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IMMERSED SURFACES IN CUBED MANIFOLDS*

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Abstract. We investigate the separability of the canonical surface immersed in cubed manifolds of non-positive curvature, partially answering the hyperbolicity question of cubed 3-manifolds. We show that the canonical surface is separable if we assume that the degrees of all edges are even. Further, the general argument also gives the same result for higher-dimensional manifolds admitting a cubing of non-positive curvature. This main result is extended to some other structures, including flying-saucer and polyhedral decompositions as well as surgeries on certain link complements; these constructions provide examples of non-Haken manifolds that are virtually Haken.

1. Introduction. Every closed PL n-manifold M^n admits a decomposition into n-dimensional cubes by further subdivision of each simplex. Endowing each cube with the geometry of a regular Euclidean cube of edge length 3 induces a length-space metric on M^n. Such a metric has non-positive curvature (in the sense of Gromov) if and only if the link of each cell, at distance 1, has no closed geodesic in the induced spherical metric of length less than 2π (this is called the *vertex condition*). Let us denote each n-cube by I^n. Throughout the paper, the term "np cubed manifold" will mean a manifold with a cubing of non-positive curvature.

Each np cubed manifold M contains a canonical immersed submanifold K^{n-1}, possibly disconnected and non-orientable, which is also an np cubed manifold, and which is totally geodesic with respect to these polyhedral metrics. K^{n-1} is obtained as the union of the n (hyper)cubes I^{n-1} within each I^n which are parallel to and equidistant from all pairs of parallel opposite faces. These decompose I^n into 2^n subcubes, which we call "corners". We will refer to K^{n-1} as the *canonical (hyper)surface* of M.

For an np cubed closed 3-manifold M^3, Mosher [Mo] has proved that the fundamental group $\pi_1(M^3)$ satisfies exactly one of the following properties:

1. $\pi_1(M^3)$ is hyperbolic in the sense of Gromov; or
2. $\pi_1(M^3)$ is toroidal, that is, there exists a free abelian subgroup of rank 2.

Cubings of 3-manifolds arise naturally from polyhedral decompositions: a generic polyhedron is an abstract polyhedron whose combinatorial type corresponds to a connected graph on a 2-sphere for which all vertices are trivalent. *Nice* generic polyhedra are those without 2-gons or 3-gons.

Consider a 3-manifold M^3 decomposed as a union of generic polyhedra. Then, the dual polyhedral decomposition is also a decomposition into generic polyhedra. These two dual cell decompositions intersect in M^3, with complementary cells being the canonical cubical decomposition from either generic polyhedral decomposition.

This cubing induces a metric of non-positive curvature if all edges of one polyhedral decomposition have degree at least 4, all faces have degree at least 4, *and* the vertex condition above is satisfied. This requires that both the polyhedral decomposition and its dual to be decompositions of M^3 into nice generic polyhedra. We will assume that the vertex condition is satisfied in the following.

Call a polyhedral decomposition *nice* if all polyhedra and dual polyhedra are nice; then, the class of nice decompositions of 3-manifolds coincides with the class of np

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