L}^\alpha\mathring{L}\varphi_\alpha - \mathring{L}c)\}.$$

   Then

$$|\mathring{L}T\mu| \lesssim M\delta^{-\varepsilon_0}\mathfrak{t}^{-3/2} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}|T\mu| \tag{5.13}$$

   holds due to $(\star)$ and (5.7).

   Integrating (5.13) along integral curves of $\mathring{L}$ and applying Gronwall's inequality yield

$$|T\mu| \lesssim M\delta^{-\varepsilon_0}. \tag{5.14}$$

Assume that for $n \le m-1$ $(m \le N-1)$,

$$|R^n T\mu| \lesssim M\delta^{-\varepsilon_0}. \tag{5.15}$$

Note that

$$\begin{aligned}\mathring{L}R^m T\mu = &\sum_{m_1+m_2=m-1} R^{m_1}\big({}^{(R)}\not\pi_{\mathring{L}}{}^X \not d_X R^{m_2}T\mu\big)\\ &+ R^m\big\{(-\not d^X\mu - 2c^{-1}\mu\tilde{T}^a\not d^X\varphi_a)\not d_X\mu + T[c^{-1}\mu(\mathring{L}^\alpha\mathring{L}\varphi_\alpha - \mathring{L}c)]\big\}.\end{aligned}$$

It follows from (5.15), Lemma 5.1 and Proposition 5.1 that

$$|\mathring{L}R^m T\mu| \lesssim M\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}|R^m T\mu| + M\delta^{-\varepsilon_0}\mathfrak{t}^{-3/2},$$

which yields (5.15) for $n=m$ by integrating along integral curves of $\mathring{L}$, and hence $|\bar{Z}^{k+1;1}\mu| \lesssim M\delta^{-\varepsilon_0}$ follows from induction. The estimates of $\not\mathcal{L}_{\bar{Z}}^{k;1}{}^{(R)}\not\pi$, $\not\mathcal{L}_{\bar{Z}}^{k;1}{}^{(T)}\not\pi$ and $\not\mathcal{L}_{\bar{Z}}^{k;1}{}^{(R)}\not\pi_{\mathring{L}}$ follow easily from using (3.23) and (3.25). Meanwhile, $\not\mathcal{L}_{\bar{Z}}^{i;1}{}^{(R)}\not\pi_T$ can be estimated by (3.25).

Analogously, (5.11) for $i \ge 2$ can be proved by induction. □

We can now improve the L^∞ estimates on derivatives of φ_γ with respect to $\bar{Z} \in \{T, R\}$.

Corollary 5.1. *Under the same assumptions as in Proposition 5.2, it holds that for $k \le N-1$,*

$$|\bar{Z}^{k;i}\mathring{\underline{L}}\varphi_\gamma(\mathfrak{t},u,\vartheta)| + |\bar{Z}^{k;i}T\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \tag{5.16}$$

$$|\bar{Z}^{k;i}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \tag{5.17}$$

here i is the number of T in $\bar{Z}^k$.

Proof. For any $k \in \mathbb{N}$ with $k+i \le N-1$, one has from (4.3) that

$$\begin{aligned} R^kT^iH_\gamma &= R^kT^i\big(\mathring{L}\mathring{\underline{L}}\varphi_\gamma + \frac{1}{2\varrho}\mathring{\underline{L}}\varphi_\gamma\big)\\ &=\mathring{L}\big(R^kT^i\mathring{\underline{L}}\varphi_\gamma\big) + \frac{1}{2\varrho}R^kT^i\mathring{\underline{L}}\varphi_\gamma - \sum_{k_1+k_2=k-1} R^{k_1}\big({}^{(R)}\not\pi_{\mathring{L}}^X\not d_X R^{k_2}T^i\mathring{\underline{L}}\varphi_\gamma\big)\\ &\quad - \sum_{i_1+i_2=i-1} R^kT^{i_1}\big({}^{(T)}\not\pi_{\mathring{L}}^X\not d_X T^{i_2}\mathring{\underline{L}}\varphi_\gamma\big) + \sum_{i_1+i_2=i, i_1\ge 1} T^{i_1}(\frac{1}{2\varrho})\cdot R^kT^{i_2}\mathring{\underline{L}}\varphi_\gamma.\end{aligned} \tag{5.18}$$

Since the last three summations above can be estimated by Lemma 5.1 and Proposition 5.2, one gets from (5.18) that

$$R^kT^iH_\gamma = \mathring{L}\big(R^kT^i\mathring{\underline{L}}\varphi_\gamma\big) + \frac{1}{2\varrho}R^kT^i\mathring{\underline{L}}\varphi_\gamma + O\big(M^2\delta^{1-i-2\varepsilon_0}\mathfrak{t}^{-2}\big). \tag{5.19}$$

On the other hand, one can also estimate $R_i^kT^iH_\gamma$ by the expression of H_γ in (4.4) as follows

$$|R^kT^iH_\gamma| \lesssim M^2\delta^{1-i-2\varepsilon_0}\mathfrak{t}^{-2}. \tag{5.20}$$

Combining (5.19) and (5.20) yields $|\mathring{L}\big(\varrho^{1/2}R^kT^i\mathring{\underline{L}}\varphi_\gamma\big)| \lesssim M^2\delta^{1-i-2\varepsilon_0}\mathfrak{t}^{-3/2}$, which can be integrated along integral curves of $\mathring{L}$ to obtain

$$|\varrho^{1/2}R^kT^i\mathring{\underline{L}}\varphi_\gamma(\mathfrak{t},u,\vartheta) - \varrho_0^{1/2}R^kT^i\mathring{\underline{L}}\varphi_\gamma(t_0,u,\vartheta)| \lesssim M^2\delta^{1-i-2\varepsilon_0}.$$

This yields that $|R^kT^i\mathring{\underline{L}}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}$ for small $\delta > 0$. This, together with the commutator relation and $\mathring{\underline{L}} = 2T + \mu\mathring{L}$, implies (5.16).

Note that (5.16) also implies $|T\bar{Z}^{k;i}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}$. Thus, (5.17) can be derived in the same way as for (4.34). □

It remains to deal with the derivatives involving the scaling vectorfield $\varrho\mathring{L}$. To this end, due to (4.15) and (3.27), one can take Lie derivatives of $\varrho\mathring{L}\check{\lambda}$ with respect to $\bar{Z}$ to estimate $\not{\mathcal{L}}_{\bar{Z}}^{k}\not{\mathcal{L}}_{\varrho\mathring{L}}\check{\lambda}$ as for (5.12). Note that the terms containing the derivatives of $\varrho\mathring{L}$ can be estimated in the same way as in the proof of Proposition 5.2. We can now state the main results as follows without tedious proofs.

Proposition 5.3. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *small, it holds that for any operate* $Z \in \{\varrho\mathring{L}, T, R\}$ *and* $k \le N-1$,

$$\begin{aligned}
&|\not{\mathcal{L}}_Z^{k+1;i,l}\not{d}x^j| \lesssim \delta^{-i}, \quad |\not{\mathcal{L}}_Z^{k;i,l}\check{\lambda}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-3/2}, \quad |\not{\mathcal{L}}_Z^{k+1;i,l}R| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{1/2},\\
&|\not{\mathcal{L}}_Z^{k;i,l}{}^{(R)}\not{\pi}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\not{\mathcal{L}}_Z^{k;i,l}{}^{(R)}\not{\pi}_{\mathring{L}}| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2},\\
&|Z^{k+1;i,l}\check{L}^j| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |Z^{k+1;i,l}\check{T}^j| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |Z^{k+1;i,l}\upsilon| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{1/2},\\
&|Z^{k+1;i,l}\mu| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\not{\mathcal{L}}_Z^{k;i,l}{}^{(R)}\not{\pi}_T| \lesssim M\delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\not{\mathcal{L}}_Z^{k;i,l}{}^{(T)}\not{\pi}| \lesssim M\delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2},
\end{aligned} \tag{5.21}$$

where the array $(k;i,l)$ *means that the number of* Z *is* k*, the number of* T *is* i *, and the number of* $\varrho\mathring{L}$ *is* l *(*$l \ge 1$*).*

Note first that (5.16) implies $|R^kT^i\underline{\mathring{L}}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}$ for $k+i \le N-1$. Assume now that the following estimates hold:

$$|R^kT^i(\varrho\mathring{L})^{l-1}\underline{\mathring{L}}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2} \quad (k+i+l-1 \le N-1, \quad l \ge 1). \tag{5.22}$$

(4.3) gives that

$$R^kT^i(\varrho\mathring{L})^l\underline{\mathring{L}}\varphi_\gamma = R^kT^i(\varrho\mathring{L})^{l-1}\big(-\frac{1}{2}\underline{\mathring{L}}\varphi_\gamma + \varrho H_\gamma\big). \tag{5.23}$$

Due to (4.4), $R^kT^i(\varrho\mathring{L})^{l-1}(\varrho H_\gamma)$ can be estimated easily by (5.1), (5.2) or (5.21). This, together with the induction assumption (5.22), shows that

$$|R^kT^i(\varrho\mathring{L})^l\underline{\mathring{L}}\varphi_\gamma| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2},\ k+i+l \le N, \tag{5.24}$$

which implies $|\frac{\partial}{\partial u}R^kT^i(\varrho\mathring{L})^l\varphi_\gamma| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}$ as for (4.35). Thus, one obtains also that for $k+i+l \le N$ and $l \ge 1$,

$$|R^kT^i(\varrho\mathring{L})^l\varphi_\gamma| \lesssim \delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{5.25}$$

Using Lemma 3.7 and rearranging the orders of derivatives in (5.24) and (5.25) lead to

Corollary 5.2. *Under the same assumptions as in Proposition 5.3, it holds that for* $k \le N-1$,

$$|Z^{k+2;i+1,l+1}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{-i-\varepsilon_0}\mathfrak{t}^{-1/2}, \tag{5.26}$$

$$|Z^{k+1;i,l+1}\varphi_\gamma(\mathfrak{t},u,\vartheta)| \lesssim \delta^{1-i-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{5.27}$$

Finally, it is noted that under the assumptions $(\star)$, we have obtained not only the estimates with M dependent bounds in Lemma 5.1, Proposition 5.1, 5.2 and 5.3, but also more refined results independent of M in Corollary 5.1 and Corollary 5.2. On the other hand, if starting with Corollary 5.1-5.2 and repeating the above analysis as in Section 5, one can improve the conclusions in Lemma 5.1, Proposition 5.1, 5.2 and 5.3 such that all the related constants are independent of M when $k \le N-3$. Therefore, from now on, we may apply these estimates without the constant M since N could be chosen large enough.

6 Energy estimates for the linearized equation

To close the bootstrap assumptions ($\star$), one needs further refined estimates than those derived in Section 5. To this end, we plan to construct some suitable energies for higher order derivatives of φ_γ. Note that φ_γ satisfies the nonlinear equation (3.41), and each derivative of φ_γ also fulfills similar equation with the same metric. Thus, in the section, we focus on the energy estimates for any smooth function Ψ solving the following linear equation

$$\mu\Box_g\Psi = \Phi \tag{6.1}$$

for a given function Φ, where Ψ and its derivatives vanish on $C_0^{\mathfrak{t}}$. The following divergence theorem in $D^{\mathfrak{t},u}$ will be used to find the related energies adapting to our problem.

Lemma 6.1. *For any vector field J, it holds that*

$$\begin{aligned}\int_{D^{\mathfrak{t},u}}\mu\mathscr{D}_\alpha J^\alpha d\nu_{\not{g}}du'd\tau =& \int_{\Sigma_{\mathfrak{t}}^u}\big(-J_T-\mu J_{\mathring{L}}\big)d\nu_{\not{g}}du' - \int_{\Sigma_{t_0}^u}\big(-J_T-\mu J_{\mathring{L}}\big)d\nu_{\not{g}}du'\\ &-\int_{C_u^{\mathfrak{t}}}J_{\mathring{L}}d\nu_{\not{g}}d\tau.\end{aligned} \tag{6.2}$$

Proof. This follows from the proof of Lemma 10.12 in [36]. □

The vectorfields J's in (6.2) will be chosen as

$$J_1 := -\varrho^{2m}g^{\alpha\kappa}Q_{\kappa\beta}\mathring{L}^\beta\partial_\alpha, \tag{6.3}$$

$$J_2 := -g^{\alpha\kappa}Q_{\kappa\beta}\mathring{\underline{L}}^\beta\partial_\alpha, \tag{6.4}$$

$$J_3 := \big(\frac{1}{2}\varrho^{2m-1}\Psi\mathscr{D}^\alpha\Psi - \frac{1}{4}\Psi^2\mathscr{D}^\alpha(\varrho^{2m-1})\big)\partial_\alpha, \tag{6.5}$$

where $m\in(\frac{1}{2},\frac{3}{4})$ is a fixed constant and Q is the *energy-momentum tensor field* of Ψ defined as

$$\begin{aligned}Q_{\alpha\beta} = Q_{\alpha\beta}[\Psi] :=& (\partial_\alpha\Psi)(\partial_\beta\Psi) - \frac{1}{2}g_{\alpha\beta}g^{\nu\lambda}(\partial_\nu\Psi)(\partial_\lambda\Psi)\\ =& (\partial_\alpha\Psi)(\partial_\beta\Psi) - \frac{1}{2}g_{\alpha\beta}\big\{|\not{d}\Psi|^2 - \mu^{-1}(\mathring{\underline{L}}\Psi)(\mathring{L}\Psi)\big\}.\end{aligned}$$

Each term in (6.2) will be analyzed respectively.

We start with the estimates of right hand side of (6.2). Note that the components of $Q_{\alpha\beta}$ relative to $\{\mathring{L},\mathring{\underline{L}},X\}$ are

$$\begin{aligned}&Q_{\mathring{L}\mathring{L}} = (\mathring{L}\Psi)^2,\quad Q_{\mathring{\underline{L}}\mathring{\underline{L}}} = (\mathring{\underline{L}}\Psi)^2,\quad Q_{\mathring{L}\mathring{\underline{L}}} = \mu|\not{d}\Psi|^2,\\ &Q_{\mathring{L}X} = (\mathring{L}\Psi)(\not{d}_X\Psi),\quad Q_{\mathring{\underline{L}}X} = (\mathring{\underline{L}}\Psi)(\not{d}_X\Psi),\\ &Q_{XX} = \frac{1}{2}(\not{d}_X\Psi)(\not{d}_X\Psi) + \frac{1}{2}\not{g}_{XX}\mu^{-1}(\mathring{\underline{L}}\Psi)(\mathring{L}\Psi).\end{aligned}$$

Then it follows from (6.2)-(6.4) that

$$\begin{aligned}\int_{D^{\mathfrak{t},u}}\mu\mathscr{D}_\alpha J_1{}^\alpha =& \int_{\Sigma_{\mathfrak{t}}^u}\frac{1}{2}\mu\varrho^{2m}\big((\mathring{L}\Psi)^2+|\not{d}\Psi|^2\big) - \int_{\Sigma_{t_0}^u}\frac{1}{2}\mu\varrho^{2m}\big((\mathring{L}\Psi)^2+|\not{d}\Psi|^2\big)\\ &+\int_{C_u^{\mathfrak{t}}}\varrho^{2m}(\mathring{L}\Psi)^2\end{aligned} \tag{6.6}$$

and

$$\int_{D^{\mathfrak{t},u}} \mu \mathscr{D}_\alpha J_2{}^\alpha = \int_{\Sigma_{\mathfrak{t}}^u} \frac{1}{2}\big((\mathring{\underline{L}}\Psi)^2 + \mu^2 |\not{d}\Psi|^2\big) - \int_{\Sigma_{t_0}^u} \frac{1}{2}\big((\mathring{\underline{L}}\Psi)^2 + \mu^2 |\not{d}\Psi|^2\big) + \int_{C_u^{\mathfrak{t}}} \mu |\not{d}\Psi|^2. \tag{6.7}$$

While (6.2) and (6.5) yield

Lemma 6.2. *It holds that*

$$\begin{aligned}\int_{D^{\mathfrak{t},u}} \mu \mathscr{D}_\alpha J_3{}^\alpha = &\int_{\Sigma_{\mathfrak{t}}^u} \big(-\frac{1}{2}\varrho^{2m-1}\Psi(\mu\mathring{L}\Psi) + \frac{1}{4}\varrho^{2m-2}\Psi^2((2m-1)(\mu-2) + \varrho\mu tr\tilde{\theta})\big) \\ &- \int_{\Sigma_{t_0}^u} \big(-\frac{1}{2}\varrho^{2m-1}\Psi(\mu\mathring{L}\Psi) + \frac{1}{4}\varrho^{2m-2}\Psi^2((2m-1)(\mu-2) + \varrho\mu tr\tilde{\theta})\big) \\ &+ \int_{C_u^{\mathfrak{t}}} \big(-\frac{1}{4} tr\lambda \cdot \varrho^{2m-1}\Psi^2 - \varrho^{2m-1}\Psi\mathring{L}\Psi\big).\end{aligned} \tag{6.8}$$

Proof. Applying (6.2) directly to J_3 yields

$$\begin{aligned}\int_{D^{\mathfrak{t},u}} \mu \mathscr{D}_\alpha J_3{}^\alpha = &\int_{\Sigma_{\mathfrak{t}}^u} \Big(-\frac{1}{2}\varrho^{2m-1}\Psi(T\Psi) - \frac{1}{2}\varrho^{2m-1}\Psi(\mu\mathring{L}\Psi) + \frac{2m-1}{4}\varrho^{2m-2}\Psi^2(\mu-1)\Big) \\ &- \int_{\Sigma_{t_0}^u} \Big(-\frac{1}{2}\varrho^{2m-1}\Psi(T\Psi) - \frac{1}{2}\varrho^{2m-1}\Psi(\mu\mathring{L}\Psi) + \frac{2m-1}{4}\varrho^{2m-2}\Psi^2(\mu-1)\Big) \\ &+ \int_{C_u^{\mathfrak{t}}} \big(-\frac{1}{2}\varrho^{2m-1}\Psi(\mathring{L}\Psi) + \frac{2m-1}{4}\varrho^{2m-2}\Psi^2\big).\end{aligned} \tag{6.9}$$

Note that Ψ vanishes on $u=0$. Then

$$\begin{aligned}\int_{S_{\mathfrak{t},u}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}} &= \int_0^u \Big(\frac{\partial}{\partial u'} \int_{S_{\mathfrak{t},u'}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}}\Big) du' \\ &= \int_{\Sigma_{\mathfrak{t}}^u} \Big(-(2m-1)\varrho^{2m-2}\Psi^2 + 2\varrho^{2m-1}\Psi T\psi + \varrho^{2m-1}\mu \mathrm{tr}\tilde{\theta}\Psi^2\Big),\end{aligned}$$

which is

$$\int_{\Sigma_{\mathfrak{t}}^u} -\frac{1}{2}\varrho^{2m-1}\Psi(T\Psi) = -\frac{1}{4}\int_{S_{\mathfrak{t},u}} \varrho^{2m-1}\Psi^2 + \frac{1}{4}\int_{\Sigma_{\mathfrak{t}}^u} \big(-(2m-1)\varrho^{2m-2}\Psi^2 + \varrho^{2m-1}\mu \mathrm{tr}\tilde{\theta}\Psi^2\big). \tag{6.10}$$

Taking $t = t_0$ in (6.10) gives

$$\int_{\Sigma_{t_0}^u} -\frac{1}{2}\varrho^{2m-1}\Psi(T\Psi) = -\frac{1}{4}\int_{S_{t_0,u}} \varrho^{2m-1}\Psi^2 + \frac{1}{4}\int_{\Sigma_{t_0}^u} \big(-(2m-1)\varrho^{2m-2}\Psi^2 + \varrho^{2m-1}\mu \mathrm{tr}\tilde{\theta}\Psi^2\big). \tag{6.11}$$

Thanks to the following identity

$$\begin{aligned}\int_{S_{\mathfrak{t},u}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}} &= \int_{S_{t_0,u}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}} + \int_{t_0}^{\mathfrak{t}} \frac{\partial}{\partial \tau}\Big(\int_{S_{\tau,u}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}}\Big) d\tau \\ &= \int_{S_{t_0,u}} \varrho^{2m-1}\Psi^2 d\nu_{\not{g}} + \int_{C_u^{\mathfrak{t}}} \Big(\mathring{L}(\varrho^{2m-1}\Psi^2) + \mathrm{tr}\lambda \cdot \varrho^{2m-1}\Psi^2\Big) d\nu_{\not{g}} d\tau,\end{aligned}$$

then (6.10) becomes

$$\begin{aligned}\int_{\Sigma_{\mathfrak{t}}^{u}}-\frac{1}{2}\varrho^{2m-1}\Psi(T\Psi)=&-\frac{1}{4}\int_{S_{t_0,u}}\varrho^{2m-1}\Psi^2-\frac{1}{4}\int_{C_u^{\mathfrak{t}}}\Big(\mathring{L}(\varrho^{2m-1}\Psi^2)+\mathrm{tr}\lambda\cdot\varrho^{2m-1}\Psi^2\Big)\\&+\frac{1}{4}\int_{\Sigma_{\mathfrak{t}}^{u}}\big(-(2m-1)\varrho^{2m-2}\Psi^2+\varrho^{2m-1}\mu\mathrm{tr}\tilde{\theta}\Psi^2\big).\end{aligned}\tag{6.12}$$

Substituting (6.12) and (6.11) into (6.9) yields (6.8). □

It follows from (6.8) and (6.6) together with (4.2) that

$$\begin{aligned}&\int_{D^{\mathfrak{t},u}}\mu(\mathscr{D}_\alpha J_1{}^\alpha-\mathscr{D}_\alpha J_3{}^\alpha)\\=&\frac{1}{2}\int_{\Sigma_{\mathfrak{t}}^{u}}\Big(\mu\varrho^{2m}|\not{d}\Psi|^2+\mu\varrho^{2m}(\mathring{L}\Psi)^2+\mu\varrho^{2m-1}\Psi(\mathring{L}\Psi)+m\varrho^{2m-2}\Psi^2+O(\delta^{-\varepsilon_0}\varrho^{2m-3/2}\Psi^2)\Big)\\&-\frac{1}{2}\int_{\Sigma_{t_0}^{u}}\Big(\mu\varrho^{2m}|\not{d}\Psi|^2+\mu\varrho^{2m}(\mathring{L}\Psi)^2+\mu\varrho^{2m-1}\Psi(\mathring{L}\Psi)+m\varrho^{2m-2}\Psi^2+O(\delta^{-\varepsilon_0}\varrho^{2m-3/2}\Psi^2)\Big)\\&+\int_{C_u^{\mathfrak{t}}}\big((\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+O(\delta^{1-\varepsilon_0}\varrho^{2m-5/2}\Psi^2)\big),\end{aligned}\tag{6.13}$$

here one has used the estimate $|\mu\mathrm{tr}\tilde{\theta}|\lesssim\delta^{-\varepsilon_0}\mathfrak{t}^{-1/2}$ due to (3.31) and the estimates in Section 5.

In order to estimate $\delta^{-\varepsilon_0}\int_{\Sigma_{\mathfrak{t}}^{u}}\varrho^{2m-3/2}\Psi^2$ and $\delta^{1-\varepsilon_0}\int_{C_u^{\mathfrak{t}}}\varrho^{2m-5/2}\Psi^2$ in (6.13), one needs the following inequality.

Lemma 6.3. *Under the assumptions* $(\star)$*, it holds that for any* $f\in C^1(D^{\mathfrak{t},u})$ *which vanishes on* C_0^t*,*

$$\int_{\Sigma_{\mathfrak{t}}^{u}}\mu f^2+\delta\int_{C_u^{\mathfrak{t}}}\varrho^{-1}f^2\lesssim\int_{\Sigma_{t_0}^{u}}\mu f^2+\int_{D^{\mathfrak{t},u}}\big(|\mathring{L}(f^2)|+\delta|T(\varrho^{-1}f^2)|\big).\tag{6.14}$$

Proof. It follows from Lemma 3.4 in [30] that for any vectorfield J,

$$\mu\mathscr{D}_\alpha J^\alpha=-\mathring{L}(\mu J_{\mathring{L}}+J_T)-T(J_{\mathring{L}})+\not{d}iv(\mu\not{J})-\mu(\mathrm{tr}\tilde{\theta}+\mathrm{tr}\lambda)J_{\mathring{L}}-\mathrm{tr}\lambda J_T,\tag{6.15}$$

where $\not{J}$ is the projection of J on $S_{\mathfrak{t},u}$. Take $J=f^2\mathring{L}$ and $J=\delta\varrho^{-1}\mu^{-1}f^2T$ in (6.15) respectively, and then use (6.2) to obtain

$$\int_{\Sigma_{\mathfrak{t}}^{u}}\mu f^2=\int_{\Sigma_{t_0}^{u}}\mu f^2+\int_{D^{\mathfrak{t},u}}\big((\mathring{L}\mu+\mu\mathrm{tr}\lambda)f^2+\mu\mathring{L}(f^2)\big),\tag{6.16}$$

$$\delta\int_{C_u^{\mathfrak{t}}}\varrho^{-1}f^2=\delta\int_{D^{\mathfrak{t},u}}\big(\mu\varrho^{-1}\mathrm{tr}\tilde{\theta}f^2+T(\varrho^{-1}f^2)\big).\tag{6.17}$$

By adding (6.16) and (6.17) together with the estimates on the coefficients in the right hand sides of (6.16)-(6.17), one arrives at

$$\int_{\Sigma_{\mathfrak{t}}^{u}}\mu f^2+\delta\int_{C_u^{\mathfrak{t}}}\varrho^{-1}f^2\lesssim\int_{\Sigma_{t_0}^{u}}\mu f^2+\int_{D^{\mathfrak{t},u}}\big(\varrho^{-1}f^2+\mu|\mathring{L}(f^2)|+\delta|T(\varrho^{-1}f^2)|\big).$$

Hence (6.14) follows directly from Gronwall's inequality. □

Let $f=\delta^{-\varepsilon_0/2}\varrho^{m-3/4}\Psi$ in (6.14). Using the facts that

$$|\mathring{L}(f^2)|\lesssim \varrho^{-1}f^2+\delta^{-\varepsilon_0}\varrho^{2m-1/2}|\mathring{L}\Psi|^2$$

and

$$|T(\varrho^{-1}f^2)|\lesssim \delta^{-1}\varrho^{-1}f^2+\delta^{1-\varepsilon_0}\varrho^{2m-5/2}|T\Psi|^2,$$

one then can get

$$\begin{aligned}&\int_{\Sigma_{\mathfrak{t}}^u}\mu\delta^{-\varepsilon_0}\varrho^{2m-3/2}\Psi^2+\int_{C_u^{\mathfrak{t}}}\delta^{1-\varepsilon_0}\varrho^{2m-5/2}\Psi^2\\ \lesssim&\int_{\Sigma_{t_0}^u}\mu\delta^{-\varepsilon_0}\varrho^{2m-3/2}\Psi^2+\int_{D^{\mathfrak{t},u}}\big(\delta^{-\varepsilon_0}\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\delta^{2-\varepsilon_0}\varrho^{2m-5/2}|T\Psi|^2\big).\end{aligned}\tag{6.18}$$

Substituting (6.18) into (6.13) yields

$$\begin{aligned}&\int_{D^{\mathfrak{t},u}}\mu\big(\mathscr{D}_\alpha J_1{}^\alpha-\mathscr{D}_\alpha J_3{}^\alpha\big)\\ \gtrsim&\int_{\Sigma_{\mathfrak{t}}^u}\big(\varrho^{2m}|\not{d}\Psi|^2+\varrho^{2m}(\mathring{L}\Psi)^2+\varrho^{2m-2}\Psi^2\big)-\int_{\Sigma_{t_0}^u}\big(|\not{d}\Psi|^2+(\mathring{L}\Psi)^2+\delta^{-\varepsilon_0}\Psi^2\big)\\ &+\int_{C_u^{\mathfrak{t}}}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2-\int_{D^{\mathfrak{t},u}}\delta^{-\varepsilon_0}\big(\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\delta^2\varrho^{2m-5/2}(\mathring{\underline{L}}\Psi)^2\big).\end{aligned}\tag{6.19}$$

By (6.19) and the identity (6.7), it is natural to define the following energies and fluxes

$$E_1[\Psi](\mathfrak{t},u):=\int_{\Sigma_{\mathfrak{t}}^u}\big(\varrho^{2m}(\mathring{L}\Psi)^2+\varrho^{2m}|\not{d}\Psi|^2+\varrho^{2m-2}\Psi^2\big),\tag{6.20}$$

$$E_2[\Psi](\mathfrak{t},u):=\int_{\Sigma_{\mathfrak{t}}^u}((\mathring{\underline{L}}\Psi)^2+|\not{d}\Psi|^2),\tag{6.21}$$

$$F_1[\Psi](\mathfrak{t},u):=\int_{C_u^{\mathfrak{t}}}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2,\tag{6.22}$$

$$F_2[\Psi](\mathfrak{t},u):=\int_{C_u^{\mathfrak{t}}}|\not{d}\Psi|^2.\tag{6.23}$$

Next, we treat the left hand side of (6.2). Set

$$V_1=\varrho^{2m}\mathring{L},\qquad V_2=\mathring{\underline{L}}.$$

Direct computations give that

$$\mu\mathscr{D}_\alpha J_1{}^\alpha=-\Phi\varrho^{2m}(\mathring{L}\Psi)-\frac{1}{2}\mu Q^{\alpha\beta}[\Psi]^{(V_1)}\pi_{\alpha\beta},\tag{6.24}$$

$$\mu\mathscr{D}_\alpha J_2{}^\alpha=-\Phi(\mathring{\underline{L}}\Psi)-\frac{1}{2}\mu Q^{\alpha\beta}[\Psi]^{(V_2)}\pi_{\alpha\beta},\tag{6.25}$$

$$\begin{aligned}\mu\mathscr{D}_\alpha J_3{}^\alpha=&\frac{1}{2}\varrho^{2m-1}\Psi\Phi+\Big(-\frac{1}{2}\varrho^{2m-1}(\mathring{L}\Psi)(\mathring{\underline{L}}\Psi)+\frac{1}{2}\mu\varrho^{2m-1}|\not{d}\Psi|^2\Big)\\ &+\frac{1}{4}\Psi^2(2m-1)\varrho^{2m-2}\Big((2m-2)\varrho^{-1}(\mu-2)+\mathring{L}\mu+\mu\mathrm{tr}\tilde{\theta}+(\mu-1)\mathrm{tr}\lambda\Big).\end{aligned}\tag{6.26}$$

Recall that the components of ${}^{(V_i)}\pi_{\alpha\beta}$ in the frame $\{\mathring{L},\underline{\mathring{L}},X\}$ have been given in (3.23)-(3.25). To compute $Q^{\alpha\beta(V_i)}\pi_{\alpha\beta}$ in (6.24)-(6.25), we can use the components of the metric in the frame $\{\mathring{L},\underline{\mathring{L}},X\}$, given in (3.38), to derive directly that

$$
\begin{aligned}
&-\frac{1}{2}\mu Q^{\alpha\beta}[\Psi]^{(V_1)}\pi_{\alpha\beta}\\
=&-\frac{1}{4}\mu^{-1(V_1)}\pi_{\mathring{L}\underline{\mathring{L}}}Q_{\mathring{L}\underline{\mathring{L}}}-\frac{1}{8}\mu^{-1(V_1)}\pi_{\underline{\mathring{L}}\mathring{L}}Q_{\mathring{L}\mathring{L}}+\frac{1}{2}{}^{(V_1)}\pi_{\underline{\mathring{L}}}^{X}Q_{\mathring{L}X}-\frac{1}{2}\mu^{(V_1)}\not\pi^{XX}Q_{XX}\\
=&(\frac{1}{2}\varrho^{2m}\mathring{L}\mu+\mu(m-\frac{1}{2})\varrho^{2m-1})|\not d\Psi|^2+\big(m(\mu-2)\varrho^{2m-1}-\frac{1}{2}\varrho^{2m}\mathring{L}\mu\big)(\mathring{L}\Psi)^2+\varrho^{2m}(\not d^X\mu\\
&+2c^{-1}\mu\tilde{T}^a\not d^X\varphi_a)(\mathring{L}\Psi)(\not d_X\Psi)-\frac{1}{2}\mu\varrho^{2m}\mathrm{tr}\check{\chi}|\not d\Psi|^2-\frac{1}{2}\varrho^{2m}\mathrm{tr}\chi(\mathring{L}\Psi)(\underline{\mathring{L}}\Psi)
\end{aligned}
\tag{6.27}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\mu Q^{\alpha\beta}[\Psi]^{(V_2)}\pi_{\alpha\beta}\\
=&\frac{1}{2}(\underline{\mathring{L}}\mu+\mu\mathring{L}\mu+c^{-1}\mu Tc+c^{-1}\mu^2\mathring{L}c+\mu^2\mathrm{tr}\lambda-2c^{-1}\mu^2\not d_X x^a\cdot\not d^X\varphi_a)|\not d\Psi|^2\\
&-(2\mu c^{-1}\tilde{T}^a\not d^X\varphi_a+\not d^X\mu)(\not d_X\Psi)(\underline{\mathring{L}}\Psi)-\mu\not d^X\mu(\mathring{L}\Psi)(\not d_X\Psi)\\
&+\big(\frac{1}{2}c^{-1}Tc+\frac{1}{2}\mu c^{-1}\mathring{L}c-c^{-1}\mu\not d^X x^a\cdot\not d_X\varphi_a+\frac{1}{2}\mu\mathrm{tr}\chi\big)(\mathring{L}\Psi)(\underline{\mathring{L}}\Psi).
\end{aligned}
\tag{6.28}
$$

We first treat the terms involving J_1 and J_3.

