LOCAL WELL-POSEDNESS OF THE INCOMPRESSIBLE CURRENT-VORTEX SHEET PROBLEMS

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Abstract. We prove the local well-posedness of the incompressible current-vortex sheet problems in standard Sobolev spaces under the surface tension or the Syrovatskij condition, which shows that both capillary forces and large tangential magnetic fields can stabilize the motion of current-vortex sheets. Furthermore, under the Syrovatskij condition, the vanishing surface tension limit is established for the motion of current-vortex sheets. These results hold without assuming the interface separating the two plasmas being a graph.

Contents

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1. Introduction

1.1. **Formulations of the problems.** We consider the free interface problems for ideal incompressible magnetohydrodynamics (MHD) equations, which describe the motions of two plasmas separating by a free interface (current-vortex sheet problems). If we denote by $\varOmega_t^\pm \subset \mathbb{R}^3$ the fluid domains at time t occupied by two kinds of plasmas respectively, the ideal incompressible MHD system can be written as

$$\text{(MHD)} \quad \begin{cases} \partial_t \mathbf{v}_\pm + (\mathbf{v}_\pm \cdot \nabla)\mathbf{v}_\pm + \dfrac{1}{\rho_\pm}\nabla p^\pm = (\mathbf{h}_\pm \cdot \nabla)\mathbf{h}_\pm & \text{in } \varOmega_t^\pm, \quad (1.1a) \\ \partial_t \mathbf{h}_\pm + (\mathbf{v}_\pm \cdot \nabla)\mathbf{h}_\pm = (\mathbf{h}_\pm \cdot \nabla)\mathbf{v}_\pm & \text{in } \varOmega_t^\pm, \quad (1.1b) \\ \nabla \cdot \mathbf{v}_\pm = 0 = \nabla \cdot \mathbf{h}_\pm & \text{in } \varOmega_t^\pm; \quad (1.1c) \end{cases}$$

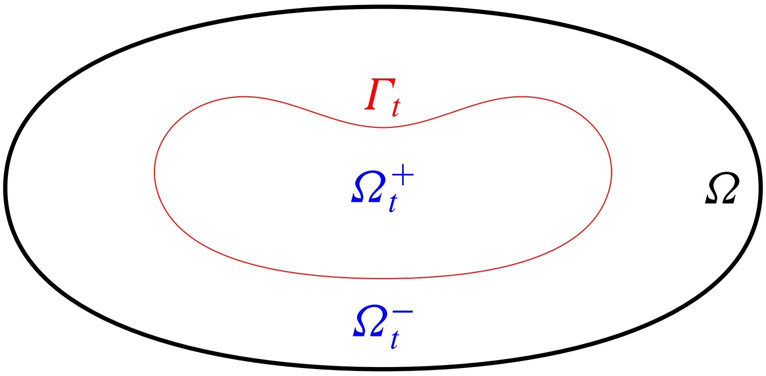
here $\rho_\pm, \mathbf{v}_\pm, \mathbf{h}_\pm, p^\pm$ are the densities, velocities, magnetic fields and effective pressures for the two plasmas respectively (c.f. [LL84] or [Dav17]). The boundary conditions are:

$$\text{(BC)} \quad \begin{cases} \mathbf{v}_+ \cdot \mathbf{N}_+ = \mathbf{v}_- \cdot \mathbf{N}_+ =: \theta & \text{on } \varGamma_t, \quad (1.2a) \\ [\![p]\!] := p^+ - p^- = \alpha^2 \kappa_+ & \text{on } \varGamma_t, \quad (1.2b) \\ \mathbf{h}_+ \cdot \mathbf{N}_+ = \mathbf{h}_- \cdot \mathbf{N}_+ = 0 & \text{on } \varGamma_t, \quad (1.2c) \\ \mathbf{v}_- \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\varOmega, \quad (1.2d) \\ \mathbf{h}_- \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\varOmega\ , \quad (1.2e) \end{cases}$$

where κ_+ is the mean curvature of $\varGamma_t$ with respect to $\mathbf{N}_+$, and $0 \le \alpha \le 1$ is a non-negative constant representing the surface tension coefficient.

Here $\varOmega \subset \mathbb{R}^3$ is a bounded domain with a fixed boundary $\partial\varOmega$, and $\varOmega = \varOmega_t^+ \cup \varGamma_t \cup \varOmega_t^-$, $\varGamma_t = \partial\varOmega_t^+$ is the moving interface with normal speed θ, and $\partial\varOmega_t^- = \partial\varOmega \cup \varGamma_t$. Denote by $\mathbf{N}_+$

the outward unit normal of $\partial\Omega_t^+ = \Gamma_t$, and $\widetilde{\mathbf{N}}$ the outward unit normal of $\partial\Omega$. Assume further that $\Gamma_t \subset \Omega$, $\Gamma_t \cap \partial\Omega = \varnothing$, and Γ_t separates Ω into two disjoint simply-connected domains $\Omega_t^\pm$.



The equations (1.1a) are the Euler equations in hydrodynamics, for which the Lorentz forces serve as the exterior body forces acting on the plasmas. Note that the displacement currents are neglected, due to the fact that the scale of the plasma velocities is much less than the speed of light. The equations (1.1b) are the combination of Faraday's Law and Ohm's Law, and (1.1c) are the incompressibility of the plasmas and Gauss's Law for magnetism. The boundary condition (1.2a) is also known as the kinematic boundary condition, which means that the free interface evolves with the two plasmas. (1.2b) is derived from the balance of momentum between two sides of the interface, and (1.2c) follows from the Gauss's Law for magnetism and physical characters of the materials. (1.2d) means that the outer plasma cannot penetrate the solid boundary, and (1.2e) follows from the assumption that the solid boundary is a perfect conductor.

1.2. **Physical background.** The motion of electrically conductive fluids (e.g., plasma, liquid metals, salt water, and electrolytes) under the influence of magnetic fields is governed by the MHD systems. The corresponding mathematical theories have numerous significant applications (e.g., drug targeting, earthquakes, sensors, and astrophysics). One of the fundamental differences between MHD and hydrodynamics is that the magnetic fields can induce currents in a moving conductive fluid, and these currents in turn polarize the fluid and change the magnetic and velocity fields in a reciprocal manner. The set of equations is a combination of those in fluid dynamics and electrodynamics, and these equations must be solved concurrently (c.f. [LL84, Dav17]). Mathematically speaking, the effect of the magnetic field is governed by the Maxwell equations and acts as a Lorentz force on the Euler system for the plasma, which can induce many nontrivial interactions and lead to rich phenomena.

The current-vortex sheet problems describe the plasma motion in a domain whose boundary evolves with the plasma itself. Such issues are significant not only because they describe numerous physical phenomena thus have significant applications in science and technology, but also since such studies give rise to profound and challenging theoretical interdisciplinary problems involving partial differential equations, differential geometry, analysis, mathematical physics, and dynamical systems.

1.3. **Previous works.** In the absence of magnetic fields, the equations are reduced to the incompressible Euler system. The free boundary problems in hydrodynamics have garnered considerable interest from the mathematical community. Although water waves are very universal in reality, from which one can see a vast diversity of phenomena, the corresponding mathematical theories are still in their infancy, because the full equations describing the motion of the waves are famously difficult to handle due to the free boundary and intrinsic nonlinearity. We refer to the works by Wu [Wu97, Wu99], Alazard-Burq-Zuily [ABZ14] for the local well-posedness of irrotational water wave problems. When the vorticity of the fluid flow is non-zero, one can refer to Christodoulou-Lindblad [CL00], Lindblad [Lin03, Lin05], Coutand-Shkoller [CS07], Cheng-Coutand-Shkoller [CCS08], Zhang-Zhang [ZZ08], Shatah-Zeng [SZ08a, SZ08b, SZ11] for the local well-posedness of the water wave and vortex sheet problems.

In contrast to the long history of the study on the water wave problems, the free-interface problems for ideal MHD equations have been studied only in recent decades. Owing to the strong coupling of the magnetic and velocity fields, it is necessary to deal with multiple hyperbolic systems simultaneously, making it difficult to establish the nonlinear well-posedness theories. In particular, how magnetic fields affect the dynamics of a plasma is an important issue. As most of fluids are electrically conductive and magnetic fields are ubiquitous, the MHD model is certainly an important physical one with similar significance as the Euler or Navier-Stokes ones. When the effect of magnetic fields is not negligible, it is significant to study the dynamics of conducting fluids. Here are some representative works on the free interface problems for the ideal incompressible MHD.

A current-vortex sheet is a hypersurface evolving with the conductive fluids, along which the magnetic and velocity fields possess tangential jumps. This sort of problems explain the motion of two conducting fluids with a free interface separating them. Around the middle of the twentieth century, Syrovatskij [Syr53] and Axford [Axf62] discovered the stability requirements for the planer incompressible current-vortex sheets and demonstrated that magnetic fields have a stabilizing influence on the plasma dynamics. The Syrovatskij stability conditions are (see Landau-Lifshitz [LL84, § 71]):

$$\rho_+|\mathbf{h}_+|^2 + \rho_-|\mathbf{h}_-|^2 > \frac{\rho_+\rho_-}{\rho_+ + \rho_-}|[\![\mathbf{v}]\!]|^2, \tag{1.3}$$

$$(\rho_+ + \rho_-)|\mathbf{h}_+ \times \mathbf{h}_-|^2 \geq \rho_+|\mathbf{h}_+ \times [\![\mathbf{v}]\!]|^2 + \rho_-|\mathbf{h}_- \times [\![\mathbf{v}]\!]|^2, \tag{1.4}$$

where $[\![\mathbf{v}]\!] := \mathbf{v}_+ - \mathbf{v}_-$ is the velocity jump. If the current-vortex sheet is assumed to be the graph of a function, there are some studies on the dynamics: Trakhinin [Tra05] proved the a priori estimate for the linearized equations under a strong stability condition:

$$|\mathbf{h}_+ \times \mathbf{h}_-| > \max\{|\mathbf{h}_+ \times [\![\mathbf{v}]\!]|, |\mathbf{h}_- \times [\![\mathbf{v}]\!]|\}. \tag{1.5}$$

Coulombel-Morando-Secchi-Trebeschi [CMST12] showed the a priori estimate without loss of derivatives for the non-linear problem under (1.5). If the original Syrovatskij condition (1.4) were replaced by the following strict one:

$$(\rho_+ + \rho_-)|\mathbf{h}_+ \times \mathbf{h}_-|^2 > \rho_+|\mathbf{h}_+ \times [\![\mathbf{v}]\!]|^2 + \rho_-|\mathbf{h}_- \times [\![\mathbf{v}]\!]|^2, \tag{1.6}$$

from which (1.3) follows, Morando-Trakhinin-Trebeschi [MTT08] derived the a priori estimates for the linearized system with loss of derivatives. The nonlinear local well-posedness result under (1.6) was first proven by Sun-Wang-Zhang [SWZ18]. The above works demonstrate that the strict Syrovatskij condition (1.6) indeed has a nonlinear stabilizing effect on the free interface (at least for a graph surface in a short time period), in contrast to the Kelvin-Helmholtz instability for pure-fluid vortex-sheet issues due to the lack of surface tension (c.f. [Ebi88] and [MB02, Chapters 9 & 11] for more detailed discussions). Recently, the methods in [SWZ18] were also applied to the study for the case with surface tension, see [LL22].

For the plasma-vacuum interface problems, if the magnetic field is parallel to the free boundary and the one in the vacuum is vanishing, we refer to Hao-Luo [HL14, HL21], Gu-Wang [GW19], and Gu-Luo-Zhang [GLZ21, GLZ22] for the local well-posedness. If the magnetic field in the vacuum is nontrivial, one can see Mordando-Trakhinin-Trebeschi [MTT14], and Sun-Wang-Zhang [SWZ19] for the local well-posedness under a stability condition. Hao-Luo [HL20] also showed the ill-posedness for the plasma-vacuum problems without the Rayleigh-Taylor sign condition, as indicated by Ebin [Ebi87] for the pure fluid-vacuum case. Concerning the global well-posedness for free-boundary incompressible inviscid MHD equations, Wang-Xin [WX21] established it for both the plasma-vacuum and the plasma-plasma problems.

Although these advances are significant, all of the aforementioned nonlinear local well-posedness results for MHD problems were founded on a crucial premise that the free interface is a graph. However, in reality, the moving surface cannot be represented simply by a graph in many significant cases. To remove these limitations seems quite challenging, even for the pure fluid problems (c.f. [CS07, SZ11]). Using the partition of unity to characterize the general interface appears feasible, but the analysis of these transition maps is rather involved due to the intense interactions between the plasmas in different local charts. In view of the strong coupling of the magnetic and velocity fields (one direct consequence of which is that the vorticity transport formula will change), MHD problems must be analyzed with greater care than the pure fluid ones. For example, one of the difficulties is that the estimates of the velocity and magnetic fields must be derived simultaneously, which is much more complex than in the case for pure fluids. More significantly, the strategies on the local dynamic motion of a general current-vortex sheet will be indispensable to the study of long-time dynamics, particularly the finite-time formation of splash singularities from a generic perturbation of a current-vortex sheet (even of a graph type).

This paper is to establish the nonlinear local well-posedness for the current-vortex sheet problems in standard Sobolev spaces, without the graph assumption on the free interface. Namely, we show that for more general physical models, both the capillary forces and large tangential magnetic fields (the Syrovatskij condition) can stabilize the motion of current-vortex sheets. In particular, our results can be applied to study the dynamics of free interfaces with turning-over points, and may be useful to construct splash singularities.

2. Main Results

For convenience, we shall use the notation $f := f_+ \mathbb{1}_{\Omega_t^+} + f_- \mathbb{1}_{\Omega_t^-}$ to represent for functions $f_\pm : \Omega_t^\pm \to \mathbb{R}$.

2.1. The stabilization effect of the surface tension.

If there exists surface tension on the free interface, the following local well-posedness result holds:

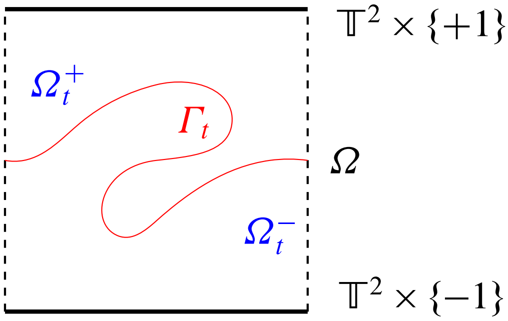
Theorem 2.1 ($\alpha = 1$ case)**.** *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a $C^1 \cap H^{\frac{3}{2}k+1}$ boundary, and $k \geq 2$ is an integer. Given the initial hypersurface $\Gamma_0 \in H^{\frac{3}{2}k+1}$ and two* *solenoidal vector fields* $\mathbf{v}_0, \mathbf{h}_0 \in H^{\frac{3}{2}k}(\Omega \setminus \Gamma_0)$*, if* Γ_0 *separates* Ω *into two disjoint simply-connected parts, then there exists a constant* $T > 0$ *so that the current-vortex sheet problem (MHD)-(BC) has a solution in the space:*

$$\Gamma_t \in C^0\Big([0,T]; H^{\frac{3}{2}k+1}\Big) \quad \text{and} \quad \mathbf{v}, \mathbf{h} \in C^0\Big([0,T]; H^{\frac{3}{2}k}(\Omega \setminus \Gamma_t)\Big).$$

Furthermore, if $k \geq 3$*, the solution is unique and it depends on the initial data continuously, i.e. the problem (MHD)-(BC) is locally well-posed.*

2.2. **The stabilization effect of the Syrovatskij condition.** In the absence of surface tension, we show that the Syrovatskij condition (1.6) can stabilize the motion of the current-vortex sheet, at least for a short time period, without the graph assumption on the interface.

Due to the hairy ball theorem, (1.6) cannot hold on a hypersurface homeomorphic to a sphere. Thereby, we assume that $\Omega = \mathbb{T}^2 \times (-1,1)$, and Γ_t is a $C^1 \cap H^2$ hypersurface diffeomorphic to $\mathbb{T}^2$ (e.g. a surface with shape "Ʊ" or "Z", or a portion of sea waves), which separates Ω into the upper and the lower parts.



Accordingly, the boundary conditions (1.2a)-(1.2e) are modified to

$$(\mathrm{BC}') \quad \begin{cases} \mathbf{v}_+ \cdot \mathbf{N}_+ = \mathbf{v}_- \cdot \mathbf{N}_+ =: \theta & \text{on } \Gamma_t, \\ [\![p]\!] := p^+ - p^- = \alpha^2 \kappa_+ & \text{on } \Gamma_t, \\ \mathbf{h}_+ \cdot \mathbf{N}_+ = \mathbf{h}_- \cdot \mathbf{N}_+ = 0 & \text{on } \Gamma_t, \\ \mathbf{v}_\pm \cdot \widetilde{\mathbf{N}}_\pm = 0 & \text{on } \mathbb{T}^2 \times \{\pm 1\}, \\ \mathbf{h}_\pm \cdot \widetilde{\mathbf{N}}_\pm = 0 & \text{on } \mathbb{T}^2 \times \{\pm 1\}, \end{cases} \tag{2.1}$$

where $\widetilde{\mathbf{N}}_\pm \equiv \pm \mathbf{e}_3$, are the outward unit normals of $\mathbb{T}^2 \times \{\pm 1\}$.

By Lemma 7.1, the strict Syrovatskij condition (1.6) implies

$$\begin{aligned} 0 < \Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!]) := \inf_{\substack{\mathbf{a} \in \mathrm{T}\Gamma_t;\, z \in \Gamma_t \\ |\mathbf{a}|=1}} \ \inf \ & \frac{\rho_+}{\rho_+ + \rho_-} |\mathbf{a} \cdot \mathbf{h}_+(z)|^2 + \frac{\rho_-}{\rho_+ + \rho_-} |\mathbf{a} \cdot \mathbf{h}_-(z)|^2 \\ & - \frac{\rho_+ \rho_-}{(\rho_+ + \rho_-)^2} |\mathbf{a} \cdot [\![\mathbf{v}]\!](z)|^2. \end{aligned}$$

We prove the following two theorems under the Syrovatskij condition (1.6):

Theorem 2.2 ($\alpha = 0$ case)**.** *Let* $k \geq 3$ *be an integer and* $\Omega := \mathbb{T}^2 \times (-1,1)$*. Suppose that* Γ_0 *is an* $H^{\frac{3}{2}k+\frac{1}{2}}$ *hypersurface diffeomorphic to* $\mathbb{T}^2$ *separating* Ω *into two parts (the upper one* *and the lower one). Assume that* $\mathbf{v}_0, \mathbf{h}_0 \in H^{\frac{3}{2}k}(\Omega \setminus \Gamma_0)$ *are two* $H^{\frac{3}{2}k}$ *solenoidal vector fields satisfying*

$$\Upsilon(\mathbf{h}_{0\pm}, [\![\mathbf{v}_0]\!]) \geq 2\mathfrak{s}_0 > 0.$$

If Γ_0 *does not touch the top or the bottom boundary, namely, for some positive constant* c_0*,*

$$\operatorname{dist}\left(\Gamma_0, \mathbb{T}^2 \times \{\pm 1\}\right) \geq 2c_0 > 0;$$

then there exists a constant $T > 0$*, so that the current-vortex sheet problem (MHD)-(BC') admits a unique solution in the space*

$$\Gamma_t \in C^0\Big([0,T]; H^{\frac{3}{2}k+\frac{1}{2}}\Big) \quad \text{and} \quad \mathbf{v}, \mathbf{h} \in C^0\Big([0,T]; H^{\frac{3}{2}k}(\Omega \setminus \Gamma_t)\Big).$$

Furthermore, for $0 \leq t \leq T$*, the solution* $(\Gamma_t, \mathbf{v}, \mathbf{h})$ *satisfies*

$$\Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!]) \geq \mathfrak{s}_0 \quad \text{and} \quad \operatorname{dist}\left(\Gamma_t, \mathbb{T}^2 \times \{\pm 1\}\right) \geq c_0,$$

and it depends on the initial data continuously. That is, the problem (MHD)-(BC') is locally well-posed.

Furthermore, the following result on the vanishing surface tension limit holds:

Theorem 2.3 ($\alpha \to 0$ limit)**.** *Assume that* $0 \leq \alpha \leq 1, k \geq 3$ *and* $\Omega = \mathbb{T}^2 \times (-1,1)$*. Fix* $\Gamma_0 \in H^{\frac{3}{2}k+1}$ *diffeomorphic to* $\mathbb{T}^2$ *with* $\operatorname{dist}(\Gamma_0, \mathbb{T}^2 \times \{\pm 1\}) \geq 2c_0 > 0$*, and solenoidal vector fields* $\mathbf{v}_{0\pm}, \mathbf{h}_{0\pm} \in H^{\frac{3}{2}k}(\Omega_0^\pm)$ *satisfying* $\Upsilon(\mathbf{h}_{0\pm}, [\![\mathbf{v}_0]\!]) \geq 2\mathfrak{s}_0 > 0$*. Then, there is a constant* $T > 0$*, independent of* α*, so that the problem, (MHD)-(BC'), is well-posed for* $t \in [0,T]$*. Furthermore, as* $\alpha \to 0$*, by passing to a subsequence, the solution to (MHD)-(BC') with surface tension converges weakly to a solution to (MHD)-(BC') with* $\alpha = 0$ *in the space* $\Gamma_t \in H^{\frac{3}{2}k+\frac{1}{2}}$ *and* $\mathbf{v}_\pm, \mathbf{h}_\pm \in H^{\frac{3}{2}k}(\Omega_t^\pm)$*.*

2.3. **Main ideas.** Inspired by the works of Shatah-Zeng [SZ11], we choose a geometric approach to analyze the problems. First of all, a reference hypersurface Γ_* diffeomorphic to the initial one is taken, and one may choose a transversal vector field $\boldsymbol{\nu}$ defined on the reference hypersurface of the same regularity and close to the unit normal in the C^1-topology. Therefore, any hypersurface near the reference one can be expressed uniquely by the height function defined on Γ_* via:

$$\Phi_\Gamma(p) = p + \gamma_\Gamma(p)\boldsymbol{\nu}(p) \quad \text{for} \quad p \in \Gamma_*.$$

Every hypersurface Γ in a small C^1-neighborhood of Γ_* is associated to a unique function γ_Γ, and the mean curvature κ of Γ can be expressed by γ_Γ. Conversely, by taking an auxiliary constant $a > 0$ determined by Γ_* and $\boldsymbol{\nu}$, the height function γ_Γ can be determined uniquely by the function $\kappa \circ \Phi_\Gamma + a^2\gamma_\Gamma : \Gamma_* \to \mathbb{R}$, whose leading order term is the mean curvature κ of Γ. Hence, the analysis of the evolution equation for the mean curvature κ can determine the evolution of the free hypersurface, which is the crucial part of the settlement of such free interface problems.

On the other hand, any vector field defined in a simply-connected domain is uniquely determined by its divergence, curl, and normal projection on the boundary (c.f. [CS17]). The evolution equation for the free interface yields the normal part of the velocity on the boundary, and the normal projection of the magnetic field is zero (for the current-vortex sheet problems considered here); thus, the incompressibility of the plasma ($\nabla \cdot \mathbf{v} = 0$) and Gauss's law for magnetism ($\nabla \cdot \mathbf{h} = 0$) imply that one only needs to determine the current ($\mathbf{j} := \nabla \times \mathbf{h}$) and vorticity ($\boldsymbol{\omega} := \nabla \times \mathbf{v}$). The evolution equations for $\boldsymbol{\omega}$ and $\mathbf{j}$ are given respectively as

$$\begin{cases}\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\mathbf{h} \cdot \nabla)\mathbf{j} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{j} \cdot \nabla)\mathbf{h}, \\ \partial_t \mathbf{j} + (\mathbf{v} \cdot \nabla)\mathbf{j} - (\mathbf{h} \cdot \nabla)\boldsymbol{\omega} = (\mathbf{j} \cdot \nabla)\mathbf{v} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{h} - 2\,\mathrm{tr}(\nabla \mathbf{v} \times \nabla \mathbf{h}).\end{cases}$$

Since $\mathbf{h}$ is parallel to the free hypersurface, as indicated in [SWZ18], it follows that $\mathbf{v} \pm \mathbf{h}$ are also evolution velocities of the interface and the fluid domain. Thus, one may use characteristic methods to transform the above equations into a system of ordinary differential equations, whose well-posedness is standard to obtain (see [CKS97]).

Once the evolution of the free interfaces, currents and vorticities are known, the original problem can be resolved via working on the div-curl systems.

Such an approach can not only resolve the free interface problems for general interfaces (in particular, with no graph assumptions) but also help to understand various stability conditions more clearly. Indeed, it will be seen in the evolution equations for the mean curvatures that the surface tension corresponds to a third order positive differential operator on the free interface, which serves as a stabilizer for the surface motion; the strict Syrovatskij condition (1.6) corresponds to a second order positive differential operator. Thus, concerning the stabilization effect, surface tensions are stronger than the strict Syrovatskij condition. Moreover, due to the existence of the unstable term in the evolution equations for the mean curvature (which is a second order differential operator resulting in the Kelvin-Helmholtz instability for the vortex sheet problems), the small tangential magnetic fields (the Syrovatskij conditions can be understood as a largeness assumption) cannot stabilize the current-vortex sheet in the absence of surface tension.

2.4. **Structure of the paper.** In § 3, we introduce some geometric relations and some analytical tools. § 4 - § 6 are devoted to the proof of Theorem 2.1. More precisely, in § 4, we rewrite the current-vortex sheet problems in a geometric manner, and derive the corresponding evolution equations. In § 5, we study the uniform linear estimates for the linearized systems; and in § 6 we consider the nonlinear problems and show the local well-posedness of the original current-vortex sheet ones. § 7 is devoted to the proof of Theorem 2.2 and 2.3. In the Appendices, we prove two technical lemmas.

3. Preliminaries

3.1. **Geometry of hypersurfaces.** For a family of hypersurfaces $\Gamma_t \subset \mathbb{R}^d$ evolving with the velocity $\mathbf{v} : \Gamma_t \to \mathbb{R}^d$, consider the local charts for the initial hypersurface $\mathbf{F} : U \to \Gamma_0 \subset \mathbb{R}^d$. Assume that Ξ_t is the flow map induced by $\mathbf{v}$, then one can take a coordinate map of Γ_t as $\mathbf{F}(t) := \Xi_t \circ \mathbf{F} : U \to \Gamma_t$. Standard geometric arguments (c.f. [Eck04]) give that the coordinate tangent vectors $\partial_i \mathbf{F}(t, z) : U \to \mathbb{R}^d, (1 \le i \le d-1)$ form a natural basis of the tangent space $\mathrm{T}_p \Gamma_t$ at $p = \mathbf{F}(t, z) \in \Gamma_t$ for each $z \in U$. The submanifold metric of $\Gamma_t \subset \mathbb{R}^d$ is given by

$$g_{ij} = \mathbf{F}_{,i} \cdot \mathbf{F}_{,j}$$

for $1 \le i, j \le d-1$, where $f_{,i}$ represents $\partial_i f$ for any function $f : U \to \mathbb{R}$. The inverse metric is defined to be

$$(g^{ij}) := (g_{ij})^{-1},$$

and the area element of Γ_t is

$$\sqrt{g} = \sqrt{\det(g_{ij})}.$$

Furthermore, there is a natural Riemannian connection on Γ_t, whose Christoffel symbols are given by

$$\Gamma_{ij}^k := g^{kl}\mathbf{F}_{,ij} \cdot \mathbf{F}_{,l} = \frac{1}{2}g^{kl}\left(g_{jl,i} + g_{il,j} - g_{ij,l}\right),$$

where the summation convention for tensors has been used. For a tangent vector $\mathbf{X} \in \mathrm{T}\Gamma_t$, one can write

$$\mathbf{X} = X^i\mathbf{F}_{,i} = g^{ij}X_j\mathbf{F}_{,i},$$

where $X_j := \mathbf{X} \cdot \mathbf{F}_{,j}$. The covariant derivative of $\mathbf{X}$ is defined to be

$$\left(\mathrm{D}_i^{\Gamma_t} X\right)^j := X_{,i}^j + \Gamma_{ik}^j X^k,$$

Hence, the divergence of $\mathbf{X}$ on Γ_t is defined to be

$$\operatorname{div}_{\Gamma_t} \mathbf{X} := g^{ij}\mathbf{X}_{,i} \cdot \mathbf{F}_{,j}. \tag{3.1}$$

One can also extend (3.1) to all vector fields defined on Γ_t, not necessarily being tangential. For a function $h : \Gamma_t \to \mathbb{R}$, the tangential gradient is defined by

$$\nabla_{\Gamma_t} h = g^{ij}h_{,i}\mathbf{F}_{,j}, \tag{3.2}$$

and the Laplace-Beltrami operator on Γ_t is given by

$$\triangle_{\Gamma_t} h = \operatorname{div}_{\Gamma_t} \nabla_{\Gamma_t} h = g^{ij}\left(h_{,ij} - \Gamma_{ij}^k h_{,k}\right) = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}h_{,j}\right). \tag{3.3}$$

For a general vector field $\mathbf{Y} : \Gamma_t \to \mathbb{R}^d$, the notation $(\mathrm{D}\mathbf{Y})^\top$ represents a (0,2)-tensor on Γ_t (here D is the covariant derivative on $\mathbb{R}^d$, and $\top$ is the tangential projection):

$$\left[(\mathrm{D}\mathbf{Y})^\top\right]_{ij} := \mathbf{Y}_{,i} \cdot \mathbf{F}_{,j}, \tag{3.4}$$

so

$$\operatorname{div}_{\Gamma_t} \mathbf{Y} = \operatorname{tr}\left[(\mathrm{D}\mathbf{Y})^\top\right].$$

Denote by $\mathbf{N} : \Gamma_t \to \mathbb{S}^{d-1}$ the unit normal vector field of Γ_t, i.e.

$$\mathbf{N} \cdot \mathbf{F}_{,i} = 0$$

for $1 \le i \le d-1$. Since $\mathbf{N} \cdot \mathbf{N} \equiv 1$, it is clear that

$$\mathbf{N}_{,i} \cdot \mathbf{N} = 0,$$

namely, $\mathbf{N}_{,i} \in \mathrm{T}\Gamma_t$ for all $1 \le i \le d-1$. The second fundamental form $\mathbf{I\!I}$ of Γ_t is a (0, 2)-tensor defined by

$$I\!I_{ij} := \mathbf{N}_{,i} \cdot \mathbf{F}_{,j} = -\mathbf{N} \cdot \mathbf{F}_{,ij}. \tag{3.5}$$

The mean curvature is defined to be the trace of $\mathbf{I\!I}$, i.e.

$$\kappa := \operatorname{tr}(\mathbf{I\!I}) = I\!I_{ij}g^{ij} = g^{ij}\mathbf{N}_{,i} \cdot \mathbf{F}_{,j} = \operatorname{div}_{\Gamma_t} \mathbf{N}. \tag{3.6}$$

Here we mention several useful identities, whose calculations can be found in [Eck04, Appendix A]:

$$\triangle_{\Gamma_t} I\!I_{ij} = \kappa_{;ij} + \kappa I\!I_i^k I\!I_{kj} - |\mathbf{I\!I}|^2 I\!I_{ij},$$

namely,

$$\triangle_{\Gamma}\mathrm{I\!I} = (\mathrm{D}_{\Gamma})^2\kappa + (\kappa\mathrm{I\!I} - |\mathrm{I\!I}|^2\mathbf{I})\cdot\mathrm{I\!I}, \tag{3.7}$$

which is called Simons' identity. Here the dot product of tensors is defined to be

$$\mathbf{C} := \mathbf{A}\cdot\mathbf{B}, \quad C_{ij} \equiv A_i^k B_{kj} = A_{il}B_{kj}g^{lk}.$$

Furthermore, it follows from the Codazzi equation that

$$\triangle_{\Gamma_t}\mathbf{N} = -|\mathrm{I\!I}|^2\mathbf{N} + \nabla_{\Gamma_t}\kappa. \tag{3.8}$$

Remark. Derivatives of functions and vector fields on hypersurfaces can also be defined in terms of the projections from $\mathbb{R}^d$ onto the tangent space of Γ_t. In particular, for a function f and a vector field $\mathbf{X}$ defined in a neighborhood of $\Gamma_t \subset \mathbb{R}^d$, the tangential gradient of f is

$$\nabla_{\Gamma_t} f = \nabla f - (\mathbf{N}\cdot\nabla f)\mathbf{N}, \tag{3.9}$$

where ∇f is the gradient in $\mathbb{R}^d$; and the tangential divergence of $\mathbf{X}$ is given by

$$\mathrm{div}_{\Gamma_t}\mathbf{X} = \mathrm{div}_{\mathbb{R}^d}\mathbf{X} - \mathbf{N}\cdot\mathrm{D}_{\mathbf{N}}\mathbf{X}, \tag{3.10}$$

where D is the covariant derivative in $\mathbb{R}^d$. The above definitions are identical to the intrinsic ones given earlier. The Laplace-Beltrami operator can be calculated in an equivalent way by:

$$\begin{aligned} \triangle_{\Gamma_t} f &= \mathrm{div}_{\Gamma_t}\nabla_{\Gamma_t} f = \mathrm{div}_{\Gamma_t}\nabla f - \mathrm{div}_{\Gamma_t}[(\mathbf{N}\cdot\nabla f)\mathbf{N}] \\ &= \triangle_{\mathbb{R}^d} f - \mathrm{D}^2 f(\mathbf{N},\mathbf{N}) - \kappa\mathbf{N}\cdot\nabla f, \end{aligned} \tag{3.11}$$

for any C^2 function defined in a neighborhood of $\Gamma_t \subset \mathbb{R}^d$.

Next, we shall derive the evolution equations. For the evolution of $\mathbf{N}$, it holds that

$$0 \equiv \partial_t(\mathbf{N}\cdot\mathbf{F}_{,i}) = \partial_t\mathbf{N}\cdot\mathbf{F}_{,i} + \mathbf{N}\cdot\mathbf{v}_{,i},$$

which, together with the fact that $\mathbf{N}$ has unit length, implies that

$$\partial_t\mathbf{N} = -g^{ij}(\mathbf{N}\cdot\mathbf{v}_{,j})\mathbf{F}_{,i}, \tag{3.12}$$

in other words,

$$\mathbb{D}_t\mathbf{N} = -\left[(\nabla\mathbf{v})^*\cdot\mathbf{N}\right]^\top, \tag{3.13}$$

here $\mathbb{D}_t$ is the material derivative along the trajectory of $\mathbf{v}$. For the metric tensor, observe that

$$\partial_t g_{ij} = \mathbf{v}_{,i}\cdot\mathbf{F}_{,j} + \mathbf{v}_{,j}\cdot\mathbf{F}_{,i} =: 2A_{ij}. \tag{3.14}$$

One can check that $\mathbf{A}$ is a tensor on Γ_t. In fact,

$$\mathbf{A} = (\mathrm{Def}\,\mathbf{v}^\top) + v^\perp\mathrm{I\!I}, \tag{3.15}$$

where "Def" represents the deformation tensor on Γ_t. In particular, the material derivative of the area element is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\det(g_{ij})}\right) = \frac{1}{2}\sqrt{\det(g_{ij})}g^{kl}\partial_t(g_{kl}) = (\mathrm{div}_{\Gamma_t}\mathbf{v})\sqrt{\det(g_{ij})},$$

i.e.,

$$\mathbb{D}_t\,\mathrm{d}S_t = \mathrm{div}_{\Gamma_t}\mathbf{v}\,\mathrm{d}S_t\,. \tag{3.16}$$

The evolution equation for the second fundamental form is

$$\partial_t I\!I_{ij} = -\mathbf{N}\cdot(\mathbf{v}_{,ij} - \Gamma_{ij}^k\mathbf{v}_{,k}), \tag{3.17}$$

in particular,

$$(\mathbb{D}_t \mathrm{I\!I})^\top = -\mathrm{D}^\top\big[(\mathrm{D}\mathbf{v})^*\mathbf{N}\big]^\top - \mathrm{I\!I}\cdot(\mathrm{D}\mathbf{v})^\top. \tag{3.18}$$

The evolution of κ is given by

$$\begin{aligned}\mathbb{D}_t\kappa &:= \partial_t(\kappa\circ\mathbf{F}) = \partial_t\Big(I\!I_{ij}g^{ij}\Big)\\ &= \mathbf{N}\cdot\Big[g^{ij}\Big(-\mathbf{v}_{,ij} + \Gamma^k_{ij}\mathbf{v}_{,k}\Big)\Big] - 2I\!I_{ij}A^{ij}\\ &= -\mathbf{N}\cdot\triangle_{\Gamma_t}\mathbf{v} - 2\langle\mathrm{I\!I}|\mathbf{A}\rangle,\end{aligned} \tag{3.19}$$

where $\langle\cdot|\cdot\rangle$ is the standard inner product of tensors defined by

$$\langle\mathbf{A}|\mathbf{B}\rangle := A_{ij}B^{ij} = A_{ij}B_{kl}g^{ik}g^{jl}.$$

The second order evolution equation is:

$$\begin{aligned}\mathbb{D}_t{}^2\kappa = &-\mathbf{N}\cdot\triangle_{\Gamma_t}(\mathbb{D}_t\mathbf{v}) + 2\mathbf{N}\cdot(\nabla\mathbf{v})\cdot(\triangle_{\Gamma_t}\mathbf{v})^\top + 4\Big\langle\mathbf{A}\Big|\mathbf{N}\cdot\big(\mathrm{D}_{\Gamma_t}\big)^2\mathbf{v}\Big\rangle - \kappa\Big|\big[(\nabla\mathbf{v})^*\cdot\mathbf{N}\big]^\top\Big|^2\\ &+ 4\langle\mathrm{I\!I}\cdot\mathbf{A} + \mathbf{A}\cdot\mathrm{I\!I}|\mathbf{A}\rangle - 2\langle\mathrm{I\!I}|\mathrm{D}_{\Gamma_t}\mathbf{v}\cdot\mathrm{D}_{\Gamma_t}\mathbf{v}\rangle - 2\Big\langle\mathrm{I\!I}\Big|[\mathrm{D}(\mathbb{D}_t\mathbf{v})]^\top\Big\rangle.\end{aligned} \tag{3.20}$$

Here we explain some terms appeared in the last expressions. If one assumes that

$$\mathbf{N} \equiv N^\alpha\mathbf{e}_{(\alpha)} \quad \text{and} \quad \mathbf{v} \equiv v^\alpha\mathbf{e}_{(\alpha)},$$

for which $\mathbf{e}_{(\alpha)}(1 \le \alpha \le d)$ is an orthonormal basis of $\mathbb{R}^d$, then

$$\mathbf{N}\cdot(\nabla\mathbf{v})\cdot(\triangle_{\Gamma_t}\mathbf{v})^\top = \sum_\alpha N^\alpha\nabla v^\alpha\cdot(\triangle_{\Gamma_t}\mathbf{v})^\top, \quad \mathbf{N}\cdot(\mathrm{D}_{\Gamma_t})^2\mathbf{v} = \sum_\alpha N^\alpha(\mathrm{D}_{\Gamma_t})^2 v^\alpha,$$

and

$$\mathrm{D}_{\Gamma_t}\mathbf{v}\cdot\mathrm{D}_{\Gamma_t}\mathbf{v} = \sum_\alpha \mathrm{D}_{\Gamma_t}v^\alpha\otimes\mathrm{D}_{\Gamma_t}v^\alpha.$$

By using the identity (3.8), one can derive an alternate formula:

$$\begin{aligned}\mathbb{D}_t{}^2\kappa = &-\triangle_{\Gamma_t}(\mathbf{N}\cdot\mathbb{D}_t\mathbf{v}) - |\mathrm{I\!I}|^2(\mathbf{N}\cdot\mathbb{D}_t\mathbf{v}) + \nabla_{\Gamma_t}\kappa\cdot\mathbb{D}_t\mathbf{v} + 2\mathbf{N}\cdot(\nabla\mathbf{v})\cdot(\triangle_{\Gamma_t}\mathbf{v})^\top\\ &+ 4\Big\langle\mathbf{A}\Big|\mathbf{N}\cdot\big(\mathrm{D}_{\Gamma_t}\big)^2\mathbf{v}\Big\rangle - \kappa\Big|\big[(\nabla\mathbf{v})^*\cdot\mathbf{N}\big]^\top\Big|^2 - 2\langle\mathrm{I\!I}|\mathrm{D}_{\Gamma_t}\mathbf{v}\cdot\mathrm{D}_{\Gamma_t}\mathbf{v}\rangle + 4\langle\mathrm{I\!I}\cdot\mathbf{A} + \mathbf{A}\cdot\mathrm{I\!I}|\mathbf{A}\rangle.\end{aligned} \tag{3.21}$$

3.2. **Reference hypersurface.** Let k be an integer with $k \ge 2$, and $\Gamma_* \subset \Omega$ a compact reference hypersurface without boundary separating Ω into two disjoint simply-connected domains $\Omega_*^\pm$. Assume that Γ_* is of Sobolev class $H^{\frac{3}{2}k+1}$. Denote by $\mathbf{N}_{*+}$ the outward unit normal of $\partial\Omega_*^+ = \Gamma_*$ and $\mathbf{N}_{*-} := -\mathbf{N}_{*+}$ the outward unit normal of $\Gamma_* \subset \partial\Omega_*^-$. Let $\mathrm{I\!I}_{*\pm}$ be the second fundamental form of Γ_* with respect to $\mathbf{N}_{*\pm}$, and $\kappa_{*\pm}$ the corresponding mean curvature.

