ASIAN J. MATH.
Vol. 1, No. 1, pp. 208–213, March 1997

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ON THE MOTIVE OF G AND THE PRINCIPAL HOMOMORPHISM $SL_2 \to \hat{G}$ *

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Abstract. Let k be a field, and let G be a connected reductive group over k. Let $\hat{G}$ be the Langlands dual group, which is a reductive group over $\mathbf{C}$. In [Gr] we attached a motive M of Artin-Tate type to G. In this paper, we relate M to the principal homomorphism $SL_2 \to \hat{G}$, which was introduced by de Siebenthal [deS] and Dynkin [D], and studied extensively by Kostant [K]. As a corollary, we relate the L-function of the dual motive $M^\vee$, when k is a local non-Archimedean field, to the Langlands L-function of the Steinberg representation of $G(k)$, with respect to the adjoint representation of the L-group. We also construct an involution θ of $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$ when $k = \mathbf{R}$.

1. The L-group (cf.[Ko], [S2]). Let k^s be a separable closure of k, and $\Gamma = \mathrm{Gal}(k^s/k)$. We define the root datum $\psi = (X^\bullet, \Delta^\bullet, X_\bullet, \Delta_\bullet)$ of G as a projective limit, as in Kottwitz [Ko, pg. 614]; the group Γ acts naturally on ψ.

A dual group $\hat{G}$ for G is, by definition, a reductive group over $\mathbf{C}$ which is furnished with

$$\text{a pinning } (\hat{G} \supset \hat{B} \supset \hat{T};\ \ e_\alpha : \mathbf{G}_a \simeq \hat{U}_\alpha) \tag{1.1}$$

$$\text{an isomorphism } i : \psi(\hat{T}, \hat{B}) \simeq \psi^\vee. \tag{1.2}$$

The description of a pinning, and the dual root datum $\psi^\vee$ can be found in Springer's survey, where a pinning is called a splitting [S2, pg. 10].

The dual group $\hat{G}$ exists, and is unique up to a unique isomorphism, by results of Chevalley (cf. [S2, pg. 9]). More precisely, if $\hat{G}'$ is another dual group with pinning $(\hat{G}' \supset \hat{B}' \supset \hat{T}'; e'_\alpha)$ and isomorphism $i' : \psi(\hat{T}', \hat{B}') \simeq \psi^\vee$, there is a unique pinned isomorphism $f : \hat{G} \to \hat{G}'$ such that the following diagram commutes

$$\begin{array}{rccc} i: & \psi(\hat{T}, \hat{B}) & \simeq & \psi^\vee \\ & \downarrow \psi(f) & & \| \\ i': & \psi(\hat{T}', \hat{B}') & \simeq & \psi^\vee \end{array} \tag{1.3}$$

As a consequence of (1.3), the group $\mathrm{Aut}(\psi) = \mathrm{Aut}(\psi^\vee)$ acts as pinned automorphisms of the group $\hat{G}$, as it acts on the set Isom $(\psi(\hat{T}, \hat{B}), \psi^\vee)$.

Since Γ acts on ψ, it acts as pinned automorphisms of $\hat{G}$. The L-group ${}^LG$ is defined as the semi-direct product

$${}^LG = \hat{G} \rtimes \Gamma. \tag{1.4}$$

2. The principal homomorphism (cf. [K], [Se], [S1]). Let Δ be the root basis determined by the pinning $\hat{T} \subset \hat{B} \subset \hat{G}$. For each α in Δ, we have a fixed isomorphism $e_\alpha : \mathbf{G}_a \simeq \hat{U}_\alpha$. Let

$$X_\alpha = \mathrm{Lie}(e_\alpha)(1) \quad \text{in } \mathrm{Lie}(\hat{U}_\alpha) \tag{2.1}$$

$$X = \sum_{\alpha \in \Delta} X_\alpha \qquad \text{in } \hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G}). \tag{2.2}$$

*Received March 10, 1997; accepted for publication (in revised form) March 30, 1997.

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