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GLUING FORMULAE FOR DONALDSON INVARIANTS FOR CONNECTED SUMS ALONG SURFACES*

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Abstract. Following our work in [18], we prove a gluing formula for the Donaldson invariants of the connected sum of two four-manifolds along surfaces of the same genus g, self-intersection zero and representing odd homology classes, solving a conjecture of Morgan and Szabó [14].

1. Introduction. This paper tries to answer the question of the behaviour of the Donaldson invariants under connected sums along surfaces of arbitrary genus $g \geq 2$ and self-intersection zero. The problem was first motivated by the computation of the basic classes of elliptic surfaces. The general elliptic surface with $b^+ > 1$ and simply connected, is constructed by the process of connected sum along embedded tori of self-intersection zero from elementary pieces (rational elliptic surfaces and homotopy $K3$-surfaces). After establishing the appropriate gluing formula for the behaviour of Donaldson invariants under the operation of connected sum along embedded tori, the information of the basic classes for some particular examples gives the basic classes of any such elliptic surface. This has been carried out by many authors [13] [3] [17] [15] [19].

The following natural case is the connected sum along embedded surfaces of genus $g = 2$. Morgan and Szabó [20] [14] treated the case when the self-intersection of one of the surfaces is 1 (and the other -1). The author [18] [17] has solved the case in which the surfaces have self-intersection zero and are odd in homology, giving a number of nice applications. For this he used the Fukaya-Floer homology as developed in [1].

The case we are going to deal with in this paper is the connected sum along embedded surfaces of higher (arbitrary) genus g, self-intersection zero and odd, giving a positive answer to the conjectures in the literature [14, conjecture 7.2] [17, conjecture 5.27].

Let X be a smooth, compact, oriented four-manifold with $b^+ > 1$ and $b^+ - b_1$ odd. For any $w \in H^2(X;\mathbb{Z})$, D^w_X will denote the corresponding Donaldson invariant [4] [12], which is defined as a linear functional on $\mathbb{A}(X) = \mathrm{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \bigwedge^* H_1(X)$ ($H_*(X)$ will always denote homology with rational coefficients, and similarly for $H^*(X)$). Let $x \in H_0(X)$ be the class of a point. Then Kronheimer and Mrowka [12] define X to be of simple type (with respect to w) when $D^w_X((x^2-4)z) = 0$ for all $z \in \mathbb{A}(X)$, and in that case define

$$\mathbb{D}^w_X(z) = D^w_X((1+\frac{x}{2})z),$$

for all $z \in \mathrm{Sym}^* H_2(X)$. The series $\mathbb{D}^w_X(e^{t\alpha})$, $\alpha \in H_2(X)$, is even or odd depending on whether

$$d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$$

is even or odd. When $b_1 = 0$ and $b^+ > 1$, X is of simple type with respect to some w if and only if it is so with respect to any w. In such case, X is just called of simple type.

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