As in [SZ11], we shall consider the evolution of hypersurfaces in a tubular neighborhood of Γ_*. Although it is natural to take normal bundle coordinates of Γ_* in classical geometric arguments, it would be better not to do so. Indeed, if Γ_* is of finite regularity, $\mathbf{N}_*$ has one less derivatives than Γ_*, hence one needs to take another transversal vector field to obtain the Fermi coordinates of the same regularity as that of Γ_*. For example, one can take a unit vector field $\boldsymbol{\nu} \in H^{\frac{3}{2}k+1}(\Gamma_*;\mathbb{R}^2)$ for which $\boldsymbol{\nu}\cdot\mathbf{N}_{*+} \ge 9/10$ by mollifying $\mathbf{N}_*$.

It follows from the implicit function theorem that there exists a constant $\delta_0 > 0$ depending on Γ_* and $\boldsymbol{\nu}$ so that

$$\begin{aligned} \varphi : \Gamma_* \times (-\delta_0, \delta_0) &\to \mathbb{R}^3 \\ (p, \gamma) &\mapsto p + \gamma \boldsymbol{\nu} \end{aligned}$$

is an $H^{\frac{3}{2}k+1}$ diffeomorphism onto a neighborhood of Γ_*. Therefore, each hypersurface Γ close to Γ_* in the C^1 topology is associated to a unique height function $\gamma_\Gamma : \Gamma_* \to \mathbb{R}$ so that

$$\Phi_\Gamma(p) := p + \gamma_\Gamma(p)\boldsymbol{\nu}(p) \tag{3.22}$$

is a diffeomorphism from Γ_* to Γ. Thus, one can use the function γ_Γ to represent the hypersurface Γ.

Definition 3.1. For $\delta > 0$ and $\frac{1+3}{2} < s \leq 1 + \frac{3}{2}k$, define $\Lambda(\Gamma_*, s, \delta)$ to be the collection of all hypersurfaces Γ close to Γ_*, whose associated coordinate functions γ_Γ satisfy $|\gamma_\Gamma|_{H^s(\Gamma_*)} < \delta$.

As $s > \frac{3-1}{2} + 1$ implies $H^s(\Gamma_*) \hookrightarrow C^1(\Gamma_*)$, $\delta \ll 1$ yields that each $\Gamma \in \Lambda$ also separates Ω into two disjoint simply-connected domains.

3.3. **Recovering a hypersurface from its mean curvature.** Here, we characterize the moving hypersurface by its mean curvature $\kappa_+ := \operatorname{tr} \mathrm{I\!I}_+$. Recall that the second fundamental form is defined by

$$\mathrm{I\!I}_+(\boldsymbol{\tau}) := \mathrm{D}_{\boldsymbol{\tau}} \mathbf{N}_+ \quad \text{for} \quad \boldsymbol{\tau} \in \mathrm{T}\Gamma. \tag{3.23}$$

For an H^s hypersurface $\Gamma \in \Lambda(\Gamma_*, s, \delta_0)$ with $s > 2$, the unit normal $\mathbf{N}_+$ has the same regularity as $\boldsymbol{\nabla}\gamma_\Gamma$. Then the mapping from $\gamma_\Gamma \in H^s(\Gamma_*)$ to the mean curvature $\kappa_+ \circ \Phi_\Gamma \in H^{s-2}(\Gamma_*)$ is smooth.

In order to establish a bijection between them, one may consider a modification

$$\mathfrak{K}[\gamma_\Gamma](p) \equiv \varkappa_a(p) := \kappa_+ \circ \Phi_\Gamma(p) + a^2 \gamma_\Gamma(p) \quad \text{for} \quad p \in \Gamma_*, \tag{3.24}$$

where a is a parameter depending only on Γ_* and $\boldsymbol{\nu}$ (c.f. [SZ11]).

For a small constant $\delta_0 > 0$, define

$$\Lambda_* := \Lambda\left(\Gamma_*, \frac{3}{2}k - \frac{1}{2}, \delta_0\right). \tag{3.25}$$

Then, the following lemma holds:

Lemma 3.2. *For $\Gamma \in \Lambda_*$ with $\kappa_+ \in H^s(\Gamma)$, $\frac{3}{2}k - \frac{5}{2} \leq s \leq \frac{3}{2}k - 1$, the following estimate holds:*

$$|\mathbf{N}_+|_{H^{s+1}(\Gamma)} + |\mathrm{I\!I}_+|_{H^s(\Gamma)} \leq C_*\big(1 + |\kappa_+|_{H^s(\Gamma)}\big), \tag{3.26}$$

for some constant C_* depending only on Λ_*.

The proof of the lemma follows from the bootstrap arguments and Simons' identity (3.7). For the details, one can refer to [SZ08a, p. 719].

If Λ_* is regarded as an open subset of $H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)$, then $\mathfrak{K}$ is a C^3-morphism from $\Lambda_* \subset H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)$ to $H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$. Furthermore, by taking $a \gg 1$, one may deduce from the positivity of $(\delta\mathfrak{K}/\delta\gamma_\Gamma)\big|_{\Gamma_*}$ that $\mathfrak{K}$ is actually a C^3 diffeomorphism, and the following proposition holds (c.f. [SZ11, Lemma 2.2]):

Proposition 3.3. *There are positive constants $C_*, \delta_0, \delta_1, a_0$ depending only on Γ_* and $\boldsymbol{\nu}$ such that for $a \geq a_0$, $\mathfrak{K}$ is a C^3 diffeomorphism from $\Lambda_* \subset H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)$ to $H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$. Denote by*

$$B_{\delta_1} := \left\{ \varkappa_a \,\middle|\, |\varkappa_a - \kappa_{*+}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} < \delta_1 \right\},$$

where κ_{*+} is the mean curvature of Γ_* with respect to $\mathbf{N}_{*+}$, then

$$\left|\mathfrak{K}^{-1}\right|_{C^3\left(B_{\delta_1}; H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)\right)} \leq C_*.$$

Furthermore, if $\varkappa_a \in B_{\delta_1} \cap H^{s-2}(\Gamma_*)$ with $\frac{3}{2}k - \frac{1}{2} \leq s \leq \frac{3}{2}k + 1$, then $\gamma_\Gamma, \Phi_\Gamma \in H^s(\Gamma_*)$, and for $\max\{s'-2, -s\} \leq s'' \leq s' \leq s$, it holds that

$$\left|\delta\mathfrak{K}^{-1}(\varkappa_a)\right|_{\mathscr{L}(H^{s''}(\Gamma_*); H^{s'}(\Gamma_*))} \leq C_* a^{s'-s''-2}\left(1 + |\varkappa_a|_{H^{s-2}(\Gamma_*)}\right), \tag{3.27}$$

where $\delta\mathfrak{K}^{-1}$ is the functional variation of $\mathfrak{K}^{-1}$.

3.4. **Harmonic coordinates and Dirichlet-Neumann operators.** Given a hypersurface $\Gamma \in \Lambda(\Gamma_*, s, \delta)$, define a map $\mathfrak{X}_\Gamma^\pm : \Omega_*^\pm \to \Omega_\Gamma^\pm$ by

$$\begin{cases} \triangle_y \mathfrak{X}_\Gamma^\pm = 0 & \text{for } y \in \Omega_*^\pm, \\ \mathfrak{X}_\Gamma^\pm(z) = \Phi_\Gamma(z) & \text{for } z \in \Gamma_*, \\ \mathfrak{X}_\Gamma^-(z') = z' & \text{for } z' \in \partial\Omega. \end{cases} \tag{3.28}$$

Then, it is clear that

$$\left\| \nabla \mathfrak{X}_\Gamma^\pm - \mathrm{Id}|_{\Omega_*^\pm} \right\|_{H^{s-\frac{1}{2}}(\Omega_*^\pm)} \leq C|\gamma_\Gamma|_{H^s(\Gamma_*)} < C\delta, \tag{3.29}$$

where $C > 0$ is uniform in $\Gamma \in \Lambda(\Gamma_*, s, \delta)$. Thus there is a constant $\delta_0 > 0$ determined by Γ_* and $\boldsymbol{\nu}$, for which $\mathfrak{X}_\Gamma^\pm$ are diffeomorphisms from $\Omega_*^\pm$ to $\Omega_\Gamma^\pm$ respectively, whenever $\delta \leq \delta_0$.

With the notations in (3.25), we list some basic inequalities, whose proofs are standard (c.f. [SZ08a, BCD11]).

Lemma 3.4. *Suppose that $\Gamma \in \Lambda_*$. Then there are constants $C_1, C_2 > 0$, depending on Λ_*, so that*

(1) If $u_\pm \in H^\sigma(\Omega_\Gamma^\pm)$ for $\sigma \in \left[-\frac{3}{2}k, \frac{3}{2}k\right]$, then

$$\frac{1}{C_1}\|u_\pm\|_{H^\sigma(\Omega_\Gamma^\pm)} \leq \left\|u_\pm \circ \mathfrak{X}_\Gamma^\pm\right\|_{H^\sigma(\Omega_*^\pm)} \leq C_1\|u_\pm\|_{H^\sigma(\Omega_\Gamma^\pm)}.$$

(2) If $f \in H^s(\Gamma)$ for $s \in \left[\frac{1}{2} - \frac{3}{2}k, \frac{3}{2}k - \frac{1}{2}\right]$, then

$$\frac{1}{C_2}|f|_{H^s(\Gamma)} \leq |f \circ \Phi_\Gamma|_{H^s(\Gamma_*)} \leq C_2|f|_{H^s(\Gamma)}.$$

Lemma 3.5. *Assume that $\Gamma \in \Lambda_*$. Then there are constants $C_1, C_2 > 0$ determined by Λ_* such that*

(1) For $u_\pm \in H^{\sigma_1}(\Omega_\Gamma^\pm)$, $w_\pm \in H^{\sigma_2}(\Omega_\Gamma^\pm)$ and $\sigma_1 \leq \sigma_2$,

$$\|u_\pm \cdot w_\pm\|_{H^{\sigma_1+\sigma_2-\frac{3}{2}}(\Omega_\Gamma^\pm)} \leq C_1\|u_\pm\|_{H^{\sigma_1}(\Omega_\Gamma^\pm)}\|w\|_{H^{\sigma_2}(\Omega_\Gamma^\pm)} \quad \textit{if} \quad \sigma_2 < \frac{3}{2}, \quad 0 < \sigma_1 + \sigma_2 \leq \frac{3}{2}k.$$

$$\|u_\pm \cdot w_\pm\|_{H^{\sigma_1}(\Omega_\Gamma^\pm)} \leq C_1\|u_\pm\|_{H^{\sigma_1}(\Omega_\Gamma^\pm)}\|w_\pm\|_{H^{\sigma_2}(\Omega_\Gamma^\pm)} \quad \textit{if} \quad \frac{3}{2} < \sigma_2 \leq \frac{3}{2}k, \quad \sigma_1 + \sigma_2 > 0.$$

(2) For $f \in H^{s_1}(\Gamma)$, $g \in H^{s_2}(\Gamma)$ and $s_1 \le s_2$,

$$|fg|_{H^{s_1+s_2-\frac{2}{2}}(\Gamma)} \le C_2 |f|_{H^{s_1}(\Gamma)} |g|_{H^{s_2}(\Gamma)} \quad \textit{if} \quad s_2 < 1, \quad 0 \le s_1 + s_2 \le \frac{3}{2}k - \frac{1}{2}.$$

$$|fg|_{H^{s_1}(\Gamma)} \le C_2 |f|_{H^{s_1}(\Gamma)} |g|_{H^{s_2}(\Gamma)} \quad \textit{if} \quad 1 < s_2 \le \frac{3}{2}k - \frac{1}{2}, \quad s_1 + s_2 > 0.$$

For any smooth function f defined on $\Gamma \in \Lambda_*$, denote by $\mathcal{H}_\pm f$ the harmonic extensions to $\Omega_\Gamma^\pm$, namely

$$\begin{cases} \triangle \mathcal{H}_+ f = 0 & \text{for} \quad x \in \Omega_\Gamma^+, \\ \mathcal{H}_+ f = f & \text{for} \quad x \in \Gamma, \end{cases} \tag{3.30}$$

and

$$\begin{cases} \triangle \mathcal{H}_- f = 0 & \text{for} \quad x \in \Omega_\Gamma^-, \\ \mathcal{H}_- f = f & \text{for} \quad x \in \Gamma, \\ \mathrm{D}_{\tilde{\mathrm{N}}} \mathcal{H}_- f = 0 & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{3.31}$$

The Dirichlet-Neumann operators are defined to be

$$\mathcal{N}_\pm f := \mathbf{N}_\pm \cdot (\nabla \mathcal{H}_\pm f)|_\Gamma. \tag{3.32}$$

Assume that $\Gamma \in \Lambda_* \subset H^{\frac{3}{2}k-\frac{1}{2}}$ and $\frac{3}{2} - \frac{3}{2}k \le s \le \frac{3}{2}k - \frac{1}{2}$. The Dirichlet-Neumann operators $\mathcal{N}_\pm : H^s(\Gamma) \to H^{s-1}(\Gamma)$ satisfy the following properties (c.f. [SZ08a, pp. 738-741]):

1. $\mathcal{N}_\pm$ are self-adjoint on $L^2(\Gamma)$ with compact resolvents;
2. $\ker(\mathcal{N}_\pm) = \{\text{const.}\}$;
3. There is a constant $C_* > 0$ uniform in $\Gamma \in \Lambda_*$ so that
$$C_* |f|_{H^s(\Gamma)} \ge |\mathcal{N}_\pm(f)|_{H^{s-1}(\Gamma)} \ge \frac{1}{C_*} |f|_{H^s(\Gamma)},$$
for any f satisfying $\int_\Gamma f \,\mathrm{d}S = 0$;
4. For $\frac{1}{2} - \frac{3}{2}k \le s_1 \le \frac{3}{2}k - \frac{1}{2}$, there is a constant C_* determined by Λ_* so that
$$\frac{1}{C_*} (\mathrm{I} - \triangle_\Gamma)^{\frac{s_1}{2}} \le (\mathrm{I} + \mathcal{N}_\pm)^{s_1} \le C_* (\mathrm{I} - \triangle_\Gamma)^{\frac{s_1}{2}},$$
i.e., the norms on $H^{s_1}(\Gamma)$ defined by interpolating $(\mathrm{I} - \triangle_\Gamma)^{\frac{1}{2}}$ and $(\mathrm{I} + \mathcal{N}_\pm)$ are equivalent;
5. For $\frac{1}{2} - \frac{3}{2}k \le s_2 \le \frac{3}{2}k - \frac{3}{2}$,
$$(\mathcal{N}_\pm)^{-1} : H_0^{s_2}(\Gamma) \to H_0^{s_2+1}(\Gamma),$$
$$H_0^{s_2}(\Gamma) = \left\{ f \in H^{s_2}(\Gamma) \,\middle|\, \int_\Gamma f \,\mathrm{d}S = 0 \right\}$$
are well-defined and bounded uniformly in $\Gamma \in \Lambda_*$.

Remark. We shall denote by $(\mathcal{N}_\pm)^{-1} := (\mathcal{N}_\pm)^{-1} \circ \mathcal{P}$ for simplicity, where

$$\mathcal{P} f := f - \fint_\Gamma f \,\mathrm{d}S \equiv f - \langle f \rangle .$$

is the projection into mean-free functions on Γ.

The following notations will be used later:

$$\bar{\mathcal{N}} := \frac{1}{\rho_+}\mathcal{N}_+ + \frac{1}{\rho_-}\mathcal{N}_-, \tag{3.33}$$

$$\widetilde{\mathcal{N}} := \left(\frac{1}{\rho_+}\mathcal{N}_+\right)\bar{\mathcal{N}}^{-1}\left(\frac{1}{\rho_-}\mathcal{N}_-\right) = \left[\left(\frac{1}{\rho_+}\mathcal{N}_+\right)^{-1} + \left(\frac{1}{\rho_-}\mathcal{N}_-\right)^{-1}\right]^{-1}. \tag{3.34}$$

At the end of this subsection, we state an important lemma (c.f. [SZ08b, p. 863]):

Lemma 3.6. *Suppose that $\Gamma \in \Lambda_*$ with $\kappa \in H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma)$. Then for $\frac{1}{2} - \frac{3}{2}k \le s \le \frac{3}{2}k - \frac{1}{2}$, one has*

$$\left|(-\triangle_\Gamma)^{\frac{1}{2}} - \mathcal{N}_\pm\right|_{\mathscr{L}(H^s(\Gamma))} \le C_*\left(1 + |\kappa|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma)}\right), \tag{3.35}$$

where the constant $C_* > 0$ is uniform in $\Gamma \in \Lambda_*$.

3.5. **Commutator estimates.** For vector fields (not necessarily solenoidal) $\mathbf{v}_\pm(t) : \Omega_t^\pm \to \mathbb{R}^3$, denote by

$$\mathbb{D}_{t\pm} := \partial_t + \mathrm{D}_{\mathbf{v}_\pm}.$$

Then one has the following lemma:

Lemma 3.7. *Suppose that $\Gamma_t \in \Lambda_*$, and $\mathbf{v}_\pm \in H^{\frac{3}{2}k}(\Omega_t^\pm)$ are the evolution velocities of $\Omega_t^\pm$, so $\mathbf{v}_\pm$ are both evolution velocities of Γ_t. Let $f(t,x) : \Gamma_t \to \mathbb{R}$ and $h(t,x) : \Omega \setminus \Gamma_t \to \mathbb{R}$ be two functions. Then the following commutator estimates hold:*

I. For $1 \le s \le \frac{3}{2}k$, $\|[\mathbb{D}_{t\pm}, \mathcal{H}_\pm]f\|_{H^s(\Omega_t^\pm)} \lesssim_{\Lambda_*} |f|_{H^{s-\frac{1}{2}}(\Gamma_t)} \cdot \|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}$;

II. For $1 \le s \le \frac{3}{2}k$, $\left\|[\mathbb{D}_{t\pm}, \triangle_\pm^{-1}]h_\pm\right\|_{H^s(\Omega_t^\pm)} \lesssim_{\Lambda_*} \|h_\pm\|_{H^{s-2}(\Omega_t^\pm)} \cdot \|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}$;

III. For $-\frac{1}{2} \le s \le \frac{3}{2}k - \frac{3}{2}$, $|[\mathbb{D}_{t\pm}, \mathcal{N}_\pm]f|_{H^s(\Gamma_t)} \lesssim_{\Lambda_*} |f|_{H^{s+1}(\Gamma_t)} \cdot |\mathbf{v}_\pm|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_t)}$;

IV. For $\frac{1}{2} \le s \le \frac{3}{2}k - \frac{1}{2}$, $\left|[\mathbb{D}_{t\pm}, \mathcal{N}_\pm^{-1}]f\right|_{H^s(\Gamma_t)} \lesssim_{\Lambda_*} |f|_{H^{s-1}(\Gamma_t)} \cdot |\mathbf{v}_\pm|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_t)}$;

V. For $-2 \le s \le \frac{3}{2}k - \frac{5}{2}$, $\left|[\mathbb{D}_{t\pm}, \triangle_{\Gamma_t}]f\right|_{H^s(\Gamma_t)} \lesssim_{\Lambda_*} |f|_{H^{s+2}(\Gamma_t)} \cdot |\mathbf{v}_\pm|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_t)}$;

VI. For $0 \le s \le \frac{3}{2}k - 1$, $\|[\mathbb{D}_{t\pm}, \mathrm{D}]h_\pm\|_{H^s(\Omega_t^\pm)} \lesssim_{\Lambda_*} \|h_\pm\|_{H^{s+1}(\Omega_t^\pm)} \cdot \|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}$.

The proof of Lemma 3.7 follows from the identities introduced in [SZ08a, pp. 709-710] and standard product estimates.

3.6. **Div-curl systems.** In this subsection, we list some basic results on div-curl systems (c.f. [CS17] for details):

Theorem 3.8. *Suppose that U is a bounded domain in $\mathbb{R}^3$ for which $\partial U \in H^{\frac{3}{2}k-\frac{1}{2}}$. Given $\mathbf{f}, g \in H^{l-1}(U)$ with $\nabla \cdot \mathbf{f} = 0$ and $h \in H^{l-\frac{1}{2}}(\partial U)$, consider the system:*

$$\begin{cases} \nabla \times \mathbf{u} = \mathbf{f} & \text{in } U, & (3.36a) \\ \nabla \cdot \mathbf{u} = g & \text{in } U, & (3.36b) \\ \mathbf{u} \cdot \mathbf{N} = h & \text{on } \partial U. & (3.36c) \end{cases}$$

If on each connected component Γ of ∂U, one has that

$$\int_\Gamma \mathbf{f} \cdot \mathbf{N} \, \mathrm{d}S = 0, \tag{3.37}$$

and the following compatibility condition holds true:

$$\int_{\partial U} h \, \mathrm{d}S = \int_U g \, \mathrm{d}x \,, \tag{3.38}$$

then for $1 \le l \le \frac{3}{2}k - 1$, *there is a solution* $\mathbf{u} \in H^l(U)$ *such that*

$$\|\mathbf{u}\|_{H^l(U)} \le C\Big(|\partial U|_{H^{\frac{3}{2}k-\frac{1}{2}}}\Big) \cdot \Big(\|\mathbf{f}\|_{H^{l-1}(U)} + \|g\|_{H^{l-1}(U)} + |h|_{H^{l-\frac{1}{2}}(\partial U)}\Big). \tag{3.39}$$

The solution is unique whenever U *is simply-connected.*

Remark. If $\mathbf{f} = \nabla \times \mathbf{u}$ for some vector field $\mathbf{u}$, then (3.37) holds naturally (see [CS17, Remark 1.2]).

4. Reformulation of the Problem

4.1. **Velocity fields on the interface.** Since the interface Γ_t separates two plasmas, and $\mathbf{v}_\pm$ have the same normal components on Γ_t, it is natural to consider the evolution of κ_+ with respect to some weighted velocity

$$\mathbf{u}_\lambda := \lambda \mathbf{v}_+ + (1-\lambda)\mathbf{v}_- \tag{4.1}$$

for some $0 \le \lambda \le 1$.

Denote by

$$\mathbb{D}_{t_\lambda} := \partial_t + \mathrm{D}_{\mathbf{u}_\lambda},$$

then

$$\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda = \big(\partial_t + \lambda\,\mathrm{D}_{\mathbf{v}_+} + (1-\lambda)\,\mathrm{D}_{\mathbf{v}_-}\big)(\lambda\mathbf{v}_+ + (1-\lambda)\mathbf{v}_-).$$

In view of (1.1a), it holds that

$$\begin{aligned}\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda = {} & \lambda^2\Big(-\frac{1}{\rho_+}\nabla p^+ + \mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+\Big) + (1-\lambda)^2\Big(-\frac{1}{\rho_-}\nabla p^- + \mathrm{D}_{\mathbf{h}_-}\mathbf{h}_-\Big) \\ & + \lambda(1-\lambda)(\mathbb{D}_{t+}\mathbf{v}_- + \mathbb{D}_{t-}\mathbf{v}_+).\end{aligned}$$

Since

$$\mathbb{D}_{t+}\mathbf{v}_- = \partial_t\mathbf{v}_- + \mathrm{D}_{\mathbf{v}_+}\mathbf{v}_- = \mathrm{D}_{\mathbf{v}_+-\mathbf{v}_-}\mathbf{v}_- - \frac{1}{\rho_-}\nabla p^- + \mathrm{D}_{\mathbf{h}_-}\mathbf{h}_-,$$

and

$$\mathbb{D}_{t-}\mathbf{v}_+ = \mathrm{D}_{\mathbf{v}_--\mathbf{v}_+}\mathbf{v}_+ - \frac{1}{\rho_+}\nabla p^+ + \mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+,$$

one may write that

$$\mathbb{D}_{t+}\mathbf{v}_- + \mathbb{D}_{t-}\mathbf{v}_+ = -\,\mathrm{D}_{\mathbf{w}}\mathbf{w} - \Big(\frac{1}{\rho_+}\nabla p^+ + \frac{1}{\rho_-}\nabla p^-\Big) + \mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ + \mathrm{D}_{\mathbf{h}_-}\mathbf{h}_-,$$

where $\mathbf{w} \in \mathrm{T}\Gamma_t$ is defined to be

$$\mathbf{w} \equiv [\![\mathbf{v}]\!] := \mathbf{v}_+ - \mathbf{v}_-. \tag{4.2}$$

Therefore,

$$\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda = -\frac{\lambda}{\rho_+}\nabla p^+ - \frac{1-\lambda}{\rho_-}\nabla p^- + \lambda\,\mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ + (1-\lambda)\,\mathrm{D}_{\mathbf{h}_-}\mathbf{h}_- - \lambda(1-\lambda)\,\mathrm{D}_{\mathbf{w}}\mathbf{w}. \tag{4.3}$$

Next, we introduce a useful decomposition of the pressure:

$$\frac{1}{\rho_\pm} p^\pm = p^\pm_{\mathbf{v},\mathbf{v}} - p^\pm_{\mathbf{h},\mathbf{h}} + \alpha^2 p^\pm_\kappa + p^\pm_b. \tag{4.4}$$

Here $p^\pm_{\mathbf{a},\mathbf{b}}$ are the solutions to the following elliptic problems respectively:

$$\begin{cases} \triangle p^+_{\mathbf{a},\mathbf{b}} = -\operatorname{tr}(\mathrm{D}\mathbf{a}_+ \cdot \mathrm{D}\mathbf{b}_+) & \text{for} \quad x \in \Omega^+_t, \\ p^+_{\mathbf{a},\mathbf{b}} = 0 \quad \text{for} \quad x \in \Gamma_t; \end{cases} \tag{4.5}$$

$$\begin{cases} \triangle p^-_{\mathbf{a},\mathbf{b}} = -\operatorname{tr}(\mathrm{D}\mathbf{a}_- \cdot \mathrm{D}\mathbf{b}_-) & \text{for} \quad x \in \Omega^-_t, \\ p^-_{\mathbf{a},\mathbf{b}} = 0 \quad \text{for} \quad x \in \Gamma_t, \\ \mathrm{D}_{\widetilde{\mathbf{N}}} p^-_{\mathbf{a},\mathbf{b}} = \widetilde{\mathrm{I\!I}}(\mathbf{a}_-, \mathbf{b}_-) & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{4.6}$$

where $\mathbf{a} = \mathbf{a}_+ \mathbb{1}_{\Omega^+_t} + \mathbf{a}_- \mathbb{1}_{\Omega^-_t}$ and $\mathbf{b} = \mathbf{b}_+ \mathbb{1}_{\Omega^+_t} + \mathbf{b}_- \mathbb{1}_{\Omega^-_t}$ are solenoidal vector fields satisfying $\mathbf{a}_- \cdot \widetilde{\mathbf{N}} = 0 = \mathbf{b}_- \cdot \widetilde{\mathbf{N}}$ on $\partial\Omega$. p_κ and p_b are given respectively by (c.f. [SZ11]):

$$p^\pm_\kappa := \frac{1}{\rho_+ \rho_-} \mathcal{H}_\pm \bar{\mathcal{N}}^{-1} \mathcal{N}_\mp \kappa_\pm, \quad \text{and} \quad p^\pm_b := \frac{1}{\rho_\pm} \mathcal{H}_\pm \mathfrak{p}, \tag{4.7}$$

where $\mathfrak{p}$ is a function defined on Γ_t whose expression will be determined later.

With this decomposition, it is routine to check that $p^\pm_{\mathbf{v},\mathbf{v}} = p^\pm_{\mathbf{h},\mathbf{h}} = 0$, $\rho_+ p^+_b = \rho_- p^-_b$, and $\rho_+ p^+_\kappa - \rho_- p^-_\kappa = \kappa_+$ hold simultaneously on Γ_t. Namely, (1.2b) is satisfied automatically.

Next, we will derive the explicit formula of $\mathfrak{p}$ by using (1.1a), (1.2a) and (1.2c). Indeed, multiplying (1.1a) by $\mathbf{N}_+$, one has

$$\mathbf{N}_+ \cdot \mathbb{D}_{t+} \mathbf{v}_+ + \frac{1}{\rho_+} \mathrm{D}_{\mathbf{N}_+} p^+ = \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_+} \mathbf{h}_+,$$

which implies that

$$\partial_t \theta + \frac{1}{\rho_+} \mathrm{D}_{\mathbf{N}_+} p^+ = \mathbf{v}_+ \cdot \mathbb{D}_{t+} \mathbf{N}_+ - \mathrm{D}_{\mathbf{v}_+} \theta + \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_+} \mathbf{h}_+.$$

It follows from (1.2c), (3.5) and (3.13) that

$$\begin{aligned} \partial_t \theta + \frac{1}{\rho_+} \mathrm{D}_{\mathbf{N}_+} p^+ &= -\mathbf{N}_+ \cdot (\mathrm{D}\mathbf{v}_+) \cdot \mathbf{v}^\top_+ - \mathrm{D}_{\mathbf{v}_+} \theta + \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_+} \mathbf{h}_+ \\ &= -\mathrm{D}_{\mathbf{v}_+} \theta - \mathrm{D}_{\mathbf{v}^\top_+} \theta + \mathrm{I\!I}_+\left(\mathbf{v}^\top_+, \mathbf{v}^\top_+\right) - \mathrm{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+). \end{aligned}$$

Similarly,

$$-\partial_t \theta + \frac{1}{\rho_-} \mathrm{D}_{\mathbf{N}_-} p^- = \mathrm{D}_{\mathbf{v}_-} \theta + \mathrm{D}_{\mathbf{v}^\top_-} \theta + \mathrm{I\!I}_-\left(\mathbf{v}^\top_-, \mathbf{v}^\top_-\right) - \mathrm{I\!I}_-(\mathbf{h}_-, \mathbf{h}_-).$$

Due to the conventions that $\mathbf{N}_+ + \mathbf{N}_- = \mathbf{0}$, $\mathrm{I\!I}_+ + \mathrm{I\!I}_- = \mathbf{0}$, and the relation that $(\mathbf{v}_+ - \mathbf{v}_-) \in \mathrm{T}\Gamma_t$, summing the above two equations yields

$$\begin{aligned} \frac{1}{\rho_+} \mathrm{D}_{\mathbf{N}_+} p^+ + \frac{1}{\rho_-} \mathrm{D}_{\mathbf{N}_-} p^- = &- \left[2\,\mathrm{D}_{\mathbf{v}^\top_+} \theta - \mathrm{I\!I}_+\left(\mathbf{v}^\top_+, \mathbf{v}^\top_+\right) + \mathrm{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+)\right] \\ &+ \left[2\,\mathrm{D}_{\mathbf{v}^\top_-} \theta - \mathrm{I\!I}_+\left(\mathbf{v}^\top_-, \mathbf{v}^\top_-\right) + \mathrm{I\!I}_+(\mathbf{h}_-, \mathbf{h}_-)\right]. \end{aligned}$$

According to the decomposition (4.4) of $p^{\pm}$ and the relation that

$$\mathrm{D}_{\mathbf{N}_+} p_\kappa^+ + \mathrm{D}_{\mathbf{N}_-} p_\kappa^- = \left(\frac{1}{\rho_+}\mathcal{N}_+\right)\bar{\mathcal{N}}^{-1}\left(\frac{1}{\rho_-}\mathcal{N}_-\right)\kappa_+ + \left(\frac{1}{\rho_-}\mathcal{N}_-\right)\bar{\mathcal{N}}^{-1}\left(\frac{1}{\rho_+}\mathcal{N}_+\right)\kappa_- = 0,$$

it holds that

$$\frac{1}{\rho_+}\mathrm{D}_{\mathbf{N}_+} p^+ + \frac{1}{\rho_-}\mathrm{D}_{\mathbf{N}_-} p^- = \mathrm{D}_{\mathbf{N}_+}\left(p_{\mathbf{v},\mathbf{v}}^+ - p_{\mathbf{h},\mathbf{h}}^+ - p_{\mathbf{v},\mathbf{v}}^- + p_{\mathbf{h},\mathbf{h}}^-\right) + \left(\frac{1}{\rho_+}\mathcal{N}_+ + \frac{1}{\rho_-}\mathcal{N}_-\right)\mathfrak{p}.$$

Therefore, one gets the formula

$$\begin{aligned}\bar{\mathcal{N}}\mathfrak{p} = &- \left[2\,\mathrm{D}_{\mathbf{v}_+^\top}\theta - \mathrm{I\!I}_+\left(\mathbf{v}_+^\top, \mathbf{v}_+^\top\right) + \mathrm{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+) + \mathrm{D}_{\mathbf{N}_+}\left(p_{\mathbf{v},\mathbf{v}}^+ - p_{\mathbf{h},\mathbf{h}}^+\right)\right] \\ &+ \left[2\,\mathrm{D}_{\mathbf{v}_-^\top}\theta - \mathrm{I\!I}_+\left(\mathbf{v}_-^\top, \mathbf{v}_-^\top\right) + \mathrm{I\!I}_+(\mathbf{h}_-, \mathbf{h}_-) + \mathrm{D}_{\mathbf{N}_+}\left(p_{\mathbf{v},\mathbf{v}}^- - p_{\mathbf{h},\mathbf{h}}^-\right)\right] \\ =: &- g^+ + g^-,\end{aligned} \tag{4.8}$$

namely,

$$\mathfrak{p} = \bar{\mathcal{N}}^{-1}(-g^+ + g^-). \tag{4.9}$$

In conclusion, if $(\varGamma_t, \mathbf{v}, \mathbf{h})$ is a solution to (MHD)-(BC) with $\varGamma_t \in \varLambda_*$, $\varGamma_t \in H^{\frac{3}{2}k+1}$ and $\mathbf{v}, \mathbf{h} \in H^{\frac{3}{2}k}(\varOmega \setminus \varGamma_t)$, then the following estimate holds:

$$\left|\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda\right|_{H^{\frac{3}{2}k-2}(\varGamma_t)} \le Q_\lambda\left(\alpha|\varkappa_a|_{H^{\frac{3}{2}k-1}(\varGamma_*)}, |\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\varGamma_*)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\varOmega\setminus\varGamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\varOmega\setminus\varGamma_t)}\right), \tag{4.10}$$

where Q_λ is a generic polynomial depending only on $\varLambda_*$ and λ. (Indeed, for $\frac{1}{2} < s \le \frac{3}{2}k - 2$, one has $|\nabla p_\kappa|_{H^s(\varGamma_t)} \lesssim_{\varLambda_*} |\kappa|_{H^{s+1}(\varGamma_t)}$, so the best estimate on $(\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda)|_{\varGamma_t}$ is its $H^{\frac{3}{2}k-2}$ norm.)

Remark. The following formula holds as long as $\mathbf{v}_\pm$ are the evolution velocities of $\varOmega_t^\pm$ (which is, in particular, independent of the MHD problems):

$$\int_{\varGamma_t} g^+ - g^- \,\mathrm{d}S = -\int_{\varOmega\setminus\varGamma_t} (\nabla\cdot\mathbf{v})^2 \,\mathrm{d}x\,. \tag{4.11}$$

So (4.8) makes sense whenever $\mathbf{v}_\pm$ are both solenoidal. Indeed,

$$\begin{aligned}&\int_{\varGamma_t} g^+ - g^- \,\mathrm{d}S_t \\ &= \int_{\varGamma_t}\left(\mathbb{D}_{t+} + \mathrm{D}_{\mathbf{v}_+^\top}\right)(\mathbf{v}_+\cdot\mathbf{N}_+) - \mathrm{I\!I}_+\left(\mathbf{v}_+^\top, \mathbf{v}_+^\top\right) + \mathrm{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+) + \mathrm{D}_{\mathbf{N}_+}\left(p_{\mathbf{v},\mathbf{v}}^+ - p_{\mathbf{h},\mathbf{h}}^+\right)\mathrm{d}S_t \\ &\quad + \int_{\varGamma_t}\left(\mathbb{D}_{t-} + \mathrm{D}_{\mathbf{v}_-^\top}\right)(\mathbf{v}_-\cdot\mathbf{N}_-) - \mathrm{I\!I}_-\left(\mathbf{v}_-^\top, \mathbf{v}_-^\top\right) + \mathrm{I\!I}_-(\mathbf{h}_-, \mathbf{h}_-) + \mathrm{D}_{\mathbf{N}_-}\left(p_{\mathbf{v},\mathbf{v}}^- - p_{\mathbf{h},\mathbf{h}}^-\right)\mathrm{d}S_t \\ &= \int_{\varOmega_t^+}\mathbb{D}_{t+}(\nabla\cdot\mathbf{v}_+)\,\mathrm{d}x + \int_{\varOmega_t^-}\mathbb{D}_{t-}(\nabla\cdot\mathbf{v}_-)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\varOmega_t^+}(\nabla\cdot\mathbf{v}_+)\,\mathrm{d}x\right) - \int_{\varOmega_t^+}(\nabla\cdot\mathbf{v}_+)^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\varOmega_t^-}(\nabla\cdot\mathbf{v}_-)\,\mathrm{d}x\right) - \int_{\varOmega_t^-}(\nabla\cdot\mathbf{v}_-)^2\,\mathrm{d}x \\ &= -\int_{\varOmega\setminus\varGamma_t}(\nabla\cdot\mathbf{v})^2\,\mathrm{d}x\,.\end{aligned}$$

4.2. **Transformation of the velocity.** As stated in the preliminary, a vector field defined in a bounded simply-connected domain is determined by its divergence, curl and appropriate boundary conditions. Since both the velocity and magnetic fields are solenoidal, they are uniquely determined by the vorticities, currents and boundary conditions.

Therefore, denote by

$$\boldsymbol{\omega}_{*\pm} := (\nabla \times \mathbf{v}_\pm) \circ \mathcal{X}_{\Gamma_t}^{\pm}, \tag{4.12}$$

then the velocity field $\mathbf{v}$ can be uniquely determined by $\varkappa_a$, θ and $\boldsymbol{\omega}_*$ via solving the following div-curl problems:

$$\begin{cases} \nabla \cdot \mathbf{v}_\pm = 0 & \text{in } \Omega_t^\pm, \\ \nabla \times \mathbf{v}_\pm = \boldsymbol{\omega}_{*\pm} \circ (\mathcal{X}_{\Gamma_t}^{\pm})^{-1} & \text{in } \Omega_t^\pm, \\ \mathbf{v}_\pm \cdot \mathbf{N}_+ = \theta & \text{on } \Gamma_t, \\ \mathbf{v}_- \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

Next, for a function $f : \Gamma_t \to \mathbb{R}$, it is natural to pull back $\mathbb{D}_{t_\lambda} f$ to Γ_* via Φ_{Γ_t}, namely, one needs to look for a vector field $\mathbf{u}_{\lambda*} : \Gamma_* \to \mathbb{R}^3$ so that

$$\mathbb{D}_{t_\lambda *}\big(f \circ \Phi_{\Gamma_t}\big) = \big(\mathbb{D}_{t_\lambda} f\big) \circ \Phi_{\Gamma_t}, \tag{4.14}$$

where

$$\mathbb{D}_{t_\lambda *} := \partial_t + \mathrm{D}_{\mathbf{u}_{\lambda*}}. \tag{4.15}$$

It is necessary that

$$(\mathrm{D}f) \circ \Phi_{\Gamma_t} \cdot \big(\partial_t \Phi_{\Gamma_t} + \mathrm{D}\Phi_{\Gamma_t} \cdot \mathbf{u}_{\lambda*}\big) = (\mathrm{D}f) \circ \Phi_{\Gamma_t} \cdot \mathbf{u}_\lambda \circ \Phi_{\Gamma_t}.$$

On the other hand, it suffices to define

$$\mathbf{u}_{\lambda*} := \big(\mathrm{D}\Phi_{\Gamma_t}\big)^{-1}\big(\mathbf{u}_\lambda \circ \Phi_{\Gamma_t} - \partial_t \Phi_{\Gamma_t}\big) = \big(\mathrm{D}\Phi_{\Gamma_t}\big)^{-1}\big[\mathbf{u}_\lambda \circ \Phi_{\Gamma_t} - (\partial_t \gamma_{\Gamma_t})\boldsymbol{\nu}\big]. \tag{4.16}$$

Since

$$\theta = \big(\partial_t \gamma_{\Gamma_t} \boldsymbol{\nu}\big) \circ \big(\Phi_{\Gamma_t}\big)^{-1} \cdot \mathbf{N}_+,$$

one has

$$\Big[\mathbf{u}_\lambda - \big(\partial_t \gamma_{\Gamma_t} \boldsymbol{\nu}\big) \circ \big(\Phi_{\Gamma_t}\big)^{-1}\Big] \cdot \mathbf{N}_+ = 0,$$

i.e., $\Big[\mathbf{u}_\lambda - \big(\partial_t \gamma_{\Gamma_t} \boldsymbol{\nu}\big) \circ \big(\Phi_{\Gamma_t}\big)^{-1}\Big] \in \mathrm{T}\Gamma_t$ and $\mathbf{u}_{\lambda*} \in \mathrm{T}\Gamma_*$.

