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THREE-MANIFOLD SUBGROUP GROWTH, HOMOLOGY OF COVERINGS AND SIMPLICIAL VOLUME*

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1. Introduction. This paper is concerned with the conjecture, communicated to the first author by A. Lubotzky and A. Shalev:

CONJECTURE 1.1. *Let M be a hyperbolic three-manifold. Let $f(d)$ denote the number of subgroups of index d in $\pi_1(M)$. There exists an absolute positive constant C_1 such that, for infinitely many d, $f(d) > \exp(C_1 d)$.*
This conjecture follows easily from the following one:

CONJECTURE 1.2. *Let M be as above. For any prime p there exists infinitely many d, for which there exists a d-sheeted covering N of M such that*

$$rank_p\,(H_1(N)) > C_2 d, \tag{1}$$

where C_2 is an absolute positive constant.

Observe that for any finitely generated group G, and a subgroup H of index d, $\mathrm{rank}_p\,(H_1(H)) \le \mathrm{const}\cdot d$, so that (1) is sharp up to a constant.

A much weaker growth rate than conjectured in (1), namely, $\mathrm{rank}_p\,(H_1(N)) > (\log d)^{2-\epsilon}$ has been proved by Shalev [Sh]. It follows from the Class Tower Theorem of [R1] that $\mathrm{rank}_p\,(H_1(N)) > (\log d)^2$.

These conjectures about the subgroup growth should be compared with the results of [Tu] and [SW] concerning the word growth of $\pi_1(M)$.

Here we prove the following result for a priori a much wider class of manifolds than hyperbolic manifolds (given the present status of the hyperbolization conjecture). Recall the definition of rich fundamental groups given in [R1]:

(R) A closed irreducible three-manifold satisfies condition (R) if either
(a) the Casson invariant $\lambda(M) > \sharp($ representations of $\pi_1(M)$ in $SL_2(\mathbb{F}_5))$ or
(b) M is hyperbolic.

MAIN THEOREM 1.1. *Suppose the three-manifold M is a rational homology sphere (that is $H_1(M,\mathbb{Q}) = 0$) satisfying (R). Then for all, but at most two, primes ℓ with $\ell \equiv 3 \pmod 4$, there exists a positive α such that for infinitely many d, there exists a d-sheeted covering N of M such that either the inequality $rank_\ell\, H_1(N) > c\, d^\alpha$, or $rank_{\mathbb{Z}}\, H_1(N) > c\, d^{1/3}$, holds.*

As a corollary we have:

THEOREM 1.2 (SUBGROUP GROWTH). *Let M be as in the Main Theorem. Then for infinitely many d, $f(d) > \exp(C\, d^\alpha)$.*

Strategy of the proof. Step 1. By Theorem 9.1 of [R1], $\pi_1(M)$ admits a Zariski dense representation to $SL_2(\mathbb{C})$. We use the strong approximation of [We] to find surjective maps from $\pi_1(M)$ onto $SL_2(\mathbb{F}_q)$, where $\mathbb{F}_q$ are residue fields of an algebraic number field K.
Step 2. If ℓ is a prime, q, s are prime powers such that ℓ divides both $|SL_2(\mathbb{F}_q)|$ and $|SL_2(\mathbb{F}_s)|$, and $1 \to \pi_1(N) \to \pi_1(M) \to SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_s) \to 1$ is a Galois covering,

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