Combining (6.24) and (6.26) and using (6.27), $\frac{1}{2}<m<\frac{3}{4}$ and $\mu<2$, one can get by using the estimates in Section 5 that

$$
\begin{aligned}
&\int_{D^{\mathfrak{t},u}}\big(\mu\mathscr{D}_\alpha J_1{}^\alpha-\mu\mathscr{D}_\alpha J_3{}^\alpha\big)\\
=&-\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)+\int_{D^{\mathfrak{t},u}}\big(\mu(m-1)\varrho^{2m-1}+\frac{1}{2}\varrho^{2m}\mathring{L}\mu-\frac{1}{2}\mu\varrho^{2m}\mathrm{tr}\check{\lambda}\big)|\not d\Psi|^2\\
&+\int_{D^{\mathfrak{t},u}}\varrho^{2m}\big(m\varrho^{-1}(\mu-2)-\frac{1}{2}\mathring{L}\mu\big)(\mathring{L}\Psi)^2+\int_{D^{\mathfrak{t},u}}\varrho^{2m}(\not d_X\mu+2c^{-1}\mu\tilde{T}^a\not d_X\varphi_a)(\mathring{L}\Psi)(\not d^X\Psi)\\
&-\int_{D^{\mathfrak{t},u}}\frac{1}{4}\Psi^2(2m-1)\varrho^{2m-2}\Big((2m-2)\varrho^{-1}(\mu-2)+\mathring{L}\mu+\mu\mathrm{tr}\tilde{\theta}+(\mu-1)\mathrm{tr}\lambda\Big)\\
&-\int_{D^{\mathfrak{t},u}}\frac{1}{2}\varrho^{2m}\mathrm{tr}\check{\lambda}(\mathring{L}\Psi)(\underline{\mathring{L}}\Psi)\\
\leq&-\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)+\int_{D^{\mathfrak{t},u}}\frac{1}{2}\varrho^{2m}\big(\mathring{L}\mu-\mu\mathrm{tr}\check{\lambda}\big)|\not d\Psi|^2-\int_{D^{\mathfrak{t},u}}\frac{1}{2}\varrho^{2m}\mathring{L}\mu(\mathring{L}\Psi)^2\\
&+\int_{D^{\mathfrak{t},u}}\varrho^{2m}(\not d_X\mu+2c^{-1}\mu\tilde{T}^a\not d_X\varphi_a)(\mathring{L}\Psi)(\not d^X\Psi)-\int_{D^{\mathfrak{t},u}}\frac{1}{2}\varrho^{2m}\mathrm{tr}\check{\lambda}(\mathring{L}\Psi)(\underline{\mathring{L}}\Psi)\\
&-\int_{D^{\mathfrak{t},u}}\frac{1}{4}(2m-1)\varrho^{2m-2}\Psi^2(\mathring{L}\mu+\mu\mathrm{tr}\tilde{\theta}+(\mu-1)\mathrm{tr}\lambda)\\
\lesssim&\Big|\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)\Big|+\int_{D^{\mathfrak{t},u}}\big(\delta^{1-\varepsilon_0}\varrho^{2m-3/2}|\not d\Psi|^2+\delta^{2-\varepsilon_0}\varrho^{2m-5/2}|\underline{\mathring{L}}\Psi|^2\big)\\
&+\int_{D^{\mathfrak{t},u}}\delta^{-\varepsilon_0}\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\int_{D^{\mathfrak{t},u}}\delta^{-\varepsilon_0}\varrho^{2m-5/2}\Psi^2.
\end{aligned}
\tag{6.29}
$$

The last integral in (6.29) can be estimated by taking $f=\varrho^{m-3/4}\Psi$ in (6.14) as

$$\begin{aligned}\delta\int_{C_u^t}\varrho^{2m-5/2}\Psi^2\lesssim&\int_{\Sigma_{t_0}^u}\Psi^2+\int_{D^{\mathfrak{t},u}}\big(\varrho^{2m-1/2}|\mathring{L}\Psi|^2+\delta^2\varrho^{2m-5/2}|\mathring{\underline{L}}\Psi|^2+\varrho^{2m-5/2}\Psi^2\big)\\ \lesssim&\int_{\Sigma_{t_0}^u}\Psi^2+\int_{D^{\mathfrak{t},u}}\big(\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\delta^2\varrho^{2m-5/2}|\mathring{\underline{L}}\Psi|^2\big)\\ &+\int_{D^{\mathfrak{t},u}}\varrho^{2m-5/2}\Psi^2.\end{aligned}$$

This, together with Gronwall's inequality, yields

$$\delta\int_{C_u^t}\varrho^{2m-5/2}\Psi^2\lesssim\int_{\Sigma_{t_0}^u}\Psi^2+\int_{D^{\mathfrak{t},u}}\big(\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\delta^2\varrho^{2m-5/2}|\mathring{\underline{L}}\Psi|^2\big).$$

Thus

$$\int_{D^{\mathfrak{t},u}}\varrho^{2m-5/2}\Psi^2\lesssim\int_{\Sigma_{t_0}^u}\Psi^2+\int_{D^{\mathfrak{t},u}}\big(\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\delta^2\varrho^{2m-5/2}|\mathring{\underline{L}}\Psi|^2\big).\tag{6.30}$$

Substituting (6.30) into (6.29), we arrive at

$$\begin{aligned}&\int_{D^{\mathfrak{t},u}}\big(\mu\mathscr{D}_\alpha J_1{}^\alpha-\mu\mathscr{D}_\alpha J_3{}^\alpha\big)\\ \lesssim&|\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)|+\int_{D^{\mathfrak{t},u}}\big(\delta^{1-\varepsilon_0}\varrho^{2m-3/2}|\not{d}\Psi|^2+\delta^{2-\varepsilon_0}\varrho^{2m-5/2}|\mathring{\underline{L}}\Psi|^2\big)\\ &+\int_{D^{\mathfrak{t},u}}\delta^{-\varepsilon_0}\varrho^{-1/2}(\varrho^m\mathring{L}\Psi+\frac{1}{2}\varrho^{m-1}\Psi)^2+\int_{\Sigma_{t_0}^u}\delta^{-\varepsilon_0}\Psi^2.\end{aligned}\tag{6.31}$$

It follows from (6.19), (6.31) and Gronwall's inequality that

$$\begin{aligned}E_1[\Psi](\mathfrak{t},u)+F_1[\Psi](\mathfrak{t},u)\lesssim&E_1[\Psi](t_0,u)+\int_{\Sigma_{t_0}}\delta^{-\varepsilon_0}\Psi^2+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}E_2[\Psi](\tau,u)d\tau\\ &+|\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)|.\end{aligned}\tag{6.32}$$

It remains to treat (6.25) involving J_2. Recalling (6.28) and estimating each coefficient in it by using Proposition 5.3, one can get

$$\begin{aligned}&-\int_{D^{\mathfrak{t},u}}\frac{1}{2}\mu Q^{\alpha\beta}[\Psi]^{(V_2)}\pi_{\alpha\beta}\\ \lesssim&\delta^{-\varepsilon_0}\int_{D^{\mathfrak{t},u}}\big(|\not{d}\Psi|^2+\delta\tau^{-1}|\mathring{\underline{L}}\Psi|\cdot|\not{d}\Psi|+\delta\tau^{-1}|\mathring{L}\Psi|\cdot|\not{d}\Psi|+\tau^{-1/2}|\mathring{L}\Psi|\cdot|\underline{\mathring{L}}\Psi|\big)\\ \lesssim&\delta^{-\varepsilon_0}\int_0^u F_2[\Psi](\mathfrak{t},u')du'+\int_{t_0}^{\mathfrak{t}}\tau^{-m-\frac{1}{2}}E_2[\Psi](\tau,u)d\tau+\delta^{-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-m-\frac{1}{2}}E_1[\Psi](\tau,u)d\tau.\end{aligned}\tag{6.33}$$

Then, it follows from (6.7), (6.21), (6.23), (6.25), (6.33) and Gronwall's inequality that

$$E_2[\Psi](t,u)+F_2[\Psi](\mathfrak{t},u)\lesssim E_2[\Psi](t_0,u)+\int_{D^{\mathfrak{t},u}}|\Phi|\cdot|\mathring{\underline{L}}\Psi|+\delta^{-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-m-\frac{1}{2}}E_1[\Psi](\tau,u)d\tau.\tag{6.34}$$

Combining (6.32) with (6.34) and using Gronwall's inequality again, due to $m\in(\frac12,\frac34)$, we obtain finally that

$$\begin{aligned}&\delta E_2[\Psi](\mathfrak{t},u)+\delta F_2[\Psi](\mathfrak{t},u)+E_1[\Psi](\mathfrak{t},u)+F_1[\Psi](\mathfrak{t},u)\\ \lesssim&\delta E_2[\Psi](t_0,u)+E_1[\Psi](t_0,u)+\int_{\Sigma_{t_0}}\delta^{-\varepsilon_0}\Psi^2+\delta\int_{D^{\mathfrak{t},u}}|\Phi\cdot\mathring{\underline{L}}\Psi|\\ &+|\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac12\varrho^{2m-1}\Psi)|.\end{aligned}\tag{6.35}$$

(6.35) will be used to derive the energy estimates for φ_γ and its derivatives. To this end, we will choose $\Psi=\Psi_\gamma^{k+1}:=Z^{k+1}\varphi_\gamma$ and then $\Phi=\Phi_\gamma^{k+1}:=\mu\Box_g\Psi_\gamma^{k+1}$ ($k\le 2N-6$) in (6.35). Note that

$$\begin{aligned}\Phi_\gamma^{k+1}&=\mu\Box_g\Psi_\gamma^{k+1}=\mu[\Box_g,Z]\Psi_\gamma^k+Z\big(\mu\Box_g\Psi_\gamma^k\big)-(Z\mu)\Box_g\Psi_\gamma^k\\&=\mu\text{div}^{(Z)}C_\gamma^k+(Z+{}^{(Z)}\Lambda)\Phi_\gamma^k,\end{aligned}\tag{6.36}$$

where

$$\begin{aligned}&{}^{(Z)}C_\gamma^k=\big({}^{(Z)}\pi^{\nu\beta}-\frac12g^{\nu\beta}\text{tr}_g{}^{(Z)}\pi\big)\partial_\nu\Psi_\gamma^k\partial_\beta,\\&{}^{(Z)}\Lambda=\frac12\text{tr}_g{}^{(Z)}\pi-\mu^{-1}Z\mu,\\&\Psi_\gamma^0=\varphi_\gamma,\quad\Phi_\gamma^0=\mu\Box_g\varphi_\gamma\end{aligned}\tag{6.37}$$

with $\text{tr}_g{}^{(Z)}\pi=g^{\alpha\beta(Z)}\pi_{\alpha\beta}$ and Φ_γ^0 being the right hand side of (3.41). Consequently, for $\Psi_\gamma^{k+1}=Z_{k+1}Z_k\cdots Z_1\varphi_\gamma$ with $Z_i\in\{\varrho\mathring{L},T,R\}$, we can derive by (6.36) and an induction argument that

$$\begin{aligned}\Phi_\gamma^{k+1}=&\sum_{j=1}^k\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)\big(\mu\text{div}^{(Z_{k+1-j})}C_\gamma^{k-j}\big)\\&+\mu\text{div}^{(Z_{k+1})}C_\gamma^k+\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_1+{}^{(Z_1)}\Lambda\big)\Phi_\gamma^0,\qquad k\ge1,\\ \Phi_\gamma^1=&\big(Z_1+{}^{(Z_1)}\Lambda\big)\Phi_\gamma^0+\mu\text{div}^{(Z_1)}C_\gamma^0.\end{aligned}\tag{6.38}$$

In order to estimate ${}^{(Z)}\Lambda$ and $\mu\text{div}^{(Z)}C_\gamma^k$, it is more convenient to rewrite

$$\text{tr}_g{}^{(Z)}\pi=-2\mu^{-1(Z)}\pi_{\mathring{L}T}+\text{tr}_{\not g}{}^{(Z)}\not\pi.$$

It follows from (3.23)-(3.25) that

$$\begin{aligned}&{}^{(T)}\Lambda=-\frac12c^{-1}Tc-\frac12c^{-1}\mu\mathring{L}c+c^{-1}\mu\not dx^a\cdot\not d\varphi_a-\mu\text{tr}_{\not g}\lambda,\\&{}^{(\varrho\mathring{L})}\Lambda=\varrho\text{tr}\check{\lambda}+2,\\&{}^{(R)}\Lambda=\upsilon\text{tr}\lambda+\frac12c^{-1}\upsilon(\mathring{L}c)-c^{-1}\upsilon\not dx^a\cdot\not d\varphi_a-\frac12c^{-1}Rc.\end{aligned}\tag{6.39}$$

In addition, in the null frame $\{\mathring{\underline{L}},\mathring{L},X\}$, the term $\mu\text{div}^{(Z)}C_\gamma^k$ can be written as

$$\begin{aligned}\mu\text{div}^{(Z)}C_\gamma^k=&-\frac12\mathring{L}\big({}^{(Z)}C^k_{\gamma,\mathring{\underline{L}}}\big)-\frac12\mathring{\underline{L}}\big({}^{(Z)}C^k_{\gamma,\mathring{L}}\big)+\not{\text{div}}\big(\mu{}^{(Z)}\not C_\gamma^k\big)\\&-\frac12\big(\mathring{L}\mu+\mu\text{tr}\lambda+2\mu\text{tr}\tilde\theta\big){}^{(Z)}C^k_{\gamma,\mathring{L}}-\frac12\text{tr}\lambda{}^{(Z)}C^k_{\gamma,\mathring{\underline{L}}},\end{aligned}\tag{6.40}$$

where

$$
\begin{aligned}
&{}^{(Z)}C^k_{\gamma,\mathring{L}} = {}^{(Z)}\not\pi_{\mathring{L}X}(\not d^X\Psi^k_\gamma) - \frac{1}{2}(\mathrm{tr}^{(Z)}\not\pi)\mathring{L}\Psi^k_\gamma,\\
&{}^{(Z)}C^k_{\gamma,\underline{\mathring{L}}} = -2({}^{(Z)}\pi_{LT} + \mu^{-1(Z)}\pi_{TT})(\mathring{L}\Psi^k_\gamma) + {}^{(Z)}\not\pi_{\underline{\mathring{L}}X}(\not d^X\Psi^k_\gamma) - \frac{1}{2}(\mathrm{tr}^{(Z)}\not\pi)\underline{\mathring{L}}\Psi^k_\gamma,\\
&\mu^{(Z)}\not C^k_{\gamma,X} = -\frac{1}{2}{}^{(Z)}\not\pi_{\underline{\mathring{L}}X}(\mathring{L}\Psi^k_\gamma) - \frac{1}{2}{}^{(Z)}\not\pi_{\mathring{L}X}(\underline{\mathring{L}}\Psi^k_\gamma) + {}^{(Z)}\pi_{\mathring{L}T}(\not d_X\Psi^k_\gamma) + \frac{1}{2}\mu\mathrm{tr}^{(Z)}\not\pi\not d_X\Psi^k_\gamma.
\end{aligned}
\tag{6.41}
$$

A direct substitution of (6.41) into (6.40) would result in a lengthy and complicated equation for $\mu\mathrm{div}^{(Z)}C^k_\gamma$. To overcome this difficulty and to get the desired estimates efficiently, we follow the ideas in [33] to decompose $\mu\mathrm{div}^{(Z)}C^k_\gamma$ as

$$
\mu\mathrm{div}^{(Z)}C^k_\gamma = {}^{(Z)}D^k_{\gamma,1} + {}^{(Z)}D^k_{\gamma,2} + {}^{(Z)}D^k_{\gamma,3},
\tag{6.42}
$$

where

$$
\begin{aligned}
{}^{(Z)}D^k_{\gamma,1} =& \frac{1}{2}\mathrm{tr}^{(Z)}\not\pi\big(\mathring{L}\underline{\mathring{L}}\Psi^k_\gamma + \frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\Psi^k_\gamma\big) - {}^{(Z)}\not\pi_{\underline{\mathring{L}}X}(\not d^X\mathring{L}\Psi^k_\gamma) - {}^{(Z)}\not\pi_{\mathring{L}X}(\not d^X\underline{\mathring{L}}\Psi^k_\gamma)\\
&+ ({}^{(Z)}\pi_{\mathring{L}T} + {}^{(Z)}\pi_{\tilde{T}T})(\mathring{L}^2\Psi^k_\gamma) + \frac{1}{2}\mu\mathrm{tr}^{(Z)}\not\pi\not\triangle\Psi^k_\gamma + {}^{(Z)}\pi_{\mathring{L}T}\not\triangle\Psi^k_\gamma,
\end{aligned}
\tag{6.43}
$$

$$
\begin{aligned}
{}^{(Z)}D^k_{\gamma,2} =& \mathring{L}({}^{(Z)}\pi_{\mathring{L}T} + {}^{(Z)}\pi_{\tilde{T}T})\mathring{L}\Psi^k_\gamma - \big(\frac{1}{2}\not\nabla^{X(Z)}\not\pi_{\underline{\mathring{L}}X} - \frac{1}{4}\underline{\mathring{L}}(\mathrm{tr}^{(Z)}\not\pi)\big)\mathring{L}\Psi^k_\gamma\\
&- (\frac{1}{2}\not\mathcal{L}_{\underline{\mathring{L}}}{}^{(Z)}\not\pi_{\mathring{L}X} - \not d_X{}^{(Z)}\pi_{\mathring{L}T})\not d^X\Psi^k_\gamma + \frac{1}{2}\not d\big(\mu\mathrm{tr}^{(Z)}\not\pi\big)\cdot\not d\Psi^k_\gamma\\
&- \frac{1}{2}(\not\nabla^{X(Z)}\not\pi_{\mathring{L}X})\underline{\mathring{L}}\Psi^k_\gamma - \frac{1}{2}(\not\mathcal{L}_{\mathring{L}}{}^{(Z)}\not\pi_{\underline{\mathring{L}}X})\not d^X\Psi^k_\gamma + \frac{1}{4}\mathring{L}(\mathrm{tr}^{(Z)}\not\pi)\underline{\mathring{L}}\Psi^k_\gamma,
\end{aligned}
\tag{6.44}
$$

$$
\begin{aligned}
{}^{(Z)}D^k_{\gamma,3} =& \Big\{\mathrm{tr}\lambda({}^{(Z)}\pi_{\mathring{L}T} + {}^{(Z)}\pi_{\tilde{T}T}) + (\frac{1}{4}\mu\mathrm{tr}\lambda + \frac{1}{2}\mu\mathrm{tr}\tilde{\theta})\mathrm{tr}^{(Z)}\not\pi - \frac{1}{2}\not d^X\mu^{(Z)}\not\pi_{\mathring{L}X}\Big\}\mathring{L}\Psi^k_\gamma\\
&+ \frac{1}{2}\Big\{(\mathrm{tr}^{(Z)}\not\pi)(\xi_X + \mu\zeta_X) + (\mu\mathrm{tr}\lambda - \mathring{L}\mu)^{(Z)}\not\pi_{\mathring{L}X} + \mathrm{tr}\lambda^{(Z)}\not\pi_{\underline{\mathring{L}}X}\Big\}\not d^X\Psi^k_\gamma.
\end{aligned}
\tag{6.45}
$$

Note that all the terms in ${}^{(Z)}D^k_{\gamma,1}$ are the products of the deformation tensor and the second order derivatives of Ψ^k_γ, except the first term containing the factor of the form $\mathring{L}\underline{\mathring{L}}\Psi^k_\gamma + \frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\Psi^k_\gamma$ (see (6.43)). It should be emphasized here that such a structure is crucial in our analysis since Ψ^k_γ is the derivative of φ_γ and by (4.3), $\mathring{L}\underline{\mathring{L}}\varphi_\gamma + \frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\varphi_\gamma = H_\gamma + \frac{1}{2}\mathrm{tr}\check{\lambda}\underline{\mathring{L}}\varphi_\gamma$ admits the better smallness and the faster time-decay rate than those for $\mathring{L}\underline{\mathring{L}}\varphi_\gamma$ and $\frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\varphi_\gamma$ separately. In addition, ${}^{(Z)}D^k_{\gamma,2}$ collects all the products of the first order derivatives of the deformation tensor and the first order derivatives of Ψ^k_γ, while ${}^{(Z)}D^k_{\gamma,3}$ denotes all the other terms.

The explicit expression for Φ^{k+1}_γ obtained from (6.38)-(6.39) and (6.42)-(6.45) will be used to estimate the corresponding last two integrals in (6.35). Due to the structure of (6.35) for $\Psi = \Psi^{k+1}_\gamma$, it is natural to define the corresponding weighted energy and flux as in [33]:

$$
E_{i,p+1}(\mathfrak{t},u) = \sum_{\gamma=0}^{2}\sum_{|\alpha|=p}\delta^{2l}E_i[Z^\alpha\varphi_\gamma](\mathfrak{t},u), \quad i=1,2,
\tag{6.46}
$$

$$
F_{i,p+1}(\mathfrak{t},u) = \sum_{\gamma=0}^{2}\sum_{|\alpha|=p}\delta^{2l}F_i[Z^\alpha\varphi_\gamma](\mathfrak{t},u), \quad i=1,2,
\tag{6.47}
$$

$$
E_{i,\le p+1}(\mathfrak{t},u) = \sum_{0\le n\le p}E_{i,n+1}(\mathfrak{t},u), \quad i=1,2,
\tag{6.48}
$$

$$F_{i,\leq p+1}(\mathfrak{t},u)=\sum_{0\leq n\leq p}F_{i,n+1}(\mathfrak{t},u),\quad i=1,2, \tag{6.49}$$

where l is the number of T in Z^α. We will treat these weighted energies in subsequent sections.

7 Higher order L^2 estimates for some quantities

In this section, we shall carry out the higher order L^2 estimates for some related quantities so that the last two terms of (6.35) can be absorbed by the left hand side, and hence the higher order energy estimates on (3.41) can be done. To this end, we first state two elementary lemmas, whose 3D analogous results can be found in Lemma 7.3 of [33] and Lemma 12.57 of [36] respectively.

Lemma 7.1. *For any function $\psi\in C^1(D^{\mathfrak{t},u})$ vanishing on C_0, it holds that for small $\delta>0$,*

$$\int_{S_{\mathfrak{t},u}}\psi^2\lesssim\delta\int_{\Sigma_{\mathfrak{t}}^u}\big(|\mathring{\underline{L}}\psi|^2+\mu^2|\mathring{L}\psi|^2\big), \tag{7.1}$$

$$\int_{\Sigma_{\mathfrak{t}}^u}\psi^2\lesssim\delta^2\int_{\Sigma_{\mathfrak{t}}^u}\big(|\mathring{\underline{L}}\psi|^2+\mu^2|\mathring{L}\psi|^2\big). \tag{7.2}$$

Therefore,

$$\int_{S_{\mathfrak{t},u}}\psi^2\lesssim\delta\big(E_2[\psi](\mathfrak{t},u)+\varrho^{-2m}E_1[\psi](\mathfrak{t},u)\big), \tag{7.3}$$

$$\int_{\Sigma_{\mathfrak{t}}^u}\psi^2\lesssim\delta^2\big(E_2[\psi](\mathfrak{t},u)+\varrho^{-2m}E_1[\psi](\mathfrak{t},u)\big). \tag{7.4}$$

Furthermore,

$$E_{1,\leq k+1}(\mathfrak{t},u)\lesssim\delta^2\varrho^{2m-2}E_{2,\leq k+2}(\mathfrak{t},u)+\delta^2\varrho^{-2}E_{1,\leq k+2}(\mathfrak{t},u). \tag{7.5}$$

Proof. For any function $f\in C^1(D^{\mathfrak{t},u})$ which vanishes on C_0, it follows from (3.23) and (3.31) that

$$\begin{aligned}\frac{\partial}{\partial u}\int_{S_{\mathfrak{t},u}}fd\nu_{\not g}&=\int_{S_{\mathfrak{t},u}}\big(Tf+\frac12\mathrm{tr}^{(T)}\not\pi\cdot f\big)d\nu_{\not g}\\&=\int_{S_{\mathfrak{t},u}}\Big(Tf+\frac12\big(-c^{-1}Tc-\mu c^{-1}\mathring{L}c+2c^{-1}\mu\not d_Xx^a(\not d^X\varphi_a)-2\mu\mathrm{tr}\lambda\big)f\Big).\end{aligned}$$

This leads to

$$\Big|\frac{\partial}{\partial u}\int_{S_{\mathfrak{t},u}}fd\nu_{\not g}\Big|\lesssim\int_{S_{\mathfrak{t},u}}(|Tf|+\delta^{-\varepsilon_0}|f|). \tag{7.6}$$

Setting $f=\psi^2$ and integrating (7.6) from 0 to u yield

$$\int_{S_{\mathfrak{t},u}}\psi^2d\nu_{\not g}\lesssim\int_{\Sigma_{\mathfrak{t}}^u}(|T\psi|\cdot|\psi|+\delta^{-\varepsilon_0}\psi^2)\lesssim\int_0^u\int_{S_{\mathfrak{t},u'}}\delta^{-1}\psi^2d\nu_{\not g}du'+\int_0^u\int_{S_{\mathfrak{t},u'}}\delta|T\psi|^2d\nu_{\not g}du'.$$

Thus Gronwall's inequality implies that

$$\int_{S_{\mathfrak{t},u}}\psi^2d\nu_{\not g}\lesssim\int_0^u\int_{S_{\mathfrak{t},u'}}\delta|T\psi|^2d\nu_{\not g}du'. \tag{7.7}$$

Since $T=\frac12\mathring{\underline{L}}-\frac12\mu\mathring{L}$, then (7.1) is proved. And hence (7.2) follow from (7.1). $\square$

Lemma 7.2. *Assume that $(\star)$ holds for small $\delta > 0$. Then for any $f \in C(D^{\mathfrak{t},u})$, $F(\mathfrak{t},u,\vartheta) := \int_{t_0}^{\mathfrak{t}} f(\tau,u,\vartheta)d\tau$ admits the following estimate:*

$$\|F\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \le (1+C\delta^{1-\varepsilon_0})\sqrt{\varrho(\mathfrak{t},u)}\int_{t_0}^{\mathfrak{t}} \frac{1}{\sqrt{\varrho(\tau,u)}}\|f\|_{L^2(\Sigma_\tau^u)}d\tau. \tag{7.8}$$

We now turn to derive L^2 estimates for higher order derivatives of $\check{\lambda}$, $\not{\pi}_{\mathring{L}X}$, $\not{\pi}_{XX}$, $\check{L}$, υ and x^j. To this end, as in Section 5, one starts with estimates for the rotational vector field.

Proposition 7.1. *Under the assumptions $(\star)$ with small $\delta > 0$, it holds that for $k \le 2N-6$,*

$$\begin{aligned}
&\|\not{\mathcal{L}}_R^k\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-3/2}\ln\mathfrak{t} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)} + \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\ln\mathfrak{t}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)},\\
&\|(R^{k+1}\check{L}^j,\ \not{\mathcal{L}}_R^{k}{}^{(R)}\not{\pi})\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \varrho^{1-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)},\\
&\|\not{\mathcal{L}}_R^{k+1}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{1/2} + \varrho^{1-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)},\\
&\|R^{k+1}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}t + \varrho^{2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)},
\end{aligned}$$

where $\tilde{E}_{i,\le k+2}(\mathfrak{t},u) = \sup_{t_0\le\tau\le\mathfrak{t}} E_{i,\le k+2}(\tau,u)$ $(i=1,2)$.

Proof. Since the L^∞ estimates for lower order derivatives of the above quantities have been given in Lemma 5.1, the corresponding rough L^2 estimates can be obtained by the fact $\|1\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \sqrt{\delta\varrho(\mathfrak{t},u)}$ (see Corollary 12.54 in [36]). In particular, one has

$$\begin{aligned}
&\|R\check{L}^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}, \quad \|{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0},\\
&\|\not{\mathcal{L}}_R\not{d}x^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{1/2}, \quad \|R\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}.
\end{aligned} \tag{7.9}$$

To treat higher order derivatives of $\check{L}^j$, one can use (3.29) for $R^{k+1}\check{L}^j$ as (5.6) in the proof of Lemma 5.1 by carrying out the following two steps:

• Keep the L^2 norms for the highest order derivatives of $\check{\lambda}$, $\check{L}$, x^j and φ, while apply the L^∞ estimates in Section 5 to treat the corresponding coefficients in these terms.

• Instead of controlling the L^2-norm of $Z^{\le k+1}\varphi$ $(Z \in \{R, \varrho\mathring{L}\})$ directly by $\varrho^{1-m}\sqrt{E_{1,\le k+1}(\mathfrak{t},u)}$, one can use (7.4) to obtain better smallness and decay rate for $||Z^{k+1}\varphi||_{L^2(\Sigma_{\mathfrak{t}}^u)}$, for example, $\|R^{\le k}\mathring{L}c\|_{L^2(\Sigma_{\mathfrak{t}}^u)}$ can be controlled by $\delta\varrho^{-1-m}\sqrt{E_{1,\le k+2}(\mathfrak{t},u)} + \delta\varrho^{-1}\sqrt{E_{2,\le k+2}(\mathfrak{t},u)}$.

Consequently, one can arrive at

$$\begin{aligned}
\|R^{k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim& \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|\not{\mathcal{L}}_R^{\le k-1}{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \mathfrak{t}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|\not{\mathcal{L}}_R^{\le k}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+ \delta\varrho^{-m}\sqrt{E_{1,\le k+2}(\mathfrak{t},u)} + \delta\sqrt{E_{2,\le k+2}(\mathfrak{t},u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|R^{\le k}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)}.
\end{aligned}$$

Using this iteratively and taking into account of (7.9), one then gets

$$\begin{aligned}
\|R^{k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim& \delta^{5/2-2\varepsilon_0}\mathfrak{t}^{-1/2} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|\not{\mathcal{L}}_R^{\le k-1}{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \mathfrak{t}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|\not{\mathcal{L}}_R^{\le k}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{-m}\sqrt{E_{1,\le k+2}} + \delta\sqrt{E_{2,\le k+2}}.
\end{aligned} \tag{7.10}$$

Similarly, one also has

$$\begin{aligned}
\|R^{k+1}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim& \delta^{1-\varepsilon_0}\mathfrak{t}^{1/2}\|\not{\mathcal{L}}_{R_i}^{\le k}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \mathfrak{t}\|R^{\le k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{1-m}\sqrt{E_{1,\le k+2}}\\
&+ \delta\varrho\sqrt{E_{2,\le k+2}},
\end{aligned} \tag{7.11}$$

$$\|\not{\mathcal{L}}_R^{k}\,{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{5/2-2\varepsilon_0}\mathfrak{t}^{-1/2} + \mathfrak{t}^{-1}\|R^{\le k}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{1/2}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{-m}\sqrt{E_{1,\le k+2}} + \delta\sqrt{E_{2,\le k+2}} + \delta^{2-2\varepsilon_0}\mathfrak{t}^{-1}\|\not{\mathcal{L}}_R^{\le k}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)}, \tag{7.12}$$

$$\|\not{\mathcal{L}}_R^{k+1}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{1/2} + \mathfrak{t}^{-1}\|R^{\le k+1}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\|R^{\le k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{2-\varepsilon_0}\varrho^{-m-1/2}\sqrt{E_{1,\le k+2}} + \delta^{2-\varepsilon_0}\varrho^{-1/2}\sqrt{E_{2,\le k+2}}. \tag{7.13}$$

It follows from (7.9)-(7.12) and a direct induction that

$$\|R^{k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \mathfrak{t}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{-m}\sqrt{E_{1,\le k+2}} + \delta\sqrt{E_{2,\le k+2}}, \tag{7.14}$$

$$\|\not{\mathcal{L}}_R^{k+1}\not{d}x\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{1/2} + \mathfrak{t}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{-m}\sqrt{E_{1,\le k+2}} + \delta\sqrt{E_{2,\le k+2}}, \tag{7.15}$$

$$\|R^{k+1}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t} + \mathfrak{t}^2\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{1-m}\sqrt{E_{1,\le k+2}} + \delta\varrho\sqrt{E_{2,\le k+2}}, \tag{7.16}$$

$$\|\not{\mathcal{L}}_R^{k}\,{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \mathfrak{t}\|\not{\mathcal{L}}_R^{\le k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta\varrho^{-m}\sqrt{E_{1,\le k+2}} + \delta\sqrt{E_{2,\le k+2}}. \tag{7.17}$$

It remains to estimate $\not{\mathcal{L}}_R^k\check{\lambda}$ in L^2 norm. To this end, as in Lemma 5.1, one needs to treat each term in $\not{\mathcal{L}}_{\mathring{L}}\not{\mathcal{L}}_R^k\check{\lambda}$ (see (5.4) and (4.15)). We treat $\not{\mathcal{L}}_R^k(\not{d}x^a\cdot\not{d}\mathring{L}\varphi_a)$ in details as an example.

