Nonlinear asymptotic stability of compressible vortex sheets with viscosity effects

Feimin Huang[1,2], Zhouping Xin[3], Lingda Xu[4,5], and Qian Yuan[1] *

[1] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China.
Email: qyuan@amss.ac.cn

[2] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China.
E-mail: fhuang@amt.ac.cn

[3] The Institute of Mathematical Sciences & Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong.
Email: zpxin@ims.cuhk.edu.hk

[4] Department of Mathematics, Yau Mathematical Sciences Center, Tsinghua University, Beijing, China.
E-mail: xulingda@tsinghua.edu.cn

[5] Yanqi Lake Beijing Institute of Mathematical Sciences And Applications, Beijing, China.

Abstract

This paper concerns the stabilizing effect of viscosity on the vortex sheets. It is found that although a vortex sheet is not a time-asymptotic attractor for the compressible Navier-Stokes equations, a viscous wave that approximates the vortex sheet on any finite time interval can be constructed explicitly, which is shown to be time-asymptotically stable in the L^∞-space with small perturbations, regardless of the amplitude of the vortex sheet. The result shows that the viscosity has a strong stabilizing effect on the vortex sheets, which are generally unstable for the ideal compressible Euler equations even for short time [26, 8, 1]. The proof is based on the L^2-energy method. In particular, the asymptotic stability of the vortex sheet under small spatially periodic perturbations is proved by studying the dynamics of these spatial oscillations. The first key point in our analysis is to construct an ansatz to cancel these oscillations. Then using the Galilean transformation, we are able to find a shift function of the vortex sheet such that an anti-derivative technique works, which plays an important role in the energy estimates. Moreover, by introducing a new variable and using the intrinsic properties of the vortex sheet, we can achieve the optimal decay rates to the viscous wave.

Keywords. Compressible Navier-Stokes equations, Vortex sheets, Nonlinear asymptotic stability

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*Corresponding author.

1 Introduction

The three-dimensional (3D) compressible isentropic Navier-Stokes (NS) equations read,

$$\begin{cases} \partial_t \rho + \text{div}\,\mathbf{m} = 0, \\ \partial_t \mathbf{m} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mu \triangle \mathbf{u} + (\mu + \lambda)\nabla \text{div}\mathbf{u}, \end{cases} \quad x \in \mathbb{R}^3, t > 0, \qquad (1.1)$$

where $\rho(x,t) > 0$ is the density, $\mathbf{m}(x,t) = \rho\mathbf{u}(x,t) \in \mathbb{R}^3$ is the momentum with $\mathbf{u}(x,t) \in \mathbb{R}^3$ being the velocity, the pressure $p(\rho)$ is the gamma law, satisfying $p(\rho) = \rho^\gamma$ with $\gamma > 1$, and the viscous coefficients μ and λ are assumed to satisfy

$$\mu > 0 \quad \text{and} \quad \mu + \lambda \geqslant 0. \qquad (1.2)$$

In the case for $\mu = \lambda = 0$, (1.1) is the 3D compressible isentropic Euler equations,

$$\begin{cases} \partial_t \rho + \text{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \text{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = 0, \end{cases} \quad x \in \mathbb{R}^3, t > 0, \qquad (1.3)$$

which admit rich wave phenomena such as shock waves, rarefaction waves and contact discontinuities, i.e. vortex sheets. A vortex sheet is an inviscid flow in which the velocity field is discontinuous in a tangential direction across a surface. In particular, a planar vortex sheet is given by,

$$(\bar{\rho}^{\textsf{vs}}, \bar{\mathbf{u}}^{\textsf{vs}})(x,t) = \begin{cases} (\bar{\rho}, \bar{\mathbf{u}}_-), & x_3 < st, \\ (\bar{\rho}, \bar{\mathbf{u}}_+), & x_3 > st, \end{cases} \qquad (1.4)$$

where $\bar{\rho} > 0$, $\bar{\mathbf{u}}_\pm = (\bar{u}_{1\pm}, \bar{u}_{2\pm}, \bar{u}_{3\pm})$ and s are constants, satisfying the Rankine-Hugoniot (RH) conditions,

$$\bar{u}_{3+} - \bar{u}_{3-} = 0 \quad \text{and} \quad -s(\bar{u}_{i+} - \bar{u}_{i-}) + \bar{u}_{3+}\bar{u}_{i+} - \bar{u}_{3-}\bar{u}_{i-} = 0 \quad \text{for } i = 1, 2, 3, \qquad (1.5)$$

which implies that $\bar{u}_{3+} = \bar{u}_{3-} = s$.

It is well known that both the compressible NS equations (1.1) and Euler equations (1.3) are invariant under the Galilean transformation, i.e. if $(\rho, \mathbf{u})$ solves (1.1) (resp. (1.3)), then so does

$$(\rho^*, \mathbf{u}^*)(x,t) = (\rho(x - \mathbf{c}t, t), \mathbf{u}(x - \mathbf{c}t, t) + \mathbf{c}) \qquad (1.6)$$

for any constant vector $\mathbf{c} \in \mathbb{R}^3$. Thus, by selecting $\mathbf{c} = -\left(\frac{\bar{u}_{1+} + \bar{u}_{1-}}{2}, \frac{\bar{u}_{2+} + \bar{u}_{2-}}{2}, s\right)$, one can assume without loss of generality that the planar vortex sheet (1.4) has the form,

$$(\bar{\rho}^{\textsf{vs}}, \bar{\mathbf{u}}^{\textsf{vs}})(x,t) = \begin{cases} (\bar{\rho}, -\bar{\mathbf{u}}), & x_3 < 0, \\ (\bar{\rho}, \bar{\mathbf{u}}), & x_3 > 0, \end{cases} \qquad (1.7)$$

where $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, 0) \neq 0$ is a constant vector.

The stability of compressible interfacial waves is an important issue in gas dynamics. Different from the nonlinear shock waves and rarefaction waves, the linear degeneracy makes the compressible contact discontinuities less stable, which may lead to various instabilities such as the Kelvin-Helmholtz instability for vortex sheets and Rayleigh-Taylor

instability for entropy waves (the other kind of contact discontinuities in which the velocity and pressure are continuous); see [7, 11]. For the 3D compressible Euler equations, the planar vortex sheets are violently unstable (see e.g. [31]). In two dimensions, the rectilinear compressible vortex sheets are also violently unstable if the Mach number M is less than $\sqrt{2}$, while the supersonic vortex sheets with Mach number $M > \sqrt{2}$ are shown to be linearly stable; see [26, 8, 30]. The nonlinear stability of the 2D supersonic vortex sheets was then established locally in time by [3, 4]; also see [27, 28] for similar stability results in the non-isentropic case. However, one cannot expect the global nonlinear stability of the compressible vortex sheets for (1.3), which was observed by [1] through the argument of nonlinear geometric optics. We also refer to [35, 2, 36, 38] and [37] for the studies of the stabilizing effects of the magnetic fields on the compressible vortex sheets and entropy waves, respectively.

In this paper, we are concerned about the viscosity effect on the stability of the compressible vortex sheets with arbitrarily large amplitude. Different from the hyperbolic theory stated above and similar to the meta-stability of the 1D entropy waves for the 1D compressible NS equations first observed in [39], we are able to obtain a time-asymptotic stability of the viscous waves associated with the vortex sheets for the compressible NS equations by the L^2-energy method. We remark here that our results hold true in both three and two dimensions; see Remark 1.4. Due to the viscosity, the discontinuous solution of (1.3) cannot govern the large-time behaviors of the classical solutions of (1.1), that is, the inviscid vortex sheet (1.7) is only a meta-stable state for the Navier-Stokes equations. Thus, the first step in our analysis is to investigate the viscous wave associated with the vortex sheet (1.7). We refer the readers to [19, 39, 14, 15, 42] for the viscous waves associated with other wave phenomena such as the shock waves and entropy waves.

Given the vortex sheet (1.7) and a fixed constant $\Lambda \geqslant 1$, we consider the Cauchy problem for (1.1) on $\{(x,t) : x \in \mathbb{R}^3, t > -\Lambda\}$ with the initial data,

$$(\rho, \mathbf{u})(x, t = -\Lambda) = \begin{cases} (\bar{\rho}, -\bar{\mathbf{u}}), & x_3 < 0, \\ (\bar{\rho}, \bar{\mathbf{u}}), & x_3 > 0. \end{cases} \tag{1.8}$$

One can observe that the solution (if exists), denoted by $(\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})$, is actually independent of the transverse variables $x_\perp = (x_1, x_2)$. Thus, regardless of the transverse derivatives, the 3D NS system (1.1) reduces to the 1D one,

$$\begin{cases} \partial_t \rho + \partial_3 m_3 = 0, \\ \partial_t m_1 + \partial_3(\rho u_1 u_3) = \mu \partial_3^2 u_1, \\ \partial_t m_2 + \partial_3(\rho u_2 u_3) = \mu \partial_3^2 u_2, \\ \partial_t m_3 + \partial_3 \left(\rho u_3^2 + p(\rho) \right) = \tilde{\mu} \partial_3^2 u_3, \end{cases} \qquad x_3 \in \mathbb{R}, t > -\Lambda, \tag{1.9}$$

where $\tilde{\mu} := 2\mu + \lambda > 0$. It is noted that (1.9) is indeed the standard 1D compressible isentropic NS system for $(\rho^{\mathsf{vs}}, u_3^{\mathsf{vs}})$,

$$\begin{cases} \partial_t \rho^{\mathsf{vs}} + \partial_3 \left(\rho^{\mathsf{vs}} u_3^{\mathsf{vs}} \right) = 0, \\ \partial_t m_3^{\mathsf{vs}} + \partial_3 \left(\rho^{\mathsf{vs}} |u_3^{\mathsf{vs}}|^2 + p(\rho^{\mathsf{vs}}) \right) = \tilde{\mu} \partial_3^2 u_3^{\mathsf{vs}}, \end{cases} \qquad x_3 \in \mathbb{R}, t > -\Lambda, \tag{1.10}$$

coupled with two scalar parabolic equations of u_i^{vs},

$$\partial_t(\rho^{\mathsf{vs}} u_i^{\mathsf{vs}}) + \partial_3(\rho^{\mathsf{vs}} u_i^{\mathsf{vs}} u_3^{\mathsf{vs}}) = \mu \partial_3^2 u_i^{\mathsf{vs}}, \qquad x_3 \in \mathbb{R}, t > -\Lambda \qquad \text{for } i = 1, 2. \tag{1.11}$$

One can first solve (1.10) with the initial data, $(\rho^{\mathsf{vs}}, u_3^{\mathsf{vs}})(x_3, t = -\Lambda) = (\bar{\rho}, 0)$, which gives the unique solution,

$$(\rho^{\mathsf{vs}}, u_3^{\mathsf{vs}})(x_3, t) = (\bar{\rho}, 0), \qquad x_3 \in \mathbb{R}, t \geqslant -\Lambda. \tag{1.12}$$

Then substituting (1.12) into (1.11) yields that

$$\partial_t u_i^{\mathsf{vs}} = \frac{\mu}{\bar{\rho}} \partial_3^2 u_i^{\mathsf{vs}}, \qquad x_3 \in \mathbb{R}, t > -\Lambda \qquad \text{for} \quad i = 1, 2. \tag{1.13}$$

By the classical parabolic theory, the Cauchy problem (1.13) with the initial data

$$u_i^{\mathsf{vs}}(x_3, t = -\Lambda) = \begin{cases} -\bar{u}_i, & x_3 < 0, \\ \bar{u}_i, & x_3 > 0, \end{cases} \tag{1.14}$$

admits a unique solution,

$$u_i^{\mathsf{vs}}(x_3, t) = \bar{u}_i \theta(x_3, t), \qquad x_3 \in \mathbb{R}, t \geqslant -\Lambda, \tag{1.15}$$

where θ is the solution of

$$\begin{cases} \partial_t \theta = \frac{\mu}{\bar{\rho}} \partial_3^2 \theta, & x_3 \in \mathbb{R}, t > -\Lambda, \\ \theta(x_3, t = -\Lambda) = \begin{cases} -1, & x_3 < 0, \\ 1, & x_3 > 0, \end{cases} \end{cases} \tag{1.16}$$

and can be computed explicitly by

$$\theta(x_3, t) = \Theta(\xi) := \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2}\sqrt{\frac{\bar{\rho}}{\mu}}\xi} e^{-\eta^2} d\eta \qquad \text{with} \quad \xi = \frac{x_3}{\sqrt{t + \Lambda}}. \tag{1.17}$$

Note that Θ is independent of Λ, satisfying that

$$\begin{cases} -2\mu\Theta''(\xi) = \bar{\rho}\xi\Theta'(\xi), & \xi \in \mathbb{R}, \\ \lim_{\xi \to \pm\infty} \Theta(\xi) = \pm 1. \end{cases} \tag{1.18}$$

Lemma 1.1. *The solution θ of (1.16) is odd and strictly increasing with respect to x_3. Moreover, for any $j = 1, 2, \cdots$, it holds that*

$$\left| \partial_3^j \theta(x_3, t) \right| \leqslant C_j (t + \Lambda)^{-\frac{j}{2}} \exp \left\{ -\frac{\bar{\rho} |x_3|^2}{8\mu(t + \Lambda)} \right\} \qquad \forall x_3 \in \mathbb{R}, t \geqslant 0. \tag{1.19}$$

Viscous wave. Combining (1.12) and (1.15), we can define the viscous wave associated with the vortex sheet (1.7) as

$$\rho^{\mathsf{vs}}(x_3, t) := \bar{\rho} \quad \text{and} \quad \mathbf{u}^{\mathsf{vs}}(x_3, t) := \theta(x_3, t)\bar{\mathbf{u}} = \Theta\left(\frac{x_3}{\sqrt{t + \Lambda}}\right)\bar{\mathbf{u}}. \tag{1.20}$$

The viscous wave (1.20) is a smooth solution (for $t \geqslant 0$) to the compressible NS equations (1.1) and connects the constant states of the vortex sheet, $(\bar{\rho}, \pm\bar{\mathbf{u}})$, as $x_3 \to \pm\infty$ for all $t \geqslant 0$. Moreover, for any $p \in [1, +\infty)$, it holds that

$$\|(\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}}) - (\bar{\rho}, \bar{\mathbf{u}}^{\mathsf{vs}})\|_{L^p(\mathbb{R}; \mathrm{d}x_3)} \leqslant C |\bar{\mathbf{u}}| [\mu(t + \Lambda)]^{\frac{1}{2p}}, \qquad t \geqslant 0.$$

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Thus, as the viscosity μ vanishes, the viscous wave (1.20) approximates the vortex sheet (1.7) on any finite time interval, but not time-asymptotically.

In this paper, we first study a spatially periodic perturbation of the viscous wave (1.20), i.e.

$$(\rho, \mathbf{m})(x, 0) = (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(x_3, 0) + (v_0, \mathbf{w}_0)(x), \quad x \in \mathbb{R}^3, \tag{1.21}$$

where $(v_0, \mathbf{w}_0)(x) = (v_0, w_{01}, w_{02}, w_{03})(x)$ is a periodic function on $\mathbb{T}^3 = [0, 1]^3$ with zero average,

$$\int_{\mathbb{T}^3} (v_0, \mathbf{w}_0)(x) dx = 0. \tag{1.22}$$

Then, we will show in Section 6 that the similar arguments can also be applied to study a localized perturbation on the domain $\Omega := \mathbb{T}^2 \times \mathbb{R}$, i.e.

$$(\rho, \mathbf{m})(x, 0) = (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(x_3, 0) + (\bar{\phi}_0, \bar{\psi}_0)(x), \quad x \in \mathbb{T}^2 \times \mathbb{R}, \tag{1.23}$$

where $(\bar{\phi}_0, \bar{\psi}_0)$ is periodic with respect to the transverse variables $x_\perp = (x_1, x_2) \in \mathbb{T}^2$ and belongs to $H^3(\mathbb{T}^2 \times \mathbb{R})$.

The compressible vortex sheets are contact discontinuities for the multi-dimensional compressible Euler equations. For the 1D viscous conservation laws with uniform viscosity, [39] was the first to prove that the inviscid contact discontinuities are meta-stable for the viscous models, and then [22, 42] constructed an asymptotic viscous contact wave and established the pointwise stability via the approximate Green function approach. For the 1D NS equations in Lagrangian coordinates, [13, 14, 15, 12] applied the basic L^2-energy method to achieve the asymptotic stability of the contact discontinuities. This paper gives the first stability results of the contact discontinuities for the compressible NS equations in Eulerian coordinates. Moreover, we can achieve the optimal decay rates; see Remark 1.6.

Note that for the problem (1.1), (1.21), although the perturbation $(\rho, \mathbf{m}) - (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})$ is periodic with respect to $x \in \mathbb{T}^3$ initially, it does not remain periodic with respect to x_3 for all positive time, due to the non-triviality of the background wave (1.20) and the nonlinearity of (1.1). Thus, it is more challenging to deal with the oscillating perturbation (1.21) than the localized one (1.23). On the other hand, the theories of the compressible Navier-Stokes equations and Euler equations are connected to each other very closely, and the study of periodic perturbations is an important and interesting topic in the hyperbolic theories. In particular, for the isentropic compressible Euler equations, the spatially periodic solutions decay to constant states as $t \to +\infty$; see [20, 9, 5, 6]. However, there appears a resonance phenomenon in the non-isentropic case; as a result, the spatially periodic solutions may oscillate in time simultaneously, which never happens neither in the isentropic case nor for the BV solutions; see [23, 33, 29, 34]. Recently, the works [41, 40, 45] also found a new phenomenon of the periodic perturbations in the stability theory of shocks, that is, a new kind of shock shifts is generated by the periodic oscillations, which generally depend on the structures of the equations, while in the case for the localized perturbations, the shock shifts depend only on the initial data.

In this paper, we show that with the initial perturbations in (1.21) and (1.23), the viscous wave associated with the vortex sheet is time-asymptotically stable in the L^∞-norm. The framework is based on an anti-derivative technique and the L^2-energy method, that is, we need to study the integrated system of the perturbation in addition. The anti-derivative technique was initiated by [19] to prove the shock stability for the 1D scalar conservation laws, and then was widely used for more 1D models to study the stability of both shocks and contact discontinuities; see [25, 10, 21, 32, 22, 14, 15, 42] for instance.

In the multi-dimensional case, as shown by [44, 43], this method is also effective to prove the stability of planar viscous shock profiles with spatially periodic perturbations for the NS equations. A key observation in [43, 44] is that, with the aid of Poincaré inequality, it suffices to use the anti-derivative technique in the estimate of the zero modes associated with the multi-dimensional perturbations. It is noted that a premise in the application of the anti-derivative technique is that the perturbations should be of zero masses for all time. In the previous works concerning localized perturbations, one can just choose constant shifts of the background waves to make the excessive masses of the perturbations vanish; and in the case for the spatially periodic perturbations, the shifts should be some specific functions of time to cancel the influence of the oscillations; see [40, 17, 46, 43]. However, for the vortex sheets with periodic perturbations, there is a new difficulty in determining the shift curves. More precisely, the previous analysis relies essentially on the assumption that crossing the interface of the background wave, all the quantities, i.e. the density, momentum and total energy, must be discontinuous. However, this assumption fails for the vortex sheets, whose density and normal momentum are continuous across the interface. To overcome this technical difficulty, in this paper we have an important observation that for the problem (1.1), (1.21), the asymptotic behaviors of the solution at the spatial infinity $x_3 \to \pm\infty$ coincide with each other after a Galilean transformation; see (2.1) and Lemma 2.2. Based on the key property, we are able to find one shift curve of the vortex sheet such that the perturbation is of zero mass for all time. This is the first novelty of this paper. The second key element in our analysis is that we introduce a new variable in terms of the anti-derivatives (see (4.10)), which plays an important role in the L^2-estimates. Furthermore, this new variable is also essential in overcoming the difficulty caused by the large amplitude of the vortex sheet and helps us achieve the optimal decay rates without the aid of the Green function.

Now we state the main results of this paper. To the best of our knowledge, this is the first result about the stabilizing effect of viscosity on the compressible vortex sheets. Also, this is the first stability result of vortex sheets in a time-asymptotic sense.

Theorem 1.2. *Let* $(\bar{\rho}^{\mathsf{vs}}, \bar{\mathbf{u}}^{\mathsf{vs}})(x_3, t)$ *be the given vortex sheet (1.7) and* $(\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})(x_3, t)$ *be the corresponding viscous wave, which is a smooth solution to the 3D compressible Navier-Stokes equations with the form (1.20). Then there exist* $\Lambda_0 \geqslant 1$ *and* $\varepsilon_0 > 0$ *such that if*

$$\Lambda \geqslant \Lambda_0 \quad and \quad \|v_0, \mathbf{w}_0\|_{H^6(\mathbb{T}^3)} \leqslant \varepsilon_0 \Lambda^{-\frac{1}{4}}, \tag{1.24}$$

then the Cauchy problem (1.1), (1.21) admits a unique classical bounded solution $(\rho, \mathbf{u})(x, t)$ *globally in time, which is periodic in the transverse variables* $x_\perp = (x_1, x_2) \in \mathbb{T}^2$. *Moreover, the solution satisfies that*

$$\sup_{x_3 \in \mathbb{R}} \left| (\rho^\flat, \mathbf{u}^\flat)(x_3, t) - (\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})(x_3, t) \right| \leqslant C\sqrt{\varepsilon_0}(t + 1)^{-\frac{3}{4}},$$
$$\sup_{x \in \mathbb{R}^3} \left| (\rho^\sharp, \mathbf{u}^\sharp)(x, t) \right| \leqslant C\varepsilon_0 e^{-\alpha_0 t}, \tag{1.25}$$

where $(\rho^\flat, \mathbf{u}^\flat)$ *and* $(\rho^\sharp, \mathbf{u}^\sharp)$ *denote the zero mode and non-zero mode of the solution, respectively, namely,*

$$(\rho^\flat, \mathbf{u}^\flat)(x_3, t) := \int_{\mathbb{T}^2} (\rho, \mathbf{u})(x_\perp, x_3, t) dx_\perp,$$
$$(\rho^\sharp, \mathbf{u}^\sharp)(x, t) := (\rho, \mathbf{u})(x, t) - (\rho^\flat, \mathbf{u}^\flat)(x_3, t); \tag{1.26}$$

and $C > 0$ *and* $\alpha_0 > 0$ *in (1.25) are some generic constants.*

Theorem 1.3. *If in Theorem 1.2, the initial data (1.21) is replaced by (1.23), then there exist $\Lambda_0 \geqslant 1$ and $\varepsilon_0 > 0$ such that if*

$$\Lambda \geqslant \Lambda_0 \quad and \quad \left\| \bar{\phi}_0, \bar{\psi}_0 \right\|_{L_3^2(\Omega)} + \left\| \nabla \bar{\phi}_0, \nabla \bar{\psi}_0 \right\|_{H^2(\Omega)} \leqslant \varepsilon_0, \tag{1.27}$$

where $\|\cdot\|_{L_3^2(\Omega)}^2 := \int_\Omega (x_3^2 + 1)^{\frac{3}{2}} |\cdot|^2 \, dx$, then for the problem (1.1), (1.23), all the results still hold true, except the decay rate of the zero mode, which is replaced by

$$\sup_{x_3 \in \mathbb{R}} \left| (\rho^\flat, \mathbf{u}^\flat)(x_3, t) - (\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})(x_3, t) \right| \leqslant C\sqrt{\varepsilon_0}(t + 1)^{-\frac{1}{2}}. \tag{1.28}$$

Remark 1.4. The results of Theorems 1.2 and 1.3 also hold true in two dimensions, since we allow the constant $\bar{u}_1$ in (1.7) to be zero and meanwhile, allow the initial perturbations to be independent of x_1.

Remark 1.5. 1) The zero-average condition (1.22) is just assumed without loss of generality. In fact, if the average, $\int_{\mathbb{T}^3}(v_0, \mathbf{w}_0)(x)dx := (\bar{v}, \bar{\mathbf{w}})$, is nonzero, then one can rewrite the initial data (1.21) as

$$(\rho_0, \mathbf{m}_0) = (\rho^{\mathsf{vs}} + \bar{v}, \mathbf{m}^{\mathsf{vs}} + \bar{\mathbf{w}}) + (v_0 - \bar{v}, \mathbf{w}_0 - \bar{\mathbf{w}}). \tag{1.29}$$

Recall that $\rho^{\mathsf{vs}} = \bar{\rho} > 0$ is a constant. Then using the Galilean transformation (1.6) with $\mathbf{c} = -\bar{\rho}^{-1}\bar{\mathbf{w}}$, the initial data (1.29) reduces to

$$(\rho_0, \mathbf{m}_0) = (\rho^{\mathsf{vs}} + \bar{v}, \mathbf{m}^{\mathsf{vs}}) + (v_0 - \bar{v}, \mathbf{w}_0 - \bar{\mathbf{w}}).$$

If $|\bar{v}|$ is suitably small, $(\rho^{\mathsf{vs}} + \bar{v}, \mathbf{m}^{\mathsf{vs}})$ is actually the viscous wave associated with the vortex sheet,

$$(\rho, \mathbf{u}) = \begin{cases} (\bar{\rho} + \bar{v}, -\frac{\bar{\rho}}{\bar{\rho}+\bar{v}}\bar{\mathbf{u}}), & x_3 < 0, \\ (\bar{\rho} + \bar{v}, \frac{\bar{\rho}}{\bar{\rho}+\bar{v}}\bar{\mathbf{u}}), & x_3 > 0. \end{cases}$$

2) In (1.27), the $L_3^2(\Omega)$-integrability of $(\bar{\phi}_0, \bar{\psi}_0)$ is to ensure that the anti-derivatives of the perturbations belong to $L^2(\mathbb{R})$; see (6.15) for the details.

Remark 1.6. In the special case that the initial perturbations in (1.21) and (1.23) are independent of the transversal variables, the Cauchy problem for (1.1) reduces to a 1D one for (1.9), whose associated hyperbolic system,

$$\begin{cases} \partial_t \rho + \partial_3 m_3 = 0, \\ \partial_t m_i + \partial_3 \left(\frac{1}{\rho} m_3 m_i \right) = 0, & i = 1, 2, \\ \partial_t m_3 + \partial_3 \left(\frac{1}{\rho} m_3^2 + p(\rho) \right) = 0, \end{cases} \tag{1.30}$$

has two linearly degenerate characteristics with eigenvalues $\lambda_1 = \lambda_2 = u_3$ (see Section 6). One can easily verify that the contact discontinuities of (1.30), associated with λ_1 or λ_2 (see [20]), just coincide with the zero mode of the planar vortex sheet (1.4) and satisfy the RH conditions (1.5). If the hyperbolic system (1.30) has uniform viscosity, [42] constructed an asymptotic ansatz that consists of a viscous contact wave and some diffusion waves propagating in the transversal characteristic fields, and established the pointwise estimates with the optimal decay rate $t^{-\frac{3}{4}}$. Here the viscous contact wave constructed in [42] is just the same as the zero mode of (1.20). Thus, this paper actually extends the stability results of [42] to a multi-dimensional and physical model. On the other hand, compared to the

results in [42], one can see that the rates in (1.25) and (1.28) are both optimal. Indeed, the spatially periodic perturbations do not generate diffusion waves in the transversal characteristic fields (see (2.7)), thus we are able to achieve the rate $t^{-\frac{3}{4}}$; while in the case for the localized perturbations, the rate is at most same as the diffusion waves, that is $t^{-\frac{1}{2}}$.

Notations. Throughout this paper, we use the following notations.

- Ω denotes the infinitely long nozzle domain

$$\Omega := \mathbb{T}^2 \times \mathbb{R}.$$

- $[\![\cdot]\!]$ denotes the difference of the states associated with the vortex sheet (1.7). For instance, $[\![\mathbf{u}]\!] = \mathbf{u}_+ - \mathbf{u}_-$ and $[\![u_1 m_1]\!] = \bar{u}_{1+}\bar{m}_{1+} - \bar{u}_{1-}\bar{m}_{1-}$, etc.

- $C \geqslant 1$ is a generic constant. The conventions $A \lesssim B$, $A \gtrsim B$, $A \sim B$ and $A = O(1)B$ mean $A \leqslant CB$, $A \geqslant C^{-1}B$, $C^{-1}B \leqslant A \leqslant CB$ and $|A| \leqslant C|B|$, respectively.

- $\mathbf{e}_i$ $(i = 1, 2, 3)$ denotes the i-th column of the 3×3 identity matrix $\mathrm{Id}_{3\times 3}$, and δ_{ij} denotes the Kronecker delta function.

- For any vector $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{v}_\perp$ denotes

$$\mathbf{v}_\perp := (v_1, v_2). \tag{1.31}$$

- For any $f(x) \in L^\infty(\mathbb{R}^3)$ that is periodic in the spatial variables $x_\perp = (x_1, x_2) \in \mathbb{T}^2$, $f^\flat$ and $f^\sharp$ denote its *zero mode*,

$$f^\flat(x_3) := \int_{\mathbb{T}^2} f(x_\perp, x_3) dx_\perp, \tag{1.32}$$

and *non-zero mode*,

$$f^\sharp(x) := f(x) - f^\flat(x_3), \tag{1.33}$$

respectively.

Outline of the paper. In Section 2, we construct the key ansatz in our analysis, which relies on the Galilean transformation and a careful choice of a shift function. In Section 3, we first formulate the perturbed system, the zero-mode system and the anti-derivative system, respectively, and then give some useful lemmas. Section 4 is devoted to the a priori estimates, consisting of three subsections showing the lower-order estimates of the zero-modes, non-zero modes and the higher-order estimates of the original perturbations, respectively. Finally, in Section 5 we prove the optimal decay rate of the perturbations and complete the proof of the main result.

2 Construction of Ansatz

As the solution $(\rho, \mathbf{m})$ to the Cauchy problem (1.1), (1.21) is periodic in the transverse directions $x_\perp \in \mathbb{T}^2$, it suffices to consider the problem on the domain $\Omega = \mathbb{T}^2 \times \mathbb{R}$. However, since the perturbation, $(\rho, \mathbf{m}) - (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})$, keeps oscillating as $|x_3| \to +\infty$, it is necessary to construct a suitable ansatz to cancel out the oscillations in order to use the energy method. Motivated by [40, 43], it is plausible that as $|x_3| \to +\infty$,

$$|(\rho, \mathbf{m}) - (\rho_\pm, \mathbf{m}_\pm)| \, (x, t) \to 0 \qquad \forall x_\perp \in \mathbb{R}^2, t > 0. \tag{2.1}$$

Here $(\rho_\pm, \mathbf{m}_\pm) = (\rho_\pm, m_{1\pm}, m_{2\pm}, m_{3\pm})(x, t)$ denotes the periodic solution of (1.1), satisfying the periodic initial data

$$(\rho_\pm, \mathbf{m}_\pm)(x, 0) = (\bar{\rho}, \bar{\mathbf{m}}_\pm) + (v_0, \mathbf{w}_0)(x), \qquad x \in \mathbb{R}^3, \tag{2.2}$$

where $(\bar{\rho}, \bar{\mathbf{m}}_\pm) = (\bar{\rho}, \pm \bar{\rho}\bar{\mathbf{u}})$ is the plus or minus constant state of the vortex sheet (1.7) and $(v_0, \mathbf{w}_0)$ is the periodic perturbation in (1.21). The global existence and exponential decay rate of $(\rho_\pm, \mathbf{m}_\pm)$ can be found in the following lemma, which has been shown in [16].