Variational estimates. In order to compute the variation of $\mathbf{u}_{\lambda*}$, one can assume that $\varkappa_a$ and $\boldsymbol{\omega}_*$ depend on a parameter β. By (4.16), it suffices to compute $\partial_\beta \mathbf{v}_{\pm*}$. Applying $\partial/\partial\beta$ to the identity

$$\big(\mathrm{D}\Phi_{\Gamma_t}\big) \cdot \mathbf{v}_{\pm*} = \mathbf{v}_\pm \circ \Phi_{\Gamma_t} - (\partial_t \gamma_{\Gamma_t})\boldsymbol{\nu},$$

one has

$$\mathrm{D}\big(\partial_\beta \gamma_{\Gamma_t} \boldsymbol{\nu}\big) \cdot \mathbf{v}_{\pm*} + (\mathrm{D}\Phi_{\Gamma_t}) \cdot \partial_\beta \mathbf{v}_{\pm*} = \partial_\beta\big(\mathbf{v}_\pm \circ \Phi_{\Gamma_t}\big) - \Big(\partial^2_{t\beta} \gamma_{\Gamma_t}\Big)\boldsymbol{\nu},$$

for which

$$\partial_\beta(\mathbf{v}_\pm \circ \Phi_{\Gamma_t}) = \Big(\partial_\beta \mathbf{v}_\pm + \mathrm{D}_{(\partial_\beta \gamma_{\Gamma_t} \boldsymbol{\nu}) \circ (\Phi_{\Gamma_t})^{-1}} \mathbf{v}_\pm\Big) \circ \Phi_{\Gamma_t}.$$

Denote by $\boldsymbol{\mu} := (\partial_\beta \gamma_{\Gamma_t} \boldsymbol{\nu}) \circ (\Phi_{\Gamma_t})^{-1}$ and use the notation

$$\mathbb{D}_\beta := \partial_\beta + \mathrm{D}_{\boldsymbol{\mu}}. \tag{4.17}$$

Then

$$\partial_\beta \mathbf{v}_{\pm *} = (\mathrm{D}\Phi_{\Gamma_t})^{-1} \cdot \Big[\big(\mathbb{D}_\beta \mathbf{v}_\pm\big) \circ \Phi_{\Gamma_t} - \big(\partial^2_{t\beta}\gamma_{\Gamma_t}\big)\boldsymbol{\nu} - \mathrm{D}(\partial_\beta \gamma_{\Gamma_t}\boldsymbol{\nu}) \cdot \mathbf{v}_{\pm *} \Big]. \tag{4.18}$$

In particular, $\big[(\mathrm{D}\Phi_{\Gamma_t}) \cdot \partial_\beta \mathbf{v}_{\pm *}\big] \circ \Phi_{\Gamma_t}^{-1} \cdot \mathbf{N}_+ \equiv 0$, so $\partial_\beta \mathbf{v}_{\pm *} \in \mathrm{T}\Gamma_*$.

Applying $\mathbb{D}_\beta$ to (4.13) and utilizing the commutator estimates together with the div-curl estimates, one can derive that for $\frac{1}{2} \le \sigma \le \frac{3}{2}k - \frac{3}{2}$,

$$\begin{aligned} \left|\partial_\beta \mathbf{v}_{\pm *}\right|_{H^\sigma(\Gamma_*)} \lesssim_{\Lambda_*} \left|\partial_\beta \gamma_{\Gamma_t}\right|_{H^{\sigma+1}(\Gamma_*)} \Big(\|\boldsymbol{\omega}_{*\pm}\|_{H^{\frac{3}{2}k-1}(\Omega_*^\pm)} + |\partial_t \gamma_{\Gamma_t}|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)} \Big) \\ + \left\|\partial_\beta \boldsymbol{\omega}_{*\pm}\right\|_{H^{\sigma-\frac{1}{2}}(\Omega_*^\pm)} + \left|\partial^2_{t\beta}\gamma_{\Gamma_t}\right|_{H^\sigma(\Gamma_*)}. \end{aligned} \tag{4.19}$$

Recalling that $\gamma_{\Gamma_t} = \mathfrak{K}^{-1}(\varkappa_a)$, one can obtain that

$$\partial_\beta \gamma_{\Gamma_t} = \delta\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta \varkappa_a], \tag{4.20}$$

and

$$\partial^2_{t\beta}\gamma_{\Gamma_t} = \delta\mathfrak{K}^{-1}(\varkappa_a)[\partial^2_{t\beta}\varkappa_a] + \delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_t\varkappa_a, \partial_\beta \varkappa_a]. \tag{4.21}$$

In conclusion, the linear relations imply the existence of six linear operators $\mathbf{B}_\pm(\varkappa_a)$, $\mathbf{F}_\pm(\varkappa_a)$ and $\mathbf{G}_\pm(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_{*\pm})$ whose ranges are all in $\mathrm{T}\Gamma_*$, so that

$$\partial_\beta \mathbf{v}_{\pm *} = \mathbf{B}_\pm(\varkappa_a)\partial^2_{t\beta}\varkappa_a + \mathbf{F}_\pm(\varkappa_a)\partial_\beta \boldsymbol{\omega}_{*\pm} + \mathbf{G}_\pm(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_{*\pm})\partial_\beta \varkappa_a. \tag{4.22}$$

Moreover, the following lemma holds, whose proof will be given in the Appendix:

Lemma 4.1. *Suppose that $a \ge a_0$ and $\varkappa_a \in B_{\delta_1} \subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$, where a_0 and B_{δ_1} are given in Proposition 3.3. If $s' - 2 \le s'' \le s' \le \frac{3}{2}k - \frac{3}{2}$, $s' \ge \frac{1}{2}$, and $\frac{1}{2} \le s \le \frac{3}{2}k - \frac{3}{2}$, then the following estimates hold:*

$$|\mathbf{B}_\pm(\varkappa_a)|_{\mathscr{L}(H^{s''}(\Gamma_*); H^{s'}(\Gamma_*; \mathrm{T}\Gamma_*))} \le C_* a^{s'-s''-2}, \tag{4.23}$$

$$|\delta\mathbf{B}_\pm(\varkappa_a)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*); \mathscr{L}\left(H^{s-2}(\Gamma_*); H^s(\Gamma_*)\right)\right]} \le C_*, \tag{4.24}$$

$$|\mathbf{F}_\pm(\varkappa_a)|_{\mathscr{L}\left(H^{s-\frac{1}{2}}(\Omega_*^\pm); H^s(\Gamma_*)\right)} \le C_*, \tag{4.25}$$

and

$$|\delta\mathbf{F}_\pm(\varkappa_a)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*); \mathscr{L}\left(H^{s-\frac{1}{2}}(\Omega_*^\pm); H^s(\Gamma_*)\right)\right]} \le C_*, \tag{4.26}$$

where C_* is a constant depending only on Λ_*.

Moreover, if $\varkappa_a \in B_{\delta_1} \cap H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)$, $\partial_t\varkappa_a \in H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$, and $\boldsymbol{\omega}_* \in H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)$, then for $\sigma' - 2 \le \sigma'' \le \sigma' \le \frac{3}{2}k + \frac{1}{2}$, $\sigma' \ge \frac{1}{2}$, and s given above, it holds that

$$|\mathbf{B}_\pm(\varkappa_a)|_{\mathscr{L}[H^{\sigma''}(\Gamma_*); H^{\sigma'}(\Gamma_*)]} \le a^{\sigma'-\sigma''-2} Q\Big(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}\Big), \tag{4.27}$$

$$|\mathbf{G}_\pm(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_{*\pm})|_{\mathscr{L}[H^{s-1}(\Gamma_*); H^s(\Gamma_*)]} \le Q\Big(|\partial_t\varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_{*\pm}\|_{H^{\frac{3}{2}k-1}(\Omega_*^\pm)}\Big), \tag{4.28}$$

and for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}k - \frac{5}{2}$,

$$
\begin{aligned}
&|\delta \mathbf{G}_{\pm}(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_{*\pm})|_{\mathscr{L}\left[H^{\sigma-1}(\Gamma_*)\times H^{\sigma-2}(\Gamma_*)\times H^{\sigma-\frac{1}{2}}(\Omega_*^{\pm});\mathscr{L}\left(H^{\sigma-1}(\Gamma_*);H^{\sigma}(\Gamma_*)\right)\right]} \\
&\quad \leq Q\left(|\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_{*\pm}\|_{H^{\frac{3}{2}k-1}(\Omega_*^{\pm})}\right),
\end{aligned}
\tag{4.29}
$$

where Q *is a generic polynomial depending only on* Λ_*.

4.3. **Evolution of the mean curvature.** Suppose that $(\Gamma_t, \mathbf{v}, \mathbf{h})$ is a solution to the interface problem (MHD)-(BC) for $0 \leq t \leq T$ with $\Gamma_t \in \Lambda_*$, $\Gamma_t \in H^{\frac{3}{2}k+1}$ and $\mathbf{v}(t), \mathbf{h}(t) \in H^{\frac{3}{2}k}(\Omega \setminus \Gamma_t)$. The hypersurface Γ_t is uniquely determined by the function $\varkappa_a(t) : \Gamma_* \to \mathbb{R}$, whose leading order term is κ_+. Then, it is natural to consider the evolution equation for κ_+ under a weighted velocity $\mathbf{u}_\lambda$. Plugging $\mathbf{u}_\lambda$ into (3.21) yields

$$
\begin{aligned}
\mathbb{D}_{t_\lambda}{}^2 \kappa_+ = &- \triangle_{\Gamma_t}(\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda \cdot \mathbf{N}_+) + \mathbb{D}_{t_\lambda}\mathbf{u}_\lambda \cdot \triangle_{\Gamma_t}\mathbf{N}_+ + 2\mathbf{N}_+ \cdot (\nabla \mathbf{u}_\lambda) \cdot (\triangle_{\Gamma_t}\mathbf{u}_\lambda)^\top \\
&+ 4\left\langle \mathbf{A}_\lambda \middle| \mathbf{N}_+ \cdot (\mathrm{D}_{\Gamma_t})^2 \mathbf{u}_\lambda \right\rangle - \kappa_+ \left|(\nabla \mathbf{u}_\lambda)^* \cdot \mathbf{N}_+\right|^2 \\
&- 2\left\langle \mathbf{I\!I}_+ \middle| \mathrm{D}_{\Gamma_t}\mathbf{u}_\lambda \cdot \mathrm{D}_{\Gamma_t}\mathbf{u}_\lambda \right\rangle + 4\left\langle \mathbf{I\!I}_+ \cdot \mathbf{A}_\lambda + \mathbf{A}_\lambda \cdot \mathbf{I\!I}_+ \middle| \mathbf{A}_\lambda \right\rangle,
\end{aligned}
\tag{4.30}
$$

where

$$
\mathbf{A}_\lambda := \frac{1}{2}\left\{(\mathrm{D}\mathbf{u}_\lambda)^\top + \left[(\mathrm{D}\mathbf{u}_\lambda)^\top\right]^*\right\}.
$$

Denoting by Q_λ a generic polynomial determined only by Λ_* and λ, one gets from (4.10), the product estimates, and Lemma 3.2 that

$$
\begin{aligned}
&\left|\mathbb{D}_{t_\lambda}{}^2 \kappa_+ + \triangle_{\Gamma_t}(\mathbb{D}_{t_\lambda}\mathbf{u}_\lambda \cdot \mathbf{N}_+)\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} \\
&\leq \begin{cases}
Q_\lambda\left(|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, \|(\mathbf{v},\mathbf{h})\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right) & \text{if} \quad k = 2, \\
Q_\lambda\left(\alpha|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, \|(\mathbf{v},\mathbf{h})\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right) & \text{if} \quad k \geq 3.
\end{cases}
\end{aligned}
\tag{4.31}
$$

Since $\mathbf{w} \equiv [\![\mathbf{v}]\!] := \mathbf{v}_+ - \mathbf{v}_- \in \mathrm{T}\Gamma_t$, it is routine to calculate

$$
\begin{aligned}
\mathbf{N}_+ \cdot \mathbb{D}_{t_\lambda}\mathbf{u}_\lambda &= \frac{1}{\rho_-}\mathrm{D}_{\mathbf{N}_-}p^- - \lambda\left(\frac{1}{\rho_+}\mathrm{D}_{\mathbf{N}_+}p^+ + \frac{1}{\rho_-}\mathrm{D}_{\mathbf{N}_-}p^-\right) \\
&\quad + \mathbf{N}_+ \cdot \left[\lambda\,\mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ + (1-\lambda)\,\mathrm{D}_{\mathbf{h}_-}\mathbf{h}_- - \lambda(1-\lambda)\,\mathrm{D}_{\mathbf{w}}\mathbf{w}\right] \\
&= -\alpha^2 \widetilde{\mathcal{N}}\kappa_+ + \left(\frac{\lambda}{\rho_+}\mathcal{N}_+ - \frac{1-\lambda}{\rho_-}\mathcal{N}_-\right)\bar{\mathcal{N}}^{-1}(g^+ - g^-) \\
&\quad - \mathrm{D}_{\mathbf{N}_+}\left[\lambda\left(p^+_{\mathbf{v},\mathbf{v}} - p^+_{\mathbf{h},\mathbf{h}}\right) + (1-\lambda)\left(p^-_{\mathbf{v},\mathbf{v}} - p^-_{\mathbf{h},\mathbf{h}}\right)\right] \\
&\quad - \lambda \mathbf{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+) - (1-\lambda)\mathbf{I\!I}_+(\mathbf{h}_-, \mathbf{h}_-) + \lambda(1-\lambda)\mathbf{I\!I}_+(\mathbf{w}, \mathbf{w}).
\end{aligned}
\tag{4.32}
$$

In order to control the $H^{\frac{3}{2}k-\frac{1}{2}}$ norm of $\left(\frac{\lambda}{\rho_+}\mathcal{N}_+ - \frac{1-\lambda}{\rho_-}\mathcal{N}_-\right)\bar{\mathcal{N}}^{-1}(g^+ - g^-)$, it suffices to have

$$
\left|\left(\frac{\lambda}{\rho_+}\mathcal{N}_+ - \frac{1-\lambda}{\rho_-}\mathcal{N}_-\right)\right|_{\mathscr{L}(H^s(\Gamma_t);H^s(\Gamma_t))} \leq Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}\right)
\tag{4.33}
$$

for $\frac{1}{2}-\frac{3}{2}k \le s \le \frac{3}{2}k-\frac{1}{2}$ and some generic polynomial Q determined by Λ_*. Thanks to Lemma 3.6, (4.33) holds as long as λ satisfies

$$\frac{\lambda}{\rho_+} = \frac{1-\lambda}{\rho_-} \iff \lambda = \frac{\rho_+}{\rho_+ + \rho_-}. \tag{4.34}$$

Denote by

$$\mathbf{u} := \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{v}_+ + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{v}_-, \quad \text{and} \quad \mathbb{D}_{\bar{t}} := \partial_t + \mathrm{D}_{\mathbf{u}}. \tag{4.35}$$

Then $\mathbf{u}\cdot\mathbf{N}_+ = \theta$ and

$$\begin{aligned}\mathbf{N}_+\cdot\mathbb{D}_{\bar{t}}\mathbf{u} = &-\alpha^2\widetilde{\mathcal{N}}\kappa_+ + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathrm{I\!I}_+(\mathbf{w},\mathbf{w})\\ &-\frac{\rho_+}{\rho_+ + \rho_-}\mathrm{I\!I}_+(\mathbf{h}_+,\mathbf{h}_+) - \frac{\rho_-}{\rho_+ + \rho_-}\mathrm{I\!I}_+(\mathbf{h}_-,\mathbf{h}_-) + \mathfrak{r}_0,\end{aligned}$$

where

$$\mathfrak{r}_0 := \frac{1}{\rho_+\rho_-}(\mathcal{N}_+ - \mathcal{N}_-)\bar{\mathcal{N}}^{-1}\left(g^+ - g^-\right) - \mathrm{D}_{\mathbf{N}_+}\left[\frac{\rho_+}{\rho_+ + \rho_-}\left(p^+_{\mathbf{v},\mathbf{v}} - p^+_{\mathbf{h},\mathbf{h}}\right) + \frac{\rho_-}{\rho_+ + \rho_-}\left(p^-_{\mathbf{v},\mathbf{v}} - p^-_{\mathbf{h},\mathbf{h}}\right)\right] \tag{4.36}$$

satisfies

$$|\mathfrak{r}_0|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_t)} \le Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right).$$

Define the following two operators:

$$\mathfrak{A}(\Gamma) := -\triangle_\Gamma\widetilde{\mathcal{N}} \tag{4.37}$$

and

$$\mathfrak{R}(\Gamma,\mathbf{J}) := (\mathrm{D}_\Gamma)_{\mathbf{J}}(\mathrm{D}_\Gamma)_{\mathbf{J}} \tag{4.38}$$

for a vector field $\mathbf{J}\in\mathrm{T}\Gamma$. It follows from Lemma 3.7 that

$$\left|\triangle_{\Gamma_t}[\mathrm{I\!I}_+(\mathbf{J},\mathbf{J})] - \mathfrak{R}(\Gamma_t,\mathbf{J})\kappa_+\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} \le Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}\right)|\mathbf{J}|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_t)}, \tag{4.39}$$

where $\mathbf{J}\in\mathrm{T}\Gamma_t$ is an $H^{\frac{3}{2}k-\frac{1}{2}}$ tangential vector field and Q is a generic polynomial determined by Λ_*.

Thus, by using the following notations:

$$\begin{aligned}\mathfrak{R}_0 := &\,\vec{\mathfrak{a}}\cdot\triangle_{\Gamma_t}\mathbf{N}_+ + 2\mathbf{N}_+\cdot(\nabla\mathbf{u})\cdot(\triangle_{\Gamma_t}\mathbf{u})^\top + 4\left\langle\mathbf{A}\middle|\mathbf{N}_+\cdot(\mathrm{D}_{\Gamma_t})^2\mathbf{u}\right\rangle\\ &-\kappa_+\left|(\nabla\mathbf{u})^*\cdot\mathbf{N}_+\right|^2 - 2\left\langle\mathrm{I\!I}_+\middle|\mathrm{D}_{\Gamma_t}\mathbf{u}\cdot\mathrm{D}_{\Gamma_t}\mathbf{u}\right\rangle + 4\left\langle\mathrm{I\!I}_+\cdot\mathbf{A} + \mathbf{A}\cdot\mathrm{I\!I}_+\middle|\mathbf{A}\right\rangle\\ &+\frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\{\mathfrak{R}(\Gamma_t,\mathbf{w})\kappa_+ - \triangle_{\Gamma_t}[\mathrm{I\!I}_+(\mathbf{w},\mathbf{w})]\}\\ &+\frac{\rho_+}{\rho_+ + \rho_-}\{\triangle_{\Gamma_t}[\mathrm{I\!I}_+(\mathbf{h}_+,\mathbf{h}_+)] - \mathfrak{R}(\Gamma_t,\mathbf{h}_+)\kappa_+\}\\ &+\frac{\rho_-}{\rho_+ + \rho_-}\{\triangle_{\Gamma_t}[\mathrm{I\!I}_+(\mathbf{h}_-,\mathbf{h}_-)] - \mathfrak{R}(\Gamma_t,\mathbf{h}_-)\kappa_+\}\\ &-\triangle_{\Gamma_t}\mathfrak{r}_0,\end{aligned} \tag{4.40}$$

with

$$\mathfrak{a} := \frac{1}{\rho_+ + \rho_-}\left(\nabla p^+ + \nabla p^-\right) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2} \mathrm{D}_{\mathbf{w}}\mathbf{w} - \frac{\rho_+}{\rho_+ + \rho_-} \mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ - \frac{\rho_-}{\rho_+ + \rho_-} \mathrm{D}_{\mathbf{h}_-}\mathbf{h}_-, \quad (4.41)$$

and p given by (4.4), (4.8) and (4.9), one can obtain the following lemma:

Lemma 4.2. *There exists a generic polynomial Q depending only on Λ_*, so that for any solution to (MHD)-(BC) for $t \in [0,T]$, $k \geq 2$, $\Gamma_t \in \Lambda_* \cap H^{\frac{3}{2}k+1}$ and $\mathbf{v}, \mathbf{h} \in H^{\frac{3}{2}k}(\Omega \setminus \Gamma_t)$, the mean curvature κ_+ satisfies the equation*

$$\begin{aligned} \mathbb{D}_{\bar{t}}^2\kappa_+ + \alpha^2 \mathfrak{A}(\Gamma_t)\kappa_+ + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathfrak{R}(\Gamma_t, \mathbf{w})\kappa_+ \\ - \frac{\rho_+}{\rho_+ + \rho_-}\mathfrak{R}(\Gamma_t, \mathbf{h}_+)\kappa_+ - \frac{\rho_-}{\rho_+ + \rho_-}\mathfrak{R}(\Gamma_t, \mathbf{h}_-)\kappa_+ = \mathfrak{R}_0, \end{aligned} \quad (4.42)$$

where $\mathbb{D}_{\bar{t}}$ is defined in (4.35), and $\mathfrak{R}_0 : \Gamma_t \to \mathbb{R}$ satisfies

$$|\mathfrak{R}_0|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} \leq Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right). \quad (4.43)$$

Furthermore, if $k \geq 3$, the estimate of $\mathfrak{R}_0$ can be refined to

$$|\mathfrak{R}_0|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} \leq Q\left(\alpha|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)} \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right). \quad (4.43')$$

4.4. **Evolution of $\varkappa_a$.** In order to compute the evolution equation for $\varkappa_a$, it is convenient to pull back $\mathfrak{A}$ and $\mathfrak{R}$ to Γ_*. More precisely, for $\Gamma \in \Lambda_*$, $\mathbf{J}_* \in \mathrm{T}\Gamma_*$, and $f : \Gamma_* \to \mathbb{R}$, define:

$$\mathscr{A}(\varkappa_a)f := \left[\mathfrak{A}(\Gamma)(f \circ \Phi_\Gamma^{-1})\right] \circ \Phi_\Gamma, \quad (4.44)$$

and

$$\mathscr{R}(\varkappa_a, \mathbf{J}_*)f := \left[\mathfrak{R}(\Gamma, \mathbf{J})(f \circ \Phi_\Gamma^{-1})\right] \circ \Phi_\Gamma, \quad (4.45)$$

where $\mathbf{J} := \mathrm{T}\Phi_\Gamma(\mathbf{J}_*) = [(\mathrm{D}\Phi_\Gamma) \cdot \mathbf{J}_*] \circ (\Phi_\Gamma)^{-1} \in \mathrm{T}\Gamma$. Furthermore, the following lemma holds (c.f. [SZ11]):

Lemma 4.3. *There are positive constants C_*, δ_1 depending only on Λ_*, so that for $\varkappa_a \in B_{\delta_1} \subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$, $\mathbf{J}_* \in H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)$, $2 \leq s \leq \frac{3}{2}k - \frac{1}{2}$, $1 \leq \sigma \leq \frac{3}{2}k - \frac{1}{2}$ and $2 \leq s_1 \leq \frac{3}{2}k - \frac{1}{2}$, the following estimates hold:*

$$|\mathscr{A}(\varkappa_a)|_{\mathscr{L}[H^s(\Gamma_*); H^{s-3}(\Gamma_*)]} \leq C_*, \quad (4.46)$$

$$|\mathscr{R}(\varkappa_a, \mathbf{J}_*)|_{\mathscr{L}[H^\sigma(\Gamma_*); H^{\sigma-2}(\Gamma_*)]} \leq C_* |\mathbf{J}_*|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)}, \quad (4.47)$$

and

$$|\delta\mathscr{A}(\varkappa_a)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*); \mathscr{L}(H^{s_1}(\Gamma_*); H^{s_1-3}(\Gamma_*))\right]} \leq C_*. \quad (4.48)$$

Furthermore, if $k \geq 3$, it holds for $2 \leq s_2 \leq \frac{3}{2}k - 1$ that

$$|\delta\mathscr{R}(\varkappa_a, \mathbf{J}_*)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-2}(\Gamma_*); \mathscr{L}(H^{s_2}(\Gamma_*); H^{s_2-2}(\Gamma_*))\right]} \leq C_*\left(1 + |\mathbf{J}_*|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)}\right). \quad (4.49)$$

Proof. (4.46) and (4.47) follow from the definitions. As for the variational estimates, suppose that $\{\varGamma_t\}_t \subset \varLambda_*$ is a family of hypersurfaces parameterized by t. Then by considering the velocity of the moving hypersurface:

$$\mathbf{v} := (\partial_t \gamma_{\varGamma_t} \nu) \circ \mathfrak{X}_{\varGamma_t}^{-1}$$

and the material derivative

$$\mathbb{D}_t := \partial_t + \mathrm{D}_{\mathbf{v}},$$

one gets

$$\frac{\partial}{\partial t}\mathscr{A}(\varkappa_a) f = \mathbb{D}_t\Big[-\triangle_{\varGamma_t}\widetilde{\mathfrak{N}}(f\circ\varPhi_{\varGamma_t}^{-1})\Big]\circ\varPhi_{\varGamma_t} = -\Big\{\Big[\mathbb{D}_t, \triangle_{\varGamma_t}\widetilde{\mathfrak{N}}\Big](f\circ\varPhi_{\varGamma_t}^{-1})\Big\}\circ\varPhi_{\varGamma_t}.$$

Thus (4.48) follows from Lemma 3.7.

As for (4.49), one may observe that

$$\frac{\partial}{\partial t}\mathscr{R}(\varkappa_a, \mathbf{J}_*) f = \mathbb{D}_t\big[\mathrm{D}_{\mathbf{J}}\,\mathrm{D}_{\mathbf{J}}(f\circ\varPhi_{\varGamma_t}^{-1})\big]\circ\varPhi_{\varGamma_t}$$

and for $\phi : \varGamma_t \to \mathbb{R}$,

$$[\mathbb{D}_t, \mathrm{D}_{\mathbf{J}}]\phi = \mathrm{D}_{(\mathbb{D}_t\mathbf{J} - \mathrm{D}_{\mathbf{J}}\mathbf{v})}\phi.$$

Hence for $0 \le s' \le \frac{3}{2}k - 2$ (here $k \ge 3$), it holds that

$$\begin{aligned}|[\mathbb{D}_t, \mathrm{D}_{\mathbf{J}}]\phi|_{H^{s'}(\varGamma_t)} &\lesssim_{\varLambda_*} |\mathrm{D}\phi\cdot(\mathbb{D}_t\mathbf{J} - \mathrm{D}_{\mathbf{J}}\mathbf{v})|_{H^{s'}(\varGamma_t)} \\ &\lesssim_{\varLambda_*} |\phi|_{H^{s'+1}(\varGamma_t)}\cdot|\mathbb{D}_t\mathbf{J} - \mathrm{D}_{\mathbf{J}}\mathbf{v}|_{H^{\frac{3}{2}k-2}(\varGamma_t)},\end{aligned}$$

which, together with Lemma 3.7, implies (4.49). □

Next, we shall derive the evolution equation for $\varkappa_a$. First, define a vector field $\mathbf{W} : \varGamma_t \to \mathbb{R}^3$ by

$$\begin{aligned}\mathbf{W} &:= \mathbb{D}_{\bar{t}}\mathbf{u} + \frac{1}{\rho_+ + \rho_-}\big(\nabla q^+ + \nabla q^-\big) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathrm{D}_{\mathbf{w}}\mathbf{w} - \frac{\rho_+}{\rho_+ + \rho_-}\mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ - \frac{\rho_-}{\rho_+ + \rho_-}\mathrm{D}_{\mathbf{h}_-}\mathbf{h}_- \\ &\equiv \mathbb{D}_{\bar{t}}\mathbf{u} + \vec{\mathfrak{b}},\end{aligned} \tag{4.50}$$

for which

$$\begin{aligned} q^\pm &:= \rho_\pm\Big(p^\pm_{\mathbf{v},\mathbf{v}} + p^\pm_{\mathbf{h},\mathbf{h}}\Big) \pm \alpha^2\frac{1}{\rho_\mp}\mathcal{H}_\pm\bar{\mathfrak{N}}^{-1}\mathfrak{N}_\mp\kappa_+ + \mathcal{H}_\pm\bar{\mathfrak{N}}^{-1}\mathfrak{q}, \\ \mathfrak{q} &:= -g^+ + g^- - \frac{1}{|\varGamma_t|}\int_{\varOmega\setminus\varGamma_t}(\nabla\cdot\mathbf{v})^2\,\mathrm{d}x\,, \\ g^\pm &:= 2\,\mathrm{D}_{\mathbf{v}^\top_\pm}\theta - \mathrm{I\!I}_+\Big(\mathbf{v}^\top_\pm, \mathbf{v}^\top_\pm\Big) + \mathrm{I\!I}_+\Big(\mathbf{h}^\top_\pm, \mathbf{h}^\top_\pm\Big) + \mathrm{D}_{\mathbf{N}_+}\Big(p^\pm_{\mathbf{v},\mathbf{v}} - p^\pm_{\mathbf{h},\mathbf{h}}\Big).\end{aligned}$$

Thus, $\mathbf{W} \equiv 0$ if $(\varGamma_t, \mathbf{v}, \mathbf{h})$ is a solution to (MHD)-(BC).

Substituting

$$\mathbb{D}_{\bar{t}}\mathbf{u} = \mathbf{W} - \frac{1}{\rho_+ + \rho_-}\big(\nabla q^+ + \nabla q^-\big) - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathrm{D}_{\mathbf{w}}\mathbf{w} + \frac{\rho_+}{\rho_+ + \rho_-}\mathrm{D}_{\mathbf{h}_+}\mathbf{h}_+ + \frac{\rho_-}{\rho_+ + \rho_-}\mathrm{D}_{\mathbf{h}_-}\mathbf{h}_-$$

into (4.30) with $\lambda = \rho_+/(\rho_+ + \rho_-)$, and pulling back it to Γ_* via (4.14), one has

$$\begin{aligned}&\mathbb{D}_{t*}{}^2(\kappa_+ \circ \Phi_{\Gamma_t}) + \alpha^2 \mathscr{A}(\varkappa_a)(\kappa_+ \circ \Phi_{\Gamma_t}) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathscr{R}(\varkappa_a, \mathbf{w}_*)(\kappa_+ \circ \Phi_{\Gamma_t})\\ &- \frac{\rho_+}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{+*})(\kappa_+ \circ \Phi_{\Gamma_t}) - \frac{\rho_-}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{-*})(\kappa_+ \circ \Phi_{\Gamma_t})\\ &= \{\mathfrak{R}_1 - \triangle_{\Gamma_t}(\mathbf{W}\cdot\mathbf{N}_+) + \mathbf{W}\cdot\triangle_{\Gamma_t}\mathbf{N}_+\} \circ \Phi_{\Gamma_t},\end{aligned} \tag{4.51}$$

where

$$\mathbf{u}_* = \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{v}_{+*} + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{v}_{-*}, \quad \mathbb{D}_{t*} := \partial_t + \mathrm{D}_{\mathbf{u}_*},$$

$$\mathbf{h}_{\pm *} := (\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot(\mathbf{h}_\pm \circ \Phi_{\Gamma_t}), \quad \mathbf{w}_* := (\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot(\mathbf{w}\circ\Phi_{\Gamma_t}) = \mathbf{v}_{+*} - \mathbf{v}_{-*},$$

and

$$\begin{aligned}\mathfrak{R}_1 :=& -\vec{\mathfrak{b}}\cdot\triangle_{\Gamma_t}\mathbf{N}_+ + 2\mathbf{N}_+\cdot(\nabla\mathbf{u})\cdot(\triangle_{\Gamma_t}\mathbf{u})^\top + 4\left\langle \mathbf{A}\middle|\mathbf{N}_+\cdot(\mathrm{D}_{\Gamma_t})^2\mathbf{u}\right\rangle\\ &-\kappa_+\left|(\nabla\mathbf{u})^*\cdot\mathbf{N}_+\right|^2 - 2\left\langle \mathbf{I\!I}_+\middle|\mathrm{D}_{\Gamma_t}\mathbf{u}\cdot\mathrm{D}_{\Gamma_t}\mathbf{u}\right\rangle + 4\left\langle \mathbf{I\!I}_+\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{I\!I}_+\middle|\mathbf{A}\right\rangle\\ &+\frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\{\mathfrak{R}(\Gamma_t, \mathbf{w})\kappa_+ - \triangle_{\Gamma_t}[\mathbf{I\!I}_+(\mathbf{w}, \mathbf{w})]\}\\ &+\frac{\rho_+}{\rho_+ + \rho_-}\{\triangle_{\Gamma_t}[\mathbf{I\!I}_+(\mathbf{h}_+, \mathbf{h}_+)] - \mathfrak{R}(\Gamma_t, \mathbf{h}_+)\kappa_+\}\\ &+\frac{\rho_-}{\rho_+ + \rho_-}\{\triangle_{\Gamma_t}[\mathbf{I\!I}_+(\mathbf{h}_-, \mathbf{h}_-)] - \mathfrak{R}(\Gamma_t, \mathbf{h}_-)\kappa_+\}\\ &-\triangle_{\Gamma_t}\mathfrak{r}_0,\end{aligned} \tag{4.52}$$

with $\mathfrak{r}_0$ given by (4.36).

Due to the relation

$$\mathbb{D}_{t*}{}^2 = \partial_{tt}^2 + \mathrm{D}_{\mathbf{u}_*}\mathrm{D}_{\mathbf{u}_*} + 2\,\mathrm{D}_{\mathbf{u}_*}\partial_t + \mathrm{D}_{\partial_t\mathbf{u}_*}$$

and (4.22), the term $\partial_t\mathbf{u}_*$ involves $\partial_{tt}^2\varkappa_a$, so (4.51) is a nonlinear equation for $\partial_{tt}^2\varkappa_a$. In order to get a equation which is linear for $\partial_{tt}^2\varkappa_a$, one may drive from (4.22) that

$$\begin{aligned}&\partial_{tt}^2(\kappa_+ \circ \Phi_{\Gamma_t}) + \mathscr{C}_\alpha(\varkappa_a, \partial_t\varkappa_a, \mathbf{v}_*, \mathbf{h}_*)(\kappa_+ \circ \Phi_{\Gamma_t})\\ &+\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t})\cdot\left[\mathbf{B}(\varkappa_a)\partial_{tt}^2\varkappa_a + \mathbf{F}(\varkappa_a)\partial_t\boldsymbol{\omega}_* + \mathbf{G}(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_*)\partial_t\varkappa_a\right]\\ &= \{\mathfrak{R}_1 - \triangle_{\Gamma_t}(\mathbf{W}\cdot\mathbf{N}_+) + \mathbf{W}\cdot\triangle_{\Gamma_t}\mathbf{N}_+\}\circ\Phi_{\Gamma_t},\end{aligned} \tag{4.53}$$

where the following notations have been used:

$$\mathbf{B}(\text{resp. }\mathbf{F}\text{ or }\mathbf{G}) := \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{B}_+(\text{resp. }\mathbf{F}_+\text{ or }\mathbf{G}_+) + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{B}_-(\text{resp. }\mathbf{F}_-\text{ or }\mathbf{G}_-), \tag{4.54}$$

and for simplicity,

$$\begin{aligned}\mathscr{C}_\alpha(\varkappa_a, \partial_t\varkappa_a, \mathbf{v}_*, \mathbf{h}_*) :=& 2\,\mathrm{D}_{\mathbf{u}_*}\partial_t + \mathrm{D}_{\mathbf{u}_*}\mathrm{D}_{\mathbf{u}_*} + \alpha^2\mathscr{A}(\varkappa_a) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathscr{R}(\varkappa_a, \mathbf{w}_*)\\ &-\frac{\rho_+}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{+*}) - \frac{\rho_-}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{-*}).\end{aligned} \tag{4.55}$$

Since $\varkappa_a = \kappa_+ \circ \Phi_{\Gamma_t} + a^2\gamma_{\Gamma_t}$, one also needs to calculate $\partial_{tt}^2\gamma_{\Gamma_t}$. Notice that for every evolution velocity $\mathbf{v} : \Gamma_t \to \mathbb{R}^3$ of Γ_t, it holds that

$$\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\cdot(\mathbf{N}_+ \circ \Phi_{\Gamma_t}) = (\mathbf{v}\cdot\mathbf{N}_+)\circ\Phi_{\Gamma_t},$$

which, together with (3.13), implies

$$(\partial_{tt}^2\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}\cdot\mathbf{N}_+ = \mathbf{N}_+\cdot \mathrm{D}_{[(\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}-\mathbf{u}]}\big[(\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}\big] + \mathbf{N}_+\cdot\Big[\mathbb{D}_{\bar t}\mathbf{u} - \mathrm{D}_{\mathbf{u}}\mathbf{u} + \mathrm{D}_{(\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}}\mathbf{u}\Big],$$

that is,

$$\begin{aligned}\partial_{tt}^2\gamma_{\Gamma_t} &= \frac{(\mathbf{N}_+\circ\Phi_{\Gamma_t})}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\cdot\big[(\mathbb{D}_{\bar t}\mathbf{u})\circ\Phi_{\Gamma_t} - \mathrm{D}_{\mathbf{u}_*}\big(\mathbf{u}\circ\Phi_{\Gamma_t}+\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\big)\big]\\ &= \frac{(\mathbf{N}_+\circ\Phi_{\Gamma_t})}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\cdot\Big[\Big(\mathbf{W}-\vec{\mathfrak{b}}\Big)\circ\Phi_{\Gamma_t} - \mathrm{D}_{\mathbf{u}_*}\big(\mathbf{u}\circ\Phi_{\Gamma_t}+\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\big)\Big].\end{aligned}\tag{4.56}$$

In particular, $\partial_{tt}^2\gamma_{\Gamma_t}$ does not involve the term $\partial_{tt}^2\varkappa_a$.