Note that $\not{\mathcal{L}}_R\not{d}x^j = \epsilon_a^j\not{d}x^a - \not{d}\big(\upsilon(\varphi_j - \check{L}^j - \frac{x^j}{\varrho})\big)$. Then

$$|\not{\mathcal{L}}_R^k\not{d}x^j| \lesssim 1 + \mathfrak{t}^{-1}|R^{\le k}\upsilon| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\big(|R^{\le k}\varphi_j| + |R^{\le k}\check{L}^j|\big).$$

This implies that

$$\begin{aligned}
&\|\not{\mathcal{L}}_R^k(\not{d}x^a\cdot\not{d}\mathring{L}\varphi_a)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\lesssim&\sum_{k_1+k_2=k}\||\not{d}R^{k_2}\mathring{L}\varphi| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}|\not{\mathcal{L}}_R^{\le k_2-1}\,{}^{(R)}\not{\pi}|\\
&+\big(\mathfrak{t}^{-1}|R^{\le k_1}\upsilon| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}(|R^{\le k_1}\varphi_j| + |R^{\le k_1}\check{L}^j|)\big)|\not{d}R^{k_2}\mathring{L}\varphi|\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\lesssim&\mathfrak{t}^{-1-m}\sqrt{E_{1,\le k+2}} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-7/2}\|R^{\le k}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{2-2\varepsilon_0}\mathfrak{t}^{-3}\|R^{\le k}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\|\not{\mathcal{L}}_R^{\le k-1}\,{}^{(R)}\not{\pi}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}.
\end{aligned}$$

The same results hold also for the L^2 norms of $\not{\mathcal{L}}_R^k(\frac{x^a}{\varrho}\not{\nabla}^2\varphi_a)$ and $\not{\mathcal{L}}_R^k(\not{d}x^a\cdot\not{d}\varphi_a)$, where $\frac{x^a}{\varrho}\not{\nabla}^2\varphi_a$ arises in the term $-c^{-1}\mathring{L}^a(\not{\nabla}^2\varphi_a) = -c^{-1}\check{L}^a(\not{\nabla}^2\varphi_a) - c^{-1}\varrho^{-1}x^a\not{\nabla}^2\varphi_a$ in (4.15).

Let $F(\mathfrak{t},u,\vartheta) = \varrho(\mathfrak{t},u)^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})(\mathfrak{t},u,\vartheta) - \varrho(t_0,u)^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})(t_0,u,\vartheta)$ in (7.8). Then

$$\|\varrho^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\sqrt{\varrho} + \sqrt{\varrho}\int_{t_0}^{\mathfrak{t}}\tau^{-1/2}\|\mathring{L}\big(\varrho^2\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\big)\|_{L^2(\Sigma_\tau^u)}d\tau. \tag{7.18}$$

Apply (5.4) and (4.15) to estimate the integrand in (7.18) to get

$$\begin{aligned}
\|\varrho^{3/2}\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim&\delta^{3/2-\varepsilon_0} + \int_{t_0}^{\mathfrak{t}}\Big(\tau^{1/2-m}\sqrt{E_{1,\le k+2}(\tau,u)}\\
&+\delta^{1-\varepsilon_0}\tau^{-1}\|R^{\le k}\check{L}^i\|_{L^2(\Sigma_\tau^u)} + \delta^{1-\varepsilon_0}\tau^{-1}\|\not{\mathcal{L}}_R^{\le k-1}\,{}^{(R)}\not{\pi}\|_{L^2(\Sigma_\tau^u)}\\
&+\delta^{1-\varepsilon_0}\tau^{-2}\|R^{\le k}\upsilon\|_{L^2(\Sigma_\tau^u)} + \delta^{1-\varepsilon_0}\|\mathrm{tr}(\not{\mathcal{L}}_R^{\le k}\check{\lambda})\|_{L^2(\Sigma_\tau^u)}\Big)d\tau.
\end{aligned}\tag{7.19}$$

Substituting the estimates (7.14)-(7.17) into (7.19) and utilizing the Gronwall's inequality yield

$$\|\mathrm{tr}(\not{\mathcal{L}}_R^k\check{\lambda})\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-3/2}\ln\mathfrak{t} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)} + \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\ln\mathfrak{t}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}. \tag{7.20}$$

Then the remaining inequalities in Proposition 7.1 follow from this and (7.14)-(7.17). □

As a consequence of ${}^{(R)}\not\pi_{\mathring{L}X}$ in (3.25) and Proposition (7.1), it is direct to get

$$\|\not{\mathcal{L}}_R^{k}\,{}^{(R)}\not\pi_{\mathring{L}}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}. \tag{7.21}$$

Next we estimate the L^2 norms of the derivatives of μ.

Proposition 7.2. *Under the same assumptions in Proposition 7.1, it holds that for $k \leq 2N-6$,*

$$\|R^{k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{1/2} + \mathfrak{t}^{1/2}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta^{2-\varepsilon_0}\mathfrak{t}^{1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}. \tag{7.22}$$

Proof. It follows from (5.8), Proposition 7.1 and the L^∞ norm estimates in Section 5 that

$$\begin{aligned}\|\mathring{L}R^{k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim & \delta^{1-\varepsilon_0}\mathfrak{t}^{-1}\|\not{\mathcal{L}}_R^{\leq k}\,{}^{(R)}\not\pi_{\mathring{L}}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\|R^{\leq k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \\ & + \mathfrak{t}^{-m}\sqrt{E_{1,\leq k+2}(\mathfrak{t},u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\|R^{\leq k+1}\check{L}^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\|R^{\leq k}\not{d}x^j\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \\ \lesssim & \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta^{2-\varepsilon_0}\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)} \\ & + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\|R^{\leq k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)}.\end{aligned} \tag{7.23}$$

Applying (7.8) to $F(\mathfrak{t},u,\vartheta) = R^{k+1}\mu(\mathfrak{t},u,\vartheta) - R^{k+1}\mu(t_0,u,\vartheta)$ and using (7.23) lead to

$$\begin{aligned}\|\mathfrak{t}^{-1/2}R^{k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim & \delta^{3/2-\varepsilon_0} + \int_{t_0}^{\mathfrak{t}}\Big\{\delta^{3/2-\varepsilon_0}\tau^{-3/2} + \tau^{-1/2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\tau,u)} \\ & + \delta^{2-\varepsilon_0}\tau^{-3/2}\sqrt{\tilde{E}_{2,\leq k+2}(\tau,u)} + \delta^{1-\varepsilon_0}\tau^{-3/2}\|\tau^{-1/2}R^{\leq k+1}\mu\|_{L^2(\Sigma_\tau^u)}\Big\}d\tau.\end{aligned} \tag{7.24}$$

This, together with Gronwall's inequality and the fact $m \in (\frac{1}{2},\frac{3}{4})$, yields (7.22). □

For any vectorfield $Z \in \{\varrho\mathring{L}, T\}$, we can also obtain similar L^2 estimates for the corresponding quantities as in Proposition 7.1 and 7.2. The main ideas and methods for this are along the same lines of establishing Proposition 5.2, 5.3 from Lemma 5.1 and Proposition 5.1, so the details are omitted. We may conclude from this and Proposition 7.1 and 7.2 that

Proposition 7.3. *Under the same assumptions in Proposition 7.1, it holds that for $k \leq 2N-6$,*

$$\delta^l\|Z^{k+1}\check{L}^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.25}$$

$$\delta^l\|Z^{k+2}x^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{3/2} + \mathfrak{t}^{2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.26}$$

$$\delta^l\|\not{\mathcal{L}}_Z^{k+1}\not{g}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2}\mathfrak{t}^{1/2} + \mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.27}$$

$$\delta^l\|Z^{k+1}\upsilon\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t} + \mathfrak{t}^{2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.28}$$

$$\delta^l\|\not{\mathcal{L}}_Z^{k}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.29}$$

$$\delta^l\|\not{\mathcal{L}}_Z^{k}\big({}^{(R)}\not\pi, {}^{(R)}\not\pi_{\mathring{L}}, {}^{(R)}\not\pi_T\big)\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \varrho^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.30}$$

$$\delta^l\|\not{\mathcal{L}}_Z^{k}\,{}^{(T)}\not\pi_{\mathring{L}}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1/2} + \varrho^{-1/2}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{-1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.31}$$

$$\delta^l\|\not{\mathcal{L}}_Z^{k}\,{}^{(T)}\not\pi\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{1/2-\varepsilon_0} + \delta^{-\varepsilon_0}\varrho^{1/2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.32}$$

$$\delta^l\|Z^{k+1}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{1/2} + \mathfrak{t}^{1/2}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}, \tag{7.33}$$

where l is the number of T in the corresponding derivatives.

8 L^2 estimates for the highest order derivatives of trλ and $\not\triangle\mu$

Note that due to (6.36) and (6.44), the top orders of derivatives of φ, λ and μ for the energy estimates in (6.35) are $2N-4$, $2N-5$ and $2N-4$ respectively. However, as shown in Proposition 7.3, the L^2 estimates for the $(2N-5)^{\text{th}}$ order derivatives of λ and $(2N-4)^{\text{th}}$ order derivatives of μ can be controlled by the $(2N-3)$ order energy of φ. So there is a mismatch here. To overcome this difficulty, we need to deal with trλ and $\not\triangle\mu$ with the corresponding top order derivatives.

As in Proposition 7.1, we also modify the associated fluxes as follows:

$$\tilde{F}_{i,p+1}(\mathfrak{t},u)=\sup_{t_0\leq\tau\leq\mathfrak{t}}\big\{F_{i,p+1}(\tau,u)\big\},\quad i=1,2,\tag{8.1}$$

$$\tilde{F}_{i,\leq p+1}(\mathfrak{t},u)=\sum_{0\leq n\leq p}\tilde{F}_{i,n+1}(\mathfrak{t},u),\quad i=1,2.\tag{8.2}$$

8.1 Estimates for the derivatives of trλ

Due to (4.13), trλ satisfies a transport equation as

$$\begin{aligned}\mathring{L}(\text{tr}\lambda)=&c^{-1}\not d x^a\cdot\not d\mathring{L}\varphi_a-c^{-1}(\mathring{L}^2\varphi_0+\varphi_a\mathring{L}^2\varphi_a)-c^{-1}\mathring{L}^\gamma\not\triangle\varphi_\gamma-\frac{1}{2}c^{-1}(\mathring{L}c)\text{tr}\lambda\\&-c^{-1}\text{tr}\lambda(\not d x^a\cdot\not d\varphi_a)-|\lambda|^2+c^{-2}f(\not d x,\mathring{L}^i,\varphi,\not g)\begin{pmatrix}\not d\varphi\cdot\not d\varphi\\(\not d x^a\cdot\not d\varphi_a)^2\\(\mathring{L}\varphi)(\mathring{L}\varphi)\\(\not d x^a\cdot\not d\varphi_a)\mathring{L}\varphi\end{pmatrix}.\end{aligned}\tag{8.3}$$

In addition, (4.3) can be rewritten as

$$\mu\not\triangle\varphi_\gamma=\mathring{L}\mathring{\underline{L}}\varphi_\gamma+\frac{1}{2\varrho}\mathring{\underline{L}}\varphi_\gamma-\tilde{H}_\gamma,\tag{8.4}$$

where $\tilde{H}_\gamma=H_\gamma-\mu\not\triangle\varphi_\gamma$. Then (8.3) can be written as

$$\mathring{L}\big(\text{tr}\lambda-E\big)=\big(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta\big)\text{tr}\lambda+\frac{1}{\varrho^2}-|\check{\lambda}|^2+e,\tag{8.5}$$

which contains only the first or zeroth order derivatives of φ_γ on the right hand side, where

$$E=c^{-1}\not d x^a\cdot\not d\varphi_a-\frac{3}{2}c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta,\tag{8.6}$$

$$e=c^{-2}f(\not d x,\mathring{L}^i,\varphi,\not g)\begin{pmatrix}\not d\varphi\cdot\not d\varphi\\(\not d x^a\cdot\not d\varphi_a)^2\\(\mathring{L}\varphi)(\mathring{L}\varphi)\\(\not d x^a\cdot\not d\varphi_a)\mathring{L}\varphi\end{pmatrix}.\tag{8.7}$$

Set $F^k=\not d Z^k\text{tr}\lambda-\not d Z^k E$ with $Z\in\{\varrho\mathring{L},T,R\}$. It then follows from (8.5) inductively that

$$\begin{aligned}\not{\mathcal{L}}_{\mathring{L}}F^k=&\big(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta\big)F^k\\&+\big(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta\big)\not d Z^k E-\not d Z^k(|\check{\lambda}|^2)+e^k,\end{aligned}\tag{8.8}$$

where for $k \geq 1$,

$$
\begin{aligned}
e^k = & \not{\mathcal{L}}_Z^k e^0 + \sum_{k_1+k_2=k-1} \not{\mathcal{L}}_Z^{k_1} \not{\mathcal{L}}_{[\mathring{L},Z]} F^{k_2} \\
& + \sum_{k_1 \leq k-1} Z^{k-k_1} \big(- \frac{2}{\varrho} - c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\beta \mathring{L} \varphi_\beta \big) \not{d} Z^{k_1} \text{tr}\lambda
\end{aligned}
\tag{8.9}
$$

and

$$
e^0 = \not{d} e + \not{d} \big(- c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\beta \mathring{L} \varphi_\beta \big) \text{tr}\lambda. \tag{8.10}
$$

Note that for any one-form κ on $S_{\mathfrak{t},u}$, it holds that

$$
\mathring{L}(\varrho^2 |\kappa|^2) = -2\varrho^2 \text{tr}\check{\lambda} |\kappa|^2 + 2\varrho^2 (\text{tr} \not{\mathcal{L}}_{\mathring{L}} \kappa) |\kappa|. \tag{8.11}
$$

Then choosing $\kappa = \varrho^2 F^k$ in (8.11) and using (8.8) lead to

$$
\begin{aligned}
\mathring{L}\big(\varrho^6 |F^k|^2\big) = & \varrho^6 \big\{ - 2\text{tr}\check{\lambda} |F^k|^2 + 2\big(- c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\beta \mathring{L} \varphi_\beta \big) |F^k|^2 + 2 e^k \cdot F^k \\
& + 2\big(- \frac{2}{\varrho} - c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\beta \mathring{L} \varphi_\beta \big) \not{d} Z^k E \cdot F^k - 2 \not{d} Z^k (|\check{\lambda}|^2) \cdot F^k \big\}.
\end{aligned}
$$

This yields,

$$
\begin{aligned}
|\mathring{L}(\varrho^3 |F^k|)| \lesssim & \varrho^3 |\check{\lambda}| \cdot |F^k| + \varrho^3 | - c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\gamma \mathring{L} \varphi_\gamma | \cdot |F^k| + \varrho^3 | - \frac{2}{\varrho} - c^{-1} \mathring{L} c \\
& + c^{-1} \mathring{L}^\gamma \mathring{L} \varphi_\gamma | \cdot |\not{d} Z^k E| + \varrho^3 |\not{d} Z^k (|\check{\lambda}|^2)| + \varrho^3 |e^k|.
\end{aligned}
\tag{8.12}
$$

Then it follows from this and (7.8) that

$$
\begin{aligned}
\delta^l \varrho^3 \|F^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)} = & \delta^l \|F^k(t_0, \cdot, \cdot)\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^l \varrho^{1/2} \int_{t_0}^{\mathfrak{t}} \Big\{ \tau^{5/2} \|\check{\lambda}\|_{L^\infty(\Sigma_\tau^u)} \cdot \|F^k\|_{L^2(\Sigma_\tau^u)} \\
& + \tau^{5/2} \| - c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\gamma \mathring{L} \varphi_\gamma \|_{L^\infty(\Sigma_\tau^u)} \cdot \|F^k\|_{L^2(\Sigma_\tau^u)} \\
& + \tau^{5/2} \| - \frac{2}{\varrho} - c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\gamma \mathring{L} \varphi_\gamma \|_{L^\infty(\Sigma_\tau^u)} \cdot \|\not{d} Z^k E\|_{L^2(\Sigma_\tau^u)} \\
& + \tau^{5/2} \|\not{d} Z^k (|\check{\lambda}|^2)\|_{L^2(\Sigma_\tau^u)} + \tau^{5/2} \|e^k\|_{L^2(\Sigma_\tau^u)} \Big\} d\tau.
\end{aligned}
$$

Thus, the Gronwall's inequality yields

$$
\begin{aligned}
\delta^l \varrho^{5/2} \|F^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \delta^l \int_{t_0}^{\mathfrak{t}} \Big\{ & \tau^{3/2} \|\not{d} Z^k E\|_{L^2(\Sigma_\tau^u)} + \tau^{5/2} \|\not{d} Z^k (|\check{\lambda}|^2)\|_{L^2(\Sigma_\tau^u)} \\
& + \tau^{5/2} \|e^k\|_{L^2(\Sigma_\tau^u)} \Big\} d\tau.
\end{aligned}
\tag{8.13}
$$

Each term in the integrand of (8.13) will be estimated as follows.

We start with $\not{d} Z^k E$. Since $E = c^{-1} \not{d} x^a \cdot \not{d} \varphi_a - \frac{3}{2} c^{-1} \mathring{L} c + c^{-1} \mathring{L}^\beta \mathring{L} \varphi_\beta$, then

$$
\delta^l \|\not{d} Z^k E\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-2} + \mathfrak{t}^{-1-m} \sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t}, u)} + \delta^{2-\varepsilon_0} \mathfrak{t}^{-5/2} \sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t}, u)}, \tag{8.14}
$$

where one has used L^∞ estimates in Section 5, Proposition 7.3 and Lemma 7.1.

Next, we treat $\not{d}Z^k(|\check{\lambda}|^2)$. Direct computations give

$$\begin{aligned}\not{d}Z^k(|\check{\lambda}|^2)=&2\mathrm{tr}\check{\lambda}(\not{d}Z^k\mathrm{tr}\lambda)+2\sum_{k_1+k_2=k,\ k_1\le k-1}Z^{k_2}\mathrm{tr}\check{\lambda}(\not{d}Z^{k_1}\mathrm{tr}\check{\lambda})\\=&2\mathrm{tr}\check{\lambda}(F^k+\not{d}Z^kE)+2\sum_{k_1+k_2=k,\ k_1\le k-1}Z^{k_2}\mathrm{tr}\check{\lambda}(\not{d}Z^{k_1}\mathrm{tr}\check{\lambda}).\end{aligned}\tag{8.15}$$

Taking L^2 norm of (8.15) on the surfaces $\Sigma_{\mathfrak{t}}^u$ directly, and applying the estimates in Section 5 and Section 7 to handle the lower and higher order derivatives respectively, one can deduce

$$\begin{aligned}\delta^l\|\not{d}Z^k(|\check{\lambda}|^2)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim&\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^l\|F^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^{5/2-2\varepsilon_0}\mathfrak{t}^{-7/2}\\&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}+\delta^{2-\varepsilon_0}\mathfrak{t}^{-7/2}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\end{aligned}\tag{8.16}$$

Finally, it remains to treat e^k.

At first, we deal with the term $\not{\mathcal{L}}_Z^{k_1}\not{\mathcal{L}}_{[\mathring{L},Z]}F^{k_2}$ in (8.9) with $k_1+k_2=k-1$. If $Z\in\{R,T\}$, then $[\mathring{L},Z]={}^{(Z)}\not{\pi}_{\mathring{L}}^XX$ by (3.42), and therefore, $\not{\mathcal{L}}_{[\mathring{L},Z]}F^{k_2}={}^{(Z)}\not{\pi}_{\mathring{L}}\cdot\not{\nabla}F^{k_2}+F^{k_2}\cdot\not{\nabla}{}^{(Z)}\not{\pi}_{\mathring{L}}$. This leads to that by Proposition 7.3,

$$\begin{aligned}\delta^l\|\not{\mathcal{L}}_Z^{k_1}\not{\mathcal{L}}_{[\mathring{L},Z]}F^{k_2}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim&\ \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^l\|F^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^{5/2-2\varepsilon_0}\mathfrak{t}^{-7/2}\\&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}+\delta^{2-\varepsilon_0}\mathfrak{t}^{-7/2}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\end{aligned}\tag{8.17}$$

If $Z=\varrho\mathring{L}$, then by $[\mathring{L},Z]=\mathring{L}$ and (8.8),

$$\not{\mathcal{L}}_Z^{k_1}\not{\mathcal{L}}_{[\mathring{L},Z]}F^{k_2}=\not{\mathcal{L}}_Z^{k_1}\big\{(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\gamma\mathring{L}\varphi_\gamma)\not{d}Z^{k_2}\mathrm{tr}\lambda-\not{d}Z^{k_2}(|\check{\lambda}|^2)+e^{k_2}\big\}.\tag{8.18}$$

Note that the estimate of $\not{\mathcal{L}}_Z^{k_1}\not{d}Z^{k_2}(|\check{\lambda}|^2)$ in (8.18) can be obtained by Proposition 7.3 immediately,

$$\begin{aligned}\delta^l\|\not{d}Z^{k-1}(|\check{\lambda}|^2)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim&\delta^{5/2-2\varepsilon_0}\mathfrak{t}^{-7/2}+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}\\&+\delta^{2-\varepsilon_0}\mathfrak{t}^{-7/2}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\end{aligned}\tag{8.19}$$

The other two terms in (8.18) can be estimated similarly as for the remaining terms in (8.9), which will be done below.

Indeed, for the term $Z^{k-k_1}(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta)\not{d}Z^{k_1}\mathrm{tr}\lambda$ in (8.9) with $k_1\le k-1$, by Proposition 7.3, one can get

$$\begin{aligned}&\delta^l\|Z^{k-k_1}(-\frac{2}{\varrho}-c^{-1}\mathring{L}c+c^{-1}\mathring{L}^\beta\mathring{L}\varphi_\beta)\not{d}Z^{k_1}\mathrm{tr}\lambda\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\\lesssim&\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-3}+\mathfrak{t}^{-2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}+\delta\mathfrak{t}^{-3}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\end{aligned}\tag{8.20}$$

It remains to estimate $\not{\mathcal{L}}_Z^ke^0$ in (8.9). By (8.10), (8.7) and Proposition 7.3, one has

$$\delta^l\|\not{\mathcal{L}}_Z^ke^0\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-3}+\mathfrak{t}^{-2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}s+\delta^{2-\varepsilon_0}\mathfrak{t}^{-7/2}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\tag{8.21}$$

Combining the estimates (8.17)-(8.21) and (8.9) yields

$$\begin{aligned}\delta^l\|e^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim&\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^l\|F^k\||_{L^2(\Sigma_{\mathfrak{t}}^u)}d\tau+\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-3}+\mathfrak{t}^{-2-m}\sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t},u)}\\&+\delta\mathfrak{t}^{-3}\sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t},u)}.\end{aligned}\tag{8.22}$$

By inserting (8.14), (8.16) and (8.22) into (8.13), one obtains from (8.5) and (8.6) that

$$\delta^l \|F^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-2} + \mathfrak{t}^{-1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{-2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}. \tag{8.23}$$

In addition, due to the definition of F^k, $\not{d}Z^k \mathrm{tr}\lambda = F^k + \not{d}Z^k E$, and hence,

$$\begin{aligned}&\delta^l\|\not{d}Z^k\mathrm{tr}\lambda\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^l\|\mathrm{d\!/\!iv}\not{\mathcal{L}}_Z^k\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\ \lesssim&\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-2} + \mathfrak{t}^{-1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{-2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}\end{aligned} \tag{8.24}$$

holds true, where one has used (8.23) and (8.14).

Remark 8.1. *Since $\not{d}Z^k tr\check{\lambda} = \not{d}Z^k tr\lambda$ by* (3.13)*, so* (8.24) *gives also the L^2 estimate for $\not{d}Z^k tr\check{\lambda}$.*

8.2 Estimates for the derivatives of $\not{\Delta}\mu$

As in Subsection 8.1, one can use (3.19) to estimate $\not{\Delta}\mu$. Indeed, it follows from $[\mathring{L}, \not{\Delta}]\mu = -2(\mathrm{tr}\check{\lambda})\not{\Delta}\mu - 2\varrho^{-1}\not{\Delta}\mu - \not{d}\mathrm{tr}\check{\lambda}\cdot\not{d}\mu$ due to Lemma 3.8 and (3.19) that

$$\begin{aligned}\mathring{L}\not{\Delta}\mu =&[\mathring{L},\not{\Delta}]\mu + \not{\Delta}\mathring{L}\mu\\ =&-2(\mathrm{tr}\check{\lambda})\not{\Delta}\mu - \frac{2}{\varrho}\not{\Delta}\mu - \not{d}\mathrm{tr}\check{\lambda}\cdot\not{d}\mu + (\not{\Delta}\mu)c^{-1}(\mathring{L}^\alpha\mathring{L}\varphi_\alpha - \mathring{L}c) + 2\not{d}\mu\cdot\not{d}(c^{-1}\mathring{L}^\alpha\mathring{L}\varphi_\alpha\\ &-c^{-1}\mathring{L}c) + \mu\{\underset{\sim\sim\sim}{\not{\Delta}(c^{-1}\mathring{L}^\alpha)}\mathring{L}\varphi_\alpha + 2\not{d}(c^{-1}\mathring{L}^\alpha)\cdot\not{d}\mathring{L}\varphi_\alpha\} + \underline{\mathring{L}(c^{-1}\mu\mathring{L}^\alpha\not{\Delta}\varphi_\alpha)}\\ &-\mathring{L}(c^{-1}\mu\mathring{L}^\alpha)\not{\Delta}\varphi_\alpha - c^{-1}\mu\mathring{L}^\alpha[\mathring{L},\not{\Delta}]\varphi_\alpha + \mu[\mathring{L},\not{\Delta}]\ln c - \underline{\mathring{L}(\mu\not{\Delta}\ln c)} + (\mathring{L}\mu)\not{\Delta}\ln c.\end{aligned} \tag{8.25}$$

To estimate the term with wavy line in (8.25), one notes that (3.29) implies

$$\not{\Delta}\check{L}^a = \not{d}\mathrm{tr}\lambda\cdot\not{d}x^a + \mathrm{tr}\check{\lambda}\not{\Delta}x^a + \not{\nabla}^X\{\frac{1}{2}c^{-1}(\mathring{L}c)\not{d}_X x^a - \frac{1}{2}c^{-1}(\not{d}_X c)\tilde{T}^a + c^{-1}\tilde{T}^b(\not{d}_X\varphi_b)\tilde{T}^a\}, \tag{8.26}$$

and the L^2 norm of each term in (8.26) can be estimated by (8.24).

Observe that the two terms with the underline in (8.25) are both derivatives with respect to $\mathring{L}$, and then can be moved to the left hand side of (8.25). Let $\tilde{E} = c^{-1}\mu\mathring{L}^\alpha\not{\Delta}\varphi_\alpha - \mu\not{\Delta}\ln c$ and $\tilde{F} = \not{\Delta}\mu - \tilde{E}$. Then (8.25) can be rewritten as

$$\begin{aligned}\mathring{L}\tilde{F} =&(-2\mathrm{tr}\check{\lambda} - 2\varrho^{-1} + c^{-1}\mathring{L}^\alpha\mathring{L}\varphi_\alpha - c^{-1}\mathring{L}c)\tilde{F}\\ &-\not{d}\mathrm{tr}\chi\cdot(\not{d}\mu - c^{-1}\mu\mathring{L}\varphi_a\not{d}x^a - c^{-1}\mu\mathring{L}^\alpha\not{d}\varphi_\alpha + \mu\not{d}\ln c) + \tilde{e}.\end{aligned}$$

Recall that $\bar{Z}$ is any vector field in $\{T, R\}$ defined in Proposition (5.2). Set $\bar{F}^k = \bar{Z}^k\not{\Delta}\mu - \bar{Z}^k\tilde{E}$, one can get by induction as for (8.8) that

$$\begin{aligned}\mathring{L}\bar{F}^k =&(-2\mathrm{tr}\check{\lambda} - 2\varrho^{-1} + c^{-1}\mathring{L}^\gamma\mathring{L}\varphi_\gamma - c^{-1}\mathring{L}c)\bar{F}^k\\ &-\not{d}\bar{Z}^k\mathrm{tr}\lambda\cdot(\not{d}\mu - c^{-1}\mu\mathring{L}\varphi_a\not{d}x^a - c^{-1}\mu\mathring{L}^\alpha\not{d}\varphi_\alpha + \mu\not{d}\ln c) + [\mathring{L},\bar{Z}]^X\not{d}_X\bar{F}^{k-1} + \bar{e}^k\end{aligned} \tag{8.27}$$

with

$$\begin{aligned}\bar{e}^k =& \sum_{\substack{k_1+k_2=k-1\\|k_1|\geq 1}}\not{\mathcal{L}}_{\bar{Z}}^{k_1}[\mathring{L},\bar{Z}]^X\not{d}_X\bar{F}^{k_2} + \bar{Z}^k\tilde{e} + \sum_{\substack{k_1+k_2=k\\|k_1|\geq 1}}\Big\{\bar{Z}^{k_1}(-2\mathrm{tr}\check{\lambda} - 2\varrho^{-1} + c^{-1}\mathring{L}^\gamma\mathring{L}\varphi_\gamma - c^{-1}\mathring{L}c)\bar{F}^{k_2}\\ &-(\not{d}\bar{Z}^{k_2}\mathrm{tr}\lambda)\cdot\not{\mathcal{L}}_{\bar{Z}}^{k_1}(\not{d}\mu - c^{-1}\mu\mathring{L}\varphi_a\not{d}x^a - c^{-1}\mu\mathring{L}^\alpha\not{d}\varphi_\alpha + \mu\not{d}\ln c)\Big\},\end{aligned}$$

where the first sum on the right hand side above vanishes when $k = 1$. Thus, applying (7.8) to $F(\mathfrak{t}, u, \vartheta) = \varrho^2 \bar{F}^k(\mathfrak{t}, u, \vartheta) - \varrho_0^2 \bar{F}^k(t_0, u, \vartheta)$ and using (8.27) lead to

$$\begin{aligned} \delta^l \varrho^{3/2} \|\bar{F}^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim & \delta^{3/2-\varepsilon_0} + \int_{t_0}^{\mathfrak{t}} \delta^{1-\varepsilon_0} \delta^l \|\bar{F}^k\|_{L^2(\Sigma_\tau^u)} d\tau \\ & + \int_{t_0}^{\mathfrak{t}} \delta^{1+l-\varepsilon_0} \tau^{1/2} \|\not{d} \bar{Z}^k \mathrm{tr}\lambda\|_{L^2(\Sigma_\tau^u)} d\tau + \int_{t_0}^{\mathfrak{t}} \tau^{3/2} \delta^l \|\bar{e}^k\|_{L^2(\Sigma_\tau^u)} d\tau. \end{aligned} \tag{8.28}$$

It follows from (8.24) and Proposition 7.3 that

$$\begin{aligned} & \int_{t_0}^{\mathfrak{t}} \delta^{1+l-\varepsilon_0} \tau^{1/2} \|\not{d} \bar{Z}^k \mathrm{tr}\lambda\|_{L^2(\Sigma_\tau^u)} d\tau + \int_{t_0}^{\mathfrak{t}} \tau^{3/2} \delta^l \|\bar{e}^k\|_{L^2(\Sigma_\tau^u)} d\tau \\ \lesssim & \delta^{3/2-\varepsilon_0} + \int_{t_0}^{\mathfrak{t}} \Big(\tau^{-1/2-m} \sqrt{\tilde{E}_{1,\le k+2}(\tau, u)} + \delta \tau^{-3/2} \sqrt{\tilde{E}_{2,\le k+2}(\tau, u)} \Big) d\tau \\ \lesssim & \delta^{3/2-\varepsilon_0} + \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)}. \end{aligned} \tag{8.29}$$

Inserting (8.29) into (8.28) and appying the Gronwall's inequality yield

$$\delta^l \varrho^{3/2} \|\bar{F}^k\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} + \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)},$$

and hence,

$$\delta^l \|\bar{Z}^k \not{\Delta} \mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-3/2} + \mathfrak{t}^{-3/2} \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \mathfrak{t}^{-3/2} \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)}.$$

The other cases, which contain at least one $\varrho \mathring{L}$ in Z^k, can be treated by using (3.19) and commutators $[\varrho \mathring{L}, \bar{Z}]$ and $[\varrho \mathring{L}, \not{\Delta}]$. Therefore, we eventually arrive at

$$\begin{aligned} & \delta^l \|Z^k \not{\Delta} \mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \\ \lesssim & \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-3/2} + \mathfrak{t}^{-3/2} \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \mathfrak{t}^{-3/2} \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)}. \end{aligned} \tag{8.30}$$

At the end of this section, we are going to improve the L^2 estimate of $\not{\mathcal{L}}_Z^k \check{\lambda}$ in (7.29). It follows from the proof of Proposition 7.1 and 7.3 that the estimate of L^2 norm of $\not{\mathcal{L}}_Z^k \check{\lambda}$ was obtained by integrating $\not{\mathcal{L}}_{\mathring{L}} \not{\mathcal{L}}_Z^k \check{\lambda}$ along integral curves of $\mathring{L}$. Such an approach leads to losses of time decay of some related terms. To avoid such a difficulty, we now make use of the estimates (8.24) and (8.30) and carry out the L^2 estimates directly by studying the equations of $\check{\lambda}$ under actions of different vectorfields.

