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SHORT TIME BEHAVIOR OF LOGARITHMIC DERIVATIVES OF THE HEAT KERNEL*

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Abstract. Let M be a compact, connected Riemannian manifold. Let $p_t(x,y)$ be the fundamental solution to Cauchy initial value problem for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_M u$, where Δ_M is the Levi-Civita Laplacian. The purpose of this note is to study the asymptotic behavior of logarithmic derivatives of $\log p_t(\,\cdot\,,y)$ at x as $t \searrow 0$. In particular, we show that a dramatic change takes place when x is at the cut-locus of y.

1. Introduction. Let M be a connected, compact, d-dimensional Riemannian manifold, and use Δ_M to denote the standard (i.e., the Levi-Civita) Laplacian on $C^\infty(M;\mathbb{R})$. Next, consider the Cauchy initial value problem for the associated heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta_M u(t,x) \quad \text{with} \quad \lim_{t\searrow 0} u(t,x) = f(x).$$

By elliptic regularity theory and the strong maximum principle, there is a unique smooth function $t \in (0,\infty) \longmapsto p_t \in C^\infty\big(M\times M;(0,\infty)\big)$ such that

$$u(t,x) = \int_M f(y)p_t(x,y)\lambda_M(dy) \quad \text{for every } f \in C(M),$$

where λ_M stands for the Riemannian measure on M. In fact, because Δ_M is essentially self-adjoint in $L^2(\lambda_M)$, $p_t(x,y) = p_t(y,x)$.

In this paper we will analyze logarithmic derivatives of $p_T(\,\cdot\,,y)$, thereby extending results in [10] about the first and second order logarithmic derivatives. To be more precise, recall Varadhan's formula (cf. [17])

$$(1.1) \qquad \lim_{T\searrow 0} T\log p_T(x,y) = -\frac{\operatorname{dist}(x,y)^2}{2} \quad \text{uniformly in } (x,y)\in M\times M,$$

where $\operatorname{dist}(x,y)$ denotes the Riemannian distance between x and y. Our goal is to examine the extent to which the limit in (1.1) can be made to commute with derivatives. Obviously, because the smoothness of $\operatorname{dist}(\,\cdot\,,y)$ breaks down at the cut-locus $\operatorname{Cut}(y)$, one expects that there should be a distinction between $x \notin \operatorname{Cut}(y)$ and $x \in \operatorname{Cut}(y)$. Indeed, this suspicion was confirmed in [10] (cf. Corollary 2.28 there), where it was shown that derivatives of first and second order commute with the limit as long as $x \notin \operatorname{Cut}(y)$, but that, in general, problems occur when $x \in \operatorname{Cut}(y)$. In fact, when $x \in \operatorname{Cut}(y)$, second order logarithmic derivatives of $p_T(\,\cdot\,,y)$ at x can diverge like T^{-2} as $T \searrow 0$ (cf. Theorem 2.34 in [10]). In the present article, we will extend these results to derivatives of all orders. In particular (cf. Theorem 4.1 below), we will show that derivatives of all orders commute with the limit in (1.1) when $x \notin \operatorname{Cut}(y)$. Further, when $x \in \operatorname{Cut}(y)$, we show that, in general, nth order logarithmic derivatives of

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