Combining (3.24) and (4.53) yields

$$\begin{aligned}&\Big[\mathrm{I}+\nabla^\top(\kappa_+\circ\Phi_{\Gamma_t})\cdot\mathbf{B}(\varkappa_a)\Big]\partial_{tt}^2\varkappa_a + \mathscr{C}_\alpha(\varkappa_a,\partial_t\varkappa_a,\mathbf{v}_*,\mathbf{h}_*)\varkappa_a\\ &+\nabla^\top(\kappa_+\circ\Phi_{\Gamma_t})\cdot[\mathbf{F}(\varkappa_a)\partial_t\boldsymbol{\omega}_* + \mathbf{G}(\varkappa_a,\partial_t\varkappa_a,\boldsymbol{\omega}_*)\partial_t\varkappa_a]\\ &+a^2\frac{\mathbf{N}_+\circ\Phi_{\Gamma_t}}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\cdot\Big[\vec{\mathfrak{b}}\circ\Phi_{\Gamma_t} + \mathrm{D}_{\mathbf{u}_*}\big(\mathbf{u}\circ\Phi_{\Gamma_t}+\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\big)\Big] - a^2\mathscr{C}_\alpha(\varkappa_a,\partial_t\varkappa_a,\mathbf{v}_*,\mathbf{h}_*)\gamma_{\Gamma_t}\\ &=\mathfrak{R}_1\circ\Phi_{\Gamma_t} + \left[-\triangle_{\Gamma_t}(\mathbf{W}\cdot\mathbf{N}_+) + \mathbf{W}\cdot\triangle_{\Gamma_t}\mathbf{N}_+ + a^2\frac{\mathbf{W}\cdot\mathbf{N}_+}{\mathbf{N}_+\cdot(\boldsymbol{\nu}\circ\Phi_{\Gamma_t}^{-1})}\right]\circ\Phi_{\Gamma_t}.\end{aligned}\tag{4.57}$$

Define a new operator:

$$\mathscr{B}(\varkappa_a) := \nabla^\top(\kappa_+\circ\Phi_{\Gamma_t})\cdot\mathbf{B}(\varkappa_a).\tag{4.58}$$

It then follows from Lemma 4.1 that

$$|\mathscr{B}(\varkappa_a)|_{\mathscr{L}[H^{s''}(\Gamma_*);H^{s'}(\Gamma_*)]} \lesssim_{\Lambda_*} a^{s'-s''-2+\epsilon}|\varkappa_a|_{H^{\frac32 k-1}(\Gamma_*)},\tag{4.59}$$

for $s'-2\le s''\le s'\le\frac32 k-2$, $s'\ge\frac12$, and $0<\epsilon\le s''-s'+2$. If $k\ge 3$, one may take $\epsilon=0$, and it holds for $\sigma'-2\le\sigma''\le\sigma'\le\frac32 k-\frac52$, $\sigma'\ge\frac12$ that

$$|\mathscr{B}(\varkappa_a)|_{\mathscr{L}[H^{\sigma''}(\Gamma_*);H^{\sigma'}(\Gamma_*)]} \lesssim_{\Lambda_*} a^{\sigma'-\sigma''-2}|\varkappa_a|_{H^{\frac32 k-\frac32}(\Gamma_*)}.\tag{4.59'}$$

Letting $s'=s''$, $0<\epsilon<\frac12$ ($\epsilon=0$ if $k\ge3$) and a_0 large enough compared to $|\varkappa_a|_{H^{\frac32 k-1}(\Gamma_*)}$ (or a_0 large compared to $|\varkappa_a|_{H^{\frac32 k-\frac32}(\Gamma_*)}$ if $k\ge 3$), one has

$$|\mathscr{B}(\varkappa_a)|_{\mathscr{L}(H^{s'}(\Gamma_*))}\le\frac12<1,\tag{4.60}$$

for $\frac12\le s'\le\frac32 k-2$ (or $\frac12\le s'\le\frac32 k-\frac52$ if $k\ge3$). Namely, $[\mathrm{I}+\mathscr{B}(\varkappa_a)]$ is an isomorphism on $H^{s'}(\Gamma_*)$. Set

$$\mathbf{j} := \nabla\times\mathbf{h} \quad\text{and}\quad \mathbf{j}_* := \mathbf{j}\circ\mathfrak{X}_{\Gamma_t}.\tag{4.61}$$

Then $\mathbf{h}$ can be recovered from $(\varkappa_a,\mathbf{j}_*)$ by solving div-curl problems. Applying the operator $[\mathrm{I}+\mathscr{B}(\varkappa_a)]^{-1}$ to (4.57), one can get the evolution equation for $\varkappa_a$ as (which is, in particular, irrelevant to the MHD system):

$$
\begin{aligned}
&\partial_{tt}^2 \varkappa_a + \mathscr{C}_\alpha(\varkappa_a, \partial_t \varkappa_a, \mathbf{v}_*, \mathbf{h}_*)\varkappa_a - \mathscr{F}(\varkappa_a)\partial_t \boldsymbol{\omega}_* - \mathscr{G}(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*) \\
&\quad = [\mathrm{I} + \mathscr{B}(\varkappa_a)]^{-1} \left\{ \left[-\triangle_{\Gamma_t}(\mathbf{W} \cdot \mathbf{N}_+) + \mathbf{W} \cdot \triangle_{\Gamma_t} \mathbf{N}_+ + a^2 \frac{\mathbf{W} \cdot \mathbf{N}_+}{\mathbf{N}_+ \cdot (\boldsymbol{\nu} \circ \Phi_{\Gamma_t}^{-1})} \right] \circ \Phi_{\Gamma_t} \right\}.
\end{aligned} \tag{4.62}
$$

The operators $\mathscr{F}$ and $\mathscr{G}$ defined above satisfy the following lemma, whose proof is given in the Appendix:

Lemma 4.4. *Assume that $a \geq a_0$ and $\varkappa_a \in B_{\delta_1}$ as in Proposition 3.3. For $\frac{1}{2} \leq s \leq \frac{3}{2}k - 2$ and $\epsilon > 0$, there are some positive constants C_* and generic polynomials Q determined by Λ_*, so that*

$$
|\mathscr{F}(\varkappa_a)|_{\mathscr{L}\left[H^{s+\epsilon-\frac{1}{2}}(\Omega \setminus \Gamma_*); H^s(\Gamma_*)\right]} \leq C_* |\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, \tag{4.63}
$$

and

$$
\begin{aligned}
&|\mathscr{G}(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*)|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \\
&\quad \leq a^2 Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)}, \|\mathbf{j}_*\|_{H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)} \right).
\end{aligned} \tag{4.64}
$$

Furthermore, if $k \geq 3$, for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}k - \frac{5}{2}$, there hold

$$
|\mathscr{F}(\varkappa_a)|_{\mathscr{L}\left[H^{\sigma-\frac{1}{2}}(\Omega \setminus \Gamma_*); H^\sigma(\Gamma_*)\right]} \leq C_* |\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, \tag{4.65}
$$

$$
\begin{aligned}
&|\mathscr{G}(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*)|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \\
&\quad \leq a^2 Q\left(\alpha |\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, |\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|(\boldsymbol{\omega}_*, \mathbf{j}_*)\|_{H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)} \right),
\end{aligned} \tag{4.66}
$$

$$
|\delta \mathscr{F}(\varkappa_a)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*); \mathscr{L}\left(H^{\frac{3}{2}k-\frac{7}{2}}(\Omega \setminus \Gamma_*); H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)\right)\right]} \leq C_*, \tag{4.67}
$$

and

$$
\begin{aligned}
&|\delta \mathscr{G}|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-4}(\Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*); H^{\frac{3}{2}k-4}(\Gamma_*)\right]} \\
&\quad \leq a^2 Q\left(|\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)}, \|\mathbf{j}_*\|_{H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)} \right).
\end{aligned} \tag{4.68}
$$

4.5. **Evolution of the current and vorticity.** We shall use the vorticity, current and the corresponding boundary conditions to recover the solenoidal vector fields $\mathbf{v}$ and $\mathbf{h}$ by solving the corresponding div-curl systems. Hence, it is necessary to consider the evolution of the vorticity and the current.

Set

$$
\boldsymbol{\omega}_\pm := \nabla \times \mathbf{v}_\pm \quad \text{and} \quad \mathbf{j}_\pm := \nabla \times \mathbf{h}_\pm.
$$

Then it follows from taking curl of the equations (1.1a) and (1.1b) that

$$
\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\mathbf{h} \cdot \nabla)\mathbf{j} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{j} \cdot \nabla)\mathbf{h}, \tag{4.69}
$$

and

$$
\partial_t \mathbf{j} + (\mathbf{v} \cdot \nabla)\mathbf{j} - (\mathbf{h} \cdot \nabla)\boldsymbol{\omega} = (\mathbf{j} \cdot \nabla)\mathbf{v} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{h} - 2\,\mathrm{tr}(\nabla \mathbf{v} \times \nabla \mathbf{h}), \tag{4.70}
$$

where in the Cartesian coordinate

$$\operatorname{tr}(\nabla \mathbf{v} \times \nabla \mathbf{h}) = \sum_{l=1}^{3} \nabla v^l \times \nabla h^l.$$

5. Linear Systems

In this section, we shall study the linearized systems deriving from (4.62), (4.69) and (4.70). More precisely, the uniform estimates will be shown, from which the local well-posedness of the linear systems follows.

5.1. **Linearized problem for $\varkappa_a$.** Suppose that $\Gamma_* \in H^{\frac{3}{2}k+1}$ ($k \geq 2$) is a reference hypersurface, and Λ_*, defined by (3.25), satisfies all the properties given in the preliminary. Now, assume that there are a t-parameterized family of hypersurfaces $\Gamma_t \in \Lambda_*$ and four tangential vector fields $\mathbf{v}_{\pm *}, \mathbf{h}_{\pm *} : \Gamma_* \to \mathrm{T}\Gamma_*$ satisfying:

$$\varkappa_a \in C^0\Big\{[0,T]; H^{\frac{3}{2}k-1}(\Gamma_*)\Big\} \cap C^1\Big\{[0,T]; B_{\delta_1} \subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\Big\}, \tag{H1}$$

and

$$\mathbf{v}_{\pm *}, \mathbf{h}_{\pm *} \in C^0\Big\{[0,T]; H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)\Big\} \cap C^1\Big\{[0,T]; H^{\frac{3}{2}k-2}(\Gamma_*)\Big\}. \tag{H2}$$

For the sake of convenience, suppose that there are positive constants L_0, L_1, L_2 so that

$$\sup_{t\in[0,T]} \left\{ |\varkappa_a(t)|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\partial_t \varkappa_a(t)|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, |(\mathbf{v}_{\pm *}(t), \mathbf{h}_{\pm *}(t))|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)} \right\} \leq L_1, \tag{5.1}$$

$$\sup_{t\in[0,T]} |(\partial_t \mathbf{v}_{\pm *}(t), \partial_t \mathbf{h}_{\pm *}(t))|_{H^{\frac{3}{2}k-2}(\Gamma_*)} \leq L_2, \tag{5.2}$$

and

$$|(\mathbf{v}_{+*}(0), \mathbf{v}_{-*}(0))|_{H^{\frac{3}{2}k-2}(\Gamma_*)} \leq L_0. \tag{5.3}$$

Using the following notations as in the previous section:

$$\mathbf{w}_* = \mathbf{v}_{+*} - \mathbf{v}_{-*}, \quad \mathbf{u}_* = \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{v}_{+*} + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{v}_{-*},$$

we consider the linear initial value problem

$$\begin{cases} \partial_{tt}^2 \mathfrak{f} + \mathscr{C}(\varkappa_a, \partial_t \varkappa_a, \mathbf{v}_*, \mathbf{h}_*)\mathfrak{f} = \mathfrak{g}, \\ \mathfrak{f}(0) = \mathfrak{f}_0, \quad \partial_t \mathfrak{f}(0) = \mathfrak{f}_1, \end{cases} \tag{5.4}$$

where $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{g}(t) : \Gamma_* \to \mathbb{R}$ are three given functions, and $\mathscr{C}$ is given by:

$$\begin{aligned}\mathscr{C}(\varkappa_a, \partial_t \varkappa_a, \mathbf{v}_*, \mathbf{h}_*) :=& 2\,\mathrm{D}_{\mathbf{u}_*}\partial_t + \mathrm{D}_{\mathbf{u}_*}\mathrm{D}_{\mathbf{u}_*} + \mathscr{A}(\varkappa_a) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathscr{R}(\varkappa_a, \mathbf{w}_*) \\ &- \frac{\rho_+}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{+*}) - \frac{\rho_-}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{-*}),\end{aligned}$$

which is exactly (4.55) with $\alpha = 1$.

We shall derive some uniform estimates, from which the uniqueness and continuous dependence on the initial data follow, while the existence can be obtained via the Galerkin approximations (or the standard semigroup theory as in [SZ11]).

In order to derive the energy estimates for (5.4), one notes first that $\mathscr{A}$ is the highest order spacial derivative term (3rd order). Then for an integer $l \in [0, k-2]$, if one multiplies (5.4) by

$$\det\big(\mathrm{D}\Phi_{\Gamma_t}\big) \cdot \Big\{\widetilde{\mathfrak{N}}\Big[\Big(\mathscr{A}^l \partial_t \mathfrak{f}\Big) \circ (\Phi_{\Gamma_t})^{-1}\Big]\Big\} \circ \Phi_{\Gamma_t}$$

and integrates on Γ_*, the leading order terms will be obtained:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_*} \partial_t \mathfrak{f} \cdot \Big\{\widetilde{\mathfrak{N}}\Big[\Big(\mathscr{A}^l \partial_t \mathfrak{f}\Big) \circ (\Phi_{\Gamma_t})^{-1}\Big]\Big\} \circ \Phi_{\Gamma_t} \\ + f \cdot \Big\{\widetilde{\mathfrak{N}}\Big[\Big(\mathscr{A}^{1+l} \mathfrak{f}\Big) \circ (\Phi_{\Gamma_t})^{-1}\Big]\Big\} \circ \Phi_{\Gamma_t} \cdot \det\big(\mathrm{D}\Phi_{\Gamma_t}\big)\, \mathrm{d}S_* \,.\end{aligned}$$

The energy is defined as:

$$\begin{aligned} E_l(t, \mathfrak{f}, \partial_t \mathfrak{f}) := \int_{\Gamma_t} & \left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\big[(\partial_t \mathfrak{f} + \mathrm{D}_{\mathbf{u}_*} \mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1}\big]\right|^2 \\ & + \left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l+1}{2}} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\big(\mathfrak{f} \circ \Phi_{\Gamma_t}^{-1}\big)\right|^2 \\ & - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\big[(\mathrm{D}_{\mathbf{w}_*} \mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1}\big]\right|^2 \\ & + \frac{\rho_+}{\rho_+ + \rho_-}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\big[\big(\mathrm{D}_{\mathbf{h}_{+*}} \mathfrak{f}\big) \circ \Phi_{\Gamma_t}^{-1}\big]\right|^2 \\ & + \frac{\rho_-}{\rho_+ + \rho_-}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathfrak{N}}^{\frac{1}{2}}\big[(\mathrm{D}_{\mathbf{h}_{-*}} \mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1}\big]\right|^2 \mathrm{d}S_t \,. \end{aligned} \tag{5.5}$$

Lemma 5.1. *For any integer $0 \le l \le k-2$, and $0 \le t \le T$, it holds that*

$$\begin{aligned} & E_l(t, \mathfrak{f}, \partial_t \mathfrak{f}) - E_l(0, \mathfrak{f}_0, \mathfrak{f}_1) \\ \le & Q(L_1, L_2)\int_0^t \left(|\mathfrak{f}(s)|_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\partial_t \mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + |\mathfrak{g}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right) \times \\ & \times \left(|\mathfrak{f}(s)|_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\partial_t \mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right) \mathrm{d}s \,, \end{aligned} \tag{5.6}$$

where Q is a generic polynomial determined by Λ_*.

Proof. Denote by

$$\mathbb{D}_t := \partial_t + \mathrm{D}_{\boldsymbol{\mu}}, \quad \text{with} \quad \boldsymbol{\mu} := (\partial_t \gamma_{\Gamma_t} \boldsymbol{\nu}) \circ \Phi_{\Gamma_t}^{-1} : \Gamma_t \to \mathbb{R}^3,$$

and for any function $f : \Gamma_* \to \mathbb{R}$ and vector field $\mathbf{a}_* : \Gamma_* \to \mathrm{T}\Gamma_*$

$$\bar{f} := f \circ \Phi_{\Gamma_t}^{-1} : \Gamma_t \to \mathbb{R}, \quad \mathbf{a} := \big(\mathrm{D}\Phi_{\Gamma_t} \cdot \mathbf{a}_*\big) \circ \Phi_{\Gamma_t}^{-1} : \Gamma_t \to \mathrm{T}\Gamma_t.$$

Thus

$$(\partial_t \mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} = \mathbb{D}_t \bar{\mathfrak{f}}, \quad (\mathrm{D}_{\mathbf{a}_*} \mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} = \mathrm{D}_{\mathbf{a}} \bar{\mathfrak{f}}.$$

For simplicity, we will use the conventions that

$$|f|_s := |f|_{H^s(\Gamma_t)},$$

and

$$\mathfrak{u} \lesssim_{L_j} \mathfrak{v}$$

if there exists a constant $C = C(\Lambda_*, L_j)$ such that $\mathfrak{u} \le C\mathfrak{v}$.

In order to commute $\mathbb{D}_t$ with the differential operators, one observes first the facts that

$$\left|\int_{\Gamma_t} g\left[\mathbb{D}_t, \widetilde{\mathcal{N}}\right] f \, \mathrm{d}S_t\right| \lesssim_{\Lambda_*} |\boldsymbol{\mu}|_{\frac{3}{2}k-\frac{1}{2}} |f|_{\frac{1}{2}} |g|_{\frac{1}{2}}, \tag{5.7}$$

and

$$\left|\int_{\Gamma_t} g\left[\mathbb{D}_t, \triangle_{\Gamma_t}\right] f \, \mathrm{d}S_t\right| \lesssim_{\Lambda_*} |\boldsymbol{\mu}|_{\frac{3}{2}k-\frac{1}{2}} |f|_{1} |g|_{1}. \tag{5.8}$$

Indeed, it follows from the properties of $\mathcal{N}_+$ that

$$\begin{aligned}\int_{\Gamma_t} g[\mathbb{D}_t, \mathcal{N}_+] f \, \mathrm{d}S_t &= \int_{\Gamma_t} \mathcal{N}_+^{-1}\mathcal{N}_+ g[\mathbb{D}_t, \mathcal{N}_+] f \, \mathrm{d}S_t + \left(\int_{\Gamma_t} g\right)\left(\int_{\Gamma_t} [\mathbb{D}_t, \mathcal{N}_+] f\right) \\ &= \int_{\Gamma_t} \mathcal{N}_+^{\frac{1}{2}} g \cdot \mathcal{N}_+^{-\frac{1}{2}} [\mathbb{D}_t, \mathcal{N}_+] f \, \mathrm{d}S_t + \left(\int_{\Gamma_t} g\right)\left(\int_{\Gamma_t} [\mathbb{D}_t, \mathcal{N}_+] f\right),\end{aligned}$$

where the last equality follows from the self-adjointness of $\mathcal{N}_+$. It follows from the following commutator formula (c.f. [SZ08a, p. 710] for the derivation):

$$[\mathbb{D}_t, \mathcal{N}_+] f = \mathrm{D}_{\mathbf{N}_+} \triangle^{-1}\left(2\,\mathrm{D}\boldsymbol{\mu} : \mathrm{D}^2 \mathcal{H}_+ f + \nabla \mathcal{H}_+ f \cdot \triangle \boldsymbol{\mu}\right) - \nabla \mathcal{H}_+ f \cdot \mathrm{D}_{\mathbf{N}_+} \mathbf{v} - \mathrm{D}_{\nabla^\top f} \boldsymbol{\mu} \cdot \mathbf{N}_+, \tag{5.9}$$

that

$$\begin{aligned}\int_{\Gamma_t} [\mathbb{D}_t, \mathcal{N}_+] f \, \mathrm{d}S_t &= \int_{\Gamma_t} f[\mathbb{D}_t, \mathcal{N}_+]1 + (1\mathcal{N}_+ f)(-\operatorname{div}_{\Gamma_t} \boldsymbol{\mu}) \, \mathrm{d}S_t \\ &= -\int_{\Gamma_t} \mathcal{N}_+(f) \cdot \operatorname{div}_{\Gamma_t} \boldsymbol{\mu} \, \mathrm{d}S_t \\ &= -\int_{\Gamma_t} f \cdot \mathcal{N}_+(\operatorname{div}_{\Gamma_t} \boldsymbol{\mu}) \, \mathrm{d}S_t \, .\end{aligned}$$

Thus, the above two relations imply that

$$\left|\int_{\Gamma_t} g[\mathbb{D}_t, \mathcal{N}_+] f \, \mathrm{d}S_t\right| \lesssim_{\Lambda_*} |\boldsymbol{\mu}|_{\frac{3}{2}k-\frac{1}{2}} |f|_{\frac{1}{2}} |g|_{\frac{1}{2}}. \tag{5.10}$$

As $\widetilde{\mathcal{N}}$ is defined via (3.34), the estimate (5.7) follows from (5.10) and Lemma 3.7. The relation (5.8) can be derived via the following formula (c.f. [SZ08a, p. 710] for the derivation):

$$\left[\mathbb{D}_t, \triangle_{\Gamma_t}\right] = -2(\mathrm{D}_{\Gamma_t})^2 f \cdot (\mathrm{D}\boldsymbol{\mu})^\top - \nabla^\top f \cdot \triangle_{\Gamma_t} \boldsymbol{\mu} + \kappa_+ \mathrm{D}_{\nabla^\top f} \boldsymbol{\mu} \cdot \mathbf{N}_+, \tag{5.11}$$

and the integration-by-parts on Γ_t.

Now, it follows from the self-adjointness of $\widetilde{\mathcal{N}}$ and $(-\triangle_{\Gamma_t})$ that

$$\begin{aligned}&\int_{\Gamma_t} \left(-\widetilde{\mathcal{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathcal{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathcal{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \cdot \left(-\widetilde{\mathcal{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathcal{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}} \widetilde{\mathcal{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \\ &\quad = \int_{\Gamma_t} \widetilde{\mathcal{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \widetilde{\mathcal{N}}^{\frac{1}{2}} (-\triangle_{\Gamma_t}) \widetilde{\mathcal{N}}^{\frac{1}{2}} \left(-\widetilde{\mathcal{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathcal{N}}^{\frac{1}{2}}\right)^{l-1} \widetilde{\mathcal{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \\ &\quad = \int_{\Gamma_t} \widetilde{\mathcal{N}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \left(-\triangle_{\Gamma_t} \widetilde{\mathcal{N}}\right)^{l} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \, .\end{aligned} \tag{5.12}$$

Observing that

$$\mathbb{D}_t \, \mathrm{d}S_t = \left(\operatorname{div}_{\Gamma_t} \boldsymbol{\mu}\right) \mathrm{d}S_t \, , \tag{5.13}$$

then, if $l=0$, one can calculate that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\quad=\int_{\Gamma_t}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\cdot\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\widetilde{\mathfrak{N}}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\Big[\mathbb{D}_t,\widetilde{\mathfrak{N}}\Big](\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(\mathrm{div}_{\Gamma_t}\boldsymbol{\mu})\,\mathrm{d}S_t\,.
\end{aligned}
\tag{5.14}
$$

It follows from the self-adjointness of $\widetilde{\mathfrak{N}}$ and (5.7) that

$$
\begin{aligned}
&\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Gamma_t}\left|\widetilde{\mathfrak{N}}^{\frac12}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t\right)-\left(\int_{\Gamma_t}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\cdot\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\right)\right|\\
&\quad\lesssim_{\Lambda_*}|\boldsymbol{\mu}|_{\frac32k-\frac12}\cdot\left(|\partial_t\mathfrak{f}|^2_{H^{\frac12}(\Gamma_*)}+|\mathfrak{f}|^2_{H^{\frac32}(\Gamma_*)}\right)
\end{aligned}
\tag{5.15}
$$

If $l=1$ (so $k\geq 3$, since $l\leq k-2$), direct computations give that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\quad=\int_{\Gamma_t}\widetilde{\mathfrak{N}}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}\Big[\mathbb{D}_t,\widetilde{\mathfrak{N}}\Big](\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\big[\mathbb{D}_t,(-\triangle_{\Gamma_t})\big]\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(-\triangle_{\Gamma_t})\Big[\mathbb{D}_t,\widetilde{\mathfrak{N}}\Big](\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\\
&\qquad+\int_{\Gamma_t}\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot(\mathrm{div}_{\Gamma_t}\boldsymbol{\mu})\,\mathrm{d}S_t\,.
\end{aligned}
\tag{5.16}
$$

Hence, one may derive from the self-adjointness of $\widetilde{\mathfrak{N}},\triangle_{\Gamma_t}$ and the estimates (5.7)-(5.8) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\left|\Big(-\widetilde{\mathfrak{N}}^{\frac12}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac12}\Big)^{\frac12}\widetilde{\mathfrak{N}}^{\frac12}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t\\
&\quad=\int_{\Gamma_t}\widetilde{\mathfrak{N}}\big(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big)\cdot(-\triangle_{\Gamma_t})\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t+r_0,
\end{aligned}
\tag{5.17}
$$

with the estimate:

$$
|r_0|\lesssim_{\Lambda_*}|\boldsymbol{\mu}|_{\frac32k-\frac12}\times\Big(|\partial_t\mathfrak{f}|^2_{H^2(\Gamma_*)}+|\mathfrak{f}|^2_{H^3(\Gamma_*)}\Big).
\tag{5.18}
$$

Therefore, by using the following relations:

$$\int_{\Gamma_t}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{2m+1}{2}}\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t$$
$$=\int_{\Gamma_t}\left(-\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\right)^m\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\left(-\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\right)^{m+1}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\,,$$

and

$$\int_{\Gamma_t}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{2m}{2}}\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t$$
$$=\int_{\Gamma_t}\left(-\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\right)^m\widetilde{\mathfrak{N}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\left(-\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\right)^{m}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\,,$$

one can argue as the cases for $l=0$ and for $l=1$ to obtain the following estimate:

$$\begin{aligned}&\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\left|\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}}\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t-I\right|\\&\leq|\boldsymbol{\mu}|_{\frac{3}{2}k-\frac{1}{2}}\cdot\left(|\mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}+|\partial_t\mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right),\end{aligned}\tag{5.19}$$

where

$$I:=\int_{\Gamma_t}\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}}\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathbb{D}_t{}^2\bar{\mathfrak{f}}+\mathbb{D}_t\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\cdot\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{l}{2}}\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}})\,\mathrm{d}S_t\,.\tag{5.20}$$

For simplicity, set

$$\mathfrak{D}:=\left(-\widetilde{\mathfrak{N}}^{\frac{1}{2}}\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}^{\frac{1}{2}}\right)^{\frac{1}{2}}.\tag{5.21}$$

The equation in (5.4) is equivalent to

$$\begin{aligned}&\mathbb{D}_t{}^2\bar{\mathfrak{f}}+2\,\mathrm{D}_{\mathbf{u}}\mathbb{D}_t\bar{\mathfrak{f}}+\mathrm{D}_{\mathbf{u}}\,\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}+\left(-\triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\right)\bar{\mathfrak{f}}+\frac{\rho_+\rho_-}{(\rho_++\rho_-)^2}\mathrm{D}_{\mathbf{w}}\,\mathrm{D}_{\mathbf{w}}\bar{\mathfrak{f}}\\&-\frac{\rho_+}{\rho_++\rho_-}\mathrm{D}_{\mathbf{h}_+}\mathrm{D}_{\mathbf{h}_+}\bar{\mathfrak{f}}-\frac{\rho_-}{\rho_++\rho_-}\mathrm{D}_{\mathbf{h}_-}\mathrm{D}_{\mathbf{h}_-}\bar{\mathfrak{f}}=\bar{\mathfrak{g}},\end{aligned}\tag{5.22}$$

which implies

$$\begin{aligned} I =&\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}} \bar{\mathfrak{g}} \,\mathrm{d}S_t \\ &+ \int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}} - 2\mathrm{D}_{\mathbf{u}}\mathbb{D}_t\bar{\mathfrak{f}} - \mathrm{D}_{\mathbf{u}}\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \,\mathrm{d}S_t \\ &- \int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\Big(-\triangle_{\Gamma_t} \widetilde{\mathcal{N}}\bar{\mathfrak{f}}\Big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \,\mathrm{d}S_t \\ &- \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathrm{D}_{\mathbf{w}}\mathrm{D}_{\mathbf{w}}\bar{\mathfrak{f}}\big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \,\mathrm{d}S_t \\ &+ \frac{\rho_+}{\rho_+ + \rho_-}\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathrm{D}_{\mathbf{h}_+}\mathrm{D}_{\mathbf{h}_+}\bar{\mathfrak{f}}\big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \,\mathrm{d}S_t \\ &+ \frac{\rho_-}{\rho_+ + \rho_-}\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathrm{D}_{\mathbf{h}_-}\mathrm{D}_{\mathbf{h}_-}\bar{\mathfrak{f}}\big) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}\big(\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\big) \,\mathrm{d}S_t \\ =:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{5.23}$$

It is clear that

$$|I_1| \lesssim_{L_1} |\mathfrak{g}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} \cdot \Big(|\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}\Big). \tag{5.24}$$

Note that for two functions $\phi, \psi : \Gamma_t \to \mathbb{R}$, the integration-by-parts formula is

$$\int_{\Gamma_t} -(\mathrm{D}_{\mathbf{u}}\phi) \cdot \psi \,\mathrm{d}S_t = \int_{\Gamma_t} (\mathrm{D}_{\mathbf{u}}\psi) \cdot \phi + \phi\psi (\mathrm{div}_{\Gamma_t} \mathbf{u}) \,\mathrm{d}S_t\,, \tag{5.25}$$

since $\mathbf{u} : \Gamma_t \to \mathrm{T}\Gamma_t$ is a tangential field and $\int_{\Gamma_t} \mathrm{div}_{\Gamma_t} (\mathbf{u}\phi\psi) \,\mathrm{d}S_t \equiv 0$.

For I_2, observe that

$$\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}) \,\mathrm{d}S_t = \int_{\Gamma_t} \widetilde{\mathcal{N}}(\mathrm{D}_{\mathbf{u}}\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \cdot (-\triangle_{\Gamma_t}\widetilde{\mathcal{N}})^l (\mathbb{D}_t\bar{\mathfrak{f}}) \,\mathrm{d}S_t\,. \tag{5.26}$$

Thus, commuting $\mathrm{D}_{\mathbf{u}}$ with $\widetilde{\mathcal{N}}$ and $\triangle_{\Gamma_t}$ via the arguments as (5.14) and (5.16), it is routine to derive that

$$\begin{aligned} &\left|\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}) + \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\mathbb{D}_t\bar{\mathfrak{f}}) \,\mathrm{d}S_t\right| \\ &\quad \lesssim_{L_1} |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)} \cdot |\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}. \end{aligned} \tag{5.27}$$

Similarly, it follows from (5.25) that

$$\left|\int_{\Gamma_t} -\mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}) \,\mathrm{d}S_t\right| \lesssim_{L_1} |\mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}, \tag{5.28}$$

and

$$\left|\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{u}}\mathbb{D}_t\bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathcal{N}}^{\frac{1}{2}}(\mathbb{D}_t\bar{\mathfrak{f}}) \,\mathrm{d}S_t\right| \lesssim_{L_1} |\partial_t\mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}. \tag{5.29}$$

Furthermore, observing that

$$\big|[\mathbb{D}_t, \mathrm{D}_{\mathbf{u}}]\bar{\mathfrak{f}}\big|_{\frac{3}{2}l+\frac{1}{2}} = \big|\mathrm{D}_{(\mathbb{D}_t\mathbf{u}-\mathrm{D}_{\mathbf{u}}\boldsymbol{\mu})}\bar{\mathfrak{f}}\big|_{\frac{3}{2}l+\frac{1}{2}} \lesssim_{Q} |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{7}{4}}(\Gamma_*)}, \tag{5.30}$$

one can deduce from (5.26)-(5.30) that

$$|I_2| \lesssim_Q |\partial_t \mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + |\mathfrak{f}|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)}. \tag{5.31}$$

Next, the estimate on I_3 can be derived via:

$$\begin{aligned} I_3 &= -\int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} \Big(-\triangle_{\Gamma_t} \widetilde{\mathfrak{N}} \Big) \bar{\mathfrak{f}} \cdot \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \\ &= -\int_{\Gamma_t} \widetilde{\mathfrak{N}} \bar{\mathfrak{f}} \cdot (-\triangle_{\Gamma_t} \widetilde{\mathfrak{N}})^{l+1} (\mathbb{D}_t \bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \,, \end{aligned} \tag{5.32}$$

which, together with the previous arguments, yield

$$\left| I_3 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} \left| \mathfrak{D}^{l+1} \widetilde{\mathfrak{N}}^{\frac{1}{2}} \bar{\mathfrak{f}} \right|^2 \mathrm{d}S_t \right| \lesssim_Q |\mathfrak{f}|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)}. \tag{5.33}$$

As for I_4, in view of the relation

$$[\mathrm{D}_{\mathbf{u}}, \mathrm{D}_{\mathbf{w}}] \bar{\mathfrak{f}} = \mathrm{D}_{[\mathbf{u},\mathbf{w}]} \bar{\mathfrak{f}}, \tag{5.34}$$

it follows from the integration-by-parts that

$$\left| \int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{u}} \bar{\mathfrak{f}}) \, \mathrm{d}S_t \right| \lesssim_{L_1} |\mathfrak{f}|^2_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}. \tag{5.35}$$

The previous arguments can be used to show

$$\begin{aligned} &\left| \int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}}) + \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \mathbb{D}_t \bar{\mathfrak{f}}) \, \mathrm{d}S_t \right| \\ &\lesssim_{L_1} |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)} \times |\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}, \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} &\left| \int_{\Gamma_t} \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \cdot \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathbb{D}_t \bar{\mathfrak{f}}) \, \mathrm{d}S_t + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} \left| \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \right|^2 \mathrm{d}S_t \right| \\ &\lesssim_Q |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)} \times \Big(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} \Big), \end{aligned} \tag{5.37}$$

that is,

$$\begin{aligned} &\left| I_4 - \frac{\rho_+ \rho_-}{2(\rho_+ + \rho_-)^2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} \left| \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{w}} \bar{\mathfrak{f}}) \right|^2 \mathrm{d}S_t \right| \\ &\lesssim_Q |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)} \times \Big(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} \Big). \end{aligned} \tag{5.38}$$

Since I_5 and I_6 have the same form as I_4, there hold

$$\begin{aligned} &\left| I_5 + \frac{\rho_+}{2(\rho_+ + \rho_-)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} \left| \mathfrak{D}^l \widetilde{\mathfrak{N}}^{\frac{1}{2}} (\mathrm{D}_{\mathbf{h}_+} \bar{\mathfrak{f}}) \right|^2 \mathrm{d}S_t \right| \\ &\lesssim_Q |\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)} \times \Big(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\partial_t \mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} \Big), \end{aligned} \tag{5.39}$$

and

$$\begin{aligned}&\left|I_6+\frac{\rho_-}{2(\rho_++\rho_-)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\left|\mathfrak{D}^l\widetilde{\mathfrak{N}}^{\frac{1}{2}}(\mathrm{D}_{\mathbf{h}_-}\bar{\mathfrak{f}})\right|^2\mathrm{d}S_t\right|\\&\lesssim_Q|\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}\times\left(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)}+|\partial_t\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right).\end{aligned}\tag{5.40}$$

In conclusion, the combination of (5.5), (5.19), (5.24), (5.31), (5.33), (5.38)-(5.40) gives that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}E_l\right|\lesssim_Q&\left(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)}+|\partial_t\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right)\times\\&\times\left(|\mathfrak{f}|_{H^{\frac{3}{2}l+2}(\Gamma_*)}+|\partial_t\mathfrak{f}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}+|\mathfrak{g}|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right),\end{aligned}\tag{5.41}$$

which completes the proof of Lemma 5.1. □

Based on Lemma 5.1, the following energy estimate for (5.4) holds:

Proposition 5.2. *There is a constant $C_*>0$ determined by Λ_* so that for any integer $0\le l\le k-2$, and $0\le t\le T$ ($T\le C$ for some constant $C=C(L_1,L_2)$), it holds that*

$$\begin{aligned}&|\mathfrak{f}(t)|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)}+|\partial_t\mathfrak{f}(t)|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\\&\le C_*\exp\left[Q(L_1,L_2)t\right]\times\\&\times\left\{|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)}+|\mathfrak{f}_1|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}+(L_0)^{12}|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}+\int_0^t|\mathfrak{g}(t')|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\,\mathrm{d}t'\right\},\end{aligned}\tag{5.42}$$

where Q is a generic polynomial determined by Λ_*.

Proof. It is clear that for some $C_*>0$, it holds that

$$\begin{aligned}&\left|\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+2}+\left|\mathbb{D}_t\bar{\mathfrak{f}}\right|_{\frac{3}{2}l+\frac{1}{2}}\\&\le C_*E_l(t,\mathfrak{f},\partial_t\mathfrak{f})+C_*\left(\left|\int_{\Gamma_t}\bar{\mathfrak{f}}\,\mathrm{d}S_t\right|^2+\left|\int_{\Gamma_t}\mathbb{D}_t\bar{\mathfrak{f}}\,\mathrm{d}S_t\right|^2\right)+C_*\left(\left|\mathrm{D}_{\mathbf{w}}\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+\frac{1}{2}}+\left|\mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+\frac{1}{2}}\right)\\&\le C_*E_l(t,\mathfrak{f},\partial_t\mathfrak{f})+C_*\left(\left|\int_{\Gamma_t}\bar{\mathfrak{f}}\,\mathrm{d}S_t\right|^2+\left|\int_{\Gamma_t}\mathbb{D}_t\bar{\mathfrak{f}}\,\mathrm{d}S_t\right|^2\right)+C_*\left(|\mathbf{v}_+|^2_{\frac{3}{2}k-2}+|\mathbf{v}_-|^2_{\frac{3}{2}k-2}\right)\left|\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+\frac{7}{4}}.\end{aligned}\tag{5.43}$$

For the last term above, it follows from the interpolation inequality that

$$\left(|\mathbf{v}_+|^2_{\frac{3}{2}k-2}+|\mathbf{v}_-|^2_{\frac{3}{2}k-2}\right)\left|\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+\frac{7}{4}}\le\frac{5}{6C_*}\left|\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+2}+\frac{(C_*)^5}{6}\left(|\mathbf{v}_+|^{12}_{\frac{3}{2}k-2}+|\mathbf{v}_-|^{12}_{\frac{3}{2}k-2}\right)\left|\bar{\mathfrak{f}}\right|^2_{\frac{3}{2}l+\frac{1}{2}},\tag{5.44}$$

with

$$\begin{aligned}|\mathbf{v}_\pm(t)|_{\frac{3}{2}k-2}&\lesssim_{\Lambda_*}|\mathbf{v}_{\pm*}(t)|_{H^{\frac{3}{2}k-2}(\Gamma_*)}\\&\lesssim_{\Lambda_*}|\mathbf{v}_{\pm*}(0)|_{H^{\frac{3}{2}k-2}(\Gamma_*)}+L_2t.\end{aligned}\tag{5.45}$$

Now, observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\bar{\mathfrak{f}}(t)\,\mathrm{d}S_t=\int_{\Gamma_t}\mathbb{D}_t\bar{\mathfrak{f}}+\bar{\mathfrak{f}}\,\mathrm{div}_{\Gamma_t}\boldsymbol{\mu}\,\mathrm{d}S_t\,.\tag{5.46}$$

Thus

$$\left|\int_{\Gamma_t} \bar{\mathfrak{f}}(t)\,\mathrm{d}S_t - \int_{\Gamma_0} \bar{\mathfrak{f}}(0)\,\mathrm{d}S_0\right| \lesssim_{L_1} \int_0^t \left|\partial_t \mathfrak{f}(t')\right|_{L^2(\Gamma_*)} + \left|\mathfrak{f}(t')\right|_{L^2(\Gamma_*)}\mathrm{d}t'\,. \tag{5.47}$$

To deal with the integral of $\mathbb{D}_t\bar{\mathfrak{f}}$, one notes that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t} \mathbb{D}_t\bar{\mathfrak{f}}\,\mathrm{d}S_t = \int_{\Gamma_t} \mathbb{D}_t{}^2\bar{\mathfrak{f}} + \left(\mathbb{D}_t\bar{\mathfrak{f}}\right)\mathrm{div}_{\Gamma_t}\boldsymbol{\mu}\,\mathrm{d}S_t\,, \tag{5.48}$$

and (5.22) implies that

$$\begin{aligned}\int_{\Gamma_t} \mathbb{D}_t{}^2\bar{\mathfrak{f}}\,\mathrm{d}S_t = &-\int_{\Gamma_t} \mathrm{D}_{\mathbf{u}}\left(2\mathbb{D}_t\bar{\mathfrak{f}} + \mathrm{D}_{\mathbf{u}}\bar{\mathfrak{f}}\right)\mathrm{d}S_t + \int_{\Gamma_t} \triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\bar{\mathfrak{f}}\,\mathrm{d}S_t \\ &- \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\int_{\Gamma_t} \mathrm{D}_{\mathbf{w}}\,\mathrm{D}_{\mathbf{w}}\bar{\mathfrak{f}}\,\mathrm{d}S_t + \frac{\rho_+}{\rho_+ + \rho_-}\int_{\Gamma_t} \mathrm{D}_{\mathbf{h}_+}\mathrm{D}_{\mathbf{h}_+}\bar{\mathfrak{f}}\,\mathrm{d}S_t \\ &+ \frac{\rho_-}{\rho_+ + \rho_-}\int_{\Gamma_t} \mathrm{D}_{\mathbf{h}_-}\mathrm{D}_{\mathbf{h}_-}\bar{\mathfrak{f}}\,\mathrm{d}S_t + \int_{\Gamma_t} \bar{\mathfrak{g}}\,\mathrm{d}S_t\,.\end{aligned} \tag{5.49}$$

Due to the fact that $\partial\Gamma_t = \varnothing$, one has

$$\int_{\Gamma_t} \triangle_{\Gamma_t}\widetilde{\mathfrak{N}}\bar{\mathfrak{f}}\,\mathrm{d}S_t \equiv 0, \tag{5.50}$$

and

$$-\int_{\Gamma_t} \mathrm{D}_{\mathbf{u}}\mathbb{D}_t\bar{\mathfrak{f}}\,\mathrm{d}S_t = \int_{\Gamma_t} \left(\mathbb{D}_t\bar{\mathfrak{f}}\right)\mathrm{div}_{\Gamma_t}\mathbf{u}\,\mathrm{d}S_t\,. \tag{5.51}$$

Thus

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t} \mathbb{D}_t\bar{\mathfrak{f}}\,\mathrm{d}S_t\right| \lesssim_{L_1} \left|\mathbb{D}_t\bar{\mathfrak{f}}\right|_0 + \left|\bar{\mathfrak{f}}\right|_1 + |\bar{\mathfrak{g}}|_0, \tag{5.52}$$

and

$$\begin{aligned}&\left|\int_{\Gamma_t} \left(\mathbb{D}_t\bar{\mathfrak{f}}\right)(t)\,\mathrm{d}S_t - \int_{\Gamma_0} \left(\mathbb{D}_t\bar{\mathfrak{f}}\right)(0)\,\mathrm{d}S_0\right| \\ &\quad\lesssim_{L_1} \int_0^t \left|\partial_t\mathfrak{f}(t')\right|_{L^2(\Gamma_*)} + \left|\mathfrak{f}(t')\right|_{H^1(\Gamma_*)} + \left|\mathfrak{g}(t')\right|_{L^2(\Gamma_*)}\mathrm{d}t'\,.\end{aligned} \tag{5.53}$$

