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# AUTOMORPHIC INDUCTION AND LEOPOLDT TYPE CONJECTURES FOR $GL(N)$*

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**1. Introduction.** Let $F$ be a number field and $G_0$ be a non-split inner form of $GL(n)_{/F}$ split at a rational prime $p$ and $\infty$. We define $G = Res_{F/\mathbb{Q}}G_0$. We start with cohomological study of automorphic forms on the reductive algebraic group $G_{/\mathbb{Q}}$ which is isomorphic to a product of copies of $GL(n)$ over a number field. In particular, under not so restrictive conditions on $G$, we shall show that the nearly $p$–ordinary (cuspidal) cohomology group

$$H = H^q_{n.ord}(X_1(Np^\infty), \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_r H^q_{n.ord}(X_1(Np^r), \mathbb{Q}_p/\mathbb{Z}_p)$$

is of co-finite type over $\mathbf{\Lambda} = \mathbb{Z}_p[[T(\mathfrak{r}_p)]]$ for a maximal (split) torus $T_{/\mathbb{Z}}$ of $GL(n)_{/\mathbb{Z}}$ (Theorem 6.2), where the injective limit is taken with respect to the restriction maps, $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for the integer ring $\mathfrak{r}$ of $F$ and $\mathbf{\Lambda}$ is the completed group algebra: $\mathbf{\Lambda} = \varprojlim_n \mathbb{Z}_p[T(\mathbb{Z}/p^n\mathbb{Z})]$. Thus for a suitable $q = r$ (that is the degree, called the bottom degree; see (BD) in 6.3 for the exact formula of $r$), we will also show the finiteness over $\mathbf{\Lambda}$ of the universal $p$–adic Hecke algebra $\mathbf{h}^{n.ord}$ defined as a subalgebra of $\mathrm{End}_{\mathbf{\Lambda}}(H)$ generated by the standard Hecke operators (Corollary 6.3). Actually we prove, under some assumptions, that the Hecke algebra of weight $\chi \in X(T)$ is obtained, up to finite error, as the specialization of the universal Hecke algebra along the algebra homomorphism $\chi : \mathbf{\Lambda} \to \mathcal{O}$ induced by $\chi$ (Control theorem: Theorem 6.5).

Here we do not disregard the torsion part of finite level cohomology groups (although in my earlier works for $GL(2)$, we restricted our study to their torsion-free part), because the Hecke algebra might be too small for non totally real or non CM fields if we kill torsion of each cohomology group.

Although the algebra $\mathbf{h}^{n.ord}$ satisfies various good properties as listed above, basic invariants in ring theory, for example, its Krull dimension, are still difficult to determine in general. The structure of $\mathbf{h}^{n.ord}$ is known to our satisfaction only for $GL(1)$ and $GL(2)$ over $\mathbb{Q}$, for which $\mathbf{h}^{n.ord}$ and the Pontryagin dual $H^*$ of $H$ are projective $\mathbf{\Lambda}$–modules of finite rank. More generally, we know the expected value of the Krull dimension for $GL(1)$ over an arbitrary number field and for all inner forms of $GL(2)$ over totally real fields $F$, assuming the Leopoldt conjecture for $(F, p)$ (see the description after Conjecture 7.1 for known information). However, we can make a reasonable guess for the dimension of $\mathbf{h}^{n.ord}$ for general $G$. Here we shall make the following conjecture predicting the Krull dimension of $\mathbf{h}^{n.ord}$:

CONJECTURE 1.1. *Suppose that $G_0$ is an inner form of $GL(n)_{/F}$. Then we have*

$$\dim(\mathbf{h}^{n.ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \le \begin{cases} m[F:\mathbb{Q}] + 1 & \text{if } n = 2m, \\ m[F:\mathbb{Q}] + r_2 + 1 & \text{if } n = 2m+1, \end{cases}$$

*where $r_2$ is the number of complex places of $F$.*

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