Lemma 2.1 ([16]). *Consider the Cauchy problem (1.1) with the periodic initial data*

$$(\rho, \mathbf{m})(x, 0) = (\bar{\rho}, \bar{\mathbf{m}}) + (v_0, \mathbf{w}_0)(x), \quad x \in \mathbb{R}^3, \tag{2.3}$$

where $(\bar{\rho}, \bar{\mathbf{m}})$ is any constant state with $\bar{\rho} > 0$ and $(v_0, \mathbf{w}_0)$ is any periodic function on $\mathbb{T}^3 = [0, 1]^3$ with zero average. Then there exists $\varepsilon_0 > 0$ such that if

$$\|v_0, \mathbf{w}_0\|_{H^{k+2}(\mathbb{T}^3)} \leqslant \varepsilon_0 \qquad for \ some \ \ k \geqslant 1,$$

then the Cauchy problem (1.1), (2.3) admits a unique global periodic solution $(\rho, \mathbf{m}) \in C\big((0, +\infty); H^{k+2}(\mathbb{T}^3)\big)$, satisfying

$$\int_{\mathbb{T}^3} (\rho - \bar{\rho}, \mathbf{m} - \bar{\mathbf{m}})(x, t) dx = 0, \qquad t \geqslant 0,$$

and

$$\|(\rho, \mathbf{m}) - (\bar{\rho}, \bar{\mathbf{m}})\|_{W^{k,\infty}(\mathbb{R}^3)} \lesssim \|v_0, \mathbf{w}_0\|_{H^{k+2}(\mathbb{T}^3)} \, e^{-ct}, \qquad t \geqslant 0.$$

Here the constant $c > 0$ is a constant, independent of $\|v_0, \mathbf{w}_0\|_{H^{k+2}(\mathbb{T}^3)}$ and t.

It follows from Lemma 2.1 that for $\varepsilon = \|v_0, \mathbf{w}_0\|_{H^6(\mathbb{T}^3)} \leqslant \varepsilon_0$ with ε_0 being suitably small, the periodic solution $(\rho_\pm, \mathbf{m}_\pm)$ to the Cauchy problem (1.1), (2.2) exists in $C(0, +\infty; W^{4,\infty}(\mathbb{R}^3))$ with the constant average $(\bar{\rho}, \bar{\mathbf{m}}_\pm)$, and for some generic constant $\alpha > 0$, the periodic perturbation,

$$(v_\pm, \mathbf{w}_\pm, \mathbf{z}_\pm) := (\rho_\pm, \mathbf{m}_\pm, \mathbf{u}_\pm) - (\bar{\rho}, \bar{\mathbf{m}}_\pm, \bar{\mathbf{u}}_\pm), \tag{2.4}$$

satisfies that

$$\|(v_\pm, \mathbf{w}_\pm, \mathbf{z}_\pm)\|_{W^{4,\infty}(\mathbb{R}^3)} \lesssim \varepsilon e^{-2\alpha t}, \qquad t \geqslant 0. \tag{2.5}$$

Now we are ready to construct the ansatz. Note that the viscous wave (1.20) can be rewritten as

$$(\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(x_3, t) = \frac{1}{2} \Big[(\bar{\rho}, \bar{\mathbf{m}}_-) \Big(1 - \Theta\Big(\frac{x_3}{\sqrt{t + \Lambda}}\Big) \Big) + (\bar{\rho}, \bar{\mathbf{m}}_+) \Big(1 + \Theta\Big(\frac{x_3}{\sqrt{t + \Lambda}}\Big) \Big) \Big]. \tag{2.6}$$

Asymptotic Ansatz. Inspired by (2.1), (2.6) and [40, 43] we set the *ansatz* as

$$(\tilde{\rho}, \tilde{\mathbf{m}})(x, t) := \frac{1}{2}\Big[(\rho_-, \mathbf{m}_-)(x, t)\Big(1 - \Theta\Big(\frac{x_3 - \sigma(t)}{\sqrt{t + \Lambda}}\Big)\Big)$$
$$+ (\rho_+, \mathbf{m}_+)(x, t)\Big(1 + \Theta\Big(\frac{x_3 - \sigma(t)}{\sqrt{t + \Lambda}}\Big)\Big)\Big], \tag{2.7}$$

where $\sigma = \sigma(t)$ is a curve to be determined later. For convenience, denote

$$\Theta_\sigma := \Theta\Big(\frac{x_3 - \sigma}{\sqrt{t + \Lambda}}\Big).$$

Define also

$$\tilde{\mathbf{u}} := \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}, \tag{2.8}$$

which satisfies that

$$\tilde{\mathbf{u}} = \frac{1}{2}\Big[\mathbf{u}_-(1 - \Theta_\sigma) + \mathbf{u}_+(1 + \Theta_\sigma)\Big]$$
$$+ \frac{1}{4\tilde{\rho}}(v_+ - v_-)(\mathbf{u}_+ - \mathbf{u}_-)\left(1 - \Theta_\sigma^2\right). \tag{2.9}$$

Note that $(\rho_\pm, \mathbf{m}_\pm)$ are solutions of (1.1). Then direct calculations yield that

$$\begin{cases} \partial_t \tilde{\rho} + \mathrm{div}\tilde{\mathbf{m}} = f_0, \\ \partial_t \tilde{\mathbf{m}} + \mathrm{div}(\tilde{\mathbf{u}} \otimes \tilde{\mathbf{m}}) + \nabla\big(p(\tilde{\rho})\big) - \mu\triangle\tilde{\mathbf{u}} - (\mu + \lambda)\nabla\mathrm{div}\tilde{\mathbf{u}} \\ \qquad = \sum\limits_{i=1}^{3} \partial_i \mathbf{F}_{1,i} + \mathbf{f}_2 := \mathbf{g} = (g_1, g_2, g_3), \end{cases} \tag{2.10}$$

where f_0, $\mathbf{F}_{1,i} = (F_{1,i1}, F_{1,i2}, F_{1,i3})$ and $\mathbf{f}_2 := (f_{2,1}, f_{2,2}, f_{2,3})$ are errors, given by

$$f_0 = \frac{\Theta_\sigma'}{2\sqrt{t + \Lambda}}\Big[-(\rho_+ - \rho_-)\Big(\sigma'(t) + \frac{x_3 - \sigma(t)}{2(t + \Lambda)}\Big) + (m_{3+} - m_{3-})\Big], \tag{2.11}$$

$$\mathbf{F}_{1,i} = \tilde{u}_i\tilde{\mathbf{m}} - \frac{1}{2}u_{i-}\mathbf{m}_-(1 - \Theta_\sigma) - \frac{1}{2}u_{i+}\mathbf{m}_+(1 + \Theta_\sigma)$$
$$+ \Big[p(\tilde{\rho}) - \frac{1}{2}p(\rho_-)(1 - \Theta_\sigma) - \frac{1}{2}p(\rho_+)(1 + \Theta_\sigma)\Big]\mathbf{e}_i$$
$$- \mu\Big[\partial_i\tilde{\mathbf{u}} - \frac{1}{2}\partial_i\mathbf{u}_-(1 - \Theta_\sigma) - \frac{1}{2}\partial_i\mathbf{u}_+(1 + \Theta_\sigma)\Big]$$
$$- (\mu + \lambda)\Big[\mathrm{div}\tilde{\mathbf{u}} - \frac{1}{2}\mathrm{div}\mathbf{u}_-(1 - \Theta_\sigma) - \frac{1}{2}\mathrm{div}\mathbf{u}_+(1 + \Theta_\sigma)\Big]\mathbf{e}_i, \tag{2.12}$$

and

$$\mathbf{f}_2 := \frac{\Theta_\sigma'}{2\sqrt{t + \Lambda}}\Big\{ -(\mathbf{m}_+ - \mathbf{m}_-)\Big(\sigma'(t) + \frac{x_3 - \sigma(t)}{2(t + \Lambda)}\Big)$$
$$+ u_{3+}\mathbf{m}_+ - u_{3-}\mathbf{m}_- - \mu\partial_3(\mathbf{u}_+ - \mathbf{u}_-)$$
$$+ \big[p(\rho_+) - p(\rho_-) - (\mu + \lambda)\mathrm{div}(\mathbf{u}_+ - \mathbf{u}_-)\big]\mathbf{e}_3\Big\}, \tag{2.13}$$

respectively.

Recall that $\Theta \to \pm 1$ as $x_3 \to \pm\infty$. Then for all $x_\perp \in \mathbb{T}^2$ and $t \geqslant 0$, it holds that

$$|\tilde{\rho} - \rho_\pm| + |\tilde{\mathbf{m}} - \mathbf{m}_\pm| + |\tilde{\mathbf{u}} - \mathbf{u}_\pm| \to 0 \quad \text{as} \quad x_3 \to \pm\infty.$$

Then for $i = 1, 2, 3$ and all $x_\perp \in \mathbb{T}^2$ and $t \geqslant 0$, it holds that

$$|\mathbf{F}_{1,i}(x,t)| \to 0 \quad \text{as} \quad x_3 \to \pm\infty,$$

which, together with the dominated convergence theorem, yields that

$$\left|\mathbf{F}^\flat_{1,i}(x_3,t)\right| \to 0 \quad \text{as} \quad x_3 \to \pm\infty.$$

Thus, by subtracting (2.10) from (1.1) and integrating the resulting equations over $\mathbb{T}^2 \times \mathbb{R}$, one can get that

$$\frac{d}{dt}\left[\int_{\mathbb{R}} \left(\rho^\flat - \tilde\rho^\flat, \mathbf{m}^\flat - \tilde{\mathbf{m}}^\flat\right)dx_3\right] = -\int_{\mathbb{R}} \left(f_0^\flat, \mathbf{f}_2^\flat\right)(x_3,t)dx_3, \qquad t > 0, \tag{2.14}$$

where one can obtain from (2.11) and (2.13) that

$$f_0^\flat = \frac{\Theta'_\sigma}{2\sqrt{t+\Lambda}}\left[-(\rho_+^\flat - \rho_-^\flat)\left(\sigma'(t) + \frac{x_3 - \sigma(t)}{2(t+\Lambda)}\right) + \left(m_{3+}^\flat - m_{3-}^\flat\right)\right], \tag{2.15}$$

and

$$\begin{aligned}
\mathbf{f}_2^\flat = \frac{\Theta'_\sigma}{2\sqrt{t+\Lambda}}\Big\{ &-\left(\mathbf{m}_+^\flat - \mathbf{m}_-^\flat\right)\left(\sigma'(t) + \frac{x_3 - \sigma(t)}{2(t+\Lambda)}\right) \\
&+ (u_{3+}\mathbf{m}_+)^\flat - (u_{3-}\mathbf{m}_-)^\flat - \mu\partial_3\left(\mathbf{u}_+^\flat - \mathbf{u}_-^\flat\right) \\
&+ \left[\left(p(\rho_+)\right)^\flat - \left(p(\rho_-)\right)^\flat - (\mu+\lambda)\partial_3\left(u_{3+}^\flat - u_{3-}^\flat\right)\right]\mathbf{e}_3\Big\}.
\end{aligned} \tag{2.16}$$

Using the Galilean transformation (1.6), one can prove that the periodic solutions $(\rho_\pm, \mathbf{u}_\pm)$ of (1.1) satisfy the following coincide-property.

Lemma 2.2. *Let* $(\bar\rho, \pm\bar{\mathbf{u}}) = (\bar\rho, \pm\bar{u}_1, \pm\bar{u}_2, 0)$ *be the constant states in the vortex sheet (1.7) and assume that (1.22) holds. Then the periodic solutions to the Cauchy problem (1.1), (2.2) satisfy that*

$$(\rho_+, \mathbf{u}_+)(x + \bar{\mathbf{u}}t, t) = (\rho_-, \mathbf{u}_-)(x - \bar{\mathbf{u}}t, t) + (0, 2\bar{\mathbf{u}}) \qquad \forall x \in \mathbb{R}^3, t \geqslant 0. \tag{2.17}$$

Moreover, the zero modes on the right-hand sides of (2.15) and (2.16) satisfy that

$$\begin{aligned}
(\rho_+^\flat, \mathbf{u}_+^\flat) &\equiv (\rho_-^\flat, \mathbf{u}_-^\flat) + (0, 2\bar{\mathbf{u}}), \\
\mathbf{m}_+^\flat &\equiv \mathbf{m}_-^\flat + 2\rho_-^\flat\bar{\mathbf{u}}, \\
(u_{3+}\mathbf{m}_+)^\flat &\equiv (u_{3-}\mathbf{m}_-)^\flat + 2m_{3-}^\flat\bar{\mathbf{u}}, \\
(p(\rho_+))^\flat &\equiv (p(\rho_-))^\flat.
\end{aligned} \tag{2.18}$$

Proof. By the Galilean invariance of the NS equations, the pairs $(\rho_+^*, \mathbf{u}_+^*)$ and $(\rho_-^*, \mathbf{u}_-^*)$ defined by

$$\begin{aligned}
(\rho_+^*, \mathbf{u}_+^*)(x,t) &:= (\rho_+, \mathbf{u}_+ - \bar{\mathbf{u}})(x + \bar{\mathbf{u}}t, t), \\
(\rho_-^*, \mathbf{u}_-^*)(x,t) &:= (\rho_-, \mathbf{u}_- + \bar{\mathbf{u}})(x - \bar{\mathbf{u}}t, t),
\end{aligned}$$

are both periodic solutions to (1.1) with the initial data

$$(\rho_\pm^*, \mathbf{u}_\pm^*)(x, t = 0) = (\bar\rho, 0, 0, 0) + (v_0, \mathbf{w}_0)(x).$$

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Thus, it follows from the uniqueness (see Lemma 2.1) that

$$(\rho_+^*, \mathbf{u}_+^*)(x,t) = (\rho_-^*, \mathbf{u}_-^*)(x,t) \qquad \forall x \in \mathbb{R}^3, t \geqslant 0,$$

which yields (2.17).

Now the identities in (2.18) can be verified one by one as follows. First, it follows from Lemma 2.1 and (2.17) that

$$\begin{aligned}
(\rho_+^\flat, \mathbf{u}_+^\flat)(x_3,t) &= \int_{\mathbb{T}^2} (\rho_+, \mathbf{u}_+)(x_\perp, x_3, t) dx_\perp \\
&= \int_{\mathbb{T}^2} (\rho_-, \mathbf{u}_-)(x_1 - 2\bar{u}_1 t, x_2 - 2\bar{u}_2 t, x_3, t) dx_\perp + (0, 2\bar{\mathbf{u}}) \\
&= (\rho_-^\flat, \mathbf{u}_-^\flat)(x_3,t) + (0, 2\bar{\mathbf{u}}) \qquad \forall x_3 \in \mathbb{R}, t \geqslant 0.
\end{aligned}$$

Recall that $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, 0)$. Then one can prove that

$$\begin{aligned}
\mathbf{m}_+^\flat(x_3,t) &= \int_{\mathbb{T}^2} (\rho_+ \mathbf{u}_+)(x,t) dx_\perp = \int_{\mathbb{T}^2} (\rho_- \mathbf{u}_-)(x - 2\bar{\mathbf{u}}t, t) dx_\perp + 2\rho_-^\flat(x_3,t)\bar{\mathbf{u}} \\
&= \mathbf{m}_-^\flat(x_3,t) + 2\rho_-^\flat(x_3,t)\bar{\mathbf{u}}, \\
(u_{3+}\mathbf{m}_+)^\flat(x_3,t) &= \int_{\mathbb{T}^2} (\rho_+ u_{3+}\mathbf{u}_+)(x_\perp, x_3, t) dx_\perp \\
&= \int_{\mathbb{T}^2} (\rho_- u_{3-}\mathbf{u}_-)(x_\perp, x_3, t) dx_\perp + 2\bar{\mathbf{u}} \int_{\mathbb{T}^2} (\rho_- u_{3-})(x_\perp, x_3, t) x_\perp \\
&= (u_{3-}\mathbf{m}_-)^\flat(x_3,t) + 2m_{3-}^\flat(x_3,t)\bar{\mathbf{u}}.
\end{aligned}$$

It is direct to prove that $(p(\rho_+))^\flat(x_3,t) = (p(\rho_-))^\flat(x_3,t)$. The proof is finished. $\qquad\square$

Using (2.18) on (2.15) and (2.16), one can get that

$$\left(f_0^\flat, f_{2,3}^\flat\right) \equiv 0, \tag{2.19}$$

and

$$f_{2,i}^\flat = \frac{\bar{u}_i \Theta_\sigma'}{\sqrt{t+\Lambda}}\left[-\rho_-^\flat\left(\sigma'(t) + \frac{x_3 - \sigma}{2(t+\Lambda)}\right) + m_{3-}^\flat \right], \qquad i = 1, 2. \tag{2.20}$$

Here recall that $(\rho_-^\flat, m_{3-}^\flat) \equiv (\rho_+^\flat, m_{3+}^\flat)$. Then integrating (2.20) on $\mathbb{R}$ yields that

$$\int_{\mathbb{R}} f_{2,i}^\flat(x_3,t) dx_3 = \bar{u}_i\left(-\mathfrak{D}(\sigma,t)\sigma'(t) + \mathfrak{N}(\sigma,t)\right), \qquad i = 1, 2, \tag{2.21}$$

where

$$\begin{aligned}
\mathfrak{D}(\sigma,t) &:= \frac{1}{\sqrt{t+\Lambda}} \int_{\mathbb{R}} \rho_-^\flat(x_3,t)\Theta'\left(\frac{x_3 - \sigma}{\sqrt{t+\Lambda}}\right) dx_3, \\
\mathfrak{N}(\sigma,t) &:= \frac{1}{\sqrt{t+\Lambda}} \int_{\mathbb{R}} \left[-\frac{x_3 - \sigma}{2(t+\Lambda)}\rho_-^\flat(x_3,t) + m_{3-}^\flat(x_3,t)\right]\Theta'\left(\frac{x_3 - \sigma}{\sqrt{t+\Lambda}}\right) dx_3.
\end{aligned} \tag{2.22}$$

Thus, if we choose the curve $\sigma(t)$ to satisfy that

$$\sigma'(t) = \frac{\mathfrak{N}(\sigma,t)}{\mathfrak{D}(\sigma,t)}, \qquad t > 0, \tag{2.23}$$

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then it holds that

$$\int_{\mathbb{R}} f_{2,1}^{\flat}(x_3,t)dx_3 = 0 \quad \text{and} \quad \int_{\mathbb{R}} f_{2,2}^{\flat}(x_3,t)dx_3 = 0 \qquad \forall t \geqslant 0. \tag{2.24}$$

This, together with (2.14) and (2.19), yields that

$$\begin{aligned}
\int_{\mathbb{R}} (\rho^{\flat} - \tilde{\rho}^{\flat}, \mathbf{m}^{\flat} - \tilde{\mathbf{m}}^{\flat})(x_3,t)dx_3 &= \int_{\mathbb{R}} (\rho^{\flat} - \tilde{\rho}^{\flat}, \mathbf{m}^{\flat} - \tilde{\mathbf{m}}^{\flat})(x_3,0)dx_3 \\
&= \int_{\mathbb{R}} \big(0, [\Theta(x_3) - \Theta(x_3 - \sigma(0))]\bar{\mathbf{m}} \big) dx_3 \\
&= (0, 2\sigma(0)\bar{\mathbf{m}}) \qquad \forall t \geqslant 0.
\end{aligned} \tag{2.25}$$

Thus, if the curve $\sigma(t)$ solves the problem (2.23) with the initial data,

$$\sigma(0) = 0, \tag{2.26}$$

then one has that

$$\int_{\mathbb{R}} (\rho^{\flat} - \tilde{\rho}^{\flat}, \mathbf{m}^{\flat} - \tilde{\mathbf{m}}^{\flat})(x_3,t)dx_3 = 0 \qquad \forall t \geqslant 0. \tag{2.27}$$

The existence and large time behavior of the curve $\sigma(t)$ can be found in the following lemma.

Lemma 2.3. *Under the assumptions of Lemma 2.2, there exists an $\varepsilon_0 > 0$ such that if $\Lambda \geqslant 1$ and $\varepsilon := \|v_0, \mathbf{w}_0\|_{H^6(\mathbb{T}^3)} \leqslant \varepsilon_0$, then the problem (2.23), (2.26), admits a unique global solution $\sigma = \sigma(t) \in C^1(0, +\infty)$, satisfying that*

$$|\sigma'(t)| \lesssim \varepsilon e^{-\alpha t}, \qquad |\sigma(t)| \lesssim \varepsilon \Lambda^{\frac{1}{2}} e^{-\alpha t}, \qquad t \geqslant 0, \tag{2.28}$$

where α are the constants in (1.20) and (2.5), respectively.

The proof of Lemma 2.3 is similar to that in [40, 43]. For readers' convenience, we also give a detailed proof in Appendix A.

Then based on Lemmas 2.1 and 2.3, the desired ansatz is well constructed by (2.7). If $\varepsilon \leqslant \varepsilon_0$ with ε_0 being suitably small, it holds that

$$\bar{\rho}/2 \leqslant \inf_{\substack{x \in \mathbb{R}^3 \\ t>0}} \tilde{\rho}(x,t) \leqslant \sup_{\substack{x \in \mathbb{R}^3 \\ t>0}} \tilde{\rho}(x,t) \leqslant 2\bar{\rho}, \tag{2.29}$$

and

$$\sup_{t>0} \|\nabla\tilde{\rho}, \nabla\tilde{\mathbf{u}}\|_{W^{3,\infty}(\mathbb{R}^3)} \lesssim \varepsilon + \Lambda^{-1/2}. \tag{2.30}$$

Moreover, the difference between the background viscous wave (2.6) and the ansatz (2.7) satisfies that

$$\begin{aligned}
\|(\tilde{\rho}, \tilde{\mathbf{m}}) - (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})\|_{W^{4,\infty}(\mathbb{R}^3)} &\lesssim \|(\tilde{\rho}, \tilde{\mathbf{m}})(\cdot, t) - (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(\cdot - \sigma, t)\|_{W^{4,\infty}(\mathbb{R}^3)} \\
&\quad + \|(\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(\cdot - \sigma, t) - (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(\cdot, t)\|_{W^{4,\infty}(\mathbb{R})} \\
&\lesssim \|(v_{\pm}, \mathbf{w}_{\pm})(\cdot, t)\|_{W^{4,\infty}(\mathbb{T}^3)} + \frac{|\sigma(t)|}{\sqrt{t + \Lambda}}
\end{aligned}$$

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$$\lesssim \varepsilon e^{-\alpha t}. \tag{2.31}$$

Similarly, one can show that

$$\|\tilde{\mathbf{u}} - \mathbf{u}^{\mathsf{vs}}\|_{W^{4,\infty}(\mathbb{R}^3)} \lesssim \varepsilon e^{-\alpha t}. \tag{2.32}$$

Since the ansatz constructed in (2.7) is time-asymptotically equivalent to the viscous wave associated with the vortex sheet in the $L^\infty(\mathbb{R}^3)$-norm, it remains to study the large time behavior of the perturbation, $(\rho, \mathbf{m}) - (\tilde{\rho}, \tilde{\mathbf{m}})$.

3 Reformulated Problems

In the following part of this paper, we always denote Λ as the positive constant in (1.20) and $\varepsilon := \|(v_0, \mathbf{w}_0)\|_{H^6(\mathbb{T}^3)}$.

With the ansatz (2.7), we denote the perturbation as

$$\phi = \rho - \tilde{\rho}, \quad \psi = (\psi_1, \psi_2, \psi_3) := \mathbf{m} - \tilde{\mathbf{m}}, \tag{3.1}$$

and

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) := \mathbf{u} - \tilde{\mathbf{u}} = \frac{\psi - \tilde{\mathbf{u}}\phi}{\rho}. \tag{3.2}$$

It follows from (1.1) and (2.10) that

$$\begin{cases} \partial_t \phi + \mathrm{div}\psi = -f_0, \\ \partial_t \psi + \mathrm{div}\left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}}\right) + \nabla(p(\rho) - p(\tilde{\rho})) \\ \qquad -\mu \triangle \left(\frac{\mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\right) - (\mu + \lambda)\nabla\mathrm{div}\left(\frac{\mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\right) = -\mathbf{g}. \end{cases} \tag{3.3}$$

Due to (1.21), (2.2), (2.7) and (2.26), the perturbation (ϕ, ψ) satisfies initially

$$(\phi, \psi)(x, 0) = (\phi_0, \psi_0)(x) \equiv 0, \qquad x \in \mathbb{R}^3. \tag{3.4}$$

On the other hand, the equivalent system for (ϕ, ζ) takes the form,

$$\begin{cases} \partial_t \phi + \rho\,\mathrm{div}\zeta + \mathbf{u} \cdot \nabla\phi + \mathrm{div}\tilde{\mathbf{u}}\phi + \nabla\tilde{\rho} \cdot \zeta = -f_0, \\ \rho\partial_t \zeta + \rho\mathbf{u} \cdot \nabla\zeta + \nabla\big(p(\rho) - p(\tilde{\rho})\big) + \rho\zeta \cdot \nabla\tilde{\mathbf{u}} + \phi(\partial_t\tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}) \\ \qquad -\mu\triangle\zeta - (\mu + \lambda)\nabla\mathrm{div}\zeta = -\mathbf{g} + f_0\tilde{\mathbf{u}}, \end{cases} \tag{3.5}$$

and (3.4) implies that

$$(\phi, \zeta)(x, 0) = (\phi_0, \zeta_0)(x) \equiv 0, \qquad x \in \mathbb{R}^3. \tag{3.6}$$

For any $T > 0$, the solution space for (3.5) can be taken as

$$\begin{aligned} \mathbb{B}(0, T) := \Big\{ &(\phi, \zeta) : (\phi, \zeta) \text{ is periodic in } x_\perp = (x_1, x_2) \in \mathbb{T}^2, \\ &(\phi, \zeta) \in C(0, T; H^3(\Omega)), \\ &\nabla\phi \in L^2(0, T; H^2(\Omega)), \nabla\zeta \in L^2(0, T; H^3(\Omega)) \Big\}. \end{aligned}$$

Theorem 3.1. *Under the assumptions of Theorem 1.2, there exist $\Lambda_0 \geqslant 1$ and $\varepsilon_0 > 0$ such that if $\Lambda \geqslant \Lambda_0$ and $\varepsilon\Lambda^{\frac{1}{4}} \leqslant \varepsilon_0$, then the Cauchy problem, (3.5), (3.6) admits a unique solution $(\phi, \zeta) \in \mathbf{B}(0, +\infty)$, satisfying that*

$$
\begin{aligned}
\left\|(\phi^\flat, \zeta^\flat)(\cdot, t)\right\|_{L^\infty(\mathbb{R})} &\leqslant C(\varepsilon\Lambda^{\frac{1}{4}})^{\frac{1}{2}}(t+1)^{-\frac{3}{4}}, \\
\left\|(\phi^\sharp, \zeta^\sharp)(\cdot, t)\right\|_{L^\infty(\mathbb{R}^3)} &\leqslant C\varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha_0 t},
\end{aligned}
\tag{3.7}
$$

where $C > 0$ and $\alpha_0 \in (0, \alpha)$ are some generic constants.