Combining (5.43)-(5.53), (5.6) and the following relation:

$$\left|\bar{\mathfrak{f}}(t)\right|_{\frac{3}{2}l+\frac{1}{2}} \lesssim_{\Lambda_*} |\mathfrak{f}(0)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + \int_0^t \left|\partial_t\mathfrak{f}(t')\right|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\mathrm{d}t'\,, \tag{5.54}$$

one can get that

$$
\begin{aligned}
&\left|\bar{\mathfrak{f}}(t)\right|^2_{\frac{3}{2}l+2} + \left|\mathbb{D}_t\bar{\mathfrak{f}}(t)\right|^2_{\frac{3}{2}l+\frac{1}{2}} \\
&\lesssim_{\Lambda_*} |\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\mathfrak{f}_1|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + (L_0)^{12}|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} \\
&\quad + \bar{Q}\cdot\int_0^t \left|\mathfrak{f}(t')\right|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)} + \left|\partial_t\mathfrak{f}(t')\right|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + \left|\mathfrak{g}(t')\right|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\,\mathrm{d}t' \\
&\quad + \left(|\mathbf{v}_{\pm*}(0)|^{12}_{H^{\frac{3}{2}k-2}(\Gamma_*)} + (L_2)^{12}t^{12}\right)\cdot C(L_1)t\int_0^t \left|\partial_t\mathfrak{f}(t')\right|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\,\mathrm{d}t' \\
&\quad + C(L_1)t\int_0^t \left|\partial_t\mathfrak{f}(t')\right|^2_{L^2(\Gamma_*)} + \left|\mathfrak{f}(t')\right|^2_{H^1(\Gamma_*)} + \left|\mathfrak{g}(t')\right|^2_{L^2(\Gamma_*)}\,\mathrm{d}t',
\end{aligned}
\tag{5.55}
$$

where $\bar{Q} = \bar{Q}(L_1, L_2)$ is the generic polynomial in the previous lemma. If $T \le Q_*(L_1, L_2)$, it follows that

$$
\begin{aligned}
&\left|\bar{\mathfrak{f}}(t)\right|^2_{\frac{3}{2}l+2} + \left|\mathbb{D}_t\bar{\mathfrak{f}}(t)\right|^2_{\frac{3}{2}l+\frac{1}{2}} \\
&\le C_*\left(|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)} + |\mathfrak{f}_1|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + (L_0)^{12}|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\right) \\
&\quad + Q\int_0^t \left|\mathfrak{f}(t')\right|^2_{H^{\frac{3}{2}l+2}(\Gamma_*)} + \left|\partial_t\mathfrak{f}(t')\right|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)} + \left|\mathfrak{g}(t')\right|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\,\mathrm{d}t',
\end{aligned}
\tag{5.56}
$$

where $Q = Q(L_1, L_2)$ is a generic polynomial determined by Λ_*. Hence (5.42) follows from Gronwall's inequality. □

Then the local well-posedness of (5.4) follows from this energy estimate:

Corollary 5.3. *For $0 \le l \le k-2$, $T \le C(L_1, L_2)$ and $\mathfrak{g} \in C^0\left([0,T]; H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)\right)$, the problem (5.4) is well-posed in $C^0\left([0,T]; H^{\frac{3}{2}l+2}(\Gamma_*)\right) \cap C^1\left([0,T]; H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)\right)$, and the estimate (5.42) holds.*

5.2. **Linearized system for the current and vorticity.** Assume still that $\Gamma_* \in H^{\frac{3}{2}k+1}$ ($k \ge 2$) is a reference hypersurface and consider a family of hypersurfaces $\{\Gamma_t\}_{0\le t\le T} \subset \Lambda_*$, for which each Γ_t separates Ω into two disjoint simply-connected domains $\Omega_t^\pm$. Suppose that $\mathbf{v}_\pm(t), \mathbf{h}_\pm(t) : \Omega_t^\pm \to \mathbb{R}^3$ are given vector fields solving

$$
\begin{cases}
\nabla\cdot\mathbf{v}_\pm = 0 = \nabla\cdot\mathbf{h}_\pm & \text{in } \Omega_t^\pm, \\
\mathbf{h}_+\cdot\mathbf{N}_+ = 0 = \mathbf{h}_-\cdot\mathbf{N}_+ & \text{on } \Gamma_t, \\
\mathbf{v}_+\cdot\mathbf{N}_+ = \mathbf{N}_+\cdot\left(\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\right)\circ\Phi_{\Gamma_t}^{-1} = \mathbf{v}_-\cdot\mathbf{N}_+ & \text{on } \Gamma_t, \\
\mathbf{v}_-\cdot\widetilde{\mathbf{N}} = 0 = \mathbf{h}_-\cdot\widetilde{\mathbf{N}} & \text{on } \partial\Omega.
\end{cases}
\tag{5.57}
$$

Assume further that there is a constant $\bar{L}_1$ so that

$$
\sup_{t\in[0,T]}\left(|\varkappa_a(t)|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, |\partial_t\varkappa_a(t)|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|(\mathbf{v}_\pm(t), \mathbf{h}_\pm(t))\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}\right) \le \bar{L}_1. \tag{5.58}
$$

Consider the following linearized current-vorticity system in $\Omega \setminus \Gamma_t$:

$$
\partial_t\widetilde{\boldsymbol{\omega}} + (\mathbf{v}\cdot\nabla)\widetilde{\boldsymbol{\omega}} - (\mathbf{h}\cdot\nabla)\widetilde{\mathbf{j}} = (\widetilde{\boldsymbol{\omega}}\cdot\nabla)\mathbf{v} - (\widetilde{\mathbf{j}}\cdot\nabla)\mathbf{h}, \tag{5.59}
$$

$$\partial_t\widetilde{\mathbf{j}} + (\mathbf{v}\cdot\nabla)\widetilde{\mathbf{j}} - (\mathbf{h}\cdot\nabla)\widetilde{\boldsymbol{\omega}} = (\widetilde{\mathbf{j}}\cdot\nabla)\mathbf{v} - (\widetilde{\boldsymbol{\omega}}\cdot\nabla)\mathbf{h} - 2\,\mathrm{tr}(\nabla\mathbf{v}\times\nabla\mathbf{h}). \tag{5.60}$$

Set

$$\boldsymbol{\xi} := \widetilde{\boldsymbol{\omega}} - \widetilde{\mathbf{j}}, \quad \boldsymbol{\eta} := \widetilde{\boldsymbol{\omega}} + \widetilde{\mathbf{j}}. \tag{5.61}$$

Then

$$\partial_t\boldsymbol{\xi} + [(\mathbf{v}+\mathbf{h})\cdot\nabla]\boldsymbol{\xi} = (\boldsymbol{\xi}\cdot\nabla)(\mathbf{v}+\mathbf{h}) + 2\,\mathrm{tr}(\nabla\mathbf{v}\times\nabla\mathbf{h}), \tag{5.62}$$

$$\partial_t\boldsymbol{\eta} + [(\mathbf{v}-\mathbf{h})\cdot\nabla]\boldsymbol{\eta} = (\boldsymbol{\eta}\cdot\nabla)(\mathbf{v}-\mathbf{h}) - 2\,\mathrm{tr}(\nabla\mathbf{v}\times\nabla\mathbf{h}). \tag{5.63}$$

Define the flow map $\mathcal{Y}^{\pm}$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y}^{\pm}(t,y) = (\mathbf{v}_{\pm}-\mathbf{h}_{\pm})\Big(t,\mathcal{Y}^{\pm}(t,y)\Big), \quad y\in\Omega_0^{\pm}. \tag{5.64}$$

As indicated in [SWZ18], due to the fact that $\mathbf{h}_{\pm}\cdot\mathbf{N}_{+}\equiv 0$, $\mathcal{Y}^{\pm}(t)$ is a bijection from $\Omega_0^{\pm}$ to $\Omega_t^{\pm}$ for small time t. Furthermore, the evolution equation for $\boldsymbol{\eta}$ can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\eta}\circ\mathcal{Y}) = [(\boldsymbol{\eta}\cdot\nabla)(\mathbf{v}-\mathbf{h}) - 2\,\mathrm{tr}(\nabla\mathbf{v}\times\nabla\mathbf{h})]\circ\mathcal{Y}, \tag{5.65}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[(\mathrm{D}\mathcal{Y})^{-1}(\boldsymbol{\eta}\circ\mathcal{Y})\Big] = -2\,\mathrm{tr}(\nabla\mathbf{v}\times\nabla\mathbf{h})\circ\mathcal{Y}, \tag{5.66}$$

which is a linear ODE system. Thus, the local well-posedness follows routinely. Similarly, the evolution equation for $\boldsymbol{\xi}$ is also locally well-posed on $[0,T]$, with the life span T depending on $\bar{L}_1$. Furthermore, the following energy estimates hold:

Proposition 5.4. *For $0\le t\le T$, it follows that*

$$\begin{aligned} &\|\widetilde{\boldsymbol{\omega}}_{\pm}(t)\|^2_{H^{\frac{3}{2}k-1}(\Omega_t^{\pm})} + \Big\|\widetilde{\mathbf{j}}_{\pm}(t)\Big\|^2_{H^{\frac{3}{2}k-1}(\Omega_t^{\pm})} \\ &\le \exp\{Q(\bar{L}_1)t\}\Big(1 + \|\widetilde{\boldsymbol{\omega}}_{\pm}(0)\|^2_{H^{\frac{3}{2}k-1}(\Omega_0^{\pm})} + \Big\|\widetilde{\mathbf{j}}_{\pm}(0)\Big\|^2_{H^{\frac{3}{2}k-1}(\Omega_0^{\pm})}\Big), \end{aligned} \tag{5.67}$$

where Q is a generic polynomial depending on Λ_*.

Proof. For $0\le s\le \frac{3}{2}k-1$, observe that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t^{\pm}}|\nabla^s\boldsymbol{\eta}_{\pm}|^2\,\mathrm{d}x \\ &= \int_{\Omega_t^{\pm}}\nabla^s\boldsymbol{\eta}_{\pm}\cdot\nabla^s\partial_t\boldsymbol{\eta}_{\pm}\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_t^{\pm}}\mathrm{D}_{(\mathbf{v}_{\pm}-\mathbf{h}_{\pm})}|\nabla^s\boldsymbol{\eta}_{\pm}|^2\,\mathrm{d}x \\ &= \int_{\Omega_t^{\pm}}\nabla^s\boldsymbol{\eta}_{\pm}\cdot\nabla^s\big[\mathrm{D}_{(\mathbf{h}_{\pm}-\mathbf{v}_{\pm})}\boldsymbol{\eta}_{\pm} + \mathrm{D}_{\boldsymbol{\eta}_{\pm}}(\mathbf{v}_{\pm}-\mathbf{h}_{\pm}) - 2\,\mathrm{tr}(\nabla\mathbf{v}_{\pm}\times\nabla\mathbf{h}_{\pm})\big]\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\Omega_t^{\pm}}\mathrm{D}_{(\mathbf{v}_{\pm}-\mathbf{h}_{\pm})}|\nabla^s\boldsymbol{\eta}_{\pm}|^2\,\mathrm{d}x \\ &= \int_{\Omega_t^{\pm}}\nabla^s\boldsymbol{\eta}_{\pm}\cdot\big[\nabla^s,\mathrm{D}_{(\mathbf{h}_{\pm}-\mathbf{v}_{\pm})}\big]\boldsymbol{\eta}_{\pm}\,\mathrm{d}x + \int_{\Omega_t^{\pm}}\nabla^s\boldsymbol{\eta}_{\pm}\cdot\nabla^s\{\mathrm{D}_{\boldsymbol{\eta}_{\pm}}(\mathbf{v}_{\pm}-\mathbf{h}_{\pm}) - 2\,\mathrm{tr}(\nabla\mathbf{v}_{\pm}\times\nabla\mathbf{h}_{\pm})\}\,\mathrm{d}x \\ &\le Q(\bar{L}_1)\Big(1+\|\boldsymbol{\eta}_{\pm}\|^2_{H^s(\Omega_t^{\pm})}\Big). \end{aligned} \tag{5.68}$$

It follows from a similar argument that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\xi}_\pm(t)\|^2_{H^s(\Omega_t^\pm)} \le Q(\bar{L}_1)\Big(1+\|\boldsymbol{\xi}_\pm(t)\|^2_{H^s(\Omega_t^\pm)}\Big). \tag{5.69}$$

Therefore, (5.67) holds due to Gronwall's inequality and (5.61). □

To show the compatibility conditions:

$$\int_{\partial\Omega} \widetilde{\boldsymbol{\omega}}_- \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} \equiv 0 \equiv \int_{\partial\Omega} \widetilde{\mathbf{j}}_- \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S}\,, \tag{5.70}$$

one observes that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} \boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} \\
&= \int_{\partial\Omega} \big(\partial_t + \mathrm{D}_{(\mathbf{v}_- - \mathbf{h}_-)}\big)\Big(\boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}}\Big) + \Big(\boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}}\Big)\,\mathrm{div}_{\partial\Omega}(\mathbf{v}_- - \mathbf{h}_-)\,\mathrm{d}\widetilde{S} \\
&= \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \mathrm{D}_{\boldsymbol{\eta}_-}(\mathbf{v}_- - \mathbf{h}_-) - 2\widetilde{\mathbf{N}} \cdot \mathrm{tr}(\nabla \mathbf{v}_- \times \nabla \mathbf{h}_-)\,\mathrm{d}\widetilde{S} \\
&\quad + \int_{\partial\Omega} -\widetilde{\mathbf{N}} \cdot \mathrm{D}_{\boldsymbol{\eta}_-^\top}(\mathbf{v}_- - \mathbf{h}_-) + (\boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}})\,\mathrm{div}_{\partial\Omega}(\mathbf{v}_- - \mathbf{h}_-)\,\mathrm{d}\widetilde{S} \\
&=: I_1 + I_2.
\end{aligned} \tag{5.71}$$

Since $\nabla \cdot (\nabla\phi \times \nabla\psi) \equiv 0$ and $\nabla \cdot (\mathbf{v}_- - \mathbf{h}_-) \equiv 0$,

$$\begin{aligned}
I_1 &= \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \mathrm{D}_{\boldsymbol{\eta}_-^\top}(\mathbf{v}_- - \mathbf{h}_-) + (\boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}})\,\mathrm{D}_{\widetilde{\mathbf{N}}}(\mathbf{v}_- - \mathbf{h}_-) \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} \\
&= \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \mathrm{D}_{\boldsymbol{\eta}_-^\top}(\mathbf{v}_- - \mathbf{h}_-) - (\boldsymbol{\eta}_- \cdot \widetilde{\mathbf{N}})\,\mathrm{div}_{\partial\Omega}(\mathbf{v}_- - \mathbf{h}_-)\,\mathrm{d}\widetilde{S} \\
&= -I_2,
\end{aligned} \tag{5.72}$$

where the geometric relation (3.10) has been used. Thus, the similar arguments yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} \boldsymbol{\xi} \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} = 0, \tag{5.73}$$

which implies the following lemma:

Lemma 5.5. *Suppose that $(\widetilde{\boldsymbol{\omega}}(t), \widetilde{\mathbf{j}}(t))$ is the solution to the linear system (5.59)-(5.60) with initial data $(\widetilde{\boldsymbol{\omega}}_0, \widetilde{\mathbf{j}}_0)$. If*

$$\int_{\partial\Omega} \widetilde{\boldsymbol{\omega}}_{0-} \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} = 0 = \int_{\partial\Omega} \widetilde{\mathbf{j}}_{0-} \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S}\,, \tag{5.74}$$

then for all t such that the solution exists, there holds

$$\int_{\partial\Omega} \widetilde{\boldsymbol{\omega}}_-(t) \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S} \equiv 0 \equiv \int_{\partial\Omega} \widetilde{\mathbf{j}}_-(t) \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\widetilde{S}\,. \tag{5.75}$$

6. Nonlinear Problems

6.1. **Initial settings.** Take a reference hypersurface $\Gamma_* \in H^{\frac{3}{2}k+1}$ and $\delta_0 > 0$ so that

$$\Lambda_* := \Lambda\left(\Gamma_*, \frac{3}{2}k - \frac{1}{2}, \delta_0\right)$$

satisfies all the properties discussed in the preliminary. We will solve the nonlinear current-vortex sheet problems by an iteration scheme based on solving the linearized problems in the space:

$$\begin{aligned}&\varkappa_a \in C^0\left([0,T]; H^{\frac{3}{2}k-1}(\Gamma_*)\right) \cap C^1\left([0,T]; B_{\delta_1} \subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\right) \cap C^2\left([0,T]; H^{\frac{3}{2}k-4}(\Gamma_*)\right);\\ &\boldsymbol{\omega}_{\pm *}, \mathbf{j}_{\pm *} \in C^0\left([0,T]; H^{\frac{3}{2}k-1}(\Omega^{\pm}_{\Gamma_*})\right) \cap C^1\left([0,T]; H^{\frac{3}{2}k-2}(\Omega^{\pm}_{\Gamma_*})\right).\end{aligned} \tag{6.1}$$

In order to construct the iteration map, we define the following function space:

Definition 6.1. For given constants $T, M_0, M_1, M_2, M_3 > 0$, define $\mathfrak{X}$ to be the collection of $(\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*)$ satisfying:

$$|\varkappa_a(0) - \kappa_{*+}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \le \delta_1,$$

$$|(\partial_t \varkappa_a)(0)|_{H^{\frac{3}{2}k-4}(\Gamma_*)}, \|\boldsymbol{\omega}_*(0)\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}, \|\mathbf{j}_*(0)\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)} \le M_0,$$

$$\sup_{t\in[0,T]}\left(|\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, |\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \|\mathbf{j}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\right) \le M_1,$$

$$\sup_{t\in[0,T]}\left(\|\partial_t \boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}, \|\partial_t \mathbf{j}_*\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}\right) \le M_2,$$

$$\sup_{t\in[0,T]}\left|\partial^2_{tt}\varkappa_a\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \le a^2 M_3 \text{ (here } a \text{ is the constant in the definition of } \varkappa_a).$$

In addition, the compatibility conditions

$$\int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \boldsymbol{\omega}_{*-} \,\mathrm{d}\widetilde{S} = \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \mathbf{j}_{*-} \,\mathrm{d}\widetilde{S} = 0 \tag{6.2}$$

hold for all $t \in [0,T]$.

For $0 < \epsilon \ll \delta_1$ and $A > 0$, the collection of initial data

$$\mathfrak{I}(\epsilon, A) := \left\{\left((\varkappa_a)_{\mathrm{I}}, (\partial_t \varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_*)_{\mathrm{I}}, (\mathbf{j}_*)_{\mathrm{I}}\right)\right\}$$

is defined by:

$$|(\varkappa_a)_{\mathrm{I}} - \kappa_{*+}|_{H^{\frac{3}{2}k-1}(\Gamma_*)} < \epsilon;$$

$$|(\partial_t \varkappa_a)_{\mathrm{I}}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \ \|(\boldsymbol{\omega}_*)_{\mathrm{I}}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \ \|(\mathbf{j}_*)_{\mathrm{I}}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)} < A,$$

and

$$\int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot (\boldsymbol{\omega}_*)_{\mathrm{I}-} \,\mathrm{d}\widetilde{S} = \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot (\mathbf{j}_*)_{\mathrm{I}-} \,\mathrm{d}\widetilde{S} = 0.$$

Thus, $\mathfrak{I}(\epsilon, A) \subset H^{\frac{3}{2}k-1}(\Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*) \times H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)$.

6.2. **Recovery of the fluid region, velocity and magnetic fields.** For $(\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*) \in \mathfrak{X}$, $\varkappa_a(t)$ induces a family of hypersurfaces $\Gamma_t \in \Lambda_*$ if $M_1 T$ is not too large. Thus Φ_{Γ_t} and $\mathcal{X}_{\Gamma_t}$ can be defined by $\varkappa_a(t)$.

For a vector field $\mathbf{Y} : \Omega \setminus \Gamma_t \to \mathbb{R}^3$, define

$$(\mathbb{P}\mathbf{Y})_\pm := \mathbf{Y}_\pm - \nabla \phi_\pm,$$

for which

$$\begin{cases} \triangle \phi_\pm = \nabla \cdot \mathbf{Y}_\pm & \text{in } \Omega_t^\pm, \\ \phi_\pm = 0 & \text{on } \Gamma_t, \\ \mathrm{D}_{\widetilde{\mathbf{N}}} \phi_- = 0 & \text{on } \partial\Omega. \end{cases}$$

Namely, $\mathbb{P}$ is the Leray projection. Set

$$\bar{\boldsymbol{\omega}} := \mathbb{P}\big(\boldsymbol{\omega}_*(t) \circ \mathcal{X}_{\Gamma_t}^{-1}\big), \quad \bar{\mathbf{j}} := \mathbb{P}\big(\mathbf{j}_*(t) \circ \mathcal{X}_{\Gamma_t}^{-1}\big). \tag{6.3}$$

Thus, $\nabla \cdot \bar{\boldsymbol{\omega}} = \nabla \cdot \bar{\mathbf{j}} = 0$ in $\Omega \setminus \Gamma_t$ and

$$\int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \bar{\boldsymbol{\omega}}_- \,\mathrm{d}S = 0 = \int_{\partial\Omega} \widetilde{\mathbf{N}} \cdot \bar{\mathbf{j}}_- \,\mathrm{d}S\,,$$

since $\mathcal{X}_{\Gamma_t}|_{\partial\Omega} = \mathrm{id}|_{\partial\Omega}$.

Now, by solving the following div-curl problems:

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \setminus \Gamma_t, \\ \nabla \times \mathbf{v} = \bar{\boldsymbol{\omega}} & \text{in } \Omega \setminus \Gamma_t, \\ \mathbf{v}_\pm \cdot \mathbf{N}_+ = \mathbf{N}_+ \cdot \big(\partial_t \gamma_{\Gamma_t} \boldsymbol{\nu}\big) \circ \Phi_{\Gamma_t}^{-1} & \text{on } \Gamma_t, \\ \mathbf{v}_- \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\Omega; \end{cases} \tag{6.4}$$

and

$$\begin{cases} \nabla \cdot \mathbf{h} = 0 & \text{in } \Omega \setminus \Gamma_t, \\ \nabla \times \mathbf{h} = \bar{\mathbf{j}} & \text{in } \Omega \setminus \Gamma_t, \\ \mathbf{h}_\pm \cdot \mathbf{N}_+ = 0 & \text{on } \Gamma_t, \\ \mathbf{h}_- \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.5}$$

one can obtain the corresponding velocity and magnetic fields $\mathbf{v}_\pm, \mathbf{h}_\pm : \Omega_t^\pm \to \mathbb{R}^3$. Furthermore, the following estimate holds thanks to Theorem 3.8:

$$\sup_{t\in[0,T]} \left(\|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}, \|\mathbf{h}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)} \right) \le Q(M_1), \tag{6.6}$$

where Q is a generic polynomial determined by Λ_*.

6.3. **Iteration map.** For $\left(\varkappa_a{}^{(n)}, \boldsymbol{\omega}_*{}^{(n)}, \mathbf{j}_*{}^{(n)}\right) \in \mathfrak{X}$ and $\big\{(\varkappa_a)_{\mathrm{I}}, (\partial_t \varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_*)_{\mathrm{I}}, (\mathbf{j}_*)_{\mathrm{I}}\big\} \in \mathfrak{I}(\epsilon, A)$, define the $(n+1)$-th step by solving the following initial value problems:

$$\begin{cases} \partial_{tt}^2 \varkappa_a{}^{(n+1)} + \mathscr{C}\left(\varkappa_a{}^{(n)}, \partial_t \varkappa_a{}^{(n)}, \mathbf{v}_*{}^{(n)}, \mathbf{h}_*{}^{(n)}\right) \varkappa_a{}^{(n+1)} \\ \qquad = \mathscr{F}\left(\varkappa_a{}^{(n)}\right) \partial_t \boldsymbol{\omega}_*{}^{(n)} + \mathscr{G}\left(\varkappa_a{}^{(n)}, \partial_t \varkappa_a{}^{(n)}, \boldsymbol{\omega}_*{}^{(n)}, \mathbf{j}_*{}^{(n)}\right) \\ \varkappa_a{}^{(n+1)}(0) = (\varkappa_a)_{\mathrm{I}}, \quad \partial_t \varkappa_a{}^{(n+1)}(0) = (\partial_t \varkappa_a)_{\mathrm{I}}; \end{cases} \tag{6.7}$$

and

$$\begin{cases} \partial_t \boldsymbol{\omega}^{(n+1)} + \mathrm{D}_{\mathbf{v}^{(n)}} \boldsymbol{\omega}^{(n+1)} - \mathrm{D}_{\mathbf{h}^{(n)}} \mathbf{j}^{(n+1)} = \mathrm{D}_{\boldsymbol{\omega}^{(n+1)}} \mathbf{v}^{(n)} - \mathrm{D}_{\mathbf{j}^{(n+1)}} \mathbf{h}^{(n)}, \\ \partial_t \mathbf{j}^{(n+1)} + \mathrm{D}_{\mathbf{v}^{(n)}} \mathbf{j}^{(n+1)} - \mathrm{D}_{\mathbf{h}^{(n)}} \boldsymbol{\omega}^{(n+1)} \\ \qquad = \mathrm{D}_{\mathbf{j}^{(n+1)}} \mathbf{v}^{(n)} - \mathrm{D}_{\boldsymbol{\omega}^{(n+1)}} \mathbf{h}^{(n)} - 2 \operatorname{tr}\left(\nabla \mathbf{v}^{(n)} \times \nabla \mathbf{h}^{(n)}\right), \\ \boldsymbol{\omega}^{(n+1)}(0) = \mathbb{P}\left((\boldsymbol{\omega}_*)_{\mathrm{I}} \circ \mathfrak{X}_{\Gamma_0{}^{(n)}}^{-1}\right), \quad \mathbf{j}^{(n+1)}(0) = \mathbb{P}\left((\mathbf{j}_*)_{\mathrm{I}} \circ \mathfrak{X}_{\Gamma_0{}^{(n)}}^{-1}\right), \end{cases} \tag{6.8}$$

where $\left(\mathbf{v}^{(n)}, \mathbf{h}^{(n)}\right)$ is induced by $\left(\varkappa_a{}^{(n)}, \boldsymbol{\omega}_*{}^{(n)}, \mathbf{j}_*{}^{(n)}\right)$ via solving (6.4)-(6.5), the tangential vector fields $\mathbf{v}_*{}^{(n)}$ and $\mathbf{h}_*{}^{(n)}$ on Γ_* are defined by

$$\begin{aligned} \mathbf{v}_{*\pm}{}^{(n)} &:= \left(\mathrm{D}\Phi_{\Gamma_t{}^{(n)}}\right)^{-1} \left[\mathbf{v}_\pm{}^{(n)} \circ \Phi_{\Gamma_t{}^{(n)}} - \left(\partial_t \gamma_{\Gamma_t{}^{(n)}}\right) \boldsymbol{\nu}\right], \\ \mathbf{h}_{*\pm}{}^{(n)} &:= \left(\mathrm{D}\Phi_{\Gamma_t{}^{(n)}}\right)^{-1} \left(\mathbf{h}_\pm{}^{(n)} \circ \Phi_{\Gamma_t{}^{(n)}}\right), \end{aligned} \tag{6.9}$$

and the current-vorticity equations are considered in the domain $\Omega \setminus \Gamma_t{}^{(n)}$.

Denoting by

$$\boldsymbol{\omega}_*{}^{(n+1)} := \boldsymbol{\omega}^{(n+1)} \circ \mathfrak{X}_{\Gamma_t{}^{(n)}}, \quad \text{and} \quad \mathbf{j}_*{}^{(n+1)} := \mathbf{j}^{(n+1)} \circ \mathfrak{X}_{\Gamma_t{}^{(n)}}, \tag{6.10}$$

we will show that $\left(\varkappa_a{}^{(n+1)}, \boldsymbol{\omega}_*{}^{(n+1)}, \mathbf{j}_*{}^{(n+1)}\right) \in \mathfrak{X}$. Indeed, in view of Lemma 4.4, there hold

$$\begin{aligned} &\left|\mathscr{F}\left(\varkappa_a{}^{(n)}\right) \partial_t \boldsymbol{\omega}_*{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \\ &\quad \le Q\left(\left|\varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-1}(\Gamma_*)}\right) \left\|\partial_t \boldsymbol{\omega}_*{}^{(n)}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)} \\ &\quad \le Q(M_1) \cdot M_2, \end{aligned}$$

and

$$\begin{aligned} &\left|\mathscr{G}\left(\varkappa_a{}^{(n)}, \partial_t \varkappa_a{}^{(n)}, \boldsymbol{\omega}_*{}^{(n)}, \mathbf{j}_*{}^{(n)}\right)\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \\ &\le a^2 Q\left(\left|\varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, \left|\partial_t \varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \left\|\left(\boldsymbol{\omega}_*{}^{(n)}, \mathbf{j}_*{}^{(n)}\right)\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\right) \\ &\le a^2 Q(M_1). \end{aligned}$$

Furthermore, by the definition of constants L_1, L_2 in § 5.1 and Lemma 4.1, one has

$$\begin{aligned} L_1 \le &\sup_{t\in[0,T]} \left(\left|\varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-1}(\Gamma_*)}, \left|\partial_t \varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\right) \\ &+ \sup_{t\in[0,T]} \left(\left|\mathbf{v}_{\pm *}{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)}, \left|\mathbf{h}_{\pm *}{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)}\right) \\ \le &\, Q(M_1), \end{aligned}$$

and

$$L_2 = \sup_{t\in[0,T]} \left(\left|\partial_t \mathbf{v}_{\pm *}{}^{(n)}\right|_{H^{\frac{3}{2}k-2}(\Gamma_*)}, \left|\mathbf{h}_{\pm *}{}^{(n)}\right|_{H^{\frac{3}{2}k-2}(\Gamma_*)}\right) \le Q\left(M_1, M_2, a^2 M_3\right).$$

Thus, by taking $l = k-2$ in (5.42), it follows that

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\left|\varkappa_a{}^{(n+1)}\right|_{H^{\frac{3}{2}k-1}(\Gamma_*)}+\left|\partial_t\varkappa_a{}^{(n+1)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\right)\\&\le C_*\exp\left\{Q\left(M_1,M_2,a^2M_3\right)T\right\}\times\left[C_*+\epsilon+A+(M_0)^{12}+T\cdot\left(a^2+M_2\right)Q(M_1)\right].\end{aligned}$$

If T is taken small enough, and M_1 is much larger than M_0 and A, then there holds

$$\sup_{t\in[0,T]}\left(\left|\varkappa_a{}^{(n+1)}\right|_{H^{\frac{3}{2}k-1}(\Gamma_*)}+\left|\partial_t\varkappa_a{}^{(n+1)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\right)\le M_1. \tag{6.11}$$

Moreover, choosing M_3 large enough compared to M_1 and M_2, one gets from (6.7) that

$$\sup_{t\in[0,T]}\left|\partial_{tt}^2\varkappa_a{}^{(n+1)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)}\le a^2M_3. \tag{6.12}$$

Similarly, by applying Proposition 5.4 to (6.8), one can deduce that

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\left\|\boldsymbol{\omega}_*{}^{(n+1)}\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)},\left\|\mathbf{j}_*{}^{(n+1)}\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\right)\\&\le Q\left(\left|\varkappa_a{}^{(n)}\right|_{C_t^0H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\right)e^{Q(M_1)T}(1+2A)\\&\le M_1,\end{aligned}\tag{6.13}$$

if T is small and $M_1\gg|\kappa_{*+}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}$. Next, by taking $M_2\gg M_1$, it holds that

$$\begin{aligned}&\left\|\partial_t\boldsymbol{\omega}_*{}^{(n+1)}(t)\right\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}\\&\le C_*\left(\left\|\partial_t\boldsymbol{\omega}^{(n+1)}\right\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_t{}^{(n)})}+\left\|\boldsymbol{\omega}^{(n+1)}\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_t{}^{(n)})}\left|\partial_t\varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\right)\\&\le Q(M_1)\le M_2\end{aligned}$$

for all $0\le t\le T$. Similarly,

$$\sup_{t\in[0,T]}\left(\left\|\partial_t\boldsymbol{\omega}_*{}^{(n+1)}\right\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)},\left\|\partial_t\mathbf{j}_*{}^{(n+1)}\right\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}\right)\le M_2. \tag{6.14}$$

The compatibility condition (6.2) follows from Lemma 5.5.

With the following notation:

$$\mathfrak{T}\left\{[(\varkappa_a)_{\rm I},(\partial_t\varkappa_a)_{\rm I},(\boldsymbol{\omega}_*)_{\rm I},(\mathbf{j}_*)_{\rm I}],\left[\varkappa_a{}^{(n)},\boldsymbol{\omega}_*{}^{(n)},\mathbf{j}_*{}^{(n)}\right]\right\}:=\left[\varkappa_a{}^{(n+1)},\boldsymbol{\omega}_*{}^{(n+1)},\mathbf{j}_*{}^{(n+1)}\right], \tag{6.15}$$

one can conclude from the previous arguments that:

Proposition 6.2. *Suppose that $k\ge 2$. For any $0<\epsilon\ll\delta_0$ and $A>0$, there are positive constants M_0,M_1,M_2,M_3, so that for small $T>0$,*

$$\mathfrak{T}\left\{[(\varkappa_a)_{\rm I},(\partial_t\varkappa_a)_{\rm I},(\boldsymbol{\omega}_*)_{\rm I},(\mathbf{j}_*)_{\rm I}],[\varkappa_a,\boldsymbol{\omega}_*,\mathbf{j}_*]\right\}\in\mathfrak{X},$$

holds for any $[(\varkappa_a)_{\rm I},(\partial_t\varkappa_a)_{\rm I},(\boldsymbol{\omega}_*)_{\rm I},(\mathbf{j}_*)_{\rm I}]\in\mathfrak{I}(\epsilon,A)$ and $[\varkappa_a,\boldsymbol{\omega}_*,\mathbf{j}_*]\in\mathfrak{X}$.

6.4. **Contraction of the iteration map.** In this subsection, it is always assumed that $k \geq 3$. Suppose that there are two one-parameter families $\left({\varkappa_a}^{(n)}(\beta), {\boldsymbol{\omega}_*}^{(n)}(\beta), {\mathbf{j}_*}^{(n)}(\beta)\right) \subset \mathfrak{X}$ and $\left((\varkappa_a)_{\mathrm{I}}(\beta), (\partial_t \varkappa_a)_{\mathrm{I}}(\beta), (\boldsymbol{\omega}_*)_{\mathrm{I}}(\beta), (\mathbf{j}_*)_{\mathrm{I}}(\beta)\right) \subset \mathfrak{I}(\epsilon, A)$ with parameter β. Define

$$\begin{aligned}&\left({\varkappa_a}^{(n+1)}(\beta), {\boldsymbol{\omega}_*}^{(n+1)}(\beta), {\mathbf{j}_*}^{(n+1)}(\beta)\right)\\ &\quad := \mathfrak{T}\left\{\left((\varkappa_a)_{\mathrm{I}}(\beta), (\partial_t \varkappa_a)_{\mathrm{I}}(\beta), (\boldsymbol{\omega}_*)_{\mathrm{I}}(\beta), (\mathbf{j}_*)_{\mathrm{I}}(\beta)\right), \left({\varkappa_a}^{(n)}(\beta), {\boldsymbol{\omega}_*}^{(n)}(\beta), {\mathbf{j}_*}^{(n)}(\beta)\right)\right\}.\end{aligned}$$

Then by applying $\frac{\partial}{\partial \beta}$ to (6.7) and (6.8) respectively, one gets

$$\begin{cases}\partial_{tt}^2 \partial_\beta {\varkappa_a}^{(n+1)} + \mathscr{C}^{(n)} \partial_\beta {\varkappa_a}^{(n+1)} = -\left(\partial_\beta \mathscr{C}^{(n)}\right) {\varkappa_a}^{(n+1)} + \partial_\beta\left(\mathscr{F}^{(n)} \partial_t {\boldsymbol{\omega}_*}^{(n)} + \mathscr{G}^{(n)}\right),\\ \partial_\beta {\varkappa_a}^{(n+1)}(0) = \partial_\beta (\varkappa_a)_{\mathrm{I}}(\beta), \quad \partial_t\left(\partial_\beta {\varkappa_a}^{(n+1)}\right)(0) = \partial_\beta (\partial_t \varkappa_a)_{\mathrm{I}}(\beta),\end{cases} \tag{6.16}$$

and

$$\begin{cases}\partial_t \mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)} + \mathrm{D}_{\mathbf{v}^{(n)}} \mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)} - \mathrm{D}_{\mathbf{h}^{(n)}} \mathbb{D}_\beta \mathbf{j}^{(n+1)} = \mathrm{D}_{\mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)}} \mathbf{v}^{(n)} - \mathrm{D}_{\mathbb{D}_\beta \mathbf{j}^{(n+1)}} \mathbf{h}^{(n)} + \vec{\mathfrak{g}}_1,\\ \partial_t \mathbb{D}_\beta \mathbf{j}^{(n+1)} + \mathrm{D}_{\mathbf{v}^{(n)}} \mathbb{D}_\beta \mathbf{j}^{(n+1)} - \mathrm{D}_{\mathbf{h}^{(n)}} \mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)} = \mathrm{D}_{\mathbb{D}_\beta \mathbf{j}^{(n+1)}} \mathbf{v}^{(n)} - \mathrm{D}_{\mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)}} \mathbf{h}^{(n)} + \vec{\mathfrak{g}}_2,\\ \mathbb{D}_\beta \boldsymbol{\omega}^{(n+1)}(0) = \mathbb{P}\left\{\left[\partial_\beta (\boldsymbol{\omega}_*)_{\mathrm{I}}\right] \circ \mathcal{X}^{-1}_{{\Gamma_0}^{(n)}(\beta)}\right\}, \quad \mathbb{D}_\beta \mathbf{j}^{(n+1)}(0) = \mathbb{P}\left\{\left[\partial_\beta (\mathbf{j}_*)_{\mathrm{I}}\right] \circ \mathcal{X}^{-1}_{{\Gamma_0}^{(n)}(\beta)}\right\},\end{cases} \tag{6.17}$$

where

$$\mathbb{D}_\beta := \frac{\partial}{\partial \beta} + \mathrm{D}_{\boldsymbol{\mu}}, \quad \boldsymbol{\mu} := \mathcal{H}\left[\left(\partial_\beta \gamma_{{\Gamma_t}^{(n)}} \boldsymbol{\nu}\right) \circ \Phi^{-1}_{{\Gamma_t}^{(n)}(\beta)}\right], \tag{6.18}$$

$$\begin{aligned}\vec{\mathfrak{g}}_1 :=& \left[\partial_t, \mathbb{D}_\beta\right] \boldsymbol{\omega}^{(n+1)} + \left[\mathrm{D}_{\mathbf{v}^{(n)}}, \mathbb{D}_\beta\right] \boldsymbol{\omega}^{(n+1)} - \left[\mathrm{D}_{\mathbf{h}^{(n)}}, \mathbb{D}_\beta\right] \mathbf{j}^{(n+1)}\\ &+ \boldsymbol{\omega}^{(n+1)} \cdot \mathbb{D}_\beta \, \mathrm{D}\mathbf{v}^{(n)} - \mathbf{j}^{(n+1)} \cdot \mathbb{D}_\beta \, \mathrm{D}\mathbf{h}^{(n)},\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\vec{\mathfrak{g}}_2 :=& -2\mathbb{D}_\beta \operatorname{tr}\left(\nabla \mathbf{v}^{(n)} \times \nabla \mathbf{h}^{(n)}\right) + \left[\partial_t, \mathbb{D}_\beta\right] \mathbf{j}^{(n+1)} + \left[\mathrm{D}_{\mathbf{v}^{(n)}}, \mathbb{D}_\beta\right] \mathbf{j}^{(n+1)}\\ &- \left[\mathrm{D}_{\mathbf{h}^{(n)}}, \mathbb{D}_\beta\right] \boldsymbol{\omega}^{(n+1)} + \mathbf{j}^{(n+1)} \cdot \mathbb{D}_\beta \, \mathrm{D}\mathbf{h}^{(n)} - \boldsymbol{\omega}^{(n+1)} \cdot \mathbb{D}_\beta \, \mathrm{D}\mathbf{h}^{(n)}.\end{aligned} \tag{6.20}$$

To estimate the Lipschitz constant for the iteration map $\mathfrak{T}$, we consider the following energy functionals:

$$\begin{aligned}\mathfrak{E}^{(n)}(\beta) := \sup_{t \in [0,T]} &\left(\left|\partial_\beta {\varkappa_a}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial_\beta \partial_t {\varkappa_a}^{(n)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \right.\\ &\left. + \left\|\partial_\beta {\boldsymbol{\omega}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*)} + \left\|\partial_\beta {\mathbf{j}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*)}\right),\end{aligned} \tag{6.21}$$

and

$$\begin{aligned}(\mathfrak{E})_{\mathrm{I}}(\beta) := &\left(\left|\partial_\beta (\varkappa_a)_{\mathrm{I}}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial_\beta (\partial_t \varkappa_a)_{\mathrm{I}}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \right.\\ &\left. + \left\|\partial_\beta (\boldsymbol{\omega}_*)_{\mathrm{I}}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*)} + \left\|\partial_\beta (\mathbf{j}_*)_{\mathrm{I}}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega \setminus \Gamma_*)}\right).\end{aligned} \tag{6.22}$$

