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ON COMPACTNESS AND COMPLETENESS OF CONFORMAL METRICS IN R^N*

CHIUN-CHUAN CHEN† AND CHANG-SHOU LIN‡

Abstract. For a smooth function $K(x)$ in $\mathbf{R}^n$, we consider the problem of finding a metric g conformal to the flat metric $|dx|^2$ such that $K(x)$ is the scalar curvature of g. Let $g = u^{\frac{4}{n-2}}|dx|^2$. Then when $n \geq 3$ the problem is equivalent to finding a positive smooth solution of the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n. \tag{0.1}$$

A solution u of (0.1) is called of slow decay if the conformal metric g can not be realized as a smooth metric on S^n. In this paper, under some conditions on K, we prove that if u is a solution of slow decay, then the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ is a complete metric in $\mathbf{R}^n$ which has bounded curvatures. As a corollary of the result, we show that under the same conditions on K as above, the Kazdan-Warner identity holds for a solution u if and only if the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ can be realized as a smooth metric on S^n. A compactness theorem is also proved.

1. Introduction. Let (M, g_0) be a Riemannian manifold of dimension n with $n \geq 3$. Given a smooth function $K(x)$ defined on M, one would like to find a metric g conformal to g_0 such that K is the scalar curvature of the new metric g. In the last several years, there have been considerable works devoted to studying this problem of prescribing scalar curvature. However, most works were only concerned with the case when M is a compact manifold; pariticularly, the standard n-dimensional unit sphere S^n. In this paper, we want to consider the case when (M, g_0) is the n-dimensional Euclidean space $\mathbf{R}^n$. When (M, g_0) is the Euclidean space and let $g = u^{\frac{4}{n-2}}g_0$ for some positive function u, then the question above is equivalent to finding positive smooth solutions of

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n \tag{1.1}$$

after an appropriate scaling, where Δ is the Laplace operator of $\mathbf{R}^n$. *In the paper, we always assume the limit $K(\infty) = \lim_{|x|\longrightarrow+\infty} K(x)$ exists and is positive.* We will discuss the case $K(\infty) = 0$ in another paper.

From the viewpoint of geometry, it is natural to ask a solution u of (1.1) such that the conformal metric $g = u^{\frac{4}{n-2}}g_0$ is a complete metric in $\mathbf{R}^n$. Following conventional notations, *a solution of (1.1) is called of slow decay if the conformal metric $u^{\frac{4}{n-2}}g_0$ can not be realized as a smooth metric on S^n.* From the viewpoint of PDE, many basic questions about solutions of equation (1.1) remain to be investigated. Three of them are listed below.

Question 1. Is every solution of equation (1.1) bounded in $\mathbf{R}^n$?

Question 2. Suppose that u is a solution of slow decay. Is the conformal metric $g = u^{\frac{4}{n-2}}g_0$ a complete metric in $\mathbf{R}^n$?

Question 3. If the conformal metric g is complete in $\mathbf{R}^n$, does g always have bounded curvatures in $\mathbf{R}^n$?

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†Department of Mathematics, National Taiwan University, Taipei, Taiwan (chchchen@math.ntu.edu.tw).

‡Department of Mathematics, National Chung Cheng University, Minghsiung, Chiayi, Taiwan (cslin@math.ccu.edu.tw).