We are going to use the L^2-energy method to prove Theorem 3.1. We first integrate (3.3) with respect to $x_\perp \in \mathbb{T}^2$ to get the zero-mode system,

$$
\begin{cases}
\partial_t \phi^\flat + \partial_3 \psi_3^\flat = 0, \\
\partial_t \psi^\flat + \partial_3\left[\left(\frac{m_3\mathbf{m}}{\rho} - \frac{\tilde{m}_3\tilde{\mathbf{m}}}{\tilde{\rho}}\right)^\flat\right] + \partial_3\left[\left(p(\rho) - p(\tilde{\rho})\right)^\flat\right]\mathbf{e}_3 \\
\quad -\mu\partial_3^2\left[\left(\frac{\mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}\right)^\flat\right] - (\mu+\lambda)\partial_3^2\left[\left(\frac{m_3}{\rho} - \frac{\tilde{m}_3}{\tilde{\rho}}\right)^\flat\right]\mathbf{e}_3 = -\mathbf{g}^\flat,
\end{cases}
\tag{3.8}
$$

where we have used (2.19), and it follows from (3.4) that

$$
(\phi^\flat, \psi^\flat)(x_3, 0) = 0, \qquad x_3 \in \mathbb{R}.
\tag{3.9}
$$

In the linearization of (3.8), there appear some zero-order terms (e.g. $\partial_3 \mathbf{u}^{\mathsf{vs}}\psi_3^\flat$) whose coefficients decay at slow rates, which make it difficult to control their L^2-estimates. Nevertheless, such difficulties can be overcome by considering the integrated system of (3.8) instead of itself. Thanks to (2.27), it is plausible to write $(\phi^\flat, \psi^\flat) = \partial_3(\Phi, \boldsymbol{\Psi})$, where

$$
(\Phi, \boldsymbol{\Psi})(x_3, t) := \int_{-\infty}^{x_3} (\phi^\flat, \psi^\flat)(y_3, t)dy_3, \qquad x_3 \in \mathbb{R}, t \geqslant 0.
\tag{3.10}
$$

To make sense of the integrated system, we first make the following a priori assumptions that, for any fixed $T > 0$, assume that $(\phi, \zeta) \in \mathbb{B}(0, T)$ is a solution to (3.5), and the associated anti-derivative variable (3.10) satisfies that

$$
(\Phi, \boldsymbol{\Psi}) \in C(0, T; L^2(\mathbb{R})).
\tag{3.11}
$$

Anti-derivative argument. With the a priori assumption (3.11), one can integrate (3.8) and (3.9) with respect to x_3 from $-\infty$ to x_3 to get that

$$
\begin{cases}
\partial_t \Phi + \partial_3 \Psi_3 = 0, \\
\partial_t \boldsymbol{\Psi} + \mathbf{u}^{\mathsf{vs}}\partial_3\Psi_3 + p'(\bar{\rho})\partial_3\Phi\mathbf{e}_3 - \frac{\mu}{\bar{\rho}}\partial_3\left(\partial_3\boldsymbol{\Psi} - \mathbf{u}^{\mathsf{vs}}\partial_3\Phi\right) - \frac{\mu+\lambda}{\bar{\rho}}\partial_3^2\Psi_3\mathbf{e}_3 \\
\quad = -\mathbf{G} + \mathbf{N},
\end{cases}
\tag{3.12}
$$

and

$$
(\Phi, \boldsymbol{\Psi})(x_3, 0) = 0, \qquad x_3 \in \mathbb{R},
\tag{3.13}
$$

where $\mathbf{G}$ denotes the anti-derivative variable of $\mathbf{g}^\flat$, i.e.

$$
\mathbf{G}(x_3, t) := \int_{-\infty}^{x_3} \mathbf{g}^\flat(y_3, t)dy_3,
\tag{3.14}
$$

and $\mathbf{N}$ is given by

$$
\mathbf{N} = -\mathbf{Q}_1^\flat - \mathbf{Q}_2^\flat + \partial_3\left(\mathbf{Q}_3^\flat + \mathbf{Q}_4^\flat\right),
\tag{3.15}
$$

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with

$$\mathbf{Q}_1 := \frac{m_3\mathbf{m}}{\rho} - \frac{\tilde{m}_3\tilde{\mathbf{m}}}{\tilde{\rho}} - \frac{\mathbf{m}}{\tilde{\rho}}\psi_3 - \frac{\tilde{m}_3}{\tilde{\rho}}\psi + \frac{\tilde{m}_3\tilde{\mathbf{m}}}{\tilde{\rho}^2}\phi + \big(p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\phi\big)\mathbf{e}_3,$$

$$\mathbf{Q}_2 := (\tilde{\mathbf{u}} - \mathbf{u}^{\mathsf{vs}})\psi_3 + \tilde{u}_3\psi - \tilde{u}_3\tilde{\mathbf{u}}\phi + \big(p'(\tilde{\rho}) - p'(\bar{\rho})\big)\phi\mathbf{e}_3,$$

$$\mathbf{Q}_3 := \mu\Big(\frac{\mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} - \frac{1}{\tilde{\rho}}\psi + \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2}\phi\Big) + (\mu + \lambda)\Big(\frac{m_3}{\rho} - \frac{\tilde{m}_3}{\tilde{\rho}} - \frac{1}{\tilde{\rho}}\psi_3 + \frac{\tilde{m}_3}{\tilde{\rho}^2}\phi\Big)\mathbf{e}_3,$$

$$\mathbf{Q}_4 := \mu\Big[\Big(\frac{1}{\tilde{\rho}} - \frac{1}{\bar{\rho}}\Big)\psi - \Big(\frac{\tilde{\mathbf{u}}}{\tilde{\rho}} - \frac{\mathbf{u}^{\mathsf{vs}}}{\bar{\rho}}\Big)\phi\Big] + (\mu + \lambda)\Big[\Big(\frac{1}{\tilde{\rho}} - \frac{1}{\bar{\rho}}\Big)\psi_3 + \frac{\tilde{u}_3}{\tilde{\rho}}\phi\Big]\mathbf{e}_3. \tag{3.16}$$

One can prove that the errors of the ansatz, i.e. the source terms $f_0, \mathbf{g}$ and $\mathbf{G}$ in (3.5) and (3.12), decay exponentially fast in time; see Appendix B.

Lemma 3.2. *Suppose that the hypotheses of Theorem 1.2 hold true. Then there exist constants $\Lambda_0 \geqslant 1$ and $\varepsilon_0 > 0$, such that if $\Lambda \geqslant \Lambda_0$ and $\varepsilon \leqslant \varepsilon_0$, then the source terms $f_0, \mathbf{g}$ and $\mathbf{G}$ in (3.5) and (3.12) satisfy that*

$$\|\mathbf{G}\|_{H^3(\mathbb{R})} + \|f_0\|_{H^3(\Omega)} + \|\mathbf{g}\|_{H^2(\Omega)} \lesssim \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha t}, \qquad t \geqslant 0, \tag{3.17}$$

where Λ and α are the constants in (1.20) and (2.5), respectively.

Now we show the local existence theorem and the a priori estimates.

Theorem 3.3 (Local existence). *Suppose that the hypotheses of Theorem 1.2 hold true, and the initial data $(\phi_0, \psi_0)(x)$ is periodic in $x_\perp$, satisfying that*

$$(\phi_0, \psi_0) \in H^3(\Omega) \quad and \quad (\Phi_0, \boldsymbol{\Psi}_0) := \int_{-\infty}^{x_3}\int_{\mathbb{T}^2}(\phi_0, \psi_0)(x_\perp, y_3)dx_\perp dy_3 \in L^2(\mathbb{R}),$$

with $\|\phi_0, \psi_0\|_{H^3(\Omega)} + \|\Phi_0, \boldsymbol{\Psi}_0\|_{L^2(\mathbb{R})} \leqslant \nu_0$. Then there exist positive constants $\Lambda_0 \geqslant 1$ and ε_0 such that if $\Lambda \geqslant \Lambda_0$ and $\varepsilon\Lambda^{\frac{1}{4}} \leqslant \varepsilon_0$, then the problem (3.5) with the initial data

$$(\phi, \psi)(x, 0) = (\phi_0, \psi_0)(x), \tag{3.18}$$

admits a unique solution $(\phi, \zeta) \in \mathbb{B}(0, T_0)$ for some $T_0 > 0$, which depends on ν_0 and ε_0. Moreover, the anti-derivative variable,

$$(\Phi, \boldsymbol{\Psi})(x_3, t) := \int_{-\infty}^{x_3}\int_{\mathbb{T}^2}(\phi, \psi)(x_\perp, y_3, t)dx_\perp dy_3,$$

exists and belongs to $C(0, T_0; L^2(\mathbb{R}))$, and it holds that

$$\sup_{t\in(0,T_0)} \big(\|\Phi, \boldsymbol{\Psi}\|^2_{L^2(\mathbb{R})} + \|\phi, \zeta\|^2_{H^3(\Omega)}\big)$$

$$\leqslant C_0\big(\|\phi_0, \psi_0\|^2_{H^3(\Omega)} + \|\Phi_0, \boldsymbol{\Psi}_0\|^2_{L^2(\mathbb{R})} + \varepsilon^2\Lambda^{\frac{1}{2}}\big), \tag{3.19}$$

where C_0 is a positive constant, independent of Λ, ε and T_0.

The local existence of $(\phi, \zeta) \in \mathbb{B}(0, T_0)$ for some small $T_0 > 0$ is standard (see [24] for instance). The existence of the anti-derivative variable $(\Phi, \Psi) \in C(0, T_0; L^2(\mathbb{R}))$ can

be derived from a linear parabolic theory. Roughly speaking, given the local solution $(\phi, \zeta) \in \mathbb{B}(0, T_0)$ to (3.5), Ψ is the unique solution to the problem,

$$\begin{cases} \partial_t \boldsymbol{\Psi} - \frac{\mu}{\bar{\rho}} \partial_3^2 \boldsymbol{\Psi} - \frac{\mu+\lambda}{\bar{\rho}} \partial_3^2 \Psi_3 \mathbf{e}_3 = \mathbf{J}^\flat + \mathbf{G} \in C(0, T_0; H^2(\mathbb{R})), \qquad x_3 \in \mathbb{R}, t > 0, \\ \boldsymbol{\Psi}(x_3, 0) = \boldsymbol{\Psi}_0(x_3), \qquad x_3 \in \mathbb{R}. \end{cases} \quad (3.20)$$

where

$$\mathbf{J} = \mu\Big(\partial_3 \zeta - \frac{\partial_3 \psi}{\bar{\rho}}\Big) + (\mu + \lambda)\Big(\partial_3 \zeta_3 - \frac{\partial_3 \psi_3}{\bar{\rho}}\Big)\mathbf{e}_3 - \big(u_3 \mathbf{m} - \tilde{u}_3 \tilde{\mathbf{m}}\big) - \big(p(\rho) - p(\tilde{\rho})\big)\mathbf{e}_3.$$

The existence of Φ can be proved similarly. We refer to [43, Section 6] for the detailed argument.

Proposition 3.4 (A priori estimates)**.** *Suppose that the hypotheses of Theorem 1.2 hold true. For any $T > 0$, suppose that $(\phi, \zeta) \in C(0, T; H^3(\Omega))$ is a solution to the problem (3.5) and satisfies the a priori assumption (3.11). Then there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if*

$$\Lambda \geqslant \Lambda_0, \quad \varepsilon\Lambda^{\frac{1}{4}} \leqslant \varepsilon_0,$$

and

$$\nu := \sup_{t \in (0,T)} \big(\|\Phi, \Psi\|_{L^2(\mathbb{R})} + \|\phi, \psi\|_{H^3(\Omega)} \big) \leqslant \nu_0, \quad (3.21)$$

then it holds that

$$\sup_{t \in (0,T)} \big(\|\Phi, \boldsymbol{\Psi}\|_{L^2(\mathbb{R})}^2 + \|\phi, \zeta\|_{H^3(\Omega)}^2 \big) + \int_0^T \Big(\|\nabla\phi\|_{H^2(\Omega)}^2 + \|\nabla\zeta\|_{H^3(\Omega)}^2 \Big) dt \quad (3.22)$$
$$\lesssim \|(\Phi, \boldsymbol{\Psi})(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + \|(\phi, \psi)(\cdot, 0)\|_{H^3(\Omega)}^2 + \varepsilon\Lambda^{\frac{1}{4}}.$$

The proof of Proposition 3.4 will be given in Section 4.

At the end of this section, we present some lemmas to be used later.

Note that the perturbations ϕ, ψ and ζ in (3.1) and (3.2) are not always of zero averages with respect to x_1 or x_2. As stated in [18], these perturbations often do not satisfy the classical 3D Gagliardo-Nirenberg (G-N) inequalities, since the 1D and 2D cases cannot be excluded. To overcome this difficulty, we use the following G-N type inequality on the domain $\Omega = \mathbb{T}^2 \times \mathbb{R}$.

Lemma 3.5 ([18], Theorem 1.4 & Lemma 3.3)**.** *Assume that $u(x)$ is a bounded function that is periodic in $x_\perp = (x_1, x_2)$. Then there exists a decomposition $u(x) = \sum_{k=1}^{3} u^{(k)}(x)$ such that*

i)

$$u^{(1)} \equiv \int_{\mathbb{T}^2} u(x_\perp, x_3) dx_\perp = u^\flat(x_3),$$
$$u^{(2)} + u^{(3)} \equiv u(x) - \int_{\mathbb{T}^2} u(x_\perp, x_3) dx_\perp = u^\sharp(x); \quad (3.23)$$

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ii) if $\nabla^l u$ belongs to $L^p(\Omega)$ with an order $l \geqslant 0$ and $p \in [1, +\infty]$, then each $u^{(k)}$ satisfies that

$$\left\| \nabla^l u^{(k)} \right\|_{L^p(\Omega)} \lesssim \left\| \nabla^l u \right\|_{L^p(\Omega)}; \tag{3.24}$$

iii) each $u^{(k)}$ satisfies the classical k-dimensional Gagliardo-Nirenberg inequality, i.e.

$$\left\| \nabla^j u^{(k)} \right\|_{L^p(\Omega)} \lesssim \left\| \nabla^m u^{(k)} \right\|_{L^r(\Omega)}^{\theta_k} \left\| u^{(k)} \right\|_{L^q(\Omega)}^{1-\theta_k}, \tag{3.25}$$

where $0 \leqslant j < m$ is any integer, $1 \leqslant p \leqslant +\infty$ is any number and $\theta_k \in \left[\frac{j}{m}, 1 \right)$ satisfies

$$\frac{1}{p} = \frac{j}{k} + \left(\frac{1}{r} - \frac{m}{k} \right) \theta_k + \frac{1}{q} \left(1 - \theta_k \right).$$

Lemma 3.6. *Suppose that $u(x)$ belongs to $W^{1,p}(\Omega)$ with $p \in [1, +\infty]$. Then its non-zero mode $u^\sharp$ satisfies that*

$$\left\| u^\sharp \right\|_{L^p(\Omega)} \lesssim \left\| \nabla_{x_\perp} u^\sharp \right\|_{L^p(\Omega)} \lesssim \left\| \nabla u \right\|_{L^p(\Omega)}. \tag{3.26}$$

Proof. Note that $u^\sharp$ has zero average on the transverse torus $\mathbb{T}^2$. Then (3.26) follows directly from the Poincaré inequality. $\qquad\square$

At last, we present an inequality in [12], which is helpful in obtaining the optimal decay rate. For a fixed constant $\beta > 0$, set

$$\kappa(x_3, t) := \frac{1}{\sqrt{t + \Lambda}} \exp \left\{ - \frac{\beta x_3^2}{t + \Lambda} \right\}, \tag{3.27}$$

and

$$\mathcal{K}(x_3, t) = \int_{-\infty}^{x_3} \kappa(y_3, t) dy_3.$$

Lemma 3.7 ([12], Lemma 1)**.** *For $0 < T \leqslant +\infty$, suppose that $h = h(x_3, t)$ satisfies*

$$h \in L^2(0, T; H^1(\mathbb{R})) \quad and \quad \partial_t h \in L^2(0, T; H^{-1}(\mathbb{R})).$$

Then it holds that

$$\int_{\mathbb{R}} h^2 \kappa^2 dx_3 + 4\beta \frac{d}{dt} \left(\int_{\mathbb{R}} h^2 \mathcal{K}^2 dx_3 \right) \lesssim \| \partial_3 h(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \langle \partial_t h, h \mathcal{K}^2 \rangle_{H^{-1}(\mathbb{R}) \times H^1(\mathbb{R})}. \tag{3.28}$$

4   A priori estimates

To prove Proposition 3.4, we first establish the H^2-estimates of the anti-derivatives and the H^1-estimates of the non-zero modes of the perturbations in Sections 4.1 and 4.2, respectively. Afterwards, we go back to the original system (3.5) in Section 4.3 to deal with the higher-order derivatives to complete the proof.

Denote

$$\nu := \sup_{t \in (0,T)} \left\{ \| \Phi, \Psi \|_{L^2(\mathbb{R})} + \| \phi, \zeta \|_{H^3(\Omega)} \right\}. \tag{4.1}$$

It follows from Lemma 3.5 that for any $h \in H^2(\Omega)$,

$$\|h\|_{L^\infty(\Omega)} \lesssim \|\nabla h\|_{L^2(\Omega)}^{\frac{1}{2}} \|h\|_{L^2(\Omega)}^{\frac{1}{2}} + \left\|\nabla^2 h\right\|_{L^2(\Omega)}^{\frac{1}{2}} \|h\|_{L^2(\Omega)}^{\frac{1}{2}} + \left\|\nabla^2 h\right\|_{L^2(\Omega)}^{\frac{3}{4}} \|h\|_{L^2(\Omega)}^{\frac{1}{4}}. \tag{4.2}$$

Then one has that

$$\sup_{t \in (0,T)} \|\phi, \zeta\|_{W^{1,\infty}(\Omega)} \lesssim \sup_{t \in (0,T)} \|\phi, \zeta\|_{H^3(\Omega)} \lesssim \nu. \tag{4.3}$$

Similarly, the perturbation of momentum, $\psi = \rho\zeta + \phi\tilde{\mathbf{u}}$, satisfies that

$$\sup_{t \in (0,T)} \|\psi\|_{W^{1,\infty}(\Omega)} \lesssim \sup_{t \in (0,T)} \|\psi\|_{H^3(\Omega)} \lesssim \sup_{t \in (0,T)} \|\phi, \zeta\|_{H^3(\Omega)} \lesssim \nu.$$

It follows from (2.29), (2.30) and (4.3) that if the positive constants Λ^{-1}, ε and ν are small, then

$$\bar{\rho}/4 \leqslant \inf_{\substack{x \in \Omega \\ t \in (0,T)}} \rho(x,t) \leqslant \sup_{\substack{x \in \Omega \\ t \in (0,T)}} \rho(x,t) \leqslant 4\bar{\rho},$$

and

$$\sup_{t \in (0,T)} \|\nabla \rho(x,t), \nabla \mathbf{u}(x,t)\|_{L^\infty(\Omega)} \lesssim \Lambda^{-1/2} + \varepsilon + \nu. \tag{4.4}$$

Besides, it follows from the Sobolev inequality and Lemma 3.5 that

$$\sup_{t \in (0,T)} \left\|\phi^\flat, \psi^\flat\right\|_{W^{2,\infty}(\mathbb{R})} \lesssim \sup_{t \in (0,T)} \left\|\phi^\flat, \psi^\flat\right\|_{H^3(\mathbb{R})} \lesssim \sup_{t \in (0,T)} \|\phi, \psi\|_{H^3(\Omega)} \lesssim \nu, \tag{4.5}$$

and

$$\sup_{t \in (0,T)} \left\|\phi^\sharp, \zeta^\sharp\right\|_{W^{1,\infty}(\Omega)} \lesssim \sup_{t \in (0,T)} \left\|\phi^\sharp, \zeta^\sharp\right\|_{H^3(\Omega)} \lesssim \sup_{t \in (0,T)} \|\phi, \zeta\|_{H^3(\Omega)} \lesssim \nu. \tag{4.6}$$

Then it holds that

$$\begin{aligned}
\sup_{t \in (0,T)} \|\Phi, \mathbf{\Psi}\|_{W^{3,\infty}(\mathbb{R})} &\lesssim \sup_{t \in (0,T)} \|\Phi, \mathbf{\Psi}\|_{H^4(\mathbb{R})} \\
&\lesssim \sup_{t \in (0,T)} \left(\|\Phi, \mathbf{\Psi}\|_{L^2(\mathbb{R})} + \left\|\phi^\flat, \psi^\flat\right\|_{H^3(\mathbb{R})} \right) \\
&\lesssim \nu.
\end{aligned} \tag{4.7}$$

Moreover, using Lemma 1.1, one can choose the positive constant β in (3.27) to be suitably small (e.g. $\beta = \frac{\bar{\rho}}{32\mu}$) such that

$$|\partial_3 \theta|^2 \lesssim \Lambda^{-\frac{1}{2}} \kappa, \quad \left|\partial_3^2 \theta\right| \lesssim \kappa^2, \quad \left|\partial_3^3 \theta\right| \lesssim \Lambda^{-\frac{1}{2}} \kappa^2, \tag{4.8}$$

and for $j = 1, 2, \cdots$,

$$\left|\partial_3^j \theta\right| \lesssim (t + \Lambda)^{-\frac{j-1}{2}} \kappa. \tag{4.9}$$

To carry out the L^2-estimate for (3.12), we may encounter a difficulty that the linear terms such as $\mathbf{u}^{\mathsf{vs}} \partial_3 \Psi_3$ and $\frac{\mu}{\rho} \mathbf{u}^{\mathsf{vs}} \partial_3^2 \Phi$ cannot get controlled due to the large amplitude of the vortex sheet. Nevertheless, we can overcome this difficulty by introducing a new variable.

New variable. Set

$$\mathbf{Z} = (Z_1, Z_2, Z_3)(x_3, t) := \mathbf{\Psi} - \mathbf{u}^{\mathsf{vs}} \Phi. \tag{4.10}$$

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It holds that

$$\sup_{t\in(0,T)}\|\mathbf{Z}\|_{W^{3,\infty}(\mathbb{R})} \lesssim \sup_{t\in(0,T)}\|\mathbf{Z}\|_{H^4(\mathbb{R})} \lesssim \sup_{t\in(0,T)}\|\Phi,\boldsymbol{\Psi}\|_{H^4(\mathbb{R})} \lesssim \nu. \tag{4.11}$$

Note that $\mathbf{u}^{\mathsf{vs}} = \bar{\mathbf{u}}\theta$ and $\partial_t\theta = \frac{\mu}{\rho}\partial_3^2\theta$. Then it follows from (3.12) that

$$\begin{cases} \partial_t\Phi + \partial_3 Z_3 = 0, \\ \partial_t\mathbf{Z} + \big(p'(\bar{\rho})\mathbf{e}_3 - \frac{\mu}{\bar{\rho}}\partial_3\theta\bar{\mathbf{u}}\big)\partial_3\Phi - \frac{\mu}{\bar{\rho}}\partial_3^2\mathbf{Z} - \frac{\mu+\lambda}{\bar{\rho}}\partial_3^2 Z_3\mathbf{e}_3 = -\mathbf{G} + \mathbf{N}. \end{cases} \tag{4.12}$$

With the aid of (4.10), we can successfully get rid of the bad linear terms in $(3.12)_2$; and the linear term, $-\frac{\mu}{\bar{\rho}}\partial_3\theta\bar{\mathbf{u}}\partial_3\Phi$ in $(4.12)_2$, has smallness due to the largeness of the constant Λ in (1.20). Moreover, the introduction of (4.10) plays an important role in achieving the optimal decay rate of the zero mode, $(\phi^\flat, \zeta^\flat)$.

Recall the notation "$\perp$" in (1.31) and the fact that $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_\perp, 0)$. For a later use, we also decompose $(4.12)_2$ into

$$\partial_t\mathbf{Z}_\perp - \frac{\mu}{\bar{\rho}}\partial_3\theta\partial_3\Phi\bar{\mathbf{u}}_\perp - \frac{\mu}{\bar{\rho}}\partial_3^2\mathbf{Z}_\perp = -\mathbf{G}_\perp + \mathbf{N}_\perp, \tag{4.13}$$

and

$$\partial_t Z_3 + p'(\bar{\rho})\partial_3\Phi - \frac{\tilde{\mu}}{\bar{\rho}}\partial_3^2 Z_3 = -G_3 + N_3 \qquad \text{with } \tilde{\mu} = 2\mu + \lambda. \tag{4.14}$$

Then we give two lemmas that will be needed in the energy estimates later. The readers may skip them at first reading.

Lemma 4.1. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then the remainders (3.16) satisfy that*

$$\begin{aligned} |\mathbf{Q}_1| + |\mathbf{Q}_3| &\lesssim |\phi|^2 + |\psi|^2, \\ |\partial_3\mathbf{Q}_1| + |\partial_3\mathbf{Q}_3| &\lesssim \big[\varepsilon e^{-\alpha t} + (t+\Lambda)^{-\frac{1}{2}}\big]\big(|\phi|^2 + |\psi|^2\big) \\ &\quad + \big(|\partial_3\phi| + |\partial_3\psi|\big)\big(|\phi| + |\psi|\big), \\ \big|\partial_3^k\mathbf{Q}_2\big| + \big|\partial_3^k\mathbf{Q}_4\big| &\lesssim \sum_{j=0}^k \varepsilon e^{-\alpha t}\big(\big|\partial_3^j\phi\big| + \big|\partial_3^j\psi\big|\big), \qquad k = 0, 1. \end{aligned} \tag{4.15}$$

Proof. The estimates $(4.15)_1$ and $(4.15)_3$ follows from (2.31), (2.32) and (3.16) directly. It suffices to estimate $\partial_3\mathbf{Q}_1$ and $\partial_3\mathbf{Q}_3$.

By (3.16), direct calculations yield that

$$\begin{aligned} \partial_3\mathbf{Q}_1 &= \partial_3\tilde{m}_3\Big(\frac{\mathbf{m}}{\rho} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} - \frac{1}{\tilde{\rho}}\psi + \frac{\tilde{m}}{\tilde{\rho}^2}\phi\Big) + \partial_3\tilde{\mathbf{m}}\Big(\frac{m_3}{\rho} - \frac{\tilde{m}_3}{\tilde{\rho}} - \frac{\psi_3}{\tilde{\rho}} + \frac{\tilde{m}_3}{\tilde{\rho}^2}\phi\Big) \\ &\quad - \partial_3\tilde{\rho}\Big(\frac{m_3\mathbf{m}}{\rho^2} - \frac{\tilde{m}_3\tilde{\mathbf{m}}}{\tilde{\rho}^2} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2}\psi_3 - \frac{\tilde{m}_3}{\tilde{\rho}^2}\psi + \frac{2\tilde{m}_3\tilde{\mathbf{m}}}{\tilde{\rho}^3}\phi\Big) \\ &\quad + \partial_3\psi_3\zeta + \zeta\partial_3\psi - \partial_3\phi(u_3\mathbf{u} - \tilde{u}_3\tilde{\mathbf{u}}) \\ &\quad + \big(p'(\rho) - p'(\tilde{\rho})\big)\partial_3\phi\mathbf{e}_3 + \big(p'(\rho) - p'(\tilde{\rho}) - p''(\tilde{\rho})\phi\big)\partial_3\tilde{\rho}\mathbf{e}_3. \end{aligned}$$

Using (2.31) and (2.32), one can get that

$$|\partial_3\mathbf{Q}_1| \lesssim \varepsilon e^{-\alpha t}\big(|\phi|^2 + |\psi|^2\big) + (t+\Lambda)^{-\frac{1}{2}}\big(|\phi|^2 + |\psi_3|^2\big)$$

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$$+ |\zeta| \, |\partial_3 \psi| + |\partial_3 \phi| \, |\zeta| + |\phi| \, |\partial_3 \phi| \, ,$$

which yields the estimate of $\partial_3 \mathbf{Q}_1$ in (4.15). The estimate of $\partial_3 \mathbf{Q}_3$ in (4.15) can be proved similarly. $\qquad\square$

Lemma 4.2. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\pm \left\| \zeta^\flat \right\|_{L^2(\Omega)} \lesssim \pm \left\| \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})} + \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})} + \nu \left\| \nabla \phi^\sharp \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}, \tag{4.16}$$

$$\pm \left\| \partial_3 \zeta^\flat \right\|_{L^2(\Omega)} \lesssim \pm \left\| \partial_3^2 \mathbf{Z} \right\|_{L^2(\mathbb{R})} + \left\| \partial_3^2 \Phi \right\|_{L^2(\mathbb{R})} + (t + \Lambda)^{-\frac{1}{2}} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})}$$
$$+ \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}, \tag{4.17}$$

$$\pm \left\| \partial_3^2 \zeta^\flat \right\|_{L^2(\mathbb{R})} \lesssim \pm \left\| \partial_3^3 \mathbf{Z} \right\|_{L^2(\mathbb{R})} + (t + \Lambda)^{-\frac{1}{2}} \left\| \partial_3^2 \Phi \right\|_{L^2(\mathbb{R})}$$
$$+ (t + \Lambda)^{-1} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})} + \nu \left\| \nabla^2 \phi, \nabla^2 \zeta^\sharp \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}, \tag{4.18}$$

and

$$\pm \left\| \nabla \psi^\sharp \right\|_{L^2(\Omega)} \lesssim \pm \left\| \nabla \zeta^\sharp \right\|_{L^2(\Omega)} + \left\| \nabla \phi^\sharp \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}, \tag{4.19}$$

$$\pm \left\| \nabla^3 \psi \right\|_{L^2(\Omega)} \lesssim \pm \left\| \nabla^3 \zeta \right\|_{L^2(\Omega)} + \left\| \nabla^3 \phi \right\|_{L^2(\Omega)} + \sum_{j=1}^{2} (t + \Lambda)^{\frac{j-4}{2}} \left\| \partial_3^j \Phi \right\|_{L^2(\mathbb{R})}$$
$$+ (\Lambda^{-\frac{1}{2}} + \nu) \left\| \nabla^2 \phi \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}. \tag{4.20}$$

The proof of Lemma 4.2 is given in Appendix C.

For convenience, in the following energy estimates, we write "$\partial_3(\cdots)$" and "$\mathrm{div}(\cdots)$" to denote the terms which vanish after integration with respect to $x_3 \in \mathbb{R}$ and $x \in \Omega$, respectively. Besides, we use $c_{i,\flat}, c_{i,\neq}$ and c_i for $i = 0, 1, 2, \cdots$ to denote some positive generic constants, which are independent of $x, t, \varepsilon, \Lambda$ and ν.

4.1 Estimates of zero-modes.

In this section, we establish the $H^2(\mathbb{R})$-estimates of the anti-derivatives, which is the key part in the a priori estimates.

