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CONJECTURAL ALGEBRAIC FORMULAS FOR REPRESENTATIONS OF GL_n†

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0. Introduction.

0.0. Let F be a non-archimedean local field. Due to the recent work of Harris and Taylor [HT], we know that the local Langlands conjecture is true. In other words, for any local field F we know the existence of the one-to-one correspondence $\phi_n : \Pi_n \to GL_n(F)\widehat{}$ between the set Π_n of n-dimensional representations of the Galois group $\mathfrak{G} = \mathrm{Gal}(\overline{F}/F)$ and the set $GL_n(F)\widehat{}$ of irreducible nondegenerate representations of the group $GL_n(F)$. In particular, one can associate an irreducible representation π_χ of the group $GL_n(F)$ to a pair (E, χ), where E is a commutative semisimple algebra over F of degree n and χ is a multiplicative character of the group E^*. However, we do not know any explicit construction for the representation π_χ. In our paper we propose an explicit "algebraic" construction for the representation π_χ at least for $n = 4$.

One can inductively characterize the correspondence ϕ_n in the following way. Suppose that we know the correspondence ϕ_{n-2}. Then for any $\sigma \in \Pi_n$ we can characterize the representation $\phi_n(\sigma)$ as the unique representation of $GL_n(F)$ such that for any representation $\rho \in \Pi_{n-2}$ we have

$$\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho)) = \Gamma(\sigma \otimes \rho^*),$$

where $\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho))$ is the Gamma function of Jacquet, Piateskii–Shapiro, Shalika [JPS] and $\Gamma(\sigma \otimes \rho^*)$ is the Gamma function of Langlands. More precisely, let $GL_n(F)\widehat{}_u \subset GL_n(F)\widehat{}$ and $\Pi_{n.u} \subset \Pi_n$ be the subsets of unitary representations. We denote by Γ_n the function on the set $GL_{n-2}(F)\widehat{}_u \times \Pi_{n,u}$ defined by $\Gamma_n(\pi, \rho) = \Gamma(\pi, \phi_{n-2}(\rho))$. To any maximal torus T in $GL_n(F)$ and any character χ of T we may associate an n-dimensional representation $\sigma_\chi \in \Pi_n$ and therefore a representation $\pi_\chi = \phi_n(\sigma_\chi) \in GL_n(F)\widehat{}$. Let (ρ_{n-2}, Wh_{n-2}) be the Whittaker representation of $GL_{n-2}(F)$. The representation of $GL_{n-2} \times T$ in the space $Wh_{n-2} \otimes L^2(T)$ decomposes in the direct integral

$$Wh_{n-2} \otimes L^2(T) = \oplus \int_{GL_{n-2}(F)\widehat{}_u \times T\widehat{}} (V_\pi \otimes \chi), \tag{0.1}$$

where V_π is the space of the representation π. Let A^T_{n-2} be the unitary operator in the space $Wh_{n-2} \otimes L^2(T)$ commuting with $GL_{n-2}(F) \times T$ that in the above decompositions is the multiplication by $\Gamma_n(\pi, \pi_\chi)$. As follows from [JPS], one can write explicit formulas for all π_χ if one knows the operator A^T_{n-2}.

The goal of this paper is to propose an algebraic formula for this operator in the case $n = 4$. More precisely, for any n we construct "algebraic" data $\boldsymbol{\mu}_n = (\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$ that define an operator $A^{T_0}_{n-2}$ corresponding to the maximal split torus

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