Thanks to Lemmas 4.1, 4.3, and 4.4, it holds that

$$\left|\left(\partial_\beta \mathscr{C}^{(n)}\right) {\varkappa_a}^{(n+1)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \leq Q(M_1) \mathfrak{E}^{(n)} \left|{\varkappa_a}^{(n+1)}\right|_{H^{\frac{3}{2}k-2}(\Gamma_*)} \leq Q(M_1) \mathfrak{E}^{(n)},$$

$$\left|\mathscr{F}\left({\varkappa_a}^{(n)}\right)\partial_\beta\partial_t{\boldsymbol{\omega}_*}^{(n)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \leq Q(M_1)\left\|\partial_\beta\partial_t{\boldsymbol{\omega}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-4}(\Omega\setminus\Gamma_*)},$$

and

$$\begin{aligned}&\left|\left(\partial_\beta\mathscr{F}^{(n)}\right)\partial_t{\boldsymbol{\omega}_*}^{(n)} + \partial_\beta\mathscr{G}^{(n)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)}\\ &\quad\leq C_*\left|\partial_\beta{\varkappa_a}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}\left\|\partial_t{\boldsymbol{\omega}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega\setminus\Gamma_*)} + a^2Q(M_1)\mathfrak{E}^{(n)}\\ &\quad\leq\left(C_*M_2 + a^2Q(M_1)\right)\mathfrak{E}^{(n)}.\end{aligned}$$

Taking $l = k-3$ in (5.42) leads to

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\left|\partial_\beta{\varkappa_a}^{(n+1)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial_\beta\partial_t{\varkappa_a}^{(n+1)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)}\right)\\ &\leq C_*\exp\left\{Q\left(M_1, M_2, a^2M_3\right)T\right\}\\ &\quad\times\left((\mathfrak{E})_{\mathrm{I}} + T\cdot\left[C_*M_2 + \left(1+a^2\right)Q(M_1)\right]\mathfrak{E}^{(n)} + T\cdot Q(M_1)\sup_{t\in[0,T]}\left\|\partial_\beta\partial_t{\boldsymbol{\omega}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-4}(\Omega\setminus\Gamma_*)}\right).\end{aligned} \tag{6.23}$$

To estimate (6.17), one can derive that

$$\left\|\vec{\mathfrak{g}}_1\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus{\Gamma_t}^{(n)})} \leq Q(M_1)\left(\left|\partial_t\partial_\beta{\varkappa_a}^{(n)}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left\|\left(\mathbb{D}_\beta\mathbf{v}^{(n)}, \mathbb{D}_\beta\mathbf{h}^{(n)}\right)\right\|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Omega\setminus{\Gamma_t}^{(n)})}\right),$$

which, together with (4.19), implies

$$\left\|\vec{\mathfrak{g}}_1\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus{\Gamma_t}^{(n)})} \leq Q(M_1)\mathfrak{E}^{(n)}.$$

Similarly, one can obtain

$$\left\|\vec{\mathfrak{g}}_2\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus{\Gamma_t}^{(n)})} \leq Q(M_1)\mathfrak{E}^{(n)}.$$

It follows from the same arguments as in Proposition 5.4 that

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\left\|\mathbb{D}_\beta\boldsymbol{\omega}^{(n+1)}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus{\Gamma_t}^{(n)})} + \left\|\mathbb{D}_\beta\mathbf{j}^{(n+1)}\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus{\Gamma_t}^{(n)})}\right)\\ &\quad\leq e^{Q(M_1)T}\left\{(\mathfrak{E})_{\mathrm{I}} + TQ(M_1)\mathfrak{E}^{(n)}\right\},\end{aligned} \tag{6.24}$$

and thus

$$\left\|\partial_t\mathbb{D}_\beta\boldsymbol{\omega}^{(n+1)}\right\|_{H^{\frac{3}{2}k-4}(\Omega\setminus{\Gamma_t}^{(n)})} \leq e^{Q(M_1)T}Q(M_1)\left\{(\mathfrak{E})_{\mathrm{I}} + TQ(M_1)\mathfrak{E}^{(n)}\right\}.$$

Since

$$\left[\partial_t\partial_\beta{\boldsymbol{\omega}_*}^{(n+1)}\right]\circ\mathfrak{X}^{-1}_{{\Gamma_t}^{(n)}} = \partial_t\mathbb{D}_\beta\boldsymbol{\omega}^{(n+1)} + \mathrm{D}_{\mathfrak{H}\left[\left(\partial_t\gamma_{{\Gamma_t}^{(n)}}\boldsymbol{\nu}\right)\circ\Phi^{-1}_{{\Gamma_t}^{(n)}}\right]}\mathbb{D}_\beta\boldsymbol{\omega}^{(n+1)},$$

one has

$$\left\|\partial_\beta\partial_t{\boldsymbol{\omega}_*}^{(n+1)}\right\|_{H^{\frac{3}{2}k-4}(\Omega\setminus\Gamma_*)} \leq e^{Q(M_1)T}Q(M_1)\left\{(\mathfrak{E})_{\mathrm{I}} + TQ(M_1)\mathfrak{E}^{(n)}\right\}. \tag{6.25}$$

Set

$$\mathfrak{F}^{(n)} := \left\|\partial_\beta\partial_t{\boldsymbol{\omega}_*}^{(n)}\right\|_{H^{\frac{3}{2}k-4}(\Omega\setminus\Gamma_*)}. \tag{6.26}$$

Then (6.23)-(6.25) imply that

$$\begin{aligned}\mathfrak{E}^{(n+1)} \leq & C_* \exp\left\{Q\left(M_1, M_2, a^2 M_3\right)T\right\} \\ & \times\left\{(\mathfrak{E})_{\mathrm{I}} + T\cdot\left[M_2\mathfrak{E}^{(n)} + \left(1+a^2\right)Q(M_1)\mathfrak{E}^{(n)} + Q(M_1)\mathfrak{F}^{(n)}\right]\right\} \\ & + e^{Q(M_1)T}Q(M_1)\left\{(\mathfrak{E})_{\mathrm{I}} + TQ(M_1)\mathfrak{E}^{(n)}\right\},\end{aligned} \tag{6.27}$$

and

$$\mathfrak{F}^{(n+1)} \leq e^{Q(M_1)T}Q(M_1)\left\{(\mathfrak{E})_{\mathrm{I}} + TQ(M_1)\mathfrak{E}^{(n)}\right\}. \tag{6.28}$$

Thus, if T is small compared to M_1, M_2, M_3 and a, then

$$\mathfrak{E}^{(n+1)} + \mathfrak{F}^{(n+1)} \leq \frac{1}{2}\left(\mathfrak{E}^{(n)} + \mathfrak{F}^{(n)}\right) + Q(M_1)(\mathfrak{E})_{\mathrm{I}}. \tag{6.29}$$

An immediate consequence is that if the initial data is fixed, the iteration map $\mathfrak{T}$ is a contraction in a space containing $\mathfrak{X}$. Since $\mathfrak{T}$ is a map from $\mathfrak{X}$ to $\mathfrak{X}$, it follows that $\mathfrak{T}$ has a unique fixed point on $\mathfrak{X}$. Namely, one obtains:

Proposition 6.3. *Assume that $k \geq 3$. For any $0 < \epsilon \ll \delta_0$ and $A > 0$, there are positive constants M_0, M_1, M_2, M_3 so that if T is small enough, then there is a map $\mathfrak{S} : \mathfrak{I}(\epsilon, A) \to \mathfrak{X}$ such that*

$$\mathfrak{T}\{\mathfrak{x}, \mathfrak{S}(\mathfrak{x})\} = \mathfrak{S}(\mathfrak{x}) \tag{6.30}$$

for each $\mathfrak{x} = \Big((\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_*)_{\mathrm{I}}, (\mathbf{j}_*)_{\mathrm{I}}\Big) \in \mathfrak{I}(\epsilon, A)$.

6.5. **The original nonlinear MHD problem.** For any given initial data $\Gamma_0 \in H^{\frac{3}{2}k+1}$ and $\mathbf{v}_0, \mathbf{h}_0 \in H^{\frac{3}{2}k}(\Omega \setminus \Gamma_0)$, one can construct

$$\Big(\varkappa_a(0), \partial_t\varkappa_a(0), \boldsymbol{\omega}_*(0), \mathbf{j}_*(0)\Big).$$

Indeed, for a reference hypersurface $\Gamma_* \in H^{\frac{3}{2}k+1}$ close enough to Γ_0 (or $\Gamma_* = \Gamma_0$) and a transversal field $\boldsymbol{\nu} \in H^{\frac{3}{2}k-1}(\Gamma_*)$, $\varkappa_a(0)$ can be given by Γ_0, and $\partial_t\varkappa_a(0)$ is determined by $\theta_0 = \mathbf{v}_{0\pm}\cdot\mathbf{N}_+$. In addition, let

$$\boldsymbol{\omega}_* := (\nabla\times\mathbf{v}_0)\circ\mathcal{X}_{\Gamma_0}^{-1}, \quad \mathbf{j}_* := (\nabla\times\mathbf{h}_0)\circ\mathcal{X}_{\Gamma_0}^{-1}. \tag{6.31}$$

Thus, $\varkappa_a(0) \in H^{\frac{3}{2}k-1}(\Gamma_*)$, $\partial_t\varkappa_a(0) \in H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)$, and $\boldsymbol{\omega}_*(0), \mathbf{j}_*(0) \in H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*)$.

Let $\{(\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_*)_{\mathrm{I}}, (\mathbf{j}_*)_{\mathrm{I}}\} := \{\varkappa_a(0), \partial_t\varkappa_a(0), \boldsymbol{\omega}_*(0), \mathbf{j}_*(0)\}$, and take the corresponding fixed point $\{\varkappa_a(t), \boldsymbol{\omega}_*(t), \mathbf{j}_*(t)\} \in \mathfrak{X}$ of the map $\mathfrak{S} : \mathfrak{I}(\epsilon, A) \to \mathfrak{X}$ given in Proposition 6.3. Thus, $(\Gamma_t, \mathbf{v}, \mathbf{h})$ can be obtained as discussed in § 6.2. We will show that the induced quantity $(\Gamma_t, \mathbf{v}, \mathbf{h})$ is a solution to the (MHD)-(BC) problem with initial data $(\Gamma_0, \mathbf{v}_0, \mathbf{h}_0)$.

Indeed, it is clear that $\Gamma(0) = \Gamma_0$, $\mathbf{v}(0) = \mathbf{v}_0$, and $\mathbf{h}(0) = \mathbf{h}_0$ by the definition and the uniqueness of div-curl systems.

First, we claim that

$$\mathbb{P}\left(\boldsymbol{\omega}_*(t)\circ\mathcal{X}_{\Gamma_t}^{-1}\right) = \boldsymbol{\omega}_*(t)\circ\mathcal{X}_{\Gamma_t}^{-1}, \quad \mathbb{P}\left(\mathbf{j}_*(t)\circ\mathcal{X}_{\Gamma_t}^{-1}\right) = \mathbf{j}_*(t)\circ\mathcal{X}_{\Gamma_t}^{-1}. \tag{6.32}$$

Indeed, taking the divergence of (6.8) and using the fact that $\nabla\cdot\mathbf{v} \equiv 0 \equiv \nabla\cdot\mathbf{h}$ yield

$$\begin{cases}\partial_t(\nabla\cdot\boldsymbol{\omega}) + \mathrm{D}_{\mathbf{v}}(\nabla\cdot\boldsymbol{\omega}) = \mathrm{D}_{\mathbf{h}}(\nabla\cdot\mathbf{j}),\\ \partial_t(\nabla\cdot\mathbf{j}) + \mathrm{D}_{\mathbf{v}}(\nabla\cdot\mathbf{j}) = \mathrm{D}_{\mathbf{h}}(\nabla\cdot\boldsymbol{\omega}),\end{cases} \tag{6.33}$$

where

$$\boldsymbol{\omega}(t) := \boldsymbol{\omega}_*(t) \circ \mathcal{X}_{\Gamma_t}^{-1}, \quad \mathbf{j}(t) := \mathbf{j}_*(t) \circ \mathcal{X}_{\Gamma_t}^{-1}.$$

Since $\nabla \cdot \boldsymbol{\omega}(0) = 0 = \nabla \cdot \mathbf{j}(0)$, it follows from the arguments in § 5.2 that $\nabla \cdot \boldsymbol{\omega} \equiv 0 \equiv \nabla \cdot \mathbf{j}$ for all t, which proves the claim.

Consequently,

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \quad \text{and} \quad \nabla \times \mathbf{h} = \mathbf{j}. \tag{6.34}$$

Next, as in (4.4), define the pressure functions via

$$p^{\pm} = \rho_{\pm}\left(p^{\pm}_{\mathbf{v},\mathbf{v}} - p^{\pm}_{\mathbf{h},\mathbf{h}} + p^{\pm}_{\kappa}\right) + \mathcal{H}_{\pm}\bar{\mathcal{N}}^{-1}(-g^{+} + g^{-}),$$

with $p^{\pm}_{\kappa}$ defined by (4.7) and

$$g^{\pm} := 2\,\mathrm{D}_{\mathbf{v}^{\top}_{\pm}}\theta - \mathrm{I\!I}_{+}\left(\mathbf{v}^{\top}_{\pm}, \mathbf{v}^{\top}_{\pm}\right) + \mathrm{I\!I}_{+}\left(\mathbf{h}^{\top}_{\pm}, \mathbf{h}^{\top}_{\pm}\right) + \mathrm{D}_{\mathbf{N}_{+}}\left(p^{\pm}_{\mathbf{v},\mathbf{v}} - p^{\pm}_{\mathbf{h},\mathbf{h}}\right).$$

Inspired by [SWZ18], define

$$\mathbf{V}_{\pm} := \partial_t \mathbf{v}_{\pm} + \mathrm{D}_{\mathbf{v}_{\pm}}\mathbf{v}_{\pm} + \frac{1}{\rho_{\pm}}\nabla p^{\pm} - \mathrm{D}_{\mathbf{h}_{\pm}}\mathbf{h}_{\pm}, \tag{6.35}$$

and

$$\mathbf{H}_{\pm} := \partial_t \mathbf{h}_{\pm} + \mathrm{D}_{\mathbf{v}_{\pm}}\mathbf{h}_{\pm} - \mathrm{D}_{\mathbf{h}_{\pm}}\mathbf{v}_{\pm}. \tag{6.36}$$

It suffices to show that $\mathbf{V} \equiv \mathbf{0} \equiv \mathbf{H}$ for $0 < t < T$. Indeed, since $\Omega_t^{\pm}$ are both assumed to be simply-connected, one only needs to verify for $\mathbf{Z} = \mathbf{V}$ or $\mathbf{H}$:

$$\begin{cases} \nabla \cdot \mathbf{Z}_{\pm} = 0 & \text{in } \Omega_t^{\pm}, \\ \nabla \times \mathbf{Z}_{\pm} = \mathbf{0} & \text{in } \Omega_t^{\pm}, \\ \mathbf{Z}_{\pm} \cdot \mathbf{N}_{+} = 0 & \text{on } \Gamma_t, \\ \mathbf{Z}_{-} \cdot \widetilde{\mathbf{N}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.37}$$

Verification of $\mathbf{V}$. Observe that

$$\nabla \cdot \mathbf{v} \equiv 0 \equiv \nabla \cdot \mathbf{h}.$$

Then it follows from the definitions of $p^{\pm}_{\mathbf{a},\mathbf{b}}$ that,

$$\nabla \cdot \mathbf{V} \equiv 0. \tag{6.38}$$

Taking curl of $\mathbf{V}$ and using (6.34) with (4.69) lead to

$$\nabla \times \mathbf{V} = \partial_t \boldsymbol{\omega} + \mathrm{D}_{\mathbf{v}}\boldsymbol{\omega} - \mathrm{D}_{\boldsymbol{\omega}}\mathbf{v} - \mathrm{D}_{\mathbf{h}}\mathbf{j} + \mathrm{D}_{\mathbf{j}}\mathbf{h} = \mathbf{0}. \tag{6.39}$$

In addition, it follows from (4.6) that on $\partial\Omega$:

$$\mathbf{V}_{-} \cdot \widetilde{\mathbf{N}} = -\widetilde{\mathrm{I\!I}}(\mathbf{v}_{-}, \mathbf{v}_{-}) + \widetilde{\mathrm{I\!I}}(\mathbf{h}_{-}, \mathbf{h}_{-}) + \mathrm{D}_{\widetilde{\mathbf{N}}}(p^{-}_{\mathbf{v},\mathbf{v}} - p^{-}_{\mathbf{h},\mathbf{h}}) = 0. \tag{6.40}$$

Thus, it remains to show that $\mathbf{V}_\pm \cdot \mathbf{N}_+ \equiv 0$. Note that for $\theta := \mathbf{v}_\pm \cdot \mathbf{N}_+$ and $\langle g^\pm \rangle := \fint_{\Gamma_t} g^\pm \,\mathrm{d}S_t$, there hold

$$
\begin{aligned}
&\mathbf{V}_+ \cdot \mathbf{N}_+ \\
&= \mathbf{N}_+ \cdot \left(\mathbb{D}_{t+} \mathbf{v}_+ - \mathrm{D}_{\mathbf{h}_+} \mathbf{h}_+ \right) + \mathrm{D}_{\mathbf{N}_+} \left(p^+_{\mathbf{v},\mathbf{v}} - p^+_{\mathbf{h},\mathbf{h}} + p^+_\kappa \right) - \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} (g^+ - g^-) \\
&= \mathbb{D}_{t+} \theta + \mathbf{N}_+ \cdot (\mathrm{D}\mathbf{v}_+) \cdot \mathbf{v}_+^\top + \mathrm{II}_+ (\mathbf{h}_+, \mathbf{h}_+) + \mathrm{D}_{\mathbf{N}_+} \left(p^+_{\mathbf{v},\mathbf{v}} - p^+_{\mathbf{h},\mathbf{h}} \right) \\
&\quad + \widetilde{\mathcal{N}} \kappa_+ - \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} (g^+ - g^-) \\
&= \mathbb{D}_{t+} \theta + \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{v}_+^\top} \mathbf{v}_+ + \mathrm{II}_+ (\mathbf{h}_+, \mathbf{h}_+) + \mathrm{D}_{\mathbf{N}_+} \left(p^+_{\mathbf{v},\mathbf{v}} - p^+_{\mathbf{h},\mathbf{h}} \right) \\
&\quad + \widetilde{\mathcal{N}} \kappa_+ + \left(\frac{1}{\rho_-} \mathcal{N}_- \right) \bar{\mathcal{N}}^{-1} g^+ + \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} g^- - g^+ + \langle g^+ \rangle \\
&= \mathbb{D}_{t+} \theta - \mathrm{D}_{\mathbf{v}_+^\top} \theta + \widetilde{\mathcal{N}} \kappa_+ + \left(\frac{1}{\rho_-} \mathcal{N}_- \right) \bar{\mathcal{N}}^{-1} g^+ + \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} g^- + \langle g^+ \rangle,
\end{aligned}
\tag{6.41}
$$

and

$$
\begin{aligned}
&\mathbf{V}_- \cdot \mathbf{N}_- \\
&\quad = - \mathbb{D}_{t-} \theta + \mathbf{N}_- \cdot (\mathrm{D}\mathbf{v}_-) \cdot \mathbf{v}_- + \mathrm{II}_- (\mathbf{h}_-, \mathbf{h}_-) + \mathrm{D}_{\mathbf{N}_-} (p^-_{\mathbf{v},\mathbf{v}} - p^-_{\mathbf{h},\mathbf{h}}) \\
&\qquad - \widetilde{\mathcal{N}} \kappa_+ - \left(\frac{1}{\rho_-} \mathcal{N}_- \right) \bar{\mathcal{N}}^{-1} (g^+ - g^-) \\
&\quad = - \mathbb{D}_{t-} \theta + \mathbf{N}_- \cdot \mathrm{D}_{\mathbf{v}_-^\top} \mathbf{v}_- + \mathrm{II}_- (\mathbf{h}_-, \mathbf{h}_-) + \mathrm{D}_{\mathbf{N}_-} (p^-_{\mathbf{v},\mathbf{v}} - p^-_{\mathbf{h},\mathbf{h}}) \\
&\qquad - \widetilde{\mathcal{N}} \kappa_+ - \left(\frac{1}{\rho_-} \mathcal{N}_- \right) \bar{\mathcal{N}}^{-1} g^+ - \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} g^- + g^- - \langle g^- \rangle \\
&\quad = - \mathbb{D}_{t-} \theta + \mathrm{D}_{\mathbf{v}_-^\top} \theta - \widetilde{\mathcal{N}} \kappa_+ - \left(\frac{1}{\rho_-} \mathcal{N}_- \right) \bar{\mathcal{N}}^{-1} g^+ - \left(\frac{1}{\rho_+} \mathcal{N}_+ \right) \bar{\mathcal{N}}^{-1} g^- - \langle g^- \rangle .
\end{aligned}
\tag{6.42}
$$

Hence,

$$
\begin{aligned}
\mathbf{V}_+ \cdot \mathbf{N}_+ + \mathbf{V}_- \cdot \mathbf{N}_- &= \mathbb{D}_{t+} \theta - \mathbb{D}_{t-} \theta - \mathrm{D}_{\mathbf{v}_+^\top} \theta + \mathrm{D}_{\mathbf{v}_-^\top} \theta + \langle g^+ - g^- \rangle \\
&= \mathrm{D}_{\mathbf{v}_+ - \mathbf{v}_-} \theta - \mathrm{D}_{\mathbf{v}_+^\top - \mathbf{v}_-^\top} \theta + \langle g^+ - g^- \rangle \\
&= 0,
\end{aligned}
\tag{6.43}
$$

where the last equality follows from (4.11) and the relation that $\mathbf{v}_+ \cdot \mathbf{N}_+ = \mathbf{v}_- \cdot \mathbf{N}_+$. Therefore, one can define

$$
\Theta := \mathbf{V}_+ \cdot \mathbf{N}_+ = \mathbf{V}_- \cdot \mathbf{N}_+ . \tag{6.44}
$$

For $\mathbf{W}$ defined via (4.50), the relation that $\nabla \cdot \mathbf{v} \equiv 0$ implies $q^\pm = p^\pm$, that is,

$$
\mathbf{W} = \frac{\rho_+}{\rho_+ + \rho_-} \mathbf{V}_+ + \frac{\rho_-}{\rho_+ + \rho_-} \mathbf{V}_- \quad \text{on} \quad \Gamma_t . \tag{6.45}
$$

Thus,

$$
\Theta = \mathbf{W} \cdot \mathbf{N}_+ . \tag{6.46}
$$

Because $\{\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*\} \in \mathfrak{X}$ is a fixed point of $\mathfrak{T}$, by (4.62), it holds that

$$-\triangle_{\Gamma_t}(\mathbf{W}\cdot\mathbf{N}_+) + \mathbf{W}\cdot\triangle_{\Gamma_t}\mathbf{N}_+ + a^2\frac{\mathbf{W}\cdot\mathbf{N}_+}{\mathbf{N}_+\cdot(\boldsymbol{\nu}\circ\Phi_{\Gamma_t}^{-1})} = 0. \tag{6.47}$$

In addition, since $\nabla\times\mathbf{V} = \mathbf{0}$, $\nabla\cdot\mathbf{V} = 0$, $\mathbf{V}_-\cdot\widetilde{\mathbf{N}} = 0$ and $\Omega_t^\pm$ are both simply-connected, there are two mean-zero functions $r^\pm(t,x') : \Gamma_t \to \mathbb{R}$ so that

$$\mathbf{V}_\pm = \frac{1}{\rho_\pm}\nabla\mathcal{H}_\pm r^\pm, \tag{6.48}$$

which implies that

$$\begin{aligned}\Theta = \mathbf{V}_+\cdot\mathbf{N}_+ &= \left(\frac{1}{\rho_+}\mathcal{N}_+\right)r^+ \\ &= -\mathbf{V}_-\cdot\mathbf{N}_- = -\left(\frac{1}{\rho_-}\mathcal{N}_-\right)r^-.\end{aligned} \tag{6.49}$$

It follows from (6.47), (6.45), and the identity (3.8) that

$$-\triangle_{\Gamma_t}\Theta + \left(\frac{a^2}{\mathbf{N}_+\cdot(\boldsymbol{\nu}\circ\Phi_{\Gamma_t}^{-1})} - |\mathrm{I\!I}_+|^2\right)\Theta + \frac{1}{\rho_+ + \rho_-}\nabla^\top\left(r^+ + r^-\right)\cdot\nabla^\top\kappa_+ = 0. \tag{6.50}$$

If a_0 is taken large enough, so that $\dfrac{a^2}{\mathbf{N}_+\cdot(\boldsymbol{\nu}\circ\Phi_{\Gamma_t}^{-1})} - |\mathrm{I\!I}_+|^2 > a$ holds for all $t\in[0,T]$ whenever $a \ge a_0$ (indeed, it holds for all $\Gamma\in\Lambda_*$), then

$$\begin{aligned}\left|\nabla^\top\Theta\right|_{L^2(\Gamma_t)}^2 + a|\Theta|_{L^2(\Gamma_t)}^2 &\le \frac{1}{\rho_+ + \rho_-}|\Theta|_{L^2(\Gamma_t)}\cdot\left|\nabla^\top\kappa_+\cdot\nabla^\top\left(r^+ + r^-\right)\right|_{L^2(\Gamma_t)} \\ &\le C_*|\Theta|_{L^2(\Gamma_t)}\left|r^+ + r^-\right|_{H^{\frac32}(\Gamma_t)}|\kappa_+|_{H^{\frac32 k - \frac52}(\Gamma_t)} \\ &\le C_{*0}\left(|\Theta|_{L^2(\Gamma_t)}^2 + \left|r^+ + r^-\right|_{H^{\frac32}(\Gamma_t)}^2\right).\end{aligned} \tag{6.51}$$

It can be deduced directly from (6.49) that

$$r^\pm = \pm\left(\frac{1}{\rho_\pm}\mathcal{N}_\pm\right)^{-1}\Theta,$$

which implies that

$$\begin{aligned}\left|r^+ + r^-\right|_{H^{\frac32}(\Gamma_t)}^2 &\le C_*|\Theta|_{H^{\frac12}(\Gamma_t)}^2 \\ &\le \frac{1}{2C_{*0}}\left|\nabla^\top\Theta\right|_{L^2(\Gamma_t)}^2 + C_*|\Theta|_{L^2(\Gamma_t)}^2.\end{aligned} \tag{6.52}$$

Thus, for a generic constant C_* determined by Λ_*, it holds that

$$\left|\nabla^\top\Theta\right|_{L^2(\Gamma_t)}^2 + (2a - C_*)|\Theta|_{L^2(\Gamma_t)}^2 \le 0. \tag{6.53}$$

If a_0 is taken large enough (depending only on Λ_*), then $\Theta = 0$, which yields $\mathbf{V}(t) \equiv \mathbf{0}$.

Verification of **H**. Similar arguments show that

$$\nabla \cdot \mathbf{H} \equiv 0, \quad \nabla \times \mathbf{H} \equiv \mathbf{0}. \tag{6.54}$$

As for the boundary terms, observe that $\mathbf{h}_\pm \cdot \mathbf{N}_+ \equiv 0$, which implies

$$\begin{aligned} \mathbf{H}_\pm \cdot \mathbf{N}_+ &= \mathbf{N}_+ \cdot \mathbb{D}_{t\pm}\mathbf{h}_\pm - \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_\pm}\mathbf{v}_\pm \\ &= -\mathbf{h}_\pm \cdot \mathbb{D}_{t\pm}\mathbf{N}_+ - \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_\pm}\mathbf{v}_\pm \\ &= \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_\pm}\mathbf{v}_\pm - \mathbf{N}_+ \cdot \mathrm{D}_{\mathbf{h}_\pm}\mathbf{v}_\pm \\ &= 0. \end{aligned} \tag{6.55}$$

In addition, one can derive from $\mathbf{v}_- \cdot \widetilde{\mathbf{N}} = 0 = \mathbf{h}_- \cdot \widetilde{\mathbf{N}}$ that

$$\mathbf{H}_- \cdot \widetilde{\mathbf{N}} = \partial_t \mathbf{h}_- \cdot \widetilde{\mathbf{N}} + [\mathbf{h}_-, \mathbf{v}_-] \cdot \widetilde{\mathbf{N}} = 0. \tag{6.56}$$

Therefore, $\mathbf{H}(t) \equiv \mathbf{0}$.

The previous arguments ensure that $(\varGamma_t, \mathbf{v}, \mathbf{h})$ is a solution to the original (MHD) system, with (BC) following from the construction. The uniqueness and the continuous dependence on the initial data of the original problem follow from those of the div-curl ones. In conclusion, Theorem 2.1 holds.

7. Stabilization Effect of the Syrovatskij Condition

7.1. **The strict Syrovatskij condition.** Since the free interface is a compact 2-D manifold, $|\mathbf{h}_+ \times \mathbf{h}_-| > 0$ on $\varGamma$ implies that $\mathbf{h}_\pm$ form a global frame of $\varGamma$. Therefore, for any tangential vector field $\mathbf{a}$ on $\varGamma$, there is a unique decomposition:

$$\mathbf{a} = a^+\mathbf{h}_+ + a^-\mathbf{h}_-. \tag{7.1}$$

Furthermore, the following relation holds:

Lemma 7.1. *If* (1.6) *holds on a compact hypersurface* $\varGamma \subset \mathbb{R}^3$, *then for any non-vanishing tangential vector field* $\mathbf{a}$ *on* $\varGamma$, *it holds that*

$$\frac{\rho_+}{\rho_+ + \rho_-}|\mathbf{a} \cdot \mathbf{h}_+|^2 + \frac{\rho_-}{\rho_+ + \rho_-}|\mathbf{a} \cdot \mathbf{h}_-|^2 - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}|\mathbf{a} \cdot [\![\mathbf{v}]\!]|^2 > 0 \quad \text{on} \quad \varGamma. \tag{7.2}$$

Proof. For simplicity, we shall use the notations:

$$g_{++} := \mathbf{h}_+ \cdot \mathbf{h}_+, \quad g_{--} := \mathbf{h}_- \cdot \mathbf{h}_-, \quad g_{+-} \equiv g_{-+} := \mathbf{h}_- \cdot \mathbf{h}_+, \tag{7.3}$$

the decomposition (7.1), and

$$[\![\mathbf{v}]\!] \equiv w^+\mathbf{h}_+ + w^-\mathbf{h}_- \quad \text{on} \quad \varGamma. \tag{7.4}$$

Thus, (1.6) is equivalent to

$$|\mathbf{h}_+ \times \mathbf{h}_-|^2 > \frac{\rho_+}{\rho_+ + \rho_-}|w^-|^2|\mathbf{h}_+ \times \mathbf{h}_-|^2 + \frac{\rho_-}{\rho_+ + \rho_-}|w^+|^2|\mathbf{h}_+ \times \mathbf{h}_-|^2, \tag{7.5}$$

namely,

$$1 > \frac{\rho_+}{\rho_+ + \rho_-}|w^-|^2 + \frac{\rho_-}{\rho_+ + \rho_-}|w^+|^2. \tag{7.6}$$

Hence, direct calculations yield

$$
\begin{aligned}
&\frac{\rho_+}{\rho_+ + \rho_-}|\mathbf{a}\cdot\mathbf{h}_+|^2 + \frac{\rho_-}{\rho_+ + \rho_-}|\mathbf{a}\cdot\mathbf{h}_-|^2 \\
&\quad = \frac{\rho_+}{\rho_+ + \rho_-}\left|a^+ g_{++} + a^- g_{+-}\right|^2 + \frac{\rho_-}{\rho_+ + \rho_-}\left|a^- g_{--} + a^+ g_{+-}\right|^2 \\
&\quad > \left(\frac{\rho_+}{\rho_+ + \rho_-}|w^-|^2 + \frac{\rho_-}{\rho_+ + \rho_-}\left|w^+\right|^2\right)\frac{\rho_+}{\rho_+ + \rho_-}\left|a^+ g_{++} + a^- g_{+-}\right|^2 \\
&\qquad + \left(\frac{\rho_+}{\rho_+ + \rho_-}|w^-|^2 + \frac{\rho_-}{\rho_+ + \rho_-}\left|w^+\right|^2\right)\frac{\rho_-}{\rho_+ + \rho_-}\left|a^- g_{--} + a^+ g_{+-}\right|^2 \\
&\quad \geq \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\left|a^+ w^+ g_{++} + a^- w^- g_{--} + a^- w^+ g_{+-} + a^+ w^- g_{+-}\right|^2 \\
&\quad = \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}|\mathbf{a}\cdot[\![\mathbf{v}]\!]|^2,
\end{aligned} \tag{7.7}
$$

which is exactly (7.2). □

Since Γ is assumed to be compact, the following corollary follows:

Corollary 7.2. *Suppose that (7.2) holds on Γ. Then it holds that*

$$
\begin{aligned}
\Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!]) := \inf_{\substack{\mathbf{a}\in \mathrm{T}\Gamma_t;\, z\in\Gamma_t \\ |\mathbf{a}|=1}} \inf \ & \frac{\rho_+}{\rho_+ + \rho_-}|\mathbf{a}\cdot\mathbf{h}_+(z)|^2 + \frac{\rho_-}{\rho_+ + \rho_-}|\mathbf{a}\cdot\mathbf{h}_-(z)|^2 \\
& - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}|\mathbf{a}\cdot[\![\mathbf{v}]\!](z)|^2 \\
=: \mathfrak{s} > 0.
\end{aligned} \tag{7.8}
$$

Equivalently, the following relation holds on Γ:

$$
\left(\frac{\rho_+}{\rho_+ + \rho_-}(\mathbf{h}_+\otimes\mathbf{h}_+) + \frac{\rho_-}{\rho_+ + \rho_-}(\mathbf{h}_-\otimes\mathbf{h}_-) - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}([\![\mathbf{v}]\!]\otimes[\![\mathbf{v}]\!])\right) \geq \mathfrak{s}_0 \mathbf{I}. \tag{7.9}
$$

7.2. **Interfaces, coordinates and div-curl systems.** From now on, Ω is assumed to be $\mathbb{T}^2\times(-1,1)$ and Ω_t^+ has a solid boundary $\mathbb{T}^2\times\{+1\}$. Hence, some statements in § 3.4 and § 3.6 need slight changes in order to be compatible to the topology of $\Omega_t^\pm$. More precisely, the harmonic coordinate maps introduced in § 3.4 are now replaced by

$$
\begin{cases}
\triangle_y \mathfrak{X}_\Gamma^\pm = 0 & \text{for } y\in\Omega_*^\pm, \\
\mathfrak{X}_\Gamma^\pm(z) = \Phi_\Gamma(z) & \text{for } z\in\Gamma_*, \\
\mathfrak{X}_\Gamma^\pm(z) = z & \text{for } z\in\mathbb{T}^2\times\{\pm1\}.
\end{cases} \tag{7.10}
$$

Similarly, the definitions of harmonic extensions of a function f defined on Γ are modified to

$$
\begin{cases}
\triangle\mathcal{H}_\pm f = 0 & \text{for } x\in\Omega_\Gamma^\pm, \\
\mathcal{H}_\pm f = f & \text{for } x\in\Gamma, \\
\mathrm{D}_{\tilde{\mathrm{N}}_\pm}\mathcal{H}_\pm f = 0 & \text{for } x\in\mathbb{T}^2\times\{\pm1\}.
\end{cases} \tag{7.11}
$$

The Dirichlet-Neumann operators are also defined by (3.32) for which $\mathcal{H}_\pm$ are given by (7.11).

Therefore, Lemmas 3.4 - 3.6, and those properties of the Dirichlet-Neumann operators introduced in § 3.4 still hold.

As for the div-curl systems, due to the different topology, we introduce the following modification of Theorem 3.8:

Theorem 7.3. *Assume that Γ is an $H^{\frac{3}{2}k-\frac{1}{2}}(k \geq 3)$ surface diffeomorphic to $\mathbb{T}^2$, with*

$$\operatorname{dist}(\Gamma, \mathbb{T}^2 \times \{\pm 1\}) \geq c_0 > 0 \tag{7.12}$$

for some positive constant c_0. Given $\mathbf{f}, g \in H^{l-1}(\Omega^+)$ and $h \in H^{l-\frac{1}{2}}(\Gamma)$ with the compatibility condition

$$\int_{\Omega^+} g \,\mathrm{d}x = \int_\Gamma h \,\mathrm{d}S \,,$$

and suppose further that $\mathbf{f}$ satisfies

$$\nabla \cdot \mathbf{f} = 0 \text{ in } \Omega^+, \quad \int_{\mathbb{T}^2 \times \{+1\}} \mathbf{f} \cdot \widetilde{\mathbf{N}}_+ \,\mathrm{d}S = 0. \tag{7.13}$$

Then, for $2 \leq l \leq \frac{3}{2}k - 1$, the following system:

$$\begin{cases} \nabla \times \mathbf{u} = \mathbf{f} & \text{in } \Omega^+, \\ \nabla \cdot \mathbf{u} = g & \text{in } \Omega^+, \\ \mathbf{u} \cdot \mathbf{N}_+ = h & \text{on } \Gamma, \\ \mathbf{u} \cdot \widetilde{\mathbf{N}}_+ = 0, \quad \int_{\mathbb{T}^2 \times \{+1\}} \mathbf{u} \,\mathrm{d}\widetilde{S} = \vec{\mathfrak{u}} & \text{on } \mathbb{T}^2 \times \{+1\} \end{cases} \tag{7.14}$$

admits a unique solution $\mathbf{u} \in H^l(\Omega^+)$ satisfying the estimate:

$$\|\mathbf{u}\|_{H^l(\Omega^+)} \leq C\left(|\Gamma|_{H^{\frac{3}{2}k-\frac{1}{2}}}, c_0\right) \times \left(\|\mathbf{f}\|_{H^{l-1}(\Omega^+)} + \|g\|_{H^{l-1}(\Omega^+)} + |h|_{H^{l-\frac{1}{2}}(\Gamma)} + \left|\vec{\mathfrak{u}}\right|\right). \tag{7.15}$$

One may refer to [CS17] and [SWZ18] for a proof of Theorem 7.3.