Lemma 4.3. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt} \mathcal{A}_{0,\flat}^{(1)} + \frac{\mu}{\bar{\rho}} \left\| \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 \lesssim (\Lambda^{-\frac{1}{2}} + \nu) \left\| \partial_3 \Phi, \Phi \kappa, \mathbf{Z} \kappa \right\|_{L^2(\mathbb{R})}^2$$
$$+ \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}. \tag{4.21}$$

where

$$\mathcal{A}_{0,\flat}^{(1)} := p'(\bar{\rho}) \left\| \Phi \right\|_{L^2(\mathbb{R})}^2 + 2 \left\| \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 - \frac{2\mu}{\bar{\rho} p'(\bar{\rho})} \int_{\mathbb{R}} \partial_3 \theta \bar{\mathbf{u}}_\perp \cdot \mathbf{Z}_\perp Z_3 dx_3. \tag{4.22}$$

Proof. Multiplying $p'(\bar{\rho})\Phi$ and $\mathbf{Z}$ on $(4.12)_1$ and $(4.12)_2$, respectively, and adding the resulting two equations together, one has that

$$\partial_t\Big[\frac{1}{2}\big(p'(\bar{\rho})\Phi^2 + |\mathbf{Z}|^2\big)\Big] + \frac{\mu}{\bar{\rho}}\,|\partial_3\mathbf{Z}|^2 + \frac{\mu+\lambda}{\bar{\rho}}\,|\partial_3 Z_3|^2$$
$$= \partial_3(\cdots) + \underbrace{(-\mathbf{G}+\mathbf{N})\cdot\mathbf{Z}}_{I_1}\underbrace{-\frac{\mu}{\bar{\rho}}\partial_3\theta\partial_3\Phi\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp}_{I_2}. \tag{4.23}$$

It follows from Lemmas 3.2 and 4.1 that

$$\int_{\mathbb{R}} |\mathbf{G}|\,|\mathbf{Z}|\,dx_3 \lesssim \|\mathbf{G}\|_{L^2(\mathbb{R})}\,\|\mathbf{Z}\|_{L^2(\mathbb{R})} \lesssim \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}, \tag{4.24}$$

and

$$\left|\int_{\mathbb{R}} \mathbf{N}\cdot\mathbf{Z}dx_3\right| \lesssim \left|\int_{\mathbb{R}}(\mathbf{Q}_1^\flat + \mathbf{Q}_2^\flat)\cdot\mathbf{Z}dx_3\right| + \left|\int_{\mathbb{R}}(\mathbf{Q}_3^\flat + \mathbf{Q}_4^\flat)\cdot\partial_3\mathbf{Z}dx_3\right|$$
$$\lesssim \|\mathbf{Z}\|_{H^1(\mathbb{R})}\,\|\mathbf{Q}_2,\mathbf{Q}_4\|_{L^2(\Omega)} + \|\mathbf{Z}\|_{W^{1,\infty}(\mathbb{R})}\,\|\mathbf{Q}_1,\mathbf{Q}_3\|_{L^1(\Omega)} \tag{4.25}$$
$$\lesssim \nu\,\|\phi,\psi\|_{L^2(\Omega)}^2 + \varepsilon\nu^2 e^{-\alpha t}.$$

Note that $\phi^\flat = \partial_3\Phi$ and $\psi^\flat = \partial_3\boldsymbol{\Psi}$. Then it follows from (3.26) that

$$\|\phi\|_{L^2(\Omega)}^2 = \big\|\phi^\flat\big\|_{L^2(\mathbb{R})}^2 + \big\|\phi^\sharp\big\|_{L^2(\Omega)}^2 \lesssim \|\partial_3\Phi\|_{L^2(\mathbb{R})}^2 + \big\|\nabla\phi^\sharp\big\|_{L^2(\Omega)}^2, \tag{4.26}$$
$$\|\psi\|_{L^2(\Omega)}^2 \lesssim \|\partial_3\boldsymbol{\Psi}\|_{L^2(\mathbb{R})}^2 + \big\|\nabla\psi^\sharp\big\|_{L^2(\Omega)}^2. \tag{4.27}$$

By (4.10), one can verify easily that

$$\|\partial_3\boldsymbol{\Psi}\|_{L^2(\mathbb{R})}^2 \lesssim \|\partial_3\mathbf{Z},\partial_3\Phi,\Phi\kappa\|_{L^2(\mathbb{R})}^2. \tag{4.28}$$

Then it follows from (4.26) to (4.28) that

$$\|\phi,\psi\|_{L^2(\Omega)}^2 \lesssim \|\partial_3\Phi,\partial_3\mathbf{Z},\Phi\kappa\|_{L^2(\mathbb{R})}^2 + \big\|\nabla\phi^\sharp,\nabla\psi^\sharp\big\|_{L^2(\Omega)}^2. \tag{4.29}$$

This, together with (4.19), yields that

$$\int_{\mathbb{R}} I_1 dx_3 \lesssim \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t} + \nu\,\|\partial_3\Phi,\partial_3\mathbf{Z},\Phi\kappa\|_{L^2(\mathbb{R})}^2 + \nu\,\big\|\nabla\phi^\sharp,\nabla\zeta^\sharp\big\|_{L^2(\Omega)}^2. \tag{4.30}$$

Now we estimate I_2. Using (4.14), one can get that

$$\frac{\bar{\rho}p'(\bar{\rho})}{\mu}I_2 = \partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp\big(G_3 - N_3\big) + \partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp\partial_t Z_3 - \frac{\tilde{\mu}}{\bar{\rho}}\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp\partial_3^2 Z_3$$
$$:= I_{2,1} + I_{2,2} + I_{2,3}.$$

First, similar to the estimate of I_1, one can prove that

$$\int_{\mathbb{R}} I_{2,1}dx_3 \lesssim \varepsilon\nu e^{-\alpha t} + \nu\,\|\partial_3\Phi,\partial_3\mathbf{Z},\Phi\kappa\|_{L^2(\mathbb{R})}^2 + \nu\,\big\|\nabla\phi^\sharp,\nabla\zeta^\sharp\big\|_{L^2(\Omega)}^2. \tag{4.31}$$

For $I_{2,2}$, it holds that

$$I_{2,2} = \partial_t\big(\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp Z_3\big) - \partial_3\partial_t\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp Z_3\underbrace{-\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\partial_t\mathbf{Z}_\perp Z_3}_{I_{2,4}}. \tag{4.32}$$

Using (4.13), one has that

$$
\begin{aligned}
I_{2,4} = {} & \partial_3\theta Z_3\bar{\mathbf{u}}_\perp \cdot (\mathbf{G}_\perp - \mathbf{N}_\perp) - \frac{\mu}{\bar\rho}|\bar{\mathbf{u}}|^2|\partial_3\theta|^2 Z_3\partial_3\Phi - \partial_3\Big(\frac{\mu}{\bar\rho}\partial_3\theta Z_3\bar{\mathbf{u}}_\perp\cdot\partial_3\mathbf{Z}_\perp\Big) \\
& + \frac{\mu}{\bar\rho}\partial_3\theta\partial_3 Z_3\bar{\mathbf{u}}_\perp\cdot\partial_3\mathbf{Z}_\perp + \frac{\mu}{\bar\rho}\partial_3^2\theta Z_3\bar{\mathbf{u}}_\perp\cdot\partial_3\mathbf{Z}_\perp.
\end{aligned}
\tag{4.33}
$$

Collecting (4.32) and (4.33), one can use (4.8) and a similar proof of (4.30) to get that

$$
\begin{aligned}
\int_{\mathbb{R}} I_{2,2}dx_3 \leqslant {} & \frac{d}{dt}\Big(\int_{\mathbb{R}}\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp Z_3 dx_3\Big) + C\big(\Lambda^{-\frac12}+\nu\big)\big\|\Phi\kappa,\mathbf{Z}\kappa,\partial_3\Phi,\partial_3\mathbf{Z}\big\|_{L^2(\mathbb{R})}^2 \\
& + C\varepsilon\nu e^{-\alpha t} + C\nu\big\|\nabla\phi^\sharp,\nabla\zeta^\sharp\big\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{4.34}
$$

Moreover, it holds that

$$
I_{2,3} = \partial_3\Big(-\frac{\tilde\mu}{\bar\rho}\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp\partial_3 Z_3\Big) + \frac{\tilde\mu}{\bar\rho}\partial_3^2\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp\partial_3 Z_3 + \frac{\tilde\mu}{\bar\rho}\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\partial_3\mathbf{Z}_\perp\partial_3 Z_3,
$$

which, together with (4.8), yields that

$$
\int_{\mathbb{R}} I_{2,3}dx_3 \lesssim \Lambda^{-\frac12}\big\|\mathbf{Z}_\perp\kappa,\partial_3\mathbf{Z}\big\|_{L^2(\mathbb{R})}^2.
\tag{4.35}
$$

Then collecting (4.30), (4.31), (4.34) and (4.35), one can obtain (4.21) if $\Lambda^{-1}>0$ and $\nu>0$ are suitably small.

$\square$

Let $\mathfrak{K}(x_3,t) := \int_{-\infty}^{x_3}\kappa^2(y,t)dy$, which satisfies that

$$
\big\|\mathfrak{K}\big\|_{L^\infty(\mathbb{R})} \lesssim (t+\Lambda)^{-\frac12} \quad\text{and}\quad \big\|\partial_t\mathfrak{K}\big\|_{L^\infty(\mathbb{R})} \lesssim (t+\Lambda)^{-\frac32}, \qquad t\geqslant 0.
\tag{4.36}
$$

Lemma 4.4. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0\geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda\geqslant\Lambda_0$, $\varepsilon\leqslant\varepsilon_0$ and $\nu\leqslant\nu_0$, then*

$$
\begin{aligned}
\big\|\Phi\kappa,\mathbf{Z}\kappa\big\|_{L^2(\mathbb{R})}^2 \lesssim {} & \frac{d}{dt}\mathcal{A}_{0,\flat}^{(2)} + \big\|\partial_3\mathbf{Z}\big\|_{L^2(\mathbb{R})}^2 + \big(\Lambda^{-\frac12}+\nu\big)\big\|\partial_3\Phi\big\|_{L^2(\mathbb{R})}^2 \\
& + \nu\big\|\nabla\phi^\sharp,\nabla\zeta^\sharp\big\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac14}e^{-\alpha t},
\end{aligned}
\tag{4.37}
$$

where $\mathcal{A}_{0,\flat}^{(2)}$ denotes a linear combination of the following terms,

$$
\int_{\mathbb{R}} Z_3\Phi\mathfrak{K}dx_3, \quad \int_{\mathbb{R}}|\mathbf{Z}_\perp|^2\mathcal{K}^2 dx_3, \quad \int_{\mathbb{R}}\partial_3\theta\bar{\mathbf{u}}_\perp\cdot\mathbf{Z}_\perp Z_3\mathcal{K}^2 dx_3.
\tag{4.38}
$$

Proof. Multiplying $\Phi\mathfrak{K}$ on (4.14) and integrating the resulting equation on $\mathbb{R}$, one has that

$$
\begin{aligned}
& \int_{\mathbb{R}}\frac12\big(p'(\bar\rho)\Phi^2 + Z_3^2\big)\kappa^2 dx_3 - \frac{d}{dt}\Big(\int_{\mathbb{R}}Z_3\Phi\mathfrak{K}dx_3\Big) \\
& = -\int_{\mathbb{R}}Z_3\Phi\partial_t\mathfrak{K}dx_3 + \frac{\tilde\mu}{\bar\rho}\int_{\mathbb{R}}\partial_3 Z_3\big(\partial_3\Phi\mathfrak{K}+\Phi\kappa^2\big)dx_3 + \int_{\mathbb{R}}(-G_3+N_3)\Phi\mathfrak{K}dx_3,
\end{aligned}
\tag{4.39}
$$

here we have used the fact that $Z_3 \partial_t \Phi \mathfrak{K} = -Z_3 \partial_3 Z_3 \mathfrak{K} = -\partial_3 \left(\frac{1}{2} Z_3^2 \mathfrak{K} \right) + \frac{1}{2} Z_3^2 \kappa^2$. Similar to the proof of (4.30), one can show that

$$\int_{\mathbb{R}} (-G_3 + N_3) \Phi \mathfrak{K} dx_3 \lesssim \varepsilon \nu e^{-\alpha t} + \nu \left\| \partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi \kappa \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla \phi^\sharp, \nabla \psi^\sharp \right\|_{L^2(\Omega)}^2. \qquad (4.40)$$

Thus, by (4.36) and (4.40), integrating (4.39) over $(0, T)$ yields that

$$\begin{aligned}
\left\| \Phi \kappa, Z_3 \kappa \right\|_{L^2(\mathbb{R})}^2 \lesssim{} & \frac{d}{dt} \left(\int_{\mathbb{R}} Z_3 \Phi \mathfrak{K} dx_3 \right) + (t + \Lambda)^{-\frac{3}{2}} \left\| \Phi, Z_3 \right\|_{L^2(\mathbb{R})}^2 \\
& + (\Lambda^{-\frac{1}{2}} + \nu) \left\| \partial_3 \Phi, \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu e^{-\alpha t}.
\end{aligned} \qquad (4.41)$$

To estimate $\int_0^T \left\| \mathbf{Z}_\perp \kappa \right\|_{L^2(\mathbb{R})}^2 dt$, it firstly follows from Lemma 3.7 that

$$\left\| \mathbf{Z}_\perp \kappa \right\|_{L^2(\mathbb{R})}^2 + \frac{d}{dt} \left(\frac{\bar{\rho}}{8\mu} \int_{\mathbb{R}} |\mathbf{Z}_\perp|^2 \mathcal{K}^2 dx_3 \right) \lesssim \left\| \partial_3 \mathbf{Z}_\perp \right\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \partial_t \mathbf{Z}_\perp \cdot \mathbf{Z}_\perp \mathcal{K}^2 \, dx_3. \qquad (4.42)$$

Here we have used that $h = |\mathbf{Z}_\perp|^2$ and $\beta = \frac{\bar{\rho}}{32\mu}$ in (3.28). Using (4.13), one can get that

$$\begin{aligned}
\int_{\mathbb{R}} & \partial_t \mathbf{Z}_\perp \cdot \mathbf{Z}_\perp \mathcal{K}^2 \, dx_3 \\
&= -\frac{\mu}{\bar{\rho}} \int_{\mathbb{R}} |\partial_3 \mathbf{Z}_\perp|^2 \mathcal{K}^2 dx_3 - \frac{2\mu}{\bar{\rho}} \int_{\mathbb{R}} \partial_3 \mathbf{Z}_\perp \cdot \mathbf{Z}_\perp \mathcal{K} \kappa dx_3 \\
&\quad + \underbrace{\frac{\mu}{\bar{\rho}} \int_{\mathbb{R}} \partial_3 \theta \partial_3 \Phi \bar{\mathbf{u}}_\perp \cdot \mathbf{Z}_\perp \mathcal{K}^2 dx_3}_{I_3} + \int_{\mathbb{R}} \left(-\mathbf{G}_\perp + \mathbf{N}_\perp \right) \cdot \mathbf{Z}_\perp \mathcal{K}^2 \, dx_3.
\end{aligned}$$

Similar to the estimate of I_2 in (4.23), one can use (4.8) and (4.14) to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} I_3 dx_3 \lesssim{} & \frac{d}{dt} \left(\int_{\mathbb{R}} \partial_3 \theta \bar{\mathbf{u}}_\perp \cdot \mathbf{Z}_\perp Z_3 \mathcal{K}^2 dx_3 \right) + \left(\Lambda^{-\frac{1}{2}} + \nu \right) \left\| \Phi \kappa, \mathbf{Z} \kappa, \partial_3 \Phi, \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 \\
& + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu e^{-\alpha t}.
\end{aligned}$$

This, together with (4.9), (4.42) and a similar proof of (4.30), yields that

$$\begin{aligned}
\left\| \mathbf{Z}_\perp \kappa \right\|_{L^2(\mathbb{R})}^2 & + \frac{d}{dt} \left(\frac{\bar{\rho}}{8\mu} \int_{\mathbb{R}} |\mathbf{Z}_\perp|^2 \mathcal{K}^2 dx_3 \right) \\
& \lesssim \left\| \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + \frac{d}{dt} \left(\int_{\mathbb{R}} \partial_3 \theta \bar{\mathbf{u}}_\perp \cdot \mathbf{Z}_\perp Z_3 \mathcal{K}^2 dx_3 \right) \\
& \quad + \left(\Lambda^{-\frac{1}{2}} + \nu \right) \left\| \partial_3 \Phi, \Phi \kappa, Z_3 \kappa \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}.
\end{aligned} \qquad (4.43)$$

Collecting (4.21), (4.41) and (4.43), one can finish the proof.

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

Lemma 4.5. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then it holds that*

$$\begin{aligned}
\frac{d}{dt} \mathcal{A}_{0,\flat}^{(3)} & + p'(\bar{\rho}) \left\| \partial_3 \Phi \right\|_{L^2(\mathbb{R})}^2 \\
& \lesssim \left\| \partial_3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \Phi \kappa \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t},
\end{aligned} \qquad (4.44)$$

where

$$\mathcal{A}_{0,\flat}^{(3)} := \frac{\tilde{\mu}}{\bar{\rho}} \|\partial_3 \Phi\|_{L^2(\mathbb{R})}^2 + 2 \int_{\mathbb{R}} Z_3 \partial_3 \Phi dx_3. \tag{4.45}$$

Proof. Multiplying $\frac{\tilde{\mu}}{\bar{\rho}} \partial_3 \Phi$ and $\partial_3 \Phi$ on $\partial_3 (4.12)_1$ and (4.14), respectively, and adding the resulting two equations together, one has that

$$\partial_t \left(\frac{\tilde{\mu}}{2\bar{\rho}} |\partial_3 \Phi|^2 + Z_3 \partial_3 \Phi \right) + p'(\bar{\rho}) |\partial_3 \Phi|^2 = \partial_3(\cdots) + |\partial_3 Z_3|^2 + (-G_3 + N_3) \partial_3 \Phi. \tag{4.46}$$

Similar to the proof of (4.30), one can prove that

$$\int_{\mathbb{R}} (-G_3 + N_3) \partial_3 \Phi dx_3 \lesssim \nu \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi \kappa\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}.$$

This, together with , can yield (4.50). $\qquad\qquad\square$

Collecting Lemmas 4.3–4.5, one has that

$$\frac{d}{dt} \mathcal{A}_{0,\flat} + c_{0,\flat} \mathcal{B}_{0,\flat} \lesssim \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}, \tag{4.47}$$

where $\mathcal{A}_{0,\flat}$ denotes a linear combination of $\mathcal{A}_{0,\flat}^{(1)}$, $\mathcal{A}_{0,\flat}^{(2)}$ and $\mathcal{A}_{0,\flat}^{(3)}$, satisfying that

$$\mathcal{A}_{0,\flat} \sim \|\Phi\|_{H^1(\mathbb{R})}^2 + \|\mathbf{Z}\|_{L^2(\mathbb{R})}^2 \, ; \tag{4.48}$$

and

$$\mathcal{B}_{0,\flat} := \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi \kappa, \mathbf{Z} \kappa\|_{L^2(\mathbb{R})}^2 . \tag{4.49}$$

Then integrating (4.47) with respect to t yields that

$$\begin{aligned}
\sup_{t \in (0,T)} & \left(\|\Phi\|_{H^1(\mathbb{R})}^2 + \|\mathbf{Z}\|_{L^2(\mathbb{R})}^2 \right) + \int_0^T \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi \kappa, \mathbf{Z} \kappa\|_{L^2(\mathbb{R})}^2 \, dt \\
& \lesssim \|\Phi_0\|_{H^1(\mathbb{R})}^2 + \|\mathbf{\Psi}_0\|_{L^2(\mathbb{R})}^2 + \nu \int_0^T \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 \, dt + \varepsilon \nu \Lambda^{\frac{1}{4}}.
\end{aligned} \tag{4.50}$$

Lemma 4.6. *Under the assumptions of Proposition 3.2, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0, \varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then it holds that*

$$\begin{aligned}
\frac{d}{dt} \mathcal{A}_{1,\flat} + c_{1,\flat} \mathcal{B}_{1,\flat} \lesssim & (t + \Lambda)^{-1} \|\partial_3 \Phi, \Phi \kappa\|_{L^2(\mathbb{R})}^2 + \nu \left\| \partial_3^2 \Phi, \partial_3^3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 \\
& + \nu \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}.
\end{aligned} \tag{4.51}$$

where

$$\mathcal{A}_{1,\flat} = p'(\bar{\rho}) \|\partial_3 \Phi\|_{L^2(\mathbb{R})}^2 + \|\partial_3 \mathbf{Z}\|_{L^2(\mathbb{R})}^2 \quad and \quad \mathcal{B}_{1,\flat} = \left\| \partial_3^2 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 . \tag{4.52}$$

Proof. Multiplying $-p'(\bar{\rho}) \partial_3^2 \Phi$ and $-\partial_3^2 \mathbf{Z}$ on $(4.12)_1$ and $(4.12)_2$, respectively and adding up the resulting two equations, one can get that

$$\begin{aligned}
\frac{d}{dt} & \left(\frac{p'(\bar{\rho})}{2} \|\partial_3 \Phi\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_3 \mathbf{Z}\|_{L^2(\mathbb{R})}^2 \right) + \frac{\mu}{\bar{\rho}} \left\| \partial_3^2 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + \frac{\mu + \lambda}{\bar{\rho}} \left\| \partial_3^2 Z_3 \right\|_{L^2(\mathbb{R})}^2 \\
& = \underbrace{-\frac{\mu}{\bar{\rho}} \int_{\mathbb{R}} \partial_3 \theta \partial_3 \Phi \bar{\mathbf{u}} \cdot \partial_3^2 \mathbf{Z} dx_3}_{I_4} + \underbrace{\int_{\mathbb{R}} (\mathbf{G} - \mathbf{N}) \cdot \partial_3^2 \mathbf{Z} dx_3}_{I_5} .
\end{aligned} \tag{4.53}$$

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One can show that

$$|I_4| \leqslant C(t+\Lambda)^{-1} \left\|\partial_3\Phi\right\|_{L^2(\mathbb{R})}^2 + \frac{\mu}{4\bar{\rho}} \left\|\partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2. \qquad (4.54)$$

It follows from Lemmas 3.2 and 4.1 that

$$\begin{aligned}
I_5 &= \int_{\mathbb{R}} \left(\mathbf{G} + \mathbf{Q}_1^\flat + \mathbf{Q}_2^\flat\right) \cdot \partial_3^2\mathbf{Z} dx_3 + \int_{\mathbb{R}} \left(\mathbf{Q}_3^\flat + \mathbf{Q}_4^\flat\right)\partial_3^3\mathbf{Z} dx_3 \\
&\lesssim \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t} + \left\|\phi, \psi\right\|_{L^4(\Omega)}^2 \left\|\partial_3^2\mathbf{Z}\right\|_{H^1(\mathbb{R})}.
\end{aligned} \qquad (4.55)$$

Using Lemma 3.5, it holds that

$$\left\|\phi^\flat\right\|_{L^4(\mathbb{R})} = \left\|\partial_3\Phi\right\|_{L^4(\mathbb{R})} \lesssim \left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^{\frac{5}{8}} \left\|\Phi\right\|_{L^2(\mathbb{R})}^{\frac{3}{8}} \lesssim \nu^{\frac{1}{2}} \left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^{\frac{1}{2}}, \qquad (4.56)$$

and similarly,

$$\left\|\psi^\flat\right\|_{L^4(\mathbb{R})} \lesssim \nu^{\frac{1}{2}} \left\|\partial_3^2\mathbf{\Psi}\right\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \qquad (4.57)$$

On the other hand, it follows from Lemma 3.5 that

$$\|h\|_{L^4(\Omega)} \lesssim \sum_{k=1}^{3} \|\nabla h\|_{L^2(\Omega)}^{\frac{k}{4}} \|h\|_{L^2(\Omega)}^{1-\frac{k}{4}} \lesssim \|h\|_{H^1(\Omega)} \qquad \forall h \in H^1(\Omega). \qquad (4.58)$$

Then using (4.56) to (4.58), one can get that

$$\begin{aligned}
\|\phi, \psi\|_{L^4(\Omega)}^2 &\lesssim \left\|\phi^\flat, \psi^\flat\right\|_{L^4(\mathbb{R})}^2 + \left\|\phi^\sharp, \psi^\sharp\right\|_{L^4(\Omega)}^2 \\
&\lesssim \nu \left\|\partial_3^2\Phi, \partial_3^2\mathbf{\Psi}\right\|_{L^2(\mathbb{R})} + \nu \left\|\phi^\sharp, \psi^\sharp\right\|_{H^1(\Omega)},
\end{aligned}$$

which, together with (3.26), (4.10) and (4.19), yields that

$$\begin{aligned}
\|\phi, \psi\|_{L^4(\Omega)}^2 &\lesssim \nu \left\|\partial_3^2\Phi, \partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})} + \nu(t+\Lambda)^{-\frac{1}{2}} \left\|\partial_3\Phi, \Phi\kappa\right\|_{L^2(\mathbb{R})} \\
&\quad + \nu \left\|\nabla\phi^\sharp, \nabla\zeta^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t}.
\end{aligned} \qquad (4.59)$$

Collecting (4.54), (4.55) and (4.59), one can finish the proof of (4.51).

$\square$

Lemma 4.7. *Under the assumptions of Proposition 3.2, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0, \varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then it holds that*

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}_{2,\flat} + c_{2,\flat}\mathcal{B}_{2,\flat} &\lesssim \mathcal{B}_{1,\flat} + \nu(t+\Lambda)^{-1} \left\|\partial_3\Phi, \Phi\kappa\right\|_{L^2(\mathbb{R})}^2 + \nu \left\|\nabla\zeta^\sharp\right\|_{L^2(\Omega)}^2 \\
&\quad + \nu \left\|\nabla^2\phi\right\|_{H^1(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}.
\end{aligned} \qquad (4.60)$$

where $\mathcal{B}_{0,0}$ and $\mathcal{B}_{1,\flat}$ are given by (4.52) and

$$\mathcal{A}_{2,\flat} = \frac{\tilde{\mu}}{2\bar{\rho}} \left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \partial_3 Z_3 \partial_3^2\Phi dx_3 \quad and \quad \mathcal{B}_{2,\flat} = \left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2. \qquad (4.61)$$

Proof. Multiplying $\frac{\tilde{\mu}}{\bar{\rho}}\partial_3^2\Phi$ and $\partial_3^2\Phi$ on $\partial_3^2(4.12)_1$ and $\partial_3(4.14)$, respectively, and adding up the resulting two equations, one can get that

$$
\frac{d}{dt}\left(\frac{\tilde{\mu}}{2\bar{\rho}}\left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}}\partial_3 Z_3\partial_3^2\Phi dx_3 \right) + p'(\bar{\rho})\left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2
$$
$$
= \left\|\partial_3^2 Z_3\right\|_{L^2(\mathbb{R})}^2 + \underbrace{\int_{\mathbb{R}}\left(G_3 - N_3\right)\partial_3^3\Phi dx_3}_{I_6}.
$$

Similar to the estimate of I_5 in (4.53), one can use Lemma 4.1 and (4.19) and (4.59) to prove that

$$
\begin{aligned}
I_6 &= \int_{\mathbb{R}}\left(G_3 + Q_{1,3}^{\flat} + Q_{2,3}^{\flat}\right)\cdot\partial_3^3\Phi dx_3 + \int_{\mathbb{R}}\left(Q_{3,3}^{\flat} + Q_{4,3}^{\flat}\right)\partial_3^4\Phi dx_3 \\
&\lesssim \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t} + \|\phi,\psi\|_{L^4(\Omega)}^2\left\|\partial_3^3\Phi\right\|_{H^1(\mathbb{R})} \\
&\lesssim \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t} + \nu\left\|\partial_3^2\Phi\right\|_{H^2(\mathbb{R})}^2 + \nu\left\|\partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 \\
&\quad + \nu(t+\Lambda)^{-1}\left\|\partial_3\Phi,\Phi\kappa\right\|_{L^2(\mathbb{R})}^2 + \nu\left\|\nabla\phi^{\sharp},\nabla\zeta^{\sharp}\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Note that $\left\|\partial_3^3\Phi\right\|_{H^1(\mathbb{R})} \lesssim \left\|\nabla^2\phi\right\|_{H^1(\Omega)}$ and $\left\|\nabla\phi^{\sharp}\right\|_{L^2(\Omega)} \lesssim \left\|\nabla^2\phi^{\sharp}\right\|_{L^2(\Omega)} \lesssim \left\|\nabla^2\phi\right\|_{L^2(\Omega)}$. Then the proof is finished.

$\square$

Lemma 4.8. *Under the assumptions of Proposition 3.2, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0, \varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then it holds that*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{A}_{3,\flat} + c_{3,\flat}\mathcal{B}_{3,\flat} &\lesssim (t+\Lambda)^{-2}\left\|\partial_3\Phi,\Phi\kappa\right\|_{L^2(\mathbb{R})}^2 + (t+\Lambda)^{-1}\left\|\partial_3^2\Phi,\partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 \\
&\quad + \nu\left\|\nabla^2\phi\right\|_{H^1(\Omega)}^2 + \nu\left\|\nabla^3\zeta\right\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}.
\end{aligned} \tag{4.62}
$$

where $\mathcal{B}_{i,\flat}$ for $i = 0, 1, 2$ are given by (4.52) and (4.61) and

$$
\mathcal{A}_{3,\flat} = p'(\bar{\rho})\left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2 + \left\|\partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 \quad and \quad \mathcal{B}_{3,\flat} = \left\|\partial_3^3\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2. \tag{4.63}
$$

Proof. Multiplying $-p'(\bar{\rho})\partial_3^3\Phi$ and $-\partial_3^3\mathbf{Z}$ on $\partial_3(4.12)_1$ and $\partial_3(4.12)_2$, respectively and adding up the resulting two equations, one can get that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{p'(\bar{\rho})}{2}\left\|\partial_3^2\Phi\right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\left\|\partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 \right) + \frac{\mu}{\bar{\rho}}\left\|\partial_3^3\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \frac{\mu+\lambda}{\bar{\rho}}\left\|\partial_3^3 Z_3\right\|_{L^2(\mathbb{R})}^2 \\
&= \underbrace{-\frac{\mu}{\bar{\rho}}\int_{\mathbb{R}}\partial_3\left(\partial_3\theta\partial_3\Phi\right)\bar{\mathbf{u}}\cdot\partial_3^3\mathbf{Z}dx_3}_{I_7} + \underbrace{\int_{\mathbb{R}}\left(\partial_3\mathbf{G} - \partial_3\mathbf{N}\right)\cdot\partial_3^3\mathbf{Z}dx_3}_{I_8}.
\end{aligned}
$$

Using (1.19), one has that

$$
I_7 \leqslant \frac{\mu}{4\bar{\rho}}\left\|\partial_3^3\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + C\sum_{j=1}^{2}(t+\Lambda)^{j-3}\left\|\partial_3^j\Phi\right\|_{L^2(\mathbb{R})}^2. \tag{4.64}
$$

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To achieve the optimal rate of $(\phi^\flat, \zeta^\flat)$, we need to deal with the term I_8 very carefully. In this respect, the subtle estimate $(4.15)_2$ plays an important role. It follows from Lemmas 3.2 and 4.1 that

$$
\begin{aligned}
I_8 &= \int_{\mathbb{R}} \left(\partial_3 \mathbf{G} + \partial_3 \mathbf{Q}_1^\flat + \partial_3 \mathbf{Q}_2^\flat \right) \cdot \partial_3^3 \mathbf{Z} dx_3 + \int_{\mathbb{R}} \left(\partial_3 \mathbf{Q}_3^\flat + \partial_3 \mathbf{Q}_4^\flat \right) \partial_3^4 \mathbf{Z} dx_3 \\
&\lesssim \left\| \partial_3^3 \mathbf{Z} \right\|_{H^1(\mathbb{R})} \left(I_{8,1} + I_{8,2} \right) + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t},
\end{aligned}
\tag{4.65}
$$

where

$$
I_{8,1} = (t + \Lambda)^{-\frac{1}{2}} \left\| \phi, \psi \right\|_{L^4(\Omega)}^2 \quad \text{and} \quad I_{8,2} = \left\| \nabla \phi, \nabla \psi \right\|_{L^4(\Omega)} \left\| \phi, \psi \right\|_{L^4(\Omega)}.
\tag{4.66}
$$

First, it follows from (4.10) that

$$
\begin{aligned}
\left\| \partial_3^4 \mathbf{Z} \right\|_{L^2(\mathbb{R})} &\lesssim \left\| \partial_3^3 \psi^\flat, \partial_3^3 \phi^\flat \right\|_{L^2(\mathbb{R})} + \Lambda^{-\frac{1}{2}} \left\| \partial_3^2 \phi^\flat \right\|_{L^2(\mathbb{R})} \\
&\quad + (t + \Lambda)^{-1} \left\| \partial_3^2 \Phi \right\|_{L^2(\mathbb{R})} + (t + \Lambda)^{-\frac{3}{2}} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})}.
\end{aligned}
\tag{4.67}
$$

Using (3.26) and (4.59), it holds that

$$
I_{8,1} \lesssim \nu (t+\Lambda)^{-\frac{1}{2}} \Big[\left\| \partial_3^2 \Phi, \partial_3^2 \mathbf{Z} \right\|_{L^2(\mathbb{R})} + (t+\Lambda)^{-\frac{1}{2}} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})} + \left\| \nabla^3 \phi^\sharp, \nabla^3 \zeta^\sharp \right\|_{L^2(\Omega)} \Big].
\tag{4.68}
$$

To estimate $I_{8,2}$, it follows from Lemma 3.5, (3.26) and (4.58) that for $j = 0, 1$,

$$
\begin{aligned}
\left\| \nabla^j \phi \right\|_{L^4(\Omega)} &\lesssim \left\| \partial_3^{j+1} \Phi \right\|_{L^4(\mathbb{R})} + \left\| \nabla^j \phi^\sharp \right\|_{L^4(\Omega)} \\
&\lesssim \left\| \partial_3^3 \Phi \right\|_{L^2(\mathbb{R})}^{\frac{3+4j}{10}} \left\| \Phi \right\|_{L^\infty(\mathbb{R})}^{\frac{7-4j}{10}} + \left\| \nabla^j \phi^\sharp \right\|_{H^1(\Omega)} \\
&\lesssim \nu^{\frac{7-4j}{10}} \big(\left\| \partial_3^3 \Phi \right\|_{L^2(\mathbb{R})}^{\frac{3+4j}{10}} + \left\| \nabla^3 \phi^\sharp \right\|_{L^2(\Omega)}^{\frac{3+4j}{10}} \big),
\end{aligned}
\tag{4.69}
$$

Similarly, one can prove that

$$
\left\| \nabla^j \psi \right\|_{L^4(\Omega)} \lesssim \nu^{\frac{7-4j}{10}} \big(\left\| \partial_3^3 \Psi \right\|_{L^2(\mathbb{R})}^{\frac{3+4j}{10}} + \left\| \nabla^3 \psi^\sharp \right\|_{L^2(\Omega)}^{\frac{3+4j}{10}} \big), \qquad j = 0, 1.
\tag{4.70}
$$

This, together with (4.69), yields that

$$
\begin{aligned}
I_{8,2} &\lesssim \nu \left\| \partial_3^3 \Phi, \partial_3^3 \Psi \right\|_{L^2(\mathbb{R})} + \nu \left\| \nabla^3 \phi, \nabla^3 \psi \right\|_{L^2(\Omega)} \\
&\lesssim \nu \left\| \partial_3^3 \mathbf{Z} \right\|_{L^2(\mathbb{R})} + \nu (t + \Lambda)^{-1} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})} \\
&\quad + \nu (t + \Lambda)^{-\frac{1}{2}} \left\| \partial_3^2 \Phi \right\|_{L^2(\mathbb{R})} + \nu \left\| \nabla^3 \phi, \nabla^3 \psi \right\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.71}
$$

Thus, collecting (4.20), (4.64), (4.67), (4.68) and (4.71), one can obtain (4.62). $\qquad \square$

Then collecting Lemmas 4.6–4.8, there exist generic numbers $M_{i,\flat} \geqslant 1$ for $i = 1, 2, 3$, such that if $\Lambda^{-1}, \varepsilon$ and ν are small, then it holds that

$$
\begin{aligned}
\frac{d}{dt} \Big(\sum_{i=1}^3 M_{i,\flat} \mathcal{A}_{i,\flat} \Big) &+ \frac{1}{2} \Big(\sum_{i=1}^3 c_{i,\flat} M_{i,\flat} \mathcal{B}_{i,\flat} \Big) \\
&\lesssim (t + \Lambda)^{-1} \left\| \partial_3 \Phi, \Phi \kappa \right\|_{L^2(\mathbb{R})}^2 + \nu \left\| \nabla^2 \phi \right\|_{H^1(\Omega)}^2 + \nu \left\| \nabla^3 \zeta \right\|_{L^2(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t},
\end{aligned}
\tag{4.72}
$$

and

$$\sum_{i=1}^{3} M_{i,\flat} \mathcal{A}_{i,\flat} \sim \left\| \partial_3 \Phi, \partial_3 \mathbf{Z} \right\|_{H^1(\mathbb{R})}^2,$$

$$\sum_{i=1}^{3} c_{i,\flat} M_{i,\flat} \mathcal{B}_{i,\flat} \sim \left\| \partial_3^2 \Phi \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_3^2 \mathbf{Z} \right\|_{H^1(\mathbb{R})}^2. \tag{4.73}$$

4.2 Estimates of non-zero modes

The estimate of the non-zero modes is similar to [43, Section 7]. However, if we follow the analysis of [43] to study the system of $(\phi^\sharp, \psi^\sharp)$, we will encounter a new difficulty in the energy estimates due to the large amplitude of the background wave (1.20). Nevertheless, this difficulty can be overcome by considering the system of $(\phi^\sharp, \zeta^\sharp)$ instead of $(\phi^\sharp, \psi^\sharp)$.

