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MOMENT MAPS AND DIFFEOMORPHISMS[†]

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Atiyah and Bott pointed out, in [1], that the curvature of a connection on a bundle over a surface can be viewed as the "momentum" corresponding to the action of the gauge group. This observation, together with various extensions, has stimulated a great deal of work and provides a conceptual framework to understand many phenomena in Yang-Mills theory. Our purpose in this paper is to explore some similar ideas in the framework of diffeomorphism groups. We begin by identifying a moment map in a rather general setting, and then see how the ideas work in some more specific situations. We hope to show that the moment map point of view is useful, both in understanding certain established results and also in suggesting new problems in geometry and analysis. While these analytical questions are the main motivation for the work, we will concentrate here on the formal aspects and will not make any serious inroads on the analysis.

1. Identification of moment maps.

1.1. Volume-preserving diffeomorphisms. Suppose a Lie group G acts on a symplectic manifold $(\mathcal{M}, \Omega)$, preserving the symplectic form. A moment, or momentum, map for the action is a map $\mu : \mathcal{M} \to \mathrm{Lie}(G)^*$ with the following property. For each element ξ in the Lie algebra $\mathrm{Lie}(G)$ the function $\langle \mu, \xi \rangle$ on $\mathcal{M}$ has derivative

$$d\langle \mu, \xi \rangle = i_{X(\xi)}(\Omega),$$

where $X(\xi)$ is the vector field on $\mathcal{M}$ defined by the infinitesimal action of ξ. This is to say that $\langle \mu, \xi \rangle$ is a Hamiltonian function for the 1-parameter subgroup generated by ξ.

Now consider the following set-up. Let S be a compact k-manifold with a fixed volume form $\sigma \in \Omega^k(S)$. Let (M, ω) be a symplectic manifold and let $\mathcal{M}$ be the infinite-dimensional space of smooth maps from S to M, in some fixed homotopy class. This may be considered as an infinite-dimensional manifold in the usual way: the tangent space to $\mathcal{M}$ at a map $f : S \to M$ is the space of sections of the bundle $f^*(TM)$ over S. (We will ignore, in this paper, any foundational questions about infinite dimensional manifolds.) Now $\mathcal{M}$ carries a natural symplectic form Ω: for sections v, w of $f^*(TM)$ we define

$$\Omega(v, w) = \int_S \omega(v, w)\sigma.$$

The Lie group $\mathcal{G}$ of volume-preserving diffeomorphisms of S acts on $\mathcal{M}$ by composition on the right, preserving Ω, so we may seek a moment map for this action. Let us suppose first that $f^*([\omega])$ is zero in the de Rham cohomology $H^2(S)$ and that $H^1(S) = 0$. Then for each $f \in \mathcal{M}$ we can make the following construction. We can, by hypothesis, choose a 1-form $a \in \Omega^1(S)$ such that $f^*(\omega) = da$. Then for any vector field ξ on S we can define a pairing

$$\langle a, \xi \rangle = \int_S a(\xi)\sigma.$$

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