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NOTES ON UNIPOTENT CLASSES*

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0. Introduction and statement of results.

0.1. Let G be a semisimple, almost simple algebraic group over $\mathbf{C}$. Let W be the Weyl group of G. For any unipotent class C of G, let ρ_C be the irreducible representation of W attached by Springer's correspondence to the pair consisting of C and the local system $\mathbf{C}$ on C. We say that C is *special* if ρ_C is a special representation of W (that is, a representation in the class $\mathcal{S}_W$ introduced in [L1]). It is known that $C \mapsto \rho_C$ is a bijection between the set of special unipotent classes of G and the set of special representations (up to isomorphism) of W.

The special unipotent classes play a key role in several problems in representation theory, such as the classification of irreducible complex representations of a reductive group over a finite field, and the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra. Unfortunately, their definition is totally ungeometrical. For this reason, special unipotent classes are often regarded as rather mysterious objects. To partially remedy this situation, we have felt the need to try to unveil some of the purely geometrical properties of special unipotent classes, or rather, of the closely connected special pieces (defined below); this has led to the present paper.

For any special unipotent class C in G, let $\tilde{C}$ be the subset of the unipotent variety $\mathcal{U}$ of G consisting of all elements in the closure $\bar{C}$ of C which are not in the closure of any special unipotent class $C' \neq C, C' \subset \bar{C}$; this definition appeared in [Sp]. Clearly, the sets $\tilde{C}$ (for various special unipotent classes C) are irreducible,locally closed subvarieties of $\mathcal{U}$; Spaltenstein [Sp] has shown that they in fact form a partition of $\mathcal{U}$. The subvarieties $\tilde{C}$ will be called *special pieces.* Note that each special piece is a union of a special class (which is open dense in the special piece) and of a certain number (possibly zero) of non-special classes. Let $\gamma(C)$ be the set of unipotent classes that are contained in the special piece $\tilde{C}$.

One of the results of this paper is:

THEOREM 0.2. *Two unipotent classes C_1, C_2 of G belong to the same special piece if and only if ρ_{C_1}, ρ_{C_2} belong to the same two-sided cell of W.*

0.3. Now let $\mathbf{c}$ be a two-sided cell of W and let $\hat{W}_{\mathbf{c}}$ be the set of irreducible representations of W (up to isomorphism) that belong to $\mathbf{c}$. To $\mathbf{c}$ we attach, as in [L3], a finite group $\mathcal{G}_{\mathbf{c}}$ and an imbedding

(a) $\hat{W}_{\mathbf{c}} \hookrightarrow M(\mathcal{G}_{\mathbf{c}})$;

here $M(\mathcal{G}_{\mathbf{c}})$ is the set of pairs (g, τ) where g is an element of $\mathcal{G}_{\mathbf{c}}$ defined up to conjugacy and τ is an irreducible representation of the centralizer of g in $\mathcal{G}_{\mathbf{c}}$ defined up to isomorphism.

THEOREM 0.4. *Let C be a special unipotent class of G and let $\mathbf{c}$ be the two sided cell of W such that ρ_C belongs to $\mathbf{c}$. For any $C_1 \in \gamma(C)$, the image under 0.3(a) of ρ_{C_1} (which belongs to $\hat{W}_{\mathbf{c}}$ by 0.2) is of the form $(g, 1)$ where g is an element of $\mathcal{G}_{\mathbf{c}}$ defined up to conjugacy. Let $\mathcal{G}'_{\mathbf{c}}$ be the subgroup of $\mathcal{G}_{\mathbf{c}}$ generated by the conjugacy classes of the*

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