To formulate the system for $(\phi^\sharp, \zeta^\sharp)$, we first subtract (3.3) by (3.8) to get that

$$\begin{cases} \partial_t \phi^\sharp + \mathrm{div}\psi^\sharp = -f_0^\sharp, \\ \partial_t \psi^\sharp + \sum_{i=1}^{3} \partial_i \big(u_i^{\mathsf{vs}} \psi^\sharp + \mathbf{m}^{\mathsf{vs}} \zeta_i^\sharp + \mathbf{n}_{1,i} \big) + \nabla \big(p'(\bar\rho)\phi^\sharp + n_2 \big) \\ \qquad\qquad = \mu \triangle \zeta^\sharp + (\mu + \lambda)\nabla \mathrm{div}\zeta^\sharp - \mathbf{g}^\sharp, \end{cases} \tag{4.74}$$

where

$$\begin{aligned} \mathbf{n}_{1,i} &= \big(u_i \mathbf{m} - \tilde{u}_i \tilde{\mathbf{m}} \big)^\sharp - u_i^{\mathsf{vs}} \psi^\sharp - \mathbf{m}^{\mathsf{vs}} \zeta_i^\sharp, \\ n_2 &= \big(p(\rho) - p(\tilde\rho) \big)^\sharp - p'(\bar\rho)\phi^\sharp. \end{aligned} \tag{4.75}$$

Note that $\psi = \tilde\rho\zeta + \tilde{\mathbf{u}}\phi + \phi\zeta$, which implies that

$$\psi^\sharp = \bar\rho\zeta^\sharp + \mathbf{u}^{\mathsf{vs}}\phi^\sharp + \mathbf{n}_3, \tag{4.76}$$

where

$$\mathbf{n}_3 = \big[(\tilde\rho - \bar\rho)\zeta + (\tilde{\mathbf{u}} - \mathbf{u}^{\mathsf{vs}})\phi \big]^\sharp + \zeta^\flat \phi^\sharp + \phi^\flat \zeta^\sharp + (\phi^\sharp \zeta^\sharp)^\sharp. \tag{4.77}$$

Then plugging (4.76) into (4.74) and using the fact that $\mathrm{div}\mathbf{u}^{\mathsf{vs}} = 0$, the system of $(\phi^\sharp, \zeta^\sharp)$ can be formulated as

$$\begin{cases} \partial_t \phi^\sharp + \bar\rho\,\mathrm{div}\zeta^\sharp + \mathbf{u}^{\mathsf{vs}} \cdot \nabla\phi^\sharp = -f_0^\sharp - \mathrm{div}\mathbf{n}_3, \\ \bar\rho\partial_t \zeta^\sharp + \bar\rho\mathbf{u}^{\mathsf{vs}} \cdot \nabla\zeta^\sharp + p'(\bar\rho)\nabla\phi^\sharp + \partial_t\mathbf{u}^{\mathsf{vs}}\phi^\sharp - \mu\triangle\zeta^\sharp - (\mu+\lambda)\nabla\mathrm{div}\zeta^\sharp \\ \qquad = \mathbf{n}_4 - \partial_t\mathbf{n}_3 + \mathbf{u}^{\mathsf{vs}} f_0^\sharp - \mathbf{g}^\sharp. \end{cases} \tag{4.78}$$

with $\mathbf{n}_4 := -\sum_{i=1}^{3} \partial_i \mathbf{n}_{1,i} - \nabla n_2 + \mathbf{u}^{\mathsf{vs}}\mathrm{div}\mathbf{n}_3 - \mathbf{u}^{\mathsf{vs}} \cdot \nabla\mathbf{n}_3$.

Combining (3.26) and (4.58), the remainder (4.77) satisfies that

$$\left\| \nabla^j \mathbf{n}_3 \right\|_{L^2(\Omega)} \lesssim \nu \left\| \nabla^j \phi^\sharp, \nabla^j \zeta^\sharp \right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t}, \qquad j = 0, 1, 2. \tag{4.79}$$

This, together with (3.26) and (4.76), yields that

$$\left\| \nabla^j \psi^\sharp \right\|_{L^2(\Omega)} \lesssim \left\| \nabla^j \zeta^\sharp, \nabla^j \phi^\sharp \right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t}, \quad j = 0, 1. \tag{4.80}$$

Note that (4.75) yields that

$$\mathbf{n}_{1,i} = \big[(\tilde{u}_i - u_i^{\mathsf{vs}})\psi + (\tilde{\mathbf{m}} - \mathbf{m}^{\mathsf{vs}})\zeta_i \big]^\sharp + \zeta_i^\sharp \psi^\flat + \zeta_i^\flat \psi^\sharp + (\zeta_i^\sharp \psi^\sharp)^\sharp,$$

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$$n_2 = \left[a_1(\tilde{\rho} - \bar{\rho})\right]^\sharp + \left(a_2 \phi^\sharp\right)^\sharp,$$

where

$$a_1 = \int_0^1 \left[p'(\bar{\rho} + \phi^\flat + s(\tilde{\rho} - \bar{\rho}) + s\phi^\sharp) - p'(\bar{\rho} + s(\tilde{\rho} - \bar{\rho})) \right] ds = O(1)\left(\left|\phi^\flat\right| + \left|\phi^\sharp\right| \right)$$

$$a_2 = \int_0^1 \left[p'(\bar{\rho} + \phi^\flat + s(\tilde{\rho} - \bar{\rho}) + s\phi^\sharp) - p'(\bar{\rho}) \right] ds = O(1)\left(\left|\tilde{\rho} - \bar{\rho}\right| + \left|\phi^\flat\right| + \left|\phi^\sharp\right| \right).$$

Thus, it holds that

$$\|\mathbf{n}_{1,i}\|_{H^1(\Omega)} \lesssim \nu \left\|\nabla \zeta^\sharp, \nabla \phi^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t} \qquad \text{for } i = 1,2,3, \tag{4.81}$$

$$\|n_2\|_{H^1(\Omega)} \lesssim \nu \left\|\nabla \phi^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t}, \tag{4.82}$$

which, together with (4.79), yields that

$$\|\mathbf{n}_4\|_{L^2(\Omega)} \lesssim \nu \left\|\nabla \phi^\sharp, \nabla \zeta^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t}. \tag{4.83}$$

Moreover, we claim that

$$\|\partial_t \mathbf{n}_3\|_{L^2(\Omega)} \lesssim \nu \left\|\nabla \phi^\sharp, \nabla^2 \zeta^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu \Lambda^{\frac{1}{4}} e^{-\alpha t}. \tag{4.84}$$

In fact, it follows from (4.78) that

$$\left\|\partial_t \phi^\sharp, \partial_t \zeta^\sharp\right\|_{L^2(\Omega)} \lesssim \left\|\nabla \phi^\sharp, \nabla^2 \zeta^\sharp\right\|_{L^2(\Omega)} + \|\partial_t \mathbf{n}_3\|_{L^2(\Omega)} + \varepsilon\Lambda^{\frac{1}{4}} e^{-\alpha t}.$$

Using the Sobolev inequality and (3.5), it holds that

$$\left\|\partial_t \zeta^\flat, \partial_t \phi^\flat\right\|_{L^\infty(\mathbb{R})} \lesssim \left\|\partial_t \zeta^\flat, \partial_t \phi^\flat\right\|_{H^1(\mathbb{R})} \lesssim \left\|\partial_t \zeta, \partial_t \phi\right\|_{H^1(\Omega)},$$

which, together with (3.5) and (4.3), yields that

$$\left\|\partial_t \zeta^\flat, \partial_t \phi^\flat\right\|_{L^\infty(\mathbb{R})} \lesssim \|\phi, \zeta\|_{H^3(\Omega)} + \|f_0, \mathbf{g}\|_{H^1(\Omega)} \lesssim \nu + \varepsilon\Lambda^{\frac{1}{4}} e^{-\alpha t}.$$

Then it follows from (4.77) that

$$\begin{aligned}
\|\partial_t \mathbf{n}_3\|_{L^2(\Omega)} &\lesssim \left\|\partial_t \zeta^\flat, \partial_t \phi^\flat\right\|_{L^\infty(\mathbb{R})} \left\|\phi^\sharp, \zeta^\sharp\right\|_{L^2(\Omega)} + \nu \left\|\partial_t \phi^\sharp, \partial_t \zeta^\sharp\right\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t} \\
&\lesssim \nu \left\|\nabla \phi^\sharp, \nabla^2 \zeta^\sharp\right\|_{L^2(\Omega)} + \nu \|\partial_t \mathbf{n}_3\|_{L^2(\Omega)} + \varepsilon\nu \Lambda^{\frac{1}{4}} e^{-\alpha t},
\end{aligned}$$

which yields (4.84).

Now we establish the a priori estimates of $(\phi^\sharp, \zeta^\sharp)$.

Lemma 4.9. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt}\mathcal{A}_{1,\sharp} + c_{1,\sharp}\mathcal{B}_{1,\sharp} \lesssim (\Lambda^{-1} + \nu) \left\|\nabla \phi^\sharp, \nabla^2 \zeta^\sharp\right\|_{L^2(\Omega)}^2 + \varepsilon\nu \Lambda^{\frac{1}{4}} e^{-\alpha t}, \tag{4.85}$$

where

$$\mathcal{A}_{1,\sharp} = \frac{p'(\bar{\rho})}{\bar{\rho}^2} \left\|\phi^\sharp\right\|_{L^2(\Omega)}^2 + \left\|\zeta^\sharp\right\|_{L^2(\Omega)}^2 \quad and \quad \mathcal{B}_{1,\sharp} = \left\|\nabla \zeta^\sharp\right\|_{L^2(\Omega)}^2.$$

Proof. Multiplying $\frac{p'(\bar{\rho})}{\bar{\rho}}\phi^{\sharp}$ and $\zeta^{\sharp}$ on $(4.78)_1$ and $(4.78)_2$, respectively, and using the fact that $\text{div}\mathbf{u}^{\text{vs}} = 0$, one can obtain that

$$\partial_t \left(\frac{p'(\bar{\rho})}{2\bar{\rho}} \left| \phi^{\sharp} \right|^2 + \frac{\bar{\rho}}{2} \left| \zeta^{\sharp} \right|^2 \right) + \mu \left| \nabla \zeta^{\sharp} \right|^2 + (\mu + \lambda) \left| \text{div}\zeta^{\sharp} \right|^2$$

$$= \text{div}(\cdots) + (\mathbf{n}_4 - \partial_t \mathbf{n}_3) \cdot \zeta^{\sharp} - \frac{p'(\bar{\rho})}{\bar{\rho}} \text{div}\mathbf{n}_3 \phi^{\sharp} - \partial_t \mathbf{u}^{\text{vs}} \cdot \zeta^{\sharp} \phi^{\sharp}$$

$$- \frac{p'(\bar{\rho})}{\bar{\rho}} f_0^{\sharp} \phi^{\sharp} + (f_0^{\sharp} \mathbf{u}^{\text{vs}} - \mathbf{g}^{\sharp}) \cdot \zeta^{\sharp}.$$

Then integrating the equality above over Ω, and using (3.17), (3.26), (4.83) and (4.84), one can get (4.85). $\qquad\square$

Lemma 4.10. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt}\mathcal{A}_{2,\sharp} + c_{2,\sharp}\mathcal{B}_{2,\sharp} \lesssim \mathcal{B}_{1,\sharp} + \nu \left\| \nabla^2 \zeta^{\sharp} \right\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}, \qquad (4.86)$$

where

$$\mathcal{A}_{2,\sharp} = \frac{\tilde{\mu}}{2\bar{\rho}^2} \left\| \nabla \phi^{\sharp} \right\|_{L^2(\Omega)}^2 + \int_{\Omega} \zeta^{\sharp} \cdot \nabla \phi^{\sharp} dx \quad and \quad \mathcal{B}_{2,\sharp} = \left\| \nabla \phi^{\sharp} \right\|_{L^2(\Omega)}^2.$$

Proof. Multiplying $\frac{\tilde{\mu}}{\bar{\rho}}\nabla\phi^{\sharp}$ and $\nabla\phi^{\sharp}$ on $\nabla(4.78)_1$ and $(4.78)_2$, respectively, leads to

$$\partial_t \left(\frac{\tilde{\mu}}{2\bar{\rho}} \left| \nabla \phi^{\sharp} \right|^2 + \bar{\rho}\zeta^{\sharp} \cdot \nabla \phi^{\sharp} \right) + p'(\bar{\rho}) \left| \nabla \phi^{\sharp} \right|^2$$

$$= \text{div}(\cdots) - \nabla\phi^{\sharp} \cdot \nabla\mathbf{u}^{\text{vs}}\nabla\phi^{\sharp} - \bar{\rho}\mathbf{u}^{\text{vs}} \cdot \nabla\zeta^{\sharp}\nabla\phi^{\sharp} - \phi^{\sharp}\partial_t\mathbf{u}^{\text{vs}} \cdot \nabla\phi^{\sharp}$$

$$+ \bar{\rho}\text{div}\zeta^{\sharp}\left(\bar{\rho}\text{div}\zeta^{\sharp} + \mathbf{u}^{\text{vs}} \cdot \nabla\phi^{\sharp} + f_0^{\sharp} + \text{div}\mathbf{n}_3 \right) \qquad (4.87)$$

$$+ \nabla\phi^{\sharp} \cdot \left(-\frac{\tilde{\mu}}{\bar{\rho}}\nabla f_0^{\sharp} + \mathbf{u}^{\text{vs}}f_0^{\sharp} - \mathbf{g}^{\sharp} \right)$$

$$+ \nabla\phi^{\sharp} \cdot (\mathbf{n}_4 - \partial_t\mathbf{n}_3) - \frac{\tilde{\mu}}{\bar{\rho}}\nabla\phi^{\sharp} \cdot \nabla\text{div}\mathbf{n}_3,$$

where we have used the fact that for any $h_0 \in H^1(\Omega), \mathbf{h} = (h_1, h_2, h_3) \in H^2(\Omega)$,

$$\nabla h_0 \cdot (\triangle\mathbf{h} - \nabla\text{div}\mathbf{h}) = \text{div}(\nabla h_0 \times \text{curl}\mathbf{h}). \qquad (4.88)$$

To estimate the term $-\frac{\tilde{\mu}}{\bar{\rho}}\nabla\phi^{\sharp} \cdot \nabla\text{div}\mathbf{n}_3$ on the right-hand side of (4.87), one first notes that

$$\left| \int_{\Omega} \nabla\phi^{\sharp} \cdot \left(\zeta^{\flat} \cdot \nabla \right) \nabla\phi^{\sharp}dx \right| = \frac{1}{2} \left| \int_{\Omega} \text{div}\zeta^{\flat} \left| \nabla\phi^{\sharp} \right|^2 dx \right| \lesssim \nu \left\| \nabla\phi^{\sharp} \right\|_{L^2(\Omega)}^2,$$

and it follows from (3.26) and (4.2) that

$$\left\| \nabla\phi^{\sharp} \cdot \left((\zeta^{\sharp} \cdot \nabla)\nabla\phi^{\sharp} \right)^{\sharp} \right\|_{L^1(\Omega)} \lesssim \left\| \nabla\phi^{\sharp} \right\|_{L^2(\Omega)} \left\| \zeta^{\sharp} \right\|_{L^{\infty}(\Omega)} \left\| \nabla^2\phi^{\sharp} \right\|_{L^2(\Omega)}$$

$$\lesssim \nu \left\| \nabla\phi^{\sharp} \right\|_{L^2(\Omega)} \left\| \zeta^{\sharp} \right\|_{H^2(\Omega)}$$

$$\lesssim \nu \left\| \nabla\phi^{\sharp} \right\|_{L^2(\Omega)} \left\| \nabla^2\zeta^{\sharp} \right\|_{L^2(\Omega)}.$$

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Thus, using (4.77), one can get that

$$\left| \int_\Omega \nabla \phi^\sharp \cdot \nabla \mathrm{div}\mathbf{n}_3 dx \right| \lesssim \nu \left\| \nabla \phi^\sharp, \nabla^2 \zeta^\sharp \right\|_{L^2(\Omega)} + \varepsilon \nu e^{-\alpha t}. \tag{4.89}$$

This, together with (4.79), (4.83) and (4.84), can yield (4.86) by integrating (4.87) over Ω. □

Lemma 4.11. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt}\mathcal{A}_{3,\sharp} + c_{3,\sharp}\mathcal{B}_{3,\sharp} \lesssim \mathcal{B}_{1,\sharp} + \mathcal{B}_{2,\sharp} + \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}, \tag{4.90}$$

where

$$\mathcal{A}_{3,\sharp} = \left\| \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 \quad and \quad \mathcal{B}_{3,\sharp} = \left\| \nabla^2 \zeta^\sharp \right\|_{L^2(\Omega)}^2.$$

Proof. Multiplying $-\triangle \zeta^\sharp$ on (4.78)₂ and using (4.88), one can obtain that

$$\begin{aligned}
\partial_t \Big(\frac{\bar{\rho}}{2} \left| \nabla \zeta^\sharp \right|^2 \Big) &+ \tilde{\mu} \left| \triangle \zeta^\sharp \right|^2 \\
&= \mathrm{div}(\cdots) + \bar{\rho}(\mathbf{u}^{\mathsf{vs}} \cdot \nabla)\zeta^\sharp \cdot \triangle \zeta^\sharp + p'(\bar{\rho})\nabla \phi^\sharp \cdot \triangle \zeta^\sharp + \phi^\sharp \partial_t \mathbf{u}^{\mathsf{vs}} \cdot \triangle \zeta^\sharp \\
&\quad - (\mathbf{n}_4 - \partial_t \mathbf{n}_3) \cdot \triangle \zeta^\sharp - (\mathbf{u}^{\mathsf{vs}} f_0^\sharp - \mathbf{g}^\sharp) \cdot \triangle \zeta^\sharp.
\end{aligned}$$

Note that $\left\| \triangle \zeta^\sharp \right\|_{L^2(\Omega)}^2 \gtrsim \left\| \nabla^2 \zeta^\sharp \right\|_{L^2(\Omega)}^2$. Then one can obtain (4.90) with the use of (3.26), (4.79) and (4.83). □

Collecting Lemmas 4.9–4.11, there exist some generic numbers $M_{i,\neq} \geqslant 1$ for $i = 1,2,3$, such that if $\Lambda^{-1}, \varepsilon$ and ν are small, then

$$\frac{d}{dt}\Big(\sum_{i=1}^3 M_{i,\neq}\mathcal{A}_{i,\neq} \Big) + \frac{1}{2}\Big(\sum_{i=1}^3 c_{i,\neq}M_{i,\neq}\mathcal{B}_{i,\neq} \Big) \lesssim \varepsilon \nu \Lambda^{\frac{1}{4}} e^{-\alpha t}, \tag{4.91}$$

and

$$\begin{aligned}
\sum_{i=1}^3 M_{i,\neq}\mathcal{A}_{i,\neq} &\sim \left\| \phi^\sharp, \zeta^\sharp \right\|_{H^1(\Omega)}^2, \\
\sum_{i=1}^3 c_{i,\neq}M_{i,\neq}\mathcal{B}_{i,\neq} &\sim \left\| \nabla \phi^\sharp \right\|_{L^2(\Omega)}^2 + \left\| \nabla \zeta^\sharp \right\|_{H^1(\Omega)}^2.
\end{aligned} \tag{4.92}$$

It follows from (3.26) that

$$\sum_{i=1}^3 M_{i,\neq}\mathcal{A}_{i,\neq} \lesssim \left\| \nabla \phi^\sharp, \nabla \zeta^\sharp \right\|_{L^2(\Omega)}^2 \lesssim \sum_{i=1}^3 c_{i,\neq}M_{i,\neq}\mathcal{B}_{i,\neq}.$$

Then there exists a generic constant $\alpha_1 \in (0, \alpha)$ such that

$$\left\| \phi^\sharp, \zeta^\sharp \right\|_{H^1(\Omega)}^2 \lesssim \big(\left\| \phi_0, \psi_0 \right\|_{H^1(\Omega)}^2 + \varepsilon \nu \Lambda^{\frac{1}{4}} \big) e^{-\alpha_1 t}. \tag{4.93}$$

4.3 Estimates of higher-order derivatives.

Now we return to the original perturbed system (3.5) to estimate the higher-order derivatives of the perturbation. It is noted that in the proof of Lemmas 4.12 and 4.13, we use only the a priori assumption that $\sup_{t\in(0,T)} \|(\phi,\zeta)\|_{H^2(\Omega)}$ is small. Thus, with a higher-order assumption (3.21), one can use similar arguments to estimate the third-order derivatives, i.e. Lemmas 4.14 and 4.15, whose proofs are omitted for convenience.

Lemma 4.12. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$
\frac{d}{dt}\mathcal{A}_1 + c_1 \mathcal{B}_1 \lesssim \sum_{j=0}^{1}(t+\Lambda)^{-2+j}\left\|\nabla^j(\phi,\zeta)\right\|_{L^2(\Omega)}^2 \tag{4.94}
$$
$$
+ \left\|\nabla^2\zeta\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla^3\zeta\right\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t},
$$

where

$$
\mathcal{A}_1 = \sum_{i=1}^{3}\int_{\Omega}\left(\frac{\tilde{\mu}}{2\rho^2}|\nabla\partial_i\phi|^2 + \nabla\partial_i\phi\cdot\partial_i\zeta\right)dx \quad and \quad \mathcal{B}_1 = \left\|\nabla^2\phi\right\|_{L^2(\Omega)}^2. \tag{4.95}
$$

Proof. For $i \in \{1,2,3\}$, taking the derivative $\nabla\partial_i$ on $(3.5)_1$ and multiplying the resulting equation by $\cdot\frac{\nabla\partial_i\phi}{\rho^2}$, one can get that

$$
\partial_t\left(\frac{|\nabla\partial_i\phi|^2}{2\rho^2}\right) + \frac{1}{\rho}\nabla\partial_i\mathrm{div}\zeta\cdot\nabla\partial_i\phi = \mathrm{div}(\cdots) + I_9, \tag{4.96}
$$

where

$$
I_9 = -\frac{1}{\rho^2}\nabla\partial_i\phi\cdot\left\{\left[\nabla\partial_i(\rho\mathrm{div}\zeta) - \rho\nabla\partial_i\mathrm{div}\zeta\right] + \left[\nabla\partial_i(\mathbf{u}\cdot\nabla\phi) - (\mathbf{u}\cdot\nabla)\nabla\partial_i\phi\right]\right.
$$
$$
\left. + \nabla\partial_i\left[\mathrm{div}\tilde{\mathbf{u}}\phi + \nabla\tilde{\rho}\cdot\zeta\right] - \frac{3}{2}\left(\mathrm{div}\tilde{\mathbf{u}} + \mathrm{div}\zeta\right)\nabla\partial_i\phi + \nabla\partial_i f_0\right\}.
$$

Using (2.31), (2.32) and (3.17), one can get that

$$
\|I_9\|_{L^1(\Omega)} \lesssim \left\|\nabla^2\phi\right\|_{L^2(\Omega)} \cdot \left[\|\nabla\phi\|_{L^4(\Omega)}\left\|\nabla^2\zeta\right\|_{L^4(\Omega)} + \left\|\nabla^2\phi\right\|_{L^2(\Omega)}\|\nabla\zeta\|_{L^\infty(\Omega)}\right.
$$
$$
\left. + \sum_{j=0}^{2}(t+\Lambda)^{-\frac{3-j}{2}}\left\|\nabla^j\phi\right\|_{L^2(\Omega)} + \varepsilon e^{-\alpha t}\|\phi,\zeta\|_{H^2(\Omega)} + \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha t}\right]. \tag{4.97}
$$

It follows from Lemma 3.5 that

$$
\|\nabla\phi\|_{L^4(\Omega)} \lesssim \sum_{k=1}^{3}\left\|\nabla^2\phi\right\|_{L^2(\Omega)}^{\frac{1}{2}+\frac{k}{8}}\|\phi\|_{L^2(\Omega)}^{\frac{1}{2}-\frac{k}{8}} \lesssim \nu^{\frac{1}{2}}\left\|\nabla^2\phi\right\|_{L^2(\Omega)}^{\frac{1}{2}},
$$
$$
\left\|\nabla^2\zeta\right\|_{L^4(\Omega)} \lesssim \sum_{k=1}^{3}\left\|\nabla^3\zeta\right\|_{L^2(\Omega)}^{\frac{k}{4}}\left\|\nabla^2\zeta\right\|_{L^2(\Omega)}^{1-\frac{k}{4}}, \tag{4.98}
$$

which yields that

$$
\|\nabla\phi\|_{L^4(\Omega)}\left\|\nabla^2\zeta\right\|_{L^4(\Omega)} \lesssim \nu\left\|\nabla^2\phi\right\|_{L^2(\Omega)} + \nu\left\|\nabla^2\zeta\right\|_{H^1(\Omega)}. \tag{4.99}
$$

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Using (4.2) with $h = \nabla\zeta$ therein, it holds that

$$\left\|\nabla^2\phi\right\|_{L^2(\Omega)}\|\nabla\zeta\|_{L^\infty(\Omega)} \lesssim \nu\left\|\nabla^2\phi\right\|_{L^2(\Omega)} + \nu\left\|\nabla^2\zeta\right\|_{H^1(\Omega)}. \tag{4.100}$$

Then combining (4.97), (4.99) and (4.100), one has that

$$\begin{aligned}
\|I_9\|_{L^1(\Omega)} &\lesssim (\Lambda^{-\frac{1}{2}} + \nu)\left\|\nabla^2\phi\right\|^2_{L^2(\Omega)} + \sum_{j=0}^{1}(t+\Lambda)^{-2+j}\left\|\nabla^j\phi\right\|^2_{L^2(\Omega)} \\
&\quad + \nu\left\|\nabla^2\zeta\right\|^2_{H^1(\Omega)} + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}.
\end{aligned} \tag{4.101}$$