Corollary 8.1. *Under the assumptions* $(\star)$ *with* $\delta > 0$ *small, it then holds that for* $k \le 2N - 6$,

$$\delta^l \|Z^k tr\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-1} + \delta \mathfrak{t}^{-1-m} \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \mathfrak{t}^{-1} \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)}, \tag{8.31}$$

where l *is the number of* T *in the corresponding derivatives. Furthermore,*

$$\delta^l \|Z^k \mathring{L} tr\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-2} + \mathfrak{t}^{-1-m} \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \mathfrak{t}^{-2} \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)} \tag{8.32}$$

and

$$\delta^{l+1} \|Z^k T tr\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0} \mathfrak{t}^{-1} + \mathfrak{t}^{-m} \sqrt{\tilde{E}_{1,\le k+2}(\mathfrak{t}, u)} + \delta \mathfrak{t}^{-1} \sqrt{\tilde{E}_{2,\le k+2}(\mathfrak{t}, u)}. \tag{8.33}$$

Proof. We first derive (8.31). Without loss of generality, $k \geq 1$ is assumed. $Z^{k-1}(\varrho\mathring{L})\mathrm{tr}\check{\lambda}$, $Z^{k-1}T\mathrm{tr}\check{\lambda}$ and $Z^{k-1}R\mathrm{tr}\check{\lambda}$ will be treated separately as follows.

Step 1. Treatment of $Z^{k-1}(\varrho\mathring{L})\mathrm{tr}\check{\lambda}$

It follows from (4.15) that

$$\begin{aligned}\delta^l|Z^{k-1}(\varrho\mathring{L})\mathrm{tr}\check{\lambda}| \lesssim &\delta^{l_1}|Z^{n_1}\mathrm{tr}\check{\lambda}| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_2}|Z^{n_2}x| + \delta^{l_2}|\mathring{L}Z^{n_2}\varphi|\\ &+ \delta^{l_2}|\mathit{d}\!\!/ Z^{n_2}\varphi| + \mathfrak{t}^{-1}\delta^{l_2}|Z^{n_2}\varphi| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|Z^{n_1}\check{L}^i|\\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(R)}\not{\pi}_{\mathring{L}}| + \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(T)}\not{\pi}_{\mathring{L}}|,\end{aligned} \tag{8.34}$$

where l_i is the number of T in Z^{n_i} $(i = 0, 1, 2)$, and $n_i \leq k - 2 + i$. (8.34) and Proposition 7.3 imply

$$\delta^l\|Z^{k-1}(\varrho\mathring{L})\mathrm{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\leq k+1}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+1}(\mathfrak{t}, u)}, \tag{8.35}$$

which, together with (7.5), yields

$$\delta^l\|Z^{k-1}(\varrho\mathring{L})\mathrm{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \delta\mathfrak{t}^{-1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t}, u)}. \tag{8.36}$$

Step 2. Treatment of $Z^{k-1}T\mathrm{tr}\check{\lambda}$

Thanks to (4.16), one has

$$\begin{aligned}\delta^{l+1}|Z^{k-1}T\mathrm{tr}\check{\lambda}| \lesssim &\delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|Z^{n_1}\mathrm{tr}^{(T)}\not{\pi}| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\delta^{l_1}|Z^{n_1}\mathrm{tr}\check{\lambda}| + \delta^{l_1+1}|Z^{n_1}\not{\Delta}\mu|\\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_2}|Z^{n_2}x| + \delta^{l_2}|\mathit{d}\!\!/ Z^{n_2}\varphi| + \delta^{l_2}|\mathring{L}Z^{n_2}\varphi| + \mathfrak{t}^{-1}\delta^{l_2}|Z^{n_2}\varphi|\\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|Z^{n_1}\check{L}^i| + \delta t^{-2}\delta^{l_2}|Z^{n_2}\mu| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(R)}\not{\pi}_{\mathring{L}}|\\ &+ \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(T)}\not{\pi}_{\mathring{L}}|,\end{aligned} \tag{8.37}$$

where the number of T in Z^{k-1} is l, l_i and n_i $(i = 0, 1, 2)$ are given as in Step 1. We now apply (8.30) to estimate $Z^{n_1}\not{\Delta}\mu$ and use Proposition 7.3 to handle the other terms in (8.37). This leads to

$$\delta^{l+1}\|Z^{k-1}T\mathrm{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\leq k+1}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{E_{2,\leq k+1}(\mathfrak{t}, u)}. \tag{8.38}$$

Thus

$$\delta^{l+1}\|Z^{k-1}T\mathrm{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} \lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \delta\mathfrak{t}^{-1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{E_{2,\leq k+2}(\mathfrak{t}, u)}. \tag{8.39}$$

Step 3. Treatment of $Z^{k-1}R\mathrm{tr}\check{\lambda}$

Due to $Z^{k-1}R\mathrm{tr}\check{\lambda} = [Z^{k-1}, R]\mathrm{tr}\check{\lambda} + RZ^{k-1}\mathrm{tr}\check{\lambda}$, then

$$\delta^l|Z^{k-1}R\mathrm{tr}\check{\lambda}| \lesssim \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(R)}\not{\pi}_{\mathring{L}}| + \delta^{2-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_0}|\not{\mathcal{L}}_Z^{n_0}{}^{(R)}\not{\pi}_T| + \mathfrak{t}\delta^{l_1}|\mathit{d}\!\!/ Z^{n_1}\mathrm{tr}\check{\lambda}|,$$

which, together with Proposition 7.3, (8.24) and (7.5), implies

$$\begin{aligned}\delta^l\|Z^{k-1}R\mathrm{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)} &\lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \mathfrak{t}^{-m}\sqrt{\tilde{E}_{1,\leq k+1}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{E_{2,\leq k+1}(\mathfrak{t}, u)}\\ &\lesssim \delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1} + \delta\mathfrak{t}^{-1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t}, u)} + \delta\mathfrak{t}^{-1}\sqrt{E_{2,\leq k+2}(\mathfrak{t}, u)}.\end{aligned} \tag{8.40}$$

Collecting (8.36), (8.39) and (8.40) leads to the desired estimate (8.31). Moreover, (8.32) and (8.33) are the direct consequences of (8.35) and (8.38) respectively. □

9 Estimates for the error terms

With the L^2 estimates in Sections 7 and 8, we are ready to handle the error terms $\delta\int_{D^{\mathfrak{t},u}}|\Phi\cdot\underline{\mathring{L}}\Psi|$ and $|\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)|$ in (6.35), and then get the final energy estimates for φ. To estimate $\int_{D^{\mathfrak{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)$ for the top order derivatives, the following two lemmas will be needed.

Lemma 9.1. *For any smooth functions f and h, it holds that*

$$\begin{aligned}\int_{S_{\mathfrak{t},u}}(fh)=&\int_{C_u^{\mathfrak{t}}}\{\mathring{L}(fh)+tr\lambda(fh)\}+\int_{S_{t_0,u}}(fh)\\=&\int_{C_u^{\mathfrak{t}}}f(\mathring{L}h+\frac{1}{2\varrho}h)+\int_{C_u^{\mathfrak{t}}}(\mathring{L}f+\frac{1}{2\varrho}f)h+\int_{C_u^{\mathfrak{t}}}tr\check{\lambda}(fh)+\int_{S_{t_0,u}}(fh),\end{aligned}\tag{9.1}$$

$$\int_{S_{\mathfrak{t},u}}(Rf)h=-\int_{S_{\mathfrak{t},u}}f(Rh)-\frac{1}{2}\int_{S_{\mathfrak{t},u}}tr^{(R)}\not\pi(fh).\tag{9.2}$$

Lemma 9.2. *If f_i $(i=1,2,3)$ are smooth functions, then it holds that*

$$\begin{aligned}-\int_{D^{\mathfrak{t},u}}\{f_1\cdot(\mathring{L}f_2+\frac{1}{2\varrho}f_2)\cdot Rf_3\}=&-\int_{D^{\mathfrak{t},u}}\{f_1\cdot Rf_2\cdot\mathring{L}f_3\}+\int_{\Sigma_{\mathfrak{t}}^u}\{f_1\cdot Rf_2\cdot f_3\}\\&+\int_{D^{\mathfrak{t},u}}Er_1+\int_{\Sigma_{\mathfrak{t}}^u}Er_2+\int_{\Sigma_{t_0}^u}Er_3\end{aligned}\tag{9.3}$$

with

$$\begin{aligned}Er_1=&(tr\check{\lambda})f_1\cdot f_2\cdot Rf_3-Rf_1\cdot f_2\cdot\mathring{L}f_3-\frac{1}{2}(tr^{(R)}\not\pi)f_1\cdot f_2\cdot\mathring{L}f_3-f_1\cdot({}^{(R)}\not\pi_{\mathring{L}X}\not d^Xf_2)\cdot f_3\\&-({}^{(R)}\not\pi_{\mathring{L}X}\not d^Xf_1)\cdot f_2\cdot f_3-(\not div^{(R)}\not\pi_{\mathring{L}})f_1\cdot f_2\cdot f_3-(\mathring{L}+\frac{1}{2\varrho})f_1\cdot Rf_2\cdot f_3\\&-R(\mathring{L}+\frac{1}{2\varrho})f_1\cdot f_2\cdot f_3-\frac{1}{2}(tr^{(R)}\not\pi)(\mathring{L}+\frac{1}{2\varrho})f_1\cdot f_2\cdot f_3,\\Er_2=&Rf_1\cdot f_2\cdot f_3+\frac{1}{2}(tr^{(R)}\not\pi)f_1\cdot f_2\cdot f_3,\\Er_3=&-f_1\cdot Rf_2\cdot f_3-Rf_1\cdot f_2\cdot f_3-\frac{1}{2}(tr^{(R)}\not\pi)f_1\cdot f_2\cdot f_3.\end{aligned}$$

Proof. Let $f=f_1\cdot Rf_3$ and $h=f_2$ in (9.1) and then integrate over $[0,u]$ to get

$$\begin{aligned}&-\int_{D^{\mathfrak{t},u}}\{f_1\cdot(\mathring{L}f_2+\frac{1}{2\varrho}f_2)\cdot Rf_3\}\\=&\int_{D^{\mathfrak{t},u}}\mathrm{tr}\check{\lambda}\{f_1\cdot f_2\cdot Rf_3\}+\int_{D^{\mathfrak{t},u}}\{(\mathring{L}+\frac{1}{2\varrho})(f_1\cdot Rf_3)\cdot f_2\}\\&-\int_{\Sigma_{\mathfrak{t}}^u}\{f_1\cdot f_2\cdot Rf_3\}+\int_{\Sigma_{t_0}^u}\{f_1\cdot f_2\cdot Rf_3\}.\end{aligned}\tag{9.4}$$

Denote the last three integrals on the right hand side of (9.4) by I, II and III respectively. Choosing $f=f_3$ and $h=f_2(\mathring{L}f_1+\frac{1}{2\varrho}f_1)$ in (9.2) yields

$$\begin{aligned}I=&-\int_{D^{\mathfrak{t},u}}R\{f_2(\mathring{L}f_1+\frac{1}{2\varrho}f_1)\}f_3-\frac{1}{2}\int_{D^{\mathfrak{t},u}}\mathrm{tr}^{(R)}\not\pi\{f_2(\mathring{L}f_1+\frac{1}{2\varrho}f_1)f_3\}\\&+\int_{D^{\mathfrak{t},u}}\{f_1\cdot f_2\cdot R\mathring{L}f_3\}+\int_{D^{\mathfrak{t},u}}\{f_1\cdot f_2\cdot({}^{(R)}\not\pi_{\mathring{L}X}\not d^Xf_3)\}.\end{aligned}\tag{9.5}$$

Treating $\int_{D^{\mathfrak{t},u}}\{f_1\cdot f_2\cdot R\mathring{L}f_3\}$ by using (9.2), one can get from (9.5) that

$$
\begin{aligned}
I=&-\int_{D^{\mathfrak{t},u}}Rf_2(\mathring{L}f_1+\frac{1}{2\varrho}f_1)f_3-\int_{D^{\mathfrak{t},u}}f_2\{R(\mathring{L}+\frac{1}{2\varrho})f_1\}f_3\\
&-\frac{1}{2}\int_{D^{\mathfrak{t},u}}\text{tr}^{(R)}\not\pi\{f_2(\mathring{L}f_1+\frac{1}{2\varrho}f_1)f_3\}-\int_{D^{\mathfrak{t},u}}f_1\cdot Rf_2\cdot\mathring{L}f_3\\
&-\int_{D^{\mathfrak{t},u}}Rf_1\cdot f_2\cdot\mathring{L}f_3-\frac{1}{2}\int_{D^{\mathfrak{t},u}}\text{tr}^{(R)}\not\pi\{f_1\cdot f_2\cdot\mathring{L}f_3\}\\
&-\int_{D^{\mathfrak{t},u}}f_1({}^{(R)}\not\pi_{\mathring{L}X}\not d^X f_2)f_3-\int_{D^{\mathfrak{t},u}}({}^{(R)}\not\pi_{\mathring{L}X}\not d^X f_1)f_2\cdot f_3\\
&-\int_{D^{\mathfrak{t},u}}(\not{\text{d}}\text{iv}^{(R)}\not\pi_{\mathring{L}})f_1\cdot f_2\cdot f_3.
\end{aligned}
\tag{9.6}
$$

Similarly,

$$
II=\int_{\Sigma_{\mathfrak{t}}^u}f_1(Rf_2)f_3+\int_{\Sigma_{\mathfrak{t}}^u}Rf_1\cdot f_2\cdot f_3+\frac{1}{2}\int_{\Sigma_{\mathfrak{t}}^u}(\text{tr}^{(R)}\not\pi)f_1\cdot f_2\cdot f_3,\tag{9.7}
$$

$$
III=-\int_{\Sigma_{t_0}^u}f_1(Rf_2)f_3-\int_{\Sigma_{t_0}^u}Rf_1\cdot f_2\cdot f_3-\frac{1}{2}\int_{\Sigma_{t_0}^u}(\text{tr}^{(R)}\not\pi)f_1\cdot f_2\cdot f_3.\tag{9.8}
$$

Thus (9.3) follows from substituting (9.6)-(9.8) into (9.4). □

Recall the notations in Section 6 that for $\Psi=Z^{k+1}\varphi_\gamma=Z_{k+1}Z_k\cdots Z_1\varphi_\gamma$, $\Phi=\Phi_\gamma^{k+1}\equiv J_1^k+J_2^k$ is given explicitly in (6.38) with J_1^k and J_2^k being the summation and the rest in (6.38) respectively. We will also use the notation that $\varphi_\gamma^n=Z_n\cdots Z_1\varphi_\gamma$ for $n\geq 1$, and $\varphi_\gamma^0=\varphi_\gamma$. Then our main task is to estimate J_1^k and J_2^k.

9.1 Estimates for J_1^k

It follows from the explicit form of J_1^k that the key is to estimate the derivatives of $\mu\text{div}^{(Z)}C_\gamma^n$ $(0\leq n\leq k)$. Due to (6.42) and the structures given in (6.43)-(6.45), the treatment involving ${}^{(Z)}D_{\gamma,2}^n$ will be given separately from those for ${}^{(Z)}D_{\gamma,1}^n$ and ${}^{(Z)}D_{\gamma,3}^n$, since the latter do not contain the top order derivatives of φ_γ.

(1) We start with the estimates involving ${}^{(Z)}D_{\gamma,1}^n$ and ${}^{(Z)}D_{\gamma,3}^n$. Substituting (3.23)-(3.25) into (6.43) and (6.45) yields

$$
\begin{aligned}
{}^{(T)}D_{\gamma,1}^n=&T\mu\mathring{L}^2\varphi_\gamma^n+\mu(\not d\mu+2c^{-1}\mu\tilde{T}^a\not d\varphi_a)\cdot\not d\mathring{L}\varphi_\gamma^n+\frac{1}{2}\text{tr}^{(T)}\not\pi(\mathring{L}\underline{\mathring{L}}\varphi_\gamma^n+\frac{1}{2}\text{tr}\chi\underline{\mathring{L}}\varphi_\gamma^n)\\
&+(\not d\mu+2c^{-1}\mu\tilde{T}^a\not d\varphi_a)\cdot\not d\underline{\mathring{L}}\varphi_\gamma^n-T\mu\not\Delta\varphi_\gamma^n+\frac{1}{2}\big(-c^{-1}Tc-c^{-1}\mu\mathring{L}c\\
&+2c^{-1}\mu\not dx^a\cdot\not d\varphi_a-2\mu\text{tr}\lambda\big)\not\Delta\varphi_\gamma^n,
\end{aligned}
\tag{9.9}
$$

$$
\begin{aligned}
{}^{(T)}D_{\gamma,3}^n=&\big\{\text{tr}\lambda T\mu+(\frac{1}{4}\mu\text{tr}\lambda+\frac{1}{2}\mu\text{tr}\tilde{\theta})\text{tr}_{\not g}{}^{(T)}\not\pi-\frac{1}{2}|\not d\mu|^2-c^{-1}\mu\tilde{T}^a\not d\mu\cdot\not d\varphi_a\big\}\mathring{L}\varphi_\gamma^n\\
&+(\frac{1}{2}\text{tr}^{(T)}\not\pi+\frac{1}{2}\mathring{L}\mu-\mu\text{tr}\lambda)(\not d\mu+2c^{-1}\mu\tilde{T}^a\not d\varphi_a)\cdot\not d\varphi_\gamma^n,
\end{aligned}
\tag{9.10}
$$

$$
\begin{aligned}
{}^{(\varrho\mathring{L})}D_{\gamma,1}^n=&(2-\mu+\varrho\mathring{L}\mu)\mathring{L}^2\varphi_\gamma^n-2\varrho(\not d\mu+2c^{-1}\mu\tilde{T}^a\not d\varphi_a)\cdot\not d\mathring{L}\varphi_\gamma^n\\
&+\varrho\text{tr}\lambda(\mathring{L}\underline{\mathring{L}}\varphi_\gamma^n+\frac{1}{2}\text{tr}\chi\underline{\mathring{L}}\varphi_\gamma^n)+\varrho(\mu\text{tr}\check{\lambda}-\mathring{L}\mu)\not\Delta\varphi_\gamma^n,
\end{aligned}
\tag{9.11}
$$

$$
\begin{aligned}
{}^{(\varrho\mathring{L})}D^n_{\gamma,3} =&\mathrm{tr}\lambda\big\{2-\mu+\varrho\mathring{L}\mu+\varrho\mu\mathrm{tr}\tilde{\theta}+\frac{1}{2}\varrho\mu\mathrm{tr}\lambda\big\}\mathring{L}\varphi^n_\gamma\\
&+2\varrho\mathrm{tr}\lambda(\not{d}\mu+2\mu c^{-1}\tilde{T}^a\not{d}\varphi_a)\cdot\not{d}\varphi^n_\gamma,
\end{aligned}\tag{9.12}
$$

$$
\begin{aligned}
{}^{(R)}D^n_{\gamma,1} =&R\mu\mathring{L}^2\varphi^n_\gamma-{}^{(R)}\not{\pi}_{\underline{\mathring{L}}}\cdot\not{d}\mathring{L}\varphi^n_\gamma+\frac{1}{2}\mathrm{tr}^{(R)}\not{\pi}(\mathring{L}\underline{\mathring{L}}\varphi^n_\gamma+\frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\varphi^n_\gamma)\\
&-{}^{(R)}\not{\pi}_{\mathring{L}}\cdot\not{d}\underline{\mathring{L}}\varphi^n_\gamma-R\mu\not{\Delta}\varphi^n_\gamma+\frac{1}{2}\mu(\mathrm{tr}^{(R)}\not{\pi})\not{\Delta}\varphi^n_\gamma,
\end{aligned}\tag{9.13}
$$

$$
\begin{aligned}
{}^{(R)}D^n_{\gamma,3} =&\big\{\mathrm{tr}\lambda R\mu+\frac{1}{2}\mu(\mathrm{tr}\tilde{\theta}+\frac{1}{2}\mathrm{tr}\lambda)\mathrm{tr}^{(R)}\not{\pi}+\frac{1}{2}\not{d}\mu\cdot{}^{(R)}\not{\pi}_{\mathring{L}}\big\}\mathring{L}\varphi^n_\gamma+\big\{\frac{1}{2}\mathrm{tr}^{(R)}\not{\pi}(\not{d}\mu\\
&+2\mu c^{-1}\tilde{T}^a\not{d}\varphi_a)-\frac{1}{2}\mathring{L}\mu^{(R)}\not{\pi}_{\mathring{L}}+\frac{1}{2}\mathrm{tr}\lambda^{(R)}\not{\pi}_{\underline{\mathring{L}}}+\frac{1}{2}\mu\mathrm{tr}\lambda^{(R)}\not{\pi}_{\mathring{L}}\big\}\cdot\not{d}\varphi^n_\gamma.
\end{aligned}\tag{9.14}
$$

Note that the term $\mathring{L}\underline{\mathring{L}}\varphi^n_\gamma+\frac{1}{2}\mathrm{tr}\lambda\underline{\mathring{L}}\varphi^n_\gamma$ appears in ${}^{(Z)}D^n_{\gamma,1}$ and J^k_1 contains at most the $(k-n)^{\mathrm{th}}$ order derivatives of ${}^{(Z)}D^n_{\gamma,i}$ $(i=1,3)$. Then it can be checked that the L^2 norms of all the terms involving ${}^{(Z)}D^n_{\gamma,i}$ $(i=1,3)$ in J^k_1 can be treated by using the L^∞-estimates in Section 5 and the related L^2 estimates of Proposition 7.3. Therefore, it holds that

$$
\begin{aligned}
&\delta^{2l+1}\,\Big|\int_{D^{\mathrm{t},u}}\sum_{j=1}^k\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big){}^{(Z_{k+1-j})}D^{k-j}_{\gamma,1}\\
&\qquad\cdot\underline{\mathring{L}}\varphi^{k+1}_\gamma\Big|\\
\lesssim&\delta^{2l+1}\int_{t_0}^{\mathrm{t}}\sum_{j=1}^k\|\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big){}^{(Z_{k+1-j})}D^{k-j}_{\gamma,1}\|_{L^2(\Sigma^u_\tau)}\\
&\cdot\|\underline{\mathring{L}}Z^{k+1}\varphi_\gamma\|_{L^2(\Sigma^u_\tau)}d\tau\\
\lesssim&\delta^{4-4\varepsilon_0}+\int_{t_0}^{\mathrm{t}}\tau^{-3/2}\tilde{E}_{1,\le k+2}(\tau,u)d\tau+\delta\int_{t_0}^{\mathrm{t}}\tau^{-3/2}\tilde{E}_{2,\le k+2}(\tau,u)d\tau.
\end{aligned}\tag{9.15}
$$

Similarly, one can get that

$$
\begin{aligned}
&\delta^{2l+1}\,\Big|\int_{D^{\mathrm{t},u}}\sum_{j=1}^k\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big){}^{(Z_{k+1-j})}D^{k-j}_{\gamma,3}\\
&\qquad\cdot\underline{\mathring{L}}Z^{k+1}\varphi_\gamma\Big|\\
\lesssim&\delta^{3-3\varepsilon_0}+\delta^{2-3\varepsilon_0}\int_{t_0}^{\mathrm{t}}\tau^{-3/2}\tilde{E}_{1,\le k+2}(\tau,u)d\tau+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathrm{t}}\tau^{-3/2}\tilde{E}_{2,\le k+2}(\tau,u)d\tau.
\end{aligned}\tag{9.16}
$$

The corresponding terms in $\delta^{2l}|\int_{D^{\mathrm{t},u}}\Phi(\varrho^{2m}\mathring{L}\Psi+\frac{1}{2}\varrho^{2m-1}\Psi)|$ can also be estimated as

$$
\begin{aligned}
&\delta^{2l}\int_{D^{\mathrm{t},u}}\Big|\sum_{j=1}^k\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)\big({}^{(Z_{k+1-j})}D^{k-j}_{\gamma,1}\\
&\qquad+{}^{(Z_{k+1-j})}D^{k-j}_{\gamma,3}\big)\big(\varrho^{2m}\mathring{L}\varphi^{k+1}_\gamma+\frac{1}{2}\varrho^{2m-1}\varphi^{k+1}_\gamma\big)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim\delta^{2l+1}\int_{D^{\mathfrak{t},u}}\varrho^{2m}\Big\{\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)\big({}^{(Z_{k+1-j})}D^{k-j}_{\gamma,1}\\
&\quad+{}^{(Z_{k+1-j})}D^{k-j}_{\gamma,3}\big)\Big\}^2+\delta^{2l-1}\int_{D^{\mathfrak{t},u}}|(\varrho^m\mathring{L}\varphi_\gamma^{k+1}+\frac{1}{2}\varrho^{m-1}\varphi_\gamma^{k+1})|^2\\
&\lesssim\delta^{4-4\varepsilon_0}+\delta\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta\int_{t_0}^{\mathfrak{t}}\tau^{-3+2m}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\
&\quad+\delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}
\tag{9.17}
$$

(2) We now estimate the terms involving ${}^{(Z)}D^n_{\gamma,2}$ $(0\leq n\leq k)$ in J_1^k. Note that the most special case is $n=0$, which corresponds to $j=k$ in J_1^k. Indeed, in this case, the order of the top derivatives in ${}^{(Z)}D^0_{\gamma,2}$ is k, which implies that ${}^{(Z)}D^0_{\gamma,2}$ contains terms involving the $(k+1)^{\text{th}}$ order derivatives of the deformation tensor. This prevents one from using Proposition 7.3 to estimate the L^2 norm of ${}^{(Z)}D^0_{\gamma,2}$ directly. Otherwise, the factors $\tilde{E}_{1,\leq k+3}(\mathfrak{t},u)$ and $\tilde{E}_{2,\leq k+3}(\mathfrak{t},u)$ will appear in the right hand side of the energy estimate (6.35), which can not be absorbed by the left hand side. Thus, we will examine carefully the expression of ${}^{(Z)}D^n_{\gamma,2}$ and apply the estimates in Section 8 to deal with the top order derivatives of $\text{tr}\lambda$ and μ. Indeed, it follows from direct computations that

$$
\begin{aligned}
{}^{(T)}D^n_{\gamma,2}=&\mathring{L}T\mu\cdot\mathring{L}\varphi^n_\gamma-\frac{1}{2}\big(\not{\mathcal{L}}_{\mathring{L}}{}^{(T)}\not{\pi}_{\underline{\mathring{L}}}\big)\cdot\not{d}\varphi^n_\gamma+\frac{1}{2}\boxed{\not{\text{div}}(\not{d}\mu}+2c^{-1}\mu\tilde{T}^a\not{d}\varphi_a)\underline{L}\varphi^n_\gamma\\
&+\big\{\frac{1}{4}\underline{\mathring{L}}\big(-c^{-1}Tc-c^{-1}\mu\mathring{L}c+2\mu c^{-1}\not{d}x^a\cdot\not{d}\varphi_a\underset{\sim}{-2\mu\text{tr}\chi}\big)+\frac{1}{2}\boxed{\not{\nabla}_X\big(\mu\not{d}^X\mu}\\
&+2c^{-1}\mu^2\tilde{T}^a\not{d}\varphi_a\big)\big\}\mathring{L}\varphi^n_\gamma-\big\{\underline{\not{d}T\mu-\frac{1}{2}\not{\mathcal{L}}_{\underline{\mathring{L}}}(\not{d}\mu}+2c^{-1}\mu\tilde{T}^a\not{d}\varphi_a)\big\}\cdot\not{d}\varphi^n_\gamma\\
&-\frac{1}{2}\underbrace{\not{d}\big(2\mu^2\text{tr}\check{\lambda}}+2\varrho^{-1}\mu^2+c^{-1}\mu Tc+c^{-1}\mu^2\mathring{L}c-2c^{-1}\mu^2(\not{d}x^a)\cdot\not{d}\varphi_a\big)\cdot\not{d}\varphi^n_\gamma\\
&+\frac{1}{4}\big(\mathring{L}\text{tr}^{(T)}\not{\pi}\big)\underline{\mathring{L}}\varphi^n_\gamma,
\end{aligned}
\tag{9.18}
$$

$$
\begin{aligned}
{}^{(\varrho\mathring{L})}D^n_{\gamma,2}=&\big\{\mathring{L}\big(-\mu+\varrho\mathring{L}\mu\big)+\underset{\sim}{\frac{1}{2}\underline{\mathring{L}}\big(\varrho\text{tr}\check{\lambda}\big)}-\boxed{\not{\nabla}_X\big(\varrho\not{d}^X\mu}+2c^{-1}\varrho\mu\tilde{T}^a\not{d}^X\varphi_a\big)\big\}\mathring{L}\varphi^n_\gamma\\
&+\frac{1}{2}\mathring{L}\big(\varrho\text{tr}\check{\lambda}\big)\underline{\mathring{L}}\varphi^n_\gamma-\big\{\not{\mathcal{L}}_{\mathring{L}}\big(\varrho\not{d}\mu+2c^{-1}\varrho\mu\tilde{T}^a\not{d}\varphi_a\big)+\not{d}\big(\mu+\varrho\mathring{L}\mu\big)\big\}\cdot\not{d}\varphi^n_\gamma\\
&+\underbrace{\not{d}\big(\varrho\mu\text{tr}\lambda\big)}\cdot\not{d}\varphi^n_\gamma,
\end{aligned}
\tag{9.19}
$$

$$
\begin{aligned}
{}^{(R)}D^n_{\gamma,2}=&(\mathring{L}R\mu)\mathring{L}\varphi^n_\gamma-\big\{\frac{1}{2}\not{\mathcal{L}}_{\mathring{L}}\big({}^{(R)}\not{\pi}_{\underline{\mathring{L}}}\big)+\boxed{\not{d}R\mu}-\frac{1}{2}\not{\mathcal{L}}_{\underline{\mathring{L}}}\underset{\sim}{(R^X\text{tr}\check{\lambda}\not{g}}-g_{ab}\epsilon^a_i\check{L}^i\not{d}x^b\\
&-\upsilon c^{-1}\tilde{T}^a\not{d}\varphi_a+\frac{1}{2}c^{-1}R^X\mathring{L}c\not{g}+\frac{1}{2}c^{-1}\upsilon\not{d}c)\big\}\cdot\not{d}\varphi^n_\gamma+\frac{1}{4}\mathring{L}\big(\text{tr}^{(R)}\not{\pi}\big)\underline{\mathring{L}}\varphi^n_\gamma\\
&+\frac{1}{4}\underline{\mathring{L}}\big(\underset{\sim}{2\upsilon\text{tr}\lambda}+c^{-1}\upsilon\mathring{L}c-2c^{-1}\upsilon\not{d}x^a\cdot\not{d}\varphi_a-c^{-1}Rc\big)\mathring{L}\varphi^n_\gamma-\frac{1}{2}\underbrace{\not{\nabla}_X\big(\mu R^X\text{tr}\check{\lambda}}\\
&+g_{ab}\mu\epsilon^b_i\check{L}^i\not{d}^Xx^a+2g_{ab}\mu\epsilon^a_i\check{T}^i\not{d}x^b+\mu\upsilon c^{-1}\tilde{T}^a\not{d}\varphi_a+\frac{1}{2}c^{-1}\mu R^X\mathring{L}c
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}c^{-1}\mu\upsilon \not{d} c+2\upsilon\not{d}\mu-2c^{-1}\mu R\varphi_a \not{d}^X x^a\big)\mathring{L}\varphi_\gamma^n+\frac{1}{2}\underbrace{\not{\nabla}_X\big(R^X\mathrm{tr}\check{\lambda}}\\
&-g_{ab}\epsilon_i^a\check{L}^i\not{d}^X x^b-\upsilon c^{-1}\tilde{T}^a\not{d}^X\varphi_a+\frac{1}{2}c^{-1}R^X\mathring{L}c+\frac{1}{2}c^{-1}\upsilon\not{d}_X c\big)\underline{\mathring{L}}\varphi_\gamma^n \qquad (9.20)\\
&+\frac{1}{2}\underbrace{\not{d}\big(2\upsilon\mu\mathrm{tr}\lambda}+c^{-1}\mu\upsilon\mathring{L}c-2c^{-1}\mu\upsilon\not{d}x^a\cdot\not{d}\varphi_a-c^{-1}\mu Rc\big)\cdot\not{d}\varphi_\gamma^n.
\end{aligned}
$$

It is emphasized that special attentions are needed for terms with underlines, wavy lines, boxes, or braces in (9.18)-(9.20). In (9.18), due to $\frac{1}{2}\not{\mathcal{L}}_{\underline{\mathring{L}}}\not{d}\mu=\not{d}T\mu+\frac{1}{2}\mu\not{d}\mathring{L}\mu$, the corresponding underline part is

$$
\not{d}T\mu-\frac{1}{2}\not{\mathcal{L}}_{\underline{\mathring{L}}}\not{d}\mu=-\frac{1}{2}\mu\not{d}\mathring{L}\mu, \qquad (9.21)
$$

which can be estimated by using (3.19). For the terms with wavy lines in (9.18)-(9.20), one can use (4.16) and (4.15) to replace $\frac{1}{2}\not{\mathcal{L}}_{\underline{\mathring{L}}}\check{\lambda}$ by $\not{\nabla}^2\mu+\cdots$ which can be handled by (8.30). We also apply (8.30) and (8.24) to estimate those terms with boxes and braces respectively. Meanwhile, one notes that in (9.18)-(9.20), there are some terms containing derivatives of the deformation tensors with respect to $\mathring{L}$. For example, $\frac{1}{2}\big(\not{\mathcal{L}}_{\mathring{L}}{}^{(T)}\not{\pi}_{\underline{\mathring{L}}}\big)\cdot\not{d}\varphi_\gamma^n$ appears in (9.18). However, these terms are not the "bad" ones and can be estimated by taking into account of (3.19), (4.15), (3.23) and (3.25).

