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A CHARACTERIZATION OF RATIONAL SINGULARITIES*

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1. Introduction. Recall that if $(R, \mathbf{m})$ is a local ring which is essentially of finite type over a field k of characteristic 0, then R is said to be of *F-injective type* (in the sense of Hochster-Roberts [HR]) if, loosely speaking, for 'almost all reductions of R modulo primes p', the Frobenius morphism acts injectively on the local cohomology modules of the reduction. If $(R, \mathbf{m})$ is normal and Cohen-Macaulay, and the top local cohomology of 'almost all' such reductions have no proper F-stable submodules, we say that R has *F-rational type* (in the sense of Huneke-Smith [HS] and Fedder-Watanabe [FW]). We will explain later in a more precise way as to what is meant by 'almost all reductions modulo primes p'; this discussion is however a little technical, and so we avoid it here.

In her thesis, Smith proved that if $(R, \mathbf{m})$ has F-rational type, then R has a *rational singularity*. Recall that this means that if $X \to \operatorname{Spec} R$ is a resolution of singularities, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, and $H^0(X, \mathcal{O}_X) = R$. She conjectured that the converse is true; this yields an intrinsic, algebraic *characterization* of rational singularities which *does not involve construction of a resolution of singularities.*

We present below a proof of the conjecture:

THEOREM 1.1. *Let $(R, \mathbf{m})$ be a local ring essentially of finite type over a field k of characteristic 0, such that R has a rational singularity. Then R has F-rational type.*

We should state right away that this result has already been obtained earlier by N. Hara [HaW], though our work was done independently, and we were not aware of Hara's work while working out our ideas.

In order to obtain the above result, we will consider below another pair of notions, related to the notions of F-injective type and F-rational type. Let R be a normal local ring, essentially of finite type over a field of characteristic 0. Let $X \to \operatorname{Spec} R$ be a resolution of singularities, such that the inverse image of the singular locus is a divisor with normal crossings. Let Z be the fibre of X over the maximal ideal of R. Since R is essentially of finite type over k, one can make sense of the following statement: for 'almost all reductions of X modulo a prime p', if $\overline{X}$ is such a reduction, and $\overline{Z} \subset \overline{X}$ is the corresponding reduction of Z, then the Frobenius morphism is injective on the local cohomology groups $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$. If this is the case, we say that $(R, \mathbf{m})$ has *resolved F-injective type.* Similarly, if $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ for $i < d = \dim X$, and $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ has no proper $\overline{R}$-submodule stable under the Frobenius (where $\overline{R}$ is the corresponding 'reduction modulo p' of R), we say that $(R, \mathbf{m})$ has *resolved F-rational type* (see §2 for a more detailed discussion of this notion).

As will be seen below (lemma 7.1), for a rational singularity, it is easy to check that these notions agree with the notions of F-injective type and F-rational type, since $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ is then identified with the top local cohomology of the corresponding local ring.

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