On the other hand, multiplying $\cdot\nabla\partial_i\phi$ on $\partial_i\left(\frac{1}{\rho}\cdot(3.5)_2\right)$, one can get that

$$\partial_t\left(\nabla\partial_i\phi\cdot\partial_i\zeta\right) + \frac{p'(\rho)}{\rho}\left|\nabla\partial_i\phi\right|^2 - \frac{\tilde{\mu}}{\rho}\nabla\partial_i\phi\cdot\nabla\text{div}\partial_i\zeta = \text{div}(\cdots) + I_{10}, \tag{4.102}$$

where

$$\begin{aligned}
I_{10} &= -\nabla\partial_i\phi\cdot\Big[\partial_i(\mathbf{u}\cdot\nabla\zeta) + \partial_i\Big(\frac{p'(\rho)}{\rho}\Big)\nabla\phi + \partial_i\Big(\frac{p'(\rho) - p'(\tilde{\rho})}{\rho}\nabla\tilde{\rho}\Big) + \partial_i(\zeta\cdot\nabla\tilde{\mathbf{u}}) \\
&\quad + \partial_i\Big(\frac{\phi}{\rho}\left(\partial_t\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}\right)\Big) + \frac{\partial_i\rho}{\rho^2}\left(\mu\triangle\zeta + (\mu+\lambda)\nabla\text{div}\zeta\right) + \partial_i\Big(\frac{\mathbf{g} - f_0\tilde{\mathbf{u}}}{\rho}\Big)\Big] \\
&\quad + \frac{\mu}{\rho^2}\sum_{j=1}^{3}\partial_j\rho\nabla\partial_i\phi\cdot\left(\partial_j\partial_i\zeta - \nabla\partial_i\zeta_j\right) - \text{div}\partial_i\zeta\partial_i\partial_t\phi.
\end{aligned}$$

Here we have used the fact that for any $(h_0, \mathbf{h}) \in H^1(\Omega) \times H^2(\Omega)$,

$$\begin{aligned}
\frac{1}{\rho}&\nabla h_0\cdot(\triangle\mathbf{h} - \nabla\text{div}\mathbf{h}) \\
&= \sum_{j=1}^{3}\partial_j\Big[\frac{1}{\rho}\nabla h_0\cdot(\partial_j\mathbf{h} - \nabla h_j)\Big] + \frac{1}{\rho^2}\sum_{j=1}^{3}\partial_j\rho\nabla h_0\cdot\left(\partial_j\mathbf{h} - \nabla h_j\right).
\end{aligned}$$

Note that $\|\phi, \zeta\|_{H^2(\Omega)} \lesssim \nu$, then it holds that

$$\begin{aligned}
\|I_{10}\|_{L^1(\Omega)} &\lesssim \left\|\nabla^2\phi\right\|_{L^2(\Omega)}\Big[\left\|\nabla^2\zeta\right\|_{L^2(\Omega)} + \|\nabla\zeta\|^2_{L^4(\Omega)} + \|\nabla\phi\|^2_{L^4(\Omega)} \\
&\quad + (t+\Lambda)^{-\frac{1}{2}}\|\nabla\phi, \nabla\zeta\|_{L^2(\Omega)} + (t+\Lambda)^{-1}\|\phi, \zeta\|_{L^2(\Omega)} \\
&\quad + \varepsilon e^{-\alpha t}\|\phi, \zeta\|_{H^2(\Omega)} + \|\nabla\phi\|_{L^4(\Omega)}\left\|\nabla^2\zeta\right\|_{L^4(\Omega)} + \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha t}\Big] \\
&\quad + \left\|\nabla^2\zeta\right\|_{L^2(\Omega)}\|\nabla\partial_t\phi\|_{L^2(\Omega)}.
\end{aligned} \tag{4.103}$$

Using $(4.98)_1$, one has that

$$\|\nabla\phi\|^2_{L^4(\Omega)} \lesssim \nu\left\|\nabla^2\phi\right\|_{L^2(\Omega)} \quad\text{and}\quad \|\nabla\zeta\|^2_{L^4(\Omega)} \lesssim \nu\left\|\nabla^2\zeta\right\|_{L^2(\Omega)}. \tag{4.104}$$

It follows from $(3.5)_1$ that

$$\begin{aligned}
\|\nabla\partial_t\phi\|_{L^2(\Omega)} &\lesssim \left\|\nabla^2\zeta\right\|_{L^2(\Omega)} + \|\nabla\phi\|_{L^4(\Omega)}\|\nabla\zeta\|_{L^4(\Omega)} + \left\|\nabla^2\phi\right\|_{L^2(\Omega)} \\
&\quad + \sum_{j=0}^{1}(t+\Lambda)^{-\frac{2-j}{2}}\left\|\nabla^j\phi\right\|_{L^2(\Omega)} + \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha t}.
\end{aligned} \tag{4.105}$$

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Then using (4.99) and (4.103) to (4.105), one can get that

$$
\begin{aligned}
\|I_{10}\|_{L^1(\Omega)} \leqslant &\left(\frac{p'(\rho)}{16\rho} + \nu\right) \left\|\nabla^2\phi\right\|_{L^2(\Omega)}^2 + C\left\|\nabla^2\zeta\right\|_{L^2(\Omega)}^2 + C\nu\left\|\nabla^3\zeta\right\|_{L^2(\Omega)}^2 \\
&+ C\sum_{j=0}^1 (t+\Lambda)^{-2+j}\left\|\nabla^j(\phi,\zeta)\right\|_{L^2(\Omega)}^2 + C\varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}.
\end{aligned}
\tag{4.106}
$$

Thus, summing (4.96) and (4.102) together and using (4.101) and (4.106), one can obtain (4.94).

$\square$

Lemma 4.13. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$
\frac{d}{dt}\mathcal{A}_2 + c_2\mathcal{B}_2 \lesssim \sum_{j=0}^1 (t+\Lambda)^{-2+j}\left\|\nabla^j(\phi,\zeta)\right\|_{L^2(\Omega)}^2 + \left\|\nabla^2\phi, \nabla^2\zeta\right\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}, \tag{4.107}
$$

where

$$
\mathcal{A}_2 = \left\|\nabla^2\zeta\right\|_{L^2(\Omega)}^2 \quad and \quad \mathcal{B}_2 = \left\|\nabla^3\zeta\right\|_{L^2(\Omega)}^2.
$$

Proof. For fixed $i \in \{1,2,3\}$, taking the derivative ∂_i on $\frac{1}{\rho}\cdot(3.5)_2$ and multiplying the resulting equation by $\cdot(-\triangle\partial_i\zeta)$, yield that

$$
\frac{1}{2}\partial_t\big(|\nabla\partial_i\zeta|^2\big) + \frac{\mu}{\rho}|\triangle\partial_i\zeta|^2 + \frac{\mu+\lambda}{\rho}|\nabla\mathrm{div}\partial_i\zeta|^2 = \mathrm{div}(\cdots) + I_{10}, \tag{4.108}
$$

where

$$
\begin{aligned}
I_{11} = &\triangle\partial_i\zeta \cdot \partial_i\Big[\mathbf{u}\cdot\nabla\zeta + \frac{p'(\rho)}{\rho}\nabla\phi + \frac{p'(\rho)-p'(\tilde{\rho})}{\rho}\nabla\tilde{\rho} + \zeta\cdot\nabla\tilde{\mathbf{u}} + \frac{\phi}{\rho}\left(\partial_t\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}\right)\Big] \\
&+ \triangle\partial_i\zeta \cdot \frac{\partial_i\rho}{\rho^2}\big[\mu\triangle\zeta + (\mu+\lambda)\nabla\mathrm{div}\zeta\big] + \triangle\partial_i\zeta \cdot \partial_i\Big(\frac{\mathbf{g}-f_0\tilde{\mathbf{u}}}{\rho}\Big) \\
&- \frac{\mu+\lambda}{\rho^2}\mathrm{div}\partial_i\zeta\nabla\rho\cdot(\triangle\partial_i\zeta - \nabla\mathrm{div}\partial_i\zeta).
\end{aligned}
$$

Here we have used the fact that for any $\mathbf{h} \in H^2(\Omega)$,

$$
\frac{1}{\rho}\nabla\mathrm{div}\mathbf{h}\cdot\big(\nabla\mathrm{div}\mathbf{h} - \triangle\mathbf{h}\big) = \mathrm{div}\Big[\frac{1}{\rho}\mathrm{div}\mathbf{h}\big(\nabla\mathrm{div}\mathbf{h} - \triangle\mathbf{h}\big)\Big] + \frac{\mathrm{div}\mathbf{h}}{\rho^2}\nabla\rho\cdot\big(\nabla\mathrm{div}\mathbf{h} - \triangle\mathbf{h}\big).
$$

Then it holds that

$$
\begin{aligned}
\|I_{11}\|_{L^1(\Omega)} \lesssim &\left\|\nabla^3\zeta\right\|_{L^2(\Omega)}\Big[\left\|\nabla^2\zeta\right\|_{L^2(\Omega)} + \left\|\nabla\zeta\right\|_{L^4(\Omega)}^2 + \left\|\nabla^2\phi\right\|_{L^2(\Omega)} + \left\|\nabla\phi\right\|_{L^4(\Omega)}^2 \\
&+ \left\|\nabla\phi\right\|_{L^4(\Omega)}\left\|\nabla^2\zeta\right\|_{L^4(\Omega)} + \sum_{j=0}^1 (t+\Lambda)^{-\frac{2-j}{2}}\left\|\nabla^j(\phi,\zeta)\right\|_{L^2(\Omega)} \\
&+ \varepsilon e^{-\alpha t}\sup_{t\in(0,T)}\left\|\phi,\zeta\right\|_{H^2(\Omega)} + \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha t}\Big],
\end{aligned}
\tag{4.109}
$$

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which, together with (4.99) and (4.104), yields that

$$\|I_{11}\|_{L^1(\Omega)} \leqslant \left(\frac{\mu}{16\rho} + \nu\right) \|\nabla^3 \zeta\|^2_{L^2(\Omega)} + C \|\nabla^2 \phi, \nabla^2 \zeta\|^2_{L^2(\Omega)}$$
$$+ C \sum_{j=0}^{1}(t + \Lambda)^{-2+j} \|\nabla^j(\phi, \zeta)\|^2_{L^2(\Omega)} + C\varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}.$$

By integration by parts, it holds that $\|\triangle\partial_i\zeta\|_{L^2(\Omega)} = \|\nabla^2\partial_i\zeta\|_{L^2(\Omega)}$. Then one can integrate (4.108) on Ω to obtain (4.107).

$\square$

With a higher-order a priori assumption that $\sup_{t\in(0,T)}\|(\phi,\zeta)\|_{H^3(\Omega)}$ is small, one can prove the estimate of the third-order derivatives of (ϕ, ζ) similarly. We omit the proofs for convenience.

Lemma 4.14. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt}\mathcal{A}_3 + c_3\mathcal{B}_3 \lesssim \sum_{j=0}^{1}(t + \Lambda)^{-3+j} \|\nabla^j(\phi, \zeta)\|^2_{L^2(\Omega)} \tag{4.110}$$
$$+ \|\nabla^2\phi\|^2_{L^2(\Omega)} + \|\nabla^2\zeta\|^2_{H^1(\Omega)} + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t},$$

where

$$\mathcal{A}_3 = \sum_{i,j=1}^{3}\int_{\Omega}\left(\frac{\tilde{\mu}}{2\rho^2}|\nabla\partial_{ij}\phi|^2 + \nabla\partial_{ij}\phi\cdot\partial_{ij}\zeta\right)dx \quad and \quad \mathcal{B}_3 = \|\nabla^3\phi\|^2_{L^2(\Omega)}.$$

Lemma 4.15. *Under the assumptions of Proposition 3.4, there exist a large constant $\Lambda_0 \geqslant 1$ and small positive constants ε_0 and ν_0 such that, if $\Lambda \geqslant \Lambda_0$, $\varepsilon \leqslant \varepsilon_0$ and $\nu \leqslant \nu_0$, then*

$$\frac{d}{dt}\mathcal{A}_4 + c_4\mathcal{B}_4 \lesssim \sum_{j=0}^{1}(t + \Lambda)^{-3+j} \|\nabla^j(\phi, \zeta)\|^2_{L^2(\Omega)} \tag{4.111}$$
$$+ \|\nabla^2\phi, \nabla^2\zeta\|^2_{H^1(\Omega)} + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t},$$

where

$$\mathcal{A}_4 = \|\nabla^3\zeta\|^2_{L^2(\Omega)} \quad and \quad \mathcal{B}_4 = \|\nabla^4\zeta\|^2_{L^2(\Omega)}.$$

Collecting Lemmas 4.12–4.15, there exist some generic numbers $M_i \geqslant 1$ for $i = 1, 2, 3, 4$ such that if $\Lambda^{-1}, \varepsilon$ and ν are small, then

$$\frac{d}{dt}\Big(\sum_{i=1}^{4}M_i\mathcal{A}_i\Big) + \frac{1}{2}\Big(\sum_{i=1}^{4}c_iM_i\mathcal{B}_i\Big)$$
$$\lesssim \|\nabla^2\zeta\|^2_{L^2(\Omega)} + \sum_{j=0}^{1}(t + \Lambda)^{-2+j} \|\nabla^j(\phi, \zeta)\|^2_{L^2(\Omega)} + \varepsilon\nu\Lambda^{\frac{1}{4}}e^{-\alpha t}, \tag{4.112}$$

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and

$$\pm \sum_{i=1}^{4} M_i \mathcal{A}_i \lesssim \pm \left\| \nabla^2 \phi, \nabla^2 \zeta \right\|_{H^1(\Omega)}^2 + \left\| \nabla \zeta \right\|_{L^2(\Omega)}^2,$$

$$\sum_{i=1}^{4} c_i M_i \mathcal{B}_i \sim \left\| \nabla^2 \phi, \nabla^3 \zeta \right\|_{H^1(\Omega)}^2. \tag{4.113}$$

Using (3.26) and (4.18), one has that

$$\frac{d}{dt} \Big(\sum_{i=1}^{4} M_i \mathcal{A}_i \Big) + \frac{1}{4} \Big(\sum_{i=1}^{4} c_i M_i \mathcal{B}_i \Big)$$
$$\lesssim (t + \Lambda)^{-1} \left\| \partial_3^2 \Phi, \partial_3^2 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + (t + \Lambda)^{-2} \left\| \partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa \right\|_{L^2(\mathbb{R})}^2 \tag{4.114}$$
$$+ \left\| \partial_3^3 \mathbf{Z} \right\|_{L^2(\mathbb{R})}^2 + \left\| \nabla^2 \zeta^\sharp \right\|_{L^2(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}} e^{-\alpha t}.$$

Proof of Proposition 3.4. Collecting (4.72), (4.73), (4.91), (4.113) and (4.114), one can choose a linear combination of $\mathcal{A}_{i,\flat}, \mathcal{A}_{i,\neq}$ and $\mathcal{A}_j$ (resp. $\mathcal{B}_{i,\flat}, \mathcal{B}_{i,\neq}$ and $\mathcal{B}_j$) for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, denoted by $\mathcal{E}_1$ (resp. $\mathcal{D}_1$), such that

$$\frac{d}{dt} \mathcal{E}_1 + \mathcal{D}_1 \lesssim (t + \Lambda)^{-1} \left\| \partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa \right\|_{L^2(\mathbb{R})}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}} e^{-\alpha t}, \tag{4.115}$$

and

$$\pm \mathcal{E}_1 \lesssim \pm \left(\left\| \phi \right\|_{H^3(\Omega)}^2 + \left\| \partial_3 \mathbf{Z} \right\|_{H^1(\mathbb{R})}^2 + \left\| \zeta^\sharp \right\|_{H^1(\Omega)}^2 + \left\| \nabla^2 \zeta \right\|_{H^1(\Omega)}^2 \right)$$
$$+ (t + \Lambda)^{-1} \left\| \Phi\kappa \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \nu^2 e^{-\alpha t}, \tag{4.116}$$
$$\mathcal{D}_1 \sim \left\| \nabla \phi \right\|_{H^2(\Omega)}^2 + \left\| \partial_3^2 \mathbf{Z} \right\|_{H^1(\mathbb{R})}^2 + \left\| \nabla \zeta^\sharp \right\|_{H^1(\Omega)}^2 + \left\| \nabla^3 \zeta \right\|_{H^1(\Omega)}^2.$$

Here we have used (4.17) with $j = 1$ in the proof of $(4.116)_1$.

With (4.50) and (4.80), integrating (4.115) with respect to $t \in (0, T)$ yields that

$$\sup_{t \in (0,T)} \mathcal{E}_1(t) + \int_0^T \mathcal{D}_1(t) dt \lesssim \left\| \Phi_0, \Psi_0 \right\|_{L^2(\mathbb{R})}^2 + \left\| \phi_0, \psi_0 \right\|_{H^3(\Omega)}^2 + \varepsilon\nu\Lambda^{\frac{1}{4}}. \tag{4.117}$$

It follows from (4.17), (4.18) and (4.116) that

$$\pm \mathcal{E}_1 \lesssim \pm \left\| \phi, \zeta \right\|_{H^3(\Omega)}^2 + \left\| \Phi\kappa \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \nu^2 e^{-\alpha t},$$
$$\pm \mathcal{D}_1 \lesssim \pm \left(\left\| \nabla \phi \right\|_{H^2(\Omega)}^2 + \left\| \nabla \zeta \right\|_{H^3(\Omega)}^2 \right) + (t + \Lambda)^{-1} \left\| \partial_3 \Phi, \Phi\kappa \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \nu^2 e^{-\alpha t}. \tag{4.118}$$

This, together with (4.50) and (4.117), yields that

$$\sup_{t \in (0,T)} \left(\left\| \Phi, \mathbf{\Psi} \right\|_{L^2(\mathbb{R})}^2 + \left\| \phi, \zeta \right\|_{H^3(\Omega)}^2 \right)$$
$$+ \int_0^T \left(\left\| \partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa, \mathbf{Z}\kappa \right\|_{L^2(\mathbb{R})}^2 + \left\| \nabla \phi \right\|_{H^2(\Omega)}^2 + \left\| \nabla \zeta \right\|_{H^3(\Omega)}^2 \right) dt \tag{4.119}$$
$$\lesssim \left\| \Phi_0, \Psi_0 \right\|_{L^2(\mathbb{R})}^2 + \left\| \phi_0, \psi_0 \right\|_{H^3(\Omega)}^2 + \varepsilon\Lambda^{\frac{1}{4}}.$$

Here we have used the fact that $\varepsilon\nu\Lambda^{\frac{1}{4}} \lesssim \varepsilon\Lambda^{\frac{1}{4}}\nu^2 + \varepsilon\Lambda^{\frac{1}{4}}$ with (4.1) and the smallness of $\varepsilon\Lambda^{\frac{1}{4}}$. Then the proof of Proposition 3.4 is finished.

$\square$

5 Decay rate

Combining Theorem 3.3 and Proposition 3.4, for the problem (3.5), (3.6), there exist $\Lambda_0 \geqslant 1$ and $\varepsilon_0 > 0$ such that, if $\Lambda \geqslant \Lambda_0$ and $\varepsilon \Lambda^{\frac{1}{4}} \leqslant \varepsilon_0$, then the global existence of the solution $(\phi, \zeta) \in \mathbb{B}(0, +\infty)$ can be proved by a standard argument. To finish the proof of Theorem 3.1, it remains to prove (3.7).

Since the estimate (4.119) holds for $T = +\infty$, then using (3.4) and (3.13), one has that

$$
\begin{aligned}
\sup_{t>0} &\left(\|\Phi, \boldsymbol{\Psi}\|_{L^2(\mathbb{R})}^2 + \|\phi, \zeta\|_{H^3(\Omega)}^2 \right) \\
&+ \int_0^{+\infty} \left(\|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa, \mathbf{Z}\kappa\|_{L^2(\mathbb{R})}^2 + \|\nabla\phi\|_{H^2(\Omega)}^2 + \|\nabla\zeta\|_{H^3(\Omega)}^2 \right) dt \lesssim \varepsilon\Lambda^{\frac{1}{4}},
\end{aligned} \tag{5.1}
$$

and

$$
\sup_{t>0} \mathcal{E}_1(t) + \int_0^{+\infty} \mathcal{D}_1(t) dt \lesssim \varepsilon\Lambda^{\frac{1}{4}}. \tag{5.2}
$$

It follows from (4.18), (4.116) and (5.1) that

$$
\mathcal{E}_1 \lesssim \mathcal{D}_1 + \|\partial_3\Phi, \partial_3\mathbf{Z}\|_{L^2(\Omega)}^2 + (t + \Lambda)^{-1} \|\Phi\kappa\|_{L^2(\mathbb{R})}^2 + \varepsilon^3\Lambda^{\frac{1}{4}}e^{-\alpha t}. \tag{5.3}
$$

With (5.1) and (5.3), multiplying $(t+1)$ on (4.115) and integrating the resulting inequality with respect to t, one can get that

$$
\begin{aligned}
(t + 1)\mathcal{E}_1 &+ \int_0^t (\tau + 1)\mathcal{D}_1 d\tau \\
&\lesssim \mathcal{E}_1(0) + \int_0^t \mathcal{E}_1 d\tau + \int_0^t \|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 d\tau + (\varepsilon\Lambda^{\frac{1}{4}})^{\frac{3}{2}} \\
&\lesssim \varepsilon\Lambda^{\frac{1}{4}}.
\end{aligned} \tag{5.4}
$$

Here we have used the fact that $\mathcal{E}_1(0) \lesssim \varepsilon^3\Lambda^{\frac{1}{4}}$, which is derived from (3.4), (3.13) and $(4.116)_1$.

On the other hand, collecting (4.62), (4.113) and (4.114), one can choose a linear combination of $\mathcal{A}_{3,\flat}$ and $\mathcal{A}_i$ (resp. $\mathcal{B}_{3,\flat}$ and $\mathcal{B}_i$) for $i = 1, 2, 3, 4$, denoted by $\mathcal{E}_2$ (resp. $\mathcal{D}_2$), such that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_2 + \mathcal{D}_2 &\lesssim (t + \Lambda)^{-2} \|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 \\
&\quad + (t + \Lambda)^{-1} \left\|\partial_3^2\Phi, \partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + (\varepsilon\Lambda^{\frac{1}{4}})^{\frac{3}{2}}e^{-\alpha t},
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
\pm\mathcal{E}_2 &\lesssim \pm\left(\left\|\partial_3^2\Phi, \partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \left\|\nabla^2\phi, \nabla^2\zeta\right\|_{H^1(\Omega)}^2 \right) \\
&\quad + (t + \Lambda)^{-1} \|\partial_3\Phi, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \varepsilon^3\Lambda^{\frac{1}{2}}e^{-\alpha t}, \\
\mathcal{D}_2 &\sim \left\|\partial_3^3\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \left\|\nabla^2\phi, \nabla^3\zeta\right\|_{H^1(\Omega)}^2.
\end{aligned} \tag{5.6}
$$

It follows from (4.18) and $(4.116)_2$ that

$$
\mathcal{E}_2 \lesssim \mathcal{D}_1 + (t + \Lambda)^{-1} \|\partial_3\Phi, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \varepsilon^3\Lambda^{\frac{1}{2}}e^{-\alpha t}.
$$

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Then multiplying $(t+1)^2$ on (5.5) and integrating the resulting inequality with respect to t, one has that

$$(t+1)^2 \mathcal{E}_2 + \int_0^t (\tau+1)^2 \mathcal{D}_2 d\tau$$
$$\lesssim \mathcal{E}_2(0) + \int_0^t (\tau+1)\mathcal{D}_1 d\tau + \int_0^t \|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa\|^2_{L^2(\mathbb{R})} d\tau + (\varepsilon\Lambda^{\frac{1}{4}})^{\frac{3}{2}}$$
$$\lesssim \varepsilon\Lambda^{\frac{1}{4}}. \tag{5.7}$$

Note that $\|\kappa\|^2_{L^\infty(\mathbb{R})} \lesssim (t+\Lambda)^{-1}$. Then it follows from (4.118), (5.1) and (5.4) that

$$\left\|\phi^\flat, \zeta^\flat\right\|^2_{L^2(\mathbb{R})} \lesssim \mathcal{E}_1 + (t+\Lambda)^{-1} \|\Phi\|^2_{L^2(\mathbb{R})} + \varepsilon^3\Lambda^{\frac{1}{4}}e^{-\alpha t}$$
$$\lesssim \varepsilon\Lambda^{\frac{1}{4}}(t+1)^{-1}. \tag{5.8}$$

In addition, it follows from (4.17) and (5.6) to (5.8) that

$$\left\|\partial_3\phi^\flat, \partial_3\zeta^\flat\right\|^2_{L^2(\mathbb{R})} \lesssim \mathcal{E}_2 + (t+\Lambda)^{-1} \left\|\phi^\flat\right\|^2_{L^2(\mathbb{R})} + (t+\Lambda)^{-2} \|\Phi\|^2_{L^2(\mathbb{R})} + \varepsilon^3\Lambda^{\frac{1}{4}}e^{-\alpha t}$$
$$\lesssim \varepsilon\Lambda^{\frac{1}{4}}(t+1)^{-2}. \tag{5.9}$$

Combining (5.8) and (5.9), one can get that

$$\left\|\phi^\flat, \zeta^\flat\right\|_{L^\infty(\mathbb{R})} \lesssim \left\|\partial_3\big(\phi^\flat, \zeta^\flat\big)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \left\|\phi^\flat, \zeta^\flat\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \lesssim (\varepsilon\Lambda^{\frac{1}{4}})^{\frac{1}{2}}(t+1)^{-\frac{3}{4}},$$

which yields $(3.7)_1$. Using (4.93) and (5.1), one can obtain $(3.7)_2$ immediately. The proof of Theorem 3.1 is finished.

Proof of Theorem 1.2. Once Theorem 3.1 is proved, it remains to prove (1.25) to complete the proof of Theorem 1.2. In fact, using (2.31), (2.32) and (3.7), one has that

$$\left\|(\rho^\flat, \mathbf{u}^\flat) - (\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})\right\|_{L^\infty(\mathbb{R})} \lesssim \left\|(\tilde\rho^\flat, \tilde{\mathbf{u}}^\flat) - (\rho^{\mathsf{vs}}, \mathbf{u}^{\mathsf{vs}})\right\|_{L^\infty(\mathbb{R})} + \left\|(\phi^\flat, \zeta^\flat)\right\|_{L^\infty(\mathbb{R})}$$
$$\lesssim (\varepsilon\Lambda^{\frac{1}{4}})^{\frac{1}{2}}(t+\Lambda)^{-\frac{3}{4}}.$$

To prove $(1.25)_2$, note that $(\rho^\sharp, \mathbf{u}^\sharp) = (\tilde\rho^\sharp, \tilde{\mathbf{u}}^\sharp) + (\phi^\sharp, \zeta^\sharp)$. Note that if $h(x) \in L^\infty(\mathbb{R}^3)$ is periodic on $\mathbb{T}^3$ with the average $\bar{h} = \int_{\mathbb{T}^3} h(x)dx$, then it holds that $h^\sharp = (h - \bar{h})^\sharp$. Then it follows from (2.7) and (2.9) that

$$\tilde\rho^\sharp = \frac{1}{2}\big[v^\sharp_-(1 - \Theta_\sigma) + v^\sharp_+(1 + \Theta_\sigma)\big],$$
$$\tilde{\mathbf{u}}^\sharp = \frac{1}{2}\big[\mathbf{z}^\sharp_-(1 - \Theta_\sigma) + \mathbf{z}^\sharp_+(1 + \Theta_\sigma)\big]$$
$$+ \Big[\frac{1}{\bar\rho}(v_+ - v_-)(\mathbf{u}_+ - \mathbf{u}_-)\Big]^\sharp (1 - \Theta_\sigma^2),$$

where $v_\pm$ and $\mathbf{z}_\pm$ defined in (2.4). Thus, it follows from (2.5) that

$$\left\|\tilde\rho^\sharp, \tilde{\mathbf{u}}^\sharp\right\|_{L^\infty(\Omega)} \lesssim \|v_-, v_+, \mathbf{z}_-, \mathbf{z}_+\|_{L^\infty(\Omega)} \lesssim \varepsilon e^{-\alpha t}. \tag{5.10}$$

This, together with $(3.7)_2$, yields that

$$\left\|(\rho^\sharp, \mathbf{u}^\sharp)\right\|_{L^\infty(\Omega)} \lesssim \left\|(\tilde\rho^\sharp, \tilde{\mathbf{u}}^\sharp)\right\|_{L^\infty(\Omega)} + \left\|(\phi^\sharp, \zeta^\sharp)\right\|_{L^\infty(\Omega)} \lesssim \varepsilon\Lambda^{\frac{1}{4}}e^{-\alpha_0 t}.$$

The proof of Theorem 1.2 is finished.

6 Localized perturbations on $\mathbb{T}^2 \times \mathbb{R}$

In this section, we outline the main ideas to solve the problem (1.1) with the initial data (1.23). In order to use the anti-derivative technique, we first follow [22, 15] to introduce some *diffusion waves* propagating along the transverse characteristics to carry the excessive mass of the perturbation $(\bar{\phi}_0, \bar{\psi}_0)$.