In summary, we can arrive at

$$
\begin{aligned}
&\delta^{2l+1}\mid\int_{D^{\mathfrak{t},u}}\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(Z_{k+1-j})}D_{\gamma,2}^{k-j}\\
&\qquad\cdot\underline{\mathring{L}}\varphi_\gamma^{k+1}\mid \qquad . \qquad (9.22)\\
\lesssim&\delta^{4-4\varepsilon_0}+\delta^{1-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-1/2-m}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta\int_{t_0}^{\mathfrak{t}}\tau^{-1/2-m}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau.
\end{aligned}
$$

It remains to deal with

$$
\delta^{2l}\mid\int_{D^{\mathfrak{t},u}}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(Z_{k+1-j})}D_{\gamma,2}^{k-j}(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1}+\frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\mid,
$$

where $j=1,2,\cdots k$, which will be done by these steps below.

(a) For $Z_{k+1-j}=T$, according (9.18), one can get from (8.24), (8.30) and Proposition 7.3 that

$$
\begin{aligned}
&\delta^{2l}\mid\int_{D^{\mathfrak{t},u}}\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(T)}D_{\gamma,2}^{k-j}\cdot(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1}\\
&\qquad+\frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\mid\\
\lesssim&\delta^{2l+1}\int_{D^{\mathfrak{t},u}}\varrho^{2m}\mid\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(T)}D_{\gamma,2}^{k-j}\mid^2 \qquad (9.23)\\
&+\delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'\\
\lesssim&\delta^{6-6\varepsilon_0}+\delta^{3-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\
&+\delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}
$$

(b) For $Z_{k+1-j} = \varrho\mathring{L}$, $Z^{k(\varrho\mathring{L})}D^0_{\gamma,2}$ contains a term $\frac{1}{2}\big(Z^k\mathring{L}(\varrho\text{tr}\check{\lambda})\big)\mathring{\underline{L}}\varphi_\gamma$. Then (4.15) implies that

$$\begin{aligned}\delta^l|Z^k\mathring{L}(\varrho\text{tr}\check{\lambda})| \lesssim&\delta^{l_2}|Z^{n_2}\text{tr}\check{\lambda}| + \delta^{l_2}|Z^{n_2}\big(\varrho(\not\triangle\varphi_0 + \mathring{L}^a\not\triangle\varphi_a)\big)| + \delta^{l_3}|\mathring{L}Z^{n_3}\varphi| \\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_3}|Z^{n_3}x| + \mathfrak{t}^{-1}\delta^{l_3}|Z^{n_3}\varphi| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_2}|Z^{n_2}\check{L}^i| \\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|\not\mathcal{L}_Z^{n_1(R)}\not\pi_{\mathring{L}}| + \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|\not\mathcal{L}_Z^{n_1(T)}\not\pi_{\mathring{L}}|,\end{aligned} \tag{9.24}$$

where l_i is the number of T in Z^{n_i} and $n_i \le k-2+i$ $(i=1,2,3)$. Due to (8.4),

$$\begin{aligned}\mu(\not\triangle\varphi_0 + \mathring{L}^a\not\triangle\varphi_a) =&\mathring{L}^\alpha\mathring{L}\mathring{\underline{L}}\varphi_\alpha + \frac{1}{2\varrho}\mathring{L}^\alpha\mathring{\underline{L}}\varphi_\alpha - \mathring{L}^\alpha\tilde{H}_\alpha \\ =&\mathring{L}(\mathring{L}^\alpha\mathring{\underline{L}}\varphi^\alpha) - (\mathring{L}\mathring{L}^a)\mathring{\underline{L}}\varphi_a + \frac{1}{2\varrho}\mathring{\underline{L}}^\alpha\mathring{L}\varphi_\alpha - \mathring{L}^\alpha\tilde{H}_\alpha.\end{aligned} \tag{9.25}$$

Substituting (3.26) and (4.4) (note that $\tilde{H}_\alpha = H_\alpha - \mu\not\triangle\varphi_\alpha$) into (9.25) yields

$$\begin{aligned}&\not\triangle\varphi_0 + \mathring{L}^a\not\triangle\varphi_a \\ =&(\mathring{L} + \frac{1}{2\varrho})\mathring{L}\varphi_0 + \sum_{a=1}^2\big\{(2\varphi_a - \mathring{L}^a)(\mathring{L} + \frac{1}{2\varrho})\mathring{L}\varphi_a + \mathring{L}(2\varphi_a - \mathring{L}^a)\mathring{L}\varphi_a\big\} \\ &+ c^{-1}(\mathring{L}^\alpha\varphi_\alpha - \mathring{L}c)\big\{\mathring{L}\varphi_0 + \sum_{a=1}^2(2\varphi_a - \mathring{L}^a)\mathring{L}\varphi_a\big\} + \frac{1}{2}c^{-1}(\tilde{T}^a\mathring{L}\varphi_a \\ &- \not dx^a\cdot\not d\varphi_a - c\text{tr}\check{\lambda} - c\varrho^{-1})\mathring{L}^\alpha\mathring{L}\varphi_\alpha + c^{-1}(\mathring{L}^\alpha\not d\varphi_\alpha + 3\tilde{T}^a\not d\varphi_a)\cdot\not dx^b(\mathring{L}\varphi_b) \\ &+ \frac{1}{2}c^{-1}(\mathring{L}c + 3\tilde{T}^a\mathring{L}\varphi_a - 3\not dx^a\cdot\not d\varphi_a + c\text{tr}\check{\lambda})(\mathring{L}^\alpha + 2\tilde{T}^\alpha)\mathring{L}\varphi_\alpha.\end{aligned} \tag{9.26}$$

It then follows from (9.26) and Proposition 7.3 that for any fixed constant $\ni\geq \frac{1}{2}$,

$$\begin{aligned}&\delta^{2l}\int_{D^{\mathfrak{t},u}}\varrho^{2m-\ni}|Z^k\big(\varrho(\not\triangle\varphi_0 + \mathring{L}^a\not\triangle\varphi_a)\big)|^2 \\ \lesssim&\delta^{2l_3}\int_{D^{\mathfrak{t},u}}\tau^{2m-\ni}|(\mathring{L} + \frac{1}{2\varrho})Z^{n_3}\varphi|^2 + \delta^{2l_3}\int_{D^{\mathfrak{t},u}}\tau^{2m-2-\ni}|Z^{n_3}\varphi|^2 \\ &+ \delta^{2-2\varepsilon_0+2l_2}\int_{D^{\mathfrak{t},u}}\tau^{2m-3-\ni}|Z^{n_2}\check{L}^i|^2 + \delta^{2-2\varepsilon_0+l_3}\int_{D^{\mathfrak{t},u}}\tau^{2m-5-\ni}|Z^{n_3}x|^2 \\ &+ \delta^{2-2\varepsilon_0+l_2}\int_{D^{\mathfrak{t},u}}\tau^{2m-1-\ni}|Z^{n_2}\text{tr}\check{\lambda}|^2 + \delta^{2-2\varepsilon_0+2l_1}\int_{D^{\mathfrak{t},u}}\tau^{2m-3-\ni}|\not\mathcal{L}_Z^{n_1(R)}\not\pi_{\mathring{L}}|^2 \\ &+ \delta^{4-2\varepsilon_0+2l_1}\int_{D^{\mathfrak{t},u}}\tau^{2m-3-\ni}|\not\mathcal{L}_Z^{n_1(T)}\not\pi_{\mathring{L}}|^2 \\ \lesssim&\int_0^u F_{1,\le k+2}(\mathfrak{t},u')du' + \delta^{3-2\varepsilon_0} + \delta^{2-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-1-\ni}\tilde{E}_{1,\le k+2}(\tau,u)d\tau \\ &+ \delta^2\int_{t_0}^{\mathfrak{t}}\tau^{2m-2-\ni}\tilde{E}_{2,\le k+2}(\tau,u)d\tau.\end{aligned} \tag{9.27}$$

In addition, (7.4) implies that for the constant $\ni\geq \frac{1}{2}$,

$$\begin{aligned}&\int_{D^{\mathfrak{t},u}}\varrho^{2m-\ni}\delta^{2l_3}|\mathring{L}Z^{n_3}\varphi|^2 \\ \lesssim&\int_{D^{\mathfrak{t},u}}\tau^{2m-\ni}\delta^{2l_3}|(\mathring{L} + \frac{1}{2\varrho})Z^{n_3}\varphi|^2 + \int_{D^{\mathfrak{t},u}}\tau^{2m-2-\ni}\delta^{2l_3}|Z^{n_3}\varphi|^2 \\ \lesssim&\int_0^u F_{1,n_3+1}(\mathfrak{t},u')du' + \delta^2\int_{t_0}^{\mathfrak{t}}\tau^{-2-\ni}E_{1,n_3+1}(\tau,u)d\tau + \delta^2\int_{D^{\mathfrak{t},u}}\tau^{2m-2-\ni}E_{2,n_3+1}.\end{aligned} \tag{9.28}$$

On the other hand, applying (8.31), (9.27) and (9.28) to estimate the first line at the right hand side of (9.24), and utilizing (7.4) and Proposition 7.3 to deal with the other terms, one then can obtain by choosing $\backepsilon = 1$ in (9.27) and (9.28) that

$$\begin{aligned}
&\delta^{2l}\int_{D^{\mathfrak{t},u}}\varrho^{2m}|Z^k\mathring{L}(\varrho\mathrm{tr}\check{\lambda})|^2|\underline{\mathring{L}}\varphi_\gamma|^2\lesssim\delta^{2l-2\varepsilon_0}\int_{D^{\mathfrak{t},u}}\varrho^{2m-1}|Z^k\mathring{L}(\varrho\mathrm{tr}\check{\lambda})|^2\\
\lesssim&\delta^{3-4\varepsilon_0}+\delta^{2-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\le k+2}(\tau,u)d\tau+\delta^{2-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\le k+2}(\tau,u)d\tau\\
&+\delta^{-2\varepsilon_0}\int_0^u F_{1,\le k+2}(\mathfrak{t},u')du'.
\end{aligned}\tag{9.29}$$

Therefore, thanks to (9.19), Proposition 7.3 and (9.29), one has

$$\begin{aligned}
&\delta^{2l}\,|\int_{D^{\mathfrak{t},u}}\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(\varrho\mathring{L})}D^{k-j}_{\gamma,2}\cdot(\varrho^{2m}\mathring{L}\varphi^{k+1}_\gamma\\
&\qquad\qquad+\frac{1}{2}\varrho^{2m-1}\varphi^{k+1}_\gamma)\,|\\
\lesssim&\delta^{2l+1}\int_{D^{\mathfrak{t},u}}\varrho^{2m}\,|\sum_{j=1}^{k}\big(Z_{k+1}+{}^{(Z_{k+1})}\Lambda\big)\ldots\big(Z_{k+2-j}+{}^{(Z_{k+2-j})}\Lambda\big)^{(\varrho\mathring{L})}D^{k-j}_{\gamma,2}\,|^2\\
&+\delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'\\
\lesssim&\delta^{4-4\varepsilon_0}+\delta^{3-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\le k+2}(\tau,u)d\tau+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\le k+2}(\tau,u)d\tau\\
&+\delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}\tag{9.30}$$

(c) Finally, we deal with the most difficult case, $Z_{k+1-j}=R$. In this case, it follows from (9.20) that $Z^{k(R)}D^0_{\gamma,2}$ contains the term $\frac{1}{2}(RZ^k\mathrm{tr}\check{\lambda})\underline{\mathring{L}}\varphi_\gamma$ whose treatment is more subtle and will be given later in Proposition 9.1. The other terms can be estimated as follows

$$\begin{aligned}
&\delta^l|Z^{k(R)}D^0_{\gamma,2}-\frac{1}{2}(RZ^k\mathrm{tr}\check{\lambda})\underline{\mathring{L}}\varphi_\gamma|\\
\lesssim&\delta^{-\varepsilon_0}\mathfrak{t}^{-1/2}\underbrace{\delta^{l_3}|\mathring{L}Z^{n_3}\varphi|}_{(9.28)}+\delta^{1-2\varepsilon_0}\underbrace{\delta^{l_2}|Z^{n_2}\mathring{L}\mathrm{tr}\check{\lambda}|}_{(8.32)}+\delta^{-\varepsilon_0}\mathfrak{t}^{-1/2}\underbrace{\delta^{l_2}|Z^{n_2}\mathrm{tr}\check{\lambda}|}_{(8.31)}\\
&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-2}\delta^{l_2}|\not{\mathcal{L}}^{n_2(R)}_Z\not{\pi}_{\mathring{L}}|+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_3}|Z^{n_3}\mu|+\delta^{2-2\varepsilon_0}\mathfrak{t}^{-3}\delta^{l_3}|Z^{n_3}\check{L}^i|\\
&+\delta^{-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_2}|Z^{n_2}\check{L}^i|+\delta^{1-2\varepsilon_0}\mathfrak{t}^{-3}\delta^{l_4}|Z^{n_4}x|+\delta^{-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_3}|Z^{n_3}\varphi|\\
&+\delta^{-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_2}|Z^{n_2}\upsilon|+\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_3}|\underline{\mathring{L}}Z^{n_3}\varphi|+\delta^{1-2\varepsilon_0}\mathfrak{t}^{-1}\delta^{l_3}|\not{d}Z^{n_3}\varphi|\\
&+\delta^{1-2\varepsilon_0}\mathfrak{t}^{-2}\delta^{l_2}|Z^{n_2}\mathrm{tr}^{(R)}\not{\pi}|+\delta^{2-2\varepsilon_0}\mathfrak{t}^{-3}\delta^{l_2}|\not{\mathcal{L}}^{n_2(R)}_Z\not{\pi}_T|+\delta^{1-2\varepsilon_0}\mathfrak{t}^{-3}\delta^{l_3}|Z^{n_3}\upsilon|\\
&+\delta^{2-2\varepsilon_0}\mathfrak{t}^{-1}\underbrace{\delta^{l_2}|Z^{n_2}T\mathrm{tr}\check{\lambda}|}_{(8.33)}+\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\underbrace{\delta^{l_2}|Z^{n_2}\not{\Delta}\mu|}_{(8.30)}+\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\underbrace{\delta^{l_2}|\not{d}Z^{n_2}\mathrm{tr}\check{\lambda}|}_{(8.24)},
\end{aligned}\tag{9.31}$$

where l_i is the number of T in Z^{n_i} and $n_i\le k-2+i$ $(i=2,3,4)$. To estimate the L^2-norm of $\delta^l|Z^{k(R)}D^0_{\gamma,2}-\frac{1}{2}(RZ^k\mathrm{tr}\check{\lambda})\underline{\mathring{L}}\varphi_\gamma|$ over $D^{\mathfrak{t},u}$, we can bound the L^2-norms of the terms underlined with braces in (9.31) by the corresponding estimates indicated bellow the braces.

While the other terms without braces can be treated by using Proposition 7.3 and (7.4). Then one can conclude that

$$\begin{aligned}
&|\ \delta^{2l}\int_{D^{\mathfrak{t},u}}\big(Z^{k(R)}D^0_{\gamma,2}-\frac{1}{2}(RZ^k\mathrm{tr}\check{\lambda})\mathring{\underline{L}}\varphi_\gamma\big)\cdot(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1}+\frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\ |\\
\lesssim&\delta^{-1}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'+\delta^{4-4\varepsilon_0}+\delta^{3-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau\\
&+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau.
\end{aligned}\tag{9.32}$$

Finally, we turn to the estimates involving the term $\frac{1}{2}(RZ^k\mathrm{tr}\check{\lambda})\mathring{\underline{L}}\varphi_\gamma$, whose L^2 norm cannot be estimated by (8.24) directly since the resulting time-decay rate is not enough to close the energy estimate (6.35) (see also the beginning of Section 10). Our strategy here is based on the structural equation (8.5). Indeed, (8.5) implies that $\mathring{L}(\mathrm{tr}\check{\lambda}-E)$ admits better rate of decay in time, which, combined with (9.3) and (9.2), will enable us to obtain the desired estimates for corresponding terms. Meanwhile, the terms involving E defined by (8.6) can be handled easily by using Proposition 7.3 and (9.28) directly. Thus we can get

Proposition 9.1. *For $\delta>0$ small, it holds that*

$$\begin{aligned}
&|\ \delta^{2l}\int_{D^{\mathfrak{t},u}}(RZ^k tr\check{\lambda})\mathring{\underline{L}}\varphi_\gamma(\varrho^{2m}\mathring{L}Z^{k+1}\varphi_\gamma+\frac{1}{2}\varrho^{2m-1}Z^{k+1}\varphi_\gamma)\ |\\
\lesssim&\delta^{3-3\varepsilon_0}+\delta^{-1}\int_0^u\tilde{F}_{1,\leq k+2}(\mathfrak{t},u')du'+\delta^{-1}\int_0^u\delta F_{2,k+2}(\mathfrak{t},u')du'\\
&+\delta^{2-3\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{m-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\
&+\delta^{2-3\varepsilon_0}\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)+\delta^{2-\varepsilon_0}\tilde{E}_{2,\leq k+2}(\mathfrak{t},u).
\end{aligned}\tag{9.33}$$

Proof. Noting (8.5) for $\mathrm{tr}\lambda$, one can bound the left hand side of (9.33) by $|\bar{I}|+|\overline{II}|$ with

$$\begin{aligned}
\bar{I}&=\delta^{2l}\int_{D^{\mathfrak{t},u}}RZ^k(\mathrm{tr}\check{\lambda}-E)\cdot\mathring{\underline{L}}\varphi_\gamma(\varrho^{2m}\mathring{L}Z^{k+1}\varphi_\gamma+\frac{1}{2}\varrho^{2m-1}Z^{k+1}\varphi_\gamma),\\
\overline{II}&=\delta^{2l}\int_{D^{\mathfrak{t},u}}(RZ^kE)\mathring{\underline{L}}\varphi_\gamma(\varrho^{2m}\mathring{L}Z^{k+1}\varphi_\gamma+\frac{1}{2}\varrho^{2m-1}Z^{k+1}\varphi_\gamma),
\end{aligned}$$

where E is defined in (8.6).

We start with the easy term $\overline{II}$. Note that RZ^kE contains $Z^k(\not{d}x^a\cdot\not{d}R\varphi_a)$ and

$$\not{d}x^a\cdot\not{d}R\varphi_a=\frac{1}{2}\mathrm{tr}^{(R)}\not{\pi}(\not{d}x^a\cdot\not{d}\varphi_a)+(Rx^a)\not{\Delta}\varphi_a.$$

Replacing $\not{\Delta}\varphi_a$ with $\mu^{-1}(\mathring{L}\mathring{\underline{L}}\varphi_a+\frac{1}{2\varrho}\mathring{\underline{L}}\varphi_a-\tilde{H}_a)$ and applying (4.4) yield

$$\begin{aligned}
&\not{d}x^a\cdot\not{d}R\varphi_a\\
=&(\mathring{L}+\frac{1}{2\varrho})R\varphi_0+\sum_{a=1}^2(2\varphi_a-\mathring{L}^a)(\mathring{L}+\frac{1}{2\varrho})R\varphi_a+\frac{1}{2}\mathrm{tr}^{(R)}\not{\pi}(\not{d}x^a\cdot\not{d}\varphi_a)\\
&-\sum_{a=1}^2(2\varphi_a-\mathring{L}^a)^{(R)}\not{\pi}_{\mathring{L}}\cdot\not{d}\varphi_a-{}^{(R)}\not{\pi}_{\mathring{L}}\cdot\not{d}\varphi_0-\frac{3}{2\varrho}R\varphi_0\\
&-\frac{1}{2\varrho}\sum_{a=1}^2(4\varphi_a-\mathring{L}^a)R\varphi_a+c^{-1}f(\mathring{L}^i,\varphi)\begin{pmatrix}\mathring{L}\varphi\\ \not{d}x^b\cdot\not{d}\varphi_b\\ \mathrm{tr}\check{\lambda}\end{pmatrix}R\varphi.
\end{aligned}\tag{9.34}$$

It then follows from (8.6) and (9.34) that

$$
\begin{aligned}
|\overline{II}| \lesssim & \delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du' + \delta^{1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-3}\delta^{2l_3}|Z^{n_3}\varphi|^2 \\
& + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-6}\delta^{2l_4}|Z^{n_4}x|^2 + \delta^{2l+1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-1}|Z^k(\not{d}x^a\cdot\not{d}R\varphi_a)|^2 \\
& + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_2}|\not{\mathcal{L}}_Z^{n_2}{}^{(R)}\not{\pi}_{\mathring{L}}|^2 + \delta^{5-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-6}\delta^{2l_1}|\not{\mathcal{L}}_Z^{n_1}{}^{(R)}\not{\pi}_T|^2 \\
& + \delta^{5-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_1}|\not{\mathcal{L}}_Z^{n_1}{}^{(T)}\not{\pi}_{\mathring{L}}|^2 + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_3}|Z^{n_3}\check{L}^i|^2 \\
& + \delta^{2l+1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-1}|\mathring{L}Z^kR\varphi|^2 \\
\lesssim & \delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du' + \delta^{1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-1}\delta^{2l}|(\mathring{L}+\frac{1}{2\varrho})Z^kR\varphi|^2 \\
& + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-6}\delta^{2l_4}|Z^{n_4}x|^2 + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_3}|Z^{n_3}\check{L}^i|^2 \\
& + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_2}|\not{\mathcal{L}}_Z^{n_2}{}^{(R)}\not{\pi}_{\mathring{L}}|^2 + \delta^{5-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-6}\delta^{2l_1}|\not{\mathcal{L}}_Z^{n_1}{}^{(R)}\not{\pi}_T|^2 \\
& + \delta^{5-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_1}|\not{\mathcal{L}}_Z^{n_1}{}^{(T)}\not{\pi}_{\mathring{L}}|^2 + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-4}\delta^{2l_2}|Z^{n_2}\mathrm{tr}^{(R)}\not{\pi}|^2 \\
& + \delta^{1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-3}\delta^{2l_3}|Z^{n_3}\varphi|^2 + \delta^{3-4\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-2}\delta^{2l_2}|Z^{n_2}\mathrm{tr}\check{\lambda}|^2 \\
& + \delta^{2l+1-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \varrho^{2m-1}|\mathring{L}Z^kR\varphi|^2,
\end{aligned}
\tag{9.35}
$$

where l_i is the number of T in Z^{n_i} and $n_i \le k-2+i$ $(i=1,2,3,4)$. Applying (9.28) with $\ni = 1$ to estimate the last term and using Proposition 7.3 to deal with the other corresponding terms, one then can obtain

$$
\begin{aligned}
|\overline{II}| \lesssim & \delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du' + \delta^{4-4\varepsilon_0} + \delta^{3-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-2}\tilde{E}_{1,\le k+2}(\tau,u)d\tau \\
& + \delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-3}\tilde{E}_{2,\le k+2}(\tau,u)d\tau.
\end{aligned}
\tag{9.36}
$$

We now treat the difficult term $\bar{I}$. Choose $f_1 = \varrho^{2m-2}\underline{\mathring{L}}\varphi_\gamma$, $f_2 = Z^{k+1}\varphi_\gamma$ and $f_3 = \varrho^2 Z^k(\mathrm{tr}\check{\lambda} - E)$ in (9.3), and define

$$
\begin{aligned}
\bar{I}_1 &= \delta^{2l}\int_{D^{\mathfrak{t},u}} \mathring{L}\big(\varrho^2 Z^k(\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\underline{\mathring{L}}\varphi_\gamma)(RZ^{k+1}\varphi_\gamma), \\
\bar{I}_2 &= \delta^{2l}\int_{\Sigma_{\mathfrak{t}}^u} (RZ^{k+1}\varphi_\gamma)(\varrho^{2m-2}\underline{\mathring{L}}\varphi_\gamma)\big(\varrho^2 Z^k(\mathrm{tr}\check{\lambda}-E)\big).
\end{aligned}
$$

It then holds that

$$
-\bar{I} = -\bar{I}_1 + \bar{I}_2 + \delta^{2l}\int_{D^{\mathfrak{t},u}} Er_1 + \delta^{2l}\int_{\Sigma_{\mathfrak{t}}^u} Er_2 + \delta^{2l}\int_{\Sigma_{t_0}^u} Er_3. \tag{9.37}
$$

Note that (8.5) implies that

$$\begin{aligned}
&\mathring{L}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)\\
=&-2\varrho E+c^{-1}\varrho^2(\mathring{L}^\alpha\mathring{L}\varphi_\alpha-\mathring{L}c)\mathrm{tr}\lambda-\varrho^2(\mathrm{tr}\check{\lambda})^2+\varrho^2 e\\
=&c^{-1}\varrho(-2\not{d}x^a\cdot\not{d}\varphi_a+2\mathring{L}c-\mathring{L}^\alpha\mathring{L}\varphi_\alpha)+c^{-1}\varrho^2(\mathring{L}^\alpha\mathring{L}\varphi_\alpha\\
&-\mathring{L}c)\mathrm{tr}\check{\lambda}-\varrho^2(\mathrm{tr}\check{\lambda})^2+\varrho^2 e,
\end{aligned}\tag{9.38}$$

so $\bar{I}_1$ can be rewritten as

$$\begin{aligned}
\bar{I}_1=&\delta^{2l}\int_{D^{\mathfrak{t},u}}[\mathring{L},Z^k]\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\mathring{\underline{L}}\varphi_\gamma)(RZ^{k+1}\varphi_\gamma)\\
&+\delta^{2l}\int_{D^{\mathfrak{t},u}}\mathring{L}\big([\varrho^2,Z^k](\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\mathring{\underline{L}}\varphi_\gamma)(RZ^{k+1}\varphi_\gamma)\\
&+\delta^{2l}\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\mathring{L}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)\mathring{\underline{L}}\varphi_\gamma(RZ^{k+1}\varphi_\gamma)\\
=:&\bar{I}_{11}+\bar{I}_{12}+\bar{I}_{13}.
\end{aligned}$$

$\bar{I}_{1i}$ $(i=1,2,3)$ will be treated separately as follows.

i. To estimate $\bar{I}_{11}$, one can use the facts that $[\mathring{L},Z^k]=\sum_{k_1+k_2=k-1}Z^{k_1}[\mathring{L},Z]Z^{k_2}$, $[\mathring{L},\varrho\mathring{L}]=\mathring{L}$ and $[\mathring{L},\bar{Z}]={}^{(\bar{Z})}\not{\pi}_{\mathring{L}}^X\not{d}_X$ with $\bar{Z}\in\{T,R\}$, and Proposition 7.3 to obtain

$$\begin{aligned}
&|\ \delta^{2l}\int_{D^{\mathfrak{t},u}}Z^{k_1}[\mathring{L},\varrho\mathring{L}]Z^{k_2}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\mathring{\underline{L}}\varphi_\gamma)(RZ^{k_1}(\varrho\mathring{L})Z^{k_2}R\varphi_\gamma)\ |\\
\lesssim&|\ \delta^{2l}\int_{D^{\mathfrak{t},u}}Z^{k_1}\mathring{L}Z^{k_2}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\mathring{\underline{L}}\varphi_\gamma)([RZ^{k_1},\varrho\mathring{L}]Z^{k_2}R\varphi_\gamma)\ |\\
&+|\ \delta^{2l}\int_{D^{\mathfrak{t},u}}Z^{k_1}\mathring{L}Z^{k_2}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)(\varrho^{2m-2}\mathring{\underline{L}}\varphi_\gamma)(\varrho\mathring{L}RZ^{k_1}Z^{k_2}R\varphi_\gamma)\ |\\
\lesssim&\delta^{-\varepsilon_0}\int_{D^{\mathfrak{t},u}}\varrho^{2m-3/2}\{\underbrace{\delta^{l_2}|Z^{n_2}\mathrm{tr}\check{\lambda}|}+\delta^{l_2}|Z^{n_2}E|\}\{\tau^{-1/2}\delta^{l_3}|Z^{n_3}\varphi|\\
&\quad+\delta^{1-\varepsilon_0}\tau^{-1/2}\delta^{l_1}(|\not{\mathcal{L}}_Z^{n_1}{}^{(R)}\not{\pi}_{\mathring{L}}|+|\not{\mathcal{L}}_Z^{n_1}{}^{(T)}\not{\pi}_{\mathring{L}}|)+\underbrace{\varrho\delta^l|\mathring{L}Z^{k+1}\varphi|}\}\\
\lesssim&\delta^{3-3\varepsilon_0}+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-5/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau\\
&+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau+\delta^{-\varepsilon_0}\int_0^u F_{1,k+2}(\mathfrak{t},u')du',
\end{aligned}\tag{9.39}$$

where the first and second term underlined by braces have been estimated by using (8.31) and (9.28) with $\ni=\frac{1}{2}$ respectively, and by (7.5), and similarly,

$$
\begin{aligned}
&| \ \delta^{2l} \int_{D^{\mathfrak{t},u}} Z^{k_1}[\mathring{L}, \bar{Z}] Z^{k_2} \big(\varrho^2(\mathrm{tr}\check{\lambda} - E)\big)(\varrho^{2m-2} \underline{\mathring{L}}\varphi_\gamma)(RZ^{k+1}\varphi_\gamma) \ | \\
\lesssim &\delta^{2-4\varepsilon_0} \int_{D^{\mathfrak{t},u}} \varrho^{4m-4} \delta^{2l_1} |\not{\mathcal{L}}_Z^{n_1(\bar{Z})} \not{\pi}_{\mathring{L}}|^2 + \delta^{2-4\varepsilon_0} \int_{D^{\mathfrak{t},u}} \varrho^{4m-2}\delta^{2l_2}|Z^{n_2}\mathrm{tr}\check{\lambda}|^2 \\
&+ \delta^{4-6\varepsilon_0} \int_{D^{\mathfrak{t},u}} \varrho^{4m-7}\delta^{2l_3}|Z^{n_3}x|^2 + \delta^{2-4\varepsilon_0} \int_{D^{\mathfrak{t},u}} \varrho^{4m-4}\delta^{2l_3}|Z^{n_3}\varphi|^2 \\
&+ \delta^{4-6\varepsilon_0} \int_{D^{\mathfrak{t},u}} \varrho^{4m-5}\delta^{2l_2}|Z^{n_2}\check{L}^i|^2 + \int_{D^{\mathfrak{t},u}} \delta^{2l}|\not{d}Z^{k+1}\varphi|^2 \\
\lesssim &\delta^{5-6\varepsilon_0} + \delta^{4-4\varepsilon_0} \int_{t_0}^{\mathfrak{t}} \tau^{2m-4}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau \\
&+ \delta^{4-4\varepsilon_0} \int_{t_0}^{\mathfrak{t}} \tau^{4m-4}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du',
\end{aligned}
\tag{9.40}
$$

where l_i is the number of T in Z^{n_i} with $n_i \leq k-2+i$ $(i=1,2,3)$.
Combining (9.39) with (9.40) yields

$$
\begin{aligned}
|\bar{I}_{11}| \lesssim &\delta^{3-3\varepsilon_0} + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-5/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau + \delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du' \\
&+ \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-\varepsilon_0}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}
\tag{9.41}
$$

ii. $|\bar{I}_{12}|$ can be handled similarly as in Case i. Indeed, due to $[\varrho^2, \varrho\mathring{L}]f = -2\varrho^2 f$ and $[\varrho^2, T]f = 2\varrho f$, then as for (9.39), one can get

$$
\begin{aligned}
&| \ \delta^{2l} \int_{D^{\mathfrak{t},u}} \mathring{L}\big(Z^{k_1}[\varrho^2, \varrho\mathring{L}]Z^{k_2}(\mathrm{tr}\check{\lambda} - E)\big)(\varrho^{2m-2}\underline{\mathring{L}}\varphi_\gamma)(RZ^{k_1}(\varrho\mathring{L})Z^{k_2}R\varphi_\gamma) \ | \\
\lesssim &\delta^{3-3\varepsilon_0} + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-5/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau \\
&+ \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-\varepsilon_0}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}
\tag{9.42}
$$

