ASIAN J. MATH.
Vol. 1, No. 2, pp. 272–292, June 1997

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THE REGULARITY OF L^p-SCALING FUNCTIONS*

KA-SING LAU† AND MANG-FAI MA‡

Abstract. The existence of L^p-scaling function and the L^p-Lipschitz exponent have been studied in [Ji] and [LW] and a criterion is given in terms of a series of product of matrices. In this paper we make some further study of the criterion. In particular we show that for p an even integer or for all $p \geq 1$ in some special cases, the criterion can be simplified to a computationally efficient form.

1. Introduction. The solution f of a 2-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n), \quad x \in \mathbb{R} \tag{1.1}$$

is called a *scaling function*. This class of functions has been studied in detail in recent literature in connection with wavelet theory [D] and constructive approximations [DGL]. The question of existence of continuous, L^1 and L^2 solutions was treated in Daubechies [D], Daubechies and Lagarias [DL1], Collela and Heil [CH], Eirola [E], Heil [H], and Micchelli and Prautzsch [MP]. The regularity of such solutions was studied, in addition to the above papers, in Cohen and Daubechies [CD], Daubechies and Lagarias [DL2,3], Herve [He], Lau, Ma and Wang [LWM] and Villemos [V1,2]. Also the existence of L^p-solutions has been characterized by Lau and Wang in [LW] and Jia[Ji].

In this paper, we will adopt the previous notations as in [CH], [DL1] and [LW]. Let $T_0 = [c_{2i-j-1}]_{1\leq i,j\leq N}$ and $T_1 = [c_{2i-j}]_{1\leq i,j\leq N}$ be the associated matrices of the coefficients $\{c_n\}$, i.e.,

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ c_4 & c_3 & c_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_1 & c_0 & 0 & \dots & 0 \\ c_3 & c_2 & c_1 & \dots & 0 \\ c_5 & c_4 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_N \end{pmatrix}.$$

It is known that if $\sum_{n=0}^{N} c_n = 2$, then 2 is always an eigenvalue of $(T_0 + T_1)$. Furthermore, if $\sum c_{2n} = \sum c_{2n+1} = 1$, then 1 is an eigenvalue of both T_0 and T_1. Let $\mathbf{v}$ be a 2-eignevector of $(T_0 + T_1)$ (which means a right eigenvector associated with the eigenvalue 2) and let

$$\tilde{\mathbf{v}} := (T_0 - I)\mathbf{v} = (I - T_1)\mathbf{v}.$$

In [LW] the following theorem was proved:

THEOREM A. *Suppose $1 \leq p < \infty$ and $\sum_{n=0}^{N} c_n = 2$. Then equation* (1.1) *has a nonzero compactly supported L^p-solution (notation: L^p_c-solution) if and only if there exists a 2-eigenvector $\mathbf{v}$ of $(T_0 + T_1)$ satisfying*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{|J|=n} \|T_J \tilde{\mathbf{v}}\|^p = 0.$$

* Received March 20, 1997; accepted for publication June 25, 1997.

† Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (kslau@math.cuhk.edu.hk).

‡ Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260.