7.3. **Reformulation of the problem.** We shall consider the free interface problem (MHD)-(BC') under the assumption that $k \geq 3$. Due to the difference between Theorems 3.8 and 7.3, the velocity and magnetic fields depend on one more boundary condition – their integrals on the bottom or the top solid boundary. Therefore, when considering the variation, one needs to assume further that the integrals of $\mathbf{v}_\pm$ and $\mathbf{h}_\pm$ on $\mathbb{T}^2 \times \{\pm 1\}$ also depend on the parameter β. More precisely, set

$$\vec{\mathfrak{v}}_\pm := \int_{\mathbb{T}^2 \times \{\pm 1\}} \mathbf{v}_\pm \,\mathrm{d}\widetilde{S} \quad \text{and} \quad \vec{\mathfrak{h}}_\pm := \int_{\mathbb{T}^2 \times \{\pm 1\}} \mathbf{h}_\pm \,\mathrm{d}\widetilde{S} \,. \tag{7.16}$$

Then, for each fixed t, $\vec{\mathfrak{v}}_\pm$ and $\vec{\mathfrak{h}}_\pm$ are constant tangential vectors on $\mathbb{T}^2 \times \{\pm 1\}$. With the same notations in § 4.2, assume that $\varkappa_a, \boldsymbol{\omega}_{*\pm}$ and $\vec{\mathfrak{v}}_\pm$ are parameterized by β. Thus, (4.22) can be rewritten as

$$\partial_\beta \mathbf{v}_{\pm *} = \mathbf{B}_\pm(\varkappa_a)\partial^2_{t\beta}\varkappa_a + \mathbf{F}_\pm(\varkappa_a)\partial_\beta \boldsymbol{\omega}_{*\pm} + \mathbf{G}_\pm(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_{*\pm})\partial_\beta \varkappa_a + \mathbf{S}_\pm(\varkappa_a)\partial_\beta \vec{\mathfrak{v}}_\pm. \tag{7.17}$$

It follows from the same arguments that Lemma 4.1 still holds with a subtle modification of indices, namely, $s, s', \sigma, \sigma' \geq \frac{3}{2}$ rather than $\geq \frac{1}{2}$. As for the new term $\mathbf{S}_\pm(\varkappa_a)\partial_\beta \vec{\mathfrak{v}}_\pm$, they are the pull-backs to Γ_* of the solutions to the following boundary value problems:

$$\begin{cases} \nabla\cdot\mathbf{y}_\pm = 0, \quad \nabla\times\mathbf{y}_\pm = \mathbf{0} & \text{in } \Omega_t^\pm, \\ \mathbf{y}_\pm\cdot\mathbf{N}_\pm = 0 & \text{on } \Gamma_t, \\ \mathbf{y}_\pm\cdot\widetilde{\mathbf{N}}_\pm = 0 & \text{on } \mathbb{T}^2\times\{\pm1\}, \\ \int_{\mathbb{T}^2\times\{\pm1\}}\mathbf{y}_\pm\,\mathrm{d}\widetilde{S} = \partial_\beta\vec{\mathfrak{v}}_\pm & \text{on } \mathbb{T}^2\times\{\pm1\}. \end{cases} \tag{7.18}$$

Therefore, since $\partial_\beta\vec{\mathfrak{v}}_\pm$ are constant on $\mathbb{T}^2\times\{\pm1\}$ for each fixed β, the following estimates hold:

$$|\mathbf{S}_\pm(\varkappa_a)|_{\mathscr{L}(\mathbb{R}^2;H^s(\Gamma_*))} \le C_* \quad \text{for} \quad \frac{3}{2}\le s\le\frac{3}{2}k-\frac{3}{2}, \tag{7.19}$$

and

$$|\delta\mathbf{S}_\pm(\varkappa_a)|_{\mathscr{L}\left(H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*);\mathscr{L}(\mathbb{R}^2;H^{s'}(\Gamma_*))\right)} \le C_* \quad \text{for} \quad \frac{3}{2}\le s'\le\frac{3}{2}k-\frac{3}{2}. \tag{7.20}$$

Due to the change of $\Omega_t^\pm$, we shall also modify the definition of $p^\pm_{\mathbf{a},\mathbf{b}}$ to:

$$\begin{cases} -\triangle p^\pm_{\mathbf{a},\mathbf{b}} = \operatorname{tr}(\mathrm{D}\mathbf{a}_\pm\cdot\mathrm{D}\mathbf{b}_\pm) & \text{in } \Omega_t^\pm, \\ p^\pm_{\mathbf{a},\mathbf{b}} = 0 & \text{on } \Gamma_t, \\ \mathrm{D}_{\widetilde{\mathbf{N}}_\pm}p^\pm_{\mathbf{a},\mathbf{b}} = \widetilde{\mathrm{I\!I}}_\pm(\mathbf{a}_\pm,\mathbf{b}_\pm) & \text{on } \mathbb{T}^2\times\{\pm1\}, \end{cases} \tag{7.21}$$

for which the solenoidal vector fields $\mathbf{a}_\pm,\mathbf{b}_\pm$ satisfy $\mathbf{a}_\pm\cdot\widetilde{\mathbf{N}}_\pm = 0 = \mathbf{b}_\pm\cdot\widetilde{\mathbf{N}}_\pm$ on $\mathbb{T}^2\times\{\pm1\}$.

With all the previous modifications on the definitions of the harmonic extensions, Lagrange multiplier pressures, and div-curl systems, it follows from the similar arguments that (4.62) can be rewritten as

$$\begin{aligned} &\partial_{tt}^2\varkappa_a + \mathscr{C}_\alpha(\varkappa_a,\partial_t\varkappa_a,\mathbf{v}_{*\pm},\mathbf{h}_{*\pm})\varkappa_a - \mathscr{F}(\varkappa_a)\partial_t\boldsymbol{\omega}_* - \mathscr{G}(\varkappa_a,\partial_t\varkappa_a,\boldsymbol{\omega}_*,\mathbf{j}_*,\vec{\mathfrak{v}},\vec{\mathfrak{h}}) - \mathscr{S}(\varkappa_a)\partial_t\vec{\mathfrak{v}} \\ &\quad= [\mathrm{I}+\mathscr{B}(\varkappa_a)]^{-1}\left\{\left[-\triangle_{\Gamma_t}(\mathbf{W}\cdot\mathbf{N}_+) + \mathbf{W}\cdot\triangle_{\Gamma_t}\mathbf{N}_+ + a^2\frac{\mathbf{W}\cdot\mathbf{N}_+}{\mathbf{N}_+\cdot(\boldsymbol{\nu}\circ\Phi_{\Gamma_t}^{-1})}\right]\circ\Phi_{\Gamma_t}\right\}. \end{aligned} \tag{7.22}$$

Since $\vec{\mathfrak{v}}_\pm$ and $\vec{\mathfrak{h}}_\pm$ are all constants, Lemma 4.4 holds with $k\ge3, \alpha=0$ and a slight change of (4.68) as:

$$\begin{aligned} &|\delta\mathscr{G}|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\times H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)\times H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)\times H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)\times\mathbb{R}^2\times\mathbb{R}^2;H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)\right]} \\ &\le a^2 Q\left(|\partial_t\varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \|\mathbf{j}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\right), \end{aligned} \tag{7.23}$$

whose proof follows from the same arguments. Furthermore, the operator $\mathscr{S}(\varkappa_a)$ satisfies

$$|\mathscr{S}(\varkappa_a)|_{\mathscr{L}\left(\mathbb{R}^2;H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\right)} \le Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}\right), \tag{7.24}$$

and

$$|\delta\mathscr{S}(\varkappa_a)|_{\mathscr{L}\left[H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*);\mathscr{L}\left(\mathbb{R}^2;H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)\right)\right]} \le Q\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}}\right). \tag{7.25}$$

Indeed, the leading order term of $\mathscr{S}(\varkappa_a)\partial_t\vec{\mathfrak{v}}$ is $\nabla^\top(\kappa_+\circ\Phi_{\Gamma_t})\cdot\mathbf{S}(\varkappa_a)\partial_t\vec{\mathfrak{v}}$, so the above estimates follow from the standard commutator and product ones.

7.4. **Linear systems.** Similar to the arguments in § 5.1, assume that $\Gamma_* \in H^{\frac{3}{2}k+\frac{1}{2}}$ $(k \geq 3)$ is a reference hypersurface, and Λ_* defined by (3.25) satisfies all the properties discussed in the preliminary. Suppose further that there are a family of hypersurfaces $\Gamma_t \in \Lambda_*$ parameterized by $t \in [0,T]$ and four tangential vector fields $\mathbf{v}_{\pm*}, \mathbf{h}_{\pm*} : \Gamma_* \to \mathrm{T}\Gamma_*$ satisfying:

$$\varkappa_a \in C^0\Big\{[0,T]; H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)\Big\} \cap C^1\Big\{[0,T]; B_{\delta_1} \subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\Big\}, \tag{H1'}$$

and

$$\mathbf{v}_{\pm*}, \mathbf{h}_{\pm*} \in C^0\Big\{[0,T]; H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)\Big\} \cap C^1\Big\{[0,T]; H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)\Big\}. \tag{H2'}$$

Moreover, assume that there are positive constants $c_0, \mathfrak{s}_0$ so that (7.9) and (7.12) hold uniformly on $[0,T]$.

The positive constants $\widetilde{L}_1$ and $\widetilde{L}_2$ are defined by:

$$\sup_{t\in[0,T]} \Big\{ |\varkappa_a(t)|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, |\partial_t \varkappa_a(t)|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, |(\mathbf{v}_{\pm*}(t), \mathbf{h}_{\pm*}(t))|_{H^{\frac{3}{2}k-\frac{1}{2}}(\Gamma_*)} \Big\} \leq \widetilde{L}_1, \tag{7.26}$$

$$\sup_{t\in[0,T]} |(\partial_t \mathbf{v}_{\pm*}(t), \partial_t \mathbf{h}_{\pm*}(t))|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)} \leq \widetilde{L}_2. \tag{7.27}$$

Consider the following linear initial value problem similar to (5.4):

$$\begin{cases} \partial_{tt}^2 \mathfrak{f} + \mathscr{C}_0(\varkappa_a, \partial_t \varkappa_a, \mathbf{v}_*, \mathbf{h}_*)\mathfrak{f} = \mathfrak{g}, \\ \mathfrak{f}(0) = \mathfrak{f}_0, \quad \partial_t \mathfrak{f}(0) = \mathfrak{f}_1, \end{cases} \tag{7.28}$$

where $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{g}(t) : \Gamma_* \to \mathbb{R}$ are three given functions, and $\mathscr{C}_0$ is given by:

$$\begin{aligned} \mathscr{C}_0(\varkappa_a, \partial_t \varkappa_a, \mathbf{v}_*, \mathbf{h}_*) :=\ & 2\,\mathrm{D}_{\mathbf{u}_*}\partial_t + \mathrm{D}_{\mathbf{u}_*}\mathrm{D}_{\mathbf{u}_*} + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2}\mathscr{R}(\varkappa_a, \mathbf{w}_*) \\ & - \frac{\rho_+}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{+*}) - \frac{\rho_-}{\rho_+ + \rho_-}\mathscr{R}(\varkappa_a, \mathbf{h}_{-*}), \end{aligned}$$

which is exactly (4.55) with $\alpha = 0$.

Thus, for $0 \leq l \leq k-2$, the energy (5.5) is replaced by

$$\begin{aligned} \widetilde{E}_l(t, \mathfrak{f}, \partial_t \mathfrak{f}) := \int_{\Gamma_t} & \bigg| \Big(-\widetilde{\mathrm{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathrm{N}}^{\frac{1}{2}} \Big)^{\frac{l}{2}} \widetilde{\mathrm{N}}^{\frac{1}{2}} \big[(\partial_t \mathfrak{f} + \mathrm{D}_{\mathbf{u}_*}\mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} \big] \bigg|^2 \\ & - \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2} \bigg| \Big(-\widetilde{\mathrm{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathrm{N}}^{\frac{1}{2}} \Big)^{\frac{l}{2}} \widetilde{\mathrm{N}}^{\frac{1}{2}} \big[(\mathrm{D}_{\mathbf{w}_*}\mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} \big] \bigg|^2 \\ & + \frac{\rho_+}{\rho_+ + \rho_-} \bigg| \Big(-\widetilde{\mathrm{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathrm{N}}^{\frac{1}{2}} \Big)^{\frac{l}{2}} \widetilde{\mathrm{N}}^{\frac{1}{2}} \big[(\mathrm{D}_{\mathbf{h}_{+*}}\mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} \big] \bigg|^2 \\ & + \frac{\rho_-}{\rho_+ + \rho_-} \bigg| \Big(-\widetilde{\mathrm{N}}^{\frac{1}{2}} \triangle_{\Gamma_t} \widetilde{\mathrm{N}}^{\frac{1}{2}} \Big)^{\frac{l}{2}} \widetilde{\mathrm{N}}^{\frac{1}{2}} \big[(\mathrm{D}_{\mathbf{h}_{-*}}\mathfrak{f}) \circ \Phi_{\Gamma_t}^{-1} \big] \bigg|^2 \mathrm{d}S_t\,. \end{aligned} \tag{7.29}$$

It follows from the same arguments as in the proof of Lemma 5.1 that there exists a generic polynomial Q determined by Λ_*, such that the following estimate holds:

$$
\begin{aligned}
&\widetilde{E}_l(t,\mathfrak{f},\partial_t\mathfrak{f})-\widetilde{E}_l(0,\mathfrak{f}_0,\mathfrak{f}_1)\\
&\quad\le Q(\widetilde{L}_1,\widetilde{L}_2)\int_0^t\Big(|\mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}+|\partial_t\mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}+|\mathfrak{g}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\Big)\times\\
&\qquad\qquad\times\Big(|\mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}+|\partial_t\mathfrak{f}(s)|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\Big)\,\mathrm{d}s\,.
\end{aligned}
\tag{7.30}
$$

Thanks to the uniform stability condition (7.9), one can derive an estimate similar to (5.42), as long as $T\le C$ for some constant $C=C(\widetilde{L}_1,\widetilde{L}_2,\mathfrak{s}_0)$:

$$
\begin{aligned}
&|\mathfrak{f}(t)|^2_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}+|\partial_t\mathfrak{f}(t)|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\\
&\le C_*e^{Q(\widetilde{L}_1,\widetilde{L}_2,\mathfrak{s}_0^{-1})t}\Big(|\mathfrak{f}_0|^2_{H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)}+|\mathfrak{f}_1|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}+\int_0^t|\mathfrak{g}(t')|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)}\,\mathrm{d}t'\Big),
\end{aligned}
\tag{7.31}
$$

for any integer $0\le l\le k-2$, $0\le t\le T$, a generic polynomial Q and a positive constant C_* depending on Λ_*. Thus, one has

Proposition 7.4. *For $0\le l\le k-2$, $T\le C(\widetilde{L}_1,\widetilde{L}_2,\mathfrak{s}_0)$ and $\mathfrak{g}\in C^0\big([0,T];H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)\big)$, the linear problem (7.28) is well-posed in $C^0\big([0,T];H^{\frac{3}{2}l+\frac{3}{2}}(\Gamma_*)\big)\cap C^1\big([0,T];H^{\frac{3}{2}l+\frac{1}{2}}(\Gamma_*)\big)$, and the energy estimate (7.31) holds.*

It is also noted that the arguments in § 5.2 are still valid for the linear systems for the current and vorticity here.

7.5. **Nonlinear problems.** As in § 6, take a reference hypersurface $\Gamma_*\in H^{\frac{3}{2}k+\frac{1}{2}}$ and $\delta_0>0$ so that

$$\Lambda_*:=\Lambda\Big(\Gamma_*,\frac{3}{2}k-\frac{1}{2},\delta_0\Big)$$

satisfies all the properties discussed in the preliminary. Furthermore, assume that there is a constant $c_0>0$ so that (7.12) holds for Γ_*. We shall solve the nonlinear problem by iterations on the linearized problems in the spaces:

$$
\begin{aligned}
&\varkappa_a\in C^0\big([0,T];H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)\big)\cap C^1\big([0,T];B_{\delta_1}\subset H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)\big)\cap C^2\big([0,T];H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)\big);\\
&\boldsymbol{\omega}_{\pm*},\mathbf{j}_{\pm*}\in C^0\big([0,T];H^{\frac{3}{2}k-1}(\Omega^\pm_{\Gamma_*})\big)\cap C^1\big([0,T];H^{\frac{3}{2}k-2}(\Omega^\pm_{\Gamma_*})\big);\\
&\vec{\mathfrak{v}}_\pm,\vec{\mathfrak{h}}_\pm\in C^1\big([0,T];\mathbb{R}^2\big).
\end{aligned}
$$

7.5.1. *Fluid region, velocity and magnetic fields.* As discussed in § 6.2, the bulk region, velocity and magnetic fields can be obtained by solving the following div-curl problems:

$$
\begin{cases}
\nabla\cdot\mathbf{v}_\pm=0,\quad \nabla\times\mathbf{v}_\pm=\bar{\boldsymbol{\omega}}_\pm & \text{in }\Omega_t^\pm,\\
\mathbf{v}_\pm\cdot\mathbf{N}_+=\mathbf{N}_+\cdot(\partial_t\gamma_{\Gamma_t}\boldsymbol{\nu})\circ\Phi_{\Gamma_t}^{-1} & \text{on }\Gamma_t,\\
\mathbf{v}_\pm\cdot\widetilde{\mathbf{N}}_\pm=0 & \text{on }\mathbb{T}^2\times\{\pm1\},\\
\int_{\mathbb{T}^2\times\{\pm1\}}\mathbf{v}_\pm\,\mathrm{d}\widetilde{S}=\vec{\mathfrak{v}}_\pm & \text{on }\mathbb{T}^2\times\{\pm1\};
\end{cases}
\tag{7.32}
$$

and

$$\begin{cases} \nabla\cdot\mathbf{h}_\pm = 0, \quad \nabla\times\mathbf{h}_\pm = \bar{\mathbf{j}}_\pm & \text{in } \Omega_t^\pm, \\ \mathbf{h}_\pm\cdot\mathbf{N}_\pm = 0 & \text{on } \Gamma_t, \\ \mathbf{h}_\pm\cdot\widetilde{\mathbf{N}}_\pm = 0 & \text{on } \mathbb{T}^2\times\{\pm1\}, \\ \int_{\mathbb{T}^2\times\{\pm1\}} \mathbf{h}_\pm \,\mathrm{d}\widetilde{S} = \vec{\mathfrak{h}}_\pm & \text{on } \mathbb{T}^2\times\{\pm1\}, \end{cases} \tag{7.33}$$

where $\bar{\boldsymbol{\omega}}_\pm$ and $\bar{\mathbf{j}}_\pm$ are given by (6.3).

7.5.2. *Iteration mapping.* In order to construct the iteration map, we consider the following function space:

Definition 7.5. For given constants $T, M_0, M_1, M_2, M_3, c_0, \mathfrak{s}_0 > 0$, define $\mathfrak{X}$ to be the collection of $\left(\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*, \vec{\mathfrak{v}}_\pm, \vec{\mathfrak{h}}_\pm\right)$ satisfying:

$$|\varkappa_a(0) - \kappa_{*+}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \le \delta_1,$$

$$|(\partial_t\varkappa_a)(0)|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)}, \|\boldsymbol{\omega}_*(0)\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}, \|\mathbf{j}_*(0)\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}, \left|\vec{\mathfrak{v}}_\pm(0)\right|, \left|\vec{\mathfrak{h}}_\pm(0)\right| \le M_0,$$

$$\sup_{t\in[0,T]}\left(|\varkappa_a|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)}, |\partial_t\varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \|(\boldsymbol{\omega}_*, \mathbf{j}_*)\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \left|\vec{\mathfrak{v}}_\pm\right|, \left|\vec{\mathfrak{h}}_\pm\right|\right) \le M_1,$$

$$\sup_{t\in[0,T]}\left(\|\partial_t\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}, \|\partial_t\mathbf{j}_*\|_{H^{\frac{3}{2}k-2}(\Omega\setminus\Gamma_*)}, \left|\partial_t\vec{\mathfrak{v}}_\pm\right|, \left|\partial_t\vec{\mathfrak{h}}_\pm\right|\right) \le M_2,$$

$$\sup_{t\in[0,T]}\left|\partial_{tt}^2\varkappa_a\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} \le a^2 M_3 \text{ (here } a \text{ is the constant in the definition of } \varkappa_a).$$

For $\Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!])$ defined by (7.8),

$$\Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!]) \ge \mathfrak{s}_0$$

holds uniformly for $0 \le t \le T$. In addition, (7.12) and the compatibility conditions

$$\int_{\mathbb{T}^2\times\{\pm1\}} \widetilde{\mathbf{N}}_\pm\cdot\boldsymbol{\omega}_{*\pm}\,\mathrm{d}\widetilde{S} = \int_{\mathbb{T}^2\times\{\pm1\}} \widetilde{\mathbf{N}}_\pm\cdot\mathbf{j}_{*\pm}\,\mathrm{d}\widetilde{S} = 0$$

hold for all $t \in [0, T]$.

As for the initial data, take $0 < \epsilon \ll \delta_1$ and $A > 0$, and consider:

$$\mathfrak{I}(\epsilon, A) := \left\{\left((\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_*)_{\mathrm{I}}, (\mathbf{j}_*)_{\mathrm{I}}\right), \left(\vec{\mathfrak{v}}_\pm\right)_{\mathrm{I}}, \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\right\},$$

where

$$|(\varkappa_a)_{\mathrm{I}} - \kappa_{*+}|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)} < \epsilon;$$

$$|(\partial_t\varkappa_a)_{\mathrm{I}}|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \ \|(\boldsymbol{\omega}_*)_{\mathrm{I}}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \ \|(\mathbf{j}_*)_{\mathrm{I}}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \ \left|\left(\vec{\mathfrak{v}}_\pm\right)_{\mathrm{I}}\right|, \ \left|\left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\right| < A,$$

$$\Upsilon(\mathbf{h}_\pm, [\![\mathbf{v}]\!]) \ge 2\mathfrak{s}_0,$$

and

$$\operatorname{dist}\left(\Gamma_{\mathrm{I}}, \mathbb{T}^2\times\{\pm1\}\right) \ge 2c_0.$$

In addition, $(\boldsymbol{\omega}_*)_{\mathrm{I}}$ and $(\mathbf{j}_*)_{\mathrm{I}}$ satisfy the following compatibility conditions:

$$\int_{\mathbb{T}^2\times\{\pm1\}} \widetilde{\mathbf{N}}_\pm \cdot (\boldsymbol{\omega}_*)_{\mathrm{I}\pm}\, \mathrm{d}\widetilde{S} = \int_{\mathbb{T}^2\times\{\pm1\}} \widetilde{\mathbf{N}}_\pm \cdot (\mathbf{j}_*)_{\mathrm{I}\pm}\, \mathrm{d}\widetilde{S} = 0.$$

Thus, $\mathfrak{I}(\epsilon, A) \subset H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*) \times H^{\frac{3}{2}k-1}(\Omega \setminus \Gamma_*) \times \mathbb{R}^4$.

Then, as in § 6.3, one can define the iteration map:

$$\begin{cases} \partial_{tt}^2 \varkappa_a{}^{(n+1)} + \mathscr{C}_0\Big(\varkappa_a{}^{(n)}, \partial_t \varkappa_a{}^{(n)}, \mathbf{v}_{*\pm}{}^{(n)}, \mathbf{h}_{*\pm}{}^{(n)}\Big)\varkappa_a{}^{(n+1)} \\ \qquad = \mathscr{F}\Big(\varkappa_a{}^{(n)}\Big)\partial_t \boldsymbol{\omega}_*{}^{(n)} + \mathscr{G}\Big(\varkappa_a{}^{(n)}, \partial_t\varkappa_a{}^{(n)}, \boldsymbol{\omega}_{*\pm}{}^{(n)}, \mathbf{j}_{*\pm}{}^{(n)}, \vec{\mathfrak{v}}_\pm^{(n)}, \vec{\mathfrak{h}}_\pm^{(n)}\Big) \\ \qquad\quad + \mathscr{S}\Big(\varkappa_a{}^{(n)}\Big)\partial_t \vec{\mathfrak{v}}_\pm^{(n)} \\ \varkappa_a{}^{(n+1)}(0) = (\varkappa_a)_{\mathrm{I}}, \quad \partial_t\varkappa_a{}^{(n+1)}(0) = (\partial_t\varkappa_a)_{\mathrm{I}}; \end{cases} \tag{7.34}$$

and

$$\begin{cases} \partial_t \boldsymbol{\omega}_\pm{}^{(n+1)} + \mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\boldsymbol{\omega}_\pm{}^{(n+1)} - \mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\mathbf{j}_\pm{}^{(n+1)} = \mathrm{D}_{\boldsymbol{\omega}_\pm{}^{(n+1)}}\mathbf{v}_\pm{}^{(n)} - \mathrm{D}_{\mathbf{j}_\pm{}^{(n+1)}}\mathbf{h}_\pm{}^{(n)}, \\ \partial_t \mathbf{j}_\pm{}^{(n+1)} + \mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\mathbf{j}_\pm{}^{(n+1)} - \mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\boldsymbol{\omega}_\pm{}^{(n+1)} \\ \qquad = \mathrm{D}_{\mathbf{j}_\pm{}^{(n+1)}}\mathbf{v}_\pm{}^{(n)} - \mathrm{D}_{\boldsymbol{\omega}_\pm{}^{(n+1)}}\mathbf{h}_\pm{}^{(n)} - 2\,\mathrm{tr}\Big(\nabla\mathbf{v}_\pm{}^{(n)} \times \nabla\mathbf{h}_\pm{}^{(n)}\Big), \\ \boldsymbol{\omega}_\pm{}^{(n+1)}(0) = \mathbb{P}\Big((\boldsymbol{\omega}_{*\pm})_{\mathrm{I}} \circ (\mathcal{X}^\pm_{\Gamma_0{}^{(n)}})^{-1}\Big), \quad \mathbf{j}_\pm{}^{(n+1)}(0) = \mathbb{P}\Big((\mathbf{j}_{*\pm})_{\mathrm{I}} \circ (\mathcal{X}^\pm_{\Gamma_0{}^{(n)}})^{-1}\Big), \end{cases} \tag{7.35}$$

where $\Big(\mathbf{v}_\pm{}^{(n)}, \mathbf{h}_\pm{}^{(n)}\Big)$ is induced by $\Big(\varkappa_a{}^{(n)}, \boldsymbol{\omega}_{*\pm}{}^{(n)}, \mathbf{j}_{*\pm}{}^{(n)}, \vec{\mathfrak{v}}_\pm^{(n)}, \vec{\mathfrak{h}}_\pm^{(n)}\Big)$ via solving (7.32)-(7.33), the tangential vector fields $\mathbf{v}_{*\pm}{}^{(n)}$ and $\mathbf{h}_{*\pm}{}^{(n)}$ on Γ_* are defined by

$$\mathbf{v}_{*\pm}{}^{(n)} := \Big(\mathrm{D}\Phi_{\Gamma_t{}^{(n)}}\Big)^{-1}\Big[\mathbf{v}_\pm{}^{(n)} \circ \Phi_{\Gamma_t{}^{(n)}} - \Big(\partial_t \gamma_{\Gamma_t{}^{(n)}}\Big)\boldsymbol{\nu}\Big],$$

$$\mathbf{h}_{*\pm}{}^{(n)} := \Big(\mathrm{D}\Phi_{\Gamma_t{}^{(n)}}\Big)^{-1}\Big(\mathbf{h}_\pm{}^{(n)} \circ \Phi_{\Gamma_t{}^{(n)}}\Big),$$

and the current-vorticity equations are considered in the domains $\Omega_t^{\pm(n)}$.

Define

$$\vec{\mathfrak{v}}_\pm^{(n+1)}(t) := (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}} + \int_0^t \int_{\mathbb{T}^2\times\{\pm1\}} -\mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\mathbf{v}_\pm{}^{(n)} - \frac{1}{\rho_\pm}\nabla p^{\pm(n)} + \mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\mathbf{h}_\pm{}^{(n)}\, \mathrm{d}\widetilde{S}\, \mathrm{d}t', \tag{7.36}$$

$$\vec{\mathfrak{h}}_\pm^{(n+1)}(t) := \Big(\vec{\mathfrak{h}}_\pm\Big)_{\mathrm{I}} + \int_0^t \int_{\mathbb{T}^2\times\{\pm1\}} \mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\mathbf{v}_\pm{}^{(n)} - \mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\mathbf{h}_\pm{}^{(n)}\, \mathrm{d}\widetilde{S}\, \mathrm{d}t', \tag{7.37}$$

and

$$\boldsymbol{\omega}_*{}^{(n+1)} := \boldsymbol{\omega}^{(n+1)} \circ \mathcal{X}_{\Gamma_t{}^{(n)}}, \quad \mathbf{j}_*{}^{(n+1)} := \mathbf{j}^{(n+1)} \circ \mathcal{X}_{\Gamma_t{}^{(n)}}, \tag{7.38}$$

where $p^{\pm(n)}$ are given by (4.4) with $\Big(\varkappa_a{}^{(n)}, \mathbf{v}_\pm{}^{(n)}, \mathbf{h}_\pm{}^{(n)}\Big)$ plugged in.

In order to show that the iteration map is a mapping from $\mathfrak{X}$ to $\mathfrak{X}$, one may first check that

$$\mathrm{dist}\,\Big(\Gamma_t^{(n+1)}, \mathbb{T}^2 \times \{\pm1\}\Big) \geq 2c_0 - CT\Big|\partial_t\varkappa_a{}^{(n+1)}\Big|_{C_t^0 H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \geq c_0, \tag{7.39}$$

and

$$\Big|\Upsilon(\mathbf{h}_\pm^{(n+1)}, [\![\mathbf{v}^{(n+1)}]\!]) - \Upsilon\big((\mathbf{h})_{\mathrm{I}\pm}, [\![(\mathbf{v})_{\mathrm{I}}]\!]\big)\Big| \leq TQ(M_1, M_2) \leq \backsim_0, \tag{7.40}$$

if T is small compared to M_1 and M_2.

Next, for $\vec{\mathfrak{v}}_\pm^{(n+1)}$ and $\vec{\mathfrak{h}}_\pm^{(n+1)}$, observe that

$$\left|\vec{\mathfrak{v}}_\pm^{(n+1)}(t)\right| + \left|\vec{\mathfrak{h}}_\pm^{(n+1)}(t)\right| \leq A + TQ(M_1) \leq M_1, \tag{7.41}$$

and

$$\left|\partial_t \vec{\mathfrak{v}}_\pm(t)\right| + \left|\partial_t \vec{\mathfrak{h}}_\pm(t)\right| \leq Q(M_1) \leq M_2, \tag{7.42}$$

provided that T is small and $M_2 \gg M_1$.

Then, with the same notation as in § 6.3, one can define

$$\begin{aligned}&\mathfrak{T}\Big(\Big[(\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}, (\mathbf{j}_{*\pm})_{\mathrm{I}}, (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}, \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\Big], \Big[\varkappa_a{}^{(n)}, \boldsymbol{\omega}_{*\pm}{}^{(n)}, \mathbf{j}_{*\pm}{}^{(n)}, \vec{\mathfrak{v}}_\pm^{(n)}, \vec{\mathfrak{h}}_\pm^{(n)}\Big]\Big)\\ &:= \left(\varkappa_a{}^{(n+1)}, \boldsymbol{\omega}_{*\pm}{}^{(n+1)}, \mathbf{j}_{*\pm}{}^{(n+1)}, \vec{\mathfrak{v}}_\pm^{(n+1)}, \vec{\mathfrak{h}}_\pm^{(n+1)}\right).\end{aligned} \tag{7.43}$$

It follows from the arguments in § 6.3 and the linear estimates in § 7.4 that the following proposition holds:

Proposition 7.6. *Suppose that $k \geq 3$. For any $0 < \epsilon \ll \delta_0$ and $A > 0$, there are positive constants $M_0, M_1, M_2, M_3, c_0, \mathfrak{s}_0$, so that for small $T > 0$,*

$$\mathfrak{T}\Big\{\Big[(\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}, (\mathbf{j}_{*\pm})_{\mathrm{I}}, (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}, \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\Big], \Big[\varkappa_a, \boldsymbol{\omega}_{*\pm}, \mathbf{j}_{*\pm}, \vec{\mathfrak{v}}_\pm, \vec{\mathfrak{h}}_\pm\Big]\Big\} \in \mathfrak{X},$$

holds for any $\left((\varkappa_a)_{\mathrm{I}}, (\partial_t\varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}, (\mathbf{j}_{*\pm})_{\mathrm{I}}, (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}, \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\right) \in \mathfrak{I}(\epsilon, A)$ *and* $\left(\varkappa_a, \boldsymbol{\omega}_{*\pm}, \mathbf{j}_{*\pm}, \vec{\mathfrak{v}}_\pm, \vec{\mathfrak{h}}_\pm\right) \in \mathfrak{X}$.

For the contraction of the iteration mapping, as in § 6.4, assume that $\left(\varkappa_a{}^{(n)}(\beta), \boldsymbol{\omega}_{*\pm}{}^{(n)}(\beta), \mathbf{j}_{*\pm}{}^{(n)}(\beta), \vec{\mathfrak{v}}_\pm^{(n)}(\beta), \vec{\mathfrak{h}}_\pm^{(n)}(\beta)\right) \subset \mathfrak{X}$ and $\left((\varkappa_a)_{\mathrm{I}}(\beta), (\partial_t\varkappa_a)_{\mathrm{I}}(\beta), (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}(\beta), (\mathbf{j}_{*\pm})_{\mathrm{I}}(\beta), (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}(\beta), \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}(\beta)\right) \subset \mathfrak{I}(\epsilon, A)$ are two families of data depending on a parameter β.

Define $\left(\varkappa_a{}^{(n+1)}(\beta), \boldsymbol{\omega}_{*\pm}{}^{(n+1)}(\beta), \mathbf{j}_{*\pm}{}^{(n+1)}(\beta), \vec{\mathfrak{v}}_\pm^{(n+1)}(\beta), \vec{\mathfrak{h}}_\pm^{(n+1)}(\beta)\right)$ to be the output of the iteration map. Then, by applying $\partial/\partial\beta$ to (7.34)-(7.37), one has the variational problems (6.17)-(6.20) as well as:

$$\begin{cases}\partial_{tt}^2\partial_\beta\varkappa_a{}^{(n+1)} + \mathscr{C}^{(n)}\partial_\beta\varkappa_a{}^{(n+1)}\\ \qquad = -\left(\partial_\beta\mathscr{C}^{(n)}\right)\varkappa_a{}^{(n+1)} + \partial_\beta\left(\mathscr{F}^{(n)}\partial_t\boldsymbol{\omega}_*{}^{(n)} + \mathscr{G}^{(n)} + \mathscr{S}^{(n)}\partial_t\vec{\mathfrak{v}}^{(n)}\right),\\ \partial_\beta\varkappa_a{}^{(n+1)}(0) = \partial_\beta(\varkappa_a)_{\mathrm{I}}(\beta), \quad \partial_t\left(\partial_\beta\varkappa_a{}^{(n+1)}\right)(0) = \partial_\beta(\partial_t\varkappa_a)_{\mathrm{I}}(\beta),\end{cases} \tag{7.44}$$

$$\begin{aligned}&\partial_t\partial_\beta\vec{\mathfrak{v}}_\pm^{(n+1)}\\ &= \partial_\beta\left(\vec{\mathfrak{v}}_\pm\right)_{\mathrm{I}} + \int_0^t\int_{\mathbb{T}^2\times\{\pm1\}}\mathbb{D}_\beta\left(-\mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\mathbf{v}_\pm{}^{(n)} - \frac{1}{\rho_\pm}\boldsymbol{\nabla}p^{\pm(n)} + \mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\mathbf{h}_\pm{}^{(n)}\right)\mathrm{d}\widetilde{S}\,\mathrm{d}t',\end{aligned} \tag{7.45}$$

and

$$\partial_t\partial_\beta\vec{\mathfrak{h}}_\pm^{(n+1)} = \partial_\beta\left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}} + \int_0^t\int_{\mathbb{T}^2\times\{\pm1\}}\mathbb{D}_\beta\left(\mathrm{D}_{\mathbf{h}_\pm{}^{(n)}}\mathbf{v}_\pm{}^{(n)} - \mathrm{D}_{\mathbf{v}_\pm{}^{(n)}}\mathbf{h}_\pm{}^{(n)}\right)\mathrm{d}\widetilde{S}\,\mathrm{d}t'. \tag{7.46}$$

Consider the energy functionals:

$$\mathfrak{E}^{(n)}(\beta) := \sup_{t\in[0,T]} \Big(\left|\partial_\beta \varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial_\beta \partial_t \varkappa_a{}^{(n)}\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} + \left\|\partial_\beta \boldsymbol{\omega}_{*\pm}{}^{(n)}\right\|_{H^{\frac{3}{2}k-2}(\Omega_*^\pm)} + \left\|\partial_\beta \mathbf{j}_{*\pm}{}^{(n)}\right\|_{H^{\frac{3}{2}k-2}(\Omega_*^\pm)} + \left\|\partial_\beta \partial_t \boldsymbol{\omega}_{*\pm}{}^{(n)}\right\|_{H^{\frac{3}{2}k-4}(\Omega_*^\pm)} + \left|\partial_\beta \vec{\mathfrak{v}}_\pm^{(n)}\right| + \left|\partial_\beta \vec{\mathfrak{h}}_\pm^{(n)}\right| \Big), \tag{7.47}$$

and

$$(\mathfrak{E})_{\mathrm{I}}(\beta) := \sup_{t\in[0,T]} \Big(\left|\partial_\beta (\varkappa_a)_{\mathrm{I}}\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial_\beta (\partial_t \varkappa_a)_{\mathrm{I}}\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} + \left\|\partial_\beta (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}\right\|_{H^{\frac{3}{2}k-2}(\Omega_*^\pm)} + \left\|\partial_\beta (\mathbf{j}_{*\pm})_{\mathrm{I}}\right\|_{H^{\frac{3}{2}k-2}(\Omega_*^\pm)} + \left|\partial_\beta (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}\right| + \left|\partial_\beta \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\right| \Big). \tag{7.48}$$

It follows from (7.45)-(7.46), (7.23)-(7.25), and the arguments in § 6.4 that

$$\mathfrak{E}^{(n+1)} \le \frac{1}{2}\mathfrak{E}^{(n)} + Q(M_1)(\mathfrak{E})_{\mathrm{I}}, \tag{7.49}$$

provided that T is sufficiently small. That is, the following proposition holds:

Proposition 7.7. *Assume that $k \ge 3$. For any $0 < \epsilon \ll \delta_0$ and $A > 0$, there are positive constants $M_0, M_1, M_2, M_3, c_0, \mathfrak{s}_0$, so that if T is small enough, then there is a map $\mathfrak{S} : \mathfrak{I}(\epsilon, A) \to \mathfrak{X}$ such that*

$$\mathfrak{T}\{\mathfrak{x}, \mathfrak{S}(\mathfrak{x})\} = \mathfrak{S}(\mathfrak{x}), \tag{7.50}$$

for each $\mathfrak{x} = \left((\varkappa_a)_{\mathrm{I}}, (\partial_t \varkappa_a)_{\mathrm{I}}, (\boldsymbol{\omega}_{*\pm})_{\mathrm{I}}, (\mathbf{j}_{*\pm})_{\mathrm{I}}, (\vec{\mathfrak{v}}_\pm)_{\mathrm{I}}, \left(\vec{\mathfrak{h}}_\pm\right)_{\mathrm{I}}\right) \in \mathfrak{I}(\epsilon, A)$.

7.5.3. *The original MHD problem.* For the fixed point given in Proposition 7.7, one can obtain $(\Gamma_t, \mathbf{v}_\pm, \mathbf{h}_\pm)$ by solving the div-curl problems (7.32)-(7.33). Observe that (7.36) and (7.37) yield

$$\int_{\mathbb{T}^2\times\{\pm1\}} \partial_t \mathbf{v}_\pm + \mathrm{D}_{\mathbf{v}_\pm}\mathbf{v}_\pm + \frac{1}{\rho_\pm}\boldsymbol{\nabla} p^\pm - \mathrm{D}_{\mathbf{h}_\pm}\mathbf{h}_\pm \,\mathrm{d}\widetilde{S} = 0 \tag{7.51}$$

and

$$\int_{\mathbb{T}^2\times\{\pm1\}} \partial_t \mathbf{h}_\pm + \mathrm{D}_{\mathbf{v}_\pm}\mathbf{h}_\pm - \mathrm{D}_{\mathbf{h}_\pm}\mathbf{v}_\pm \,\mathrm{d}\widetilde{S} = 0, \tag{7.52}$$

which, together with the arguments in § 6.5, shows that $(\Gamma_t, \mathbf{v}_\pm, \mathbf{h}_\pm)$ is the unique solution to the (MHD)-(BC') problem.

In particular, Theorem 2.2 follows.

7.6. **Vanishing surface tension limit.** In this subsection, it is always assumed that $\Omega = \mathbb{T}^2 \times \{\pm1\}$, $k \ge 3$ and the initial data satisfy the assumptions of Theorem 2.2. To derive the uniform-in-α estimates, we consider the following four parts of the energies:

$$\mathcal{E}_0(t) := \frac{1}{2}\int_{\Omega_t^+} \rho_+\left(|\mathbf{v}_+|^2 + |\mathbf{h}_+|^2\right)\mathrm{d}x + \frac{1}{2}\int_{\Omega_t^-} \rho_-\left(|\mathbf{v}_-|^2 + |\mathbf{h}_-|^2\right)\mathrm{d}x + \int_{\Gamma_t} \alpha^2 \,\mathrm{d}S_t\,, \tag{7.53}$$

$$\mathcal{E}_1(t) := |\mathbb{D}_{\bar{t}}\kappa_+|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} + \alpha^2|\kappa_+|^2_{H^{\frac{3}{2}k-1}(\Gamma_t)} + |\kappa_+|^2_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \tag{7.54}$$

$$\mathcal{E}_2(t) := \|\boldsymbol{\omega}_\pm\|^2_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)} + \|\mathbf{j}_\pm\|^2_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)}, \tag{7.55}$$

and

$$\mathcal{E}_3(t) := \left|\vec{\mathfrak{v}}_\pm\right|^2 + \left|\vec{\mathfrak{h}}_\pm\right|^2. \tag{7.56}$$

It follows from the (MHD)-(BC') system that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0(t) \equiv 0. \tag{7.57}$$

Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t^+} \rho_+\left(|\mathbf{v}_+|^2 + |\mathbf{h}_+|^2\right)\mathrm{d}x &= \int_{\Omega_t^+} \rho_+ \mathrm{D}_{\mathbf{h}_+}(\mathbf{v}_+ + \mathbf{h}_+) - \mathbf{v}_+ \cdot \nabla p^+ \,\mathrm{d}x \\ &= \int_{\Omega_t^+} -\mathbf{v}_+ \cdot \nabla p^+ + \rho_+ \mathrm{D}_{\mathbf{h}_+}(\mathbf{h}_+ \cdot \mathbf{v}_+)\,\mathrm{d}x \\ &= \int_{\Omega_t^+} -\nabla\cdot\left(p^+\mathbf{v}_+\right) + \rho_+\nabla\cdot(\mathbf{h}_+[\mathbf{h}_+\cdot\mathbf{v}_+])\,\mathrm{d}x \\ &= \int_{\Gamma_t} -p^+\theta \,\mathrm{d}S_t\,.\end{aligned} \tag{7.58}$$

Similarly, one can calculate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t^-} \rho_-\left(|\mathbf{v}_-|^2 + |\mathbf{h}_-|^2\right)\mathrm{d}x = \int_{\Gamma_t} p_-\theta\,\mathrm{d}S_t\,. \tag{7.59}$$

It follows from (3.6) and (3.16) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\mathrm{d}S_t &= \int_{\Gamma_t} \mathrm{div}_{\Gamma_t}\mathbf{v}\,\mathrm{d}S_t \\ &= \int_{\Gamma_t} \mathrm{div}_{\Gamma_t}(\theta\mathbf{N}_+) + \mathrm{div}_{\Gamma_t}(\mathbf{v} - \theta\mathbf{N}_+)\,\mathrm{d}S_t \\ &= \int_{\Gamma_t} \theta\,\mathrm{div}_{\Gamma_t}\mathbf{N}_+ + \nabla^\top\theta\cdot\mathbf{N}_+\,\mathrm{d}S_t \\ &= \int_{\Gamma_t} \theta\kappa_+\,\mathrm{d}S_t\,.\end{aligned} \tag{7.60}$$

Thus, (7.57) follows from the above relations and the boundary condition that $p^+ - p^- = \alpha^2\kappa_+$ on Γ_t.