Meanwhile, it follows from (8.31) and Proposition 7.3 that

$$
\begin{aligned}
&| \ \delta^{2l} \int_{D^{\mathfrak{t},u}} \mathring{L}\big(Z^{k_1}[\varrho^2, T]Z^{k_2}(\mathrm{tr}\check{\lambda} - E)\big)(\varrho^{2m-2}\underline{\mathring{L}}\varphi_\gamma)(RZ^{k+1}\varphi_\gamma) \ | \\
\lesssim &\int_{D^{\mathfrak{t},u}} \delta^{2l}|\not{d}Z^{k+1}\varphi|^2 + \delta^{2-2\varepsilon_0}\int_{D^{\mathfrak{t},u}} \tau^{4m-3}\delta^{2l_2}\big(|Z^{n_2}\mathrm{tr}\check{\lambda}|^2 + |Z^{n_2}E|^2\big) \\
\lesssim &\delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du' + \delta^{5-4\varepsilon_0} + \delta^{4-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau \\
&+ \delta^{4-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{4m-5}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau.
\end{aligned}
\tag{9.43}
$$

Therefore,

$$
\begin{aligned}
|\bar{I}_{12}| \lesssim &\delta^{3-3\varepsilon_0} + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-5/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau + \delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du' \\
&+ \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-\varepsilon_0}\int_0^u F_{1,k+2}(\mathfrak{t},u')du'.
\end{aligned}
\tag{9.44}
$$

iii. We will use (9.38) to estimate $\bar{I}_{13}$. Due to (9.2) and (9.34), one has

$$\begin{aligned}
&\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\big(c^{-1}\varrho(2\not{d}x^a\cdot\not{d}\varphi_a-2\mathring{L}c+\mathring{L}^\alpha\mathring{L}\varphi_\alpha)\big)\underline{\mathring{L}}\varphi_\gamma(RZ^{k+1}\varphi_\gamma)\Big|\\
\leq&\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}RZ^k\big(c^{-1}\varrho(2\not{d}x^a\cdot\not{d}\varphi_a-2\mathring{L}c+\mathring{L}^\alpha\mathring{L}\varphi_\alpha)\big)\underline{\mathring{L}}\varphi_\gamma(Z^{k+1}\varphi_\gamma)\Big|\\
&+\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\big(c^{-1}\varrho(2\not{d}x^a\cdot\not{d}\varphi_a-2\mathring{L}c+\mathring{L}^\alpha\mathring{L}\varphi_\alpha)\big)R\underline{\mathring{L}}\varphi_\gamma(Z^{k+1}\varphi_\gamma)\Big|\\
&+\frac{1}{2}\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\big(c^{-1}\varrho(2\not{d}x^a\cdot\not{d}\varphi_a-2\mathring{L}c+\mathring{L}^\alpha\mathring{L}\varphi_\alpha)\big)\underline{\mathring{L}}\varphi_\gamma(Z^{k+1}\varphi_\gamma)\text{tr}{}^{(R)}\not{\pi}\Big|\\
\lesssim&\int_{D^{\mathfrak{t},u}}\delta^{1-\varepsilon_0}\varrho^{2m-3/2}\big\{\delta^{1-\varepsilon_0+l_4}\tau^{-5/2}|Z^{n_4}x|+\tau^{-1}\delta^{l_3}|Z^{n_3}\varphi|+\delta^l|Z^k(\mathring{L}+\frac{1}{2\varrho})R\varphi|\\
&+\delta^{1-\varepsilon_0+l_2}\tau^{-3/2}(|Z^{n_2}\text{tr}{}^{(R)}\not{\pi}|+|\not{\mathcal{L}}_Z^{n_2}{}^{(R)}\not{\pi}_{\mathring{L}}|+\delta|\not{\mathcal{L}}_Z^{n_2}{}^{(T)}\not{\pi}_{\mathring{L}}|+\tau|Z^{n_2}\text{tr}\check{\lambda}|)\\
&+\delta^{2-\varepsilon_0+l_1}\tau^{-5/2}|\not{\mathcal{L}}_Z^{n_1}{}^{(R)}\not{\pi}_T|+\underbrace{\delta^l|\mathring{L}RZ^k\varphi|}+\delta^{1-\varepsilon_0+l_3}\tau^{-3/2}|Z^{n_3}\check{L}^i|\big\}\delta^l|Z^{k+1}\varphi|,
\end{aligned}$$

where l_i is the number of T in Z^{n_i} with $n_i\leq k-2+i$ $(i=1,2,3,4)$, and the term underline with brace can be estimated by using (9.28) for $\backepsilon=\frac{1}{2}$.
This, together with Lemma 7.1, Proposition 7.3 and (9.28), yields

$$\begin{aligned}
&\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\big(c^{-1}\varrho(2\not{d}x^a\cdot\not{d}\varphi_a-2\mathring{L}c+\mathring{L}^\alpha\mathring{L}\varphi_\alpha)\big)\underline{\mathring{L}}\varphi_\gamma(RZ^{k+1}\varphi_\gamma)\Big|\\
\lesssim&\delta^{4-2\varepsilon_0}+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-3/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau\\
&+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau+\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'.
\end{aligned}\tag{9.45}$$

In addition, for l_i and n_i defined in (9.45), one can get

$$\begin{aligned}
&\delta^{2l}\,\Big|\int_{D^{\mathfrak{t},u}}\varrho^{2m-2}Z^k\big(c^{-1}\varrho^2(\mathring{L}^\alpha\mathring{L}\varphi_\alpha-\mathring{L}c)\text{tr}\check{\lambda}-\varrho^2(\text{tr}\check{\lambda})^2+\varrho^2e\big)\underline{\mathring{L}}\varphi_\gamma(RZ^{k+1}\varphi_\gamma)\Big|\\
\lesssim&\int_{D^{\mathfrak{t},u}}\delta^{2l}|\not{d}Z^{k+1}\varphi|^2+\int_{D^{\mathfrak{t},u}}\delta^{2-4\varepsilon_0}\tau^{4m-4}\delta^{2l_3}|Z^{n_3}\varphi|^2+\int_{D^{\mathfrak{t},u}}\delta^{4-6\varepsilon_0}\tau^{4m-5}\delta^{2l_2}|Z^{n_2}L^i|^2\\
&+\int_{D^{\mathfrak{t},u}}\delta^{4-6\varepsilon_0}\tau^{4m-7}\delta^{2l_3}|Z^{n_3}x|^2+\int_{D^{\mathfrak{t},u}}\delta^{2-4\varepsilon_0}\tau^{4m-2}\delta^{2l_2}|Z^{n_2}\text{tr}\check{\lambda}|^2\\
\lesssim&\delta^{5-6\varepsilon_0}+\delta^{4-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-4}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{4-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{4m-4}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\
&+\delta^{-1}\int_0^u\delta F_{2,k+2}(\mathfrak{t},u')du',
\end{aligned}\tag{9.46}$$

where the last integral in the second inequality has been estimated by (8.31). Collecting (9.38), (9.45) and (9.46) yields

$$\begin{aligned}
|\bar{I}_{13}|\lesssim&\delta^{4-2\varepsilon_0}+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-3/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{-1}\int_0^u\delta F_{2,k+2}(\mathfrak{t},u')du'\\
&+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau+\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'.
\end{aligned}\tag{9.47}$$

We conclude from (9.41), (9.44) and (9.47) that

$$\begin{aligned}|\bar{I}_1| \lesssim &\delta^{3-3\varepsilon_0} + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-3/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau + \delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du'\\ &+ \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-\varepsilon_0}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'.\end{aligned} \tag{9.48}$$

Next we deal with $\bar{I}_2$. For any vectorfield $\bar{Z}\in\{R,T\}$, it follows from equation (9.38) that

$$\begin{aligned}&\delta^l|\mathring{L}\big(\varrho^2 R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)\big)|\\ \leq&\delta^l|[\mathring{L},R\bar{Z}^k]\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)| + \delta^l|\mathring{L}R[\varrho^2,\bar{Z}^k](\mathrm{tr}\check{\lambda}-E)| + \delta^l|R\bar{Z}^k\mathring{L}\big(\varrho^2(\mathrm{tr}\check{\lambda}-E)\big)|\\ \lesssim&\delta^{1-\varepsilon_0}\varrho^{1/2}\delta^{l_2}|R\bar{Z}^{n_2}(\mathrm{tr}\check{\lambda}-E)|\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\delta^{l_2}|\not{\mathcal{L}}_{\bar{Z}}^{n_2(\bar{Z})}\not{\pi}_{\mathring{L}}| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_4}|Z^{n_4}x|\\ &+ \delta^{1-\varepsilon_0}\varrho^{1/2}\delta^{l_3}|\not{d}Z^{n_3}\varphi| + \delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\delta^{l_3}|\bar{Z}^{n_3}\check{L}^i| + \varrho\delta^{l_2}|\bar{Z}^{n_2}(\not{d}x^a\cdot\not{d}R\varphi_a)|\\ &+ \delta^{l_3}|Z^{n_3}\varphi| + \mathfrak{t}\delta^{l_3}|\mathring{L}Z^{n_3}\varphi| + \delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}|\not{\mathcal{L}}_Z^{n_1(R)}\not{\pi}_T| + \delta^{1-\varepsilon_0}\mathfrak{t}^{1/2}\delta^{l_2}|Z^{n_2}\mathrm{tr}\check{\lambda}|\\ &+ \delta\varrho\delta^{l_2}|\bar{Z}^{n_2}\mathring{L}\mathrm{tr}\check{\lambda}|.\end{aligned} \tag{9.49}$$

This, together with (7.3) and (8.32), yields

$$\begin{aligned}&\delta^l\|\mathring{L}\big(\varrho^2 R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)\big)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\ \lesssim&\delta^{1-\varepsilon_0}\varrho^{1/2}\delta^l\|R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + \delta^{3/2-\varepsilon_0} + \delta^{1-\varepsilon_0}\mathfrak{t}^{1/2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)}\\ &+ \delta\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)} + \varrho\delta^{l_2}\|\bar{Z}^{n_2}(\not{d}x^a\cdot\not{d}R\varphi_a)\|_{L^2(\Sigma_{\mathfrak{t}}^u)} + t\delta^{l_3}\|\mathring{L}Z^{n_3}\varphi\|_{L^2(\Sigma_{\mathfrak{t}}^u)}.\end{aligned} \tag{9.50}$$

On the other hand, applying (7.8) to $F=\varrho^2R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)(t,u,\vartheta)-\varrho_0^2R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)(t_0,u,\vartheta)$, one can get from (9.50) and (9.34) that

$$\begin{aligned}&\varrho^{3/2}\delta^l\|R\bar{Z}^k(\mathrm{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\ \lesssim&\delta^{3/2-\varepsilon_0}\mathfrak{t}^{1/2} + \delta^{1-\varepsilon_0}\mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}\\ &+ \int_{t_0}^{\mathfrak{t}}\tau^{1/2}\big\{\delta^{l_2}\|\bar{Z}^{n_2}(\not{d}x^a\cdot\not{d}R\varphi_a)\|_{L^2(\Sigma_\tau^u)} + \delta^{l_3}\|\mathring{L}Z^{n_3}\varphi\|_{L^2(\Sigma_\tau^u)}\big\}d\tau\\ \lesssim&\int_{t_0}^{\mathfrak{t}}\tau^{1/2}\delta^{l_3}\|(\mathring{L}+\frac{1}{2\varrho})\bar{Z}^{n_3}\varphi\|_{L^2(\Sigma_\tau^u)}d\tau + \delta^{3/2-\varepsilon_0}\mathfrak{t}^{1/2}\\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}\\ \lesssim&\mathfrak{t}^{1-m}\Big(\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'\Big)^{1/2} + \delta^{3/2-\varepsilon_0}\mathfrak{t}^{1/2}\\ &+ \delta^{1-\varepsilon_0}\mathfrak{t}^{1-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)} + \delta\mathfrak{t}^{1/2}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}.\end{aligned} \tag{9.51}$$

When there is at least one $\varrho\mathring{L}$ in Z^k, that is, $Z^k=Z^{k_1}(\varrho\mathring{L})\bar{Z}^{k_2}$ for $k_1+k_2=k-1$, according to (8.5), Proposition 7.3, (8.24) and (7.5), one can deduce that

$$\begin{aligned}
&\delta^l\|RZ^{k_1}(\varrho\mathring{L})\bar{Z}^{k_2}(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\leq&\delta^l\|RZ^{k_1}[\varrho\mathring{L},\bar{Z}^{k_2}](\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^l\|RZ^{k_1}\bar{Z}^{k_2}(\varrho\mathring{L})(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\lesssim&\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_1}\|\not{\mathcal{L}}_Z^{n_1(\bar{Z})}\not{\pi}_{\mathring{L}}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^{l_1}\|RZ^{n_1}\text{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\mathfrak{t}^{-1}\delta^{l_3}\|Z^{n_3}\varphi\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2}\delta^{l_1}\|Z^{n_1}\text{tr}\check{\lambda}\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^{1-\varepsilon_0}\mathfrak{t}^{-3/2}\delta^{l_2}\|Z^{n_2}\check{L}^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-5/2}\delta^{l_3}\|Z^{n_3}x^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\lesssim&\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1}+\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)}+\delta\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)}.
\end{aligned}\tag{9.52}$$

Thus, it follows from (5.1) and (5.2) that

$$\begin{aligned}
\delta^l\|RZ^k(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\lesssim&\,\mathfrak{t}^{-1/2-m}\Big(\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'\Big)^{1/2}+\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1}\\
&+\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2-m}\sqrt{\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)}+\delta\mathfrak{t}^{-1}\sqrt{\tilde{E}_{2,\leq k+2}(\mathfrak{t},u)},
\end{aligned}\tag{9.53}$$

which, together with (9.2) and (8.31), yields

$$\begin{aligned}
|\bar{I}_2|=&\delta^{2l}|\int_{\Sigma_{\mathfrak{t}}^u}(Z^{k+1}\varphi_\gamma)\varrho^{2m}R\big\{\underline{\mathring{L}}\varphi_\gamma Z^k(\text{tr}\check{\lambda}-E)\big\}dx\\
&+\frac{1}{2}\int_{\Sigma_{\mathfrak{t}}^u}\varrho^{2m}(Z^{k+1}\varphi_\gamma)\underline{\mathring{L}}\varphi_\gamma Z^k(\text{tr}\check{\lambda}-E)\cdot\text{tr}^{(R)}\not{\pi}dx|\\
\lesssim&\delta^{-\varepsilon_0}\varrho^{2m-1/2}\delta^l\|Z^{k+1}\varphi\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\cdot\delta^l\|RZ^k(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&+\delta^{-\varepsilon_0}\varrho^{2m-1/2}\delta^l\|Z^{k+1}\varphi\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\cdot\delta^l\|Z^k(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
\lesssim&\delta^{3-3\varepsilon_0}\varrho^{2m-3/2}+\delta^{-\varepsilon_0}\varrho^{-1/2}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'\\
&+\delta^{2-3\varepsilon_0}\varrho^{-1/2}\tilde{E}_{1,\leq k+2}(\mathfrak{t},u)+\delta^{2-\varepsilon_0}\varrho^{2m-3/2}\tilde{E}_{2,\leq k+2}(\mathfrak{t},u).
\end{aligned}\tag{9.54}$$

Recall (9.37). It remains to estimate terms involving Er_i ($i=1,2,3$) given in Lemma 9.2. First, one has

$$\begin{aligned}
&\delta^{2l}\,|\int_{D^{\mathfrak{t},u}}Er_1\,|\\
\lesssim&\delta^{-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-1}\delta^l\|Z^{k+1}\varphi\|_{L^2(\Sigma_\tau^u)}\Big\{\delta^{1-\varepsilon_0+l}\|\not{d}Z^k(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_\tau^u)}\\
&+\tau^{-1/2}\delta^{l_2}\|Z^{n_2}\text{tr}\check{\lambda}\|_{L^2(\Sigma_\tau^u)}+\delta^{1-\varepsilon_0}\tau^{-2}\delta^{l_1}\|\not{\mathcal{L}}_Z^{n_1(\bar{Z})}\not{\pi}_{\mathring{L}}\|_{L^2(\Sigma_\tau^u)}\\
&+\tau^{-3/2}\delta^{l_3}\|Z^{n_3}\varphi\|_{L^2(\Sigma_\tau^u)}+\delta^{1-\varepsilon_0}\tau^{-2}\delta^{l_2}\|Z^{n_2}\check{L}^i\|_{L^2(\Sigma_\tau^u)}\\
&+\delta^{1-\varepsilon_0}\tau^{-3}\delta^{l_3}\|Z^{n_3}x^i\|_{L^2(\Sigma_\tau^u)}\Big\}d\tau\\
&+\delta^{1-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-1}\delta^l\|\not{d}Z^{k+1}\varphi\|_{L^2(\Sigma_\tau^u)}\delta^l\|Z^k(\text{tr}\check{\lambda}-E)\|_{L^2(\Sigma_\tau^u)}d\tau\\
&+\int_{t_0}^{\mathfrak{t}}|\int_{\Sigma_{\mathfrak{t}}^u}\delta^{2l}(RZ^{k+1}\varphi_\gamma)\varrho^{2m-1}\underline{\mathring{L}}\varphi_\gamma Z^k(\text{tr}\check{\lambda}-E)dx\,|\,d\tau,
\end{aligned}\tag{9.55}$$

here one has used the identity $(\mathring{L}+\frac{1}{2\varrho})f_1=(2m-2)\varrho^{2m-3}\underline{\mathring{L}}\varphi_\gamma+\varrho^{2m-2}(\mathring{L}+\frac{1}{2\varrho})\underline{\mathring{L}}\varphi_\gamma=(2m-2)\varrho^{2m-3}\underline{\mathring{L}}\varphi_\gamma+O(\delta^{1-2\varepsilon_0}\varrho^{2m-4})$ due to (4.3). Notice that the last term on the right hand side of (9.55) is just $\int_{t_0}^{\mathfrak{t}} \varrho^{-1}|\bar{I}_2|d\tau$ which can be estimated by (9.54), while the other terms can be estimated by using (8.24), (8.31), (7.4) and Proposition 7.3. Thus we conclude that

$$\begin{aligned}\delta^{2l} \,|\int_{D^{\mathfrak{t},u}} Er_1 \,|\lesssim \delta^{3-3\varepsilon_0} + \delta^{2-3\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{m-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau \\ + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau + \delta^{-1}\int_0^u \tilde{F}_{1,\leq k+2}(\mathfrak{t},u')du'.\end{aligned} \tag{9.56}$$

Next, $\delta^{2l}\int_{\Sigma_{\mathfrak{t}}^u} Er_2$ and $\delta^{2l}\int_{\Sigma_{t_0}^u} Er_3$ can be bounded by applying (8.31) and Proposition 7.3 directly as

$$\delta^{2l}\int_{\Sigma_{\mathfrak{t}}^u} Er_2 + \delta^{2l}\int_{\Sigma_{t_0}^u} Er_3 \lesssim \delta^{3-3\varepsilon_0} + \delta^{2-3\varepsilon_0}\tilde{E}_{1,\leq k+2}(\mathfrak{t},u) + \delta^{2-\varepsilon_0}\tilde{E}_{2,\leq k+2}(\mathfrak{t},u). \tag{9.57}$$

Then the estimate for $\bar{I}$ follows from (9.48), (9.54), (9.56), (9.57) and (9.37).This and (9.36) complete the estimate (9.33) in Proposition 9.1. □

Based on (9.32) and (9.33), we can end this subsection with

$$\begin{aligned}&\delta^{2l}\,|\int_{D^{\mathfrak{t},u}} \sum_{j=1}^{k} \big(Z_{k+1} + {}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_{k+2-j} + {}^{(Z_{k+2-j})}\Lambda\big){}^{(R)}D_{\gamma,2}^{k-j}\cdot(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1} \\ &\qquad + \frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\,| \\ &\lesssim |\,\delta^{2l}\int_{D^{\mathfrak{t},u}} Z^{k\ (R)}D_{\gamma,2}^{0}(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1} + \frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\,| + \delta^{-1}\int_0^u F_{1,k+2}(\mathfrak{t},u')du' \\ &\quad + \sum_{p=0}^{k-1}\delta^{1+2l}\int_{D^{\mathfrak{t},u}} |Z^{p\ (R)}D_{\gamma,2}^{k-j}|^2\cdot|Z^{\leq k-1-p\ (Z)}\Lambda|^2 \\ &\quad + \delta^{1+2l}\sum_{j=1}^{k-1}\int_{D^{\mathfrak{t},u}} |\sum_{j=1}^{k}\big(Z_{k+1} + {}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_{k+2-j} + {}^{(Z_{k+2-j})}\Lambda\big){}^{(R)}D_{\gamma,2}^{k-j}\,|^2 \\ &\lesssim \delta^{3-3\varepsilon_0} + \delta^{-1}\int_0^u \tilde{F}_{1,\leq k+2}(\mathfrak{t},u')du' + \delta^{-1}\int_0^u \delta F_{2,k+2}(\mathfrak{t},u')du' \\ &\quad + \delta^{1-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{m-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau + \delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-5/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau \\ &\quad + \delta^{2-3\varepsilon_0}\tilde{E}_{1,\leq k+2}(\mathfrak{t},u) + \delta^{2-\varepsilon_0}\tilde{E}_{2,\leq k+2}(\mathfrak{t},u).\end{aligned} \tag{9.58}$$

9.2 Estimates for J_2^k

Recall that $J_2^k \equiv \mu \mathrm{div}^{(Z_{k+1})}C_\gamma^k + \big(Z_{k+1} + {}^{(Z_{k+1})}\Lambda\big)\dots\big(Z_1 + {}^{(Z_1)}\Lambda\big)\Phi_\gamma^0$ does not contain the top order derivatives of $\mathrm{tr}\lambda$ and $\not{\Delta}\mu$. Therefore, according to Proposition 7.3 and the expressions of ${}^{(Z)}D_{\gamma,j}^k$ in (9.9)-(9.14) and (9.18)-(9.20), one can get

$$\begin{aligned}&\delta^{2l+1}\int_{D^{\mathfrak{t},u}} |\sum_{j=1}^{3} {}^{(Z_{k+1})}D_{\gamma,j}^k\cdot \mathring{\underline{L}}\varphi_\gamma^{k+1}| \\ &\lesssim \delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}} \tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau + \delta\int_{t_0}^{\mathfrak{t}}\tau^{-2m}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau \\ &\quad + \delta^{-1}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'\end{aligned} \tag{9.59}$$

and

$$\begin{aligned}&\delta^{2l}\Big|\int_{D^{\mathfrak{t},u}}\sum_{j=1}^{3}{}^{(Z_{k+1})}D_{\gamma,j}^{k}\cdot(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1}+\frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\Big|\\ \lesssim&\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\ &+\delta^{-1}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'.\end{aligned}\tag{9.60}$$

In addition, Φ_γ^0 is just $\mu\Box_g\varphi_\gamma$ which has the explicit form in (3.41). Then, Proposition 7.3 and (6.39) lead to

$$\begin{aligned}&\delta^{2l+1}\int_{D^{\mathfrak{t},u}}|(Z_{k+1}+{}^{(Z_{k+1})}\Lambda)\cdots(Z_1+{}^{(Z_1)}\Lambda)\Phi_\gamma^0\cdot\underline{\mathring{L}}\varphi_\gamma^{k+1}|\\ \lesssim&\delta^{3-3\varepsilon_0}+\delta^{2-3\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2m-1/2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{2-\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-3/2}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\ &+\delta^{-1}\int_0^u F_{1,\leq k+2}(\mathfrak{t},u')du'\end{aligned}\tag{9.61}$$

and

$$\begin{aligned}&\delta^{2l}\Big|\int_{D^{\mathfrak{t},u}}(Z_{k+1}+{}^{(Z_{k+1})}\Lambda)\cdots(Z_1+{}^{(Z_1)}\Lambda)\Phi_\gamma^0\cdot(\varrho^{2m}\mathring{L}\varphi_\gamma^{k+1}+\frac{1}{2}\varrho^{2m-1}\varphi_\gamma^{k+1})\Big|\\ \lesssim&\delta^{4-4\varepsilon_0}+\delta^{3-4\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{-2}\tilde{E}_{1,\leq k+2}(\tau,u)d\tau+\delta^{3-2\varepsilon_0}\int_{t_0}^{\mathfrak{t}}\tau^{2m-3}\tilde{E}_{2,\leq k+2}(\tau,u)d\tau\\ &+\delta^{-1}\int_0^u F_{1,\leq n+1}(\mathfrak{t},u')du'.\end{aligned}\tag{9.62}$$

10 The global existence near C_0

We are now ready to prove the global existence of the smooth solution ϕ to the equation (1.6) with initial data (1.7) near C_0. Indeed, substituting (9.15)-(9.17), (9.22), (9.23), (9.30) and (9.58)-(9.62) into (6.35), and using the Gronwall's inequality, one can get that under the assumptions $(\star)$ with small $\delta>0$ for $\frac{1}{2}<m<\frac{3}{4}$,

$$\delta\tilde{E}_{2,\leq 2N-4}(\mathfrak{t},u)+\delta\tilde{F}_{2,\leq 2N-4}(\mathfrak{t},u)+\tilde{E}_{1,\leq 2N-4}(\mathfrak{t},u)+\tilde{F}_{1,\leq 2N-4}(\mathfrak{t},u)\lesssim\delta^{2-2\varepsilon_0}.\tag{10.1}$$

Based on (10.1), we are ready to close the bootstrap assumptions $(\star)$ in Section 4. To this end, one needs the following Sobolev type embedding formula.

Lemma 10.1. *For any function $f\in H^2(S_{\mathfrak{t},u})$, under the assumptions $(\star)$ for $\delta>0$ small, it holds that*

$$\|f\|_{L^\infty(S_{\mathfrak{t},u})}\lesssim\frac{1}{\sqrt{\mathfrak{t}}}\sum_{a\leq 1}\|R^af\|_{L^2(S_{\mathfrak{t},u})}.\tag{10.2}$$

Proof. This follows from Proposition 18.10 of [36]. □

It follows from (10.2), (10.1) and (7.1) that for $k\leq 2N-6$,

$$\delta^l|Z^k\varphi_\gamma|\lesssim\frac{\delta^l}{\sqrt{\mathfrak{t}}}\sum_{a\leq 1}\|R^aZ^k\varphi_\gamma\|_{L^2(S_{\mathfrak{t},u})}\lesssim\frac{\delta^{1/2}}{\sqrt{\mathfrak{t}}}\big(\sqrt{E_{1,\leq 2N-4}}+\sqrt{E_{2,\leq 2N-4}}\big)\lesssim\delta^{1-\varepsilon_0}\mathfrak{t}^{-1/2},\tag{10.3}$$

which is independent of M. This closes the bootstrap assumptions ($\star$), and hence the existence of the solution ϕ to equation (1.6) with (1.7) in the domain $D^{\mathfrak{t},4\delta}$ can be proved by the standard continuity argument (see Figure 3 in Section 3.1).

Finally, let $\Gamma\in\{(t+r)L,\underline{L},\Omega\}$ defined in the end of Section 1. For any $(t,x)\in\tilde{C}_{2\delta}$, we will refine the estimate on $|\Gamma^\alpha\phi|$ with better smallness $O(\delta^{2-\varepsilon_0})$

$$|\Gamma^\alpha\phi(t,x)|\lesssim\delta^{2-\varepsilon_0}t^{-1/2},\quad|\alpha|\le 2N-9,\tag{10.4}$$

which will be crucial to prove the global existence of the solution ϕ to (1.6) in $B_{2\delta}$.

First, we improve the estimates on derivatives of $\mathring{L}^\alpha\mathring{L}\varphi_\alpha$, $\mathring{L}^\alpha\varphi_\alpha$ and $u-(\mathfrak{t}-r)$.

Using (10.2) again, one can get by (7.1) that

$$\begin{aligned}
&\|\varrho\delta^l Z^\beta(\mathring{L}^\alpha\varphi_\alpha)\|_{L^\infty(S_{\mathfrak{t},u})}\\
\lesssim&\delta^{l+1/2}\mathfrak{t}^{-1/2}\Big(\|\mathring{\underline{L}}(\varrho R^{\le 1}Z^\beta(\mathring{L}^\alpha\varphi_\alpha))\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\|\mathring{L}(\varrho R^{\le 1}Z^\beta(\mathring{L}^\alpha\varphi_\alpha))\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\Big)\\
\lesssim&\delta^{l+1/2}\mathfrak{t}^{-1/2}\Big\{\|R^{\le 1}Z^\beta(\mathring{L}^\alpha\varphi_\alpha)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\varrho\|[\mathring{\underline{L}},R]Z^\beta(\mathring{L}^\alpha\varphi_\alpha)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\\
&\qquad+\varrho\|R^{\le 1}[\mathring{\underline{L}},Z^\beta](\mathring{L}^\alpha\varphi_\alpha)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\varrho\|R^{\le 1}Z^\beta\mathring{\underline{L}}(\mathring{L}^\alpha\varphi_\alpha)\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\Big\}.
\end{aligned}$$

Since $\mathring{\underline{L}}(\mathring{L}^\alpha\varphi_\alpha)=(\mathring{\underline{L}}\mathring{L}^\alpha)\varphi_\alpha+\mathring{\underline{L}}^\alpha\mathring{L}\varphi_\alpha$ and $\mathring{\underline{L}}\mathring{L}^\alpha$ is a combination of (3.26) and (3.32), for $|\beta|\le 2N-7$, one then has

$$\begin{aligned}
&\|\varrho\delta^l Z^\beta(\mathring{L}^\alpha\varphi_\alpha)\|_{L^\infty(S_{\mathfrak{t},u})}\\
\lesssim&\delta^{1/2}\mathfrak{t}^{-1/2}\big\{\delta^{l_2}\|Z^{n_2}\varphi\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\delta^{1-\varepsilon_0+l_1}\mathfrak{t}^{-1/2}\|Z^{n_1}\check{L}^i\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\big\}\\
&+\delta^{3/2-\varepsilon_0+l_2}\mathfrak{t}^{-1}\big\{\|Z^{n_2}\mu\|_{L^2(\Sigma_{\mathfrak{t}}^u)}+\mathfrak{t}^{-1}\|Z^{n_2}x^a\|_{L^2(\Sigma_{\mathfrak{t}}^u)}\big\}\\
&+\delta^{3/2-\varepsilon_0+l_0}\mathfrak{t}^{-1}\big\{\|\not{\mathcal{L}}_Z^{n_0(\bar{Z})}\not{\pi}_{\mathring{L}}\|_{L^2(\Sigma_\tau^u)}+\|\not{\mathcal{L}}_Z^{n_0(R)}\not{\pi}_T\|_{L^2(\Sigma_\tau^u)}\big\}\\
\lesssim&\delta^{2-\varepsilon_0}\mathfrak{t}^{-1/2}+\delta^{3/2-\varepsilon_0}\mathfrak{t}^{-1/2}\sqrt{\tilde{E}_{1,\le 2N-4}}+\delta^{3/2}\mathfrak{t}^{-1/2}\sqrt{\tilde{E}_{2,\le 2N-4}}\\
\lesssim&\delta^{2-\varepsilon_0}\mathfrak{t}^{-1/2},
\end{aligned}\tag{10.5}$$

where l_i is the number of T in Z^{n_i} and $n_i\le|\beta|+i$ $(i=0,1,2)$. (10.5) implies that in the domain $D^{\mathfrak{t},u}$, $\mathring{L}^\alpha\varphi_\alpha$ can be estimated more accurately as

$$\delta^l|Z^\beta(\mathring{L}^\alpha\varphi_\alpha)|\lesssim\delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2},\quad|\beta|\le 2N-7.\tag{10.6}$$

Similarly, it holds that

$$\delta^l|Z^\beta(\mathring{L}^\alpha\mathring{L}\varphi_\alpha)|\lesssim\delta^{2-\varepsilon_0}\mathfrak{t}^{-5/2},\quad|\beta|\le 2N-8.\tag{10.7}$$

In addition, it follows from (3.27) and (10.7) that for $|\beta|\le 2N-8$,

$$\delta^l|Z^\beta\check{L}^a|\lesssim\delta^{2-\varepsilon_0}\mathfrak{t}^{-1},\tag{10.8}$$

which leads to

$$|Z^\beta\big(1-\frac{r}{\varrho}\big)|\lesssim\delta^{2-\varepsilon_0}\mathfrak{t}^{-1},\quad|\beta|\le 2N-8\tag{10.9}$$

by (4.41) and the fact $\varphi_0+\varphi_i\omega^i=\mathring{L}^\alpha\varphi_\alpha-\varphi_i\check{L}^i+\varphi_i\omega^i(1-\frac{r}{\varrho})=O(\delta^{2-\varepsilon_0}\mathfrak{t}^{-3/2})+\varphi_i\omega^i(1-\frac{r}{\varrho})$. It follows from (10.9) that the distance between C_0 and $C_{4\delta}$ on the hypersurface $\Sigma_{\mathfrak{t}}$ is $4\delta+O(\delta^{2-\varepsilon_0})$ and the characteristic surface C_u $(0\le u\le 4\delta)$ is almost straight with the error $O(\delta^{2-\varepsilon_0})$ from the corresponding outgoing conic surface.

