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THE WULFF SHAPE AS THE ASYMPTOTIC LIMIT OF A GROWING CRYSTALLINE INTERFACE*

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Abstract. We present a proof of a conjecture made in the field of crystal growth. Namely, for an initial state consisting of any number of growing crystals moving outwards with normal velocity given to be $\gamma(\vec{n})$, for $\vec{n}$ the unit outwards normal, then the asymptotic growth shape is a Wulff crystal, appropriately scaled in time. This shape minimizes the surface energy, which is the surface integral of $\gamma(\vec{n})$, for a given volume. The proof works in any number of dimensions. Additionally, we develop a new approach for obtaining the Wulff shape by minimizing the surface energy divided by the enclosed volume to the $\frac{1}{d}$ power in R^d. We show that if we evolve a convex surface (not enclosing a Wulff shape) under the motion described above, that the quantity to be minimized strictly decreases to its minimum as time increases. We have thus discovered a link between this surface evolution and this (generally nonconvex) energy minimization. A generalized Huyghen's principle is obtained. Finally, given the asymptotic shape we also obtain the associated (unique) convex $\gamma(\vec{n})$. The key technical tool is the level set method and the theory and characterization of viscosity solutions to Hamilton-Jacobi equations.

1. Introduction.

The study of an anisotropic crystal growing in a melt gives rise to an equation relating the normal velocity of the motion to both the orientation of the crystal and to its curvature. In this paper we present a very simple and straightforward proof of a statement frequently made in the crystal literature – see e.g. Chernov [4], p. 215. Namely, in the case when the outwards normal velocity is equal to $\gamma(\vec{n})$, for γ the surface tension and $\vec{n}$ the unit outwards normal, then the asymptotic growth shape is precisely the celebrated Wulff crystal, appropriately scaled in time. This shape minimizes the surface energy for a given volume.

The proof, which works in any number of space dimensions, is constructive, giving asymptotic in time estimates. Moreover, the initial state can consist of any number of growing crystals, some of which may even contain holes. As time increases the individuals will merge into a growing crystal whose asymptotic limit is a single Wulff shape.

The associated energy function need not be convex. Thus singularities in the shape, i.e. jumps in the normal direction, may develop not only in time but also in the asymptotic limit. Facets and other jumps in normal direction can be characterized precisely with the help of the theory of viscosity solutions [5,6].

Numerical results using the localized level set method [21] and the high order essentially nonoscillatory approximations to Hamilton-Jacobi equations developed in [14,15] validate our theoretical results. We shall discuss this in future work with D. Peng.

In a parallel work, Osher, Merriman, Zhao and Peng, [13], have developed a connection between the static shape of crystalline materials in the plane, and the propagation of shock waves. They show that there is a precise sense in which any two dimensional crystalline form can be described in terms of rarefaction waves and contact discontinuities.

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