As the viscosity in (1.9) vanishes, the resulting hyperbolic system takes the form,

$$\partial_t \begin{pmatrix} \rho \\ \mathbf{m} \end{pmatrix} + A(\rho, \mathbf{u}) \partial_3 \begin{pmatrix} \rho \\ \mathbf{m} \end{pmatrix} = 0, \tag{6.1}$$

where

$$A(\rho, \mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -u_1 u_3 & u_3 & 0 & u_1 \\ -u_2 u_3 & 0 & u_3 & u_2 \\ p'(\rho) - u_3^2 & 0 & 0 & 2u_3 \end{pmatrix}. \tag{6.2}$$

This matrix has four real eigenvalues,

$$\lambda_0 = u_3 - \sqrt{p'(\rho)}, \quad \lambda_1 = \lambda_2 = u_3, \quad \lambda_3 = u_3 + \sqrt{p'(\rho)}, \tag{6.3}$$

with the associated right eigenvectors,

$$\mathbf{r}_0(\rho, \mathbf{u}) = \begin{pmatrix} 1 \\ u_1 \\ u_2 \\ \lambda_0 \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_3(\rho, \mathbf{u}) = \begin{pmatrix} 1 \\ u_1 \\ u_2 \\ \lambda_3 \end{pmatrix}. \tag{6.4}$$

For the constant states $(\bar{\rho}, \pm\bar{\mathbf{u}})$ in (1.7), we denote

$$\lambda_i^\pm := \lambda_i(\bar{\rho}, \pm\bar{\mathbf{u}}) \quad \text{and} \quad \mathbf{r}_i^\pm := \mathbf{r}_i(\bar{\rho}, \pm\bar{\mathbf{u}}), \quad i = 0, 3.$$

Diffusion waves. Define $\vartheta(x_3, t) = \frac{1}{2\sqrt{\pi(t+1)}} \exp\left\{-\frac{x_3^2}{4(t+1)}\right\}$ and

$$\begin{aligned} \vartheta_-(x_3, t) &:= \vartheta\big(x_3 - \lambda_0^-(t+1), t\big), \\ \vartheta_+(x_3, t) &:= \vartheta\big(x_3 - \lambda_3^+(t+1), t\big). \end{aligned} \tag{6.5}$$

Then it holds that

$$\partial_t \vartheta_- + \lambda_0^- \partial_3 \vartheta_- = \partial_3^2 \vartheta_-, \qquad \partial_t \vartheta_+ + \lambda_3^+ \partial_3 \vartheta_+ = \partial_3^2 \vartheta_+,$$

and

$$\int_{\mathbb{R}} \vartheta_-(x_3, t) dx_3 = \int_{\mathbb{R}} \vartheta_+(x_3, t) dx_3 = 1 \qquad \forall t \geqslant 0.$$

Denote

$$\epsilon := \left\|\bar{\phi}_0, \bar{\psi}_0\right\|_{L_3^2(\Omega)} + \left\|\nabla\bar{\phi}_0, \nabla\bar{\psi}_0\right\|_{H^2(\Omega)}. \tag{6.6}$$

Decompose the mass, $\int_\Omega (\bar{\phi}_0, \bar{\psi}_0) dx$, into

$$\int_\Omega (\bar{\phi}_0, \bar{\psi}_0) dx = \alpha_0 \mathbf{r}_0^- + \alpha_1 \mathbf{r}_1 + \alpha_2 \mathbf{r}_2 + \alpha_3 \mathbf{r}_3^+, \tag{6.7}$$

where each $\alpha_i \in \mathbb{R}$ is a constant, satisfying that

$$\max_{i=0,\cdots,3} |\alpha_i| \lesssim \|\bar{\phi}_0, \bar{\psi}_0\|_{L^1(\Omega)} \lesssim \|\bar{\phi}_0, \bar{\psi}_0\|_{L^2_3(\Omega)} \lesssim \epsilon.$$

Ansatz. In the case for the $H^3(\Omega)$-perturbations, we set the ansatz as

$$\begin{aligned}(\tilde{\rho}, \tilde{\mathbf{m}})(x_3, t) &:= (\rho^{\mathsf{vs}}, \mathbf{m}^{\mathsf{vs}})(x_3, t) + \alpha_0 \mathbf{r}_0^- \vartheta_-(x_3, t) + \alpha_3 \mathbf{r}_3^+ \vartheta_+(x_3, t) \\ &\quad + (\alpha_1 \mathbf{r}_1 + \alpha_2 \mathbf{r}_2)\vartheta(x_3, t),\end{aligned} \quad (6.8)$$

and

$$\tilde{\mathbf{u}}(x_3, t) := \frac{\tilde{\mathbf{m}}(x_3, t)}{\tilde{\rho}(x_3, t)}.$$

Equivalently, (6.8) reduces to

$$\begin{aligned}\tilde{\rho} &= \bar{\rho} + \alpha_0 \vartheta_- + \alpha_3 \vartheta_+, \\ \tilde{m}_i &= \bar{\rho}\bar{u}_i\theta + \bar{u}_i(\alpha_3 \vartheta_+ - \alpha_0 \vartheta_-) + \alpha_i \vartheta, \qquad i = 1, 2, \\ \tilde{m}_3 &= \lambda_0^- \alpha_0 \vartheta_- + \lambda_3^+ \alpha_3 \vartheta_+.\end{aligned} \quad (6.9)$$

Remark 6.1. The construction (6.8) is different from [15], since we do not shift the background vortex sheet to cancel the excessive mass in the linearly degenerate characteristic fields. In fact, we find that the shift is not essential for the contact discontinuities, since after shifting, the difference between two contact discontinuities decays just same as the diffusion waves. However, it should be mentioned that the shifts are an essential point in the stability theory of shocks.

It follows from (6.9) that

$$\begin{cases} \partial_t \tilde{\rho} + \operatorname{div} \tilde{\mathbf{m}} = \partial_3 F, \\ \partial_t \tilde{\mathbf{m}} + \operatorname{div}\left(\frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}}\right) + \nabla p(\tilde{\rho}) = \mu \triangle \tilde{\mathbf{u}} + (\mu + \lambda)\nabla \operatorname{div}\tilde{\mathbf{u}} + \partial_3 \mathbf{G}, \end{cases} \quad (6.10)$$

where $F = F(x_3, t) \in \mathbb{R}$ and $\mathbf{G} = \mathbf{G}(x_3, t) = (G_1, G_2, G_3)(x_3, t) \in \mathbb{R}^3$ are given by

$$\begin{aligned}F &= \alpha_0 \partial_3 \vartheta_- + \alpha_3 \partial_3 \vartheta_+, \\ G_i &= \alpha_i \partial_3 \vartheta + \left[\frac{\tilde{m}_3 \tilde{m}_i}{\tilde{\rho}} - \bar{u}_i(\alpha_3 \lambda_3^+ \vartheta_+ - \alpha_0 \lambda_0^- \vartheta_-)\right] \\ &\quad + \bar{u}_i(\alpha_3 \partial_3 \vartheta_+ - \alpha_0 \partial_3 \vartheta_-) - \mu \partial_3(\tilde{u}_i - \bar{u}_i \theta), \qquad i = 1, 2, \\ G_3 &= \frac{\tilde{m}_3^2}{\tilde{\rho}} + \left[p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})(\tilde{\rho} - \bar{\rho})\right] - \tilde{\mu}\partial_3 \tilde{u}_3 \\ &\quad + \lambda_0^- \alpha_0 \partial_3 \vartheta_- + \lambda_3^+ \alpha_3 \partial_3 \vartheta_+.\end{aligned} \quad (6.11)$$

In the following part of this section, we let

$$\mathfrak{E} = e^{-\frac{c_0|x_3|^2}{1+t}} + e^{-\frac{c_0\left|x_3 - \lambda_0^-(t+1)\right|^2}{1+t}} + e^{-\frac{c_0\left|x_3 - \lambda_3^+(t+1)\right|^2}{1+t}} \quad (6.12)$$

for some generic constant $c_0 > 0$.

Lemma 6.2. *For $j = 0, 1, 2, \cdots$, it holds that*

$$\left|\partial_3^j F\right| + \left|\partial_3^j \mathbf{G}\right| \lesssim \epsilon(t+1)^{-1-\frac{j}{2}} \mathfrak{E}. \quad (6.13)$$

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Proof. We prove (6.13) for $j = 0$ only, since the case for higher orders is similar. For convenience, we write $A \cong B$ if $|A - B| \lesssim \epsilon(t+1)^{-1}\mathfrak{E}$. First, it is direct to show that $|\partial_3 \vartheta| + |\vartheta|^2 \cong 0$, which yields that $|F| + |G_3| \cong 0$. To prove that $G_i \cong 0$ for $i = 1, 2$, it suffices to show that

$$I := \frac{\tilde{m}_3 \tilde{m}_i}{\tilde{\rho}} - \bar{u}_i \left(\alpha_3 \lambda_3^+ \vartheta_+ - \alpha_0 \lambda_0^- \vartheta_- \right) \cong 0.$$

In fact, it follows from (6.9) that

$$I \cong \bar{u}_i \theta \left(\lambda_0^- \alpha_0 \vartheta_- + \lambda_3^+ \alpha_3 \vartheta_+ \right) + \bar{u}_i \left(\alpha_0 \lambda_0^- \vartheta_- - \alpha_3 \lambda_3^+ \vartheta_+ \right)$$
$$= \bar{u}_i \left[\alpha_0 \lambda_0^- \vartheta_- (\theta + 1) + \alpha_3 \lambda_3^+ \vartheta_+ (\theta - 1) \right].$$

Note that $\lambda_0^- = -\sqrt{p'(\bar{\rho})} < 0$. Thus, if $x_3 \geqslant 0$, then

$$\vartheta_-(\theta + 1) \lesssim \vartheta_- \lesssim e^{-\frac{\left| \lambda_0^- \right|^2 t}{8}} e^{-\frac{\left| x_3 - \lambda_0^-(t+1) \right|^2}{8(t+1)}}.$$

If $x_3 < 0$, then by (4.9) and the fact that $\int_{-\infty}^{-|A|} e^{-y^2} dy \lesssim e^{-\frac{A^2}{2}}$, one has that

$$0 < \vartheta_-(\theta + 1) = \vartheta_- \int_{-\infty}^{x_3} \partial_{y_3} \theta(y_3, t) dy_3$$
$$\lesssim \vartheta_- \int_{-\infty}^{x_3} \vartheta(y_3, t) dy_3$$
$$\lesssim e^{-\frac{\left| x_3 - \lambda_0^-(t+1) \right|^2}{4(t+1)}} e^{-\frac{\alpha x_3^2}{2(t+1)}}$$
$$\lesssim e^{-c_1 |x_3| - c_1 t},$$

for some $c_1 > 0$ which depends only on $\left| \lambda_0^- \right|$ and α. Similarly, one can prove that

$$0 < \vartheta_+(1 - \theta) \lesssim e^{-c_0 t} \left(e^{-\frac{c_1 |x_3|^2}{1+t}} + e^{-\frac{c_1 \left| x_3 - \lambda_3^+(t+1) \right|^2}{1+t}} \right), \qquad x_3 \in \mathbb{R}, t \geqslant 0.$$

$\square$

It follows from (1.1) and (6.10) that $\frac{d}{dt} \left\{ \int_\Omega [(\rho, \mathbf{m}) - (\tilde{\rho}, \tilde{\mathbf{m}})] dx \right\} = 0$. This, together with (6.7), yields that

$$\int_\Omega [(\rho, \mathbf{m}) - (\tilde{\rho}, \tilde{\mathbf{m}})] dx \equiv \int_\Omega [(\rho, \mathbf{m}) - (\tilde{\rho}, \tilde{\mathbf{m}})]|_{t=0} dx = 0. \qquad (6.14)$$

Using the same notations (3.1), (3.10), (3.16) and (4.10), the systems for (ϕ, ζ) and (Φ, Ψ) are almost same as (3.5) and (3.12), respectively, except the source terms on the right-hand sides, in which the new ones satisfy (6.13) instead of (3.17). Denote

$$(\phi_0, \psi_0)(x) := (\phi, \psi)(x, 0)$$
$$= (\bar{\phi}_0, \bar{\psi}_0)(x) + \alpha_0 \mathbf{r}_0^- \vartheta_-(x_3, 0) + \alpha_3 \mathbf{r}_3^+ \vartheta_+(x_3, 0) + (\alpha_1 \mathbf{r}_1 + \alpha_2 \mathbf{r}_2) \vartheta(x_3, 0),$$
$$(\Phi_0, \Psi_0)(x_3) := (\Phi, \Psi)(x_3, 0)$$
$$= \int_{-\infty}^{x_3} (\phi_0^\flat, \psi_0^\flat)(y_3) dy_3 = -\int_{x_3}^{+\infty} (\phi_0^\flat, \psi_0^\flat)(y_3) dy_3.$$

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Then it follows from (1.27) that $(\phi_0, \psi_0) \in H^3(\Omega)$ and $(\Phi_0, \Psi_0) \in L^2(\mathbb{R})$. In fact,

$$
\begin{aligned}
\|\Phi_0\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^0 \Big(\int_{-\infty}^{x_3} \phi_0^\flat(y_3) dy_3 \Big)^2 dx_3 + \int_0^{+\infty} \Big(\int_{x_3}^{+\infty} \phi_0^\flat(y_3) dy_3 \Big)^2 dx_3 \\
&\lesssim \|\bar\phi_0\|_{L_3^2(\Omega)}^2 \int_0^{+\infty} \int_{x_3}^{+\infty} (y_3^2 + 1)^{-\frac{3}{2}} dy_3 dx_3 + \epsilon \\
&\lesssim \epsilon.
\end{aligned}
\tag{6.15}
$$

And the L^2-estimate of Ψ_0 is similar.

Now we establish the a priori estimates. First, we assume a priorily that for a fixed $T > 0$,

$$
(\Phi, \Psi) \in C(0, T; L^2(\mathbb{R})) \quad \text{and} \quad (\phi, \zeta) \in C(0, T; H^3(\Omega)).
$$

Denote

$$
\nu := \sup_{t \in (0,T)} \big(\|\Phi, \Psi\|_{L^\infty(\mathbb{R})} + \|\phi, \psi\|_{H^3(\Omega)} \big).
\tag{6.16}
$$

Remark 6.3. The definition of ν here is different from (4.1). In fact, same as in [15], due to the diffusion waves, the L^2-norm of the anti-derivatives, $\|\Phi, \Psi\|_{L^2(\mathbb{R})}$, grows actually at rates $(t+1)^{\frac{1}{4}}$. Nevertheless, we will show that the L^2-norm of their derivatives, $\|\partial_3 \Phi, \partial_3 \Psi\|_{L^2(\mathbb{R})}$, decays at rates $(t+1)^{-\frac{1}{4}}$, which ensures the uniform boundedness of the L^∞-norm, $\|\Phi, \Psi\|_{L^\infty(\mathbb{R})}$.

It is noted that in the anti-derivative system, the estimates of $\mathbf{Q}_i$ for $i = 1, 2, 3, 4$ are slightly different from (4.15), which read

$$
\begin{aligned}
|\mathbf{Q}_1| + |\mathbf{Q}_3| &\lesssim |\phi|^2 + |\psi|^2, \\
|\partial_3 \mathbf{Q}_1| + |\partial_3 \mathbf{Q}_3| &\lesssim \big[\epsilon(t+1)^{-1} + (t+\Lambda)^{-\frac{1}{2}} \big] \big(|\phi|^2 + |\psi_3|^2 \big) \\
&\quad + \big(|\partial_3 \phi| + |\partial_3 \psi| \big) \big(|\phi| + |\psi| \big), \\
|\nabla^k \mathbf{Q}_2| + |\nabla^k \mathbf{Q}_4| &\lesssim \sum_{j=0}^k \epsilon(t+1)^{-\frac{k-j+1}{2}} \big(|\nabla^j \phi| + |\nabla^j \psi| \big), \qquad k = 0, 1.
\end{aligned}
\tag{6.17}
$$

In addition, the estimates (4.16) to (4.20) still hold true with $\varepsilon = 0$.

Lemma 6.4. If Λ^{-1}, ϵ and ν are small enough, then

$$
\frac{d}{dt} \mathcal{A}_{0,\flat} + c_{0,\flat} \mathcal{B}_{0,\flat} \lesssim \nu \|\nabla \phi^\sharp, \nabla \zeta^\sharp\|_{L^2(\Omega)}^2 + \epsilon(t+1)^{-1} \|\Phi, \mathbf{Z}\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{1}{2}},
\tag{6.18}
$$

$$
\begin{aligned}
\frac{d}{dt} \big(\mathcal{A}_{1,\flat} &+ \frac{\tilde\mu}{4\bar\rho} \mathcal{A}_{2,\flat} \big) + c_{1,\flat} \mathcal{B}_{1,\flat} + \frac{\tilde\mu}{4\bar\rho} c_{2,\flat} \mathcal{B}_{2,\flat} \\
&\lesssim (\epsilon + \nu) \big(\|\partial_3^3 \mathbf{Z}\|_{L^2(\mathbb{R})}^2 + \|\nabla \zeta^\sharp\|_{L^2(\Omega)}^2 + \|\nabla^2 \phi\|_{H^1(\Omega)}^2 \big) \\
&\quad + (t+1)^{-1} \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{3}{2}},
\end{aligned}
\tag{6.19}
$$

$$
\begin{aligned}
\frac{d}{dt} \mathcal{A}_{3,\flat} &+ c_{3,\flat} \mathcal{B}_{3,\flat} \\
&\lesssim (\epsilon + \nu) \big(\|\nabla^2 \phi\|_{H^1(\Omega)}^2 + \|\nabla^3 \zeta\|_{L^2(\Omega)}^2 \big) + (t+1)^{-1} \|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\|_{L^2(\mathbb{R})}^2 \\
&\quad + (t+1)^{-2} \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \epsilon(1+t)^{-\frac{5}{2}},
\end{aligned}
\tag{6.20}
$$

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where $\mathcal{A}_{i,\flat}$ and $\mathcal{B}_{i,\flat}$ for $i = 0, \cdots, 3$, denote the same notations as in Section 4.1, satisfying that

$$\mathcal{A}_{0,\flat} \sim \|\Phi\|_{H^1(\mathbb{R})}^2 + \|\mathbf{Z}\|_{L^2(\mathbb{R})}^2, \quad \mathcal{B}_{0,\flat} \sim \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa, \mathbf{Z}\kappa\|_{L^2(\mathbb{R})}^2,$$

$$\mathcal{A}_{1,\flat} + \frac{\tilde{\mu}}{4\bar{\rho}} \mathcal{A}_{2,\flat} \sim \|\partial_3 \Phi\|_{H^1(\mathbb{R})}^2 + \|\partial_3 \mathbf{Z}\|_{L^2(\mathbb{R})}^2, \quad \mathcal{B}_{1,\flat} + \frac{\tilde{\mu}}{4\bar{\rho}} \mathcal{B}_{2,\flat} \sim \left\|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\right\|_{L^2(\mathbb{R})}^2,$$

$$\mathcal{A}_{3,\flat} \sim \left\|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\right\|_{L^2(\mathbb{R})}^2, \quad \mathcal{B}_{3,\flat} \sim \left\|\partial_3^3 \mathbf{Z}\right\|_{L^2(\mathbb{R})}^2.$$

Proof. The proof is almost same as those of Lemmas 4.3–4.8. Indeed, the only different ingredients are the estimates related to the terms $(F, \mathbf{G})$ and $(\mathbf{Q}_2, \mathbf{Q}_4)$.

For instance, in (4.24) and (4.25), we use the different estimates,

$$\|F, \mathbf{G}\|_{L^2(\mathbb{R})} \|\Phi, \mathbf{Z}\|_{L^2(\mathbb{R})} \lesssim \epsilon(t+1)^{-\frac{3}{4}} \|\Phi, \mathbf{Z}\|_{L^2(\mathbb{R})}$$
$$\lesssim \epsilon(t+1)^{-1} \|\Phi, \mathbf{Z}\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{1}{2}},$$

and

$$\|\mathbf{Q}_2, \mathbf{Q}_4\|_{L^2(\Omega)} \|\mathbf{Z}\|_{H^1(\mathbb{R})} \lesssim \epsilon(t+1)^{-1} \|\mathbf{Z}\|_{H^1(\mathbb{R})}^2 + \epsilon \|\phi, \psi\|_{L^2(\Omega)}^2$$
$$\lesssim \epsilon(t+1)^{-1} \|\mathbf{Z}\|_{L^2(\mathbb{R})}^2 + \epsilon \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \epsilon \left\|\nabla\phi^\sharp, \nabla\zeta^\sharp\right\|_{L^2(\Omega)}^2;$$

and in the estimate of (4.65), we use the different estimates,

$$\left\|\partial_3^2 F, \partial_3^2 \mathbf{G}\right\|_{L^2(\mathbb{R})} \left\|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\right\|_{L^2(\mathbb{R})} \lesssim \epsilon(t+1)^{-1} \left\|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{5}{2}},$$

$$\left\|\partial_3 \mathbf{Q}_2, \partial_3 \mathbf{Q}_4\right\|_{L^2(\Omega)} \left\|\partial_3^3 \mathbf{Z}\right\|_{H^1(\mathbb{R})} \lesssim \epsilon \left\|\partial_3^3 \mathbf{Z}\right\|_{H^1(\mathbb{R})}^2 + \epsilon(t+1)^{-2} \|\phi, \psi\|_{L^2(\Omega)}^2$$
$$+ \epsilon(t+1)^{-1} \|\nabla\phi, \nabla\psi\|_{L^2(\Omega)}^2.$$

Note that except these estimates, we do not use the boundedness of $\|\Phi, \mathbf{\Psi}\|_{L^2(\mathbb{R})}$ elsewhere, that is, the remaining proof are exactly the same as in Section 4.1. $\qquad\square$

Lemma 6.5. *If Λ^{-1}, ϵ and ν are small enough, then*

$$\frac{d}{dt}\mathcal{A}_{\neq} + \mathcal{B}_{\neq} \leqslant 0, \tag{6.21}$$

where $\mathcal{A}_\sharp$ and $\mathcal{B}_\sharp$ are two energy functionals, satisfying that

$$\mathcal{A}_{\neq} \sim \left\|\phi^\sharp, \zeta^\sharp\right\|_{H^1(\Omega)}^2, \qquad \mathcal{B}_{\neq} \sim \left\|\nabla\phi^\sharp\right\|_{L^2(\Omega)}^2 + \left\|\nabla\zeta^\sharp\right\|_{H^1(\Omega)}^2.$$

Proof. Note that the ansatz (6.8) depends only on x_3, and so are the error terms F and $\mathbf{G}$, which implies that $(\partial_3 F)^\sharp = 0$ and $(\partial_3 \mathbf{G})^\sharp = 0$. Then the proof is similar as in Section 4.2. $\qquad\square$

Lemma 6.6. *If Λ^{-1}, ϵ and ν are small enough, then*

$$\frac{d}{dt}\mathcal{A} + c\mathcal{B} \lesssim \left\|\partial_3^3 \mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \left\|\nabla^2 \zeta^\sharp\right\|_{L^2(\Omega)}^2 + (t+1)^{-1} \left\|\partial_3^2 \Phi, \partial_3^2 \mathbf{Z}\right\|_{L^2(\Omega)}^2$$
$$+ (t+1)^{-2} \|\partial_3 \Phi, \partial_3 \mathbf{Z}, \Phi\kappa\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{7}{2}}, \tag{6.22}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two energy functionals, satisfying that

$$\pm\mathcal{A} \lesssim \pm\left\|\nabla^2 \phi, \nabla^2 \zeta\right\|_{H^1(\Omega)}^2 + \|\nabla\zeta\|_{L^2(\Omega)}^2, \qquad \mathcal{B} \sim \left\|\nabla^2 \phi, \nabla^3 \zeta\right\|_{H^1(\Omega)}^2. \tag{6.23}$$

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Proof. Note that

$$\left|\partial_3^j \tilde\rho\right| + \left|\partial_3^j \tilde m_3\right| + \left|\partial_3^j (\tilde{\mathbf{m}}_\perp - \mathbf{m}_\perp^{\mathsf{vs}})\right| \lesssim \epsilon(t+1)^{-\frac{j+1}{2}}, \qquad j = 1, 2, \cdots.$$

The proof is similar to Section 4.3. □

Now we show the decay rate. First, it follows from Lemma 6.5 that

$$\left\|\phi^\sharp, \zeta^\sharp\right\|_{H^1(\Omega)} \lesssim \epsilon e^{-\alpha_1 t}. \tag{6.24}$$

This, together with (6.18), yields that

$$\frac{d}{dt}\mathcal{A}_{0,\flat} + c_\flat \mathcal{B}_{0,\flat} \leqslant C_0 \epsilon(t+1)^{-1}\mathcal{A}_{0,\flat} + C_0\epsilon(t+1)^{-\frac{1}{2}}. \tag{6.25}$$

By Gronwall's inequality, multiplying $(t+1)^{-C_0\epsilon}$ on (6.25) yields that if ϵ is suitably small, then

$$\mathcal{A}_{0,\flat} + \int_0^t \mathcal{B}_{0,\flat}d\tau \lesssim \epsilon(t+1)^{\frac{1}{2}}. \tag{6.26}$$

Similar to (4.115), one can choose a linear combination of $\mathcal{A}_{i,\flat}$ for $i = 1, 2, 3$, $\mathcal{A}_\sharp$ and $\mathcal{A}$ (resp. $\mathcal{B}_{i,\flat}$ for $i = 1, 2, 3$, $\mathcal{B}_\sharp$ and $\mathcal{B}$), denoted by $\mathcal{E}_1$ (resp. $\mathcal{D}_1$), such that

$$\frac{d}{dt}\mathcal{E}_1 + \mathcal{D}_1 \lesssim (t+1)^{-1}\left\|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa\right\|_{L^2(\mathbb{R})}^2 + \epsilon(t+1)^{-\frac{3}{2}}. \tag{6.27}$$

and

$$\pm\mathcal{E}_1 \lesssim \pm\left(\|\phi\|_{H^3(\Omega)}^2 + \|\partial_3\mathbf{Z}\|_{H^1(\mathbb{R})}^2 + \left\|\zeta^\sharp\right\|_{H^1(\Omega)}^2 + \left\|\nabla^2\zeta\right\|_{H^1(\Omega)}^2\right)$$
$$+ (t+\Lambda)^{-1}\left\|\Phi\kappa\right\|_{L^2(\mathbb{R})}^2, \tag{6.28}$$
$$\mathcal{D}_1 \sim \|\nabla\phi\|_{H^2(\Omega)}^2 + \left\|\partial_3^2\mathbf{Z}\right\|_{H^1(\mathbb{R})}^2 + \left\|\nabla\zeta^\sharp\right\|_{H^1(\Omega)}^2 + \left\|\nabla^3\zeta\right\|_{H^1(\Omega)}^2.$$

Then it holds that

$$(t+1)\mathcal{E}_1 + \int_0^t (\tau+1)\mathcal{D}_1 d\tau \lesssim \mathcal{E}_1(0) + \int_0^t \mathcal{B}_{0,\flat}d\tau + \epsilon(t+1)^{\frac{1}{2}}, \tag{6.29}$$

which, together with (6.26), yields that

$$\mathcal{E}_1 \lesssim \epsilon(t+1)^{-\frac{1}{2}}, \qquad \int_0^t (\tau+1)\mathcal{D}_1 d\tau \lesssim \epsilon(t+1)^{\frac{1}{2}}. \tag{6.30}$$

Also, similar to (5.5) and (5.6), one can choose a linear combination of $\mathcal{A}_{3,\flat}$ $\mathcal{A}_\sharp$ and $\mathcal{A}$ (resp. $\mathcal{B}_{3,\flat}$, $\mathcal{B}_\sharp$ and $\mathcal{B}$), denoted by $\mathcal{E}_2$ (resp. $\mathcal{D}_2$), such that

$$\frac{d}{dt}\mathcal{E}_2 + \mathcal{D}_2 \lesssim (t+1)^{-1}\left\|\partial_3^2\Phi, \partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2$$
$$+ (t+1)^{-2}\left\|\partial_3\Phi, \partial_3\mathbf{Z}, \Phi\kappa\right\|_{L^2(\mathbb{R})}^2 + \epsilon(1+t)^{-\frac{5}{2}}, \tag{6.31}$$

and

$$\pm\mathcal{E}_2 \lesssim \pm\left(\left\|\partial_3^2\Phi, \partial_3^2\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \left\|\nabla^2\phi, \nabla^2\zeta\right\|_{H^1(\Omega)}^2\right) + (t+\Lambda)^{-1}\left\|\partial_3\Phi, \Phi\kappa\right\|_{L^2(\mathbb{R})}^2,$$
$$\mathcal{D}_2 \sim \left\|\partial_3^3\mathbf{Z}\right\|_{L^2(\mathbb{R})}^2 + \left\|\nabla^2\phi, \nabla^3\zeta\right\|_{H^1(\Omega)}^2. \tag{6.32}$$

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Note that $\mathcal{E}_2 \lesssim \mathcal{D}_1 + (t + \Lambda)^{-2} \|\partial_3\Phi, \Phi\kappa\|^2_{L^2(\mathbb{R})}$. Then one has

$$
\begin{aligned}
(t+1)^2\mathcal{E}_2 &+ \int_0^t (\tau+1)^2 \mathcal{D}_2 d\tau \\
&\lesssim \int_0^t (\tau+1)\mathcal{D}_1 d\tau + \int_0^t \mathcal{B}_{0,\flat} d\tau + \epsilon(t+1)^{\frac{1}{2}} \\
&\lesssim \epsilon(t+1)^{\frac{1}{2}},
\end{aligned}
\tag{6.33}
$$

which yields that

$$
\mathcal{E}_2 \lesssim \epsilon(t+1)^{-\frac{3}{2}}.
\tag{6.34}
$$

Then combining (6.26) and (6.30), one has that

$$
\|\Phi, \boldsymbol{\Psi}\|_{L^\infty(\mathbb{R})} \lesssim \|\partial_3\Phi, \partial_3\boldsymbol{\Psi}\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|\Phi, \boldsymbol{\Psi}\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \lesssim \mathcal{E}_1^{\frac{1}{4}} \mathcal{A}_{0,\flat}^{\frac{1}{4}} \lesssim 1.
$$

Using (4.16) and (4.17) with $\varepsilon = 0$ yields that

$$
\left\|\phi^\flat, \zeta^\flat\right\|^2_{L^2(\mathbb{R})} \lesssim \mathcal{E}_1 + (t+\Lambda)^{-1}\mathcal{A}_{0,\flat} \lesssim \epsilon(t+1)^{-\frac{1}{2}},
$$

and

$$
\left\|\partial_3\phi^\flat, \partial_3\zeta^\flat\right\|^2_{L^2(\mathbb{R})} \lesssim \mathcal{E}_2 + (t+1)^{-1}\mathcal{E}_1 + (t+1)^{-2}\mathcal{A}_{0,\flat} \lesssim \epsilon(t+1)^{-\frac{3}{2}}.
$$

Finally, one has that

$$
\left\|\phi^\flat, \zeta^\flat\right\|_{L^\infty(\mathbb{R})} \lesssim \left\|\partial_3(\phi^\flat, \zeta^\flat)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \left\|\phi^\flat, \zeta^\flat\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{1}{2}}(t+1)^{-\frac{1}{2}}.
\tag{6.35}
$$

Note that the diffusion waves propagating along the transverse characteristics, i.e. $\vartheta, \vartheta_\pm$ in (6.8), also decay at the rate $(t+1)^{-\frac{1}{2}}$. Hence, one can obtain $(1.28)_1$ immediately. The proof of Theorem 1.3 is complete.

A Proof of Lemma 2.3

Proof of Lemma 2.3. If ε is small, then it follows from Lemma 2.1 that the periodic solutions $(\rho_\pm, \mathbf{m}_\pm)$ to the problem (1.1), (2.2) belong to the $C(0, +\infty; W^{4,\infty}(\mathbb{T}^3))$ space. Then the existence and uniqueness of (2.23) can be derived from the Cauchy-Lipschitz theorem. It remains to show (2.28). In fact, using (2.5) and Cauchy's inequality, one has that $\rho^\flat_- = \bar{\rho} + O(1)\varepsilon e^{-\alpha t}$ and $m^\flat_{3-} = O(1)\varepsilon e^{-\alpha t}$. Then it follows from (2.22) that

$$
\mathfrak{D} = 2\bar{\rho} + O(1)\varepsilon e^{-\alpha t},
\tag{A.1}
$$

and

$$
\mathfrak{N} = -\frac{\bar{\rho}}{2\sqrt{t+\Lambda}} \int_\mathbb{R} \frac{x_3 - \sigma}{t+\Lambda} \Theta'\left(\frac{x_3 - \sigma}{\sqrt{t+\Lambda}}\right) dx_3 + O(1)\varepsilon e^{-\alpha t} = O(1)\varepsilon e^{-\alpha t}.
\tag{A.2}
$$

Here in (A.2) we have used the fact that $\xi\Theta'(\xi)$ is odd with respect to ξ (see Lemma 1.1). Thus, it holds that $|\sigma'(t)| \lesssim \varepsilon e^{-\alpha t}$.