On the other hand, note that

$$\begin{aligned} L &= \mathring{L} + \mu^{-1}\{(1-\frac{r}{\varrho}) - g_{ij}\check{L}^i\tilde{T}^j + (\frac{\varrho}{r}-1)g_{ij}\check{T}^i\tilde{T}^j\}T + \frac{\varrho}{r}g_{ij}\check{T}^i(\not{d}^X x^j)X, \\ \underline{L} &= \mathring{L} + \mu^{-1}\{1+\frac{\varrho}{r} - c^{-1}\varphi_i\tilde{T}^i - \frac{\varrho}{r}g_{ij}\check{T}^i\tilde{T}^j\}T - \frac{\varrho}{r}g_{ij}\check{T}^i(\not{d}^X x^j)X, \\ \Omega &= R - \mu^{-1}g_{ab}\epsilon_i^a x^i\check{L}^b T + \mu^{-1}c^{-1}\epsilon_i^a x^i\varphi_a T. \end{aligned} \tag{10.10}$$

When $|\beta| \le 2N-7$,

$$\begin{aligned} TZ^\beta(\epsilon_i^a x^i\varphi_a) =& [T, Z^\beta](\epsilon_i^a x^i\varphi_a) + Z^\beta T(\epsilon_i^a x^i\varphi_a) \\ =& \sum_{|\beta_1|+|\beta_2|=|\beta|-1} Z^{\beta_1}[T,Z]Z^{\beta_2}(\epsilon_i^a x^i\varphi_a) + Z^\beta(\mu\epsilon_i^a\tilde{T}^i\varphi_a) \\ &+ Z^\beta\big(T^\alpha R\varphi_\alpha + g_{mn}\epsilon_j^m x^j\check{T}^n\tilde{T}^a T\varphi_a\big), \end{aligned}$$

and then $|TZ^\beta(\epsilon_i^a x^i\varphi_a)| \lesssim \delta^{1-2\varepsilon_0-l}$ (l is the number of T in Z^β), which implies that

$$|\underline{L}Z^\beta(\epsilon_i^a x^i\varphi_a)| \lesssim \delta^{1-2\varepsilon_0-l}, \quad |\beta| \le 2N-7 \tag{10.11}$$

due to the second equation in (10.10) and (10.3). Integrate (10.11) along integral curves of $\underline{L}$, and use the zero boundary value on C_0 to get

$$|Z^\beta(\epsilon_i^a x^i\varphi_a)| \lesssim \delta^{2-2\varepsilon_0-l}, \quad |\beta| \le 2N-7. \tag{10.12}$$

Therefore, in $D^{\mathfrak{t},u}$, collecting (10.10), (10.9), (10.8), (10.12) and (10.3) yields

$$|\ \Gamma^\alpha\varphi_\gamma\ |\lesssim \delta^{1-l-\varepsilon_0}\mathfrak{t}^{-1/2}, \quad |\alpha| \le 2N-7, \tag{10.13}$$

where l is the number of $\underline{L}$ in Γ^α.

Recall that ϕ is the solution of (1.6) and $\varphi_\gamma = \partial_\gamma\phi$. Thus (10.13) implies that for $|\alpha| \le 2N-7$,

$$|\ \underline{L}\Gamma^\alpha\phi\ |\lesssim \delta^{1-l-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{10.14}$$

Hence, as for (10.12), one can get that in $D^{\mathfrak{t},4\delta}$, for $|\alpha| \le 2N-7$,

$$|\ \Gamma^\alpha\phi\ |\lesssim \delta^{2-l-\varepsilon_0}\mathfrak{t}^{-1/2}. \tag{10.15}$$

For any point $P(t^0, x^0) \in \tilde{C}_{2\delta}$, there is an integral line of L across this point and the initial point is denoted by $P_0(t_0, x_0)$ on Σ_{t_0} with $|x_0| = 1$. It follows from (2.11) that

$$|\ L(r^{1/2}\partial^\alpha\underline{L}\phi)\ |\lesssim \delta^{2-2\varepsilon_0-|\alpha|}t^{-3/2}, \quad |\alpha| \le 1. \tag{10.16}$$

Integrating (10.16) along integral curves of L and applying (2.4) to show that on $\tilde{C}_{2\delta}$,

$$|\ \partial^\alpha\underline{L}\phi\ |\lesssim \delta^{2-2\varepsilon_0-|\alpha|}t^{-1/2}, \quad |\alpha| \le 1. \tag{10.17}$$

Using (10.17) and (2.11) again gives $|\ L(r^{1/2}\underline{L}\phi)\ |\lesssim \delta^{2-\varepsilon_0}t^{-3/2}$, which implies in turn that

$$|\underline{L}\phi| \lesssim \delta^{2-\varepsilon_0}t^{-1/2},$$

and hence, $|\ L\underline{L}\phi\ |\lesssim \delta^{2-\varepsilon_0}t^{-3/2}$ holds by (2.11). By an induction argument and (2.11), one can show that on $\tilde{C}_{2\delta}$,

$$|\ \Gamma^\alpha\phi\ |\lesssim \delta^{2-\varepsilon_0}t^{-1/2}, \quad |\alpha| \le 2N-9.$$

Thus, (10.4) is proved.

11 Global existence inside $B_{2\delta}$ and the proof of Theorem 1.1

In this section, we prove the existence of the solution ϕ to (1.6) inside $B_{2\delta}$. To this end, define

$$D_t := \{(\bar{t}, x) : \bar{t} - |x| \geq 2\delta, t_0 \leq \bar{t} \leq t\} \subset B_{2\delta}$$

to be the shaded part in Figure 4 below. Different from the small value problem inside $B_{2\delta}$ in [8] and [16], the solution ϕ to (1.6) in D_t remains large here due to its initial data on time t_0 (see Theorem 2.1). Note that for $\delta > 0$ small, the L^∞ norm of ϕ and its first order derivatives are small on the boundary $\tilde{C}_{2\delta}$ of $B_{2\delta}$ (especially, $\Gamma^\alpha \phi$ admits the better smallness $O(\delta^{2-\varepsilon_0})$ on $\tilde{C}_{2\delta}$, see (10.4)).

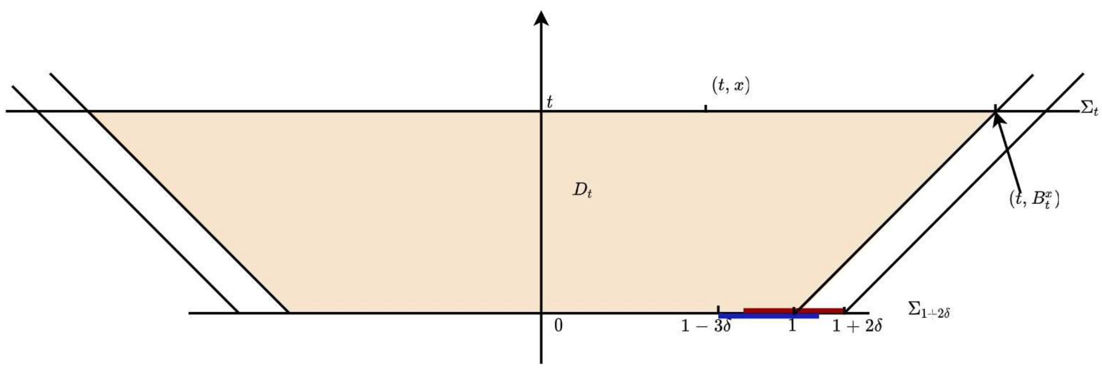

Figure 4. The domain D_t inside $B_{2\delta}$

We will adopt the energy method to get the global existence of ϕ in $B_{2\delta}$. To this end, getting suitable rate of time decay of ϕ by the Klainerman-Sobolev type $L^\infty - L^2$ inequality is crucial. However, since D_t has finite lateral boundary, then the classical Klainerman-Sobolev type $L^\infty - L^2$ inequality cannot be used directly in D_t. Inspired by the works in [15], [26] and [32], we intend to establish the following modified Klainerman-Sobolev inequalities.

Lemma 11.1. *For any function $f(t,x) \in C^\infty(\mathbb{R}^{1+2})$, $t \geq 1$, $(t,x) \in D_T = \{(t,x) : t - |x| \geq 2\delta, t_0 \leq t \leq T\}$, the following inequalities hold:*

$$| f(t,x) | \lesssim \sum_{i=0}^{2} t^{-1} \delta^{(i-1)s} \|\bar{\Gamma}^i f(t, \cdot)\|_{L^2(r \leq t/2)}, \ |x| \leq \frac{1}{4} t, \tag{11.1}$$

$$| f(t,x) | \lesssim | f(t, B_t^x) | + \sum_{a \leq 1, |\beta| \leq 1} t^{-1/2} \|\Omega^a \partial^\beta f(t, \cdot)\|_{L^2(t/4 \leq r \leq t - 2\delta)}, \ |x| \geq \frac{1}{4} t, \tag{11.2}$$

where $\bar{\Gamma} \in \{S, H_i, \Omega\}$, (t, B_t^x) is the intersection point of the boundary $\tilde{C}_{2\delta}$ and the ray crossing (t,x) which emanates from $(t,0)$, and s is the any nonnegative constant in (11.1).

It should be remarked that though this Lemma and its proof are similar to those of Proposition 3.1 in [32], yet the refined inner estimate, (11.1) is new (the appearance of factor $\delta^{(i-1)s}$) and is crucial for the treatment of the short pulse initial data here that is not needed for the small data case in [15, 26, 32].

Proof. The basic approach here is similar to those for Proposition 3.1 in [32] by separating the inner estimate from the outer ones. However, to get the refined inner estimate (11.1), we will use a δ-dependent scaling. Let $\chi \in C_c^\infty(\mathbb{R}_+)$ be a nonnegative cut-off function such that $\chi(r) \equiv 1$ fro $r \in [0, \frac{1}{4}]$ and $\chi(r) \equiv 0$ for $r \geq \frac{1}{2}$.

Define $f_1(t,x)=\chi(\frac{|x|}{t})f(t,x)$ and $f_2=f-f_1$. Then supp $f_1\subset\{(t,x):|x|\le\frac{t}{2}\}$ and supp $f_2\subset\{(t,x):|x|\ge\frac{t}{4}\}$. One can obtain the inner estimate (11.1) and the outer estimate (11.2) separately as follows.

First, for any point (t,x) satisfying $|x|\le\frac14 t$, $f_1(t,x)=f(t,x)$. One can rescale the variable as $x=t\delta^s y$, and then use the Sobolev embedding theorem for y to get

$$\begin{aligned}|f(t,x)|=|f_1(t,t\delta^s y)|&\lesssim\sum_{|\alpha|\le2}\Big(\int_{\mathbb{R}^2}|\partial_y^\alpha\big(f(t,t\delta^s y)\chi(\delta^s y)\big)|^2dy\Big)^{1/2}\\&\lesssim\sum_{|\alpha|\le2}\Big(\int_{|\delta^s y|\le1/2}(t\delta^s)^{2|\alpha|}|(\partial_x^\alpha f)(t,t\delta^s y)|^2dy\Big)^{\frac12}\\&\lesssim\sum_{|\alpha|\le2}\Big(\int_{|z|\le\frac12 t}(t\delta^s)^{2|\alpha|-2}|\partial_z^\alpha f(t,z)|^2dz\Big)^{1/2}.\end{aligned}\tag{11.3}$$

Note that

$$\partial_i=-\frac{1}{t-r}\big(\frac{x^i}{t+r}S-\frac{t}{t+r}H_i+\frac{x_\perp^i}{t+r}\Omega\big)\quad\text{with } x_\perp=(-x^2,x^1)\tag{11.4}$$

and $t\sim t-|z|$ in the domain $\{(t,z):|z|\le\frac12 t\}$. Then, it holds that

$$\begin{aligned}&|t\partial_z f(t,z)|\lesssim|(t-|z|)\partial_z f(t,z)|\lesssim|\bar\Gamma f(t,z)|,\\&|t^2\partial_z^2 f(t,z)|\lesssim\sum_{|\alpha|\le2}|\bar\Gamma^\alpha f(t,z)|.\end{aligned}\tag{11.5}$$

Therefore, substituting (11.5) into (11.3) yields (11.1).

Next, for (t,x) satisfying $|x|\ge\frac14 t$, by the Newton-Leibnitz formula and the Sobolev embedding theorem on the circle S_t^ρ with radius ρ and center at the origin on Σ_t, one has

$$\begin{aligned}f^2(t,x)&=f^2(t,B_t^x)-\int_{|x|}^{t-2\delta}\partial_\rho\big(f^2(t,\rho\omega)\big)d\rho\\&\lesssim f^2(t,B_t^x)+\int_{|x|}^{t-2\delta}\frac1\rho\sum_{a,b\le1}\|\Omega^a f\|_{L^2(S_t^\rho)}\|\Omega^b\partial f\|_{L^2(S_t^\rho)}d\rho\\&\lesssim f^2(t,B_t^x)+\sum_{a\le1,|\beta|\le1}t^{-1}\|\Omega^a\partial^\beta f(t,\cdot)\|^2_{L^2(t/4\le r\le t-2\delta)}.\end{aligned}$$

which implies (11.2). Thus Lemma 11.1 is verified. □

We also need the following inequality which is similar to Lemma 2.3 for 3D case in [10].

Lemma 11.2. *For $f(t,x)\in C^\infty(\mathbb{R}^{1+2})$ and $t\ge1$, it holds that for $1\le\bar t\le t-2\delta$,*

$$\|\frac{f(t,\cdot)}{1+t-|\cdot|}\|_{L^2(\bar t\le|x|\le t-2\delta)}\lesssim t^{1/2}\|f(t,B_t^{\cdot})\|_{L^\infty(\bar t\le|x|\le t-2\delta)}+\|\partial f(t,\cdot)\|_{L^2(\bar t\le|x|\le t-2\delta)}.\tag{11.6}$$

We will apply the energy method to prove the global existence of solution ϕ to (1.6) in $B_{2\delta}$. To this end, motivated by the works on global existence of solutions with small data to 3D nonlinear wave equations satisfying the first null condition in [9, 25], we define the energy as

$$E_{k,l}(t)=\|\partial\tilde\Gamma^k\Omega^l\phi(t,\cdot)\|^2_{L^2(\Sigma_t\cap D_t)}+\iint_{D_t}\frac{|\tilde Z\tilde\Gamma^k\Omega^l\phi|^2(t',x)}{1+(t'-|x|)^{3/2}}dxdt',\tag{11.7}$$

where $\tilde{Z} \in \{\tilde{Z}_i = \omega^i \partial_t + \partial_i, i = 1, 2\}$, $\tilde{\Gamma} \in \{\partial, S, H_1, H_2\}$. Based on the estimate (2.3) on Σ_{t_0}, one can make the following bootstrap assumptions:

For $t \geq t_0$, there exists a uniform constant M_0 such that

$$E_{k,l}(t) \leq {M_0}^2 \delta^{2a_k}, \quad k + l \leq 5 \tag{11.8}$$

with $a_0 = a_1 = 2 - \varepsilon_0$, $a_2 = \frac{9}{8} - \varepsilon_0$, $a_3 = \frac{3}{8} - \varepsilon_0$, $a_4 = -\frac{1}{2} - \varepsilon_0$ and $a_5 = -\frac{3}{2} - \varepsilon_0$.

According to Lemma 11.1, 11.2, and assumptions (11.8), we can obtain the following L^∞ estimates.

Proposition 11.1. *Under the assumptions* (11.8)*, when $\delta > 0$ is small, it holds that*

$$\begin{aligned} &|\tilde{Z}\Omega^{\leq 3}\phi| \lesssim M_0 \delta^{25/16-\varepsilon_0} t^{-3/2}(1 + t - r), \qquad |\tilde{Z}\tilde{\Gamma}\Omega^{\leq 2}\phi| \lesssim M_0 \delta^{9/8-\varepsilon_0} t^{-3/2}(1 + t - r), \\ &|\tilde{Z}\tilde{\Gamma}^2\Omega^{\leq 1}\phi| \lesssim M_0 \delta^{5/16-\varepsilon_0} t^{-3/2}(1 + t - r), \qquad |\tilde{Z}\tilde{\Gamma}^3\phi| \lesssim M_0 \delta^{-9/16-\varepsilon_0} t^{-3/2}(1 + t - r), \end{aligned} \tag{11.9}$$

and

$$\begin{aligned} &|\partial\Omega^{\leq 3}\phi| \lesssim M_0 \delta^{25/16-\varepsilon_0} t^{-1/2}, \qquad |\partial\tilde{\Gamma}\Omega^{\leq 2}\phi| \lesssim M_0 \delta^{9/8-\varepsilon_0} t^{-1/2}, \\ &|\partial\tilde{\Gamma}^2\Omega^{\leq 1}\phi| \lesssim M_0 \delta^{5/16-\varepsilon_0} t^{-1/2}, \qquad |\partial\tilde{\Gamma}^3\phi| \lesssim M_0 \delta^{-9/16-\varepsilon_0} t^{-1/2}. \end{aligned} \tag{11.10}$$

Proof. First, for $|x| \leq \frac{t}{4}$, one gets from (11.1) that

$$\begin{aligned} &|\tilde{Z}\Omega^{\leq 3}\phi| + |\partial\Omega^{\leq 3}\phi| \\ \lesssim & t^{-1}\Big\{\delta^{-s}\|\partial\Omega^{\leq 3}\phi\|_{L^2(\Sigma_t \cap D_t)} + \|\tilde{\Gamma}\partial\Omega^{\leq 3}\phi\|_{L^2(\Sigma_t \cap D_t)} + \delta^s\|\tilde{\Gamma}^2\partial\Omega^{\leq 3}\phi\|_{L^2(\Sigma_t \cap D_t)}\Big\}. \end{aligned} \tag{11.11}$$

Choosing $s = \frac{7}{16}$ in (11.11) and utilizing assumptions (11.8) yield

$$|\tilde{Z}\Omega^{\leq 3}\phi| + |\partial\Omega^{\leq 3}\phi| \lesssim M_0 \delta^{25/16-\varepsilon_0} t^{-1}. \tag{11.12}$$

Next, for $\frac{t}{4} \leq |x| \leq t - 2\delta$, choosing $f(t, x) = (1 + t - |x|)^{-1}|\tilde{Z}\Omega^{\leq 3}\phi(t, x)|$ in (11.2), and applying (11.6) to $\Omega^{\leq 1}\tilde{Z}\Omega^{\leq 3}\phi(t, x)$, one can get

$$\begin{aligned} &\frac{|\tilde{Z}\Omega^{\leq 3}\phi(t, x)|}{1 + t - |x|} \\ \lesssim & |\tilde{Z}\Omega^{\leq 3}\phi(t, B_t^x)| + t^{-1/2}\|\frac{\Omega^{\leq 1}\partial^{\leq 1}\tilde{Z}\Omega^{\leq 3}\phi}{1 + t - r}\|_{L^2(t/4 \leq r \leq t - 2\delta)} \\ \lesssim & \|\Omega^{\leq 1}\tilde{Z}\Omega^{\leq 3}\phi(t, B_t^{\cdot})\|_{L^\infty(t/4 \leq r \leq t-2\delta)} + t^{-1/2}\|\partial\Omega^{\leq 1}\tilde{Z}\Omega^{\leq 3}\phi\|_{L^2(t/4 \leq r \leq t - 2\delta)}. \end{aligned} \tag{11.13}$$

Since $\omega^i\partial_t + \partial_i = \omega^i L + \frac{1}{r}\omega_\perp^i \Omega$, $|\Omega^{\leq 1}\tilde{Z}\Omega^{\leq 3}\phi(t, B_t^x)| \lesssim \delta^{2-\varepsilon_0} t^{-3/2}$ due to (10.4), then it follows from (11.13) and (11.8) that

$$\frac{|\tilde{Z}\Omega^{\leq 3}\phi(t, x)|}{1 + t - |x|} \lesssim M_0 \delta^{2-\varepsilon_0} t^{-3/2}. \tag{11.14}$$

In addition, (11.2) implies directly that

$$|\partial\Omega^{\leq 3}\phi(t, x)| \lesssim M_0 \delta^{2-\varepsilon_0} t^{-1/2}. \tag{11.15}$$

Thus it follows from (11.12), (11.14) and (11.15) that

$$\begin{aligned} &|\tilde{Z}\Omega^{\leq 3}\phi| \lesssim M_0 \delta^{25/16-\varepsilon_0} t^{-3/2}(1 + t - r), \\ &|\partial\Omega^{\leq 3}\phi| \lesssim M_0 \delta^{25/16-\varepsilon_0} t^{-1/2}. \end{aligned}$$

Finally, the other cases can be treated similarly as above except choosing different s such as $s = \frac{13}{16}$ for $\tilde{Z}\tilde{\Gamma}\Omega^{\leq 2}\phi$ and $\tilde{Z}\tilde{\Gamma}^2\Omega^{\leq 1}\phi$, while $s = \frac{15}{16}$ for $\tilde{Z}\tilde{\Gamma}^3\phi$. Details are omitted.

□

Corollary 11.1. *Under the same conditions in Proposition 11.1, it holds that*

$$
\begin{aligned}
&|\partial^2\Omega^{\le 2}\phi| \lesssim M_0\delta^{9/8-\varepsilon_0}t^{-1/2}(1+t-r)^{-1},\\
&|\partial^2\tilde\Gamma\Omega^{\le 1}\phi| \lesssim M_0\delta^{5/16-\varepsilon_0}t^{-1/2}(1+t-r)^{-1},\\
&|\partial^2\tilde\Gamma^2\phi| \lesssim M_0\delta^{-9/16-\varepsilon_0}t^{-1/2}(1+t-r)^{-1},
\end{aligned}
\tag{11.16}
$$

and

$$
\begin{aligned}
&|\tilde Z\partial\Omega^{\le 2}\phi| \lesssim M_0\delta^{9/8-\varepsilon_0}t^{-3/2},\\
&|\tilde Z\partial\tilde\Gamma\Omega^{\le 1}\phi| \lesssim M_0\delta^{5/16-\varepsilon_0}t^{-3/2},\\
&|\tilde Z\partial\tilde\Gamma^2\phi| \lesssim M_0\delta^{-9/16-\varepsilon_0}t^{-3/2}.
\end{aligned}
\tag{11.17}
$$

Proof. These results follow from (11.4) and (11.9)-(11.10) directly. □

We are now ready to carry out the energy estimates in D_t using the ghost weight method in [2]. To this end, choosing a multiplier $W\partial_t v$ with $W = e^{2(1+t-r)^{-1/2}}$ and integrating $W\partial_t v g^{\alpha\beta}(\partial\phi)\partial^2_{\alpha\beta}v$ over D_t yield

$$
\begin{aligned}
&\int_{\Sigma_t\cap D_t}\frac12\Big\{(\partial_t v)^2+(1+2\partial_t\phi+|\nabla\phi|^2)|\nabla v|^2-\sum_{i,j}\partial_i\phi\partial_j\phi\partial_i v\partial_j v\Big\}W\\
&\quad+\iint_{D_t}\frac12 W\sum_i\frac{|\tilde Z_i v|^2}{(1+\tau-r)^{3/2}}\\
&=\int_{\Sigma_{t_0}\cap D_t}\frac12\Big\{(\partial_t v)^2+(1+2\partial_t\phi+|\nabla\phi|^2)|\nabla v|^2-\sum_{i,j}\partial_i\phi\partial_j\phi\partial_i v\partial_j v\Big\}W\\
&\quad+\frac{\sqrt2}{2}\int_{\tilde C_{2\delta}\cap D_t}\Big\{\frac12(1+2\partial_t\phi+|\nabla\phi|^2)\big((Lv)^2+\frac{(\Omega v)^2}{r^2}\big)-\big(L\phi+\frac{(\Omega\phi)^2}{2r^2}\big)(\partial_t v)^2\\
&\qquad-\frac12\sum_{i,j}\partial_i\phi\partial_j\phi\tilde Z_i v\tilde Z_j v\Big\}W\\
&\quad+\iint_{D_t}\frac12\Big\{\sum_{i,j}(-2\partial_t\phi\delta^{ij}+\partial_i\phi\partial_j\phi-|\nabla\phi|^2\delta^{ij})\tilde Z_i v\tilde Z_j v-(\omega^i\omega^j\\
&\qquad\quad-\delta^{ij})\tilde Z_i\phi\tilde Z_j\phi(\partial_t v)^2+2\omega^i(\tilde Z_i\phi)(\partial_t v)^2\Big\}\frac{W}{(1+\tau-r)^{3/2}}\\
&\quad-\iint_{D_t}W\partial_t v g^{\alpha\beta}\partial^2_{\alpha\beta}v+\iint_{D_t}\Big\{O(\tilde Z\partial\phi\cdot\partial v\cdot\partial v)+O(\partial\phi\cdot\tilde Z\partial\phi\cdot\partial v\cdot\partial v)\Big\}W\\
&\quad+\iint_{D_t}\partial_t^2\phi\Big\{(1-\omega^k\partial_k\phi)\sum_i(\tilde Z_i v)^2+\sum_{i,j}\omega^i\partial_j\phi\tilde Z_i v\tilde Z_j v\Big\}W.
\end{aligned}
\tag{11.18}
$$

Note that due to (11.10), the integrand of $\int_{\Sigma_t\cap D_t}$ in (11.18) is equivalent to

$$
\big((\partial_t v)^2+|\nabla v|^2\big)W,
\tag{11.19}
$$

and the integrand of $\int_{\tilde C_{2\delta}\cap D_t}$ in (11.18) can be controlled by

$$
\big\{(Lv)^2+\frac{1}{r^2}(\Omega v)^2+\delta^{2-\varepsilon_0}\tau^{-3/2}(\partial_t v)^2\big\}W
\tag{11.20}
$$

with the help of (10.4) and $\tilde{Z}_i = \omega^i L + \frac{\omega_\perp^i}{r}\Omega$, here we have neglected the constant coefficients.

Inserting (11.19) and (11.20) into (11.18), and utilizing Proposition 11.1 and Corollary 11.1, one then can get by Gronwall's inequality that for small $\delta > 0$,

$$\begin{aligned} &\int_{\Sigma_t\cap D_t}\big((\partial_t v)^2+|\nabla v|^2\big)W+\iint_{D_t}\sum_i\frac{|\tilde{Z}_i v|^2}{(1+\tau-r)^{3/2}}W\\ \lesssim&\int_{\Sigma_{t_0}\cap D_t}\big((\partial_t v)^2+|\nabla v|^2\big)W+\iint_{D_t}|\,W\partial_t v g^{\alpha\beta}\partial^2_{\alpha\beta}v\,| \qquad (11.21)\\ &+\int_{\tilde{C}_{2\delta}\cap D_t}\big\{(Lv)^2+\frac{1}{r^2}(\Omega v)^2+\delta^{2-\varepsilon_0}\tau^{-3/2}(\partial_t v)^2\big\}W. \end{aligned}$$

To close the bootstrap assumptions (11.8), we apply (11.21) to $v = \tilde{\Gamma}^k\Omega^l\phi$ $(k+l\le 6)$. By (10.4), $|(L\tilde{\Gamma}^k\Omega^l\phi)^2 + \frac{1}{r^2}(\Omega\tilde{\Gamma}^k\Omega^l\phi)^2 + \delta^{2-\varepsilon_0}\tau^{-3/2}(\partial_t\tilde{\Gamma}^k\Omega^l\phi)^2| \lesssim \delta^{4-2\varepsilon_0}\tau^{-5/2}$ holds on $\tilde{C}_{2\delta}$. Therefore,

$$\int_{\tilde{C}_{2\delta}\cap D_t}\big\{(Lv)^2+\frac{1}{r^2}(\Omega v)^2+\delta^{2-\varepsilon_0}\tau^{-3/2}(\partial_t v)^2\big\}W\lesssim\delta^{4-2\varepsilon_0}.$$

In addition, on the initial hypersurface $\Sigma_{t_0}\cap D_t$, it holds that $|\partial\Omega^l\phi| \lesssim \delta^{2-\varepsilon_0}$ and $|\partial\tilde{\Gamma}^k\Omega^l\phi| \lesssim \delta^{3-k-\varepsilon_0}$ for $1\le k\le 6-l$ by (2.3). Hence, (11.21) gives that

$$E_{k,l}(t)\lesssim\delta^{4-2\varepsilon_0}+\iint_{D_t}|\,(\partial_t\tilde{\Gamma}^k\Omega^l\phi)(g^{\alpha\beta}\partial^2_{\alpha\beta}\tilde{\Gamma}^k\Omega^l\phi)\,|,\quad k\le\min\{1,6-l\} \qquad (11.22)$$

$$E_{k,l}(t)\lesssim\delta^{7-2k-2\varepsilon_0}+\iint_{D_t}|\,(\partial_t\tilde{\Gamma}^k\Omega^l\phi)(g^{\alpha\beta}\partial^2_{\alpha\beta}\tilde{\Gamma}^k\Omega^l\phi)\,|,\quad 2\le k\le 6-l. \qquad (11.23)$$

It remains to estimate $\iint_{D_t}|\,(\partial_t\tilde{\Gamma}^k\Omega^l\phi)(g^{\alpha\beta}\partial^2_{\alpha\beta}\tilde{\Gamma}^k\Omega^l\phi)\,|$ in (11.22) and (11.23).

Theorem 11.1. *Under the assumptions* (11.8) *with* $\delta > 0$ *small, it holds that*

$$E_{k,l}(t)\lesssim\delta^{2a_k}t^{2\iota},\quad k+l\le 6, \qquad (11.24)$$

where a_k $(k=0,1,\cdots,5)$ *are those constants defined in* (11.8)*,* $a_6 = -\frac{5}{2}-\varepsilon_0$*, and* ι *is some constant multiple of* δ^{ς} *with* $0<\varsigma<\frac{5}{8}-2\varepsilon_0$.

Proof. Acting the operator $\tilde{\Gamma}^k\Omega^l$ on (1.6) and commuting it with $g^{\alpha\beta}\partial^2_{\alpha\beta}$ yield

$$\begin{aligned} g^{\alpha\beta}\partial^2_{\alpha\beta}\tilde{\Gamma}^k\Omega^l\phi=&\sum_{\substack{k_1+k_2\le k,\,l_1+l_2\le l\\ k_2+l_2<k+l}}G_1(\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi,\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi)\\ &+\sum_{\substack{k_1+k_2+k_3\le k\\ l_1+l_2+l_3\le l\\ k_3+l_3<k+l}}G_2(\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi,\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi,\partial^2\tilde{\Gamma}^{k_3}\Omega^{l_3}\phi), \end{aligned}$$

where G_1 is a generic quadratic form satisfying the first null condition, and G_2 is a generic cubic form satisfying the second null condition. Hence it follows from Lemma 2.2 in [20] that

$$\begin{aligned} &\iint_{D_t}|\,(\partial_t\tilde{\Gamma}^k\Omega^l\phi)(g^{\alpha\beta}\partial^2_{\alpha\beta}\tilde{\Gamma}^k\Omega^l\phi)\,|\\ \lesssim&\iint_{D_t}|\partial\tilde{\Gamma}^k\Omega^l\phi|\Big\{\sum_{\substack{k_1+k_2\le k,\,l_1+l_2\le l\\ k_2+l_2<k+l}}\big(|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big)\\ &+\sum_{\substack{k_1+k_2+k_3\le k\\ l_1+l_2+l_3\le l\\ k_3+l_3<k+l}}\big(|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_3}\Omega^{l_3}\phi|+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_3}\Omega^{l_3}\phi|\big)\Big\}. \end{aligned} \qquad (11.25)$$

Next we estimate the right hand side of the inequality (11.25). In fact, as it can be checked easily that the last summation in (11.25) has better smallness and rate of time-decay, one needs only to pay attention to the term involving the first summation in (11.25). This will be carried out case by case below.

