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ON QUANTUM COHOMOLOGY RINGS OF FANO MANIFOLDS AND A FORMULA OF VAFA AND INTRILIGATOR*

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0. Introduction. Quantum multiplications on the cohomology of symplectic manifolds were first proposed by the physicist Vafa [Va2], based on Witten's topological sigma models [Wi1]. In [RuTi], Ruan and the second named author gave a mathematical construction of quantum multiplications on cohomology groups of *positive* symplectic manifolds (cf. Chapter 1). The definition uses certain symplectic invariants, called Gromov-Witten (GW-) invariants, that were previously defined by Ruan for (semi-) positive symplectic manifolds [Ru]. A large class of such manifolds is provided by Fano manifolds (complex manifolds with ample anti-canonical bundle). Examples are low degree complete intersections and compact complex homogeneous spaces like Grassmann manifolds. If M is a Fano (or positive symplectic) manifold the quantum cohomology $QH^*_{[\omega]}(M)$ is just the cohomology space $H^*(M,\mathbb{C})$ with a (non-homogeneous) associative, graded commutative multiplication, the quantum multiplication. This multiplication depends on the choice of a (complexified) Kähler class $[\omega]$ on M. Its homogeneous part (the "weak coupling limit" $\lambda\cdot[\omega]$, $\lambda\to\infty$) is the usual cup product.

In this note we observe that quantum cohomology rings have a nice description in terms of generators and relations: If $H^*(M,\mathbb{C})=\mathbb{C}[X_1,\dots,X_N]/(f_1,\dots,f_k)$ is a presentation of the cohomology ring (for simplicity we assume $\deg X_i$ even for the moment) then $QH^*_{[\omega]}(M)=\mathbb{C}[X_1,\dots,X_N]/(f_1^{[\omega]},\dots,f_k^{[\omega]})$, where $f_1^{[\omega]},\dots,f_k^{[\omega]}$ are just the polynomials $f_1,\dots,f_k$ evaluated in the quantum ring associated to $[\omega]$ (Theorem 2.2). The $f_i^{[\omega]}$ are real-analytic in $[\omega]$ and thus have a natural analytic extension to $H^{1,1}(M)$, and the quantum cohomology rings fit together into a flat analytic family over $H^{1,1}(M)$.

As an application of this observation we shall compute the quantum cohomology of the Grassmannians. The calculations for $\mathrm{G}(k,n)$ reduce to the single quantum product $c_k\wedge_{\mathrm{Q}} s_{n-k}$ of the top non-vanishing Chern respectively Segre class of the tautological k-bundle. We derive in Chapter 3:

THEOREM 0.1. *Let S be the tautological bundle over the Grassmann manifold $\mathrm{G}(k,n)$ of complex k-planes in $\mathbb{C}^n$, $[\omega]=-\lambda\cdot c_1(S)$ a $(1,1)$-class on $\mathrm{G}(k,n)$ ($\lambda\in\mathbb{R}_{>0}\Leftrightarrow[\omega]$ Kähler). Then*

$$QH^*_{[\omega]}(\mathrm{G}(k,n))=\mathbb{C}[X_1,\dots,X_k]/(Y_{n-k+1},\dots,Y_{n-1},Y_n+(-1)^ke^{-\lambda})\,,$$

where X_i corresponds to the i-th Chern class $c_i(S)$ and the Y_j (corresponding to the j-th Segre class of S) are given recursively by ($X_i=0$ for $i>k$)

$$Y_j=-Y_{j-1}\cdot X_1-\ldots-Y_1\cdot X_{j-1}-X_j\,.$$

This is in fact the form previously derived by Vafa using arguments from Quantum Field Theory [Va1], [Va2] (but note the sign of the quantum contribution). These

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