Now we compute the limit of $\sigma(t)$ as $t \to +\infty$. For $i = 1$ or 2, it follows from $(2.10)_2$ that

$$\partial_t \tilde{m}_i^\flat + \partial_3 \big[(\tilde{u}_3 \tilde{m}_i)^\flat + (p(\tilde{\rho}))^\flat \big] = \mu \partial_3^2 \tilde{u}_i^\flat + g_i^\flat. \tag{A.3}$$

For any fixed $t > 0$, integer $N > 0$ and constant $a \in \mathbb{T} = (0,1)$, define the bounded domain

$$\Omega_{a,N}(t) := \{ (x_3, \tau) : \sigma(\tau) - N + a < x_3 < \sigma(\tau) + N + a, \ 0 < \tau < t \}.$$

Integrating (A.3) over $B_{a,N}(t)$ yields that

$$
\begin{aligned}
\int_{\Omega_{a,N}(t)} g_i^\flat(x_3, \tau)\, dx_3 d\tau &= \underbrace{\int_{\sigma(t)-N+a}^{\sigma(t)+N+a} \tilde{m}_i^\flat(x_3, t) dx_3 - \int_{-N+a}^{N+a} \tilde{m}_i^\flat(x_3, 0) dx_3}_{I_1} \\
&\quad \underbrace{- \int_0^t \big[(\tilde{u}_3 \tilde{m}_i)^\flat + (p(\tilde{\rho}))^\flat - \mu \partial_3 \tilde{u}_i^\flat - \sigma'(\tau) \tilde{m}_i^\flat \big] (\sigma(\tau) + N + a, \tau) d\tau}_{I_2} \\
&\quad \underbrace{+ \int_0^t \big[(\tilde{u}_3 \tilde{m}_i)^\flat + (p(\tilde{\rho}))^\flat - \mu \partial_3 \tilde{u}_i^\flat - \sigma'(\tau) \tilde{m}_i^\flat \big] (\sigma(\tau) - N + a, \tau) d\tau}_{I_3}.
\end{aligned}
\tag{A.4}
$$

Denote $\xi = \frac{x_3}{\sqrt{t+\Lambda}}$. Then it holds that

$$
\begin{aligned}
I_1 &= \underbrace{\frac{1}{2} \int_{-N+a}^{N+a} \big[w_{i-}^\flat(x_3 + \sigma, t)(1 - \Theta(\xi)) + w_{i+}^\flat(x_3 + \sigma, t)(1 + \Theta(\xi)) \big] dx_3}_{I_{1,1}} \\
&\quad + \underbrace{\bar{m}_i \int_{-N+a}^{N+a} \Theta(\xi) dx_3 - \bar{m}_i \int_{-N+a}^{N+a} \Theta\Big(\frac{x_3}{\sqrt{\Lambda}} \Big) dx_3}_{I_{1,2}}.
\end{aligned}
\tag{A.5}
$$

Since $w_{i\pm}(\cdot, t)$ have zero averages on $\mathbb{T}^3$, then

$$
\begin{aligned}
I_{1,1} = \frac{1}{2} \Big[&\int_a^{N+a} w_{i-}^\flat(x_3 + \sigma, t)(1 - \Theta(\xi)) dx_3 - \int_{-N+a}^a w_{i-}^\flat(x_3 + \sigma, t)(1 + \Theta(\xi)) dx_3 \\
&+ \int_{-N+a}^a w_{i+}^\flat(x_3 + \sigma, t)(1 + \Theta(\xi)) dx_3 - \int_a^{N+a} w_{i+}^\flat(x_3 + \sigma, t)(1 - \Theta(\xi)) dx_3 \Big].
\end{aligned}
$$

By (2.18), one has that $w_{i+}^\flat \equiv w_{i-}^\flat + 2\bar{u}_i v_-^\flat$. Thus, it holds that

$$I_{1,1} = \bar{u}_i \Big[\int_{-N+a}^a v_-^\flat(x_3 + \sigma, t)(1 + \Theta(\xi)) dx_3 - \int_a^{N+a} v_-^\flat(x_3 + \sigma, t)(1 - \Theta(\xi)) dx_3 \Big],$$

which yields that

$$
\begin{aligned}
\lim_{N \to +\infty} \int_0^1 I_{1,1} da = \bar{u}_i \int_0^1 \Big[&\int_{-\infty}^a v_-^\flat(x_3 + \sigma, t)(1 + \Theta(\xi)) dx_3 \\
&- \int_a^{+\infty} v_-^\flat(x_3 + \sigma, t)(1 - \Theta(\xi)) dx_3 \Big] da.
\end{aligned}
\tag{A.6}
$$

On the other hand, since $\Theta(\cdot)$ is an odd function, then it holds that

$$\int_{-N+a}^{N+a} \Theta(\xi)dx_3 = \int_N^{N+a} \Theta(\xi)dx_3 + \int_{-N}^N \Theta(\xi)dx_3 - \int_{-N}^{-N+a} \Theta(\xi)dx_3$$

$$= \int_N^{N+a} \Theta(\xi)dx_3 - \int_{-N}^{-N+a} \Theta(\xi)dx_3.$$

This, together with the fact that $\Theta(\xi) \to \pm 1$ as $\xi \to \pm\infty$ and the dominated convergence theorem, yields that

$$\lim_{N\to+\infty} \int_0^1 \int_{-N+a}^{N+a} \Theta(\xi)dx_3 da = \int_0^1 \left[\lim_{N\to+\infty} \int_N^{N+a} \Theta(\xi)dx_3 - \lim_{N\to+\infty} \int_{-N}^{-N+a} \Theta(\xi)dx_3 \right] da$$

$$= 1.$$

Similarly, one can show that

$$\lim_{N\to+\infty} \int_0^1 \int_{-N+a}^{N+a} \Theta\left(\frac{x_3}{\sqrt{\Lambda}}\right) dx_3 da = 1.$$

Thus, it holds that

$$\lim_{N\to+\infty} \int_0^1 I_{1,2} da = 0. \tag{A.7}$$

For I_2, it holds that

$$I_2 = -\int_0^t \left[(u_{3+}m_{i+})^\flat + (p(\rho_+))^\flat - \mu\partial_3 u_{i+}^\flat - \sigma'(t)m_{i+}^\flat \right](\sigma(\tau) + a, \tau)d\tau$$

$$- \int_0^t I_{2,1}^\flat(\sigma(\tau) + N + a, \tau)d\tau,$$

where $I_{2,1} = \left[\tilde{u}_3\tilde{m}_i - u_{3+}m_{i+} + p(\tilde{\rho}) - p(\rho_+) - \mu\partial_3(\tilde{u}_i - u_{i+}) - \sigma'(\tau)(\tilde{m}_i - m_{i+}) \right]$, which satisfies that $\left| I_{2,1}^\flat(\sigma(\tau) + N + a, \tau) \right| \lesssim 1 - \Theta(\frac{N+a}{\sqrt{\tau+\Lambda}})$. Then one has that

$$\lim_{N\to+\infty} \int_0^1 I_2 da = -\int_0^t \int_0^1 \left[(u_{3+}m_{i+})^\flat + (p(\rho_+))^\flat \right](\sigma(\tau) + a, \tau)da d\tau + \bar{m}_i\sigma(t).$$

Similarly, one can prove that

$$\lim_{N\to+\infty} \int_0^1 I_3 da = \int_0^t \int_0^1 \left[(u_{3-}m_{i-})^\flat + (p(\rho_-))^\flat \right](\sigma(\tau) + a, \tau)da d\tau + \bar{m}_i\sigma(t).$$

Using (2.18) and (2.26), one has that

$$\lim_{N\to+\infty} \int_0^1 (I_2 + I_3) da = -2\bar{u}_i \int_0^t \int_0^1 m_{3-}^\flat(\sigma(\tau) + a, \tau)da d\tau + 2\bar{m}_i\sigma(t) = 2\bar{m}_i\sigma(t). \tag{A.8}$$

Using (2.12) and (2.13), one can get that $g_i^\flat \in L^\infty(0, t; L^1(\mathbb{R}))$. Then it follows from the dominated convergence theorem and (2.24) that

$$\lim_{N\to+\infty} \int_0^1 \int_{B_{a,N}(t)} g_i^\flat(x_3, t)dx_3 dt da = \int_0^t \int_\mathbb{R} g_i^\flat(x_3, t)dx_3 dt = \int_0^t \int_\mathbb{R} f_{2,i}^\flat(x_3, t)dx_3 dt = 0.$$

This, together with (A.6) to (A.8), yields that

$$\sigma(t) = \frac{1}{2\bar{\rho}} \int_0^1 \left[\int_a^{+\infty} v_-^\flat(x_3 + \sigma, t)(1 - \Theta(\xi))dx_3 - \int_{-\infty}^a v_-^\flat(x_3 + \sigma, t)(1 + \Theta(\xi))dx_3 \right] da$$

$$= O(1)\varepsilon\Lambda^{1/2}e^{-\alpha t}.$$

The proof is finished. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

B Proof of Lemma 3.2

Proof of Lemma 3.2. For convenience, we use the convention $A \approx B$ to denote that $\|A - B\|_{L^2(\Omega)} \lesssim \varepsilon \Lambda^{1/4} e^{-\alpha t}$.

It follows from (2.9) that

$$\tilde{\mathbf{u}} \approx \frac{1}{2}\big[\mathbf{u}_-(1 - \Theta_\sigma) + \mathbf{u}_+(1 + \Theta_\sigma)\big], \tag{B.1}$$

which, together with (2.7), yields that

$$\tilde{u}_3\tilde{\mathbf{m}} \approx \frac{1}{4}\big[u_{3-}\mathbf{m}_-(1 - \Theta_\sigma)^2 + u_{3+}\mathbf{m}_+(1 + \Theta_\sigma)^2\big]$$

$$\approx \frac{1}{2}\big[u_{3-}\mathbf{m}_-(1 - \Theta_\sigma) + u_{3+}\mathbf{m}_+(1 + \Theta_\sigma)\big]. \tag{B.2}$$

Similarly, one can verify that

$$p(\tilde{\rho}) - \frac{1}{2}p(\rho_-)(1 - \Theta_\sigma) - \frac{1}{2}p(\rho_+)(1 + \Theta_\sigma)$$

$$= \frac{1}{2}\int_0^1 p'(\rho_- + r(\tilde{\rho} - \rho_-))dr(\tilde{\rho} - \rho_-)(1 - \Theta_\sigma)$$

$$+ \frac{1}{2}\int_0^1 p'(\rho_+ + r(\tilde{\rho} - \rho_+))dr(\tilde{\rho} - \rho_+)(1 + \Theta_\sigma)$$

$$\approx 0. \tag{B.3}$$

1) We now prove the estimate of $\mathbf{G}$. By (2.19) and (2.24), one has that for all $x_3 \in \mathbb{R}$ and $t \geqslant 0$,

$$G_i(x_3, t) = F_{1,3i}^\flat(x_3, t) + \int_{-\infty}^{x_3} f_{2,i}^\flat(y_3, t)dy_3$$

$$= -F_{1,3i}^\flat(x_3, t) - \int_{x_3}^{+\infty} f_{2,i}^\flat(y_3, t)dy_3 \quad \text{for} \quad i = 1, 2,$$

$$G_3(x_3, t) = F_{1,33}^\flat(x_3, t).$$

With the aid of (B.1) to (B.3) and the fact that

$$\partial_3\tilde{\mathbf{u}} \approx \frac{1}{2}\big[\partial_3\mathbf{u}_-(1 - \Theta_\sigma) + \partial_3\mathbf{u}_+(1 + \Theta_\sigma)\big] + \bar{\mathbf{u}}\partial_3\Theta_\sigma,$$

$$\text{div}\tilde{\mathbf{u}} \approx \frac{1}{2}\big[\text{div}\mathbf{u}_-(1 - \Theta_\sigma) + \text{div}\mathbf{u}_+(1 + \Theta_\sigma)\big],$$

it can follow from (2.12) that

$$\mathbf{F}_{1,3}^\flat \approx -\mu\bar{\mathbf{u}}\partial_3\Theta_\sigma = \frac{-\mu\bar{\mathbf{u}}}{\sqrt{t + \Lambda}}\Theta'\Big(\frac{x_3 - \sigma}{\sqrt{t + \Lambda}}\Big). \tag{B.4}$$

Then one has that $G_3 = F_{1,33}^\flat \approx 0$. For $i = 1$ or 2, it follows from (2.20) that

$$\int_{\pm\infty}^{x_3} f_{2,i}^\flat(y_3, t)dy_3 = \frac{-\bar{\rho}\bar{u}_i}{2\sqrt{t + \Lambda}}\int_{\pm\infty}^{\frac{x_3 - \sigma}{\sqrt{t + \Lambda}}} \xi\Theta'(\xi)d\xi + R_{i\pm}, \tag{B.5}$$

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where

$$R_{i\pm} = \frac{\bar{u}_i}{\sqrt{t + \Lambda}} \int_{\pm\infty}^{x_3} \left[-(\rho_-^\flat - \bar{\rho})\frac{x_3 - \sigma}{2(t + \Lambda)} - \rho_-^\flat \sigma'(t) + m_{3-}^\flat \right] \Theta_\sigma' dy_3.$$

Since $\Theta' > 0$, then it holds that $|R_{i\pm}| \lesssim \varepsilon e^{-\alpha t}(1 \mp \Theta_\sigma)$. It follows from (1.18) that

$$\mu\Theta'(\xi) = -\frac{\bar{\rho}}{2} \int_{-\infty}^{\xi} \eta\Theta'(\eta)d\eta = \frac{\bar{\rho}}{2} \int_{\xi}^{+\infty} \eta\Theta'(\eta)d\eta \qquad \forall \xi \in \mathbb{R}.$$

This, together with (B.4) and (B.5), yields that

$$F_{1,3i}^\flat + \int_{-\infty}^{x_3} f_{2,i}dy_3 \approx R_{i-} \quad \text{and} \quad -F_{1,3i}^\flat - \int_{x_3}^{+\infty} f_{2,i}^\flat dy_3 \approx R_{i+},$$

which implies that $G_i \approx R_{i-}$ and meanwhile, $G_i \approx R_{i+}$. Thus, it holds that

$$\|G_i\|_{L^2(\mathbb{R})} \lesssim \varepsilon\Lambda^{1/4}e^{-\alpha t} + \left(\int_{-\infty}^{\sigma} |R_{i-}|^2 dx_3 \right)^{\frac{1}{2}} + \left(\int_{\sigma}^{+\infty} |R_{i+}|^2 dx_3 \right)^{\frac{1}{2}} \lesssim \varepsilon\Lambda^{1/4}e^{-\alpha t}.$$

2) Using (2.11) and the fact that $(\rho_+ - \rho_-, m_{3+} - m_{3-}) = O(1)\varepsilon e^{-\alpha t}$, one can easily obtain that $\|f_0\|_{L^2(\Omega)} \lesssim \varepsilon\Lambda^{-\frac{1}{4}}e^{-\alpha t}$. Then we estimate $\mathbf{g}$. It follows from (2.7) and (2.9) that for $j = 1, 2, 3$,

$$\partial_j(\tilde{u}_j\tilde{\mathbf{m}}) \approx \frac{1}{2}\left[\partial_j(u_{j-}\mathbf{m}_-)(1 - \Theta_\sigma) + \partial_j(u_{j+}\mathbf{m}_+)(1 + \Theta_\sigma)\right],$$

$$\partial_j p(\tilde{\rho}) \approx \frac{1}{2}\partial_j p(\rho_-)(1 - \Theta_\sigma) + \frac{1}{2}\partial_j p(\rho_+)(1 + \Theta_\sigma),$$

$$\partial_j^2\tilde{\mathbf{u}} \approx \frac{1}{2}\partial_j^2\mathbf{u}_-(1 - \Theta_\sigma) + \frac{1}{2}\partial_j^2\mathbf{u}_+(1 + \Theta_\sigma) + \frac{\delta_{j3}\bar{\mathbf{u}}}{t + \Lambda}\Theta_\sigma'',$$

$$\partial_j\text{div}\tilde{\mathbf{u}} \approx \frac{1}{2}\partial_j\text{div}\mathbf{u}_-(1 - \Theta_\sigma) + \frac{1}{2}\partial_j\text{div}\mathbf{u}_+(1 + \Theta_\sigma).$$

Using the relations above in (2.12), one can get that

$$\sum_{j=1}^{3} \partial_j\mathbf{F}_{1,j} \approx -\frac{\mu\bar{\mathbf{u}}}{t + \Lambda}\Theta''\left(\frac{x_3 - \sigma}{\sqrt{t + \Lambda}}\right). \tag{B.6}$$

On the other hand, it follows from (2.13) that

$$\mathbf{f}_2 \approx -\frac{\bar{\mathbf{m}}}{2(t + \Lambda)} \cdot \frac{x_3 - \sigma}{\sqrt{t + \Lambda}}\Theta'\left(\frac{x_3 - \sigma}{\sqrt{t + \Lambda}}\right),$$

which, together with (1.18), yields that $\mathbf{g} = \sum_{j=1}^{3} \partial_j\mathbf{F}_{1,j} + \mathbf{f}_2 \approx 0$. The estimates of the derivatives of f_0 and $\mathbf{g}$ can be proved similarly. We omit the details. $\qquad\square$

C Proof of Lemma 4.2

Proof. 1) Note that

$$\zeta^\flat = \frac{1}{\bar\rho}\psi^\flat - \frac{\mathbf{u}^{\mathsf{vs}}}{\bar\rho}\phi^\flat + \Big[\Big(\frac{1}{\tilde\rho} - \frac{1}{\bar\rho}\Big)\psi\Big]^\flat - \Big[\Big(\frac{\tilde{\mathbf{u}}}{\tilde\rho} - \frac{\mathbf{u}^{\mathsf{vs}}}{\bar\rho}\Big)\phi\Big]^\flat - \Big(\frac{1}{\tilde\rho}\phi\zeta\Big)^\flat. \tag{C.1}$$

Then it holds that

$$\pm\big\|\zeta^\flat\big\|_{L^2(\Omega)} \lesssim \pm\big\|\psi^\flat\big\|_{L^2(\Omega)} + \big\|\phi^\flat\big\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t} + \nu\,\|\phi\|_{L^2(\Omega)}\,,$$

which yields (4.16) directly. Similarly, one has that

$$\begin{aligned}
\pm\big\|\partial_3\zeta^\flat\big\|_{L^2(\Omega)} \lesssim{}& \pm\big\|\partial_3\psi^\flat\big\|_{L^2(\Omega)} + \big\|\partial_3\phi^\flat\big\|_{L^2(\Omega)} + (t+\Lambda)^{-\frac{1}{2}}\big\|\phi^\flat\big\|_{L^2(\Omega)}\\
&+ \varepsilon\nu e^{-\alpha t} + \nu\,\|\partial_3\phi, \partial_3\zeta\|_{L^2(\Omega)}\,,
\end{aligned}$$

which gives (4.17). Moreover, it holds that

$$\begin{aligned}
\pm\big\|\partial_3^2\zeta^\flat\big\|_{L^2(\mathbb{R})} \lesssim{}& \pm\big\|\partial_3^2\psi^\flat\big\|_{L^2(\mathbb{R})} + \sum_{j=0}^{2}(t+\Lambda)^{-1+\frac{j}{2}}\big\|\partial_3^j\phi^\flat\big\|_{L^2(\mathbb{R})} + \varepsilon\nu e^{-\alpha t}\\
&+ \|\nabla\phi\|_{L^4(\Omega)}\|\nabla\zeta\|_{L^4(\Omega)} + \nu\,\big\|\nabla^2\phi, \nabla^2\zeta\big\|_{L^2(\Omega)}\,,
\end{aligned}$$

which, together with (4.104), yields (4.18).

2) It follows from (3.26), (4.76) and (4.79)$_3$ that

$$\begin{aligned}
\big\|\nabla^2\psi^\sharp\big\|_{L^2(\Omega)} \lesssim{}& \big\|\nabla^2\zeta^\sharp\big\|_{L^2(\Omega)} + \big\|\phi^\sharp\big\|_{H^2(\Omega)} + \big\|\nabla^2\mathbf{r}_3\big\|_{L^2(\Omega)}\\
\lesssim{}& \big\|\nabla^2\zeta^\sharp\big\|_{L^2(\Omega)} + \nu\,\big\|\nabla^2\phi^\sharp, \nabla^2\zeta^\sharp\big\|_{L^2(\Omega)} + \varepsilon\nu e^{-\alpha t},
\end{aligned}$$

which yields (4.19).

Using the identity that

$$\psi = \bar\rho\zeta + \mathbf{u}^{\mathsf{vs}}\phi + \phi\zeta + \big[(\tilde\rho - \bar\rho)\zeta + (\tilde{\mathbf{u}} - \mathbf{u}^{\mathsf{vs}})\phi\big],$$

one has that

$$\begin{aligned}
\big\|\nabla^3\psi\big\|_{L^2(\Omega)} \lesssim{}& \big\|\nabla^3\zeta\big\|_{L^2(\Omega)} + \delta\sum_{j=0}^{1}(t+1)^{\frac{j-3}{2}}\big\|\partial_3^j\phi^\flat\big\|_{L^2(\mathbb{R})} + \delta\,\big\|\phi^\sharp\big\|_{H^1(\Omega)}\\
&+ (\delta + \nu)\big\|\nabla^2\phi\big\|_{H^1(\Omega)} + \|\nabla\phi\|_{L^4(\Omega)}\big\|\nabla^2\zeta\big\|_{L^4(\Omega)}\\
&+ \nu\,\big\|\nabla^3\zeta\big\|_{L^2(\Omega)} + \varepsilon e^{-\alpha t}\,\|\phi, \zeta\|_{H^3(\Omega)}\,.
\end{aligned} \tag{C.2}$$

It follows from (4.98)$_1$ that

$$\|\nabla\phi\|_{L^4(\Omega)} \lesssim \|\phi\|_{H^2(\Omega)}^{\frac{1}{2}}\big\|\nabla^2\phi\big\|_{L^2(\Omega)}^{\frac{1}{2}} \lesssim \nu^{\frac{1}{2}}\big\|\nabla^2\phi\big\|_{L^2(\Omega)}^{\frac{1}{2}}\,,$$

and similarly,

$$\big\|\nabla^2\zeta\big\|_{L^4(\Omega)} \lesssim \|\nabla\zeta\|_{H^2(\Omega)}^{\frac{1}{2}}\big\|\nabla^3\zeta\big\|_{L^2(\Omega)}^{\frac{1}{2}} \lesssim \nu^{\frac{1}{2}}\big\|\nabla^3\zeta\big\|_{L^2(\Omega)}^{\frac{1}{2}}\,.$$

Applying these two inequalities to (C.2) and using (3.26), one can obtain (4.20).

$\square$

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References

[1] Artola, M., Majda, A.: Nonlinear development of instabilities in supersonic vortex sheets. I. The basic kink modes. *Phys. D* **28**(3), 253–281 (1987)

[2] Chen, G.-Q., Wang, Y.-G.: Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **187**(3), 369–408 (2008)

[3] Coulombel, J. F., Secchi, P.: The stability of compressible vortex sheets in two space dimensions, *Indiana Univ. Math. J.* **53**: no. 4, 941–1012 (2004)

[4] Coulombel, J. F., Secchi, P.: Nonlinear compressible vortex sheets in two space dimensions. *Ann. Sci. Éc. Norm. Supér (4)* **4**: no. 1, 85–139 (2008)

[5] Dafermos, C. M.: Large time behavior of periodic solutions of hyperbolic systems of conservation laws. *J. Differential Equations* **121**(1), 183–202 (1995)

[6] _____ : Long time behavior of periodic solutions to scalar conservation laws in several space dimensions. *SIAM J. Math. Anal.* **45**(4), 2064–2070 (2013)

[7] Ebin, D.: Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids. *Comm. Partial Differential Equations* **13**(10), 1265–1295 (1988)

[8] Fejer, J. A. and Miles, J. W.: On the stability of a plane vortex sheet with respect to three-dimensional disturbances, *J. Fluid Mech.* **15**, 335–336 (1963)

[9] Glimm, J., Lax, P. D.: Decay of solutions of systems of nonlinear hyperbolic conservation laws, Memoirs of the American Mathematical Society, no. 101 *American Mathematical Society, Providence, R.I.* (1970)

[10] Goodman J.: Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Ration. Mech. Anal.* **95**(4), 325–344 (1986)

[11] Guo, Y., Tice, I.: Compressible, inviscid Rayleigh-Taylor instability. *Indiana Univ. Math. J.* **60**(2), 677–712 (2011)

[12] Huang, F., Li, J., Matsumura, A.: Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, *Arch. Ration. Mech. Anal.* **197**(1), 89–116 (2010)

[13] Huang, F., Matsumura, A., Shi, X.: On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary, *Osaka J. Math.* **41**(1), 193–210 (2004)

[14] Huang, F., Matsumura, A., Xin, Z.: Stability of contact discontinuities for the 1-d compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **179**(1), 55–77 (2006)

[15] Huang, F., Xin, Z., Yang, T.: Contact discontinuity with general perturbations for gas motions, *Adv. Math.* **219**(4), 1246–1297 (2008)

[16] Huang, F., Xu, L., Yuan, Q.: Asymptotic stability of planar rarefaction waves under periodic perturbations for 3-d Navier-Stokes equations, *Adv. Math.* **404**, part B, Paper No. 108452, 27 pp. (2022)

[17] Huang, F., Yuan, Q.: Stability of large-amplitude viscous shock under periodic perturbation for 1-d isentropic Navier-Stokes equations. *Comm. Math. Phys.* **387**(3), 1655–1679 (2021)

[18] ⸺: Stability of planar rarefaction waves for scalar viscous conservation law under periodic perturbations, *Methods Appl. Anal.*, **28**(3), 337–353 (2021)

[19] Il'in, A. M., Oleĭnik, O. A.: Asymptotic behavior of solutions of the Cauchy problem for some quasi-linear equations for large values of the time, *Mat. Sb. (N.S.)* **51**(93) 191–216 (1960)

[20] Lax, P. D.: Hyperbolic systems of conservation laws ii, *Comm. Pure Appl. Math* **10**(4), 537–566 (1957)

[21] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.* **56** (1985)

[22] Liu, T.-P., Xin, Z.: Pointwise decay to contact discontinuities for systems of viscous conservation laws, *Asian J. Math.* **1**(1), 34–84 (1997)

[23] Majda, A., Rosales, R.: Resonantly interacting weakly nonlinear hyperbolic waves. I. A single space variable, *Stud. Appl. Math.* **71**(2), 149–179 (1984)

[24] Matsumura A., Nishida T.: The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20**(1), 67–104 (1980)

[25] Matsumura A., Nishihara K.: On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **2**(1), 17–25 (1985)

[26] Miles, J. W.: On the disturbed motion of a plane vortex sheet. *J. Fluid Mech.* **4**, 538–552 (1958)

[27] Morando, A., Trebeschi, P.: Two-dimensional vortex sheets for the nonisentropic Euler equations: linear stability. *J. Hyperbolic Differ. Equ.* **5**(3) 487–518 (2008)

[28] Morando, A., Trebeschi, P., Wang, T.: Two-dimensional vortex sheets for the nonisentropic Euler equations: nonlinear stability. *J. Differential Equations* **266**(9), 5397–5430 (2019)

[29] Qu, P., Xin, Z.: Long time existence of entropy solutions to the one-dimensional non-isentropic Euler equations with periodic initial data. *Arch. Ration. Mech. Anal.* **216**(1), 221–259 (2015)

[30] Serre, D.: Systems of conservation laws. 2. *Cambridge University Press, Cambridge,* xii+269 pp., (2000)

[31] _____: Systems of Conservation Laws. 1. *Cambridge University Press, Cambridge,* xxii+263 pp. (1999)

[32] Szepessy A., Xin Z.: Nonlinear stability of viscous shock waves, *Arch. Ration. Mech. Anal.* **122**(1), 53–103 (1993)

[33] Temple, B., Young, R.: The large time stability of sound waves. *Comm. Math. Phys.* **179**(2), 417–466 (1996)

[34] _____: The Nonlinear Theory of Sound. *ArXiv:2305.15623*, 76 pages (2023)

[35] Trakhinin, Y.: Existence of compressible current-vortex sheets: variable coefficients linear analysis. *Arch. Ration. Mech. Anal.* **177**(3), 331–366 (2005)

[36] Trakhinin, Y.: The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **191**(2), 245–310 (2009)

[37] Wang, Y., Xin, Z.: Existence of multi-dimensional contact discontinuities for the ideal compressible magnetohydrodynamics. *arXiv:2112.08580v3*, to appear in *Comm. Pure Appl. Math.*

[38] Wang, Y.-G., Yu, F.: Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets. *Arch. Ration. Mech. Anal.* **208**(2), 341–389 (2013)

[39] Xin, Z.: On nonlinear stability of contact discontinuities, *Hyperbolic problems: theory, numerics, applications, World Sci. Publ., River Edge, NJ,* 249–257 (1996)

[40] Xin, Z., Yuan, Q., Yuan, Y.: Asymptotic stability of shock profiles and rarefaction waves under periodic perturbations for 1-D convex scalar viscous conservation laws. *Indiana Univ. Math. J.* **70**(6), 2295–2349 (2021)

[41] _____: Asymptotic stability of shock waves and rarefaction waves under periodic perturbations for 1-d convex scalar conservation laws, *SIAM J. Math. Anal.* **51**(4), 2971–2994 (2019)

[42] Xin, Z., Zeng, H.: Pointwise stability of contact discontinuity for viscous conservation laws with general perturbations. *Comm. Partial Differential Equations* **35**(7), 1326–1354 (2010)

[43] Yuan, Q.: Nonlinear asymptotic stability of planar viscous shocks for 3D compressible Navier-Stokes equations with periodic perturbations. *arXiv:2212.13414* (2022)

[44] _____: Planar viscous shocks with periodic perturbations for scalar multi-dimensional viscous conservation laws, *SIAM J. Math. Anal.* **55**(3), 1499–1523.(2023)

[45] Yuan, Q., Yuan, Y.: On Riemann solutions under different initial periodic perturbations at two infinities for 1-d scalar convex conservation laws, *J. Differential Equations* **268**(9), 5140–5155 (2019)

[46] _____ : Periodic perturbations of a composite wave of two viscous shocks for 1-d full compressible Navier-Stokes equations, *SIAM J. Math. Anal.* **54**(3), 2876–2905 (2022)