Case 1: $k = 0$ **and** $l \leq 6$**. There are two subcases:** $l_1 \leq l_2$ **and** $l_1 > l_2$**.**

If $l_1 \leq l_2$, then $l_1 \leq 3$, $l_2 \leq l-1$. Then Proposition 11.1 implies that

$$|\tilde{Z}\Omega^{l_1}\phi| \lesssim M_0\delta^{25/16-\varepsilon_0}t^{-3/2}(1+t-r), \quad |\partial\Omega^{l_1}\phi| \lesssim M_0\delta^{25/16-\varepsilon_0}t^{-1/2}.$$

Thus, one has by (11.4) that

$$\begin{aligned}
&\iint_{D_t} |\partial\Omega^l\phi|(|\tilde{Z}\Omega^{l_1}\phi|\cdot|\partial^2\Omega^{l_2}\phi| + |\partial\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\Omega^{l_2}\phi|) \\
\lesssim& \int_{t_0}^{\mathrm{t}} M_0\delta^{25/16-\varepsilon_0}\tau^{-3/2}\|\partial\Omega^l\phi\|_{L^2(\Sigma_\tau\cap D_t)}\|(1+\tau-r)\partial^2\Omega^{l_2}\phi\|_{L^2(\Sigma_\tau\cap D_t)}d\tau \\
&+ \int_{t_0}^{t} M_0\delta^{25/16-\varepsilon_0}\tau^{-1/2}\|\partial\Omega^l\phi\|_{L^2(\Sigma_\tau\cap D_t)}\|\tilde{Z}\partial\Omega^{l_2}\phi\|_{L^2(\Sigma_\tau\cap D_t)}d\tau \\
\lesssim& \int_{t_0}^{t} \tau^{-3/2}\|\partial\Omega^{\leq 6}\phi\|^2_{L^2(\Sigma_\tau\cap D_t)}d\tau + \int_{t_0}^{t} M_0^2\delta^{25/8-2\varepsilon_0}\tau^{-3/2}\|\partial\tilde{\Gamma}\Omega^{\leq 5}\phi\|^2_{L^2(\Sigma_\tau\cap D_t)}d\tau \\
\lesssim& \sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}E_{0,a}(\tau)d\tau + M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{5}\int_{t_0}^{t}\tau^{-3/2}E_{1,a}(\tau)d\tau.
\end{aligned} \tag{11.26}$$

If $l_1 > l_2$, then $l_2 \leq 2$. Then Corollary 11.1 implies that

$$|\partial^2\Omega^{l_2}\phi| \lesssim M_0\delta^{9/8-\varepsilon_0}t^{-1/2}(1+t-r)^{-1}, \quad |\tilde{Z}\partial\Omega^{l_2}\phi| \lesssim M_0\delta^{9/8-\varepsilon_0}t^{-3/2}.$$

Thus,

$$\begin{aligned}
&\iint_{D_t} |\partial\Omega^l\phi|(|\tilde{Z}\Omega^{l_1}\phi|\cdot|\partial^2\Omega^{l_2}\phi| + |\partial\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\Omega^{l_2}\phi|) \\
\lesssim& M_0\delta^{9/8-\varepsilon_0}\Big\{\iint_{D_t}\tau^{-1/2}|\partial\Omega^l\phi|\frac{|\tilde{Z}\Omega^{l_1}\phi|}{1+\tau-r} + \iint_{D_t}\tau^{-3/2}|\partial\Omega^l\phi|\cdot|\partial\Omega^{l_1}\phi|\Big\} \\
\lesssim& \delta^{\varsigma}\int_{t_0}^{t}\tau^{-1}\|\partial\Omega^l\phi\|^2_{L^2(\Sigma_\tau\cap D_t)}d\tau + M_0^2\delta^{9/4-2\varepsilon_0-\varsigma}\iint_{D_t}\frac{|\tilde{Z}\Omega^{l_1}\phi|^2}{1+(\tau-r)^2} \\
&+ M_0\delta^{9/8-\varepsilon_0}\sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}\|\partial\Omega^a\phi\|^2_{L^2(\Sigma_\tau\cap D_t)}d\tau \\
\lesssim& \delta^{\varsigma}\int_{t_0}^{t}\tau^{-1}E_{0,l}(\tau)d\tau + M_0^2\delta^{9/4-2\varepsilon_0-\varsigma}E_{0,l_1}(t) + M_0\delta^{9/8-\varepsilon_0}\sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}E_{0,a}(\tau)d\tau.
\end{aligned} \tag{11.27}$$

Inserting (11.26) and (11.27) into (11.22) yields

$$\sum_{l=0}^{6}E_{0,l}(t) \lesssim \delta^{4-2\varepsilon_0} + M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{5}\int_{t_0}^{t}\tau^{-3/2}E_{1,a}(\tau)d\tau + \delta^{\varsigma}\int_{t_0}^{t}\tau^{-1}E_{0,l}(\tau)d\tau, \tag{11.28}$$

where one has used $\varsigma \in (0, \frac{5}{8} - 2\varepsilon_0)$ and Grownwall's inequality.

Case 2: $k=1$ **and** $l\le 5$**. There are also two subcases below.**

If $k_1+l_1\le k_2+l_2$, then $k_1+l_1\le 3$. Since the case $k_1=0$ can be treated as in Case 1, we assume that $k_1=1$. Then applying Proposition 11.1 gives

$$|\tilde{Z}\tilde{\Gamma}\Omega^{l_1}\phi|\lesssim M_0\delta^{9/8-\varepsilon_0}t^{-3/2}(1+t-r),\quad |\partial\tilde{\Gamma}\Omega^{l_1}\phi|\lesssim M_0\delta^{9/8-\varepsilon_0}t^{-1/2}.$$

Since $k_2+l_2\le 5$, thus for $\delta>0$ small, one can get that

$$\begin{aligned}
&\iint_{D_t}|\partial\tilde{\Gamma}\Omega^l\phi|\big\{|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big\}\\
\lesssim&\iint_{D_t}|\partial\tilde{\Gamma}\Omega^l\phi|\big\{|\tilde{Z}\tilde{\Gamma}\Omega^{l_1}\phi|\cdot|\partial^2\Omega^{l_2}\phi|+|\tilde{Z}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{\le 1}\Omega^{l_2}\phi|\\
&\qquad+|\partial\tilde{\Gamma}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\Omega^{l_2}\phi|+|\partial\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{\le 1}\Omega^{l_2}\phi|\big\}\\
\lesssim&\sum_{a=0}^{5}\int_{t_0}^{t}\tau^{-3/2}E_{1,a}(\tau)d\tau+M_0^2\delta^{9/4-2\varepsilon_0}\sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}E_{0,a}(\tau)d\tau\\
&+M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{4}\int_{t_0}^{t}\tau^{-3/2}E_{2,a}(\tau)d\tau.
\end{aligned}\tag{11.29}$$

If $k_1+l_1>k_2+l_2$, then $k_2+l_2\le 2$. Note that the case $k_2=0$ can be treated as in Case 1. Thus we assume $k_2=1$. Then Corollary 11.1 implies that

$$|\partial^2\tilde{\Gamma}\Omega^{l_2}\phi|\lesssim M_0\delta^{5/16-\varepsilon_0}t^{-1/2}(1+t-r)^{-1},\quad |\tilde{Z}\partial\tilde{\Gamma}\Omega^{l_2}\phi|\lesssim M_0\delta^{5/16-\varepsilon_0}t^{-3/2}.$$

It hence follows from (11.8) that

$$\begin{aligned}
&\iint_{D_t}|\partial\tilde{\Gamma}\Omega^l\phi|\big\{|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big\}\\
\lesssim&\delta^{\varsigma}\int_{t_0}^{t}\tau^{-1}E_{1,l}(\tau)d\tau+M_0^2\delta^{5/8-2\varepsilon_0-\varsigma}\sum_{a=0}^{6}E_{0,a}(t)+\sum_{a=0}^{5}\int_{t_0}^{t}\tau^{-3/2}E_{1,a}(\tau)d\tau\\
&+M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}E_{0,a}(\tau)d\tau+M_0^2\delta^{9/4-2\varepsilon_0-\varsigma}\sum_{a=0}^{5}E_{1,a}(t).
\end{aligned}\tag{11.30}$$

Similarly as in Case 1, one can substitute (11.29) and (11.30) into (11.22) to get

$$\begin{aligned}
&\sum_{l=0}^{5}E_{1,l}(t)\\
\lesssim&\delta^{4-2\varepsilon_0}+M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{6}\int_{t_0}^{t}\tau^{-3/2}E_{0,a}(\tau)d\tau+\delta^{\varsigma}\int_{t_0}^{t}\tau^{-1}E_{1,l}(\tau)d\tau\\
&+M_0^2\delta^{5/8-2\varepsilon_0-\varsigma}\sum_{a=0}^{6}E_{0,a}(t)+M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{4}\int_{t_0}^{t}\tau^{-3/2}E_{2,a}(\tau)d\tau.
\end{aligned}\tag{11.31}$$

Case 3: $2\le k\le 5$ **and** $l\le 6-k$**. One then needs also to separate two subcases.**

If $k_1 + l_1 \le k_2 + l_2$, then $k_1 + l_1 \le 3$. Then Proposition 11.1 implies directly

$$\begin{aligned}
&\iint_{D_t} |\partial\tilde{\Gamma}^k\Omega^l\phi|\big\{|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi| + |\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big\}\\
\lesssim&\sum_{a=0}^{6-k}\int_{t_0}^t \tau^{-3/2}E_{k,a}(\tau)d\tau + M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a=0}^{8-k}\int_{t_0}^t \tau^{-3/2}E_{k-2,a}(\tau)d\tau\\
&+ M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{7-k}\int_{t_0}^t \tau^{-3/2}E_{k-1,a}(\tau)d\tau + M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{5-k}\int_{t_0}^t \tau^{-3/2}E_{k+1,a}(\tau)d\tau.
\end{aligned} \tag{11.32}$$

If $k_1 + l_1 > k_2 + l_2$, then $k_2 + l_2 \le 2$. It then follows from Corollary 11.1 that

$$\begin{aligned}
&\iint_{D_t} |\partial\tilde{\Gamma}^k\Omega^l\phi|\big\{|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi| + |\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big\}\\
\lesssim&\delta^{\varsigma}\int_{t_0}^t \tau^{-1}E_{k,l}(\tau)d\tau + M_0^2\delta^{5/8-2\varepsilon_0-\varsigma}\sum_{a=0}^{7-k}E_{k-1,a}(t) + \sum_{a=0}^{6-k}\int_{t_0}^t \tau^{-3/2}E_{k,a}(\tau)d\tau\\
&+ M_0^2\delta^{-9/8-2\varepsilon_0-\varsigma}\sum_{a+b\le 6, a\le k-2}E_{a,b}(t) + M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{7-k}\int_{t_0}^t \tau^{-3/2}E_{k-1,a}(\tau)d\tau\\
&+ M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a+b\le 6, a\le k-2}\int_{t_0}^t \tau^{-3/2}E_{a,b}(\tau)d\tau + M_0^2\delta^{9/4-2\varepsilon_0-\varsigma}\sum_{a=0}^{6-k}E_{k,a}(t).
\end{aligned} \tag{11.33}$$

Substituting (11.32) and (11.33) into (11.23) yields

$$\begin{aligned}
&\sum_{l=0}^{6-k}E_{k,l}(t)\\
\lesssim&\delta^{7-2k-2\varepsilon_0} + M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a+b\le 6, a\le k-2}\Big\{\int_{t_0}^t \tau^{-3/2}E_{a,b}(\tau)d\tau + \delta^{-\varsigma}E_{a,b}(t)\Big\}\\
&+ M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{7-k}\Big\{\int_{t_0}^t \tau^{-3/2}E_{k-1,a}(\tau)d\tau + \delta^{-\varsigma}E_{k-1,a}(t)\Big\}\\
&+ M_0^2\delta^{25/8-2\varepsilon_0}\sum_{a=0}^{5-k}\int_{t_0}^t \tau^{-3/2}E_{k+1,a}(\tau)d\tau + \delta^{\varsigma}\sum_{l=0}^{6-k}\int_{t_0}^t \tau^{-1}E_{k,l}(\tau)d\tau.
\end{aligned} \tag{11.34}$$

Case 4: $k = 6$ **and** $l = 0$**.**

We start with the subcase $k_1 \le k_2$. Then $k_1 \le 3$. Hence similar arguments as for (11.32) give that

$$\begin{aligned}
&\iint_{D_t} |\partial\tilde{\Gamma}^6\phi|\big\{|\tilde{Z}\tilde{\Gamma}^{k_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\phi| + |\partial\tilde{\Gamma}^{k_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\phi|\big\}\\
\lesssim&\int_{t_0}^t \tau^{-3/2}E_{6,0}(\tau)d\tau + M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a+b\le 6, a\le 4}\int_{t_0}^t \tau^{-3/2}E_{a,b}(\tau)d\tau\\
&+ M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{1}\int_{t_0}^t \tau^{-3/2}E_{5,a}(\tau)d\tau.
\end{aligned} \tag{11.35}$$

In the subcase $k_1 > k_2$, one can get from Proposition 11.1 directly that

$$\begin{aligned}
&\iint_{D_t} |\partial\tilde{\Gamma}^6\phi|\{|\tilde{Z}\tilde{\Gamma}^{k_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\phi| + |\partial\tilde{\Gamma}^{k_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\phi|\}\\
\lesssim&\delta^{\varsigma}\int_{t_0}^t \tau^{-1}E_{6,0}(\tau)d\tau + M_0^2\delta^{5/8-2\varepsilon_0-\varsigma}\sum_{a=0}^{1}E_{5,a}(t) + \int_{t_0}^t \tau^{-3/2}E_{6,0}(\tau)d\tau\\
&+ M_0^2\delta^{-9/8-2\varepsilon_0-\varsigma}\sum_{a+b\le 6, a\le 4}E_{a,b}(t) + M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{1}\int_{t_0}^t\tau^{-3/2}E_{5,a}(\tau)d\tau\\
&+ M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a+b\le 6, a\le 4}\int_{t_0}^t\tau^{-3/2}E_{a,b}(\tau)d\tau + M_0^2\delta^{9/4-2\varepsilon_0-\varsigma}E_{6,0}(t).
\end{aligned}\tag{11.36}$$

Collecting (11.35) and (11.36), and applying (11.22), one gets that

$$\begin{aligned}
E_{6,0}(t) \lesssim&\delta^{-5-2\varepsilon_0} + M_0^2\delta^{-9/8-2\varepsilon_0}\sum_{a+b\le 6, a\le 4}\int_{t_0}^t\tau^{-3/2}E_{a,b}(\tau)d\tau\\
&+ M_0^2\delta^{5/8-2\varepsilon_0}\sum_{a=0}^{1}\int_{t_0}^t\tau^{-3/2}E_{5,a}(\tau)d\tau + \delta^{\varsigma}\int_{t_0}^t\tau^{-1}E_{6,0}(\tau)d\tau\\
&+ M_0^2\delta^{5/8-2\varepsilon_0-\varsigma}\sum_{a=0}^{1}E_{5,a}(t) + M_0^2\delta^{-9/8-2\varepsilon_0-\varsigma}\sum_{a+b\le 6, a\le 4}E_{a,b}(t).
\end{aligned}\tag{11.37}$$

Let $b_0 = b_1 = 0$, $b_2 = \frac{7}{4}$, $b_3 = \frac{13}{4}$ and $b_k = 2k-3$ for $k = 4, 5, 6$. As $0 < \varsigma < \frac{5}{8} - 2\varepsilon_0$, we sum up the inequalities (11.28), (11.31), (11.34) and (11.37) with the weigh δ^{b_k} for $E_{k,l}$ $(k+l \le 6)$, and use the Grownwall's inequality to obtain

$$\sum_{k+l\le 6}\delta^{b_k}E_{k,l}(t) \lesssim \delta^{4-2\varepsilon_0}t^{2\iota},$$

which proves (11.24). □

As for the M_0-independent energy estimate in (11.24), we can also obtain the M_0 independent L^∞ estimates of ϕ and its derivatives corresponding to Proposition (11.1) and Corollary (11.1).

Corollary 11.2. *When $\delta > 0$ is small, it holds that in the domain $B_{2\delta}$,*

$$\begin{aligned}
&|\tilde{Z}\Omega^{\le 3}\phi| \lesssim \delta^{25/16-\varepsilon_0}t^{-3/2+\iota}(1+t-r), && |\tilde{Z}\tilde{\Gamma}\Omega^{\le 2}\phi| \lesssim \delta^{9/8-\varepsilon_0}t^{-3/2+\iota}(1+t-r),\\
&|\tilde{Z}\tilde{\Gamma}^2\Omega^{\le 1}\phi| \lesssim \delta^{5/16-\varepsilon_0}t^{-3/2+\iota}(1+t-r), && |\tilde{Z}\tilde{\Gamma}^3\phi| \lesssim \delta^{-9/16-\varepsilon_0}t^{-3/2+\iota}(1+t-r),\\
&|\partial\Omega^{\le 3}\phi| \lesssim \delta^{25/16-\varepsilon_0}t^{-1/2+\iota}, && |\partial\tilde{\Gamma}\Omega^{\le 2}\phi| \lesssim \delta^{9/8-\varepsilon_0}t^{-1/2+\iota},\\
&|\partial\tilde{\Gamma}^2\Omega^{\le 1}\phi| \lesssim \delta^{5/16-\varepsilon_0}t^{-1/2+\iota}, && |\partial\tilde{\Gamma}^3\phi| \lesssim \delta^{-9/16-\varepsilon_0}t^{-1/2+\iota},
\end{aligned}\tag{11.38}$$

and

$$\begin{aligned}
&|\partial^2\Omega^{\le 2}\phi| \lesssim \delta^{9/8-\varepsilon_0}t^{-1/2+\iota}(1+t-r)^{-1}, && |\partial^2\tilde{\Gamma}\Omega^{\le 1}\phi| \lesssim \delta^{5/16-\varepsilon_0}t^{-1/2+\iota}(1+t-r)^{-1},\\
&|\partial^2\tilde{\Gamma}^2\phi| \lesssim \delta^{-9/16-\varepsilon_0}t^{-1/2+\iota}(1+t-r)^{-1}, && |\tilde{Z}\partial\Omega^{\le 2}\phi| \lesssim \delta^{9/8-\varepsilon_0}t^{-3/2+\iota},\\
&|\tilde{Z}\partial\tilde{\Gamma}\Omega^{\le 1}\phi| \lesssim \delta^{5/16-\varepsilon_0}t^{-3/2+\iota}, && |\tilde{Z}\partial\tilde{\Gamma}^2\phi| \lesssim \delta^{-9/16-\varepsilon_0}t^{-3/2+\iota}.
\end{aligned}\tag{11.39}$$

Compared with (11.8), Proposition 11.1 and Corollary 11.1, though the estimates in (11.24) and Corollary 11.2 do not depend on M_0, yet they contain increasing time factors $t^{2\iota}$ or t^{ι}. To overcome this difficulty and close the assumptions (11.8), we now study the equation on the difference between ϕ and ϕ_a, where ϕ_a satisfies

$$\begin{cases} -\partial_t^2\phi_a + \triangle\phi_a = 0,\\ \phi_a(t_0,x) = \phi(t_0,x),\\ \partial_t\phi_a(t_0,x) = \partial_t\phi(t_0,x). \end{cases} \tag{11.40}$$

Proposition 11.2. *ϕ_a defined by* (11.40) *satisfies the following estimate on the hypersurface $\Sigma_t\cap D_t$:*

$$\int_{\Sigma_t\cap D_t}|\partial\tilde{\Gamma}^k\Omega^l\phi_a|^2dx + \iint_{D_t}\frac{|\tilde{Z}\tilde{\Gamma}^k\Omega^l\phi_a(\tau,x)|^2}{(1+\tau-|x|)^{3/2}}dxd\tau \lesssim \begin{cases} \delta^{4-2\varepsilon_0}, & k=0,1,\\ \delta^{7-2k-2\varepsilon_0}, & k\geq 2.\end{cases} \tag{11.41}$$

Proof. Let

$$D^0 := \{(\bar{t},x) : 0\leq \bar{t}-|x|\leq 2\delta, t_0\leq\bar{t}\leq t\}.$$

One can estimate ϕ_a in D^0 by the energy method. Indeed, it is easy to check that

$$\int_{\Sigma_t\cap D^0}|\partial\hat{Z}^\alpha\phi_a|^2 + \frac{\sqrt{2}}{2}\int_{\tilde{C}_{2\delta}\cap D^0}\big(|L\hat{Z}^\alpha\phi_a|^2 + \frac{1}{r^2}|\Omega\hat{Z}^\alpha\phi_a|^2\big) = \int_{\Sigma_{t_0}\cap D^0}|\partial\hat{Z}^\alpha\phi_a|^2, \tag{11.42}$$

where $\hat{Z}\in\{\partial,\Omega,S,H_1,H_2\}$. The initial data of ϕ_a in (11.40) satisfy $|\partial Z^\alpha\phi_a|_{\Sigma_{t_0}\cap D^0}\lesssim\delta^{1-\varepsilon_0-l}$ with l being the number of ∂ in Z^α. Therefore, (11.42) implies

$$\int_{\Sigma_t\cap D^0}|\partial\hat{Z}^\alpha\phi_a|^2 + \frac{\sqrt{2}}{2}\int_{\tilde{C}_{2\delta}\cap D^0}\big(|L\hat{Z}^\alpha\phi_a|^2 + \frac{1}{r^2}|\Omega\hat{Z}^\alpha\phi_a|^2\big) \lesssim \delta^{3-2\varepsilon_0-2l}. \tag{11.43}$$

By the following Sobolev's imbedding theorem on the circle $\mathbb{S}_r^1$ (with center at the origin and radius r)

$$|w(t,x)|\lesssim\frac{1}{\sqrt{r}}\|\Omega^{\leq 1}w\|_{L^2(\mathbb{S}_r^1)},$$

it then follows that for any point (t,x) in D^0,

$$|\hat{Z}^\alpha\phi_a(t,x)|\lesssim t^{-1/2}\|\Omega^{\leq 1}\hat{Z}^\alpha\phi_a\|_{L^2(\mathbb{S}_r)}\lesssim t^{-1/2}\delta^{1/2}\|\partial\Omega^{\leq 1}\hat{Z}^\alpha\phi_a\|_{L^2(\Sigma_t\cap D^0)}\lesssim\delta^{2-\varepsilon_0-l}t^{-1/2}. \tag{11.44}$$

In addition, $\bar{\Gamma}^\alpha\phi_a$ solves

$$L\big(r^{1/2}\underline{L}\bar{\Gamma}^\alpha\phi_a\big) = \frac{1}{2}r^{-1/2}L\bar{\Gamma}^\alpha\phi_a,$$

which implies $|L\big(r^{1/2}\underline{L}\bar{\Gamma}^\alpha\phi_a\big)|\lesssim\delta^{2-\varepsilon_0}t^{-2}$ by (11.44) since $\bar{\Gamma}\in\{S,H_1,H_2,\Omega\}$. And hence, on the surface $\tilde{C}_{2\delta}$, $|\underline{L}\bar{\Gamma}^\alpha\phi_a|\lesssim\delta^{2-\varepsilon_0}t^{-1/2}$ holds after integrating $L\big(r^{1/2}\underline{L}\bar{\Gamma}^\alpha\phi_a\big)$ along integrate curves of L on $\tilde{C}_{2\delta}$. Then,

$$|L\tilde{\Gamma}\Omega^l\phi_a| + \frac{1}{r}|\Omega\tilde{\Gamma}\Omega^l\phi_a|\lesssim\delta^{2-\varepsilon_0}t^{-3/2}\quad\text{on }\tilde{C}_{2\delta}.$$

By an induction argument, one can get

$$|L\tilde{\Gamma}^k\Omega^l\phi_a| + \frac{1}{r}|\Omega\tilde{\Gamma}^k\Omega^l\phi_a|\lesssim\delta^{2-\varepsilon_0}t^{-3/2}\quad\text{on }\tilde{C}_{2\delta}. \tag{11.45}$$

Therefore,

$$\int_{\tilde{C}_{2\delta}\cap D_t}\big(|L\tilde{\Gamma}^k\Omega^l\phi_a|^2 + \frac{1}{r^2}|\Omega\tilde{\Gamma}^k\Omega^l\phi_a|^2\big)\lesssim\delta^{4-2\varepsilon_0}. \tag{11.46}$$

It follows the property of the weight function W in (11.18) and (11.40) that

$$\begin{aligned}&\int_{\Sigma_t\cap D_t} W|\partial\tilde{\Gamma}^k\Omega^l\phi_a|^2+\iint_{D_t} W\frac{|\tilde{Z}\tilde{\Gamma}^k\Omega^l\phi_a|^2}{(1+\tau-r)^{3/2}}\\ =&\int_{\Sigma_{t_0}\cap D_t} W|\partial\tilde{\Gamma}^k\Omega^l\phi_a|^2+\frac{\sqrt{2}}{2}\int_{\tilde{C}_{2\delta}\cap D_t} W\big(|L\tilde{\Gamma}^k\Omega^l\phi_a|^2+\frac{1}{r^2}|\Omega\tilde{\Gamma}^k\Omega^l\phi_a|^2\big).\end{aligned}\tag{11.47}$$

Then the estimate (11.41) is a direct consequence of (11.47), (2.3) and (11.46). □

Similarly as for Proposition 11.1, one can use Lemma 11.1, 11.2 and Proposition 11.2 to get the L^∞ estimate of $\phi-\phi_a$ in D_t. Indeed, let $\dot{\phi}=\phi-\phi_a$. Then $\dot{\phi}$ solves

$$\begin{cases}-\partial_t^2\dot{\phi}+\triangle\dot{\phi}=2\sum_{i=1}^{2}\partial_i\phi\partial_t\partial_i\phi-2\partial_t\phi\triangle\phi+\sum_{i,j=1}^{2}\partial_i\phi\partial_j\phi\partial_i\partial_j\phi-|\nabla\phi|^2\triangle\phi,\\ \dot{\phi}(t_0,x)=\partial_t\dot{\phi}(t_0,x)=0.\end{cases}\tag{11.48}$$

Proposition 11.3. *If $\delta>0$ is small and $k+l\le 5$, then*

$$\|\partial\tilde{\Gamma}^k\Omega^l\dot{\phi}\|^2_{L^2(\Sigma_t\cap D_t)}+\iint_{D_t}\frac{|\tilde{Z}\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2(t',x)}{1+(t'-|x|)^{3/2}}dxdt'\lesssim\delta^{2a_k}.\tag{11.49}$$

Proof. By commuting the operator $\tilde{\Gamma}^k\Omega^l$ with $-\partial_t^2+\triangle$, and noting that the right hand side of the equation in (11.48) satisfies the first and second null conditions, one then has from (11.38) and (11.39) that

$$\begin{aligned}&|(-\partial_t^2+\triangle)\tilde{\Gamma}^k\Omega^l\dot{\phi}|\\ \lesssim&\sum_{\substack{k_1+k_2\le k\\ l_1+l_2\le l}}\big(|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big)\\ &+\sum_{\substack{k_1+k_2+k_3\le k\\ l_1+l_2+l_3\le l}}\big(|\tilde{Z}\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\cdot|\partial^2\tilde{\Gamma}^{k_3}\Omega^{l_3}\phi|\\ &\qquad+|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\cdot|\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\cdot|\tilde{Z}\partial\tilde{\Gamma}^{k_3}\Omega^{l_3}\phi|\big)\\ \lesssim&\sum_{\substack{k_1+k_2\le k\\ l_1+l_2\le l\\ k_1+l_1\le 2}}\big\{\delta^{\varpi_{k_1}}t^{-3/2+\iota}(1+t-r)|\partial^2\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|+\delta^{\varpi_{k_1}}t^{-1/2+\iota}|\tilde{Z}\partial\tilde{\Gamma}^{k_2}\Omega^{l_2}\phi|\big\}\\ &+\sum_{\substack{k_1+k_2\le k\\ l_1+l_2\le l\\ k_2+l_2\le 2}}\big\{\delta^{\varpi_{k_2+1}}t^{-1/2+\iota}(1+t-r)^{-1}|\tilde{Z}\Gamma^{k_1}\Omega^{l_1}\phi|+\delta^{\varpi_{k_2+1}}t^{-3/2+\iota}|\partial\tilde{\Gamma}^{k_1}\Omega^{l_1}\phi|\big\},\end{aligned}\tag{11.50}$$

where $\varpi_0=\frac{25}{16}-\varepsilon_0$, $\varpi_1=\frac{9}{8}-\varepsilon_0$, $\varpi_2=\frac{5}{16}-\varepsilon_0$ and $\varpi_3=-\frac{9}{16}-\varepsilon_0$. And hence,

$$\begin{aligned}&\|(-\partial_t^2+\triangle)\tilde{\Gamma}^k\Omega^l\dot{\phi}\|_{L^2(\Sigma_t\cap D_t)}\\ \lesssim&\sum_{s=0}^{\min\{2,k\}}\delta^{\varpi_s}t^{-3/2+\iota}\sum_{\substack{p+q\le 6\\ p\le k-s+1}}\sqrt{E_{p,q}(t)}\\ &+\sum_{s=1}^{\min\{k+1,3\}}\delta^{\varpi_s}t^{-1/2+\iota}\sum_{\substack{p+q\le 5\\ p\le k-s+1}}\Big\{\|\frac{\tilde{Z}\tilde{\Gamma}^p\Omega^q\phi}{1+t-r}\|_{L^2(\Sigma_t\cap D_t)}+t^{-1}\sqrt{E_{p,q}(t)}\Big\}.\end{aligned}\tag{11.51}$$

On the other hand, by Lemma 11.2 and Theorem 11.1, one can obtain that

$$\|\frac{\tilde{Z}\tilde{\Gamma}^p\Omega^q\phi}{1+t-r}\|_{L^2(\Sigma_t\cap D_t)}\lesssim\delta^{2-\varepsilon_0}t^{-1}+\delta^{a_{p+1}}t^{-1+\iota}. \tag{11.52}$$

Substituting (11.52) and (11.24) into (11.51) yields

$$\begin{aligned}\|(-\partial_t^2+\triangle)\tilde{\Gamma}^k\Omega^l\dot{\phi}\|_{L^2(\Sigma_t\cap D_t)}\lesssim&\sum_{s=0}^{\min\{2,k\}}\delta^{\varpi_s}\delta^{a_{k-s+1}}t^{-3/2+2\iota}+\sum_{s=1}^{\min\{k+1,3\}}\delta^{\varpi_s}\delta^{a_{k-s+2}}t^{-3/2+2\iota}\\ \lesssim&\delta^{a_k+1/8-\varepsilon_0}t^{-3/2+2\iota}.\end{aligned} \tag{11.53}$$

Integrate $W(\partial_t\tilde{\Gamma}^k\Omega^l\dot{\phi})\big((-\partial_t^2+\triangle)\tilde{\Gamma}^k\Omega^l\dot{\phi}\big)$ over domain D_t with $W=e^{2(1+t-r)^{-1/2}}$ as in (11.18). Then it follows (11.53) that

$$\begin{aligned}&\int_{\Sigma_t\cap D_t}W|\partial\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2+\iint_{D_t}\frac{W}{(1+\tau-r)^{3/2}}|\tilde{Z}\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2\\ \lesssim&\int_{\Sigma_{t_0}\cap D_t}|\partial\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2+\iint_{D_t}W|\partial_t\tilde{\Gamma}^k\Omega^l\dot{\phi}|\cdot|(-\partial_t^2+\triangle)\tilde{\Gamma}^k\Omega^l\dot{\phi}|\\ &+\int_{\tilde{C}_{2\delta}\cap D_t}\big(|L\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2+\frac{1}{r^2}|\Omega\tilde{\Gamma}^k\Omega^l\dot{\phi}|^2\big)\\ \lesssim&\int_{t_0}^t\tau^{-3/2+2\iota}\|\partial\tilde{\Gamma}^k\Omega^l\dot{\phi}\|^2_{L^2(\Sigma_\tau\cap D_t)}d\tau+\delta^{1/4-2\varepsilon_0+2a_k}\\ &+\begin{cases}\delta^{4-2\varepsilon_0}, & k\le\min\{1,5-l\},\\ \delta^{7-2k-2\varepsilon_0}, & 2\le k\le 5-l.\end{cases}\end{aligned} \tag{11.54}$$

Therefore, (11.49) follows from (11.54) directly since $\iota>0$ is small enough and $\varepsilon_0\le\frac{1}{8}$ holds. □

Theorem 11.2. *When $\delta>0$ is small, there exists a smooth solution ϕ to* (1.6) *in $B_{2\delta}$ for $t\ge t_0$.*

Proof. Since $\phi=\phi_a+\dot{\phi}$, then Proposition 11.2 and 11.3 imply that

$$E_{k,l}(t)\lesssim\delta^{2a_k}\quad\text{for } k+l\le 5, \tag{11.55}$$

which is independent of M_0, then the assumptions (11.8) are then improved. □

Finally, we prove Theorem 1.1.

Proof. Theorem 2.1 gives the local existence of smooth solution ϕ to (1.6) with (1.7). On the other hand, the global existence of the solution in $A_{2\delta}$ and in $B_{2\delta}$ has been established in Section 10 and Theorem 11.2 respectively. Then it follows from the uniqueness of the smooth solution to (1.6) that the proof of $\phi\in C^\infty([1,+\infty)\times\mathbb{R}^2)$ is finished. In addition, $|\nabla\phi|\lesssim\delta^{1-\varepsilon_0}t^{-1/2}$ follows from (2.1), (2.2), (10.3), and the first inequality in (11.10). Thus Theorem 1.1 is proved.

□

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