Furthermore, applying the arguments in the proof of Lemma 5.1 to (4.42), (4.43') and (7.9) yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)\right| \le Q\left(\alpha|\kappa_+|_{H^{\frac{3}{2}k-1}(\Gamma_t)}, |\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right) \tag{7.61}$$

for some generic polynomial Q determined by Λ_*, c_0, and $\mathfrak{s}_0$. Similarly, (4.69)-(4.70) and the arguments in § 5.2 lead to:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t)\right| \le Q\left(|\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right). \tag{7.62}$$

As for $\mathcal{E}_3$, it follows from (7.36) and (7.37) that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_3(t)\right| \le Q\left(|\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \|\mathbf{v}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}, \|\mathbf{h}\|_{H^{\frac{3}{2}k}(\Omega\setminus\Gamma_t)}\right). \tag{7.63}$$

On the other hand, if T is small compared to $\|(\mathbf{v}_{0\pm}, \mathbf{h}_{0\pm})\|_{H^{\frac{3}{2}k}(\Omega_0^\pm)}$, one has

$$\operatorname{dist}\left(\Gamma_t, \mathbb{T}^2\times\{\pm1\}\right) \ge c_0.$$

Then it follows from the estimates of div-curl systems and Lemma 3.2 that

$$\begin{aligned}\|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)} \le& Q\left(|\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, c_0\right)\times\\ &\times\left(\|\boldsymbol{\omega}_\pm\|_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)} + |\mathbf{N}_+\cdot\triangle_{\Gamma_t}\mathbf{v}_\pm|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} + \left|\vec{\mathfrak{v}}_\pm\right|\right),\end{aligned} \tag{7.64}$$

and

$$\begin{aligned}\|\mathbf{h}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)} \le& Q\left(|\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, c_0\right)\times\\ &\times\left(\|\mathbf{j}_\pm\|_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)} + |\mathbf{N}_+\cdot\triangle_{\Gamma_t}\mathbf{h}_\pm|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} + \left|\vec{\mathfrak{h}}_\pm\right|\right).\end{aligned} \tag{7.65}$$

In addition, (3.19) implies

$$-\mathbf{N}_+\cdot\triangle_{\Gamma_t}\mathbf{v}_\pm = \mathbb{D}_{t\pm}\kappa_+ + 2\left\langle \mathrm{I\!I}_+ \middle| (\mathrm{D}\mathbf{v}_\pm)^\top\right\rangle = \mathbb{D}_{\bar{t}}\kappa_+ + \mathrm{D}_{(\mathbf{v}_\pm-\mathbf{u})}\kappa_+ + 2\left\langle \mathrm{I\!I}_+ \middle| (\mathrm{D}\mathbf{v}_\pm)^\top\right\rangle.$$

Thus, one has

$$\begin{aligned}&|\mathbf{N}_+\cdot\triangle_{\Gamma_t}\mathbf{v}_\pm|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)}\\ &\quad\le C_*\left(|\mathbb{D}_{\bar{t}}\kappa_+|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)} + |\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}|(\mathbf{v}_+, \mathbf{v}_-)|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}\right).\end{aligned} \tag{7.66}$$

It follows from the interpolation inequalities of Sobolev norms that

$$\begin{aligned}&\|\mathbf{v}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)}\\ &\le Q\left(|\mathbb{D}_{\bar{t}}\kappa_+|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_t)}, |\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \|\boldsymbol{\omega}_\pm\|_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)}, \left|\vec{\mathfrak{v}}_\pm\right|, \|\mathbf{v}_\pm\|_{L^2(\Omega_t^\pm)}\right),\end{aligned} \tag{7.67}$$

where Q is a generic polynomial determined by Λ_* and c_0. Similarly,

$$-\mathbf{N}_+\cdot\triangle_{\Gamma_t}\mathbf{h}_\pm = \mathrm{D}_{\mathbf{h}_\pm}\kappa_+ + 2\left\langle \mathrm{I\!I}_+ \middle| (\mathrm{D}\mathbf{h}_\pm)^\top\right\rangle \tag{7.68}$$

implies that

$$\|\mathbf{h}_\pm\|_{H^{\frac{3}{2}k}(\Omega_t^\pm)} \le Q\left(|\kappa_+|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_t)}, \|\mathbf{j}_\pm\|_{H^{\frac{3}{2}k-1}(\Omega_t^\pm)}, \left|\vec{\mathfrak{h}}_\pm\right|, \|\mathbf{h}_\pm\|_{L^2(\Omega_t^\pm)}\right). \tag{7.69}$$

In conclusion, setting

$$\mathcal{E}(t) := \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \tag{7.70}$$

then one can deduce that,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)\right| \le Q[\mathcal{E}(t)], \tag{7.71}$$

where Q is a generic polynomial depending on $\varLambda_*, \eqsim_0$ and c_0 (in particular, independent of α and t).

Thus, Theorem 2.3 follows from the above energy estimates.

Appendix A. Proof of Lemma 4.1

Proof. (4.23), (4.25), and (4.27)-(4.28) follow from (4.19)-(4.21) and Proposition 3.3. As for the variational estimates, let $f, h : \varGamma_* \to \mathbb{R}$, $\mathbf{g} : \varOmega \setminus \varGamma_* \to \mathbb{R}^3$ be given quantities, and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 : \varOmega \setminus \varGamma_t \to \mathbb{R}^3$ the solutions to the following div-curl problems respectively:

$$\begin{cases} \nabla \cdot \mathbf{w}_1 = [\nabla \cdot, \mathrm{D}_\mathrm{f}]\mathbf{v} & \text{in } \varOmega \setminus \varGamma_t, \\ \nabla \times \mathbf{w}_1 = [\nabla \times, \mathrm{D}_\mathrm{f}]\mathbf{v} + [\mathrm{D}_\mathrm{f}, \mathbb{P}]\Big[\boldsymbol{\omega}_* \circ \big(\mathfrak{X}_{\varGamma_t}\big)^{-1}\Big], & \text{in } \varOmega \setminus \varGamma_t, \\ \mathbf{N}_+ \cdot \mathbf{w}_{1\pm} = \mathbf{N}_+ \cdot \mathbf{b}_\pm(\varkappa_a, \partial_t \varkappa_a, f) & \text{on } \varGamma_t, \\ \widetilde{\mathbf{N}} \cdot \mathbf{w}_{1-} = 0 & \text{on } \partial\varOmega, \end{cases} \tag{A.1}$$

where

$$\begin{aligned} \mathbf{b}_\pm = {} & \mathrm{D}\big[\big(\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\big) \circ (\varPhi_{\varGamma_t})^{-1}\big] \cdot \big[\mathbf{v}_\pm|_{\varGamma_t} - \big(\delta\mathfrak{K}^{-1}(\varkappa_a)[\partial_t \varkappa_a]\boldsymbol{\nu}\big) \circ (\varPhi_{\varGamma_t})^{-1}\big] \\ & + \big[\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_t \varkappa_a, f]\boldsymbol{\nu}\big] \circ (\varPhi_{\varGamma_t})^{-1}, \end{aligned} \tag{A.2}$$

$$\mathrm{f}_\pm := \mathcal{H}_\pm\big[\big(\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\big) \circ (\varPhi_{\varGamma_t})^{-1}\big]; \tag{A.3}$$

$$\begin{cases} \nabla \cdot \mathbf{w}_2 = 0 & \text{in } \varOmega \setminus \varGamma_t, \\ \nabla \times \mathbf{w}_2 = \mathbb{P}\big(\mathbf{g} \circ (\mathfrak{X}_{\varGamma_t})^{-1}\big) & \text{in } \varOmega \setminus \varGamma_t, \\ \mathbf{N}_+ \cdot \mathbf{w}_{2\pm} = 0 & \text{on } \varGamma_t, \\ \widetilde{\mathbf{N}} \cdot \mathbf{w}_{2-} = 0 & \text{on } \partial\varOmega; \end{cases} \tag{A.4}$$

and

$$\begin{cases} \nabla \cdot \mathbf{w}_3 = 0 & \text{in } \varOmega \setminus \varGamma_t, \\ \nabla \times \mathbf{w}_3 = 0 & \text{in } \varOmega \setminus \varGamma_t, \\ \mathbf{N}_+ \cdot \mathbf{w}_{3\pm} = \big[\delta\mathfrak{K}^{-1}(\varkappa_a)[h]\boldsymbol{\nu}\big] \circ (\varPhi_{\varGamma_t})^{-1} \cdot \mathbf{N}_+ & \text{on } \varGamma_t, \\ \widetilde{\mathbf{N}} \cdot \mathbf{w}_{3-} = 0 & \text{on } \partial\varOmega. \end{cases} \tag{A.5}$$

Then

$$\mathbb{D}_\beta \mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3, \tag{A.6}$$

with the substitution $f = \partial_\beta \varkappa_a$, $\mathbf{g} = \partial_\beta \boldsymbol{\omega}_*$ and $h = \partial^2_{t\beta} \varkappa_a$.

By the definitions, there hold

$$\mathbf{B}_\pm(\varkappa_a)h = (\mathrm{D}\varPhi_{\varGamma_t})^{-1}\big\{\mathbf{w}_{3\pm}|_{\varGamma_t} \circ \varPhi_{\varGamma_t} - \delta\mathfrak{K}^{-1}(\varkappa_a)[h]\boldsymbol{\nu}\big\}, \tag{A.7}$$

$$\mathbf{F}_\pm(\varkappa_a)\mathbf{g} = (\mathrm{D}\varPhi_{\varGamma_t})^{-1}\big(\mathbf{w}_{2\pm}|_{\varGamma_t} \circ \varPhi_{\varGamma_t}\big), \tag{A.8}$$

and

$$\begin{aligned} \mathbf{G}_\pm(\varkappa_a, \partial_t \varkappa_a, \boldsymbol{\omega}_{*\pm})f = {} & (\mathrm{D}\varPhi_{\varGamma_t})^{-1}\big\{\mathbf{w}_{1\pm}|_{\varGamma_t} \circ (\varPhi_{\varGamma_t}) - \delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_t \varkappa_a, f]\boldsymbol{\nu}\big\} \\ & - (\mathrm{D}\varPhi_{\varGamma_t})^{-1}\big\{\mathrm{D}\big(\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\big) \cdot \mathbf{v}_{\pm *}\big\}. \end{aligned} \tag{A.9}$$

Therefore, if $\varkappa_a$ and $\boldsymbol{\omega}_*$ depend on a parameter β, then applying $\partial/\partial\beta$ to (A.7) yields that

$$\begin{aligned}&\delta\mathbf{B}_\pm(\varkappa_a)[\partial_\beta\varkappa_a]h\\ &= -(\mathrm{D}\Phi_{\Gamma_t})^{-1}\,\mathrm{D}(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})(\mathrm{D}\Phi_{\Gamma_t})^{-1}\big\{\mathbf{w}_{3\pm}|_{\Gamma_t}\circ\Phi_{\Gamma_t}-\delta\mathfrak{K}^{-1}(\varkappa_a)[h]\boldsymbol{\nu}\big\}\\ &\quad+(\mathrm{D}\Phi_{\Gamma_t})^{-1}\big\{(\mathbb{D}_\beta\mathbf{w}_{3\pm})|_{\Gamma_t}\circ\Phi_{\Gamma_t}-\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta\varkappa_a,h]\boldsymbol{\nu}\big\},\end{aligned}\tag{A.10}$$

where

$$\mathbb{D}_\beta := \partial_\beta + \mathrm{D}_{\mathfrak{H}(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\mathfrak{X}_{\Gamma_t})^{-1}}.\tag{A.11}$$

Thus, one can derive from the commutator and div-curl estimates that for $\frac{1}{2}\le s\le\frac{3}{2}k-\frac{3}{2}$,

$$\big|\delta\mathbf{B}_\pm(\varkappa_a)[\partial_\beta\varkappa_a]h\big|_{H^s(\Gamma_*)}\le C|h|_{H^{s-2}(\Gamma_*)}\cdot\big|\partial_\beta\varkappa_a\big|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)},\tag{A.12}$$

namely, (4.24) holds. (4.26) can be shown in a similar way.

The proof of (4.29) is a little more complicated. Indeed, denote by

$$Q = Q\Big(|\partial_t\varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)},\|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\Big),$$

and note that

$$\begin{aligned}&\frac{\partial}{\partial\beta}[\mathbf{G}_\pm(\varkappa_a,\partial_t\varkappa_a,\boldsymbol{\omega}_{*\pm})f]\\ &= -(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\mathrm{D}(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\mathbf{a}_\pm+(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\partial_\beta\mathbf{a}_\pm,\end{aligned}\tag{A.13}$$

where

$$\mathbf{a}_\pm := \mathbf{w}_{1\pm}|_{\Gamma_t}\circ(\Phi_{\Gamma_t})-\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_t\varkappa_a,f]\boldsymbol{\nu}-\mathrm{D}\big(\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\big)\cdot\mathbf{v}_{\pm*}.\tag{A.14}$$

It follows from (A.1), (4.16), and Proposition 3.3 that

$$\begin{aligned}|\mathbf{a}_\pm|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} &\lesssim_Q \|\mathbf{w}_{1\pm}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_t)}+|\partial_t\varkappa_a|_{H^{\frac{3}{2}k-\frac{9}{2}}(\Gamma_*)}\cdot|f|_{H^{\sigma-\frac{5}{2}}(\Gamma_*)}\\ &\quad+\|\mathbf{v}_{\pm*}\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\cdot|f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\\ &\lesssim_Q |f|_{H^{\sigma-\frac{5}{2}}(\Gamma_*)}.\end{aligned}\tag{A.15}$$

Therefore,

$$\big|(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\mathrm{D}(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\mathbf{a}_\pm\big|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)}\lesssim_Q\big|\partial_\beta\varkappa_a\big|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)}|f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}.\tag{A.16}$$

Next, by observing that

$$\begin{aligned}\partial_\beta\mathbf{a}_\pm &= (\mathbb{D}_\beta\mathbf{w}_{1\pm})|_{\Gamma_t}\circ\Phi_{\Gamma_t}-\delta^3\mathfrak{K}^{-1}(\varkappa_a)\big[\partial_\beta\varkappa_a,\partial_t\varkappa_a,f\big]\boldsymbol{\nu}\\ &\quad-\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial^2_{t\beta}\varkappa_a,f]\boldsymbol{\nu}-\mathrm{D}\big\{\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta\varkappa_a,f]\boldsymbol{\nu}\big\}\cdot\mathbf{v}_{\pm*}\\ &\quad-\mathrm{D}\big\{\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\big\}\cdot\partial_\beta\mathbf{v}_{\pm*},\end{aligned}\tag{A.17}$$

one can deduce from (4.19) that

$$\begin{aligned}&\big|\partial_\beta\mathbf{a}_\pm-(\mathbb{D}_\beta\mathbf{w}_{1\pm})|_{\Gamma_t}\circ\Phi_{\Gamma_t}\big|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)}\\ &\lesssim_Q |f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\Big(\big|\partial_\beta\varkappa_a\big|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)}+\big|\partial^2_{t\beta}\varkappa_a\big|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}+\|\partial_\beta\boldsymbol{\omega}_{*\pm}\|_{H^\sigma(\Omega_*^\pm)}\Big).\end{aligned}\tag{A.18}$$

As for the estimate of $\mathbb{D}_\beta \mathbf{w}_{1\pm}$, first note that

$$\begin{aligned}\left|\mathbb{D}_\beta \mathbf{w}_{1\pm}\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_t)} &\lesssim_Q \left\|\mathbb{D}_\beta \mathbf{w}_{1\pm}\right\|_{H^\sigma(\Omega_t^\pm)} \\ &\lesssim_Q \left\|\nabla\cdot\left(\mathbb{D}_\beta \mathbf{w}_{1\pm}\right)\right\|_{H^{\sigma-1}(\Omega_t^\pm)} + \left\|\nabla\times\left(\mathbb{D}_\beta \mathbf{w}_{1\pm}\right)\right\|_{H^{\sigma-1}(\Omega_t^\pm)} \\ &\quad + \left|\mathbf{N}_+\cdot\left(\mathbb{D}_\beta \mathbf{w}_{1\pm}\right)\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_t)}.\end{aligned} \tag{A.19}$$

The commutator estimates yield that

$$\begin{aligned}&\left\|\nabla\cdot\left(\mathbb{D}_\beta \mathbf{w}_{1\pm}\right)\right\|_{H^{\sigma-1}(\Omega_t^\pm)} + \left\|\nabla\times\left(\mathbb{D}_\beta \mathbf{w}_{1\pm}\right)\right\|_{H^{\sigma-1}(\Omega_t^\pm)} \\ &\lesssim_Q |f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\cdot\left(\left|\partial_\beta \varkappa_a\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} + \left\|\partial_\beta \boldsymbol{\omega}_{*\pm}\right\|_{H^\sigma(\Omega_*^\pm)}\right).\end{aligned} \tag{A.20}$$

For the boundary estimate of $\mathbb{D}_\beta \mathbf{w}_{1\pm}$, due to the relations that

$$\begin{aligned}&\mathbf{N}_+\cdot(\mathbb{D}_\beta \mathbf{w}_{1\pm}) - \mathbf{N}_+\cdot\mathbb{D}_\beta \mathbf{b}_\pm \\ &= \mathbf{N}_+\cdot \mathrm{D}\left[(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}\right]\cdot(\mathbf{w}_{1\pm}^\top - \mathbf{b}_\pm^\top),\end{aligned} \tag{A.21}$$

and

$$\begin{aligned}&\left|\mathbf{N}_+\cdot \mathrm{D}\left[(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\circ(\Phi_{\Gamma_t})^{-1}\right]\cdot(\mathbf{w}_{1\pm}^\top - \mathbf{b}_\pm^\top)\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_t)} \\ &\lesssim_Q |f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\cdot\left|\partial_\beta\varkappa_a\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)},\end{aligned} \tag{A.22}$$

it is direct to compute that

$$\begin{aligned}&(\mathbb{D}_\beta \mathbf{b}_\pm)\circ\Phi_{\Gamma_t} = \partial_\beta(\mathbf{b}_\pm\circ\Phi_{\Gamma_t}) \\ &= \mathrm{D}\left\{\delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta\varkappa_a, f]\boldsymbol{\nu}\right\}\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\left\{\mathbf{v}_\pm|_{\Gamma_t}\circ\Phi_{\Gamma_t} - \delta\mathfrak{K}^{-1}(\varkappa_a)[\partial_t\varkappa_a]\boldsymbol{\nu}\right\} \\ &\quad - \mathrm{D}\left\{\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\right\}\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot\mathrm{D}\left(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu}\right)\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot \\ &\qquad\cdot\left\{\mathbf{v}_\pm|_{\Gamma_t}\circ\Phi_{\Gamma_t} - \delta\mathfrak{K}^{-1}(\varkappa_a)[\partial_t\varkappa_a]\boldsymbol{\nu}\right\} \\ &\quad + \mathrm{D}\left\{\delta\mathfrak{K}^{-1}(\varkappa_a)[f]\boldsymbol{\nu}\right\}\cdot(\mathrm{D}\Phi_{\Gamma_t})^{-1}\cdot \\ &\qquad\cdot\left\{(\mathbb{D}_\beta\mathbf{v}_\pm)\circ\Phi_{\Gamma_t} - \delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta\varkappa_a, \partial_t\varkappa_a]\boldsymbol{\nu} - \delta\mathfrak{K}^{-1}(\varkappa_a)[\partial_{t\beta}^2\varkappa_a]\boldsymbol{\nu}\right\} \\ &\quad - \delta^2\mathfrak{K}^{-1}(\varkappa_a)[\partial_{t\beta}^2\varkappa_a, f]\boldsymbol{\nu} - \delta^3\mathfrak{K}^{-1}(\varkappa_a)[\partial_\beta\varkappa_a, \partial_t\varkappa_a, f]\boldsymbol{\nu}.\end{aligned} \tag{A.23}$$

Therefore, it follows from (4.19) and Proposition 3.3 that

$$\left|\partial_\beta(\mathbf{b}_\pm\circ\Phi_{\Gamma_t})\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} \lesssim_Q |f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\cdot\left(\left|\partial_\beta\varkappa_a\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} + \left|\partial_{t\beta}^2\varkappa_a\right|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\right). \tag{A.24}$$

In conclusion,

$$\begin{aligned}&\left|\frac{\partial}{\partial\beta}\left[\mathbf{G}_\pm(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_{*\pm})f\right]\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} \\ &\lesssim_Q |f|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)}\cdot\left(\left|\partial_\beta\varkappa_a\right|_{H^{\sigma-\frac{1}{2}}(\Gamma_*)} + \left|\partial_{t\beta}^2\varkappa_a\right|_{H^{\sigma-\frac{3}{2}}(\Gamma_*)} + \left\|\partial_\beta\boldsymbol{\omega}_*\right\|_{H^\sigma(\Omega\setminus\Gamma_*)}\right),\end{aligned} \tag{A.25}$$

which is exactly (4.29). □

Appendix B. Proof of Lemma 4.4

Proof. Note that

$$\mathscr{F}(\varkappa_a)\mathbf{g} = [\mathrm{I} + \mathscr{B}(\varkappa_a)]^{-1}\Big[\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot \mathbf{F}(\varkappa_a)\mathbf{g}\Big]. \tag{B.1}$$

It follows from (4.25) that

$$\begin{aligned} |\mathscr{F}(\varkappa_a)\mathbf{g}|_{H^s(\Gamma_*)} &\lesssim_{\Lambda_*} \left|\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot \mathbf{F}(\varkappa_a)\mathbf{g}\right|_{H^s(\Gamma_*)} \\ &\lesssim_{\Lambda_*} \left|\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t})\right|_{H^{\frac{3}{2}k-2}(\Gamma_*)} |\mathbf{F}(\varkappa_a)\mathbf{g}|_{H^{s+\epsilon}(\Gamma_*)} \\ &\lesssim_{\Lambda_*} |\varkappa_a|_{H^{\frac{3}{2}k-1}(\Gamma_*)} \|\mathbf{g}\|_{H^{s+\epsilon-\frac{1}{2}}(\Omega\setminus\Gamma_*)}, \end{aligned} \tag{B.2}$$

for $\frac{1}{2} \le s \le \frac{3}{2}k - 2$, which is exactly (4.63). For $k \ge 3$ and $\frac{1}{2} \le \sigma \le \frac{3}{2}k - \frac{5}{2}$, similar arguments lead to (4.65).

To estimate $\mathscr{G}$, one observes that

$$\begin{aligned} &[\mathrm{I} + \mathscr{B}(\varkappa_a)]\mathscr{G}(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*) \\ &= -a^2\frac{\mathbf{N}_+ \circ \Phi_{\Gamma_t}}{\boldsymbol{\nu} \cdot (\mathbf{N}_+ \circ \Phi_{\Gamma_t})} \cdot \Big[\vec{\mathfrak{b}} \circ \Phi_{\Gamma_t} + \mathrm{D}_{\mathbf{u}_*}\big(\mathbf{u} \circ \Phi_{\Gamma_t} + \partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\big)\Big] \\ &\quad + a^2\mathscr{C}_\alpha\gamma_{\Gamma_t} + \mathscr{B}\mathscr{C}_\alpha\varkappa_a - \nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot [\mathbf{G}\partial_t\varkappa_a] + \mathfrak{R}_1 \circ \Phi_{\Gamma_t}, \end{aligned} \tag{B.3}$$

with $\mathfrak{R}$ given by (4.52). Thus, (4.64) and (4.66) follow from (4.50), (4.43), (4.43'), (4.59), (4.59'), Lemma 4.3 and Theorem 3.8.

Next, we consider the variational estimates under the assumption that $k \ge 3$. Suppose that $\xi(\varkappa_a)$ and $\eta(\varkappa_a)$ are two functionals so that

$$\xi = (\mathrm{I} + \mathscr{B})\eta. \tag{B.4}$$

Then, when computing the variation formula, the following relation holds:

$$\frac{\partial\xi}{\partial\beta} = (\mathrm{I} + \mathscr{B})\frac{\partial\eta}{\partial\beta} + \frac{\partial}{\partial\beta}(\mathscr{B})\eta. \tag{B.5}$$

Therefore, if $\varkappa_a$ is parameterized by β, then

$$\delta\eta\,(\varkappa_a)\big[\partial_\beta\varkappa_a\big] = [\mathrm{I} + \mathscr{B}(\varkappa_a)]^{-1}\big\{\delta\xi\,(\varkappa_a)\big[\partial_\beta\varkappa_a\big] - \delta\mathscr{B}\,(\varkappa_a)\big[\partial_\beta\varkappa_a\big]\eta(\varkappa_a)\big\}, \tag{B.6}$$

where $\delta\mathscr{B}$ has the following estimate (by Lemma 4.1):

$$|\delta\mathscr{B}(\varkappa_a)|_{\mathscr{L}\Big[H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*);\mathscr{L}\big(H^{s-2}(\Gamma_*);H^s(\Gamma_*)\big)\Big]} \le C_*, \tag{B.7}$$

for $\frac{1}{2} \le s \le \frac{3}{2}k - \frac{7}{2}$ and a constant $C_* > 0$ depending on Λ_* and s.

Similarly, it follows form (B.1)-(B.5) and (4.26) that

$$\begin{aligned} &\left|\delta\{(\mathrm{I} + \mathscr{B}(\varkappa_a))\mathscr{F}(\varkappa_a)\}\,[\partial_\beta\varkappa_a]\mathbf{g}\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} \\ &\lesssim_{\Lambda_*} \left|\nabla^\top\partial_\beta(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot \mathbf{F}(\varkappa_a)\mathbf{g}\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} + \left|\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot \delta\mathbf{F}(\varkappa_a)[\partial_\beta\varkappa_a]\mathbf{g}\right|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Gamma_*)} \\ &\lesssim_{\Lambda_*} \left|\partial_\beta\varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \|\mathbf{g}\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega\setminus\Gamma_*)}, \end{aligned} \tag{B.8}$$

which, together with (B.6), (B.7), (B.1) and Lemma 4.1, yields (4.67).

To derive the variational estimate of $\mathscr{G}$, we suppose that $\varkappa_a$, $\boldsymbol{\omega}_*$, and $\mathbf{j}_*$ depend on a parameter β. With the same notations as in (4.17), the variational estimate of $\mathfrak{R}_1 \circ \Phi_{\Gamma_t}$ can be given by:

$$\left| \frac{\partial}{\partial\beta} (\mathfrak{R}_1 \circ \Phi_{\Gamma_t}) \right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \lesssim_{\Lambda_*} \left| \mathbb{D}_\beta \mathfrak{R}_1 \right|_{H^{\frac{3}{2}k-4}(\Gamma_t)}. \tag{B.9}$$

For simplicity, from now on, we shall use notations $|\cdot|_s \equiv |\cdot|_{H^s(\Gamma_t)}$, $\|\phi_\pm\|_s \equiv \|\phi_\pm\|_{H^s(\Omega_t^\pm)}$, and

$$\vec{\mathfrak{b}} := \frac{1}{\rho_+ + \rho_-}(\nabla q^+ + \nabla q^-) + \frac{\rho_+\rho_-}{(\rho_+ + \rho_-)^2} \mathrm{D}_{\mathbf{w}} \mathbf{w} - \frac{\rho_+}{\rho_+ + \rho_-} \mathrm{D}_{\mathbf{h}_+} \mathbf{h}_+ - \frac{\rho_-}{\rho_+ + \rho_-} \mathrm{D}_{\mathbf{h}_-} \mathbf{h}_-.$$

First note that

$$\left| \mathbb{D}_\beta \left(\vec{\mathfrak{b}} \cdot \triangle_{\Gamma_t} \mathbf{N}_+ \right) \right|_{\frac{3}{2}k-4} \lesssim_{\Lambda_*} |\mathbb{D}_\beta \vec{\mathfrak{b}}|_{\frac{3}{2}k-\frac{7}{2}} |\triangle_{\Gamma_t} \mathbf{N}_+|_{\frac{3}{2}k-\frac{7}{2}} + \left| \mathbb{D}_\beta \triangle_{\Gamma_t} \mathbf{N}_+ \right|_{\frac{3}{2}k-\frac{7}{2}} |\vec{\mathfrak{b}}|_{\frac{3}{2}k-\frac{7}{2}}, \tag{B.10}$$

and one can derive from Lemma 3.7 and (3.13) that

$$\left| \mathbb{D}_\beta \triangle_{\Gamma_t} \mathbf{N}_+ \right|_{\frac{3}{2}k-4} \lesssim_{\Lambda_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}. \tag{B.11}$$

As for the term $\mathbb{D}_\beta \vec{\mathfrak{b}}$, Lemma 3.7, (4.19) and (4.50) lead to

$$\begin{aligned} \left| \mathbb{D}_\beta \vec{\mathfrak{b}} \right|_{\frac{3}{2}k-4} &\lesssim_{\Lambda_*} \left| \mathbb{D}_\beta \nabla q \right|_{\frac{3}{2}k-4} + \left| \mathbb{D}_\beta \mathrm{D}_{\mathbf{h}} \mathbf{h} \right|_{\frac{3}{2}k-4} + \left| \mathbb{D}_\beta \mathrm{D}_{\mathbf{w}} \mathbf{w} \right|_{\frac{3}{2}k-4} \\ &\lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-3}(\Gamma_*)} + \left| \partial^2_{t\beta} \varkappa_a \right|_{H^{\frac{3}{2}k-5}(\Gamma_*)} + \left\| \partial_\beta \boldsymbol{\omega}_* \right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega \setminus \Gamma_*)} \\ &\quad + \left\| \partial_\beta \mathbf{j}_* \right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega \setminus \Gamma_*)}, \end{aligned} \tag{B.12}$$

where Q_* is a generic polynomial depending on Λ_* of the quantities $|\partial_t \varkappa_a|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}$, $\|\boldsymbol{\omega}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}$, and $\|\mathbf{j}_*\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}$. Next, observe that

$$\begin{aligned} \left| \mathbb{D}_\beta \left[\mathbf{N} \cdot \mathrm{D}\mathbf{u} \cdot (\mathrm{D}_{\Gamma_t})^2 \mathbf{u} \right] \right|_{\frac{3}{2}k-4} &\lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left| \mathbb{D}_\beta \mathbf{u} \right|_{\frac{3}{2}k-2} \\ &\lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left| \partial^2_{t\beta} \varkappa_a \right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left\| \partial_\beta \boldsymbol{\omega}_* \right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}, \end{aligned} \tag{B.13}$$

and

$$\left| \mathbb{D}_\beta \mathbf{I\!I}_+ \right|_{\frac{3}{2}k-4} \lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-4}(\Gamma_*)}. \tag{B.14}$$

Thus, the variational estimates of the first six terms of (4.52) follow easily. To deal with the terms involving $\triangle_{\Gamma_t} \mathbf{I\!I} - \mathfrak{R}$, one can deduce from (3.7), (4.19) and Lemma 3.7 that

$$\begin{aligned} &\left| \mathbb{D}_\beta \{ \triangle_{\Gamma_t} [\mathbf{I\!I}_+(\mathbf{w},\mathbf{w})] - \mathfrak{R}(\Gamma_t,\mathbf{w})\kappa_+ \} \right|_{\frac{3}{2}k-4} \\ &\lesssim_{Q_*} \left| \mathbb{D}_\beta \mathbf{w} \right|_{\frac{3}{2}k-2} + \left| \mathbb{D}_\beta \kappa_+ \right|_{\frac{3}{2}k-3} + \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} \\ &\lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left| \partial^2_{t\beta} \varkappa_a \right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left\| \partial_\beta \boldsymbol{\omega}_* \right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}. \end{aligned} \tag{B.15}$$

Similar arguments yield that

$$\begin{aligned} &\left| \mathbb{D}_\beta \{ \triangle_{\Gamma_t} [\mathbf{I\!I}_+(\mathbf{h}_\pm,\mathbf{h}_\pm)] - \mathfrak{R}(\Gamma_t,\mathbf{h}_\pm)\kappa_+ \} \right|_{\frac{3}{2}k-4} \\ &\lesssim_{Q_*} \left| \partial_\beta \varkappa_a \right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left\| \partial_\beta \mathbf{j}_* \right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}. \end{aligned} \tag{B.16}$$

For the last term of (4.52), one can deduce from (4.36) and Lemma 3.7 that

$$
\begin{aligned}
&\left|\mathbb{D}_\beta \triangle_{\Gamma_t} \mathfrak{r}_0\right|_{\frac{3}{2}k-4} \\
&\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\mathbb{D}_\beta \mathfrak{r}_0\right|_{\frac{3}{2}k-2} \\
&\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\mathbb{D}_\beta (g^+ - g^-)\right|_{\frac{3}{2}k-3} + \left\|\mathbb{D}_\beta \left(p_{\mathbf{v},\mathbf{v}} - p_{\mathbf{h},\mathbf{h}}\right)\right\|_{\frac{3}{2}k-\frac{1}{2}} \\
&\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left\|\mathbb{D}_\beta \mathbf{v}\right\|_{\frac{3}{2}k-\frac{3}{2}} + \left\|\mathbb{D}_\beta \mathbf{h}\right\|_{\frac{3}{2}k-\frac{3}{2}} \\
&\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial^2_{t\beta} \varkappa_a\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left\|\partial_\beta \boldsymbol{\omega}_*\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)} + \left\|\partial_\beta \mathbf{j}_*\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}.
\end{aligned}
\tag{B.17}
$$

Thus, the variational estimate of the last term of (B.3) has been obtained.

Next, for the variation of the first term on the right hand side of (B.3), observe that

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\left(\frac{\mathbf{N}_+ \circ \Phi_{\Gamma_t}}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\cdot\left[\vec{\mathfrak{b}}\circ\Phi_{\Gamma_t} + \mathrm{D}_{\mathbf{u}_*}\left(\mathbf{u}\circ\Phi_{\Gamma_t} + \partial_t\gamma_{\Gamma_t}\boldsymbol{\nu}\right)\right]\right) \\
&\quad = \frac{\partial}{\partial\beta}\left(\frac{\mathbf{N}_+ \circ \Phi_{\Gamma_t}}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\right)\cdot\left\{\vec{\mathfrak{b}}\circ\Phi_{\Gamma_t} - \mathrm{D}_{\mathbf{u}_*}\left[\left(\mathbf{u}\circ\Phi_{\Gamma_t}\right) + \left(\partial_t\gamma_{\Gamma_t}\right)\boldsymbol{\nu}\right]\right\} \\
&\qquad + \frac{\mathbf{N}_+ \circ \Phi_{\Gamma_t}}{\boldsymbol{\nu}\cdot(\mathbf{N}_+\circ\Phi_{\Gamma_t})}\cdot\frac{\partial}{\partial\beta}\left(\vec{\mathfrak{b}}\circ\Phi_{\Gamma_t} - \mathrm{D}_{\mathbf{u}_*}\left[\left(\mathbf{u}\circ\Phi_{\Gamma_t}\right) + \left(\partial_t\gamma_{\Gamma_t}\right)\boldsymbol{\nu}\right]\right).
\end{aligned}
\tag{B.18}
$$

Due to the relations that

$$
\left|\partial_\beta(\mathbf{N}_+\circ\Phi_{\Gamma_t})\right|_{\frac{3}{2}k-\frac{3}{2}} \lesssim_{\Lambda_*} \left|\mathrm{D}(\partial_\beta\gamma_{\Gamma_t}\boldsymbol{\nu})\right|_{H^{\frac{3}{2}k-\frac{3}{2}}(\Gamma_*)} \lesssim_{\Lambda_*} \left|\partial_\beta\varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \tag{B.19}
$$

and

$$
\begin{aligned}
&\left|\frac{\partial}{\partial\beta}\left(\mathrm{D}_{\mathbf{u}_*}\left[(\mathbf{u}\circ\Phi_{\Gamma_t}) + (\partial_t\gamma_{\Gamma_t})\boldsymbol{\nu}\right]\right)\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \\
&\quad = \left|\frac{\partial}{\partial\beta}\left(\mathrm{D}_{\mathbf{u}_*}\left[\mathrm{D}\Phi_{\Gamma_t}\cdot\mathbf{u}_* + 2(\partial_t\gamma_{\Gamma_t})\boldsymbol{\nu}\right]\right)\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \\
&\lesssim_{Q_*} \left|\partial_\beta\mathbf{u}_*\right|_{H^{\frac{3}{2}k-3}(\Gamma_*)} + \left|\partial^2_{t\beta}\gamma_{\Gamma_t}\right|_{H^{\frac{3}{2}k-3}(\Gamma_*)} + \left|\partial_\beta\varkappa_a\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \\
&\lesssim_{Q_*} \left|\partial_\beta\varkappa_a\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left|\partial^2_{t\beta}\varkappa_a\right|_{H^{\frac{3}{2}k-5}(\Gamma_*)} + \left\|\partial_\beta\boldsymbol{\omega}_*\right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega\setminus\Gamma_*)} + \left\|\partial_\beta\mathbf{j}_*\right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega\setminus\Gamma_*)},
\end{aligned}
\tag{B.20}
$$

the estimate of (B.18) can be deduced via (B.12).

Since $\mathbf{h}$ can be recovered from $(\varkappa_a, \mathbf{j}_*)$ by solving the div-curl problems, the operator $\mathscr{C}_\alpha$ can also be expressed as $\mathscr{C}_\alpha(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*)$. It follows form (4.55), (4.19), (4.22), Lemma 4.1 and Lemma 4.3 that for $2 \le s' \le \frac{3}{2}k - 1$, $\mathscr{V} := H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*) \times H^{\frac{3}{2}k-4}(\Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*) \times H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)$, it holds that

$$
\begin{aligned}
&\left|\delta\mathscr{C}_\alpha\left(\varkappa_a, \partial_t\varkappa_a, \boldsymbol{\omega}_*, \mathbf{j}_*\right)\right|_{\mathscr{L}[\mathscr{V};\mathscr{L}(H^{s'}(\Gamma_*);H^{s'-3}(\Gamma_*))]} \\
&\quad \le Q_*\left(\left|\partial_t\varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)}, \left\|\boldsymbol{\omega}_*\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}, \left\|\mathbf{j}_*\right\|_{H^{\frac{3}{2}k-1}(\Omega\setminus\Gamma_*)}\right).
\end{aligned}
\tag{B.21}
$$

In particular, by letting $s' := \frac{3}{2}k - 1$, one can deduce that

$$\begin{aligned}&\left|\frac{\partial}{\partial\beta}\left(\mathscr{C}_\alpha \gamma_{\Gamma_t} + \mathscr{B}\mathscr{C}_\alpha \varkappa_a\right)\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \\ &\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Gamma_*)} + \left|\partial^2_{t\beta}\varkappa_a\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} + \left\|\partial_\beta \boldsymbol{\omega}_*\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)} \\ &\quad + \left\|\partial_\beta \mathbf{j}_*\right\|_{H^{\frac{3}{2}k-\frac{5}{2}}(\Omega\setminus\Gamma_*)}.\end{aligned} \tag{B.22}$$

Furthermore, one may derive from Lemma 4.1 that

$$\begin{aligned}&\left|\frac{\partial}{\partial\beta}\left\{\nabla^\top(\kappa_+ \circ \Phi_{\Gamma_t}) \cdot [\mathbf{G}\partial_t \varkappa_a]\right\}\right|_{H^{\frac{3}{2}k-4}(\Gamma_*)} \\ &\lesssim_{Q_*} \left|\partial_\beta \varkappa_a\right|_{H^{\frac{3}{2}k-3}(\Gamma_*)} + \left|\partial^2_{t\beta}\varkappa_a\right|_{H^{\frac{3}{2}k-5}(\Gamma_*)} + \left\|\partial_\beta \boldsymbol{\omega}_*\right\|_{H^{\frac{3}{2}k-\frac{7}{2}}(\Omega\setminus\Gamma_*)}.\end{aligned} \tag{B.23}$$

In conclusion, (4.68) follows from (B.4)-(B.7), (B.3) and (B.9)-(B.23). □

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